The history of alternating sign matrices

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The birth of alternating sign matrices

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Definition

An alternating sign matrix is a square matrix consisting of 0's, 1's and (-1)'s such that, ignoring 0's, along each row and each column one reads $1, -1, 1, \dots, -1, 1$ (that is, 1's and (-1)'s alternate, and at the beginning and at the end there stands a 1).

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

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Determinants and alternating sign matrices, Adv. Math. **62** (1986), 169–184

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Let M be an $n \times n$ matrix. Denote the submatrix of M in which rows $i_1, i_2, \dots, i_k$ and columns $j_1, j_2, \dots, j_k$ are omitted by $M_{i_1,i_2,\dots,i_k}^{j_1,j_2,\dots,j_k}$. Then the (ordinary) determinant satisfies Jacobi's formula

$$\det M \cdot \det M_{1,n}^{1,n} = \det M_1^1 \cdot \det M_n^n - \det M_1^n \cdot \det M_n^1.$$

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$$\det M \cdot \det M_{1,n}^{1,n} = \det M_1^1 \cdot \det M_n^n - \det M_1^n \cdot \det M_n^1.$$

Definition (Robbins and Rumsey $\sim$ 1980)

The λ-determinant of a square matrix M is recursively defined by

$$\det_\lambda M = \frac{\det_\lambda M_1^1 \cdot \det_\lambda M_n^n + \lambda \det_\lambda M_1^n \cdot \det_\lambda M_n^1}{\det_\lambda M_{1,n}^{1,n}},$$

with initial conditions $\det_\lambda \big(()\big) = 1$ and $\det_\lambda \big((m)\big) = m$.

Theorem

For an $n \times n$ matrix M, we have

$$\det_\lambda M = \sum_A \lambda^{\operatorname{inv} A - \operatorname{neg} A}(1+\lambda)^{\operatorname{neg} A} \prod_{1 \le i,j \le n} M_{i,j}^{A_{i,j}},$$

where the sum runs over all $n \times n$ alternating sign matrices A, and $\operatorname{neg} A$ *is the number of (-1)s in A and*

$$\operatorname{inv} A = \sum_{\substack{i,j,i',j' \\ i<i',\ j>j'}} A_{i,j} A_{i',j'}.$$

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Theorem

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This is the first instance of Fomin and Zelevinsky's (later) Laurent phenomenon (2001).

The alternating sign matrix conjecture

1×1 alternating sign matrices: 1

2×2 alternating sign matrices: $\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \qquad \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$

3×3 alternating sign matrices:

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix}$$

$$\begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{matrix}$$

Let $A(n)$ denote the number of all $n \times n$ alternating sign matrices.

n	1	2	3	4	5	6
$A(n)$	1	2	7	42	429	7436

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Conjecture (Mills, Robbins, and Rumsey $\sim$ 1980)

The number of $n \times n$ alternating sign matrices equals

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

Definition (ANDREWS $\sim$ 1978)

A descending plane partition is an array π of positive integers of the form

$$\begin{array}{ccccccc}
\pi_{1,1} & \pi_{1,2} & & \cdots\cdots\cdots & & & \pi_{1,\lambda_1} \\
 & \pi_{2,2} & & \cdots\cdots & & \pi_{2,\lambda_2} & \\
 & & \ddots & & \iddots & & \\
 & & \pi_{k,k} & \cdots & \pi_{k,\lambda_k} & &
\end{array}$$

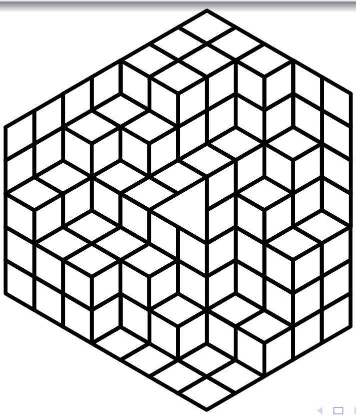
such that

1. $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k \ge k$,
2. the entries along rows are weakly decreasing,
3. the entries along columns are strictly decreasing,
4. the first entry in each row does not exceed the number of entries in the preceding row but is greater than the number of entries in its own row.

Descending plane partitions

Proposition

Descending plane partitions of order n are in bijection with rhombus tilings of a hexagon with side lengths $n-1$, $n+1$, $n-1$, $n+1$, $n-1$, $n+1$ with a triangle of size 2 removed from the centre, which are invariant under a rotation by 120°.



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Theorem (Andrews 1979)

The number of descending plane partition of a hexagon with side lengths $n-1$, $n+1$, $n-1$, $n+1$, $n-1$, $n+1$ equals

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

Why descending plane partitions?

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Andrews introduced descending plane partitions while trying to prove Macdonald's conjecture (1977) on the enumeration of cyclically symmetric plane partitions.

Plane partitions

Definition

A plane partition in an $a \times b \times c$ box can be seen as a rhombus tiling of a hexagon with side lengths a, b, c, a, b, c.

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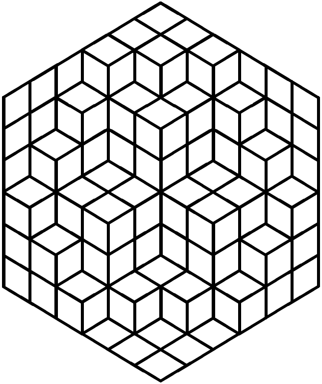

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Plane partitions can have several symmetries.

There are 10 possible ways to combine these symmetries and accordingly 10 symmetry classes of plane partitions.

Stanley proposed the programme of enumeration of the 10 symmetry classes of plane partitions.

Rather surprisingly, it turned out that for all the symmetry classes there are nice compact product formulae, already proved or conjectured.

Class 1: *Unrestricted Plane Partitions*

Theorem (MacMahon $\sim$ 1900)

The number of all plane partitions contained in an $a \times b \times c$ box is given by

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}.$$

Symmetry classes of plane partitions

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The number of all plane partitions contained in an $a \times b \times c$ box is given by

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Class 2: *Symmetric Plane Partitions*

Theorem (Andrews 1978)

The number of all symmetric plane partitions contained in an $a \times a \times c$ box is given by

$$\prod_{i=1}^{a} \frac{c+2i-1}{2i-1} \prod_{1 \le i < j \le a} \frac{c+i+j-1}{i+j-1}.$$

Class 3: *Cyclically Symmetric Plane Partitions*

Theorem (ANDREWS $\sim$ 1979)

The number of all cyclically symmetric plane partitions contained in an $a \times a \times a$ box is given by

$$\prod_{i=1}^{a} \frac{3i-1}{3i-2} \prod_{1 \le i < j \le a} \frac{2i+j-1}{2i+j-2} \prod_{1 \le i < j,k \le a} \frac{i+j+k-1}{i+j+k-2}.$$

Symmetry classes of plane partitions

Class 3: *Cyclically Symmetric Plane Partitions*

Theorem (Andrews $\sim$ 1979)

The number of all cyclically symmetric plane partitions contained in an $a \times a \times a$ box is given by

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A *q*-analogue exists which was open at the time.

Class 4: *Totally Symmetric Plane Partitions*

Conjecture (MACDONALD, ROBBINS $\sim$ 1980)

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$$\prod_{1 \le i \le j \le k \le a} \frac{i+j+k-1}{i+j+k-2}.$$

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Class 5: *Self-Complementary Plane Partitions*

Conjecture (Robbins, Stanley $\sim$ 1980)

The number $P_5(a,b,c)$ *of all self-complementary plane partitions contained in an* $a \times b \times c$ *box is given by*

$$P_5(2a,2b,2c) = P_1(a,b,c)^2,$$
$$P_5(2a+1,2b,2c) = P_1(a,b,c)P_1(a+1,b,c),$$
$$P_5(2a+1,2b+1,2c) = P_1(a+1,b,c)P_1(a,b+1,c),$$

where $P_1(a,b,c)$ *is the number of unrestricted plane partitions in an* $a \times b \times c$ *box.*

Class 6: *Transpose-Complementary Plane Partitions*

Conjecture (Stanley ∼ 1980)

The number of all transpose-complementary plane partitions contained in an $a \times a \times c$ box is given by

$$\binom{c+a-1}{a-1} \prod_{1 \le i \le j \le a-2} \frac{2c+i+j+1}{i+j+1}.$$

Class 6: *Transpose-Complementary Plane Partitions*

Conjecture (Stanley ∼ 1980)

The number of all transpose-complementary plane partitions contained in an $a \times a \times c$ box is given by

$$\binom{c+a-1}{a-1} \prod_{1 \le i \le j \le a-2} \frac{2c+i+j+1}{i+j+1}.$$

Class 7: *Symmetric Self-Complementary Plane Partitions*

Conjecture (Stanley ∼ 1980)

The number $P_7(a,b,c)$ of all symmetric self-complementary plane partitions contained in an $a \times b \times c$ box is given by

$$P_7(2a,2a,2c) = P_1(a,a,c),$$
$$P_7(2a+1,2a+1,2c) = P_1(a+1,a,c).$$

Class 8: *Cyclically Symmetric Transpose-Complementary Plane Partitions*

Conjecture (Stanley ∼ 1980)

The number of all cyclically symmetric transpose-complementary plane partitions contained in an $2a \times 2a \times 2a$ box is given by

$$\prod_{i=0}^{a-1} \frac{(3i+1)!\,(6i)!\,(2i)!}{(4i+1)!\,(4i)!}.$$

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The number of all cyclically symmetric transpose-complementary plane partitions contained in an $2a \times 2a \times 2a$ box is given by

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Class 9: *Cyclically Symmetric Self-Complementary Plane Partitions*

Conjecture (Stanley ∼ 1980)

The number of all cyclically symmetric self-complementary plane partitions contained in an $2a \times 2a \times 2a$ box is given by

$$\prod_{i=0}^{a-1} \frac{(3i+1)!^2}{(a+i)!^2}.$$

Class 10: *Totally Symmetric Self-Complementary Plane Partitions*

Conjecture (Andrews, Robbins $\sim$ 1980)

The number of all totally symmetric self–complementary plane partitions contained in an $2a \times 2a \times 2a$ box is given by

$$\prod_{i=0}^{a-1} \frac{(3i+1)!}{(a+i)!}.$$

Class 10: *Totally Symmetric Self-Complementary Plane Partitions*

Conjecture (Andrews, Robbins $\sim$ 1980)

The number of all totally symmetric self–complementary plane partitions contained in an $2a \times 2a \times 2a$ box is given by

$$\prod_{i=0}^{a-1} \frac{(3i+1)!}{(a+i)!}.$$

The alternating sign matrix numbers again!

Symmetry classes of alternating sign matrices

Symmetry classes of alternating sign matrices

If there are these theorems and conjectures for symmetry classes of plane partitions, it may be worthwhile to look at symmetry classes of alternating sign matrices, said Mills, Robbins, and Rumsey.

It turns out that, by combining the symmetries of the square, there are 8 symmetry classes of alternating sign matrices.

Symmetry classes of alternating sign matrices

Class 1: *Unrestricted Alternating Sign Matrices*

Conjecture (Robbins ∼ 1980)

The number of all $n \times n$ alternating sign matrices is given by

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Class 2: *Diagonally Symmetric Alternating Sign Matrices*

No nice product formula exists.

Symmetry classes of alternating sign matrices

Class 1: *Unrestricted Alternating Sign Matrices*

Conjecture (ROBBINS ∼ 1980)

The number of all $n \times n$ alternating sign matrices is given by

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

Class 2: *Diagonally Symmetric Alternating Sign Matrices*

No nice product formula exists.

Class 3: *Vertically Symmetric Alternating Sign Matrices*

Conjecture (MILLS ∼ 1980)

The number $A_3(n)$ of all $n \times n$ vertically symmetric alternating sign matrices satisfies $A_3(2n) = 0$ and

$$\frac{A_3(2n+1)}{A_3(2n-1)} = \frac{\binom{6n-2}{2n}}{2\binom{4n-1}{2n}}.$$

Class 4: *Half-Turn Symmetric Alternating Sign Matrices*

Conjecture (Robbins $\sim$ 1980)

The number $A_4(n)$ of all $n \times n$ half-turn symmetric alternating sign matrices satisfies

$$\frac{A_4(2n+1)}{A_4(2n)} = \frac{\binom{3n}{n}}{\binom{2n}{n}},$$

$$\frac{A_4(2n)}{A_4(2n-1)} = \frac{4\binom{3n}{n}}{3\binom{2n}{n}}.$$

Class 5: *Quarter-Turn Symmetric Alternating Sign Matrices*

Conjecture (ROBBINS ∼ 1980)

The number $A_5(n)$ of all $n \times n$ quarter-turn symmetric alternating sign matrices satisfies $A_5(4n-2) = 0$ and

$$A_5(4n) = A_4(2n) A_1^2(n),$$
$$A_5(4n+1) = A_4(2n+1) A_1^2(n),$$
$$A_5(4n-1) = A_4(2n-1) A_1^2(n),$$

where $A_1(n)$ is the number of all unrestricted $n \times n$ alternating sign matrices.

Class 6: *Diagonally and Anti-Diagonally Symmetric Alternating Sign Matrices*

Conjecture (Robbins ∼ 1980)

The number $A_6(n)$ of all $n \times n$ diagonally and anti-diagonally symmetric alternating sign matrices satisfies

$$\frac{A_6(2n+1)}{A_6(2n-1)} = \frac{\binom{3n}{n}}{\binom{2n-1}{n}}.$$

There exists no nice product formula for $A_6(n)$.

Class 7: *Vertically and Horizontally Symmetric Alternating Sign Matrices*

Conjecture (Robbins $\sim$ 1980)

The number $A_7(n)$ of all $n \times n$ vertically and horizontally symmetric alternating sign matrices satisfies

$$\frac{A_7(4n+1)}{A_7(4n-1)} = \frac{(3n-1)}{(4n-1)} \frac{\binom{6n-3}{2n-1}}{\binom{4n-2}{2n-1}},$$

$$\frac{A_7(4n+3)}{A_7(4n+1)} = \frac{(3n+1)}{(4n+1)} \frac{\binom{6n}{2n}}{\binom{4n}{2n}}.$$

There exists no nice product formula for $A_7(2n)$.

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$$\frac{A_7(4n+3)}{A_7(4n+1)} = \frac{(3n+1)}{(4n+1)} \frac{\binom{6n}{2n}}{\binom{4n}{2n}}.$$

There exists no nice product formula for $A_7(2n)$.

Class 8: *Totally Symmetric Alternating Sign Matrices*

There exists no nice product formula.

Progress

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In his attempts to prove this conjecture, Andrews had introduced a parametric family of plane partitions, let us call them μ-descending plane partitions. For $\mu = 2$ these are the descending plane partitions, and for $\mu = 0$ they are in easy bijection with cyclically symmetric plane partitions. He was able to prove a nice product formula for μ-descending plane partitions for all non-negative integers μ. However, a (nice) q-analogue seems to exist only for $\mu = 0$, that is, for cyclically symmetric plane partitions, and for $\mu = 2$, that is, for descending plane partitions.

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Mills, Robbins, and Rumsey proved both q-analogues "in one stroke."

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1994: Andrews proves the conjectured formula for Class 10 (totally self-complementary symmetric plane partitions).

1995: Stembridge proves the enumeration formula for Class 4 (totally symmetric plane partitions).

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2011: Koutschan, Kauers, and Zeilberger prove the q-analogue for Class 4.

The alternating sign matrix theorem

PROOF OF THE ALTERNATING SIGN MATRIX CONJECTURE

Doron Zeilberger [1]

Abstract: The number of $n \times n$ matrices whose entries are either -1, 0, or 1, whose row- and column- sums are all 1, and such that in every row and every column the non-zero entries alternate in sign, is proved to be $[1!4!\dots(3n-2)!]/[n!(n+1)!\dots(2n-1)!]$, as conjectured by Mills, Robbins, and Rumsey.

INTRODUCTION

The number of permutations ("houses") that can be made using n objects ("stones"), for $n \leq 7$, is given in *Sefer Yetsira* (Ch. IV, v. 12), a Cabalistic text written more than 1700 years ago. The general formula, $n!$, was stated and proved about 1000 years later by Rabbi Levi Ben Gerson ("Ralbag"). The Cabala, which is a combinatorial Theory Of Everything (both physical and spiritual), was interested in this problem because $n!$ is the number of inverse images $\tau^{-1}(w)$ of a generic $n-$lettered word w under the canonical homomorphism τ:

$$\tau : \{\aleph, \dots, tav\}^* \rightarrow \{\aleph, \dots, tav\}^* / \{\aleph(bet) = (bet)\aleph, \dots, (shin)(tav) = (tav)(shin)\} \quad .$$

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$$\tau : \{\aleph, \dots, tav\}^* \to \{\aleph, \dots, tav\}^* / \{\aleph\,(bet) = (bet)\,\aleph, \dots, (shin)\,(tav) = (tav)\,(shin)\} \quad .$$

The homomorphism τ is of considerable interest, since two words w_1, and w_2 are *temura-equivalent* (anagrams) if $\tau(w_1) = \tau(w_2)$.

A weaker, but just as important, equivalence relation on Hebrew words and sentences is the one

The alternating sign matrix theorem

Since 3 is the cardinality of the set $\{\aleph, mem, shin\}$, the "mother-letters", dear to the authors of Sefer Yetsira, they would have most likely enthusiastically approved of the generalization of permutation matrices, *alternating sign matrices*, introduced by Robbins and Rumsey[RR] in their study of a determinant-evaluation rule due to yet another wizard, the Rev. Charles Dodgson. Rather than only use the two symbols 0, 1 as entries, an alternating sign matrix is allowed the use of the three symbols $\{-1, 1, 0\}$ (corresponding to guilt, innocence, and the tongue of the law, respectively). The row- and column- sums have still to be 1, and in addition, in every row and every column, the non-zero elements, $1, -1$ (right and wrong), have to alternate.

Mills, Robbins, and Rumsey[MRR1] discovered that, like their predecessors the permutations, that are enumerated by the beautiful formula $n!$, these new mysterious objects seem to be enumerated by an almost equally simple formula:

$$A_n := \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} \quad ,$$

the now famous[R][Z3] sequence $1, 2, 7, 42, 429, \ldots$, first encountered by George Andrews[A1].

The second ingredient is my third love, *constant term* identities introduced to me by Dick Askey. Dennis Stanton[Stant] and John Stembridge[Ste] showed me how to crack them[Z4][Z5]. The *Stanton-Stembridge trick* (see below) was indeed crucial.

The third and last ingredient, which is not mentioned explicitly, but without which this proof could never have come to be, is my current love: computer algebra and Maple. Practically every lemma, sublemma, subsublemma ..., was first conjectured with the aid of, and then tested by, Maple. I thank the Maple team for creating Maple, Shalosh B. Ekhad for its diligent computations, and James C.T. Pool for his generous permission to use the Drexel computing facilities. I am also indebted to Russ de Flavia, our dedicated local Unix guru, for his constant technical support.

Finally, this paper would have been little more than a curiosity if not for George Andrews'[A2] recent brilliant proof of another conjecture of Mills, Robbins, and Rumsey[MRR3] (conj. 2 of [Stanl]), that the number of so-called Totally Symmetric, Self- Complementary Plane Partitions (TSSCPP) is also enumerated by A_n. All that I show is, that the sequence enumerating ASMs is the same as the one enumerating TSSCPPs, and then I take a free ride on Andrews'[A2] result that the later is indeed given by $1, 2, 7, 42, \ldots$.

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How To Read The Proof: The proof should be read the same way as it was conceived. First with $k = 2$, then with $k = 3$, then with $k = 4$, and finally with general k.

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How To Referee This Paper: The editor should appoint a chief referee who is responsible for checking that the sublemmas imply the lemma, and who should also appoint reliable subreferees, one for each sublemma. Now each subreferee acts recursively.

The alternating sign matrix theorem

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PROOF OF THE ALTERNATING SIGN MATRIX CONJECTURE [1]

Doron ZEILBERGER[2]

Submitted: May 1, 1995; Accepted: July 25, 1995

Checked by[3]: David Bressoud and

Gert Almkvist, Noga Alon, George Andrews, Anonymous, Dror Bar-Natan, Francois Bergeron, Nantel Bergeron, Gaurav Bhatnagar, Anders Björner, Jonathan Borwein, Mireille Bousquest-Mélou, Francesco Brenti, E. Rodney Canfield, William Chen, Chu Wenchang, Shaun Cooper, Kequan Ding, Charles Dunkl, Richard Ehrenborg, Leon Ehrenpreis, Shalosh B. Ekhad, Kimmo Eriksson, Dominique Foata, Omar Foda, Aviezri Fraenkel, Jane Friedman, Frank Garvan, George Gasper, Ron Graham, Andrew Granville, Eric Grinberg, Laurent Habsieger, Jim Haglund, Han Guo-Niu, Roger Howe, Warren Johnson, Gil Kalai, Viggo Kann, Marvin Knopp, Don Knuth, Christian Krattenthaler, Gilbert Labelle, Jacques Labelle, Jane Legrange, Pierre Leroux, Ethan Lewis, Daniel Loeb, John Majewicz, Steve Milne, John Noonan, Kathy O'Hara, Soichi Okada, Craig Orr, Sheldon Parnes, Peter Paule, Bob Proctor, Arun Ram, Marge Readdy, Amitai Regev, Jeff Remmel, Christoph Reutenauer, Bruce Reznick, Dave Robbins, Gian-Carlo Rota, Cecil Rousseau, Bruce Sagan, Bruno Salvy, Isabella Sheftel, Rodica Simion, R. Jamie Simpson, Richard Stanley, Dennis Stanton, Volker Strehl, Walt Stromquist, Bob Sulanke, X.Y. Sun, Sheila Sundaram, Raphaële Supper, Nobuki Takayama, Xavier G. Viennot, Michelle Wachs, Michael Werman, Herb Wilf, Celia Zeilberger, Hadas Zeilberger, Tamar Zeilberger, Li Zhang, Paul Zimmermann .

Dedicated to my Friend, Mentor, and Guru, Dominique Foata.

Lemma 1: For $n \geq k \geq 1$, the number of $n \times k$-Gog trapezoids equals the number of $n \times k$-Magog trapezoids.

[The number of n by k Magog trapezoids, for specific n and k, is obtained by typing `b(k,n);` while the number of n by k Gog trapezoids is given by `m(k,n);`. To verify lemma 1, type `S1(k,n):`.]

This would imply, by setting $n = k$, that,

Corollary 1': For $n \geq 1$, the number of n-Gog triangles equals the number of n-Magog triangles.

Since n-Gog triangles are equi-numerous with $n \times n$ alternating sign matrices, and n-Magog triangles are equi-numerous with TSSCPPs bounded in $[0, 2n]^3$, this would imply, together with Andrews's[A2] affirmative resolution of the TSCCPP conjecture, the following result, that was conjectured in [MRR1].

The Alternating Sign Matrix Theorem: The number of $n \times n$ alternating sign matrices, for $n \geq 1$, is:

$$\frac{1!4!\dots(3n-2)!}{n!(n+1)!\dots(2n-1)!} = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} \quad .$$

We now need the following $(sub)^6$ lemma:

Subsubsubsubsubsublemma 1.2.1.2.1.1.1: Let U_j, $j = 1, \ldots, l$, be quantities in an associative algebra, then:

$$1 - \prod_{j=1}^{l} U_j = \sum_{j=1}^{l} \left\{ \prod_{h=1}^{j-1} U_h \right\} (1 - U_j) \quad .$$

Proof of the ASM Conjecture-References

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The alternating sign matrix theorem

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The alternating sign matrix theorem

Zeilberger's method of proof is constant term identities, and he crucially uses Andrews' result on the enumeration of totally symmetric self-complementary plane partitions.

First he finds a constant-term expression for the number of $n \times n$ alternating sign matrices. (Actually, for something more general.)

Subsequently he finds a constant-term expression for the number of totally symmetric self-complementary plane partitions in a $2n \times 2n \times 2n$ box. (Actually, again for something more general.)

These two constant terms do not have the same form.

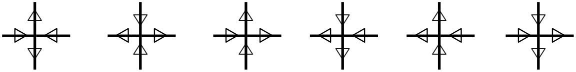
He symmetrises both constant terms.

Finally he shows that, after symmetrisation, both constant terms can be manipulated into the same expression.

The six vertex model

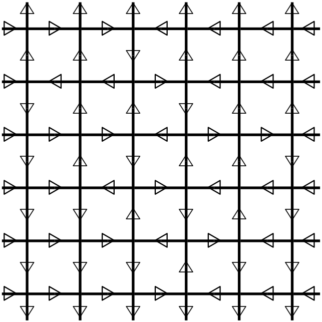
In 1996, Kuperberg presented a completely different proof of the alternating sign matrix theorem, based on a recently discovered bijection between alternating sign matrices and configurations in the six vertex model with domain wall boundary conditions.

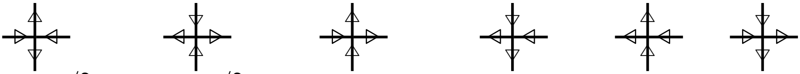
The six vertex model



$1 \qquad -1 \qquad 0 \qquad 0 \qquad 0 \qquad 0$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

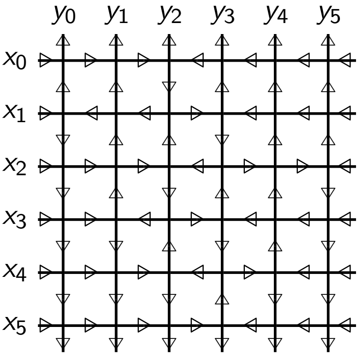


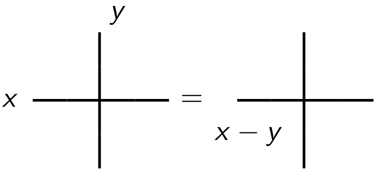


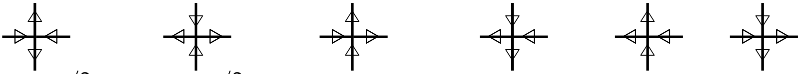
$-q^{-x/2} \qquad -q^{x/2} \qquad [x-1] \qquad [x-1] \qquad [x] \qquad [x]$

where $[x] = (q^{x/2} - q^{-x/2})/(q^{1/2} - q^{-1/2})$.

The six vertex model







$-q^{-x/2}$ $\quad -q^{x/2}$ $\quad [x-1]$ $\quad [x-1]$ $\quad [x]$ $\quad [x]$

where $[x] = (q^{x/2} - q^{-x/2})/(q^{1/2} - q^{-1/2})$.

Define $Z(x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1}) = \sum \text{weight}(C)$, where the sum is over all six vertex configurations of the $n \times n$ square.

The six vertex model

Write $Z(n; \mathbf{x}, \mathbf{y})$ for $Z(x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1})$.

Write $Z(n;\mathbf{x},\mathbf{y})$ for $Z(x_0,\dots,x_{n-1},y_0,\dots,y_{n-1})$.

Lemma

The function $Z(n;\mathbf{x},\mathbf{y})$ is symmetric in the x_i's and in the y_i's.

The proof is based on an instance of the Yang–Baxter equation (star-triangle relation).

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If $x_i = y_j + 1$ then

$$Z(n;\mathbf{x},\mathbf{y}) = -q^{-1/2}\left(\prod_{k\neq i}[x_i - y_k]\right)\left(\prod_{k\neq j}[x_k - y_j]\right) Z(n-1;\mathbf{x}\backslash x_i,\mathbf{y}\backslash y_j)$$

The six vertex model

Write $Z(n; \mathbf{x}, \mathbf{y})$ for $Z(x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1})$.

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If $x_i = y_j + 1$ then

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Lemma

The quantity $Z(n; \mathbf{x}, \mathbf{y})$ is a polynomial in q^{x_0} of degree at most $n-1$.

Theorem (Izergin 1987)

The partition function $Z(n;\mathbf{x},\mathbf{y})$ is given by

$$(-1)^n \frac{\left(\prod_{i=0}^{n-1} q^{(y_i-x_i)/2}\right)\prod_{0\le i,j<n}[x_i-y_j][x_i-y_j-1]}{\left(\prod_{0\le j<i<n}[x_i-x_j]\right)\left(\prod_{0\le i<j<n}[y_i-y_j]\right)} \det M,$$

where

$$M_{i,j}=\frac{1}{[x_i-y_j][x_i-y_j-1]}.$$

The six vertex model

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where

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Now one would like to specialise $x_i=\frac{1}{2}$ and $y_j=0$ because with this choice each configuration has the same weight.

In order to overcome the singularities in Izergin's formula for this choice, one considers instead $x_i=\frac{1}{2}+(i+1)\varepsilon$ and $y_j=-j\varepsilon$.

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The six vertex model

Now one would like to specialise $x_i = \frac{1}{2}$ and $y_j = 0$ because with this choice each configuration has the same weight.

In order to overcome the singularities in Izergin's formula for this choice, one considers instead $x_i = \frac{1}{2} + (i+1)\varepsilon$ and $y_j = -j\varepsilon$.

For this choice, the determinant can be computed by means of Cauchy's determinant evaluation

$$\det_{1\le i,j\le n}\left(\frac{1}{X_i + Y_j}\right) = \frac{\prod_{1\le i<j\le n}(X_i - X_j)(Y_i - Y_j)}{\prod_{1\le i,j\le n}(X_i + Y_j)}.$$

After this, the limit $\varepsilon \to 0$ can be safely done.

Kuperberg had seen a paper by Tsuchiya (1998) which contained a determinant formula for a class of alternating sign matrices now called "U-turn six vertx configurations". This gave him the idea that, with an appropriate choice of weights, the partition functions for the six vertex configurations corresponding to the alternating sign matrices in symmetry classes might also be expressible as a determinant, or a Pfaffian.

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In

Symmetry classes of alternating-sign matrices under one roof, Ann. Math. **156** (2002), 835–866

he carried out that programme and settled:

- Class 3 (vertically symmetric alternating sign matrices)
- Class 4, even case (half-turn symmetric alternating sign matrices)
- Class 5, even case (quarter-turn symmetric alternating sign matrices)

Symmetry classes of alternating sign matrices

2006: Okada proves the formulae for Class 7 (vertically and horizontally symmetric alternating sign matrices)

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2017: Behrend, Fischer, and Konvalinka prove the last remaining case, Class 6 (diagonally and anti-diagonally symmetric alternating sign matrices)

In 2006, Fischer developed a new approach to the enumeration of alternating sign matrices, represented in terms of monotone triangles.

The operator formula

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$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 5 & 6 & \\ 2 & 3 & 5 & 6 & & \\ 2 & 4 & 5 & & & \\ 3 & 5 & & & & \\ 4 & & & & & \end{matrix}$$

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Definition

A *monotone triangle* is an array of positive integers of the form

$$\begin{array}{ccccc} a_{11} & a_{12} & \cdots\cdots & & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} & \\ & \cdots\cdots & & & \\ a_{n1} & & & & \end{array}$$

such that entries along rows are strictly increasing, entries along columns are weakly increasing, and entries along diagonals from lower-left to upper-right are weakly increasing.

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Theorem

The number of monotone triangles with top row $(k_1, k_2, \ldots, k_n)$ is equal to

$$\prod_{1 \le s < t \le n} (id - E_{k_s} + E_{k_s} E_{k_t}) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i},$$

where $E_x p(x) := p(x+1)$.

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Using this formula, Fischer gave a new proof of the (refined) alternating sign matrix theorem, found further refinements, and also applied similar ideas to symmetry classes of alternating sign matrices. Another important aspect of this (and other) operator formulae is that they can be translated into constant term fomulae and thus build the bridge to Zeilberger's original approach.

Alternating sign triangles

Alternating sign triangles

Definition

An alternating sign triangle of order n is a triangular array of the form

$$\begin{array}{ccccccc} a_{1,1} & a_{1,2} & a_{1,3} & \dots & \dots & \dots & a_{1,2n-1} \\ & a_{2,2} & a_{2,3} & \dots & \dots & a_{2,2n-2} & \\ & & \dots & \dots & \dots & & \\ & & & a_{n,n} & & & \end{array}$$

with $a_{i,j} \in \{0, 1, -1\}$ such that:

- The non-zero entries alternate in each row and each column.
- All row sums are 1.
- The topmost non-zero entry of each column is 1.

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- The non-zero entries alternate in each row and each column.
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Theorem (Ayyer, Fischer, Behrend 2016)

The number of alternating sign triangles of order n equals

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}.$$

Bijections

Bijections

Theorem

The following objects are counted by the numbers

$$\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}:$$

1. $n \times n$ *alternating sign matrices;*
2. *descending plane partitions of order* n*;*
3. *totally symmetric self-complementary plane partitions in a* $(2n) \times (2n) \times (2n)$ *box;*
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What about bijections?

In 2019, Fischer and Konvalinka constructed an algorithmic bijection between alternating sign matrices and descending plane partitions, using "sijections".

Refinements

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Theorem (Zeilberger 1996)

The number of all $n \times n$ alternating sign matrices with the unique 1 in row 1 in the j-th column is given by

$$\frac{(j)_{n-1}\,(n-i+1)_{n-1}}{(n-1)!}\prod_{i=0}^{n-2}\frac{(3i+1)!}{(n+i)!},$$

where $(\alpha)_m = \alpha(\alpha+1)\cdots(\alpha+m-1)$ for $m \ge 1$ and $(\alpha)_0 = 1$.

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In their papers from 1983, Mills, Robbins, and Rumsey had proposed several conjectures predicting that alternating sign matrices with several statistics fixed would be counted by the same numbers as descending plane partitions with certain other statistics fixed. Their paper from 1986 contains similar conjectures for alternating sign matrices and totally symmetric self-complementary plane partitions.

Refinements

In 2012, Behrend, Di Francesco, and Zinn-Justin defined four statistics ν, μ, ρ_1, ρ_2 for alternating sign matrices and for descending plane partitions and the corresponding partition functions

$$Z_n^{\mathrm{ASM}}(x, y, z_1, z_2) = \sum_{A \in \mathrm{ASM}(n)} x^{\nu(A)} y^{\mu(A)} z_1^{\rho_1(A)} z_2^{\rho_2(A)}$$

$$Z_n^{\mathrm{DPP}}(x, y, z_1, z_2) = \sum_{D \in \mathrm{DPP}(n)} x^{\nu(D)} y^{\mu(D)} z_1^{\rho_1(D)} z_2^{\rho_2(D)}.$$

Theorem (Behrend, Di Francesco, and Zinn-Justin 2012)

For all positive integers n,

$$Z_n^{ASM}(x, y, z_1, z_2) = Z_n^{DPP}(x, y, z_1, z_2).$$

Refinements

Here, for $A \in \mathrm{ASM}(n)$,

$$\begin{aligned}
\nu(A) &= \mathrm{inv}(A),\\
\mu(A) &= \#(-1)\text{s in } A,\\
\rho_1(A) &= (\text{position of the 1 in the first row of } A) - 1,\\
\rho_2(A) &= n - (\text{position of the 1 in the last row of } A),
\end{aligned}$$

and

$$\begin{aligned}
\nu(D) &= \#\text{parts } D_{ij} \text{ in } D \text{ for which } D_{ij} > j - i,\\
\mu(D) &= \#\text{parts } D_{ij} \text{ in } D \text{ for which } D_{ij} \le j - i,\\
\rho_1(D) &= \# n\text{'s in } D,\\
\rho_2(D) &= (\#(n-1)\text{'s in } D) + (\#\text{rows of length } n-1 \text{ in } D).
\end{aligned}$$

Refinements

Many different refined enumeration results have been proved, by — in addition to already mentioned authors — Gangl, Fonseca, Höngesberg, Koutschan, Riegler, Saikia, Schreier-Aigner.

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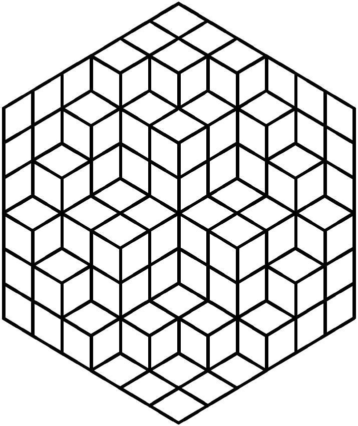
Many different refined enumeration results have been proved, by — in addition to already mentioned authors — Gangl, Fonseca, Höngesberg, Koutschan, Riegler, Saikia, Schreier-Aigner.

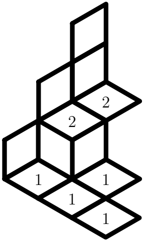
Most of the refined conjectures of Mills, Robbins, and Rumsey remain unresolved, though.

A Gog–Magog conjecture

A Gog–Magog conjecture

TSSCPPs and Magog triangles





$$\longrightarrow \quad \begin{array}{ccc} 1 & 2 & 2 \\ 1 & 1 & \\ 1 & & \end{array}$$

A Gog–Magog conjecture

Definition

An (m, n, k)-*Magog trapezoid* is an array of positive integers of the form

$$\begin{array}{cccccc} b_{11} & b_{12} & & \dots\dots\dots & & b_{1n} \\ b_{21} & b_{22} & \dots & & b_{2,n-1} & \\ & \dots\dots\dots & & & \iddots & \\ b_{k1} & \dots & b_{k,n-k+1} & & & \end{array}$$

such that entries along rows are weakly increasing, entries along columns are weakly decreasing, and such that the entries in the first row are bounded by $b_{11} \le m+1$, $b_{12} \le m+2$, $\dots$, $b_{1n} \le m+n$.

A Gog–Magog conjecture

ASMs and monotone triangles

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 5 & 6 & \\ 2 & 3 & 5 & 6 & & \\ 2 & 4 & 5 & & & \\ 3 & 5 & & & & \\ 4 & & & & & \end{matrix}$$

A Gog–Magog conjecture

Definition

An (m,n,k)-*Gog trapezoid* is an array of positive integers of the form

$$\begin{array}{ccccc} a_{11} & a_{12} & \dots & a_{1,k} & \\ a_{21} & a_{22} & \dots & a_{2,k} & \\ & \dots\dots\dots & & & \\ a_{n+1-k,1} & & \dots & a_{n+1-k,k} & \\ & \dots\dots\dots & & \iddots & \\ a_{n1} & & & & \end{array}$$

such that entries along rows are strictly increasing, entries along columns are weakly increasing, and entries along diagonals from lower-left to upper-right are weakly increasing, and such that the entries in the right-most column are bounded by $a_{1k} \le m+k$, $a_{2k} \le m+k+1$, $\dots$, $a_{n+1-k,k} \le m+n$.

A Gog–Magog conjecture

A Gog–Magog conjecture

Conjecture

The number of (m,n,k)*-Magog trapezoids with* s *Maxima in the first row and* t *Minima in the last row equals the number of* (m,n,k)*-Gog trapezoids with* t *Maxima in the right-most column and* s *Minima in the left-most column. Here, a Maximum is an entry that is equal to its upper bound, whereas a Minimum is an entry that is* 1*.*

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(3) For $k = 1$: we want to show that the number of sequences of positive integers $(a_1, a_2, \ldots, a_n)$ with $a_k \leq m + k$ and with s Maxima and t Minima is exactly the same as the number of arrays of the same type but with t Maxima and s Minima.

A Gog–Magog conjecture

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Start with the sequence. At each step look for the left-most entry with minimal difference to its corresponding upper-bound (i.e., find k minimal such that $(m + k) - a_k$ is minimal). Remove it (i.e., remove a_k), and record this difference $+ 1$ (i.e., record $(m + k) - a_k + 1$).

A Gog–Magog conjecture

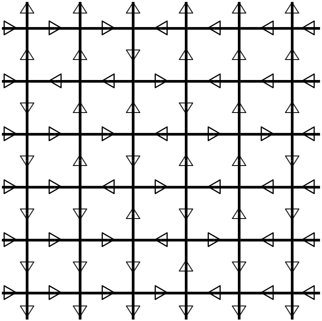
Start with the sequence. At each step look for the left-most entry with minimal difference to its corresponding upper-bound (i.e., find k minimal such that $(m+k)-a_k$ is minimal). Remove it (i.e., remove a_k), and record this difference $+\ 1$ (i.e., record $(m+k)-a_k+1$).

EXAMPLE. $n=6$, $m=1$, $s=2$, $t=1$: The first line indicates the upper bounds. The second line is the sequence with which we start. The subsequent lines show the rest of the sequence after each step. In the right-most column the difference $(m+k)-a_k$ is displayed.

$$
\begin{array}{cccccc|c}
2 & 3 & 4 & 5 & 6 & 7 & \\
\hline
1 & 3 & 3 & 5 & 5 & 6 & 1 \\
 & 1 & 3 & 5 & 5 & 6 & 1 \\
 & & 1 & 3 & 5 & 6 & 2 \\
 & & & 1 & 3 & 6 & 2 \\
 & & & & 1 & 3 & 5 \\
 & & & & & 1 & 7
\end{array}
$$

The "summary"

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$



1	2	3	4	5	6
1	2	4	5	6	
2	3	5	6		
2	4	5			
3	5				
4					