Domino tilings of generalised Aztec triangles

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CK: Evaluate a determinant.

. . . .
SC: *Di Francesco's conjectured determinant evaluation would follow as a special case.*

Di Francesco's determinant

In 2021, in the context of counting certain configurations in the 20-vertex model, Di Francesco came up with the following conjecture:

Conjecture

For all positive integers n, we have

$$\det_{0\le i,j\le n-1}\left(2^i\binom{i+2j+1}{2j+1}+\binom{-i+2j+1}{2j+1}\right) = 2^{\binom{n}{2}+1}\prod_{i=0}^{n-1}\frac{(4i+2)!}{(n+2i+1)!}.$$

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More precisely, Di Francesco observed that the number of domino tilings of certain regions that he called AZTEC TRIANGLES is the same as the number of these 20-vertex configurations. He showed that the number of domino tilings is given by one half of the above determinant.

Di Francesco's determinant

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Di Francesco's determinant, plus a generalisation

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$$\det_{0\le i,j\le n-1}\left(2^i\binom{x+i+2j+1}{2j+1}+\binom{x-i+2j+1}{2j+1}\right)$$
$$=2^{\binom{n}{2}+1}\prod_{i=0}^{n-1}\frac{i!}{(2i+1)!}\prod_{i=0}^{\lfloor n/2\rfloor}(x+4i+1)_{n-2i}\prod_{i=0}^{\lfloor (n-1)/2\rfloor}(x-2i+3n)_{n-2i-1},$$

where $(\alpha)_m:=\alpha(\alpha+1)\cdots(\alpha+m-1)$ *for* $m\ge 1$, *and* $(\alpha)_0:=1$.

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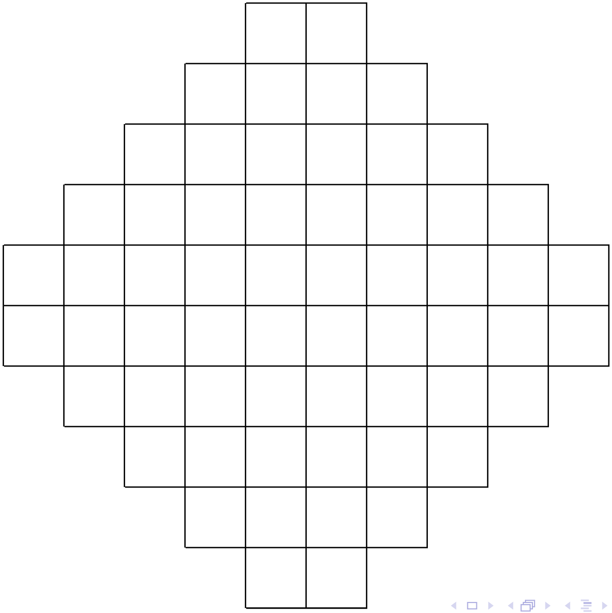
$$\det_{0\le i,j\le n-1}\left(2^i\binom{x+i+2j+1}{2j+1}+\binom{x-i+2j+1}{2j+1}\right)$$
$$=2^{\binom n2+1}\prod_{i=0}^{n-1}\frac{i!}{(2i+1)!}\prod_{i=0}^{\lfloor n/2\rfloor}(x+4i+1)_{n-2i}\prod_{i=0}^{\lfloor (n-1)/2\rfloor}(x-2i+3n)_{n-2i-1},$$

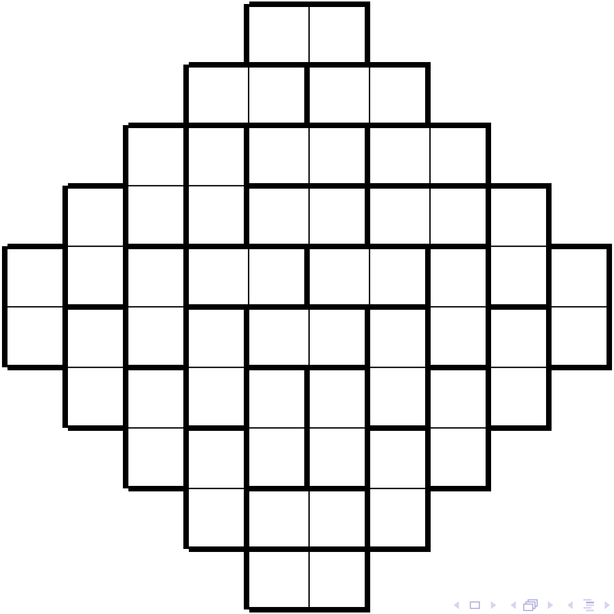
where $(\alpha)_m := \alpha(\alpha+1)\cdots(\alpha+m-1)$ *for* $m\ge 1$*, and* $(\alpha)_0 := 1$*.*

Another CK (Christoph Koutschan) proved Di Francesco's determinant evaluation using Zeilberger's holonomic Ansatz (and heavy computer calculations).

The Aztec diamond



Domino tilings of the Aztec diamond



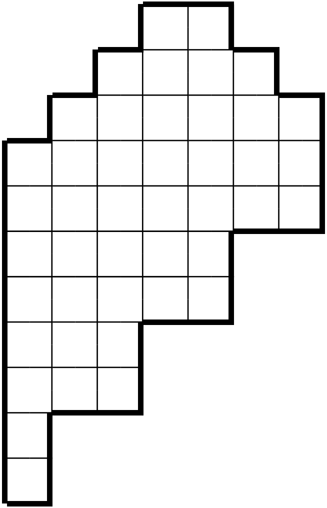
The Aztec diamond theorem

Theorem (Elkies, Kuperberg, Larsen, Propp 1992)

The number of domino tilings of the Aztec diamond of size n is

$$2^{\binom{n+1}{2}}.$$

The Aztec triangle of Di Francesco



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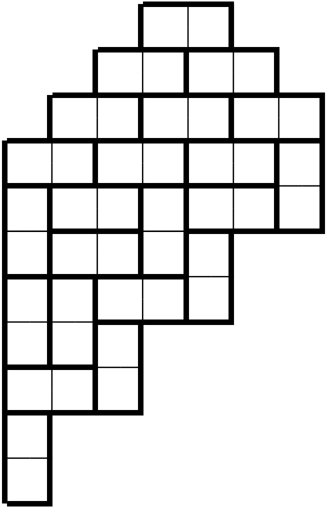


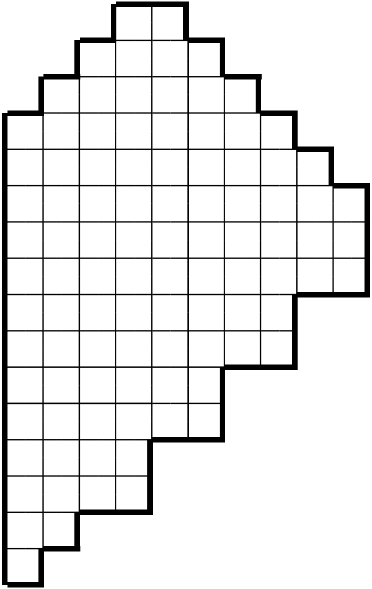
The Aztec triangle of Di Francesco

Theorem (Di Francesco + Koutschan)

The number of domino tilings of the Aztec triangle of size n is

$$2^{\binom{n}{2}} \prod_{i=0}^{n-1} \frac{(4i+2)!}{(n+2i+1)!}.$$

A generalised Aztec triangle



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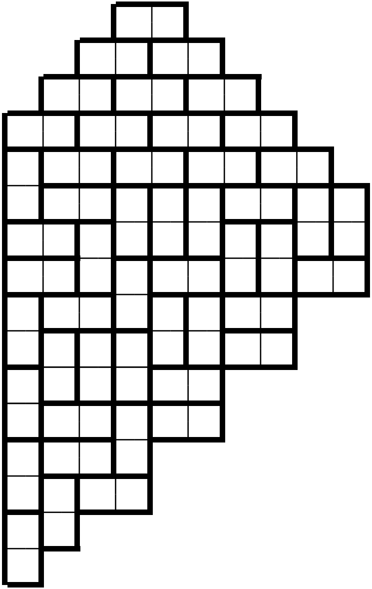
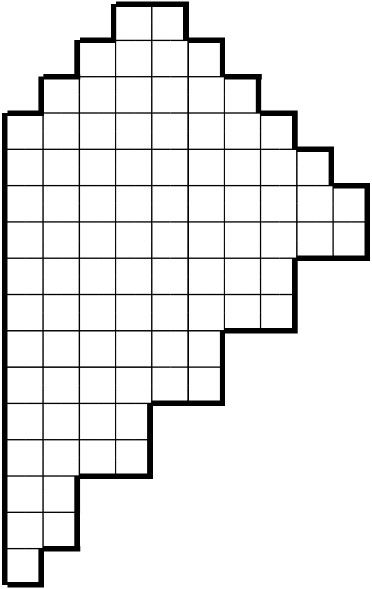


Another generalised Aztec triangle



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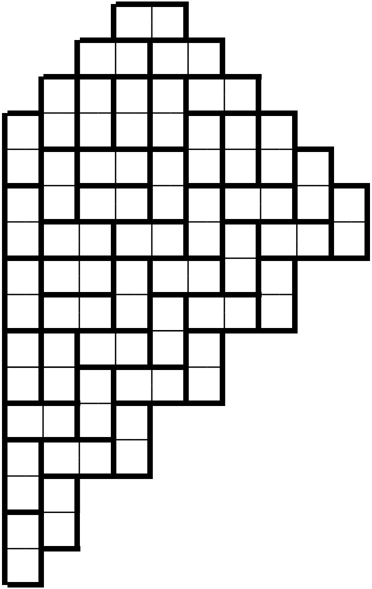

Enumeration of generalised Aztec triangles

Conjecture

The number of domino tilings of the (n,k)-Aztec triangle of type I is

$$\prod_{i\ge 0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s)\prod_{s=k-2i}^{2k-4i-2}(2n+s)\right)\Bigg/\prod_{i=1}^{k-1}(2i+1)^{k-i}.$$

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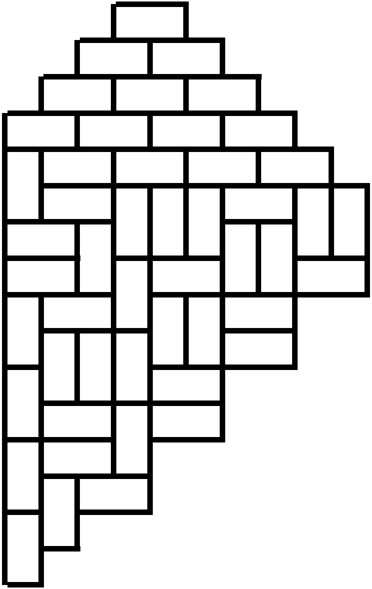
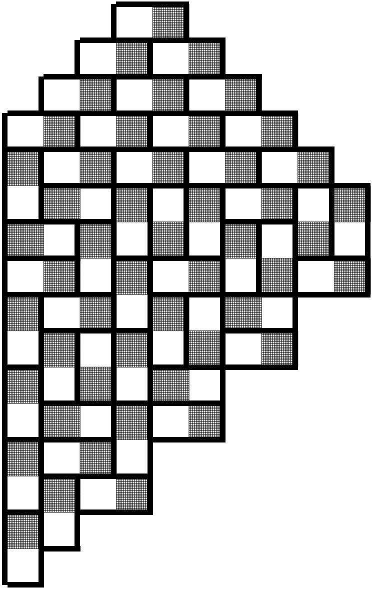
Conjecture

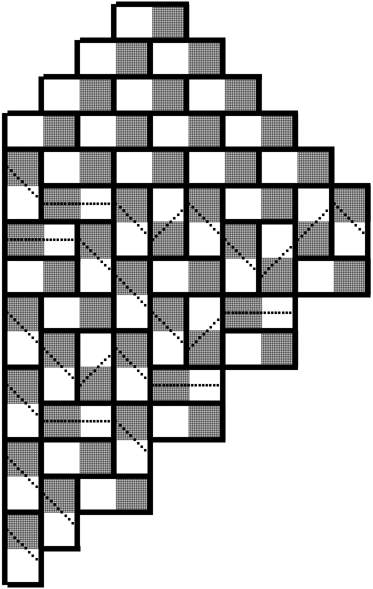
The number of domino tilings of the (n,k)-Aztec triangle of type II is

$$\prod_{i\ge 0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s+1)\prod_{s=k-2i}^{2k-4i-2}(2n+s+1)\right)\Big/\prod_{i=1}^{k-1}(2i+1)^{k-i}.$$



Generalised Aztec triangles and non-intersecting paths





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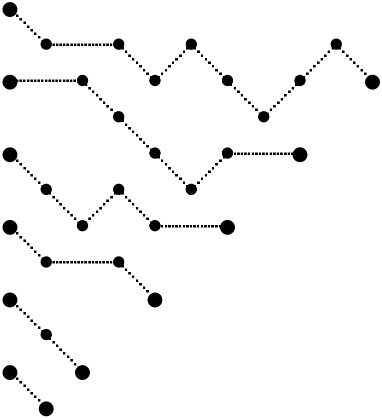

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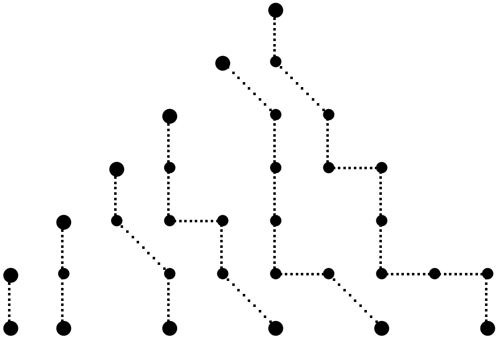

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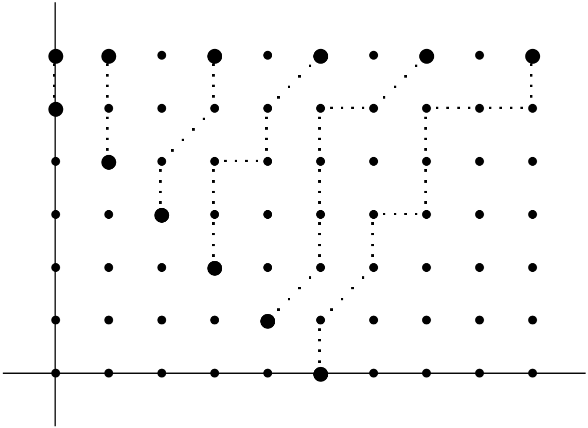

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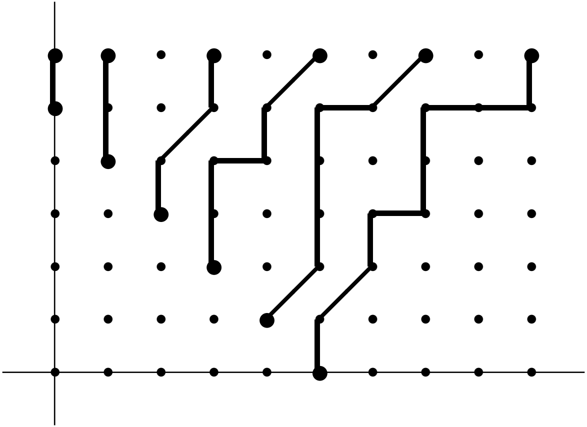
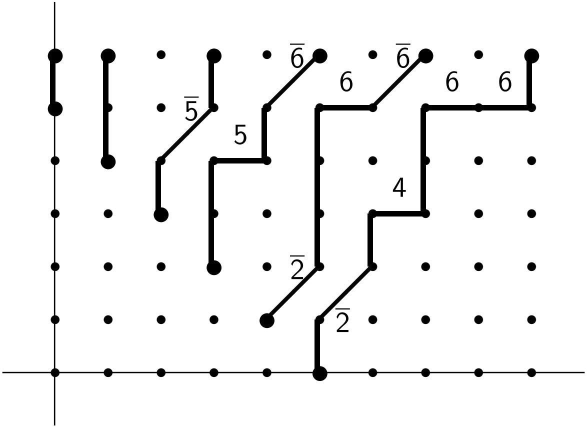
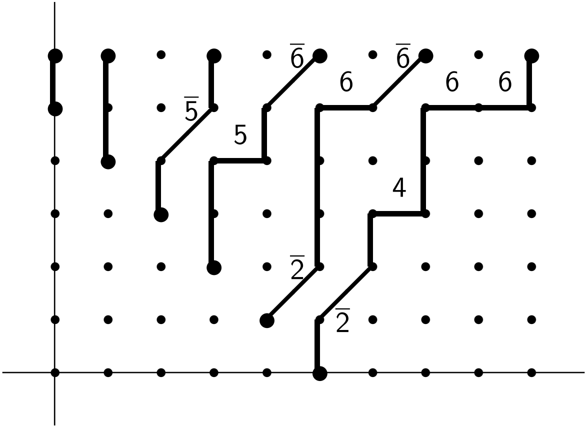


Generalised Aztec triangles and super symplectic tableaux



Generalised Aztec triangles and super symplectic tableaux

$\overline{2}$	4	6	6
$\overline{2}$	6	$\overline{6}$	
5	$\overline{6}$		
$\overline{5}$			



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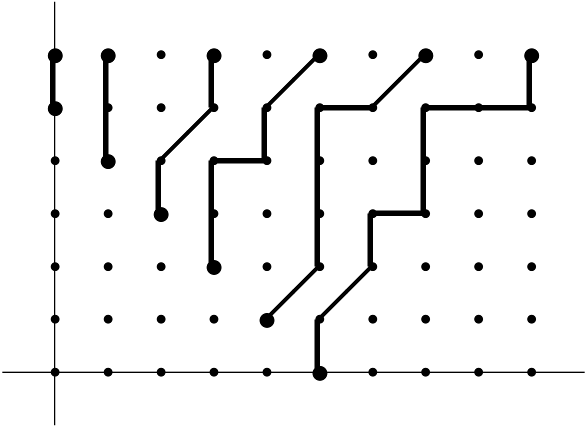

Hence:
The number of domino tilings of the (n, k)-Aztec triangle of type I equals $\det D_1(k; n)$, *where*

$$D_1(k; n) = \left(D(2j - i, i + n - k - 1)\right)_{1 \le i,j \le k},$$

with $D(m, n)$ a Delannoy number, i.e., the number of paths from $(0, 0)$ to (m, n) consisting of steps $(1, 0)$, $(0, 1)$, and $(1, 1)$.

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Furthermore, the number of domino tilings of the (n,k)*-Aztec triangle of type II equals* $\det D_2(k;n)$*, where*

$$D_2(k;n) = \left(D(2j-i, i+n-k-1) + D(2j-i-1, i+n-k-1)\right)_{1\le i,j\le k}.$$

A determinant evaluation

We need to show:

$$\det\left(D(2j-i,i+n-k-1)\right)_{1\le i,j\le k}$$
$$=\prod_{i\ge0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s)\prod_{s=k-2i}^{2k-4i-2}(2n+s)\right)\Bigg/\prod_{i=1}^{k-1}(2i+1)^{k-i},$$

and also:

$$\det\left(D(2j-i,i+n-k-1)+D(2j-i-1,i+n-k-1)\right)_{1\le i,j\le k}$$
$$=\prod_{i\ge0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s+1)\prod_{s=k-2i}^{2k-4i-2}(2n+s+1)\right)\Bigg/\prod_{i=1}^{k-1}(2i+1)^{k-i}.$$

The Delannoy numbers

We have

$$D(m,n) = \langle u^m v^n \rangle \frac{1}{1-u-v-uv}$$
$$= \sum_{\ell=0}^{m} \binom{m}{\ell} \binom{n}{\ell} 2^{\ell}$$
$$= \sum_{\ell=0}^{m} \binom{m+n-\ell}{m-\ell,\, n-\ell,\, \ell}.$$

In particular, $D(m,n)$ is a polynomial in n of degree m.

Determinant evaluations: Identification of factors

A short proof of the Vandermonde determinant evaluation

$$\det\left(X_i^{j-1}\right)_{1\le i,j\le n}=\prod_{1\le i<j\le n}(X_j-X_i).$$

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PROOF.

- If $X_{i_1} = X_{i_2}$ with $i_1 \neq i_2$, then the determinant vanishes. Hence,

$$\prod_{1\le i<j\le n}(X_j - X_i) \text{ divides } \det_{1\le i,j\le n}\left(X_i^{j-1}\right)$$

as a polynomial in $X_1, X_2, \dots, X_n$.

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- The degree of the product is $\binom{n}{2}$.

The degree of the determinant is at most $\binom{n}{2}$.

Consequently,

$$\det_{1\le i,j\le n}\left(X_i^{j-1}\right) = \text{const.} \times \prod_{1\le i<j\le n}(X_j - X_i).$$

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- One can compute the constant by comparing coefficients of

What are the essential steps?

Determinant evaluations: Identification of factors

What are the essential steps?

(1) Identification of factors
(2) Comparison of degrees
(3) Evaluation of the constant

Objection: This works because there are so many (to be precise: n) variables at our disposal.

What, if there is, say, only one *variable* μ, *and you want to prove that* $(\mu + a)^E$ *divides the determinant*?

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What, if there is, say, only one variable μ, and you want to prove that $(\mu + a)^E$ divides the determinant?

Important fact:

> *For proving that $(\mu + a)^E$ divides the determinant, we put $\mu = -a$ in the matrix and find E linearly independent vectors in the kernel of the matrix.*

Our determinant

Here is our wanted determinant evaluation, rewritten:

$$\det D_1(k;n) = \det\big(D(k-2i+j,n-j-1)\big)_{0\le i,j\le k-1}$$

$$= 2^{k^2}\prod_{i=1}^{k}\frac{1}{(i)_i}\prod_{s=0}^{k-2}(n-s-1)^{\min\{\lfloor(s+1+\chi(k\text{ even}))/2\rfloor,\lfloor(k-s)/2\rfloor\}}$$

$$\cdot\prod_{s=0}^{k-1}(n-s-\tfrac{1}{2})^{\min\{\lfloor(s+1+\chi(k\text{ odd}))/2\rfloor,\lfloor(k-s+1)/2\rfloor\}}$$

$$\cdot\prod_{s=0}^{k-2}(n+k-s-1)^{\min\{\lfloor(s+2)/2\rfloor,\lfloor(k-s-\chi(k\text{ odd}))/2\rfloor\}}$$

$$\cdot\prod_{s=1}^{k-2}(n+k-s-\tfrac{1}{2})^{\min\{\lfloor(s+1)/2\rfloor,\lfloor(k-s-\chi(k\text{ even}))/2\rfloor\}}.$$

Here, $\chi(\mathcal{S})=1$ if $\mathcal{S}$ is true and $\chi(\mathcal{S})=0$ otherwise.

Our determinant

STEP 1: *The term* $(n-s-1)^{\min\{\lfloor (s+1+\chi(k \text{ even}))/2\rfloor, \lfloor (k-s)/2\rfloor\}}$ *is a factor of* $\det D_1(k;n)$.

Our determinant

STEP 1: *The term* $(n-s-1)^{\min\{\lfloor (s+1+\chi(k\text{ even}))/2\rfloor,\lfloor (k-s)/2\rfloor\}}$ *is a factor of* $\det D_1(k;n)$.

Claim: For $0\le a\le s$, $k\ge 2s-a+2$, and $k\equiv a \pmod 2$, we have

$$\sum_{j=a}^{2s-a+1}\binom{2s-2a+1}{j-a}\cdot\big(\text{column } j \text{ of } D_1(k;s+1)\big)=0.$$

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Concentrate on row i and use one of our formulae for the Delannoy numbers:

$$\sum_{j=a}^{2s-a+1}\binom{2s-2a+1}{j-a}\sum_{\ell=0}^{n-2i+j}\binom{n-2i+s-\ell}{n-2i+j-\ell,\,s-j-\ell,\,\ell}.$$

The sum over j van be simplified using the Chu–Vandermonde convolution formula.

Our determinant

STEP 2: *The term* $(n - s - \frac{1}{2})^{\min\{\lfloor (s+1+\chi(k \text{ odd}))/2 \rfloor, \lfloor (k-s+1)/2 \rfloor\}}$ *is a factor of* $\det D_1(k; n)$.

Our determinant

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Claim: For $0\le a\le s$, $k\ge 2s-a+1$, and $k\not\equiv a \pmod 2$, we have

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This is done similarly as in Step 1.

Our determinant

STEP 3: *The term* $(n+k-s-1)^{\min\{\lfloor (s+2)/2\rfloor, \lfloor (k-s-\chi(k \text{ odd}))/2\rfloor\}}$ *is a factor of* $\det D_1(k;n)$.

Our determinant

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Claim: For $0 \le 2a \le s$ and $k \ge 2s-2a+2$, we have

$$\sum_{i=a}^{s+1-a} (-1)^i \binom{s+1-2a}{i-a} \cdot \big(\text{row } i \text{ of } D_1(k;-k+s+1)\big)$$
$$-\sum_{i=s+1-a}^{k-1} 2^{2s+2-4a} \binom{i-a-1}{s-2a} \cdot \big(\text{row } i \text{ of } D_1(k;-k+s+1)\big) = 0.$$

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Equivalently:

$$\sum_{i=a}^{s+1-a} (-1)^i \binom{s+1-2a}{i-a} \cdot D(k-2i+j,-k+s-j)$$

$$= \sum_{i=s+1-a}^{k-1} 2^{2s+2-4a} \binom{i-a-1}{s-2a} \cdot D(k-2i+j,-k+s-j).$$

Interlude: Some General Philosophy on Combinatorial Analysis

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John Riordan (*Combinatorial Identities*, 1968)

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Interlude: Some General Philosophy on Combinatorial Analysis

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others, to my horror, use contour integrals, differential equations, and other resources of mathematical analysis.

Interlude: Some General Philosophy on Combinatorial Analysis

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Our determinant

Express the Delannoy number as a contour integral:

$$\begin{aligned} D(k,n) &= \sum_{\ell=0}^{k} \binom{k}{\ell}\binom{n}{\ell} 2^{\ell} \\ &= \frac{1}{(2\pi \mathbf{i})^2} \int_{\mathcal{C}_x} \int_{\mathcal{C}_y} \sum_{\ell \geq 0} 2^{\ell} \frac{(1+x)^k (1+y)^n}{(xy)^{\ell+1}} \, dx \, dy \\ &= \frac{1}{(2\pi \mathbf{i})^2} \int_{\mathcal{C}_x} \int_{\mathcal{C}_y} \frac{(1+x)^k (1+y)^n}{xy-2} \, dx \, dy, \end{aligned}$$

where

$$\mathcal{C}_x = \{5e^{\mathbf{i}t} : 0 \leq t \leq 2\pi\} \quad \text{and} \quad \mathcal{C}_y = \{0.5e^{\mathbf{i}w} : 0 \leq w \leq 2\pi\}.$$

Our determinant

We start with the left-hand side of our identity:

$$\sum_{i=a}^{s+1-a} (-1)^i \binom{s+1-2a}{i-a} D(k-2i+j, -k+s-j)$$

$$= \sum_{i=a}^{s+1-a} (-1)^i \binom{s+1-2a}{i-a} \frac{1}{(2\pi \mathbf{i})^2} \int_{\mathcal{C}_x} \int_{\mathcal{C}_y} \frac{(1+x)^{k-2i+j}(1+y)^{-k+s-j}}{xy-2}\, dx\, dy$$

$$= \frac{1}{(2\pi \mathbf{i})^2} \int_{\mathcal{C}_x} \int_{\mathcal{C}_y} \frac{(1+x)^{k-2a+j}(1+y)^{-k+s-j}}{xy-2} \left(1 - \frac{1}{(1+x)^2}\right)^{s+1-2a} dx\, dy$$

$$= \frac{1}{(2\pi \mathbf{i})^2} \int_{\mathcal{C}_x} \int_{\mathcal{C}_y} \frac{(x(2+x))^{s+1-2a}(1+x)^{k+2a+j-2s-2}(1+y)^{-k+s-j}}{xy-2}\, dx\, dy.$$

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$$= \frac{1}{(2\pi \mathbf{i})^2} \int_{\mathcal{C}_x} \int_{\mathcal{C}_y} \frac{(x(2+x))^{s+1-2a}(1+x)^{k+2a+j-2s-2}(1+y)^{-k+s-j}}{xy-2}\, dx\, dy.$$

Now apply the residue theorem with respect to y.

Our determinant

We obtain:

$$\frac{1}{2\pi\mathbf{i}}\int_{\mathcal{C}_x}\frac{(x(2+x))^{s+1-2a}(1+x)^{k+2a+j-2s-2}(1+\frac{2}{x})^{-k+s-j}}{x}\,dx$$
$$=\frac{1}{2\pi\mathbf{i}}\int_{\mathcal{C}_x}x^{k-2a+j}(2+x)^{1-2a-k+2s-j}(1+x)^{k+2a+j-2s-2}\,dx.$$

Our determinant

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$$= \frac{1}{2\pi \mathbf{i}} \int_{\mathcal{C}_x} x^{k-2a+j}(2+x)^{1-2a-k+2s-j}(1+x)^{k+2a+j-2s-2}\,dx.$$

We perform the same procedure with the expression on the right-hand side and obtain:

$$\frac{2^{2s+2-4a}}{2\pi \mathbf{i}} \int_{\mathcal{C}_x} x^{k-2s+j+2a-2}(2+x)^{-k-j+2a-1}(1+x)^{k-2a+j}\,dx.$$

We obtain:

$$\frac{1}{2\pi\mathbf{i}}\int_{\mathcal{C}_x}\frac{(x(2+x))^{s+1-2a}(1+x)^{k+2a+j-2s-2}(1+\frac{2}{x})^{-k+s-j}}{x}\,dx$$
$$=\frac{1}{2\pi\mathbf{i}}\int_{\mathcal{C}_x}x^{k-2a+j}(2+x)^{1-2a-k+2s-j}(1+x)^{k+2a+j-2s-2}\,dx.$$

We perform the same procedure with the expression on the right-hand side and obtain:

$$\frac{2^{2s+2-4a}}{2\pi\mathbf{i}}\int_{\mathcal{C}_x}x^{k-2s+j+2a-2}(2+x)^{-k-j+2a-1}(1+x)^{k-2a+j}\,dx.$$

Here we do the substitution $x\mapsto-\frac{2(1+x)}{2+x}$, and we obtain the first expression.

Our determinant

STEP 4: *The term* $(n+k-s-\frac{1}{2})^{\min\{\lfloor (s+1)/2\rfloor,\lfloor (k-s-\chi(k \text{ even}))/2\rfloor\}}$ *is a factor of* $\det D_1(k;n)$.

Our determinant

STEP 4: *The term* $(n+k-s-\frac{1}{2})^{\min\{\lfloor (s+1)/2\rfloor, \lfloor (k-s-\chi(k \text{ even}))/2\rfloor\}}$ *is a factor of* $\det D_1(k;n)$.

Claim A: For $0 \le a < s$ and $k \ge 4s-2a-1$, we have

$$\sum_{i=a}^{k-1} c_1(s-a, i-a)\cdot \big(\text{row } i \text{ of } D_1(k; -k+2s-\tfrac{1}{2})\big) = 0.$$

Claim B: For $0 \le a < s$ and $k \ge 4s-2a+1$, we have

$$\sum_{i=a}^{k-1} c_2(s-a, i-a)\cdot \big(\text{row } i \text{ of } D_1(k; -k+2s+\tfrac{1}{2})\big) = 0.$$

Our determinant

Here,

$$c_1(s,l) = -\frac{(4l-4s+1)\,(-1)^{s-1}(1-l)_{s-1}\,\left(\frac{1}{2}\right)_s\,\left(\frac{1}{2}\right)_{l-s}}{(2l-4s+1)\,l!\,(s-1)!}$$
$$-(4l-4s+1)\sum_{r=1}^{s}\frac{2^{4r-3}(2r-\frac{1}{2})_{s-r}\,(2r-\frac{1}{2})_{l-s-r}}{(s-r)!\,(l-s-r+1)!}$$

and

$$c_2(s,l) = \frac{(4l-4s-1)\,(-1)^{s-1}(1-l)_{s-1}\,\left(\frac{1}{2}\right)_{s+1}\,\left(\frac{1}{2}\right)_{l-s-1}}{(2l-4s-1)\,l!\,(s-1)!}$$
$$-(4l-4s-1)\sum_{r=1}^{s}\frac{2^{4r-1}(2r+\frac{1}{2})_{s-r}\,(2r+\frac{1}{2})_{l-s-r-1}}{(s-r)!\,(l-s-r)!}.$$

Our determinant

We consider the proof of the first (triple sum) identity. Concentrating on the j-th column, we must show

$$\sum_{i=a}^{k-1} c_1(s-a, i-a) \cdot D(k-2i+j, -k+2s-\tfrac{3}{2}-j) = 0.$$

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One can convince oneself, that it is sufficient to prove this for $j=0$, that is,

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Using contour integrals again, the triple sum can be simplified to a double sum.

Our determinant

Once worked out, we must prove the double sum identity

$$\sum_{i=0}^{k-1}\sum_{\ell=0}^{k-2i}\frac{(4i-4s+1)\,(-1)^{s-1}(1-i)_{s-1}\left(\frac{1}{2}\right)_s\left(\frac{1}{2}\right)_{i-s}}{(2i-4s+1)\,i!\,(s-1)!}$$

$$\cdot\binom{k-2i}{\ell}\binom{-k+2s-\frac{3}{2}}{\ell}2^{\ell}$$

$$+\sum_{r=1}^{s}\sum_{\ell=0}^{k-2r-2s+2}(-1)^k2^{k-2r-2s+3-\ell}\frac{2^{4r-3}(2r-\frac{1}{2})_{s-r}}{(s-r)!}$$

$$\cdot\frac{(k-2r-2s+1)!\,(k+2r-2s-1)!}{\ell!^2\,(k-2r-2s+2-\ell)!\,(k+2r-2s-\ell)!}$$

$$\cdot(-\ell^2+2k+k^2+4r-4r^2-4s-4ks+4s^2)=0,$$

for all integers k and s with $0<s$ and $k\geq 4s-1$.

Our determinant

How to prove this crazy identity?

Our determinant

How to prove this crazy identity?

THE SHORT VERSION: Using Christoph Koutschan's *Mathematica* package `HolonomicFunctions`, this is a routine task.

Our determinant

Step 5: *The determinant* $D_1(k;n)$ *is a polynomial in* n *of degree at most* $\binom{k+1}{2}$.

Our determinant

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Recall:

$$D_1(k;n) = \big(D(k-2i+j, n-j-1)\big)_{0\le i,j\le k-1}.$$

Hence, the degree of $\det D_1(k;n)$ is at most

$$\max_{\sigma\in S_k}\left(\sum_{i=0}^{k-1}(k-2i+\sigma(i))\right) = \max_{\sigma\in S_k}\left(k^2-\sum_{i=0}^{k-1} i\right) = \frac{k(k+1)}{2}.$$

Our determinant

STEP 5: *The determinant $D_1(k;n)$ is a polynomial in n of degree at most $\binom{k+1}{2}$.*

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Moreover, it is easy to see directly that the claimed product formula has degree $\binom{k+1}{2}$ as a polynomial in n.

Our determinant

STEP 6: *The coefficient of* $n^{k(k+1)/2}$ *in* $D_1(k;n)$ *is* $2^{k^2}\prod_{i=1}^{k}\frac{1}{(i)_i}$.

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We must compute the determinant of leading coefficients:

$$\det_{0\le i,j\le k-1}\left(\frac{2^{k-2i+j}}{(k-2i+j)!}\right)$$

$$= 2^{\binom{k+1}{2}}\prod_{i=0}^{k-1}\frac{1}{(2k-2i-1)!}\det_{0\le i,j\le k-1}\left((k-2i+j+1)_{k-j-1}\right)$$

$$= 2^{\binom{k+1}{2}}\prod_{i=0}^{k-1}\frac{1}{(2k-2i-1)!}\det_{0\le i,j\le k-1}\left((-2i)^{k-j-1}\right)$$

$$= 2^{\binom{k+1}{2}}\prod_{i=0}^{k-1}\frac{1}{(2k-2i-1)!}\prod_{0\le i<j\le k-1}((-2i)-(-2j))$$

$$= 2^{\binom{k+1}{2}+\binom{k}{2}}\prod_{i=0}^{k-1}\frac{i!}{(2k-2i-1)!} = 2^{k^2}\prod_{i=1}^{k}\frac{1}{(i)_i}.$$

Our determinant

The number of domino tilings of the generalised Aztec triangle of type I

Theorem

The number of domino tilings of the (n, k)-Aztec triangle of type I is

$$\prod_{i\ge0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s)\prod_{s=k-2i}^{2k-4i-2}(2n+s)\right)\Bigg/\prod_{i=1}^{k-1}(2i+1)^{k-i}\ .$$

The second determinant

Recall that there was also the determinant of

$$D_2(k;n)$$
$$= \big(D(2j-i, i+n-k-1) + D(2j-i-1, i+n-k-1)\big)_{1\le i,j\le k}\,.$$

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$$D_2(k;n) \\ = \left(D(2j-i, i+n-k-1) + D(2j-i-1, i+n-k-1)\right)_{1\le i,j\le k}.$$

Lemma

We have

$$\det D_1(k; n+\tfrac{1}{2}) = \det D_2(k;n).$$

Proof.

In fact, we have

$$D_1(k; n+\tfrac{1}{2}) = \left(\frac{(\frac{1}{2})_{j-i}}{(j-i)!}\right)_{0\le i,j\le k-1} \cdot D_2(k;n).$$

□

The number of domino tilings of the generalised Aztec triangle of type II

Theorem

The number of domino tilings of the (n, k)-Aztec triangle of type II is

$$\prod_{i\ge 0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s+1)\prod_{s=k-2i}^{2k-4i-2}(2n+s+1)\right)\Bigg/\prod_{i=1}^{k-1}(2i+1)^{k-i}.$$

Yet another determinant

What about

$$\det_{0\le i,j\le n-1}\left(2^i\binom{x+i+2j+1}{2j+1}+\binom{x-i+2j+1}{2j+1}\right)$$
$$\overset{?}{=} 2^{\binom{n}{2}+1}\prod_{i=0}^{n-1}\frac{i!}{(2i+1)!}\prod_{i=0}^{\lfloor n/2\rfloor}(x+4i+1)_{n-2i}\prod_{i=0}^{\lfloor (n-1)/2\rfloor}(x-2i+3n)_{n-2i-1}\,?$$

Yet another determinant

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Let us call this determinant $D_3(n;x)$.

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Let us call this determinant $D_3(n;x)$.

Theorem

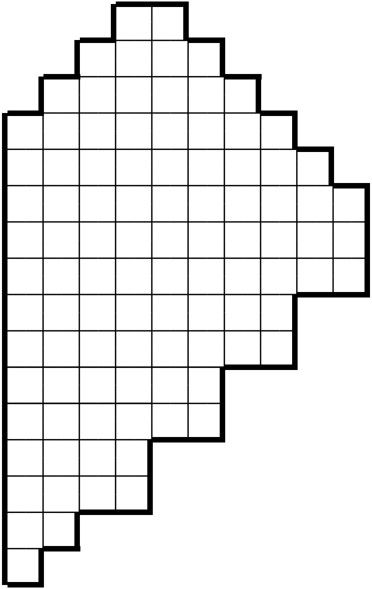
We have

$$\det D_1(n;y+n-1)=\frac{1}{2}D_3(n;2y).$$

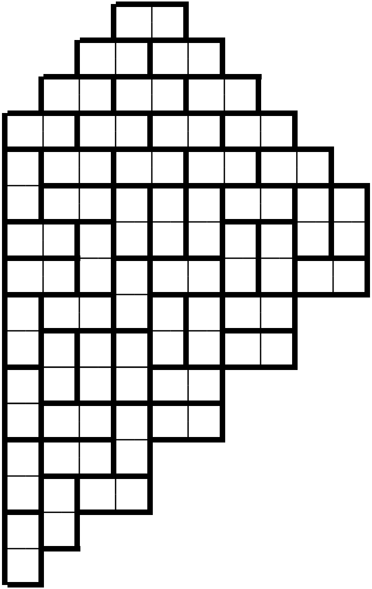
Proof.

Di Francesco proves this for $x=0$. Consequently, one has to adapt his arguments to this more general situation, and also add one step at some point. □

A generalised Aztec triangle



A generalised Aztec triangle



A generalised Aztec triangle