Identities for Cylindric Schur Functions

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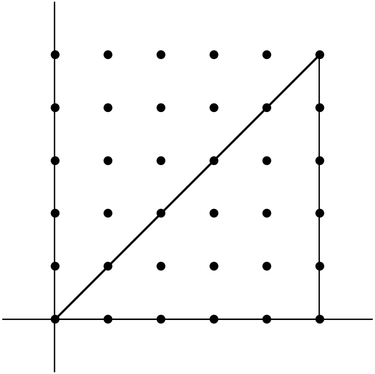

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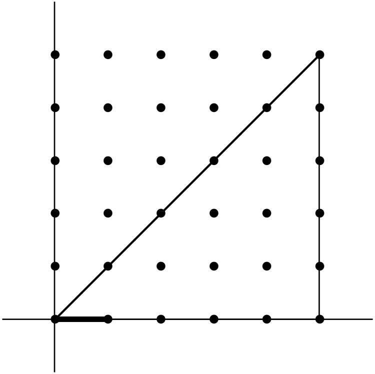

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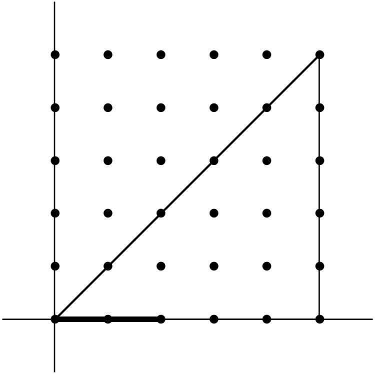

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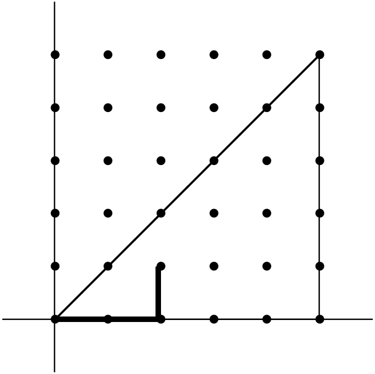

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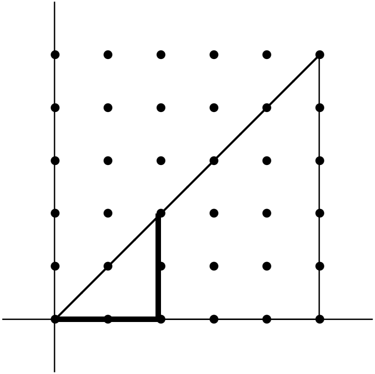

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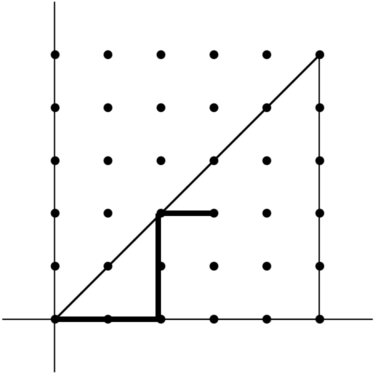

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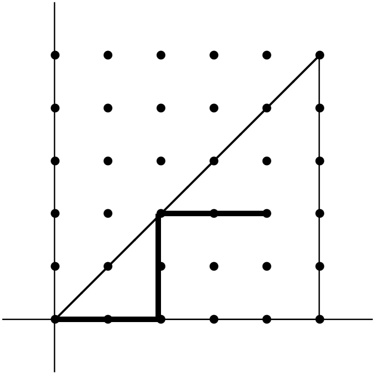

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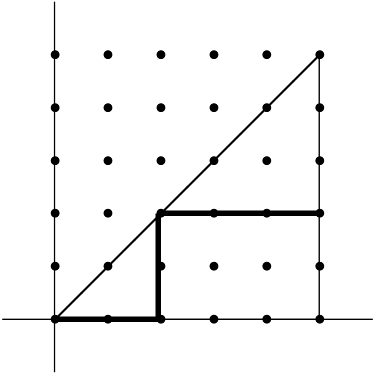

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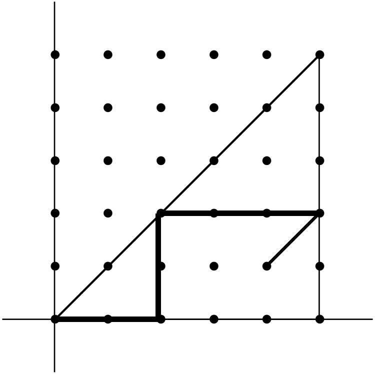

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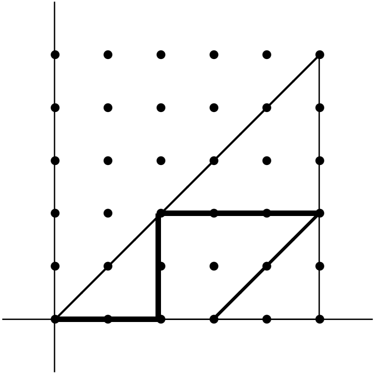

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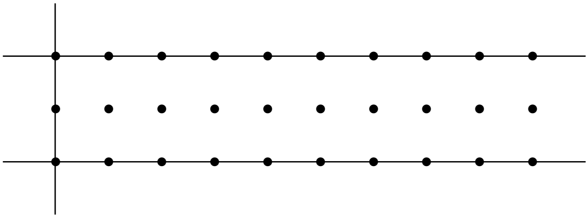

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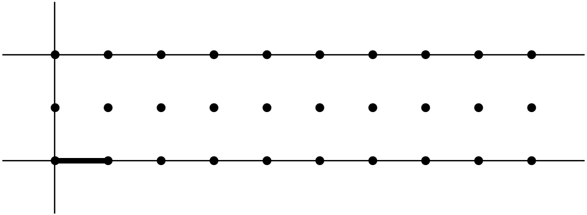

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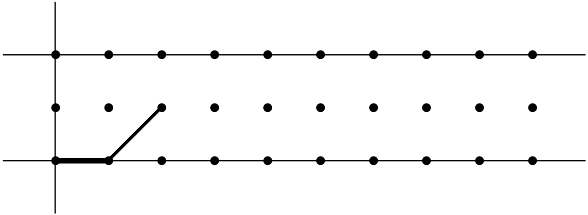

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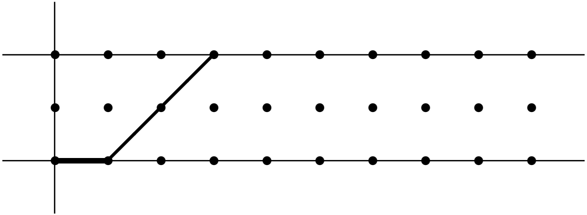

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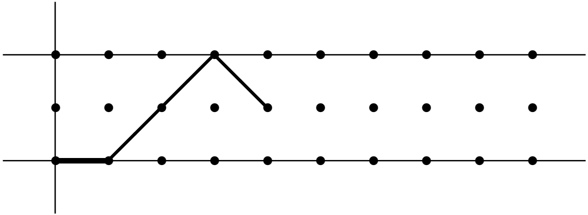

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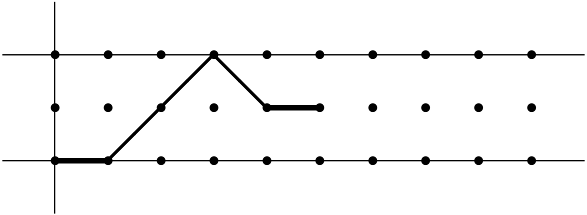

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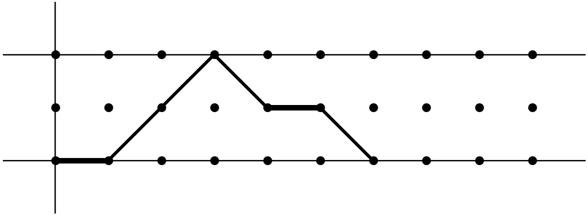

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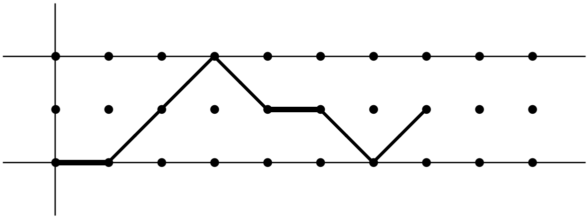

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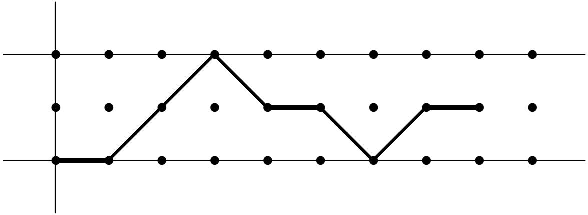

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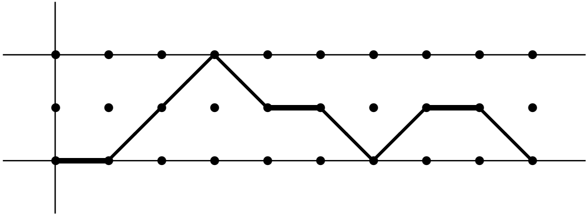

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Mortimer and Prellberg use generating function techniques (the so-called kernel method) to prove their result.

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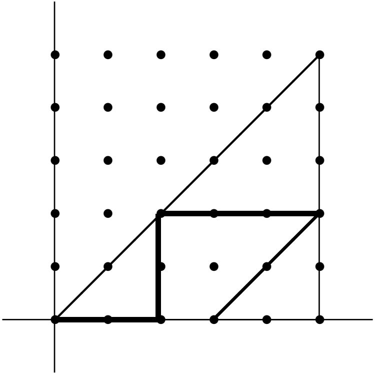

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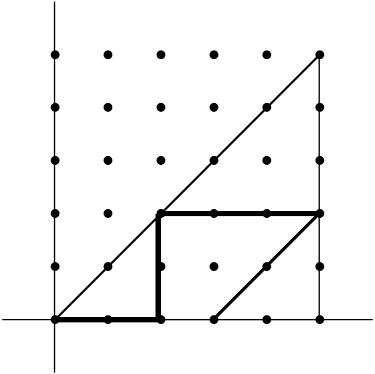

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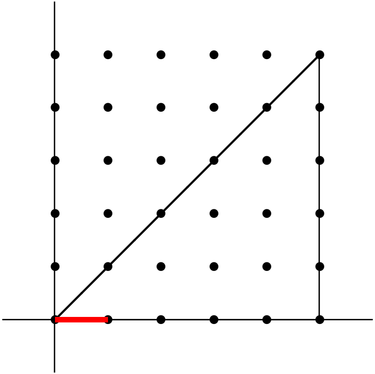

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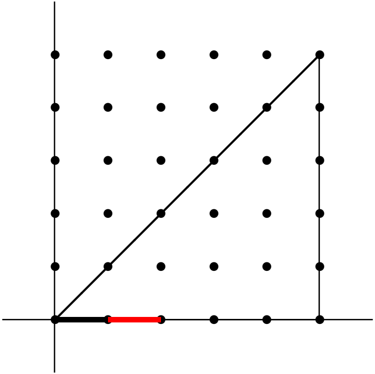

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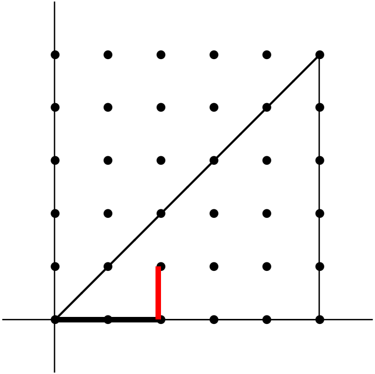

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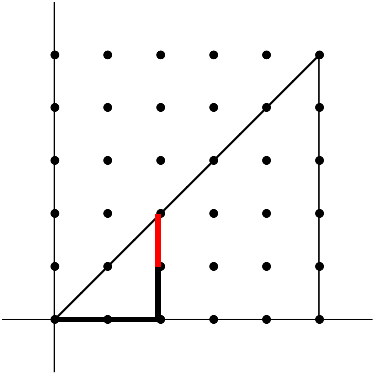

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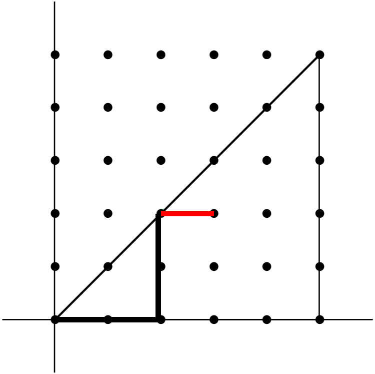

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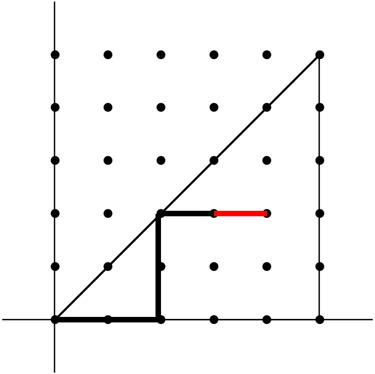

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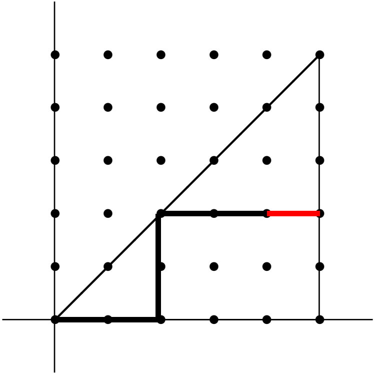

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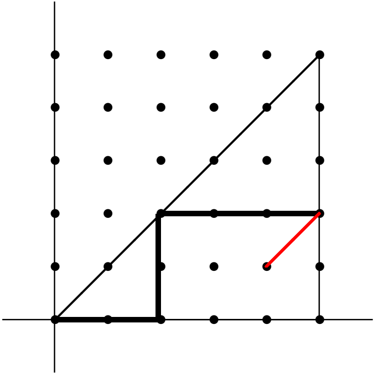

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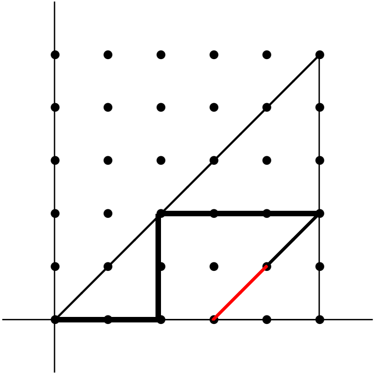

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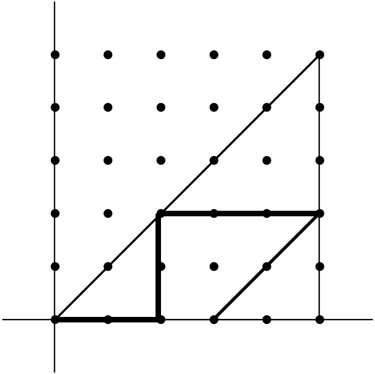

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A corresponding standard Young tableau

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A sequence of vectors $(x_1, x_2, x_3) \in \mathbb{Z}^3$ with
$m = 5 \geq x_1 \geq x_2 \geq x_3 \geq 0$:

$$(1,0,0),\ (2,0,0),\ (2,1,0),\ (2,2,0),\ (3,2,0),\ (4,2,0),\ (5,2,0),\ (5,2,1),\ (5,2,2).$$

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1

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1	2
3	

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$$\begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}$$

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A corresponding standard Young tableau

$$\begin{matrix} 1 & 2 & 5 \\ 3 & 4 & \end{matrix}$$

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$$\begin{matrix} 1 & 2 & 5 & 6 \\ 3 & 4 & & \end{matrix}$$

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$$\begin{matrix} 1 & 2 & 5 & 6 & 7 \\ 3 & 4 \end{matrix}$$

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A corresponding standard Young tableau

1	2	5	6	7
3	4			
8				

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1	2	5	6	7
3	4			
8	9			

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A corresponding standard Young tableau

$$\begin{matrix} 1 & 2 & 5 & 6 & 7 \\ 3 & 4 & & & \\ 8 & 9 & & & \end{matrix}$$

A reformulation

Hence: the walks P of length n from $(0,0)$ to any point in $\mathbb{Z}^2$ consisting of steps in $\{(1,0),(0,1),(-1,-1)\}$ such that P is contained in the triangular region

$$\{(x_1,x_2) \in \mathbb{R}^2 : m \geq x_1 \geq x_2 \geq 0\}$$

are in bijection with standard Young tableaux of size n with 3 rows (rows can also be empty) such that each subdiagram consisting of entries $1,2,\ldots,i$ has the property that the difference of the length of the first row and the length of the last row is at most m. Let us call this difference the relative width.

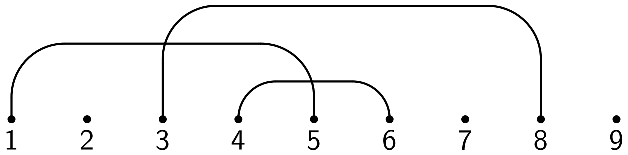
A reformulation

A reformulation

There exists a bijection between the Motzkin paths of length n and of height at most h and matchings on $[n] := \{1, 2, \dots, n\}$ without a 2-crossing and without an $(h+1)$-nesting.

A reformulation

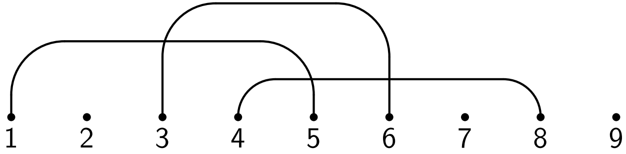
A matching on $[n] := \{1, 2, \ldots, n\}$ is a (partial) pairing of the elements $1, 2, \ldots, n$.



A reformulation

A *k-crossing* in a matching are k pairs $(i_1, j_1), \ldots, (i_k, j_k)$ such that $i_1 < \cdots < i_k < j_1 < \cdots < j_k$.

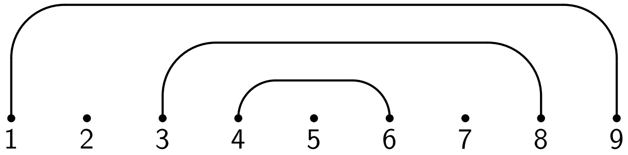
For example, here is a 3-crossing:



A reformulation

A *k-nesting* in a matching are k pairs $(i_1, j_1), \ldots, (i_k, j_k)$ such that $i_1 < \cdots < i_k < j_k < \cdots < j_1$.

For example, here is a 3-nesting:



A reformulation

A reformulation

Let $\mathrm{SYT}_n(h, w)$ denote the set of standard Young tableaux of size n with height at most h and "relative width" at most w.

Let $\mathrm{NCNN}_n(r, s)$ be the set of r-noncrossing and s-nonnesting matchings on $[n]$.
The earlier bijections show in fact:

Corollary

For all non-negative integers n and w, we have

$$|\,\mathrm{SYT}_n(3, 2w + 1)| = |\,\mathrm{NCNN}_n(2, w + 1)|.$$

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Maybe:

Speculation

For all non-negative integers n, h and w, we have

$$|\,\mathrm{SYT}_n(2h+1, 2w+1)| = |\,\mathrm{NCNN}_n(h+1, w+1)|.$$

A reformulation

Speculation

For all non-negative integers n, h and w, we have

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A reformulation

Speculation

For all non-negative integers n, h and w, we have

$$|\,\mathrm{SYT}_n(2h+1, 2w+1)| = |\,\mathrm{NCNN}_n(h+1, w+1)|.$$

THE LEFT-HAND SIDE:

The standard tableaux of size n with height at most $2h+1$ and "relative width" at most $2w+1$ can be seen as lattice paths in $\mathbb{Z}^{2h+1}$ starting at zero, consisting of positive unit steps in coordinate directions, and staying in the region

$$\big\{(x_1, x_2, \dots, x_{2h+1}) : x_1 \ge x_2 \ge \cdots \ge x_{2h+1} \ge x_1 - (2w+1)\big\}.$$

A reformulation

The standard tableaux of size n with height at most $2h+1$ and "relative" width at most $2w+1$ can be seen as lattice paths in $\mathbb{Z}^{2h+1}$ starting at zero, consisting of positive unit steps in coordinate directions, and staying in the region

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If the end point is also given, say $(\lambda_1, \lambda_2, \dots, \lambda_{2h+1})$, then there is a formula which gives the number of such paths due to Filaseta (which turned out to be a special case of the more general random-walks-in Weyl-chambers formula of Gessel and Zeilberger):

$$\sum_{\substack{\lambda \vdash n \\ \lambda_1 - \lambda_{2h+1} \leq 2w+1}} \sum_{\substack{k_1 + \cdots + k_{2h+1} = 0 \\ k_1, \dots, k_{2h+1} \in \mathbb{Z}}} n! \cdot \det_{1 \leq i,j \leq 2h+1} \left(\frac{1}{(\lambda_i - i + j + (2h+2w+2)k_i)!} \right).$$

A reformulation

The notation $\lambda \vdash n$ means that $\lambda = (\lambda_1, \ldots, \lambda_{2h+1})$ is a partition of n, that is, $\lambda_1 \geq \cdots \geq \lambda_{2h+1} \geq 0$ and $\lambda_1 + \cdots + \lambda_{2h+1} = n$.

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A reformulation

Speculation

For all non-negative integers n, h and h, we have

$$|\,\mathrm{SYT}_n(2h+1, 2w+1)| = |\,\mathrm{NCNN}_n(h+1, w+1)|.$$

A reformulation

Speculation

For all non-negative integers n, h and h, we have

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THE RIGHT-HAND SIDE:

The matchings without $(h+1)$-crossing and without $(w+1)$-nesting are in bijection with vacillating tableaux $\emptyset = \rho_0, \rho_1, \dots, \rho_{n-1}, \rho_n = \emptyset$ (here, "vacillating" means that two successive partitions in this sequence differ by at most one cell), where each ρ_i has at most h rows and at most w columns. (This is seen by a Robinson–Schensted-like bijection, which is best presented in terms of growth diagrams.)

A reformulation

The right-hand side:

The matchings without $(h+1)$-crossing and without $(w+1)$-nesting are in bijection with vacillating tableaux $\emptyset = \rho_0, \rho_1, \dots, \rho_{n-1}, \rho_n = \emptyset$ (here, "vacillating" means that two successive partitions in this sequence differ by at most one cell), where each ρ_i has at most h rows and at most w columns.

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THE RIGHT-HAND SIDE:

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In turn, these vacillating tableaux can be seen as lattice paths in $\mathbb{Z}^h$ starting at and returning to the origin, consisting of positive *and negative* unit steps in coordinate directions *and zero steps*, and staying in the region

$$\{(x_1, x_2, \dots, x_h) : w \geq x_1 \geq x_2 \geq \cdots \geq x_h \geq 0\}.$$

A reformulation

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$$\left\{(x_1, x_2, \ldots, x_h) : w \geq x_1 \geq x_2 \geq \cdots \geq x_h \geq 0\right\}.$$

For these paths there exists a formula (also following from the result of Gessel and Zeilberger):

$$\sum_{m\geq 0} \binom{n}{2m}$$

$$\cdot \left\langle \frac{x^{2m}}{(2m)!} \right\rangle \sum_{k_1,\ldots,k_h \in \mathbb{Z}} \det_{1\leq i,j\leq h} \left(I_{-i+j+(2h+2w+2)k_i}(2x) - I_{i+j+(2h+2w+2)k_i}(2x) \right),$$

where

$$I_\alpha(x) = \sum_{\ell\geq 0} \frac{(x/2)^{2\ell+\alpha}}{\ell!\,(\ell+\alpha)!}.$$

A conjecture

In summary: the

Speculation

For all non-negative integers n, h and w, we have

$$|\,\mathrm{SYT}_n(2h+1, 2w+1)| = |\,\mathrm{NCNN}_n(h+1, w+1)|.$$

is equivalent to ...

A conjecture

$$\sum_{\substack{\lambda\vdash n\\ \lambda_1-\lambda_{2h+1}\le 2w+1}} \sum_{\substack{k_1+\cdots+k_{2h+1}=0\\ k_1,\dots,k_{2h+1}\in\mathbb{Z}}} n!$$

$$\cdot \det_{1\le i,j\le 2h+1}\left(\frac{1}{(\lambda_i-i+j+(2h+2w+2)k_i)!}\right)$$

$$= \sum_{m\ge 0}\binom{n}{2m}$$

$$\cdot\left\langle \frac{x^{2m}}{(2m)!}\right\rangle \sum_{k_1,\dots,k_h\in\mathbb{Z}} \det_{1\le i,j\le h}\left(I_{-i+j+(2h+2w+2)k_i}(2x)-I_{i+j+(2h+2w+2)k_i}(2x)\right).$$

Here, $\lambda\vdash n$ means that $\lambda=(\lambda_1,\dots,\lambda_{2h+1})$ is a partition of n, that is, $\lambda_1\ge\cdots\ge\lambda_{2h+1}\ge 0$ and $\lambda_1+\cdots+\lambda_{2h+1}=n$.

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We need here the elementary symmetric functions

$$e_m(\mathbf{x}) := \sum_{1 \le i_1 < i_2 < \cdots < i_m} x_{i_1} x_{i_2} \cdots x_{i_m}.$$

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The multinomial coefficient is a product of elementary symmetric functions in disguise:

$$\frac{n!}{m_1! \cdots m_k!} = \langle x_1 x_2 \cdots x_n \rangle \, e_{m_1}(\mathbf{x}) \cdots e_{m_k}(\mathbf{x}).$$

A conjecture

$$\sum_{\substack{\lambda \vdash n \\ \lambda_1 - \lambda_{2h+1} \le 2w+1}} \sum_{\substack{k_1+\cdots+k_{2h+1}=0 \\ k_1,\dots,k_{2h+1}\in\mathbb{Z}}} n!$$

$$\cdot \det_{1\le i,j\le 2h+1}\left(\frac{1}{(\lambda_i - i + j + (2h+2w+2)k_i)!}\right)$$

$$= \sum_{m\ge 0}\binom{n}{2m}$$

$$\cdot \left\langle \frac{x^{2m}}{(2m)!}\right\rangle \sum_{k_1,\dots,k_h\in\mathbb{Z}} \det_{1\le i,j\le h}\left(I_{-i+j+(2h+2w+2)k_i}(2x) - I_{i+j+(2h+2w+2)k_i}(2x)\right),$$

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$$I_\alpha(x) = \sum_{\ell\ge 0}\frac{(x/2)^{2\ell+\alpha}}{\ell!\,(\ell+\alpha)!}.$$

Here, $\lambda \vdash n$ means that $\lambda = (\lambda_1, \dots, \lambda_{2h+1})$ is a partition of n, that is, $\lambda_1 \ge \cdots \ge \lambda_{2h+1} \ge 0$ and $\lambda_1 + \cdots + \lambda_{2h+1} = n$.

A more general conjecture

$$\sum_{\substack{\lambda:\ell(\lambda)\le 2h+1\\ \lambda_1-\lambda_{2h+1}\le 2w+1}} \sum_{\substack{k_1+\cdots+k_{2h+1}=0\\ k_1,\dots,k_{2h+1}\in\mathbb{Z}}} \det_{1\le i,j\le 2h+1}\left(e_{\lambda_i-i+j+(2h+2w+2)k_i}(\mathbf{x})\right)$$

$$= \sum_{k\ge 0} e_k(\mathbf{x}) \sum_{k_1,\dots,k_h\in\mathbb{Z}} \det_{1\le i,j\le h}\left(f_{-i+j+(2h+2w+2)k_i}(\mathbf{x}) - f_{i+j+(2h+2w+2)k_i}(\mathbf{x})\right),$$

where

$$f_\alpha(\mathbf{x}) = \sum_{\ell\ge 0} e_\ell(\mathbf{x}) e_{\ell+\alpha}(\mathbf{x}).$$

An even more general conjecture

$$\sum_{\substack{\lambda:\ell(\lambda)\le 2h+1\\ \lambda_1-\lambda_{2h+1}\le w}} \sum_{\substack{k_1+\cdots+k_{2h+1}=0\\ k_1,\dots,k_{2h+1}\in\mathbb{Z}}} \det_{1\le i,j\le 2h+1}\left(e_{\lambda_i-i+j+(2h+w+1)k_i}(\mathbf{x})\right)$$

$$=\sum_{k\ge 0} e_k(\mathbf{x}) \sum_{k_1,\dots,k_h\in\mathbb{Z}} \det_{1\le i,j\le h}\left(f_{-i+j+(2h+w+1)k_i}(\mathbf{x}) - f_{i+j+(2h+w+1)k_i}(\mathbf{x})\right),$$

where

$$f_\alpha(\mathbf{x}) = \sum_{\ell\ge 0} e_\ell(\mathbf{x}) e_{\ell+\alpha}(\mathbf{x}).$$

A companion conjecture

$$\sum_{\substack{\lambda:\ell(\lambda)\le 2h\\ \lambda_1-\lambda_{2h}\le w}} \sum_{\substack{k_1+\cdots+k_{2h}=0\\ k_1,\dots,k_{2h}\in\mathbb{Z}}} \det_{1\le i,j\le 2h}\left(e_{\lambda_i-i+j+(2h+w)k_i}(\mathbf{x})\right)$$

$$= \sum_{k_1,\dots,k_h\in\mathbb{Z}} (-1)^{\sum_{i=1}^h k_i} \det_{1\le i,j\le h}\left(f_{-i+j+(2h+w)k_i}(\mathbf{x})+f_{i+j-1+(2h+w)k_i}(\mathbf{x})\right),$$

where

$$f_\alpha(\mathbf{x}) = \sum_{\ell\ge 0} e_\ell(\mathbf{x})e_{\ell+\alpha}(\mathbf{x}).$$

The context: the $w = \infty$ case

The first identity with $w = \infty$:

$$\sum_{\lambda:\ell(\lambda)\le 2h+1} \det_{1\le i,j\le 2h+1} \left(e_{\lambda_i-i+j}(\mathbf{x})\right) = \sum_{k\ge 0} e_k(\mathbf{x}) \det_{1\le i,j\le h} \left(f_{-i+j}(\mathbf{x}) - f_{i+j}(\mathbf{x})\right),$$

or, equivalently, using Schur functions,

$$\sum_{\lambda:\lambda_1\le 2h+1} s_\lambda(\mathbf{x}) = \sum_{k\ge 0} e_k(\mathbf{x}) \det_{1\le i,j\le h} \left(f_{-i+j}(\mathbf{x}) - f_{i+j}(\mathbf{x})\right),$$

where

$$f_\alpha(\mathbf{x}) = \sum_{\ell\ge 0} e_\ell(\mathbf{x}) e_{\ell+\alpha}(\mathbf{x}).$$

Schur functions

For a given partition $(\lambda_1, \ldots, \lambda_k)$, the Schur function $s_\lambda(\mathbf{x})$ can be defined by the Jacobi–Trudi determinant

$$s_\lambda(\mathbf{x}) = \det_{1 \le i,j \le k} \left(e_{\lambda'_i - i + j}(\mathbf{x}) \right).$$

They can also be defined combinatorially as a generating function for semistandard tableaux of the shape λ.

Example. $\lambda = (5, 3, 3)$ (or: $\lambda' = (3, 3, 3, 1, 1)$)

$$T = \begin{matrix} 1 & 1 & 3 & 4 & 4 \\ 2 & 3 & 4 & & \\ 4 & 4 & 5 & & \end{matrix}$$

The associated monomial: $\mathbf{x}^T = x_1^2 x_2 x_3^2 x_4^5 x_5$

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They can also be defined combinatorially as a generating function for semistandard tableaux of the shape λ.

Combinatorial definition:

$$s_\lambda(\mathbf{x}) = \sum_T \mathbf{x}^T,$$

where the sum is over all semistandard tableaux of shape λ.

The context: the $w = \infty$ case

The first identity with $w = \infty$:

$$\sum_{\lambda:\ell(\lambda)\le 2h+1} \det_{1\le i,j\le 2h+1} \left(e_{\lambda_i-i+j}(\mathbf{x})\right) = \sum_{k\ge 0} e_k(\mathbf{x}) \det_{1\le i,j\le h} \left(f_{-i+j}(\mathbf{x}) - f_{i+j}(\mathbf{x})\right),$$

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where

$$f_\alpha(\mathbf{x}) = \sum_{\ell\ge 0} e_\ell(\mathbf{x}) e_{\ell+\alpha}(\mathbf{x}).$$

The context: the $w = \infty$ case

The second identity with $w = \infty$:

$$\sum_{\lambda:\ell(\lambda)\le 2h} \det_{1\le i,j\le 2h}\left(e_{\lambda_i-i+j}(\mathbf{x})\right) = \det_{1\le i,j\le h}\left(f_{-i+j}(\mathbf{x}) + f_{i+j-1}(\mathbf{x})\right),$$

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$$\sum_{\lambda:\ell(\lambda)\le 2h} \det_{1\le i,j\le 2h}\left(e_{\lambda_i-i+j}(\mathbf{x})\right) = \det_{1\le i,j\le h}\left(f_{-i+j}(\mathbf{x}) + f_{i+j-1}(\mathbf{x})\right),$$

or, equivalently,

$$\sum_{\lambda:\lambda_1\le 2h} s_\lambda(\mathbf{x}) = \det_{1\le i,j\le h}\left(f_{-i+j}(\mathbf{x}) + f_{i+j-1}(\mathbf{x})\right),$$

where

$$f_\alpha(\mathbf{x}) = \sum_{\ell\ge 0} e_\ell(\mathbf{x})e_{\ell+\alpha}(\mathbf{x}).$$

These two identities play an important role in the enumeration of plane partitions and in the representation theory of $GL_n(\mathbb{C})$ and $SL_n(\mathbb{C})$.

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FIRST STEP. Apply the minor summation formula of Ishikawa and Wakayama. One obtains a Pfaffian of size $2h$.

SECOND STEP. Using an identity due to Gordon, reduce the Pfaffian to a determinant of size h.

THIRD STEP. Do some row and column manipulations to arrive at the final result.

The minor summation formula

Theorem (Ishikawa, Wakayama (special case))

Let n, p be integers such that $0 \le 2n \le p$. Let M be any $(2n) \times p$ matrix. Then we have

$$\sum_K \det\left(M_K\right) = \mathrm{Pf}\left(\sum_{1\le a<b\le p}\left(M_{i,a}M_{j,b} - M_{i,b}M_{j,a}\right)\right)_{1\le i<j\le 2n},$$

where K runs over all $(2n)$-element subsets of $[1,p]$, and where M_K denotes the minor of M consisting of the columns indexed by K.

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The first identity again: combinatorial interpretation?

$$\sum_{\substack{\lambda:\ell(\lambda)\le 2h+1\\ \lambda_1-\lambda_{2h+1}\le w}} \sum_{\substack{k_1+\cdots+k_{2h+1}=0\\ k_1,\dots,k_{2h+1}\in\mathbb{Z}}} \det_{1\le i,j\le 2h+1}\left(e_{\lambda_i-i+j+(2h+w+1)k_i}(\mathbf{x})\right)$$

$$=\sum_{k\ge 0} e_k(\mathbf{x}) \sum_{k_1,\dots,k_h\in\mathbb{Z}} \det_{1\le i,j\le h}\left(f_{-i+j+(2h+w+1)k_i}(\mathbf{x}) - f_{i+j+(2h+w+1)k_i}(\mathbf{x})\right),$$

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The first identity again: combinatorial interpretation?

Theorem

The coefficient of $\mathbf{x}^{\mathbf{m}}$ *in*

$$\sum_{\substack{\lambda:\ell(\lambda)\le h\\ \lambda_1-\lambda_h\le w}}\ \sum_{\substack{k_1+\cdots+k_h=0\\ k_1,\dots,k_h\in\mathbb{Z}}} \det\left(e_{\lambda_i-i+j+(h+w)k_i}(\mathbf{x})\right)_{1\le i,j\le h}$$

equals the number of cylindric *semistandard tableaux of content* $\mathbf{m}$ *with at most* h *columns and with "relative height" at most* w*. Alternatively, this coefficient equals the number of paths in* $\mathbb{Z}^h$ *starting at the origin and staying in the region*

$$\{(x_1,x_2,\dots,x_h): x_1\ge x_2\ge\cdots\ge x_h\ge x_1-w\},$$

where the i*-th step is a vector with* m_i *coordinates equal to* 1 *and* $h-m_i$ *coordinates equal to* 0*.*

This follows from the main theorem on cylindric partitions.

The first identity again: combinatorial interpretation?

Indeed, for fixed λ, the summand

$$\sum_{\substack{k_1+\cdots+k_h=0\\ k_1,\dots,k_h\in\mathbb{Z}}} \det\left(e_{\lambda_i-i+j+(h+w)k_i}(\mathbf{x})\right)_{1\le i,j\le h}$$

appears in work of Postnikov in Schubert calculus under the name of cylindric Schur polynomial, and the cylindric semistandard tableaux appear also in work of Goodman and Wenzl in a Hecke algebra context.

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Here is the first identity again:

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$$=\sum_{k\ge 0} e_k(\mathbf{x}) \sum_{k_1,\dots,k_h\in\mathbb{Z}} \det_{1\le i,j\le h}\left(f_{-i+j+(2h+w+1)k_i}(\mathbf{x})-f_{i+j+(2h+w+1)k_i}(\mathbf{x})\right),$$

where

$$f_\alpha(\mathbf{x})=\sum_{\ell\ge 0} e_\ell(\mathbf{x})e_{\ell+\alpha}(\mathbf{x}).$$

The right-hand side can be interpreted as a (certain) generating function for up-down tableaux.

Open question

Here is the first identity again:

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Does this identity have a representation-theoretic or geometric meaning?