Hypergeometrics in action!

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Hypergeometric Series

Definition

A *hypergeometric series* is a series of the form

$$ {}_rF_s\left[\begin{matrix}\alpha_1,\dots,\alpha_r\\ \beta_1,\dots,\beta_s\end{matrix};z\right]=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\,\cdots(\alpha_r)_k}{k!\,(\beta_1)_k\cdots(\beta_s)_k}z^k, $$

where $(\alpha)_k := \alpha(\alpha+1)\cdots(\alpha+k-1)$.

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where $(\alpha)_k:=\alpha(\alpha+1)\cdots(\alpha+k-1)$.

It is routine to decide whether a given series can be written in hypergeometric form or not: if t_k denotes the k-th summand in the sum above, then

$$ \frac{t_{k+1}}{t_k}=\frac{(\alpha_1+k)\cdots(\alpha_r+k)}{(k+1)(\beta_1+k)\cdots(\beta_s+k)}z. $$

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Hence: a series can be written in hypergeometric form if and only if the ratio of its $(k+1)$-st by its k-th summand is a *rational function in* k.
Moreover, the conversion into hypergeometric notation is completely *automatic*. (*Maple* and *Mathematica* do it, for example.)

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$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = {}_0F_1\left[\begin{matrix}-\\\frac{1}{2}\end{matrix}; -\frac{x^2}{4}\right].$$

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Chebyshev polynomials of the second kind:

$$U_n(x) = \sum_{k\ge 0} (-1)^k \binom{n-k}{k} (2x)^{n-2k} = (2x)^n\, {}_2F_1\left[\begin{matrix}-\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}\\ -n\end{matrix}; x^{-2}\right].$$

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For example, the sum

$$\sum_{k=0}^{L} \binom{M}{k} \binom{N}{L-k}$$

can be written in the form

$$\binom{N}{L} {}_2F_1\left[\begin{matrix} -M, -L \\ N-L+1 \end{matrix}; 1\right].$$

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The "classical" treatment of hypergeometric series

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The *theory of hypergeometric series* has a very long tradition, with names such as *Euler, Gauß, Kummer, Thomae, Whipple, Sears, Bailey*, etc. associated to it.

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where n is a non-negative integer,

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where n is a non-negative integer, and *transformation formulae* such as

$$ {}_4F_3\left[\begin{matrix} a, b, c, -n \\ e, f, 1+a+b+c-e-f-n \end{matrix}; 1\right] \\ = \frac{(a)_n\,(e+f-a-b)_n\,(e+f-a-c)_n}{(e)_n\,(f)_n\,(e+f-a-b-c)_n} \\ \times {}_4F_3\left[\begin{matrix} -n, e-a, f-a, e+f-a-b-c \\ e+f-a-b, e+f-a-c, 1-a-n \end{matrix}; 1\right], $$

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Returning to our earlier binomial sum:

$$\sum_{k=0}^{L} \binom{M}{k} \binom{N}{L-k} = \binom{N}{L} {}_2F_1\left[\begin{matrix} -M, -L \\ N-L+1 \end{matrix}; 1\right].$$

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The ${}_2F_1$-series can be summed by means of the Chu–Vandermonde identity, so that

$$\sum_{k=0}^{L}\binom{M}{k}\binom{N}{L-k}=\binom{N}{L}\frac{(N+M-L+1)_L}{(N-L+1)_L}$$

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$$\sum_{k=0}^{L} \binom{M}{k} \binom{N}{L-k} = \binom{N}{L} \frac{(N+M-L+1)_L}{(N-L+1)_L} = \binom{M+N}{L}.$$

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        If 'L' is a natural number, then:

Out[2]= (-1-M-N) SUM[N]+(1-L+M+N) SUM[1+N]==0
```

Some papers of Volker Strehl

Volker Strehl.
Identities of Rothe-Abel-Schläfli-Hurwitz-type.
Discrete Math., 99:321–340, 1992.

P. Lisoněk, Peter Paule, and Volker Strehl.
Improvement of the degree setting in Gosper's algorithm.
J. Symbolic Comput., 16(3):243–258, 1993.

Volker Strehl.
Recurrences and Legendre transform.
In *Séminaire Lotharingien de Combinatoire*, 33:81–100, 1993.

Volker Strehl.
Binomial identities—combinatorial and algorithmic aspects.
Discrete Math., 136(1-3):309–346, 1994.

Roberto Pirastu and Volker Strehl.
Rational summation and Gosper-Petkovšek representation.
J. Symbolic Comput., 20(5-6):617–635, 1995.

Asymptotics of a Selberg integral

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In a recent paper in *random scattering theory* (*"random matrix approach to quantum transport in chaotic cavities"*), Carré, Deneufchâtel, Luque and Vivo consider the Selberg-type integral

$$S_k(a,b) = \frac{1}{N!}\int_{[0,1]^N} x_1^k \left(\prod_{1\le i<j\le N}(x_i - x_j)^2\right)\left(\prod_{i=1}^N x_i^{a-1}(1-x_i)^{b-1}\,dx_i\right),$$

and they aim at determining its asymptotic behaviour when N, a, b all tend to infinity so that $a \sim a_1 N$ and $b \sim b_1 N$.

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and they aim at determining its asymptotic behaviour when N, a, b all tend to infinity so that $a \sim a_1 N$ and $b \sim b_1 N$.

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For $k = 0$, this is exactly Selberg's famous integral.
The *Selberg integral* can be evaluated in closed form, and the result is a product/quotient of gamma functions.
Consequently, the asymptotics of $S_0(a,b)$ is easily determined by means of known asymptotic formulae for the Barnes G-function.

We may therefore restrict our attention to

$$J_k = \frac{S_k(a,b)}{S_0(a,b)}.$$

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Using classical identities in the theory of symmetric functions and the evaluation of Selberg-like integrals, it is not too difficult to derive that

$$J_k = \frac{1}{N \cdot k!} \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \frac{(N-i)_k \, (a+N-i-1)_k}{(a+b+2N-i-2)_k}.$$

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Carré, Deneufchâtel, Luque and Vivo develop a difference calculus over several pages in order to approach the problem.
However: this is a hypergeometric series! Namely,

$$J_k = \frac{(N+1)_{k-1} \, (a+N-1)_k}{k! \, (2N+a+b-2)_k} \times {}_4F_3 \left[\begin{matrix} 1-N, 1-k, 2-a-N, 3-a-b-k-2N \\ 2-a-k-N, 1-k-N, 3-a-b-2N \end{matrix} ; 1 \right].$$

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So, the theory of hypergeometric series should do it!
And it does ...
The sum of the upper parameters equals

$$(1-N)+(1-k)+(2-a-N)+(3-a-b-k-2N) = 7-2a-b-2k-4N,$$

while the sum of the lower parameters equals

$$2-a-k-N, 1-k-N, 3-a-b-2N = 6-2a-b-2k-4N.$$

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Let us check our earlier horrendous transformation formula:

$${}_4F_3\left[\begin{matrix}a, b, c, -n \\ e, f, 1+a+b+c-e-f-n\end{matrix}; 1\right] = \frac{(a)_n\,(e+f-a-b)_n\,(e+f-a-c)_n}{(e)_n\,(f)_n\,(e+f-a-b-c)_n} \times {}_4F_3\left[\begin{matrix}-n, e-a, f-a, e+f-a-b-c \\ e+f-a-b, e+f-a-c, 1-a-n\end{matrix}; 1\right].$$

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To do this kind of "operation," the hypergeometric literature offers *contiguous relations*. An example is

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{}_4F_3\left[\begin{matrix}A,B,C,D\\E,F,G\end{matrix};z\right]=z\frac{BCD}{EFG}\,{}_4F_3\left[\begin{matrix}A,B+1,C+1,D+1\\E+1,F+1,G+1\end{matrix};z\right]
+{}_4F_3\left[\begin{matrix}A-1,B,C,D\\E,F,G\end{matrix};z\right].
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$$

If we iterate this contiguous relation, then we arrive at

$$
{}_4F_3\left[\begin{matrix}A,B,C,D\\E,F,G\end{matrix};z\right]=z^r\frac{(B)_r\,(C)_r\,(D)_r}{(E)_r\,(F)_r\,(G)_r}\,{}_4F_3\left[\begin{matrix}A,B+r,C+r,D+r\\E+r,F+r,G+r\end{matrix};z\right]
+\sum_{s=0}^{r-1}z^s\frac{(B)_s\,(C)_s\,(D)_s}{(E)_s\,(F)_s\,(G)_s}\,{}_4F_3\left[\begin{matrix}A-1,B+s,C+s,D+s\\E+s,F+s,G+s\end{matrix};z\right].
$$

Asymptotics of a Selberg integral

We apply the iterated contiguous relation to our series:

$$J_k = \sum_{s=0}^{k-1} \frac{(N+1)_{k-1}\,(a+N-1)_k\,(1-k)_s}{k!\,(a+b+2N-2)_k\,(2-a-k-N)_s} \cdot \frac{(2-a-N)_s\,(3-a-b-k-2N)_s}{(1-k-N)_s\,(3-a-b-2N)_s} \cdot {}_4F_3\left[\begin{matrix} -N, 1-k+s, 2-a-N+s, 3-a-b-k-2N+s \\ 2-a-k-N+s, 1-k-N+s, 3-a-b-2N+s \end{matrix}; 1\right].$$

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The sum of the upper parameters:

$$(-N)+(1-k+s)+(2-a-N+s)+(3-a-b-k-2N+s) = 6-2a-b-2k-4N+3s.$$

The sum of the lower parameters:

$$(-N)+(1-k+s)+(2-a-N+s)+(3-a-b-k-2N+s) = 6-2a-b-2k-4N+3s.$$

We apply the iterated contiguous a second time:

$$J_k = \sum_{s=0}^{k-1} \sum_{t=0}^{k-s-1} \frac{(N+1)_{k-1}\,(a+N-1)_k\,(1-k)_{s+t}}{k!\,(a+b+2N-2)_k}$$

$$\cdot \frac{(2-a-N)_{s+t}\,(3-a-b-k-2N)_{s+t}}{(2-a-k-N)_{s+t}\,(1-k-N)_{s+t}\,(3-a-b-2N)_{s+t}}$$

$$\cdot {}_4F_3\left[\begin{matrix}3-a-b-k-2N+s+t, -1-N,\\ 3-a-b-2N+s+t,\end{matrix}\right.$$

$$\left.\begin{matrix}2-a-N+s+t, 1-k+s+t\\ 1-k-N+s+t, 2-a-k-N+s+t\end{matrix}; 1\right].$$

Asymptotics of a Selberg integral

We apply the iterated contiguous a second time:

$$J_k = \sum_{s=0}^{k-1} \sum_{t=0}^{k-s-1} \frac{(N+1)_{k-1}\,(a+N-1)_k\,(1-k)_{s+t}}{k!\,(a+b+2N-2)_k} \cdot \frac{(2-a-N)_{s+t}\,(3-a-b-k-2N)_{s+t}}{(2-a-k-N)_{s+t}\,(1-k-N)_{s+t}\,(3-a-b-2N)_{s+t}} \cdot {}_4F_3\left[\begin{matrix} 3-a-b-k-2N+s+t, -1-N, \\ 3-a-b-2N+s+t, \end{matrix} \begin{matrix} 2-a-N+s+t, 1-k+s+t \\ 1-k-N+s+t, 2-a-k-N+s+t \end{matrix}; 1\right].$$

Now the sum of the upper parameters is by one less than the sum of the lower parameter!

Our horrendous transformation formula can be applied, and, after some simplification, the resulting expression collapses to

$$J_k = \sum_{s=0}^{k-1} \sum_{t=0}^{k-s-1} \frac{(a-1)_{k-s-t-1}\,(1-a-N)_{s+t+1}}{k!} \cdot \frac{(k-s-t)_{s+t}\,(s+t+2)_{k-s-t-1}}{(2-a-b-2N)_k} \cdot {}_4F_3\left[\begin{matrix}1-k+s+t, k, a+b+N-2, a+N\\ s+t+2, a-1, a+b+2N-1\end{matrix};1\right]$$

$$= \sum_{m=0}^{k-1} \frac{(a-1)_{k-m-1}\,(1-a-N)_{m+1}\,(k-m)_m\,(m+1)_{k-m}}{k!\,(2-a-b-2N)_k} \cdot {}_4F_3\left[\begin{matrix}1-k+m, k, a+b+N-2, a+N\\ m+2, a-1, a+b+2N-1\end{matrix};1\right].$$

Explicitly,

$$J_k = \sum_{m=0}^{k-1} \frac{(a-1)_{k-m-1}\,(1-a-N)_{m+1}\,(k-m)_m\,(m+1)_{k-m}}{k!\,(2-a-b-2N)_k} \cdot \sum_{i=0}^{k-m-1} \frac{(-k+m+1)_i\,(k)_i\,(a+b+N-2)_i\,(a+N)_i}{i!\,(m+2)_i\,(a-1)_i\,(a+b+2N-1)_i}.$$

Explicitly,

$$J_k = \sum_{m=0}^{k-1} \frac{(a-1)_{k-m-1}\,(1-a-N)_{m+1}\,(k-m)_m\,(m+1)_{k-m}}{k!\,(2-a-b-2N)_k} \cdot \sum_{i=0}^{k-m-1} \frac{(-k+m+1)_i\,(k)_i\,(a+b+N-2)_i\,(a+N)_i}{i!\,(m+2)_i\,(a-1)_i\,(a+b+2N-1)_i}.$$

The limit $N, a, b \to \infty$ so that $a \sim a_1 N$ and $b \sim b_1 N$ can now be safely done in each summand separately.

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$$J_k = \sum_{m=0}^{k-1} \frac{(a-1)_{k-m-1}\,(1-a-N)_{m+1}\,(k-m)_m\,(m+1)_{k-m}}{k!\,(2-a-b-2N)_k}$$

$$\cdot \sum_{i=0}^{k-m-1} \frac{(-k+m+1)_i\,(k)_i\,(a+b+N-2)_i\,(a+N)_i}{i!\,(m+2)_i\,(a-1)_i\,(a+b+2N-1)_i}.$$

The limit $N, a, b \to \infty$ so that $a \sim a_1 N$ and $b \sim b_1 N$ can now be safely done in each summand separately.

The result is:

$$\lim_{N\to\infty} J_k = \sum_{m=0}^{k-1} (-1)^{k-m-1} \binom{k-1}{m} \left(\frac{a_1}{a_1+b_1+2}\right)^k \left(\frac{a_1+1}{a_1}\right)^{m+1}$$

$$\cdot \sum_{\ell=0}^{k-m-1} (-1)^\ell \binom{k-m-1}{\ell} \frac{(k+\ell-1)!(m+1)!}{(k-1)!(m+\ell+1)!} \left(\frac{(a_1+1)(a_1+b_1+1)}{a_1(a_1+b_1+2)}\right)^\ell.$$

Asymptotics of a Selberg integral

Doing some more *"hypergeometrics,"* one arrives at the more compact statement:

Theorem

The limit of the quantity J_k as $N, a, b \to \infty$ such that $a \sim a_1 N$ and $b \sim b_1 N$ is equal to

$$\lim_{N\to\infty} J_k = \frac{1}{k}\sum_{j=0}^{k-1}(-1)^j \binom{k+j-1}{j} \frac{(a_1+1)^{j+1}}{(a_1+b_1+2)^{k+j}} \cdot \sum_{i=0}^{k-j-1} \binom{k}{i}\binom{k}{i+j+1}(a_1+1)^i.$$

Non-crossing partitions on an annulus

Non-crossing partitions on an annulus

m-divisible non-crossing partitions on the (A, B)-annulus are *set partitions* of $\{1, 2, \ldots, A+B\}$ all of whose block sizes are divisible which can be drawn in a non-crossing fashion inside an (A, B)-annulus.

Non-crossing partitions on an annulus

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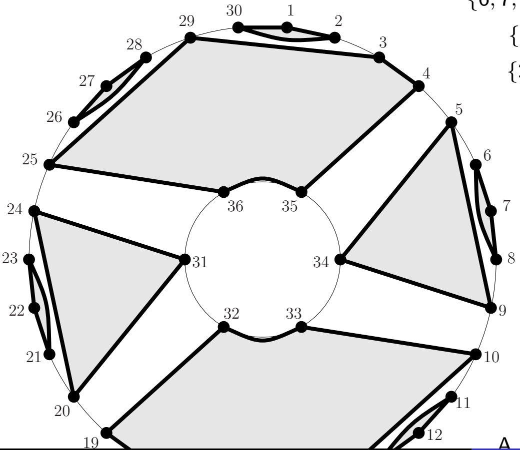

$\{\{1, 2, 30\}, \{3, 4, 35, 36, 25, 29\}, \{5, 9, 34\},$
$\{6, 7, 8\}, \{10, 14, 18, 19, 32, 33\}, \{11, 12, 13\},$
$\{15, 16, 17\}, \{20, 24, 31\},$
$\{21, 22, 23\}, \{26, 27, 28\}\}$

A 3-divisible non-crossing partition

Non-crossing partitions on an annulus

(m-divisible) non-crossing partitions on an annulus have arisen in various contexts: in *statistical physics*, in *free probability*, and in *Coxeter group theory*.

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Question: *How many m-divisible non-crossing partitions on the (A, B)-annulus are there?*

Non-crossing partitions on an annulus

Using

Non-crossing partitions on an annulus

Using a *combinatorial decomposition*,

Non-crossing partitions on an annulus

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Non-crossing partitions on an annulus

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$$\sum_{t\ge(A+1)/m}(mt-A)\binom{A+t-1}{t}\binom{B+\frac{A+B}{m}-t-1}{\frac{A+B}{m}-t}$$
$$-\sum_{t\ge(A+1)/m}\frac{B(mt-A)(mt-A+1)}{B+1}\binom{A+t-1}{t}\binom{B+\frac{A+B}{m}-t}{\frac{A+B}{m}-t}$$
$$+\sum_{t\ge(A+2)/m}\frac{A(mt-A-1)(mt-A)}{A+1}\binom{A+t}{t}\binom{B+\frac{A+B}{m}-t-1}{\frac{A+B}{m}-t}.$$

Theorem?

The number of m-divisible non-crossing partitions on the (A, B)-annulus is equal to

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Definition (H. Wilf)

An *enumeration formula* is an expression which is computable in time less than needed for generating all the objects that we want to count.

Non-crossing partitions on an annulus

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Non-crossing partitions on an annulus

Our Favourite Theorem

The number of m-divisible non-crossing partitions on the (A, B)-annulus is equal to[a]

$$\langle NICE \rangle .$$

[a] © Doron Zeilberger

A miracle (??)

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If one programs this in computer algebra and says

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Factor[%]
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then one obtains:

For $A = 2a$, $B = 2b$, $m = 2$:

$$\frac{1}{3}\binom{3a}{a+1}\binom{3b}{b+1}\frac{(a+1)(b+1)(4ab-a-b+1)}{(2a+1)(2b+1)(a+b)};$$

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For $A = 3a-1$, $B = 3b-2$, $m = 3$:

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For $A = 3a-1$, $B = 3b-2$, $m = 3$:

$$\frac{1}{3}\binom{4a-2}{a+1}\binom{4b-3}{b+1}\frac{(a+1)b(b+1)}{(a+b-1)(3b-1)};$$

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Factor[%]
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Etc.

The “explanation”

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The first sum,

$$\sum_{t\ge(A+1)/m}(mt-A)\binom{A+t-1}{t}\binom{B+\frac{A+B}{m}-t-1}{\frac{A+B}{m}-t},$$

is a *telescoping* sum!

The "explanation"

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is a *telescoping* sum! Namely,

$$(mt-A)\binom{A+t-1}{t}\binom{B+\frac{A+B}{m}-t-1}{\frac{A+B}{m}-t}=G(t+1)-G(t)$$

with

$$G(t)=-\frac{mAB}{A+B}\binom{A+t-1}{t-1}\binom{B+\frac{A+B}{m}-t}{\frac{A+B}{m}-t}.$$

The "explanation"

The second and third sum *together*,

$$-\sum_{t\ge(A+1)/m}\frac{B(mt-A)(mt-A+1)}{B+1}\binom{A+t-1}{t}\binom{B+\frac{A+B}{m}-t}{\frac{A+B}{m}-t}$$

$$+\sum_{t\ge(A+2)/m}\frac{A(mt-A-1)(mt-A)}{A+1}\binom{A+t}{t}\binom{B+\frac{A+B}{m}-t-1}{\frac{A+B}{m}-t},$$

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The "explanation"

Namely,

$$-\frac{B(mt-A)(mt-A+1)}{B+1}\binom{A+t-1}{t}\binom{B+\frac{A+B}{m}-t}{\frac{A+B}{m}-t}$$
$$+\frac{A(mt-A-1)(mt-A)}{A+1}\binom{A+t}{t}\binom{B+\frac{A+B}{m}-t-1}{\frac{A+B}{m}-t}$$
$$=H(t+1)-H(t)$$

with

$$H(t)=-\frac{(mt-A+1)(mt-A-m-1)}{(A+1)(B+1)}$$
$$\times\frac{(A+t-1)!\,(B+\frac{A+B}{m}-t)!}{(t-1)!\,(A-1)!\,(\frac{A+B}{m}-t)!\,(B-1)!}.$$

The "explanation"

After all hard thinking (= "classical" hypergeometrics) did not lead to anything, in despair (and lack of other ideas) one tries the *Gosper algorithm*.

The “explanation”

After all hard thinking (= “classical” hypergeometrics) did not lead to anything, in despair (and lack of other ideas) one tries the *Gosper algorithm*.

The Gosper algorithm *decides* whether a $G(t)$ exists such that

$$(mt-A)\binom{A+t-1}{t}\binom{B+\frac{A+B}{m}-t-1}{\frac{A+B}{m}-t}=G(t+1)-G(t),$$

and if it does, it *finds* it!

The "explanation"

So:

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So:

```
In[1]:= <<zb.m
```

The "explanation"

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        Fast Zeilberger Package by Peter Paule,
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The "explanation"

So:

```
In[1]:= <<zb.m
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In[2]:= Gosper[(m*t-A)Binomial[A+t-1,t]
              Binomial[B+(A+B)/m-t-1,(A+B)/m-t],t]
```

The "explanation"

So:

```
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In[2]:= Gosper[(m*t-A)Binomial[A+t-1,t]
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Out[2]= {-(A-m t) Binomial[-1+B+(A+B)/m-t,(A+B)/m-t]
                  Binomial[-1+A+t,t]==
          Δt[(1/(A+B))t (-A-B-B m+m t)
                  Binomial[-1+B+(A+B)/m-t,(A+B)/m-t]
                  Binomial[-1+A+t,t]]}
```

Here, Δ_t is the standard difference operator $\Delta_t G(t) := G(t+1) - G(t)$.

Non-crossing partitions on an annulus

Theorem

The number of m-divisible non-crossing partitions on the (A,B)-annulus is equal to

$$\frac{AB\big(mAB-((A\ mod\ m)\cdot(B\ mod\ m)+1)(A+B)+m\big)}{(\chi(A\equiv B\equiv 0\ mod\ m)m+1)(A+1)(B+1)(A+B)} \times \binom{\lfloor\frac{m+1}{m}A\rfloor}{A}\binom{\lfloor\frac{m+1}{m}B\rfloor}{B},$$

where $(A\ mod\ m)$ is the remainder of the division of A by m, and $\chi(\mathcal{A})=1$ if $\mathcal{A}$ is true and $\chi(\mathcal{A})=0$ otherwise.

A Happy Retirement!