Boundary dents, the arctic circle and the arctic ellipse

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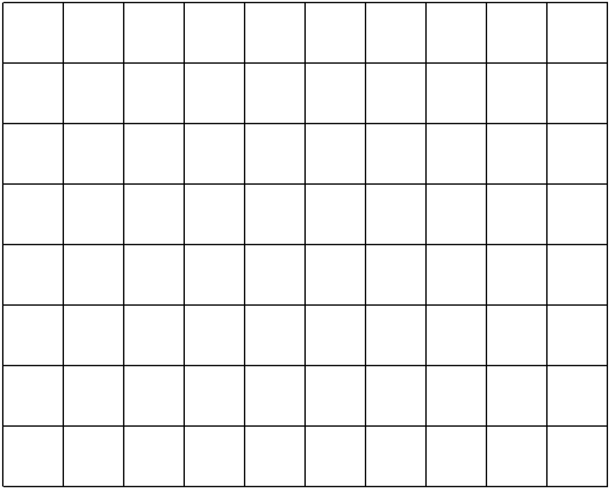
Domino tilings of a rectangle



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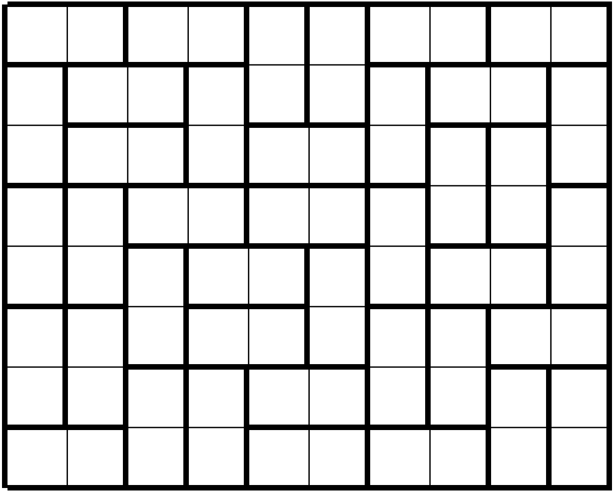

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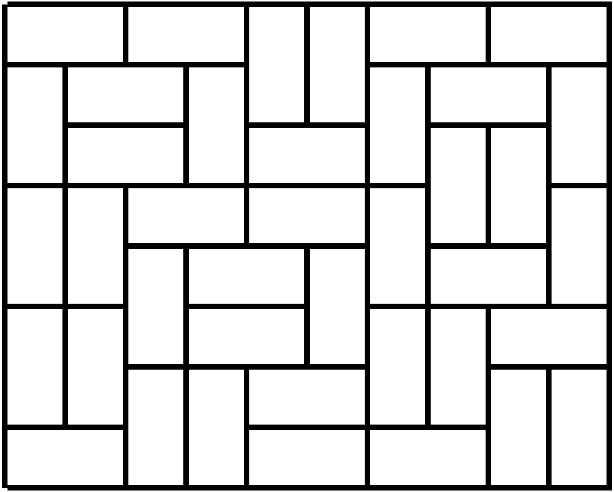


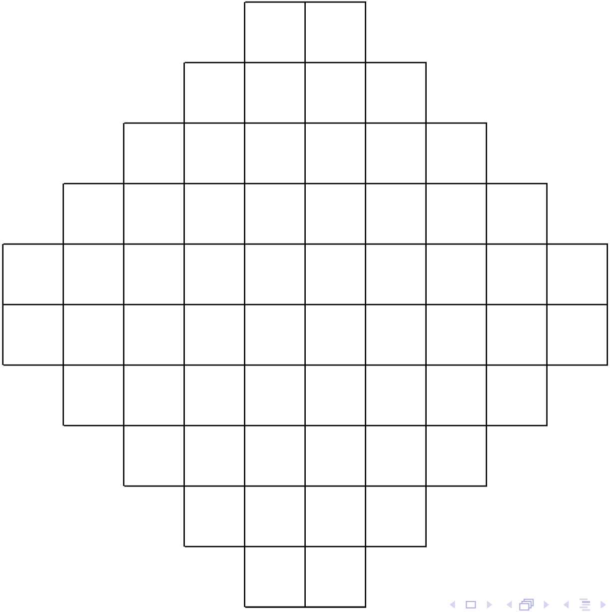
Domino tilings of a rectangle

Theorem (Kasteleyn 1961, Temperley and Fisher 1961)

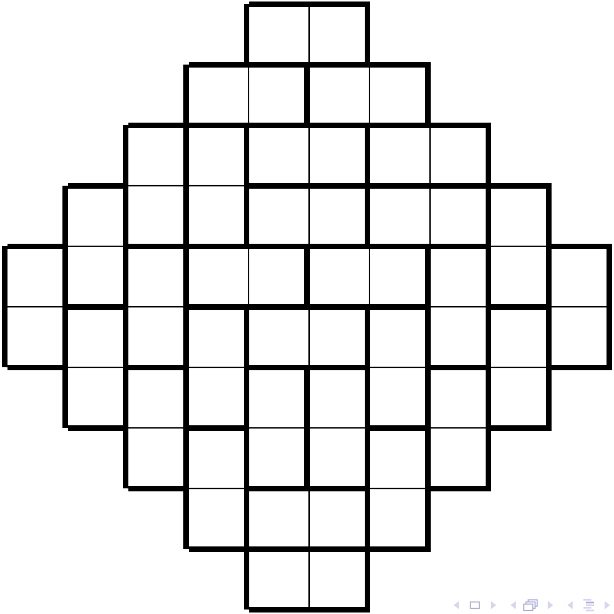
Let m and n be positive integers, with m being even. The number of domino tilings of an $m \times n$-rectangle is given by

$$2^{mn/2}\prod_{i=1}^{m/2}\prod_{j=1}^{n}\left(\cos^2\frac{\pi i}{m+1}+\cos^2\frac{\pi j}{n+1}\right)^{1/2}.$$

The Aztec diamond



Domino tilings of the Aztec diamond



The Aztec diamond theorem

Theorem (Elkies, Kuperberg, Larsen, Propp 1992)

The number of domino tilings of the Aztec diamond of size n is

$$2^{\binom{n+1}{2}}.$$

The Arctic Circle Theorem (Jockusch, Propp, Shor 1995)

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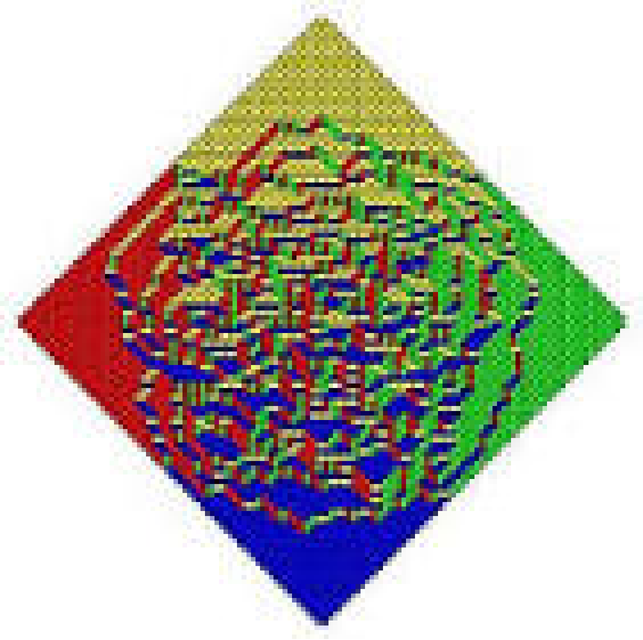

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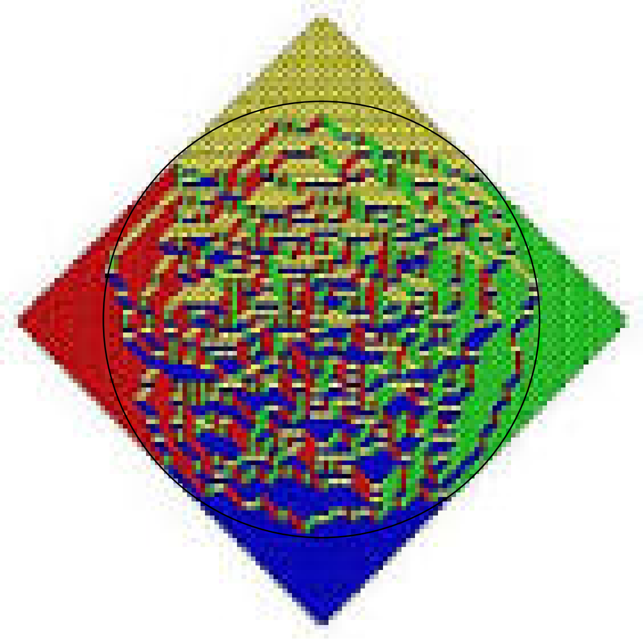

A question of Jim Propp

Is it possible to increase the number of tilings by poking holes?

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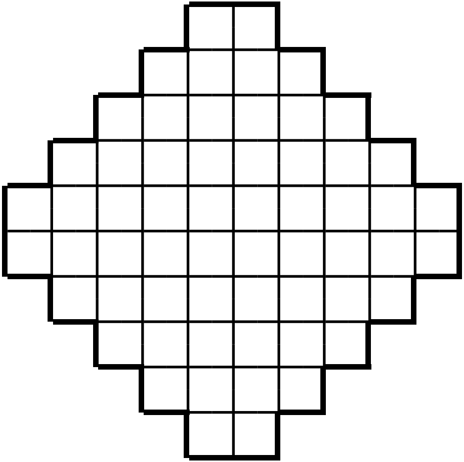

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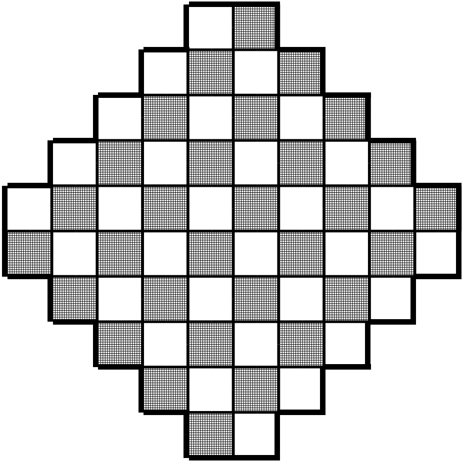

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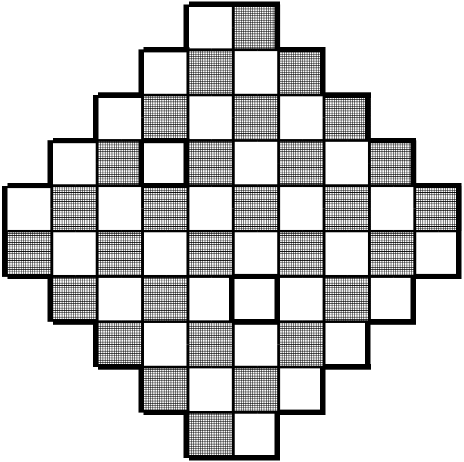
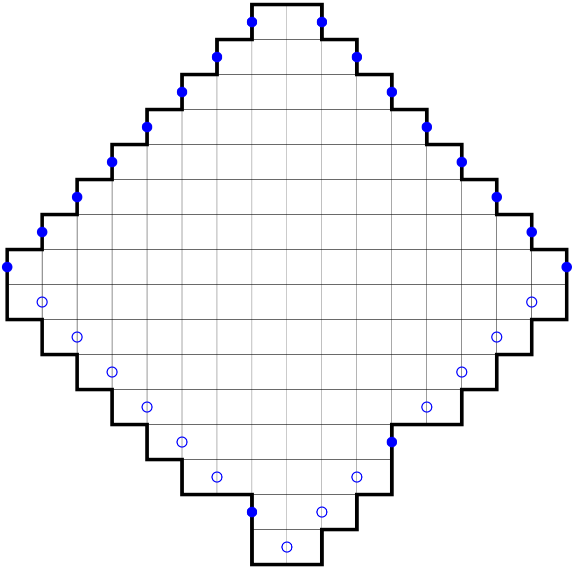


A dented Aztec diamond



Some notation

We write AD_n for the Aztec diamond of size n.

We let $AD_n^{i,j}$ denote the region obtained from AD_n by removing the ith unit square on its southwestern boundary and the jth unit square on its southeastern boundary (both counted from bottom to top).

For a region R we write $\mathrm{M}(R)$ for the number of domino tilings of R.

Main results

Theorem

For $1 \leq i, j \leq n$, we have:

(a).

$$\frac{\mathrm{M}(AD_n^{i,j})}{\mathrm{M}(AD_n)} = \sum_{k=0}^{n-1} \binom{k}{i-1} \binom{k}{j-1} \frac{1}{2^{k+1}}.$$

(b).

$$\lim_{n\to\infty} \frac{\mathrm{M}(AD_n^{i,j})}{\mathrm{M}(AD_n)} = \sum_{k=0}^{\infty} \binom{k}{i-1} \binom{k}{j-1} \frac{1}{2^{k+1}}.$$

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We compute the bivariate generating function:

$$\begin{aligned}\sum_{i,j\ge 0}x^iy^j\sum_{k=0}^{\infty}\binom{k}{i}\binom{k}{j}\frac{1}{2^{k+1}}&=\sum_{k=0}^{\infty}\sum_{i,j\ge 0}x^iy^j\binom{k}{i}\binom{k}{j}\frac{1}{2^{k+1}}\\&=\sum_{k=0}^{\infty}\frac{1}{2^{k+1}}(1+x)^k(1+y)^k\\&=\frac{1}{2}\cdot\frac{1}{1-\frac{(1+x)(1+y)}{2}}\\&=\frac{1}{1-x-y-xy}.\end{aligned}$$

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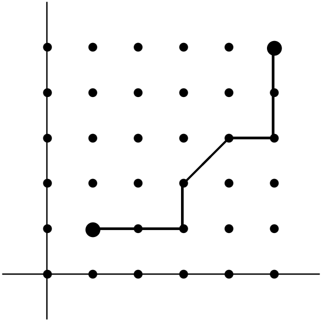
This is the generating function of Delannoy numbers!

Delannoy paths

Delannoy paths are paths on $\mathbb{Z}^2$ using only steps $(1,0)$, $(0,1)$ or $(1,1)$.



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(b).

$$\lim_{n\to\infty} \frac{\mathrm{M}(AD_n^{i,j})}{\mathrm{M}(AD_n)} = D(i-1, j-1),$$

where $D(k,l)$ is the Delannoy number, defined to be the number of paths on $\mathbb{Z}^2$ from $(0,0)$ to (k,l) using only steps $(1,0)$, $(0,1)$ or $(1,1)$.

Main results

Theorem

Let C be the circle inscribed in the unit square. Then as $n, i, j \to \infty$ so that $i/n \to a$ and $j/n \to b$, where $0 < a, b < 1$, we have

$$\lim_{n\to\infty} \frac{\mathrm{M}(AD_n^{i,j})}{\mathrm{M}(AD_n)D(i-1,j-1)} = \begin{cases} 1, & \text{if the segment } [(a,0),(0,b)] \text{ is outside } C, \\ 0, & \text{if the segment } [(a,0),(0,b)] \text{ crosses } C. \end{cases}$$

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Remark. The condition "the segment $[(a,0),(0,b)]$ is outside C" is equivalent to

$$(1-a)(1-b) > \frac{1}{2}.$$

Main results

Theorem

As $n, i, j \to \infty$ *so that* $i/n \to a$, $j/n \to b$, *we have*

$$\frac{\mathrm{M}(AD_n^{i,j})}{\mathrm{M}(AD_n)} \sim \begin{cases} \dfrac{(a+b+\sqrt{a^2+b^2})\left(\frac{(a+\sqrt{a^2+b^2})^b(b+\sqrt{a^2+b^2})^a}{a^a b^b}\right)^n}{2\sqrt{2\pi n}\sqrt{ab\sqrt{a^2+b^2}}}, & \text{if } (1-a)(1-b) > 1/2, \\[2ex] \dfrac{\sqrt{(1-a)(1-b)}\left(2a^a b^b (1-a)^{1-a}(1-b)^{1-b}\right)^{-n}}{4\pi n\sqrt{ab}\left(\frac{1}{2}-(1-a)(1-b)\right)}, & \text{if } (1-a)(1-b) < 1/2. \end{cases}$$

Main results

Corollary

As $n, i, j \to \infty$ so that $i/n \to a$, $j/n \to b$, we have

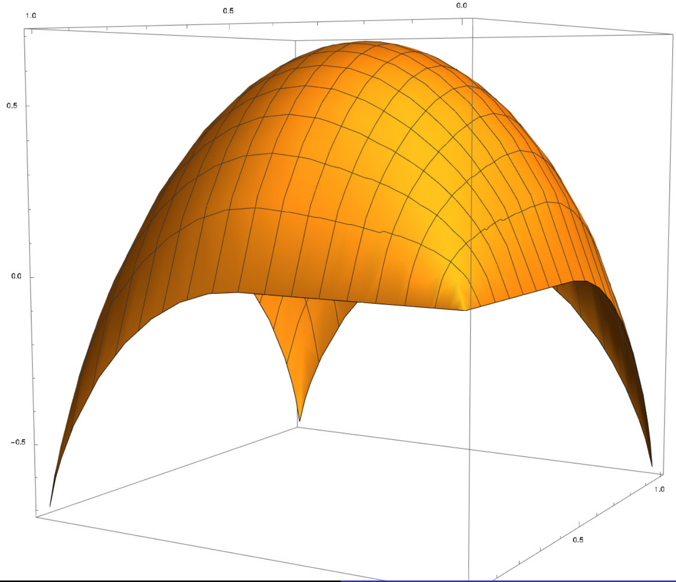
$$\frac{1}{n} \log \frac{\mathrm{M}(AD_n^{i,j})}{\mathrm{M}(AD_n)} \to f(a,b),$$

where the function $f(a,b)$ is defined by

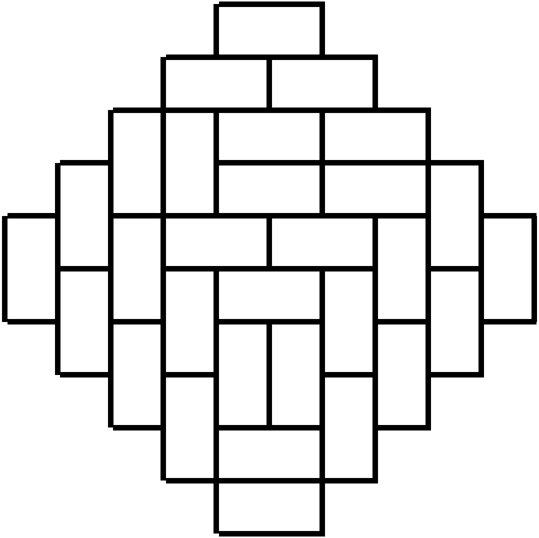
$$f(a,b) = \begin{cases} \log \left(\frac{a}{b} + \sqrt{1 + \frac{a^2}{b^2}} \right)^b \left(\frac{b}{a} + \sqrt{1 + \frac{b^2}{a^2}} \right)^a, & (1-a)(1-b) > \frac{1}{2}, \\ -\log 2a^a b^b (1-a)^{1-a} (1-b)^{1-b}, & (1-a)(1-b) < \frac{1}{2}. \end{cases}$$

Main results

The function $f(a, b)$



The simplest proof of the Aztec diamond theorem



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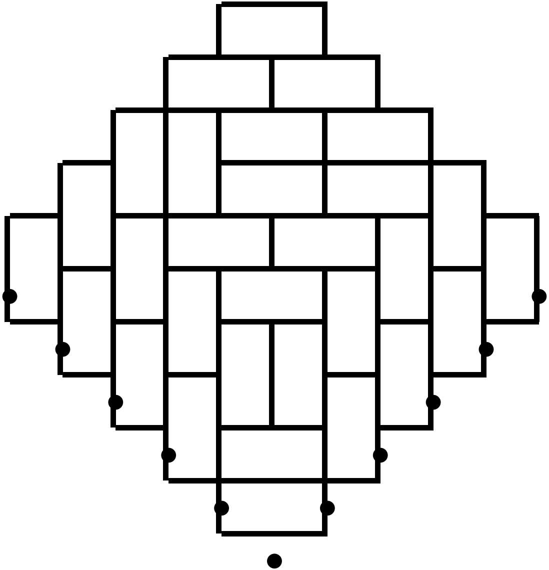

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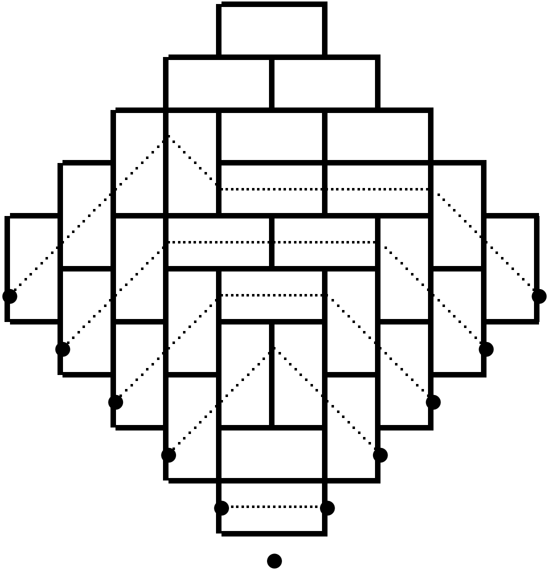

The simplest proof of the Aztec diamond theorem

By the Lindström–Gessel–Viennot theorem on non-intersecting lattice paths, we obtain

$$\mathrm{M}(AD_n) = \det\left(D(i,j)\right)_{0\le i,j\le n}.$$

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Claim. We have

$$\big(D(i,j)\big)_{0\le i,j\le n} = L_n \begin{pmatrix} 1 & & & & \\ & 2 & & & \\ & & 4 & & \\ & & & \ddots & \\ & & & & 2^n \end{pmatrix} L_n^t,$$

with $L_n = \left(\binom{i}{j}\right)_{0\le i,j\le n}$.

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The claim is equivalent to $D(i,j) = \sum_{k\ge 0} \binom{i}{k}\binom{j}{k} 2^k$.

The Aztec diamond theorem

Theorem (Elkies, Kuperberg, Larsen, Propp 1992)

The number of domino tilings of the Aztec diamond of size n is

$$2^{\binom{n+1}{2}}.$$

Proof of the first main result

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For $1 \leq i, j \leq n$, we have:

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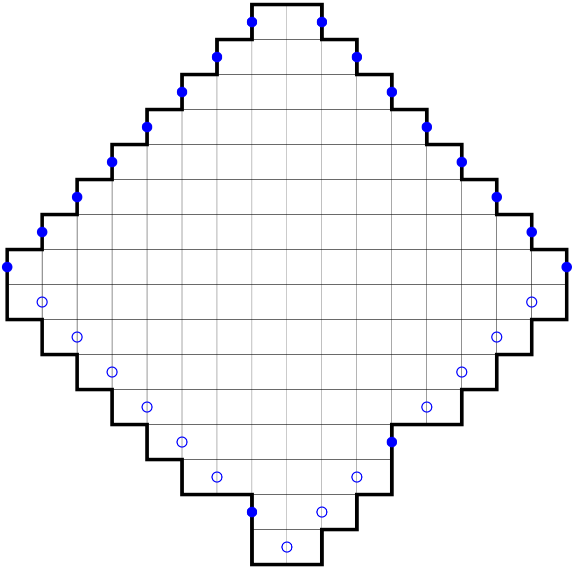

Proof of the first main result

We must compute the determinant of the Delannoy matrix

$$\det\left(D(r,s)\right)_{0\le r,s\le n,\ r\ne i,\ s\ne j}\ ,$$

where the *i*th row and *j*th column is omitted.

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However, this is (up to sign) the (j,i)-entry of the inverse matrix multiplied by the determinant of the complete matrix, that is, $2^{\binom{n+1}{2}}$.

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The inverse matrix is easily computed from the factorisation

$$\left(D(i,j)\right)_{0\le i,j\le n} = L_n \begin{pmatrix} 1 &&&& \\ & 2 &&& \\ && 4 && \\ &&& \ddots & \\ &&&& 2^n \end{pmatrix} L_n^t,$$

with $L_n = \left(\binom{i}{j}\right)_{0\le i,j\le n}$.

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If everything is put together, we obtain:

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Theorem

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$$\sim \begin{cases} \dfrac{(a+b+\sqrt{a^2+b^2})\left(\frac{(a+\sqrt{a^2+b^2})^b(b+\sqrt{a^2+b^2})^a}{a^a b^b}\right)^n}{2\sqrt{2\pi n}\sqrt{ab\sqrt{a^2+b^2}}}, & \text{if } (1-a)(1-b) > 1/2, \\[2ex] \dfrac{\sqrt{(1-a)(1-b)}\left(2a^a b^b(1-a)^{1-a}(1-b)^{1-b}\right)^{-n}}{4\pi n\sqrt{ab}\left(\frac{1}{2}-(1-a)(1-b)\right)}, & \text{if } (1-a)(1-b) < 1/2. \end{cases}$$

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We must asymptotically estimate the sum

$$\sum_{k=0}^{n-1} \binom{k}{i-1} \binom{k}{j-1} \frac{1}{2^{k+1}}$$

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Let

$$\begin{aligned} F(a,b;n,k) &:= \binom{k}{an}\binom{k}{bn}2^{-k-1} \\ &= \frac{\Gamma^2(k+1)}{\Gamma(an+1)\,\Gamma(k-an+1)\,\Gamma(bn+1)\,\Gamma(k-bn+1)}2^{-k-1}. \end{aligned}$$

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Let

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We determine the maximum, writing $k=nt$:

$$\frac{\partial}{\partial t}\frac{\Gamma^2(nt+1)}{\Gamma(an+1)\,\Gamma(tn-an+1)\,\Gamma(bn+1)\,\Gamma(tn-bn+1)}2^{-tn-1}=0.$$

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With $\psi(x)$ denoting the digamma function $\Gamma'(x)/\Gamma(x)$, this leads to

$$2\psi(tn+1)-\psi(tn-an+1)-\psi(tn-bn+1)-\log 2=0.$$

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With $\psi(x)$ denoting the digamma function $\Gamma'(x)/\Gamma(x)$, this leads to

$$2\psi(tn+1) - \psi(tn-an+1) - \psi(tn-bn+1) - \log 2 = 0.$$

We have $\psi(x) \sim \log x$ as $x \to \infty$. Hence, in a first approximation,

$$2\log(tn) - \log(tn-an) - \log(tn-bn) - \log 2 = 0,$$

or, equivalently,

$$\frac{t^2}{(t-a)(t-b)} = 2.$$

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The solutions are $t = a + b \pm \sqrt{a^2 + b^2}$, the maximum point is $t = a + b + \sqrt{a^2 + b^2}$.

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We recall that $k = nt$.

The important question is:

Is $t_0 = a + b + \sqrt{a^2 + b^2}$ smaller than 1 or not?

Proof of the second main result

The maximum point is $t = a + b + \sqrt{a^2 + b^2}$.

We recall that $k = nt$.

The important question is:

Is $t_0 = a + b + \sqrt{a^2 + b^2}$ smaller than 1 or not?

It turns out that

$$a + b + \sqrt{a^2 + b^2} < 1$$

is equivalent to

$$(1 - a)(1 - b) > \frac{1}{2}.$$

Proof of the second main result

Case 1. $(1-a)(1-b) > \frac{1}{2}$. By Stirling's approximation for the gamma function, we get

$$F(a,b;n,nt_0+l) = \frac{1}{2\pi\sqrt{2ab}\,n} \\ \times \exp\left(d(a,b)n + e(a,b)\frac{l}{n} - f(a,b)\frac{l^2}{n} + O\left(\frac{l^3}{n^2}\right)\right),$$

where

$$d(a,b) = -a\log a - b\log b + b\log\left(a+\sqrt{a^2+b^2}\right) \\ + a\log\left(b+\sqrt{a^2+b^2}\right),$$

$$e(a,b) = -\frac{\sqrt{a^2+b^2}}{\left(a+b+\sqrt{a^2+b^2}\right)^2},$$

$$f(a,b) = \frac{\sqrt{a^2+b^2}}{\left(\sqrt{a^2+b^2}+a+b\right)^2}.$$

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We use this approximation in the range $|l| < n^{3/5}$. It is not difficult to see that this range provides tthe main contribution to the sum $\sum_l F(a,b;n,nt_0+l)$, and that the other ranges are asymptotically negligible. In other words, we obtain

$$\sum_{k=0}^{n-1} F(a,b;n,k) \sim \frac{e^{d(a,b)n}}{2\pi\sqrt{2ab}\,n} \sum_{|l|<n^{3/5}} \exp\left(-f(a,b)\frac{l^2}{n}\right).$$

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Rewrite the sum as

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Case 2. $(1-a)(1-b) < \frac{1}{2}$. We have

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$$= F(a,b;n,n-1) \sum_{l=0}^{n-1} \frac{F(a,b;n,n-1-l)}{F(a,b;n,n-1)}.$$

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Using again Stirling's approximation, we get

$$\frac{F(a,b;n,n-1-l)}{F(a,b,n,n-1)} = \exp\Bigg(l \cdot \log\big(2(1-a)(1-b)\big)$$
$$+ \left(\frac{l}{n} + \frac{l^2}{n}\right) \frac{2ab-a-b}{2(1-a)(1-b)} + O\left(\frac{l^3}{n^2}\right)\Bigg)$$

and

$$F(a,b,n,n-1) = \frac{\sqrt{(1-a)(1-b)}}{2\pi\sqrt{ab}\,n}$$
$$\times \left(2a^a b^b (1-a)^{1-a} (1-b)^{1-b}\right)^{-n} \left(1 + O\left(\frac{1}{n}\right)\right).$$

Proof of the second main result

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✓

Conditional proof of the Arctic Circle Theorem

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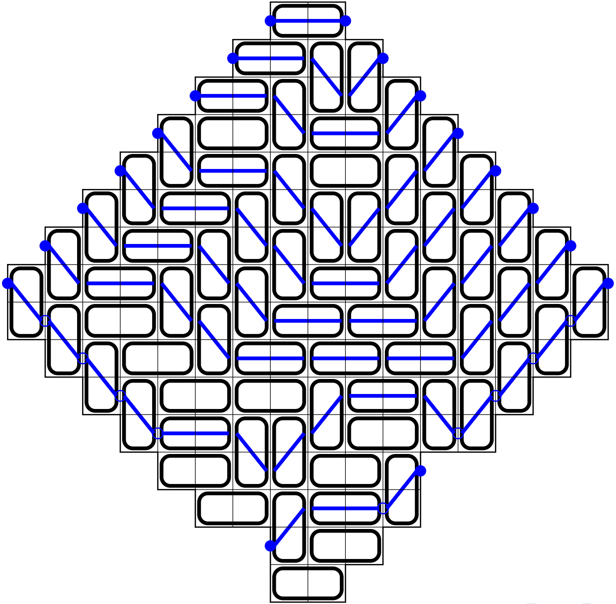

Epilogue

We have completely analogous results for rhombus tilings of hexagons.