The poset of bipartitions

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Let $X$ be a totally ordered alphabet, $w$ a word of length $n$ with letters from $X$. The *major index* of $w$ is

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**Theorem (MacMahon)**

*The inversion number and major index statistics are equidistributed over each rearrangement class $R(a_1, a_2, \ldots, a_k)$.  

$R(a_1, a_2, \ldots, a_k)$ is the set of all words consisting of $a_i$ letters $i$, $1 \leq i \leq k$.  

Gábor Hetyei and Christian Krattenthaler*
Generalization to relations

Let $U \subseteq X \times X$ be a relation, $w$ a word of length $n$ with letters from $X$. We define

$$\text{maj}_U(w) = n - 1 \sum_{i=1}^n \chi((w_i, w_{i+1}) \in U)$$

and

$$\text{inv}_U(w) = \sum_{1 \leq i < j \leq n} \chi((w_i, w_j) \in U).$$

Theorem (Foata–Zeilberger)

The statistics $\text{maj}_U(w)$ and $\text{inv}_U(w)$ are equidistributed over each rearrangement class $R(a_1, a_2, \ldots, a_k)$ if and only if $U$ is a bipartitional relation.
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The definition of a bipartitional relation
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Definition (G.-N. Han, rephrased)
A relation $U \subseteq X \times X$ on a finite set $X$ is a *bipartitional relation*, if both $U$ and $(X \times X) \setminus U$ are transitive.
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**Definition (Foata-Zeilberger)**

Let $(B_1, B_2, \ldots, B_k)$ be an ordered partition of $X$, and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) \in \{0, 1\}^k$. The bipartitional relation $U$ represented as $U(B_{\varepsilon_1}^1, B_{\varepsilon_2}^2, \ldots, B_{\varepsilon_k}^k)$ is

$$(x, y) \in U \iff \begin{cases} x \in B_i \text{ and } y \in B_j \text{ for some } i < j, \\ \text{or} \\ x, y \in B_i \text{ for some } i \text{ satisfying } \varepsilon_i = 1. \end{cases}$$
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**Example**

\[U = \{(3, 1), (3, 2), (1, 1), (1, 2), (2, 1), (2, 2)\}\]

is represented as

\[U(\{3\}^0, \{1, 2\}^1), \quad \text{or as} \quad U(\{3\}, \{1, 2\}).\]
We denote the set of bipartitional relations on $X$ by $\text{Bip}(X)$. 
Generalizing MacMahon's equidistribution result

The poset of bipartitions

Order complex of a poset and Möbius function

Discrete Morse theory and the Babson–Hersh result

Putting it all together

\[ \text{Bip}({1, 2}) \]
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Bip(\{1, 2\})

U(\{1, 2\})

U(\{1\}, \{2\})

U(\{1\}, \{2\})

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Quick facts about Bip($X$)
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- It is a graded poset with unique minimum element 
  \( U(\{1, 2, \ldots, n\}) = \emptyset \) and unique maximum element 
  \( U(\{1, 2, \ldots, n\}) = X \times X. \)
Quick facts about $\text{Bip}(X)$

- It is a graded poset with unique minimum element $U(\{1, 2, \ldots, n\}) = \emptyset$ and unique maximum element $U(\{1, 2, \ldots, n\}) = X \times X$.

- It is self-dual under complementation:
  $$X \times X \setminus U(B_1^{\varepsilon_1}, B_2^{\varepsilon_2}, \ldots, B_k^{\varepsilon_k}) = U(B_k^{1-\varepsilon_{k-1}}, B_{k-1}^{1-\varepsilon_{k-1}}, \ldots, B_1^{1-\varepsilon_1}).$$
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- It is a lattice. (Join $U \lor V$ is transitive closure of $U \cup V \cup U \circ V$)
Quick facts about \( \text{Bip}(X) \)

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- It is a lattice. (Join \( U \lor V \) is transitive closure of \( U \cup V \cup U \circ V \))

- It is not even modular.
The Möbius function is an important invariant of a poset. It is a function $\mu$ which assigns to each interval in a poset an integer. By definition, it is the inverse with respect to convolution of the so-called zeta function. In simple terms, this is

$$\mu([x, x]) = 1 \quad \text{for all } x,$$

$$\sum_{z: x \leq z \leq y} \mu([x, z]) = 0 \quad \text{for all } x < y.$$
An annoyingly simple conjecture

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Conjecture (1996)

The Möbius function of any interval in Bip($X$) is 0, 1 or $-1$. 

No idea how to prove it for over a decade.

Corollary (2009)

The above conjecture is true.

The precise theorems will be stated at the end of this talk.

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Order complex of a poset

Definition

Let $P$ be a partially ordered set (poset). The order complex $\Delta(P)$ of $P$ is the simplicial complex $\Delta(P) = \{\{x_1, \ldots, x_k\} : x_1 < \cdots < x_k \text{ in } P\}$.

Theorem (P. Hall)
Let $P$ be a partially ordered set (poset) with minimum $\hat{0}$ and maximum $\hat{1}$. We have $\mu([\hat{0}, \hat{1}]) = \tilde{\chi}(\Delta(P) \setminus \{\hat{0}, \hat{1}\})$, where $\tilde{\chi}$ denotes the reduced Euler characteristic.
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**Theorem (P. Hall)**

Let $P$ be a partially ordered set (poset) with minimum $\hat{0}$ and maximum $\hat{1}$. Then we have

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\mu ([\hat{0}, \hat{1}]) = \tilde{\chi}(\Delta(P \setminus \{\hat{0}, \hat{1}\}))
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where $\tilde{\chi}(\ . \ )$ denotes the reduced Euler characteristics.

The reduced Euler characteristics $\tilde{\chi}(\Delta)$ of a simplicial complex $\Delta$ is

$$
-1 + f_0 - f_1 + f_2 - + \cdots,
$$

where $f_i$ denotes the number of faces (cells) of $\Delta$ of dimension $i$ (i.e., containing $i + 1$ elements).
Forman’s discrete Morse theory for engineers
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- Unmatched cells are called *critical*.
- If there is no critical cell then the complex is contractible.
- If there is exactly one critical cell, then the complex is homotopy equivalent to a sphere of the dimension of that critical cell.

\[ \text{In particular: If } \{\hat{0}, \hat{1}\} \text{ is contractible, then } \mu([\hat{0}, \hat{1}]) = 0. \]
\[ \text{If } \{\hat{0}, \hat{1}\} \text{ is homotopy equivalent to a sphere of dimension } m, \text{ then } \mu([\hat{0}, \hat{1}]) = (-1)^m. \]
Forman’s discrete Morse theory for engineers

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- Unmatched cells are called critical.
- If there is no critical cell then the complex is contractible.
- If there is exactly one critical cell, then the complex is homotopy equivalent to a sphere of the dimension of that critical cell.

In particular: If $P$ is a poset with minimum $\hat{0}$ and maximum $\hat{1}$, then:
- if $\Delta(P \setminus \{\hat{0}, \hat{1}\})$ is contractible, then $\mu([\hat{0}, \hat{1}]) = 0$.
- if $\Delta(P \setminus \{\hat{0}, \hat{1}\})$ is homotopy equivalent to a sphere of dimension $m$, then $\mu([\hat{0}, \hat{1}]) = (-1)^m$. 
The Babson–Hersh implementation for order complexes
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- for each chain \( C_i \), one reads off a system \( I(C_i) \) of intervals contained in \( \{1, 2, \ldots, n\} \);
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- from the interval systems, one can construct an acyclic matching on the face poset of the order complex;
- properties of the interval system of a maximal chain $C_i$ tell one which cells are the critical cells under this matching.
The main result for Bip(X)
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**Theorem**

$\Delta(\text{Bip}(X) \setminus \{\emptyset, X \times X\})$ is homotopy equivalent to a sphere of dimension $|X| - 2$. 
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$\Delta(\text{Bip}(X) \setminus \{\emptyset, X \times X\})$ is homotopy equivalent to a sphere of dimension $|X| - 2$.

**Corollary**

We have $\mu([\emptyset, X \times X]) = (-1)^{|X|}$. 
Generalization to regular intervals

Definition
We say that an interval \([U, V]\) is regular if for every \(x\) belonging to a nonunderlined block in \(U\) and to an underlined block in \(V\), the block containing \(x\) in \(U\) is equal to the block containing \(x\) in \(V\). Otherwise we call \([U, V]\) irregular.
**Generalization to regular intervals**

**Definition**

We say that an interval \([U, V] \subseteq \text{Bip}(X)\) is *regular* if for every \(x\) belonging to a nonunderlined block in \(U\) and to an underlined block in \(V\), the block containing \(x\) in \(U\) is equal to the block containing \(x\) in \(V\). Otherwise we call \([U, V]\) *irregular.*
The case $|X| = 2$
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**Proposition**

Every regular interval \([U, V] \subseteq \text{Bip}(X)\) is isomorphic to a direct product of Boolean lattices and lattices of the form \(\text{Bip}(B)\), where each \(B\) is a block in the ordered bipartition representation of \(U\) and of \(V\) such that \(B\) is nonunderlined in \(U\) and underlined in \(B\).

**Corollary**

If \([U, V] \subseteq \text{Bip}\{1, 2, \ldots, n\}\) is regular, then \(\mu([U, V]) = (-1)^{\text{rk}(U) - \text{rk}(V)}\).
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Corollary

If \([U, V] \subseteq \text{Bip}\{1, 2, \ldots, n\}\) is regular, then
\[
\mu([U, V]) = (-1)^{rk(U) - rk(V)}.
\]
Generalization to irregular intervals

Theorem
If $[U, V] \subseteq \text{Bip} (\{1, 2, \ldots, n\})$ is not regular, then the order complex $\Delta ([U, V] \setminus \{U, V\})$ is contractible.

Corollary
If $[U, V] \subseteq \text{Bip} (\{1, 2, \ldots, n\})$ is not regular, then $\mu ([U, V]) = 0$. 

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