Chen Wang's proof of the Borwein Conjecture

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Prologue

The “birth” of the Borwein Conjecture

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September 1993: Workshop on *"Symbolic Computation in Combinatorics"*, Cornell University, USA (organised by Peter Paule and Volker Strehl)

George Andrews gave a two-part lecture on *"AXIOM and the Borwein Conjecture"*.

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Then the sign pattern of the coefficients in the expansion of this polynomial is $+--+--+--\cdots$.

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Example. $n=3$:

$$\begin{aligned}(1-q)(1-q^2)&(1-q^4)(1-q^5)(1-q^7)(1-q^8)\\ = {}& 1-q-q^2+q^3-q^4+2q^6-q^7-q^8\\ &+3q^9-q^{10}-q^{11}+2q^{12}-2q^{13}-2q^{14}+2q^{15}-q^{16}-q^{17}\\ &+3q^{18}-q^{19}-q^{20}+2q^{21}-q^{23}+q^{24}-q^{25}-q^{26}\\ &+q^{27}\end{aligned}$$

The "birth" of the Borwein Conjecture

More formally:

Let

$$(a;q)_m := \prod_{i=0}^{m-1} (1 - aq^i).$$

Conjecture (Peter Borwein)

Let the polynomials $A_n(q)$, $B_n(q)$ and $C_n(q)$ be defined by the relationship

$$\frac{(q;q)_{3n}}{(q^3;q^3)_n} = A_n(q^3) - qB_n(q^3) - q^2 C_n(q^3).$$

Then these polynomials have non-negative coefficients.

What did we know?

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There is a nice trick which allows one to use the q-binomial theorem in order to find elegant formulae for $A_n(q)$, $B_n(q)$, $C_n(q)$:

$$\begin{aligned}
&(1-q)(1-q^2)(1-q^4)(1-q^5)\cdots(1-q^{3n-2})(1-q^{3n-1})\\
&\quad=(1-q)(1-q^4)\cdots(1-q^{3n-2})\cdot(1-q^2)(1-q^5)\cdots(1-q^{3n-1})\\
&\quad=(-1)^n q^{(3n+1)n/2}(1-q^{-3n+1})\cdots(1-q^{-5})(1-q^{-2})\\
&\qquad\qquad\cdot(1-q)(1-q^4)\cdots(1-q^{3n-2})\\
&\quad=(-1)^n q^{(3n+1)n/2}\,(q^{-3n+1};q^3)_{2n}.
\end{aligned}$$

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We found

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Here we need the q-binomial theorem:

$$(z;q)_N = (1-z)(1-qz)\cdots(1-q^{N-1}z) \\ = \sum_{k=0} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} N \\ k \end{bmatrix}_q z^k$$

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Thus, we obtain

$$(1-q)(1-q^2)(1-q^4)(1-q^5)\cdots(1-q^{3n-2})(1-q^{3n-1}) \\ = \sum_{j=-n}^{n} (-1)^j q^{(3j+1)j/2} \begin{bmatrix} 2n \\ n+j \end{bmatrix}_{q^3}.$$

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$$=\sum_{j=-n}^{n}(-1)^j q^{(3j+1)j/2}\begin{bmatrix}2n\\ n+j\end{bmatrix}_{q^3}.$$

Since the q-binomial coefficient is on base q^3, it is easy to separate the terms with exponent $\equiv s$ modulo 3, $s=0,1,2$:

$$A_n(q)=\sum_{j=-\infty}^{\infty}(-1)^j q^{j(9j+1)/2}\begin{bmatrix}2n\\ n+3j\end{bmatrix}_q,$$

$$B_n(q)=\sum_{j=-\infty}^{\infty}(-1)^j q^{j(9j-5)/2}\begin{bmatrix}2n\\ n+3j-1\end{bmatrix}_q,$$

$$C_n(q)=\sum_{j=-\infty}^{\infty}(-1)^j q^{j(9j+7)/2}\begin{bmatrix}2n\\ n+3j+1\end{bmatrix}_q.$$

What did we know?

Compare with:

Theorem (Andrews, Baxter, Bressoud, Burge, Forrester, Viennot)

Let K be a positive integer, and m, n, α, β be non-negative integers, satisfying $\alpha + \beta < 2K$ and $\beta - K \le n - m \le K - \alpha$. Then the polynomial

$$\sum_{j \in \mathbb{Z}} (-1)^j q^{jK \frac{j(\alpha+\beta)+\alpha-\beta}{2}} \begin{bmatrix} m+n \\ n-Kj \end{bmatrix}_q$$

is the generating function for partitions inside an $m \times n$ rectangle that satisfy some so-called "hook difference conditions" specified by α, β and K.

What did we know?

In order to apply this theorem to the Borwein Conjecture, we have to choose $m = n$, $\alpha = 5/3$, $\beta = 4/3$ and $K = 3$.

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Alas, α and β are not integers!

Many people have tried to adapt the (combinatorial) arguments of Andrews et al. in order to cope with this situation, to no avail.

What did we know?

David Bressoud extended the mystery by making the following much more general conjecture.

Conjecture (David Bressoud)

Let m and n be positive integers, α and β be positive rational numbers, and K be a positive integer such that αK and βK are integers. If $1 \le \alpha + \beta \le 2K + 1$ (with strict inequalities if $K = 2$) and $\beta - K \le n - m \le K - \alpha$, then the polynomial

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{j(K(\alpha+\beta)j + K(\alpha-\beta))/2} \begin{bmatrix} m+n \\ m-Kj \end{bmatrix}$$

has non-negative coefficients.

What did we know?

Moderate progress on this generalised conjecture has been made. Alexander Berkovich and Ole Warnaar proved Bressoud's conjecture for several infinite families around 2005.

What did we know?

A partial result is:

Proposition (ANDREWS)

The power series $A_\infty(q)$, $B_\infty(q)$, $C_\infty(q)$ have non-negative coefficients. More precisely, we have

$$A_\infty(q) = \frac{(q^4, q^5, q^9; q^9)_\infty}{(q;q)_\infty},$$

$$B_\infty(q) = \frac{(q^2, q^7, q^9; q^9)_\infty}{(q;q)_\infty},$$

$$C_\infty(q) = \frac{(q^1, q^8, q^9; q^9)_\infty}{(q;q)_\infty},$$

where we use the short notation

$$(a_1, a_2, \dots, a_k; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_k; q)_\infty.$$

What did we know?

The proof uses Jacobi's triple product identity

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} z^k = (q;q)_\infty \, (z;q)_\infty \, (q/z;q)_\infty ,$$

a special case of which is Euler's pentagonal number theorem

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$$(q;q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2}.$$

Namely, we have

$$\frac{(q;q)_\infty}{(q^3;q^3)_\infty} = \frac{\sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2}}{(q^3;q^3)_\infty}.$$

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What did we know?

Even more generally:

Theorem (Andrews, P. Borwein and Garvan)

For any prime number p, if

$$\frac{(q;q)_\infty}{(q^p;q^p)_\infty} = \sum_{j=0}^{\infty} c_p(j) q^j,$$

then $c_p(j)$ and $c_p(j+p)$ have the same sign for all j.

Preliminaries

November 2017:

November 2017: Chen Wang tells me that he wants to prove the Borwein Conjecture.

Preliminaries

His starting point is another set of formulae of Andrews:

Theorem (Andrews)

Let, as before,

$$\frac{(q;q)_{3n}}{(q^3;q^3)_n} = A_n(q^3) - qB_n(q^3) - q^2C_n(q^3).$$

Then

$$A_n(q) = \sum_{j=0}^{n/3} \frac{q^{3j^2}(1-q^{2n})(q^3;q^3)_{n-j-1}(q;q)_{3j}}{(q;q)_{n-3j}(q^3;q^3)_{2j}(q^3;q^3)_j},$$

$$B_n(q) = \sum_{j=0}^{(n-1)/3} \frac{q^{3j^2+3j}(1-q^{3j+2}+q^{n+1}-q^{n+3j+2})(q^3;q^3)_{n-j-1}(q;q)_{3j}}{(q;q)_{n-3j-1}(q^3;q^3)_{2j+1}(q^3;q^3)_j},$$

$$C_n(q) = \sum_{j=0}^{(n-1)/3} \frac{q^{3j^2+3j}(1-q^{3j+1}+q^{n}-q^{n+3j+2})(q^3;q^3)_{n-j-1}(q;q)_{3j}}{(q;q)_{n-3j-1}(q^3;q^3)_{2j+1}(q^3;q^3)_j}.$$

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Wang had experimentally observed that, in this sum, the term for $j=0$ gives the main contribution to the coefficients in the polynomial, while the other terms contribute much less.

$$A_n(q) = \sum_{j=0}^{n/3} \frac{q^{3j^2}(1-q^{2n})(q^3;q^3)_{n-j-1}(q;q)_{3j}}{(q;q)_{n-3j}(q^3;q^3)_{2j}(q^3;q^3)_j}.$$

Wang had experimentally observed that, in this sum, the term for $j=0$ gives the main contribution to the coefficients in the polynomial, while the other terms contribute much less.

His idea hence was to estimate the contributions of the terms and show — at least for large n — that indeed the first term dominated the other terms.

$$A_n(q) = \sum_{j=0}^{n/3} \frac{q^{3j^2}(1-q^{2n})(q^3;q^3)_{n-j-1}(q;q)_{3j}}{(q;q)_{n-3j}(q^3;q^3)_{2j}(q^3;q^3)_j}.$$

Preliminaries

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For comparison:

$$A_n(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(9j+1)/2} \begin{bmatrix} 2n \\ n+3j \end{bmatrix}_q .$$

In this formula, the terms for which j is in a large range around 0 all contribute roughly the same. In other words, a large amount of cancellation happens which makes estimations difficult.

Fact: It "suffices" to prove non-negativity of the coefficients of

$$B_n(q) = \sum_{j=0}^{(n-1)/3} \frac{q^{3j^2+3j}(1 - q^{3j+2} + q^{n+1} - q^{n+3j+2})(q^3;q^3)_{n-j-1}(q;q)_{3j}}{(q;q)_{n-3j-1}(q^3;q^3)_{2j+1}(q^3;q^3)_j}.$$

Preliminaries

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Why?

- Because of the symmetry of Borwein's polynomial, we have

$$C_n(q) = q^{\deg B_n} B_n(1/q).$$

- We have

$$A_n(q) = (1+q^{2n-1})A_{n-1}(q) + q^n(B_{n-1}(q) + C_{n-1}(q)).$$

Further fact: The first n coefficients "are okay" (i.e., have the predicted sign pattern)!

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Why?

- Recall that it is a theorem that the infinite product
$$\frac{(q;q)_\infty}{(q^3;q^3)_\infty}$$
has the sign pattern $+--+--\cdots$.
- What is the difference between this and Borwein's polynomial?
$$\frac{(q;q)_{3n}}{(q^3;q^3)_n}=\frac{(q;q)_\infty}{(q^3;q^3)_\infty}\cdot\frac{(q^{3n+3};q^3)_\infty}{(q^{3n+1};q)_\infty}=\frac{(q;q)_\infty}{(q^3;q^3)_\infty}\cdot\left(1+O(q^{3n+1})\right).$$
Consequently, the first $3n$ coefficients (and hence also the $3n$ last coefficients) of the two polynomials agree!

Summary: It "suffices" to prove that

$$\langle q^m \rangle\, B_n(q)$$

is non-negative for $n \le m \le n^2 - 1 - n$.

Start of the proof

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The expression to be analysed:

$$B_n(q) = \sum_{j=0}^{(n-1)/3} \frac{q^{3j^2+3j}(1 - q^{3j+2} + q^{n+1} - q^{n+3j+2})(q^3;q^3)_{n-j-1}(q;q)_{3j}}{(q;q)_{n-3j-1}(q^3;q^3)_{2j+1}(q^3;q^3)_j}.$$

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Write

$$B_n(q) = \sum_{j=0}^{(n-1)/3} B_{n,j}(q).$$

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Write

$$B_n(q) = \sum_{j=0}^{(n-1)/3} B_{n,j}(q).$$

The term

$$1 - q^{3j+2} + q^{n+1} - q^{n+3j+2}$$

in the above sum is somewhat troublesome. Therefore, we split the summand $B_{n,j}(q)$ into smaller pieces.

Start of the proof

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Let

$$D_{n,j}(q) := \frac{q^{3j^2+3j}(q^3;q^3)_{n-j-1}(q;q)_{3j+1}}{(q;q)_{n-3j-1}(q^3;q^3)_{2j+1}(q^3;q^3)_j},$$

$$E_{n,j}(q) := \frac{q^{3j^2+3j}(1-q)(q^3;q^3)_{n-j-1}(q;q)_{3j}}{(q;q)_{n-3j-1}(q^3;q^3)_{2j+1}(q^3;q^3)_j}.$$

Then we have

$$B_{n,j}(q) = q(1+q^n)D_{n,j}(q) + E_{n,j}(q).$$

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Then we get

$$B_n(q) = q(1 + q^n)D_n(q) + E_n(q).$$

Thus, we must show non-negativity of the coefficients of $D_n(q)$ and $E_n(q)$.

For these two series, the same observation holds: the terms $D_{n,0}(q)$ and $E_{n,0}(q)$ (seem to) give the respective main contributions, while the terms $D_{n,j}(q)$ and $E_{n,j}(q)$ with $j \geq 1$ contribute much less.

How to do the estimations?

Cauchy's formula:

$$\langle q^m \rangle P_n(q) = \frac{1}{2\pi i}\int_\Gamma P_n(q)\frac{dq}{q^{m+1}} = \sum_{j=0}^{(n-1)/3} \frac{1}{2\pi i}\int_\Gamma P_{n,j}(q)\frac{dq}{q^{m+1}},$$

where $P_n(q)$ is either $D_n(q)$ or $E_n(q)$, and $P_{n,j}(q)$ is either $D_{n,j}(q)$ or $E_{n,j}(q)$.

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We choose as contour Γ a circle of radius r, where r has to be chosen so that it runs through the saddle point of $P_{n,j}(q)$. After substitution $q = re^{i\theta}$, we obtain

$$\langle q^m\rangle P_n(q) = \sum_{j=0}^{(n-1)/3}\frac{r^{-m}}{2\pi}\int_{-\pi}^{\pi} P_{n,j}(re^{i\theta})e^{-mi\theta}\,d\theta.$$

Outline of the proof

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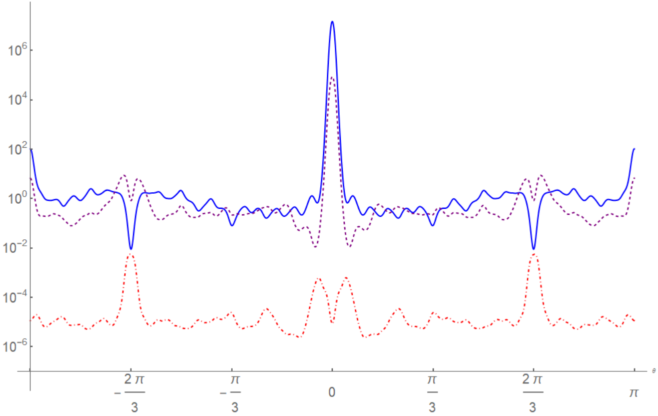

Modulus of $D_{36,0}(0.95e^{i\theta})$ (blue), of $D_{36,2}(0.95e^{i\theta})$ (purple, dashed), and of $D_{36,8}(0.95e^{i\theta})$ (red, dot-dashed). The vertical axis has a logarithmic scale.

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We need to cut the summation range (j) and the integration domain (θ) into pieces: to this end, we choose (appriate) cut-offs j_0 and θ_0. The following vocabulary "resonates" the strategy behind:

- The term *primary peak* refers to the part where $j = 0$ and $|\theta| \le \theta_0$.
- The term *secondary peaks* refers to the parts where $1 \le j \le j_0$ and $|\theta| \le \theta_0$.
- The term *tails* refers to the parts where $0 \le j \le j_0$ and $\theta_0 < |\theta| \le \pi$.
- Finally, the term *remainders* refers to the parts where $j > j_0$.

Outline of the proof

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The cut-offs are chosen as follows:

$$\theta_0 = \frac{1}{3} \frac{1-r}{1-r^n},$$
$$j_0 = \lfloor \log_2 n \rfloor,$$

where r is the value of the saddle point given by the unique solution to the saddle point equation

$$\frac{rP'_{n,0}(r)}{P_{n,0}(r)} = m.$$

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Lemma

For all $P \in \{D, E\}$, all integers $n \geq 1$, and $m \in (0, \deg P_n)$, the saddle point equation

$$\frac{d}{dr}\left(r^{-m} P_{n,0}(r)\right) = 0$$

has a unique solution $r \in \mathbb{R}^+$. Moreover, if $n \leq m \leq (\deg P_n)/2$, then we have $r_0 < r \leq 1$ where

$$r_0 = e^{-\sqrt{\alpha/n}},$$

and $\alpha = 2/\sqrt{3}$ is the maximum value of the function $x \mapsto \frac{1+2x}{1+x+x^2}$ on $[0, 1]$.

Outline of the proof

$$\langle q^m \rangle P_n(q) = \sum_{j=0}^{(n-1)/3} \frac{r^{-m}}{2\pi} \int_{-\pi}^{\pi} P_{n,j}(re^{i\theta}) e^{-mi\theta} \, d\theta.$$

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- The *primary peak* is estimated by a Gaußian integral. A relative error of $\varepsilon_{0,P}(n,r)$ occurs.
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Wang "tweaks" his estimations so that, for $n > 7000$ and $n \le m \le n^2/2$, these relative errors can be bounded above by:

P	$\varepsilon_{0,P} \le$	$\varepsilon_{1,P} \le$	$\varepsilon_{2,P} \le$	$\varepsilon_{3,P} \le$	Sum
D		0.197	0.237	0.004	0.982
E	0.544	0.046	0.266	0.008	0.864

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Theorem (Chen Wang)

Let the polynomials $A_n(q)$, $B_n(q)$ and $C_n(q)$ be defined by the relationship

$$\frac{(q;q)_{3n}}{(q^3;q^3)_n} = A_n(q^3) - qB_n(q^3) - q^2 C_n(q^3).$$

Then these polynomials have non-negative coefficients.

Some of the nasty details

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Lemma

Suppose that $u, v \in \mathbb{R}^+$. *Then*

$$\int_0^{\frac{3}{4}\frac{u}{v}} e^{-ux^2}\left(e^{vx^3}-1\right)\,dx < 1.1 \times \frac{v}{u^2}.$$

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$$\int_0^{\frac{3}{4\sqrt{2}}\sqrt{\frac{u}{v}}} e^{-ux^2}\left(e^{vx^4}-1\right)\,dx < \frac{1}{3\sqrt{3}}\frac{v^{1/2}}{u^{3/2}}.$$

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Lemma

For all $x \in [-1,1]$ and all $n \in \mathbb{Z}^+$, we have

$$T_n(x) \geq \frac{-n^2(1-x)(2x+3)+3(1+x)}{n^2(1-x)+3(1+x)},$$

where $T_n(x)$ is the n-th Chebyshev polynomial of the first kind.

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Lemma

Let $a, b \in \mathbb{Z}^+$ such that $b \ge 2$, and $r \in [0,1]$. Then we have

$$\sum_{m=a}^{a+b-1} r^{m-a}\sin(m\theta/2)^2 \ge \frac{1}{2}\frac{1-r^b}{1-r}\left(1-\sqrt{\frac{1+\kappa\frac{(1+r^b)^2}{(1-r^b)^2}\tan^2\frac{\theta}{2}}{1+\kappa\frac{(1+r)^2}{(1-r)^2}\tan^2\frac{\theta}{2}}}\right),$$

where $\kappa = \frac{(1-r^b)(1-r^{b/3})}{(1+r^b)(1+r^{b/3})}$.

Epilogue

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Conjecture (First Borwein Conjecture)

Let the polynomials $A_n(q)$, $B_n(q)$ and $C_n(q)$ be defined by the relationship

$$\frac{(q;q)_{3n}}{(q^3;q^3)_n} = A_n(q^3) - qB_n(q^3) - q^2 C_n(q^3).$$

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Conjecture (Second Borwein Conjecture)

Let the polynomials $\alpha_n(q)$, $\beta_n(q)$ and $\gamma_n(q)$ be defined by the relationship

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Conjecture (Third Borwein Conjecture)

Let the polynomials $\nu_n(q)$, $\phi_n(q)$, $\chi_n(q)$, $\psi_n(q)$ and $\omega_n(q)$ be defined by the relationship

$$\frac{(q;q)_{5n}}{(q^5;q^5)_n} = \nu_n(q^5) - q\phi_n(q^5) - q^2\chi_n(q^5) - q^3\psi_n(q^5) - q^4\omega_n(q^5),$$

Then these polynomials have non-negative coefficients.

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Conjecture (Chen Wang)

Let the polynomials $\widetilde{\alpha}_n(q)$, $\widetilde{\beta}_n(q)$ and $\widetilde{\gamma}_n(q)$ be defined by the relationship

$$\frac{(q;q)_{3n}^3}{(q^3;q^3)_n^3} = \widetilde{\alpha}_n(q^3) - q\widetilde{\beta}_n(q^3) - q^2\widetilde{\gamma}_n(q^3).$$

Then these polynomials have non-negative coefficients.

Can Wang's ideas be used to prove also these conjectures?

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PROBLEM: There are no reasonable explicit formulae for the coefficients $\alpha_n(q)$, $\beta_n(q)$, etc. in these conjectures. In particular, there is no analogue of Andrews'

$$B_n(q) = \sum_{j=0}^{(n-1)/3} \frac{q^{3j^2+3j}(1-q^{3j+2}+q^{n+1}-q^{n+3j+2})(q^3;q^3)_{n-j-1}(q;q)_{3j}}{(q;q)_{n-3j-1}(q^3;q^3)_{2j+1}(q^3;q^3)_j}.$$

Thus, there does not seem to be a starting point.

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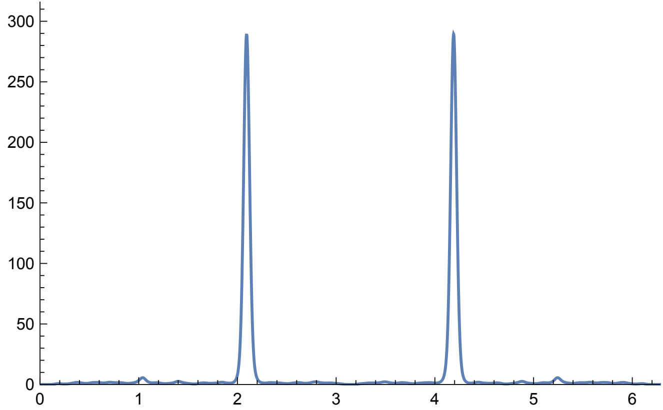

Modulus of $(q;q)_{30}/(q^3;q^3)_{10}$ at $q = .95e^{i\theta}$ at logarithmic scale

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Is the Borwein Conjecture (and its variations) about Combinatorics or Asymptotics?

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Not so clear . . .

For example, it seems that the "Borwein polynomial"

$$\frac{(q;q)_{pn}}{(q^p;q^p)_n}$$

has the sign pattern $+\underbrace{-\cdots-}_{p-1 \text{ times}}$ for coefficients of q^m in the "middle range" $pn \le m \le \binom{p}{2}n^2 - pn$ for n large enough.

Gaurav Bhatnagar and Michael Schlosser made several conjectures of "Borwein type" which are also "asymptotic" conjectures.