Non-intersecting Lattice Paths, Classical Group Characters, and Multivariate Hypergeometric Series

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Outline

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- Message of Ole Warnaar: "discrete Mehta-type integrals"

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- Classical group characters

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- Non-intersecting lattice paths
- Classical group characters
- Basic hypergeometric series

Discrete Mehta-type integrals?

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Ole Warnaar:

`Together with Richard Brent, I have recently been looking at sums of the form`

$$\sum_{k_1,\dots,k_r\in\mathbb{Z}} \left| \prod_{1\le i<j\le r} (k_i^\alpha - k_j^\alpha) \right|^\gamma \prod_{i=1}^{r} |k_i|^\delta \binom{2n}{n+k_i},$$

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`At least, for` $\alpha,\gamma\in\{1,2\}$ `and small` δ`, we believe that these sums can be evaluated in closed form.`

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$$(2\pi)^{-r/2}\int_{\mathbb{R}^r}\left|\prod_{1\le i<j\le r}(t_i-t_j)\right|^{\gamma}\prod_{i=1}^{r}e^{-t_i^2/2}\,dt_1\cdots dt_r = \prod_{i=1}^{r}\frac{\Gamma(1+i\gamma/2)}{\Gamma(1+\gamma/2)}.$$

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$$\sum_{k_1,\dots,k_r\in\mathbb{Z}}\prod_{1\le i<j\le r}(k_i-k_j)^2\prod_{i=1}^{r}\binom{2n}{n+k_i}\binom{2m}{m+k_i}$$
$$=\prod_{i=1}^{r}\binom{m+n}{i-1}^2\binom{2n}{i-1}\binom{2m}{i-1}(2m+2n-i-r+2)!\,(i-1)!^5$$

can be proved in various ways, one of which is by the use of Schur functions, as I pointed out in a paper 15 years ago.

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But, say,

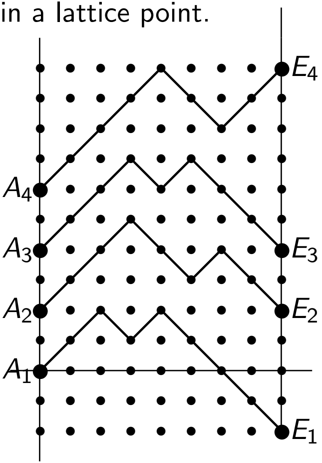
$$\sum_{k_1,\dots,k_r\in\mathbb{Z}} \prod_{1\le i<j\le r} (k_i^2-k_j^2)^2 \prod_{i=1}^{r} |k_i|\binom{2n}{n+k_i}\binom{2m}{m+k_i} = ??$$

We shall be concerned with paths in the integer lattice consisting of up-steps $(1,1)$ and down-steps $(1,-1)$.

Non-intersecting lattice paths

We shall be concerned with paths in the integer lattice consisting of up-steps $(1, 1)$ and down-steps $(1, -1)$.

A family of lattice paths is called *non-intersecting* if no two paths in the family meet in a lattice point.



Non-intersecting lattice paths

Theorem (Karlin–McGregor, Lindström, Gessel–Viennot, Fisher, John–Sachs, Gronau–Just–Schade–Scheffler–Wojciechowski)

Let G be an acyclic, directed graph, and let $A_1, A_2, \ldots, A_r$ and $E_1, E_2, \ldots, E_r$ be vertices in the graph with the property that, for $i < j$ and $k < l$, any (directed) path from A_i to E_l intersects with any path from A_j to E_k. Then the number of families $(P_1, P_2, \ldots, P_r)$ of non-intersecting (directed) paths, where the i-th path P_i runs from A_i to E_i, $i = 1, 2, \ldots, r$, is given by

$$\det_{1\le i,j\le r}\left(|\mathcal{P}(A_j \to E_i)|\right),$$

where $\mathcal{P}(A \to E)$ denotes the set of paths from A to E.

Non-intersecting lattice paths

Let $A_i = (0, 2(i-1))$ and $E_i = (n, k_i)$, $i = 1, 2, \ldots, r$, with $k_i \equiv n \pmod 2$. Then the number of families $(P_1, P_2, \ldots, P_r)$ of non-intersecting lattice paths, where P_i connects A_i with E_i, $i = 1, 2, \ldots, r$, is given by

$$\det_{1\le i,j\le r}\left(\binom{n}{j-1+\frac{1}{2}(n-k_i)}\right)$$

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$$\det_{1\le i,j\le r}\left(\binom{n}{j-1+\frac{1}{2}(n-k_i)}\right)$$
$$= \frac{\prod_{1\le i<j\le r}\left(\frac{1}{2}(k_j-k_i)\right)}{\prod_{i=1}^{r}\left(\frac{1}{2}(m-k_i)+r-1\right)!}\,\frac{\prod_{i=1}^{r}(m+i-1)!}{\prod_{i=1}^{r}\left(\frac{1}{2}(m+k_i)\right)!}.$$

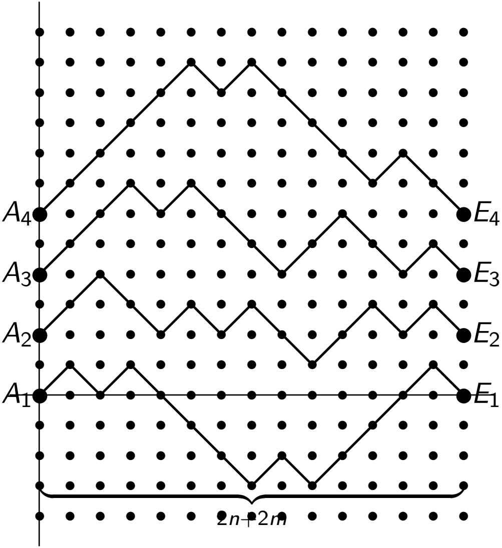
(ADC1, Theorem 26; *hook-content formula* in disguise)

We are ready to prove

$$\sum_{k_1,\dots,k_r\in\mathbb{Z}} \prod_{1\le i<j\le r} (k_i - k_j)^2 \prod_{i=1}^{r} \binom{2n}{n+k_i} \binom{2m}{m+k_i}$$
$$= \prod_{i=1}^{r} \binom{m+n}{i-1}^2 \binom{2n}{i-1} \binom{2m}{i-1} (2m+2n-i-r+2)!\,(i-1)!^5,$$

and the proof consists in one picture!

Non-intersecting lattice paths



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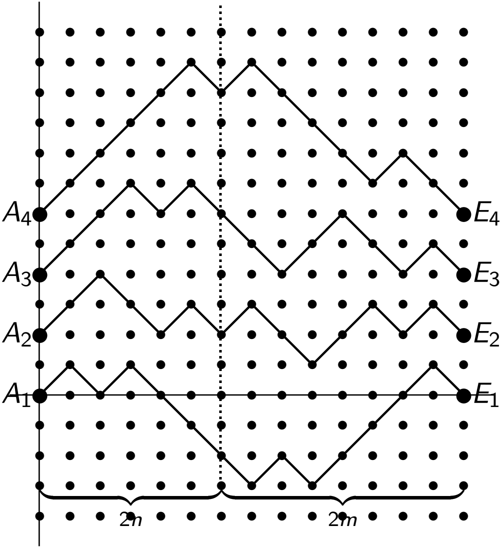

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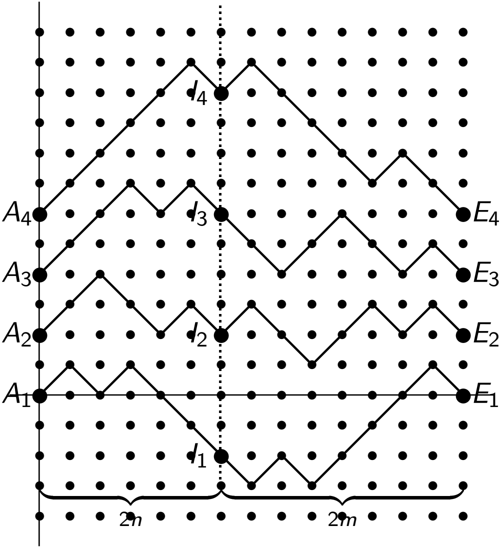

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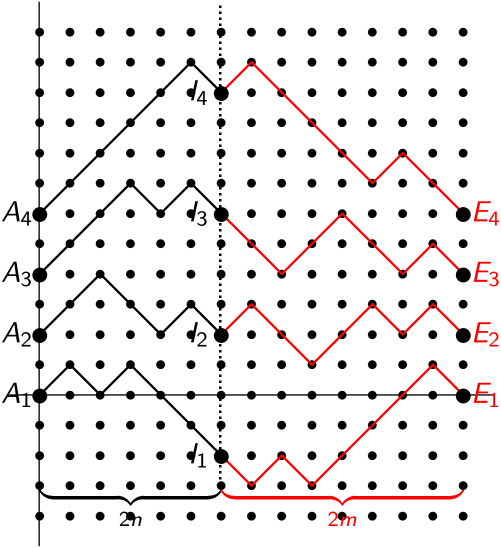

Non-intersecting lattice paths

If everything is worked out, then the previous picture does indeed prove

$$\sum_{k_1,\dots,k_r\in\mathbb{Z}} \prod_{1\le i<j\le r} (k_i-k_j)^2 \prod_{i=1}^{r} \binom{2n}{n+k_i}\binom{2m}{m+k_i}$$
$$= \prod_{i=1}^{r} \binom{m+n}{i-1}^2 \binom{2n}{i-1}\binom{2m}{i-1}(2m+2n-i-r+2)!\,(i-1)!^5.$$

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$$\sum_{k_1,\dots,k_r\in\mathbb{Z}} \prod_{1\le i<j\le r} (k_i^2-k_j^2)^2 \prod_{i=1}^{r} |k_i|^2 \binom{2n}{n+k_i}\binom{2m}{m+k_i} = ??$$

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The above sum is equivalent to

$$2^r\, r! \sum_{0\le k_1<\cdots<k_r} \prod_{1\le i<j\le r} (k_i^2-k_j^2)^2 \prod_{i=1}^{r} k_i^2 \binom{2n}{n+k_i}\binom{2m}{m+k_i} = ??$$

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$$\prod_{1\le i<j\le r}(k_i^2-k_j^2)\prod_{i=1}^{r}k_i=\prod_{1\le i<j\le r}(k_i-k_j)(k_i+k_j)\prod_{i=1}^{r}k_i\ ?$$

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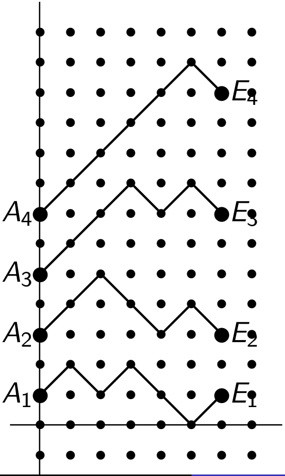
By non-intersecting lattice paths again!

Non-intersecting lattice paths

Let $A_i = (0, 2i-1)$ and $E_i = (n, k_i - 1)$, $i = 1, 2, \ldots, r$, with $k_i \equiv n \pmod 2$. Here, the non-intersecting lattice paths that we consider have the the additional property that paths never run below the x-axis.



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By the main theorem on non-intersecting lattice paths, the number of families of these non-intersecting lattice paths is again given by a determinant. The individual entries are obtained by the reflection principle:

$$\det_{1 \le i,j \le r} \left(\binom{n}{j + \frac{1}{2}(n - k_i)} - \binom{n}{-j + 1 + \frac{1}{2}(n - k_i)} \right).$$

And, once again, this determinant

$$\det_{1\le i,j\le r}\left(\binom{n}{j+\frac{1}{2}(n-k_i)}-\binom{n}{-j+1+\frac{1}{2}(n-k_i)}\right)$$

And, once again, this determinant can be evaluated:

$$\det_{1\le i,j\le r}\left(\binom{n}{j+\frac{1}{2}(n-k_i)}-\binom{n}{-j+1+\frac{1}{2}(n-k_i)}\right)$$
$$=\prod_{1\le i<j\le r}\left(\tfrac{1}{2}(k_j-k_i)\right)\left(\tfrac{1}{2}(k_j+k_i-2)\right)$$
$$\times\prod_{i=1}^{r}\frac{(k_i-1)\,(n+2i-2)!}{\left(\frac{1}{2}(n-k_i)+r\right)!\,\left(\frac{1}{2}(n+k_i)+r-1\right)!}.$$

(ADC1, Theorem 30; dimension formula for irreducible representations of $Sp_{2n}(\mathbb{C})$ in disguise)

And, once again, this determinant can be evaluated:

$$\det_{1\le i,j\le r}\left(\binom{n}{j+\frac{1}{2}(n-k_i)}-\binom{n}{-j+1+\frac{1}{2}(n-k_i)}\right)$$

$$=\prod_{1\le i<j\le r}\left(\tfrac{1}{2}(k_j-k_i)\right)\left(\tfrac{1}{2}(k_j+k_i-2)\right)$$

$$\times\prod_{i=1}^{r}\frac{(k_i-1)\,(n+2i-2)!}{\left(\frac{1}{2}(n-k_i)+r\right)!\,\left(\frac{1}{2}(n+k_i)+r-1\right)!}.$$

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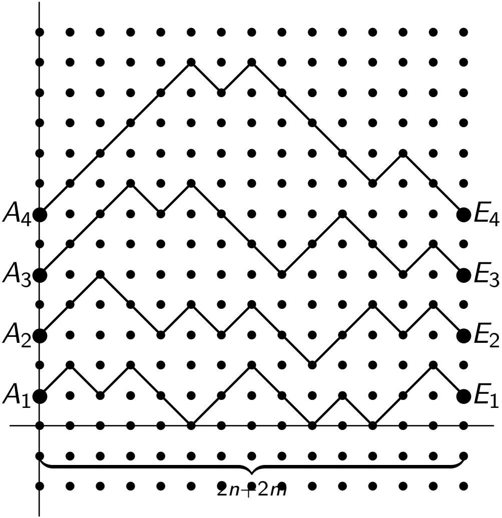
One can "smell" the type B Vandermonde product: one only needs to replace k_i by $2k_i+1$ (which you need to take if n is odd).

Here is the one-picture proof of

$$\sum_{0\le k_1<\cdots<k_r}\prod_{1\le i<j\le r}(k_i^2-k_j^2)^2\prod_{i=1}^{r}k_i^2\binom{2n}{n+k_i}\binom{2m}{m+k_i}$$
$$=2^{(m+n)r-3\binom{r+1}{2}}\prod_{i=1}^{r}\frac{(2n)!}{(2n-2i+1)!}\frac{(2m)!}{(2m-2i+1)!}$$
$$\cdot\frac{(2i-1)!\,(2m+2n-2i-2r+1)!!}{(m+n-i+1)!}.$$

Non-intersecting lattice paths



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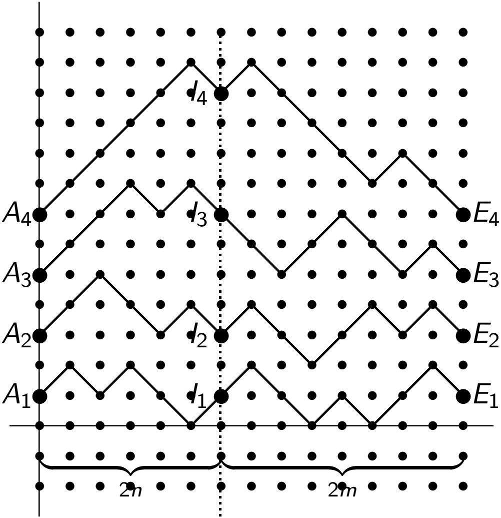
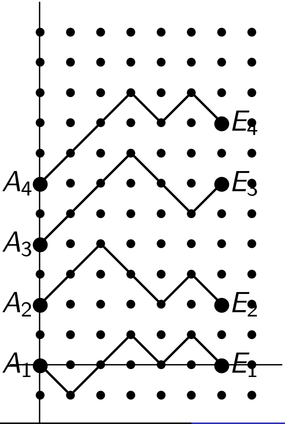


Non-intersecting lattice paths

There is a third scenario, where this idea works:

Let $A_i = (0, 2(i-1))$ and $E_i = (n, k_i)$, $i = 1, 2, \ldots, r$, with $k_i \equiv n \pmod 2$. Here, we consider families of non-intersecting lattice paths with the property that the family remains non-intersecting if any of the paths are reflected in the x-axis.



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Non-intersecting lattice paths

Let $A_i = (0, 2(i-1))$ and $E_i = (n, k_i)$, $i = 1, 2, \ldots, r$, with $k_i \equiv n \pmod 2$. Here, we consider families of non-intersecting lattice paths with the property that the family remains non-intersecting if any of the paths are reflected in the x-axis.

By a combination of the Lindström–Gessel–Viennot involution and path reflections, it can be shown that the number of the above families is given by the determinant

$$\frac{1}{2}\det_{1\le i,j\le r}\left(\binom{m}{j-1+\frac{1}{2}(m-k_i)}+\binom{m}{-j+1+\frac{1}{2}(m-k_i)}\right),$$

which equals the closed form product

$$\prod_{1\le i<j\le r}\left(\tfrac{1}{2}(k_j-k_i)\right)\left(\tfrac{1}{2}(k_j+k_i)\right)\prod_{i=1}^{r}\frac{(m+2i-2)!}{\left(\frac{1}{2}(m-k_i)+r-1\right)!\,\left(\frac{1}{2}(m+k_i)+r-1\right)!}.$$

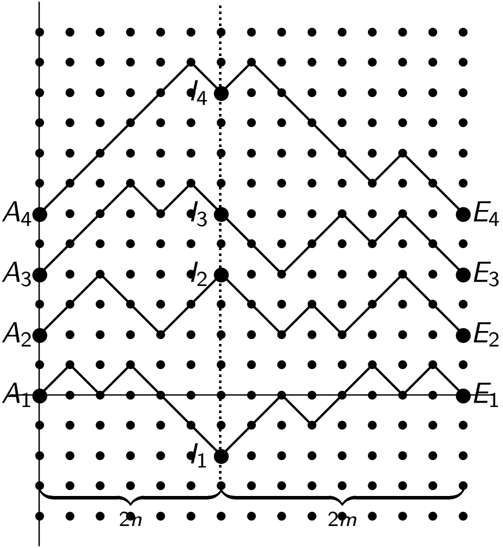
(ADC1, Theorem 31; dimension formula for irreducible representations of $O_{2n}(\mathbb{C})$ in disguise)

Consequently, there is a one-picture proof of another identity, namely

$$\sum_{k_1,\dots,k_r\in\mathbb{Z}} \prod_{1\le i<j\le r} (k_i^2-k_j^2)^2 \prod_{i=1}^{r} \binom{2n}{n+k_i}\binom{2m}{m+k_i}$$

$$= r!\,2^{(m+n+2)r-\binom{r}{2}} \prod_{i=1}^{r} \frac{(2n)!}{(2n-2i+2)!}\,\frac{(2m)!}{(2m-2i+2)!} \cdot \frac{(2i-2)!\,(2m+2n-2i-2r+3)!!}{(m+n-i+1)!}.$$

Non-intersecting lattice paths



Classical group characters

Classical group characters

The classical group characters of interest here are:

- *Schur functions* $s_N(\lambda;\mathbf{x})$;
- *symplectic characters* $sp_{2n}(\lambda;\mathbf{x}^{\pm 1})$;
- *orthogonal characters* $o_N(\lambda;\mathbf{x}^{\pm 1})$.

They are indexed by *partitions* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, i.e., integer sequences with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$.

The sets of variables used here are

$$\mathbf{x} = (x_1, \ldots, x_N)$$
$$\mathbf{x}^{\pm 1} = \begin{cases} (x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}), & \text{if } N = 2n, \\ (x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}, 1), & \text{if } N = 2n+1. \end{cases}$$

Classical group characters

Let $e_k(x_1, \ldots, x_N)$ denote the k-th *elementary symmetric functions*

$$e_k(x_1, \ldots, x_N) = \sum_{1 \le i_1 < \cdots < i_k \le N} x_{i_1} \cdots x_{i_k}.$$

Furthermore, let λ' denote the partition conjugate to λ.

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Furthermore, let λ' denote the partition conjugate to λ.

- *Schur functions*

$$s_N(\lambda;\mathbf{x}) = \det_{1\le i,j\le\lambda_1}\left(e_{\lambda'_i-i+j}(\mathbf{x})\right).$$

- *symplectic characters*

$$sp_{2n}(\lambda;\mathbf{x}) = \det_{1\le i,j\le\lambda_1}\left(e_{\lambda'_i-i+j}(\mathbf{x}^{\pm1}) - e_{\lambda'_i-i-j}(\mathbf{x}^{\pm1})\right).$$

- *orthogonal characters*

$$o_N(\lambda;\mathbf{x}) = \frac{1}{2}\det_{1\le i,j\le\lambda_1}\left(e_{\lambda'_i-i+j}(\mathbf{x}^{\pm1}) + e_{\lambda'_i-i-j+2}(\mathbf{x}^{\pm1})\right).$$

Combinatorial interpretation of Schur functions

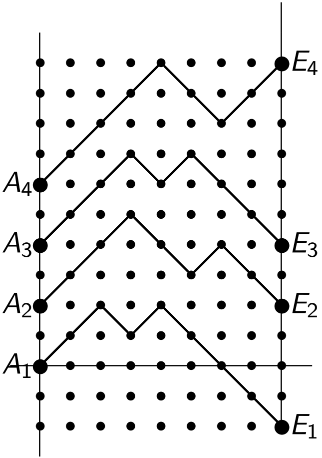
Let $A_i = (0, 2(i-1))$ and $E_i = (N, k_i)$, $i = 1, 2, \ldots, r$. The *Schur function* corresponding to these data is

$$s_N(\lambda; \mathbf{x}) = \sum_F w(F),$$

where F ranges over families of non-intersecting lattice paths which connect A_i to E_i, $i = 1, 2, \ldots, r$.

(*semistandard tableaux* in disguise)

Combinatorial interpretation of Schur functions



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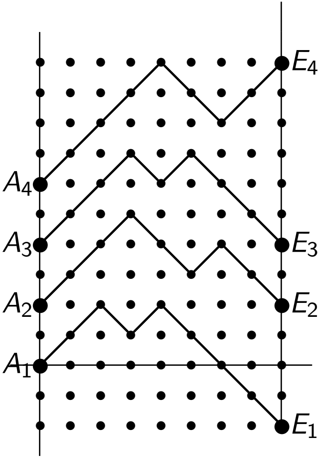
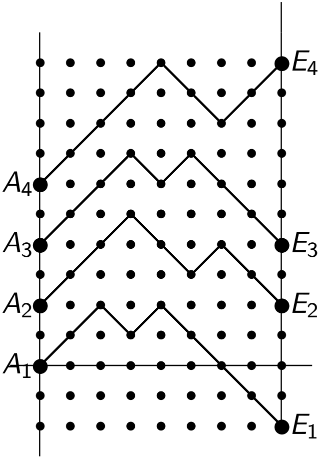


Combinatorial interpretation of Schur functions

$\lambda' = (5, 4, 4, 2)$

$\lambda = (4, 4, 3, 3, 1)$



$$\lambda'_i = \#(\text{down-steps between } A_i \text{ and } E_i)$$

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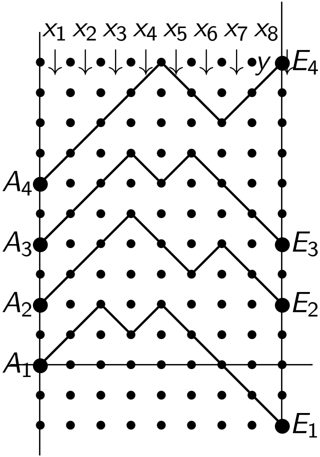


$w(F) = x_3 x_4^2 x_5^3 x_6^3 x_7^3 x_8^3$

$$\lambda'_i = \#(\text{down-steps between } A_i \text{ and } E_i)$$

$$w(F) = \prod x_i^{\#(\text{down-steps in ``column } i\text{''})}$$

Combinatorial interpretation of symplectic characters

Let $A_i = (0, 2i - 1)$ and $E_i = (2n, k_i - 1)$, $i = 1, 2, \dots, r$. Here, the non-intersecting lattice paths that we consider have the the additional property that paths never run below the x-axis.

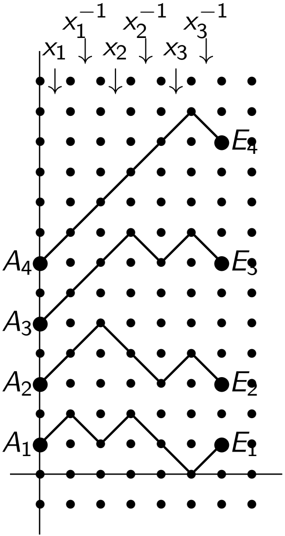
The *symplectic character* corresponding to these data is

$$sp_{2n}(\lambda; \mathbf{x}) = \sum_F w'(F),$$

where F ranges over the above families of non-intersecting lattice paths which connect A_i to E_i, $i = 1, 2, \dots, r$.

(*King and El-Sharkaway's symplectic tableaux* in disguise)

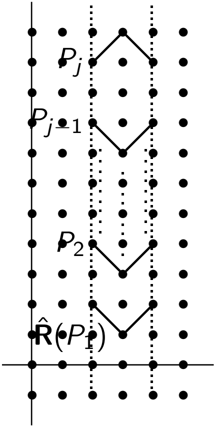
Combinatorial interpretation of symplectic characters

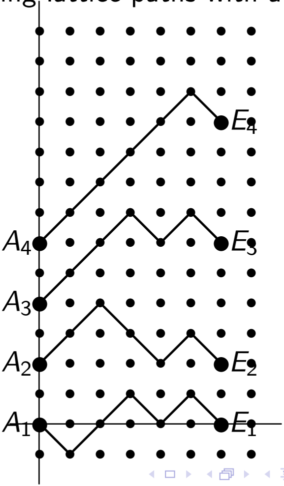


$$w'(F) = x_1^{-1}x_2^{-2}x_3^{-2}$$

Combinatorial interpretation of orthogonal characters

Let $A_i = (0, 2(i-1))$ and $E_i = (N, k_i)$, $i = 1, 2, \ldots, r$. Here, we consider families of non-intersecting lattice paths with a technical additional property.





Combinatorial interpretation of orthogonal characters

The *orthogonal character* corresponding to these data is

$$o_N(\lambda;\mathbf{x}) = \sum_F w'(F),$$

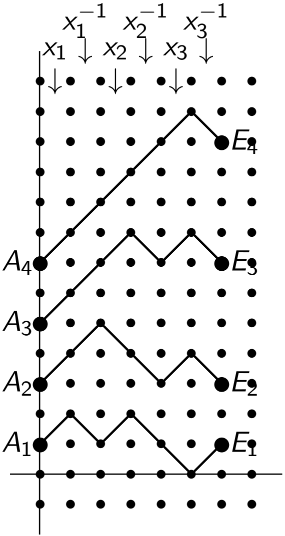
where F ranges over families of non-intersecting lattice paths which connect A_i to E_i, $i = 1, 2, \ldots, r$, and the weight $w'(F)$ is the same as for symplectic characters.

(*Proctor's orthogonal tableaux* of the first kind in disguise)

Why do these combinatorial interpretations work?

For example, in the symplectic case,

Combinatorial interpretation of symplectic characters



$$w'(F) = x_1^{-1} x_2^{-2} x_3^{-2}$$

Why do these combinatorial interpretations work?

For example, in the symplectic case, the (weighted) Lindström–Gessel–Viennot theorem implies that

$$\sum_F w'(F) = \det\left(\sum_{P:A_j\to E_i,\ \text{pos.}} w'(P)\right).$$

So, we need to compute

$$\sum_{P:A\to E,\ \text{pos.}} w'(P)$$

for given lattice points A and E.

Why do these combinatorial interpretations work?

So, we need to compute

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for given lattice points $A = (0, a)$ and $E = (2n, e)$.

A modified reflection principle

So, we need to compute

$$\sum_{P:A\to E,\ \text{pos.}} w'(P) = \mathrm{GF}(\text{all paths}) - \mathrm{GF}(\text{bad paths})$$

for given lattice points $A = (0, a)$ and $E = (2n, e)$.

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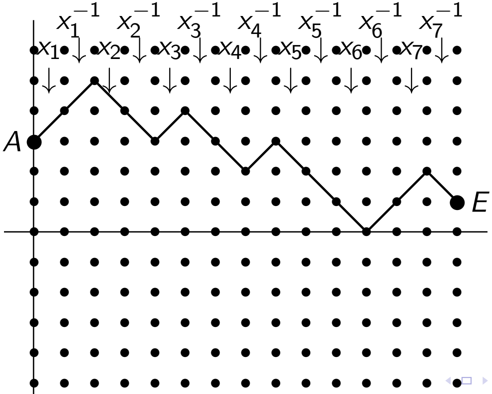
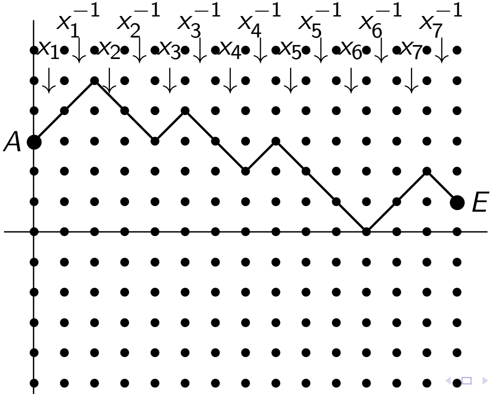


A modified reflection principle

So, we need to compute

$$\sum_{P:A\to E,\ \text{pos.}} w'(P) = e_{(n+a-e)/2}(\mathbf{x}^{\pm 1}) - \text{GF(bad paths)}$$

for given lattice points $A = (0, a)$ and $E = (2n, e)$.



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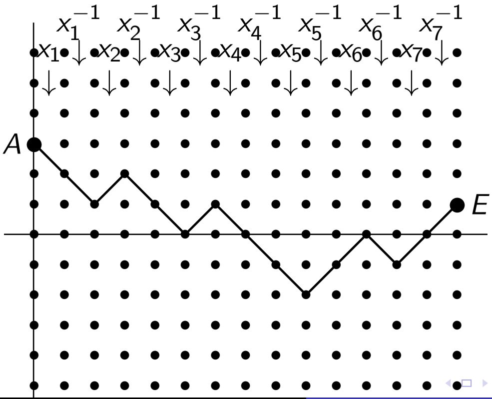

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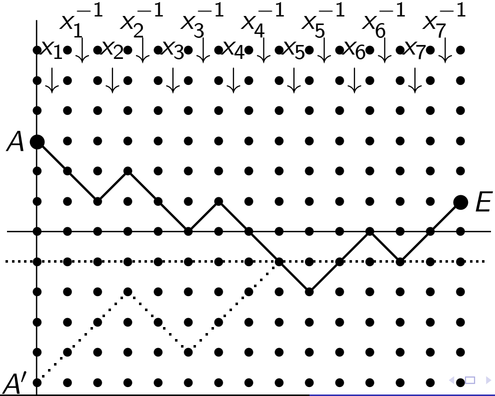

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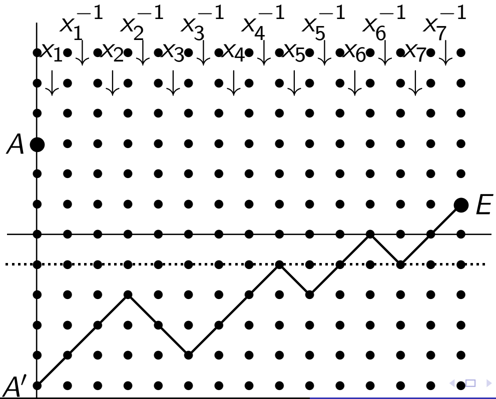
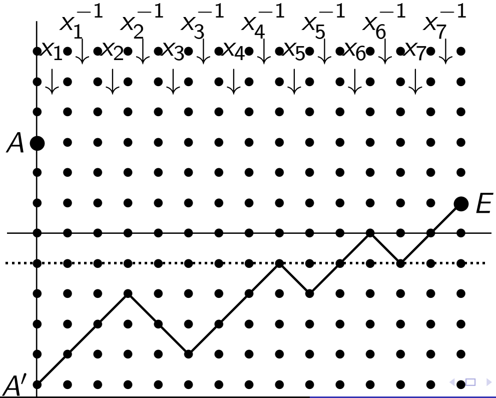


A modified reflection principle

So, we need to compute

$$\sum_{P:A\to E,\ \text{pos.}} w'(P) = e_{(n+a-e)/2}(\mathbf{x}^{\pm 1}) - e_{(n-a-e-2)/2}(\mathbf{x}^{\pm 1})$$

for given lattice points $A = (0, a)$ and $E = (2n, e)$.



A modified reflection principle

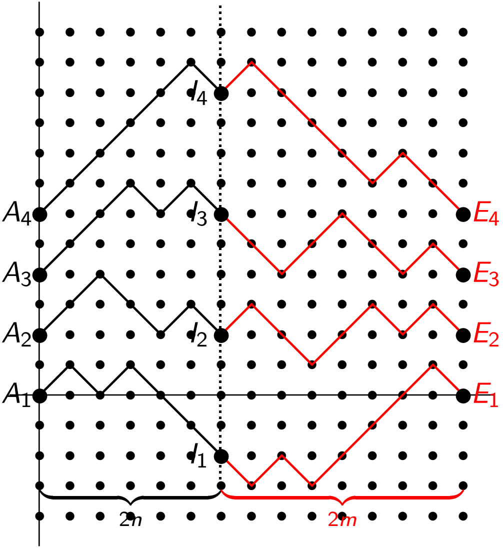
If the previous finding is substituted back in

$$\sum_F w'(F) = \det \left(\sum_{P: A_j \to E_i,\ \text{pos.}} w'(P) \right),$$

we obtain

$$\sum_F w'(F) = \det_{1 \le i,j \le \lambda_1} \left(e_{\lambda'_i - i + j}(\mathbf{x}^{\pm 1}) - e_{\lambda'_i - i - j}(\mathbf{x}^{\pm 1}) \right) = sp_{2n}(\lambda; \mathbf{x}).$$

Classical group characters



Classical group characters

Theorem

For all non-negative integers m and n with $m \le n$, we have

$$\sum_{\lambda:\lambda_1\le r} s_m\big(\lambda;\mathbf{x}\big)s_n\big((r^{n-m},\lambda);\mathbf{y}\big) = s_{m+n}\big((r^{m+n});\mathbf{x},\mathbf{y}\big).$$

Classical group characters

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For all non-negative integers m and n with $m \le n$, we have

$$\sum_{\lambda:\lambda_1 \le r} sp_{2m}\left(\lambda; \mathbf{x}^{\pm 1}\right) sp_{2n}\left((r^{n-m}, \lambda); \mathbf{y}^{\pm 1}\right) = sp_{2(m+n)}\left((r^{m+n}); \mathbf{x}^{\pm 1}, \mathbf{y}^{\pm 1}\right).$$

and

$$\sum_{\lambda:\lambda_1 \le r} sp_{2m+1}\left(\lambda; \mathbf{x}^{\pm 1}; z\right) sp_{2n+1}\left((r^{n-m}, \lambda); \mathbf{y}^{\pm 1}; z\right) = z^r \, sp_{2(m+n+1)}\left((r^{m+n}); \mathbf{x}^{\pm 1}, \mathbf{y}^{\pm 1}, z^{\pm 1}\right).$$

The symplectic characters on the left-hand side are the "odd symplectic characters" of Proctor.

Classical group characters

Theorem

For all non-negative integers m and n with $m \le n$, we have

$$\sum_{\lambda:\lambda_1\le r} o_{2m}\left(\lambda;\mathbf{x}^{\pm1}\right) o_{2n}\left((r^{n-m},\lambda);\mathbf{y}^{\pm1}\right) = o_{2(m+n)}\left((r^{m+n});\mathbf{x}^{\pm1},\mathbf{y}^{\pm1}\right).$$

and

$$\sum_{\lambda:\lambda_1\le r} o_{2m+1}\left(\lambda;\mathbf{x}^{\pm1},1\right) o_{2n+1}\left((r^{n-m},\lambda);\mathbf{y}^{\pm1},1\right) = o_{2(m+n+1)}\left((r^{m+n});\mathbf{x}^{\pm1},\mathbf{y}^{\pm1},1^{\pm1}\right).$$

Classical group characters

As a matter of fact, these identities had been found earlier by Soichi Okada, except for one, the identity

$$\sum_{\lambda:\lambda_1\le r} sp_{2m+1}\left(\lambda;\mathbf{x}^{\pm1};z\right) sp_{2n+1}\left((r^{n-m},\lambda);\mathbf{y}^{\pm1};z\right) = z^r\, sp_{2(m+n+1)}\left((r^{m+n});\mathbf{x}^{\pm1},\mathbf{y}^{\pm1},z^{\pm1}\right)$$

for odd symplectic characters. He used heavy determinant calculations for proving his formulae.

q-analogues?

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Yes, for example,

$$\sum_{k_1,\dots,k_r\in\mathbb{Z}} \prod_{1\le i<j\le r} [k_j-k_i]_q^2\,[k_i+k_j]_q^2 \prod_{i=1}^{r} q^{k_i^2-(2i-1)k_i} \left|[k_i]_{q^2}\right| \begin{bmatrix} 2n \\ n+k_i \end{bmatrix}_q \begin{bmatrix} 2m \\ m+k_i \end{bmatrix}$$

$$= r!\left(\frac{2}{1+q}\right)^r q^{-2\binom{r+1}{3}} \prod_{i=1}^{r} \frac{[2n]_q!}{[n-i+1]_q!\,[n-i]_q!}\,\frac{[2m]_q!}{[m-i+1]_q!\,[m-i]_q!} \cdot \frac{[i-1]_q!^2\,[m+n-i-r+1]_q!}{[m+n-i+1]_q!},$$

where

$$[\alpha]_q = \frac{1-q^\alpha}{1-q},$$

$$[n]_q! = [n]_q\,[n-1]_q\cdots[1]_q,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}.$$

Yes, also for those where the Vandermonde-type product is not squared, for example

$$\sum_{k_1,\dots,k_r\in\mathbb{Z}} \prod_{1\le i<j\le r} [k_j-k_i]_q\,[k_i+k_j]_q \prod_{i=1}^{r} q^{\binom{k_i-r+i}{2}} |[k_i]_q| \begin{bmatrix} 2n \\ n+k_i \end{bmatrix}_q$$

$$= 2^r\, r! \prod_{i=1}^{r} (-q;\sqrt{q})_{2n-2i} \frac{(q^{3/2};q)_\infty\,(q^{2n-i+2};q)_\infty\,(q^{n-i+1};q)_\infty}{(q^i;q)_\infty\,(q^{2n+1};q)_\infty\,(q^{n-i+\frac{3}{2}};q)_\infty},$$

but . . .


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