Robinson–Schensted–Knuth correspondence, jeu de taquin, and growth diagrams

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The RSK correspondence

The Robinson–Schensted correspondence and its generalisation by Knuth, the Robinson–Schensted–Knuth correspondence, are very fundamental bijections between permutations (words) and pairs of standard Young tableaux (semistandard tableaux).

They appear prominently in combinatorics (of course), representation theory, algebraic geometry, commutative algebra, probability theory, among others.

Standard Young tableaux

A *standard Young tableau* is a left-justified array of the integers $1, 2, \ldots, n$, such that entries are increasing along rows and columns.

Given a standard Young tableau T, the vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where λ_i is the length of the i-th row of T is called the *shape* of T.

Example

$n = 12$, $\lambda = (4, 3, 2, 2, 1)$.

$$\begin{matrix} 1 & 3 & 4 & 8 \\ 2 & 6 & 7 \\ 5 & 10 \\ 9 & 12 \\ 11 \end{matrix}$$

Robinson–Schensted correspondence

The Robinson–Schensted correspondence is a bijection

$$\pi \quad \longleftrightarrow \quad (P, Q)$$

between *permutations* of $\{1, 2, \dots, n\}$ and pairs (P, Q) of *standard Young tableaux of the same shape*.

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Hence,

$$n! = \sum_{\lambda \vdash n} f_\lambda^2,$$

where f_λ denotes the number of standard Young tableaux of shape λ.

Robinson–Schensted correspondence

Example

$\pi = 5137426$:

$$\emptyset \longrightarrow 5 \longrightarrow \begin{array}{l}1\\5\end{array} \longrightarrow \begin{array}{l}13\\5\end{array} \longrightarrow \begin{array}{l}137\\5\end{array}$$

$$\longrightarrow \begin{array}{l}134\\57\end{array} \longrightarrow \begin{array}{l}124\\37\\5\end{array} \longrightarrow \begin{array}{l}1246\\37\\5\end{array}$$

Robinson–Schensted correspondence

Example

$\pi = 5137426$:

$$(\emptyset\ ,\ \emptyset) \longrightarrow (5\ ,\ 1) \longrightarrow \begin{pmatrix} 1 & & 1 \\ 5 & , & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 13 & & 13 \\ 5 & , & 2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 137 & & 134 \\ 5 & , & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 134 & & 134 \\ 57 & , & 25 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 124 & & 134 \\ 37 & , & 25 \\ 5 & & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1246 & & 1347 \\ 37 & , & 25 \\ 5 & & 6 \end{pmatrix}$$

Growth diagrams

Growth diagrams

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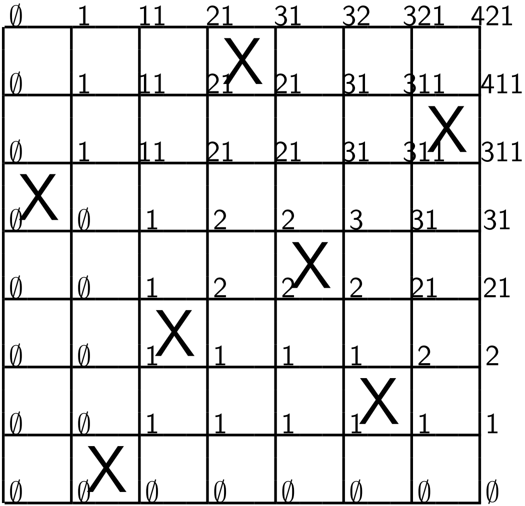

Informally:

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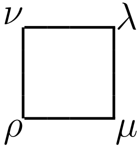
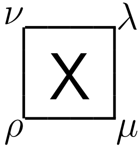
The corners of cells are labelled by partitions.

Along each row or column of a diagram, partitions are contained in each other, growing by at most one at a time. Growth by one can only happen if in the row/column to the left/below there is an X.



a. A cell without cross



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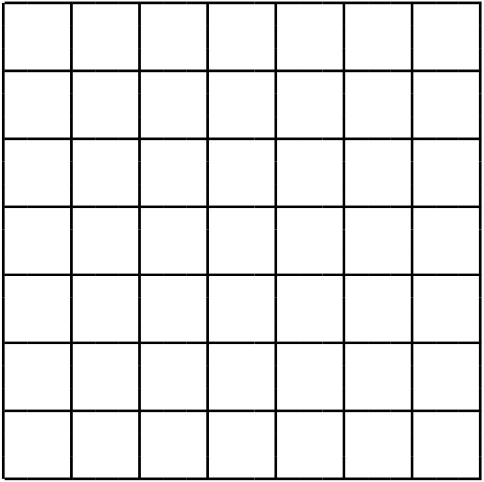

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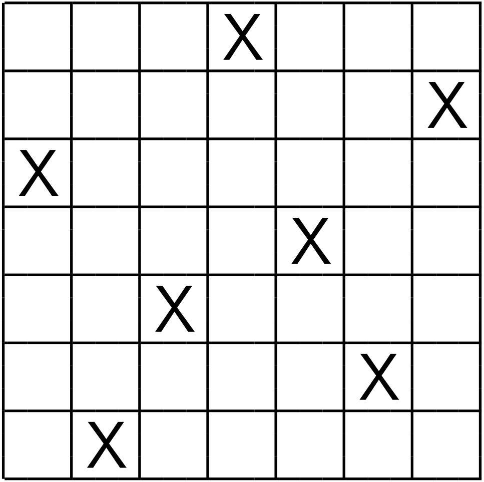

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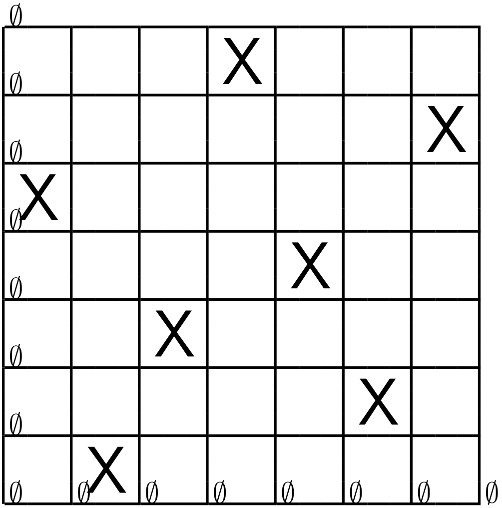

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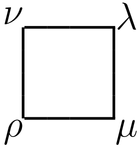

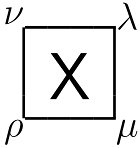

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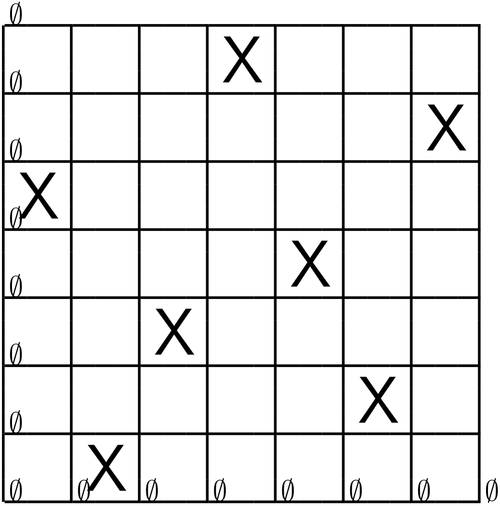

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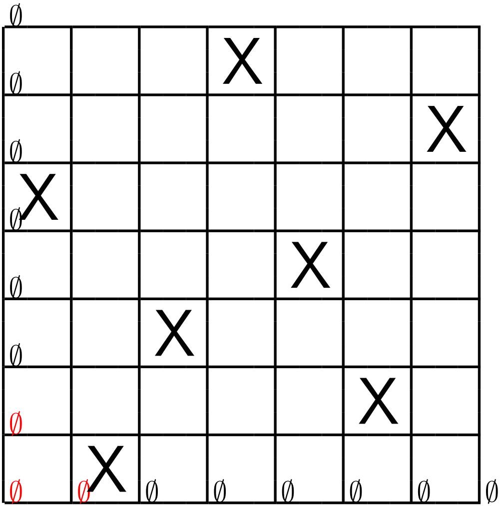

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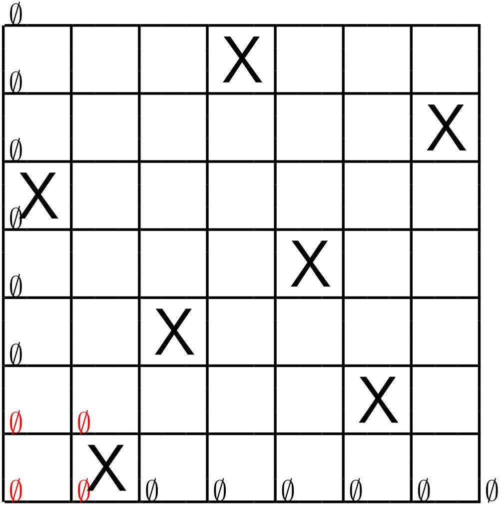

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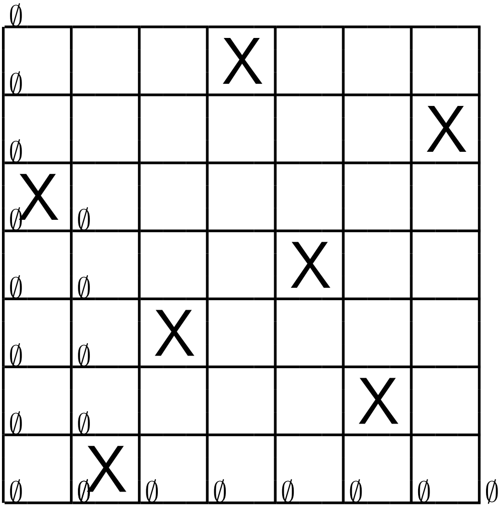

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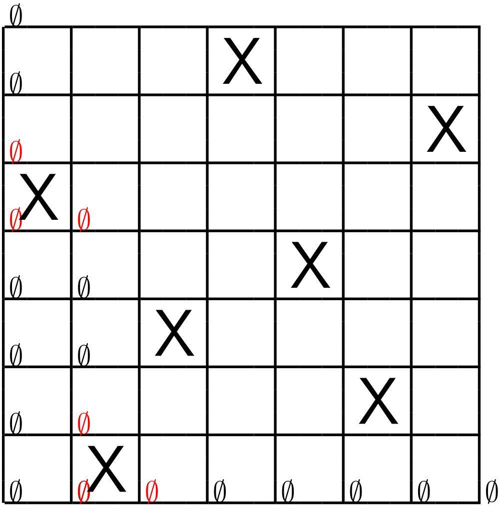

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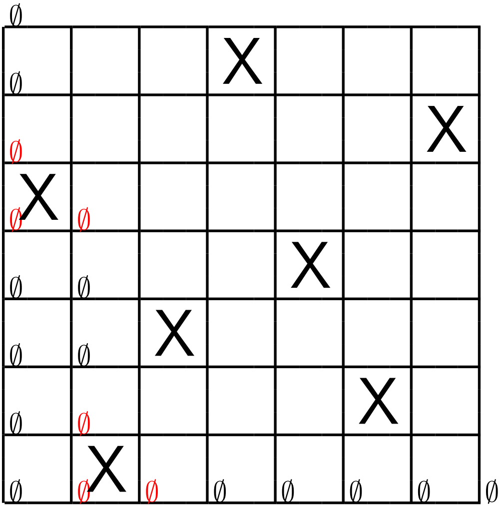

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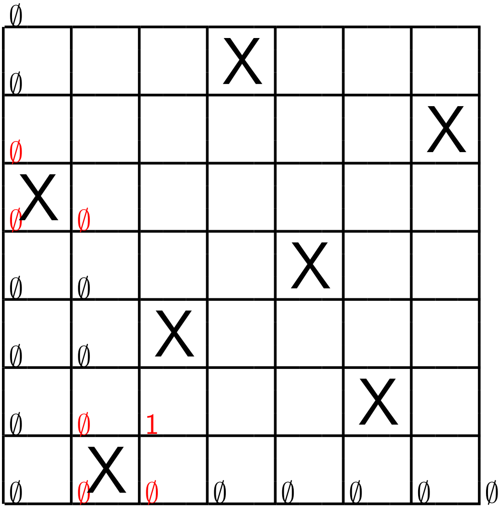

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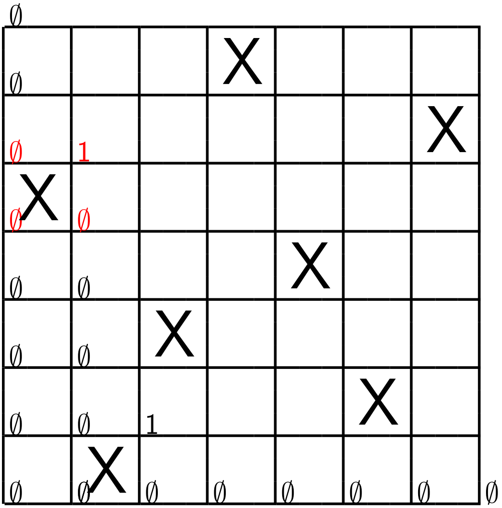

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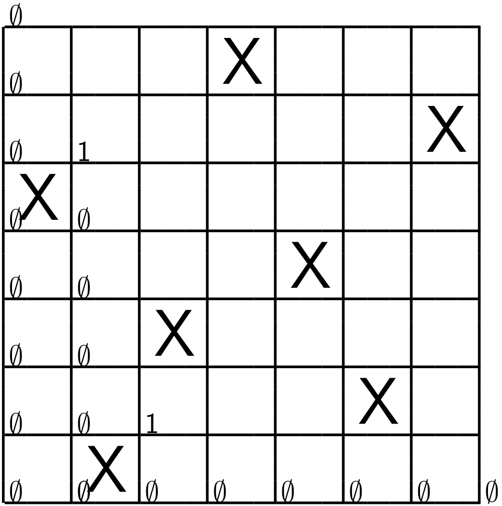

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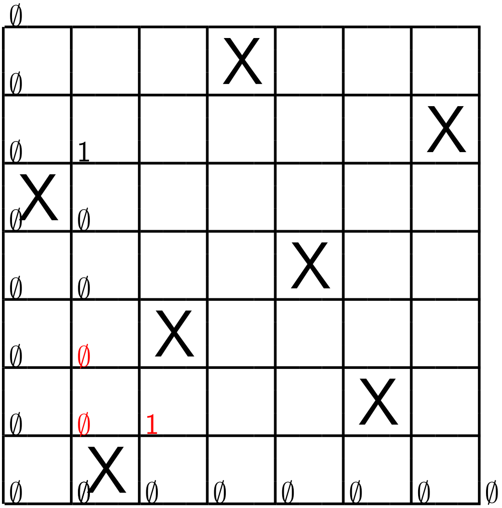

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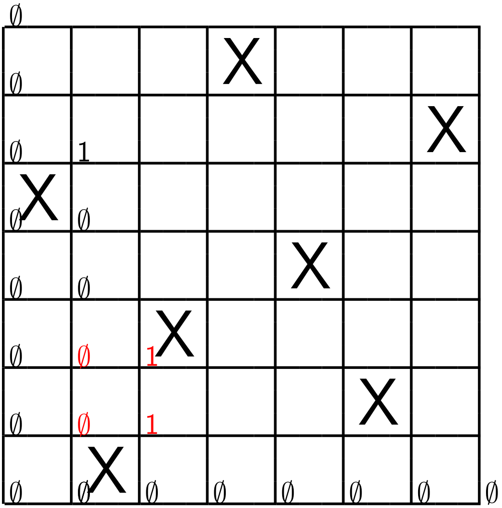

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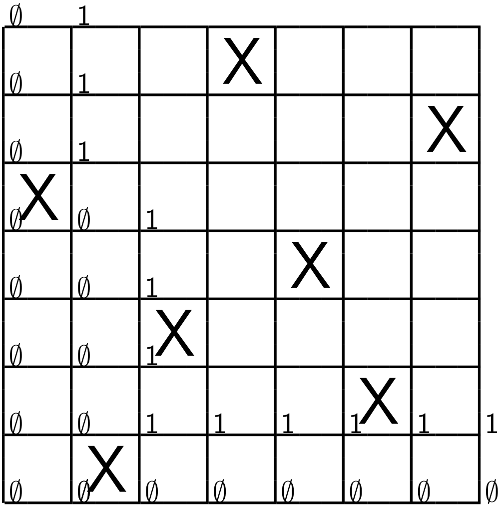

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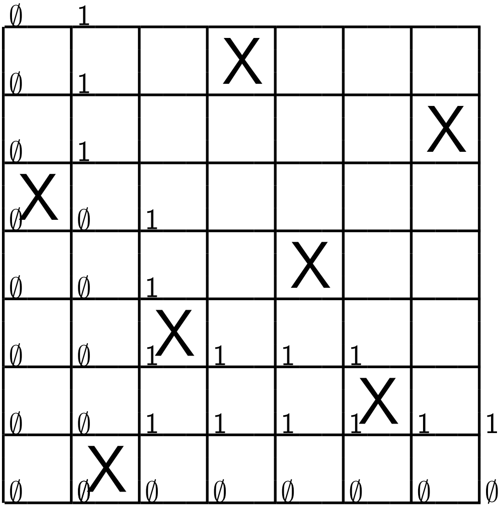

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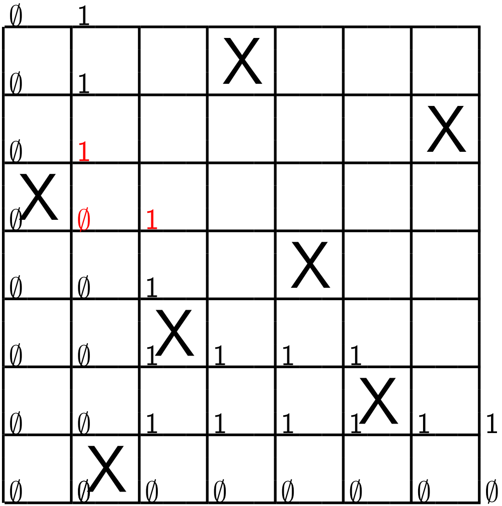

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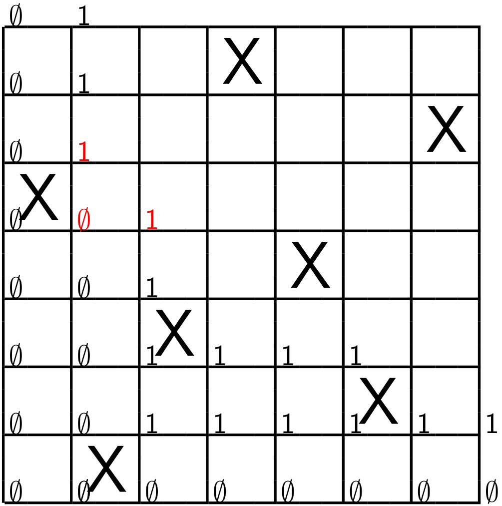

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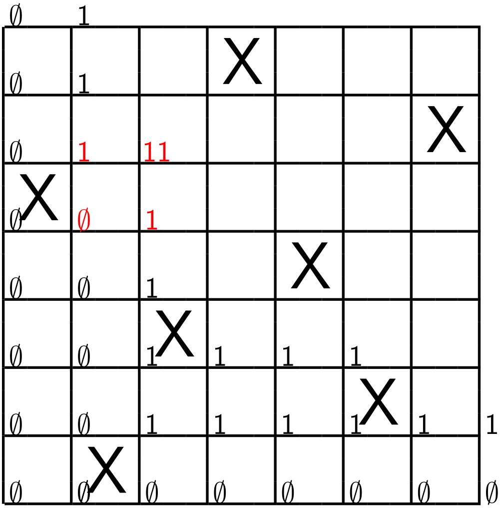

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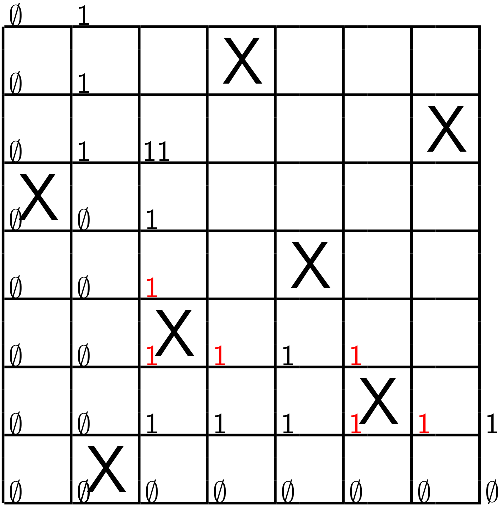

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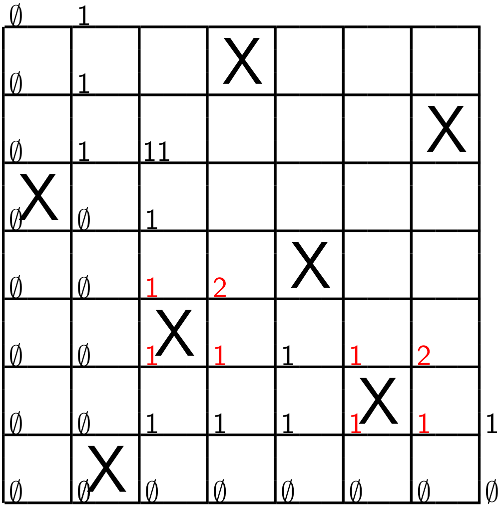

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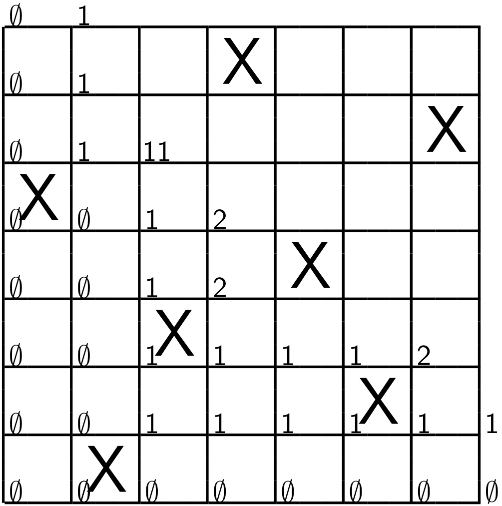

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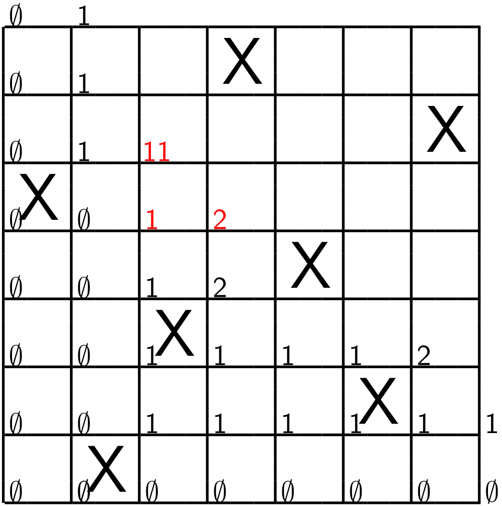

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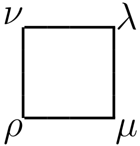

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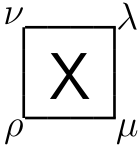

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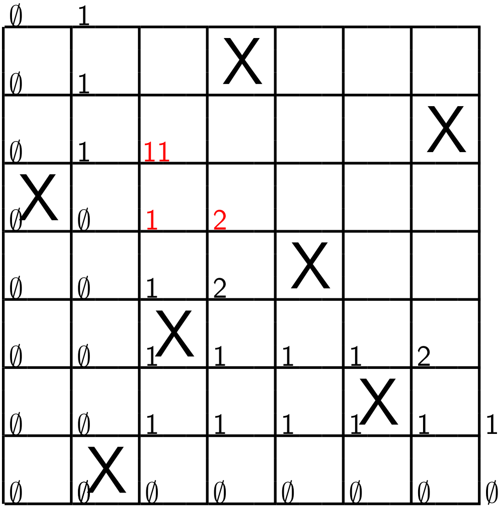

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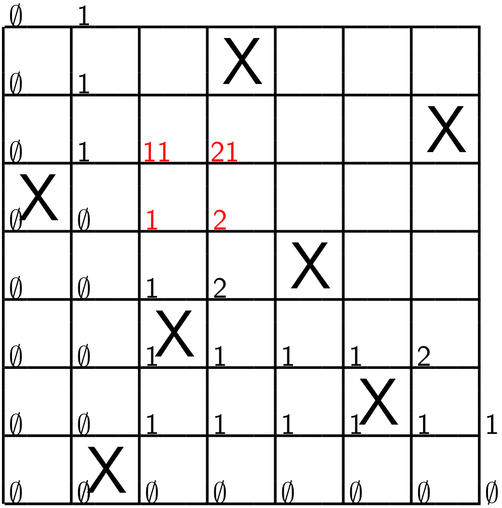

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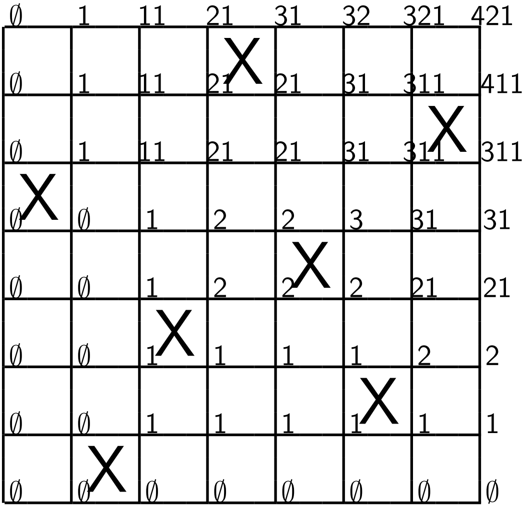

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Along the right border of the diagram we read:

$$\emptyset,\ 1,\ 2,\ 21,\ 31,\ 311,\ 411,\ 421.$$

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Along the top border of the diagram we read:

$$\emptyset,\ 1,\ 11,\ 21,\ 31,\ 32,\ 321,\ 421.$$

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Along the right border of the diagram we read:

$$\emptyset,\ 1,\ 2,\ 21,\ 31,\ 311,\ 411,\ 421.$$

Along the top border of the diagram we read:

$$\emptyset,\ 1,\ 11,\ 21,\ 31,\ 32,\ 321,\ 421.$$

These two sequences of shapes correspond to the pair

$$\begin{pmatrix} \begin{matrix} 1246 \\ 37 \\ 5 \end{matrix} & , & \begin{matrix} 1347 \\ 25 \\ 6 \end{matrix} \end{pmatrix}.$$

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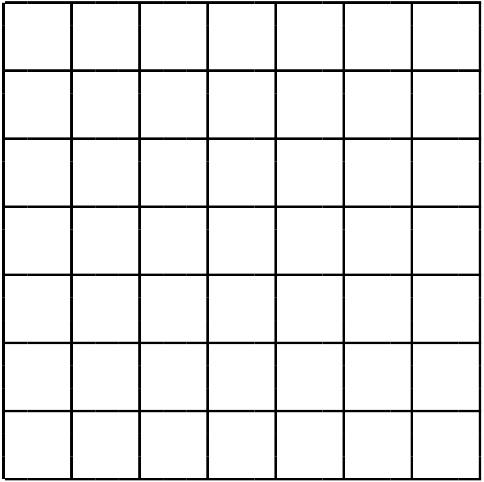

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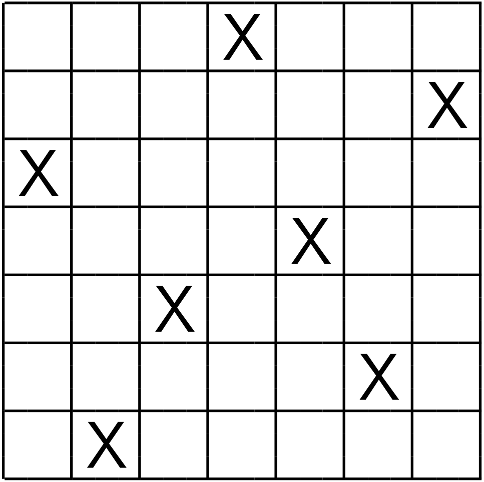

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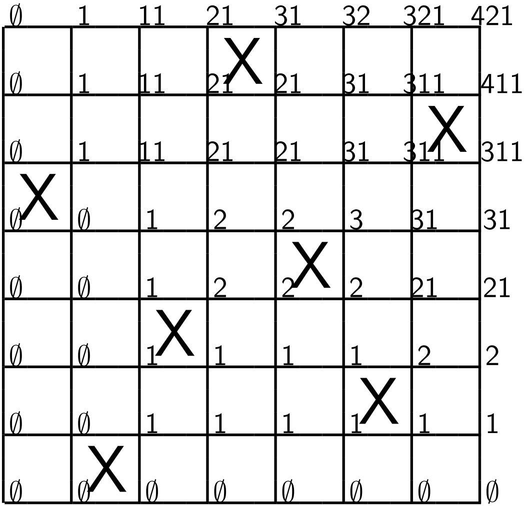
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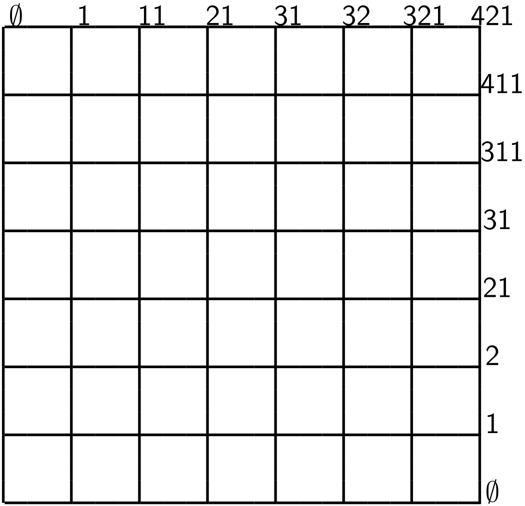

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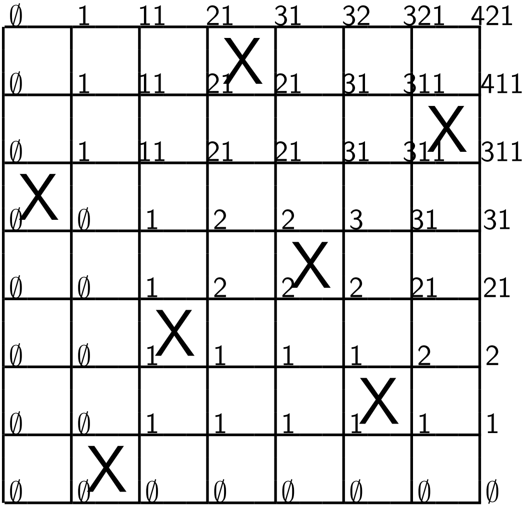

Growth diagrams — global view

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Theorem (Greene)

Given a growth diagram with empty partitions labelling all the corners along the left side and the bottom side of the Ferrers shape, the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ labelling corner c satisfies the following two properties:

(G1) *For any k, the maximal cardinality of the union of k increasing chains situated in the rectangular region to the left and below of c is equal to $\lambda_1 + \lambda_2 + \dots + \lambda_k$.*

(G2) *For any k, the maximal cardinality of the union of k decreasing chains situated in the rectangular region to the left and below of c is equal to $\lambda'_1 + \lambda'_2 + \dots + \lambda'_k$, where λ' denotes the partition conjugate to λ.*

In particular, λ_1 is the length of the longest increasing chain in the rectangular region to the left and below of c, and λ'_1 is the length of the longest decreasing chain in the same rectangular region.

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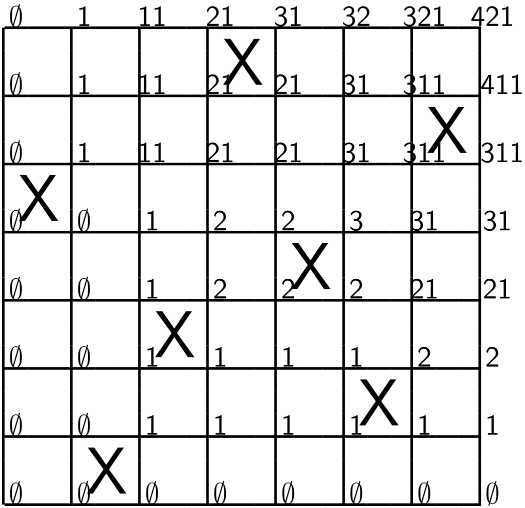

Conjecture (Burrill)

Let n and k be given non-negative integers. The number of oscillating tableaux of length n with at most k columns, starting at $\emptyset$ and ending at some one-column shape is the same as the number of standard Young tableaux of size n with all columns of length at most $2k$.

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An *oscillating tableau* is a sequence $\emptyset = \lambda^0, \lambda^1, \dots, \lambda^n = \lambda$ of shapes such that λ^i and λ^{i+1} differ by exactly one cell.

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(A standard Young tableau is an oscillating tableau where λ^{i+1} is always by one cell larger than λ^i.)

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Theorem

Let n, m, k be given non-negative integers. The number of oscillating tableaux of length n with at most k columns, starting at $\emptyset$ and ending at the one-column shape (1^m), is equal to the number of standard Young tableaux of size n with m columns of odd length, all columns of length at most $2k$.

Example

$n = 12,\ k = 3,\ m = 2.$

$$\begin{array}{llll} 1 & 3 & 4 & 8 \\ 2 & 6 & 7 & \\ 5 & 10 & & \\ 9 & 12 & & \\ 11 & & & \end{array}$$

$$\updownarrow$$

$$\emptyset,\ 1,\ 11,\ 21,\ 22,\ 21,\ 31,\ 311,\ 3111,\ 311,\ 31,\ 21,\ 11.$$

The bijection

The bijection

STEP 1. *Jeu de taquin* (Schützenberger).

1	3	4	8
2	6	7	
5	10		
9	12		
11			

The bijection

STEP 1. *Jeu de taquin* (Schützenberger).

1	3	4	8
2	6	7	*II*
5	10		
9	12		
11			
I			

Fill in $I, II, \ldots$ so that all columns have even length.

The bijection

STEP 1. *Jeu de taquin* (Schützenberger).

1	3	4	8
2	6	7	*II*
5	10		
9	12		
11			
I			

Fill in $I, II, \ldots$ so that all columns have even length.

Slide I, II, III, ..., in this order, to the top-left of the tableau. I.e., if s is the entry that is slided to the top-left, s is exchanged with the larger of its top- and left-neighbour, as often as possible.

The bijection

STEP 1. *Jeu de taquin* (Schützenberger).

1	3	4	8
2	6	7	*II*
5	10		
9	12		
I			
11			

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1	3	4	8
2	6	7	*II*
5	10		
I	12		
9			
11			

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2	6	7	*II*
I	10		
5	12		
9			
11			

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Slide *I*, *II*, *III*, ..., in this order, to the top-left of the tableau. I.e., if *s* is the entry that is slided to the top-left, *s* is exchanged with the larger of its top- and left-neighbour, as often as possible.

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STEP 1. *Jeu de taquin* (Schützenberger).

1	3	4	8
I	6	7	*II*
2	10		
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11			

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The bijection

STEP 1. *Jeu de taquin* (Schützenberger).

$$\begin{matrix} I & 3 & 4 & 8 \\ 1 & 6 & 7 & II \\ 2 & 10 \\ 5 & 12 \\ 9 \\ 11 \end{matrix}$$

Fill in $I, II, \dots$ so that all columns have even length.

Slide I, II, III, ..., in this order, to the top-left of the tableau. I.e., if s is the entry that is slided to the top-left, s is exchanged with the larger of its top- and left-neighbour, as often as possible.

The bijection

STEP 1. *Jeu de taquin* (Schützenberger).

I	3	4	*II*
1	6	7	8
2	10		
5	12		
9			
11			

Fill in $I, II, \dots$ so that all columns have even length.

Slide *I*, *II*, *III*, ..., in this order, to the top-left of the tableau. I.e., if s is the entry that is slided to the top-left, s is exchanged with the larger of its top- and left-neighbour, as often as possible.

The bijection

STEP 1. *Jeu de taquin* (Schützenberger).

I	3	II	4
1	6	7	8
2	10		
5	12		
9			
11			

Fill in $I, II, \dots$ so that all columns have even length.

Slide I, II, III, . . . , in this order, to the top-left of the tableau. I.e., if s is the entry that is slided to the top-left, s is exchanged with the larger of its top- and left-neighbour, as often as possible.

The bijection

STEP 1. *Jeu de taquin* (Schützenberger).

I	II	3	4
1	6	7	8
2	10		
5	12		
9			
11			

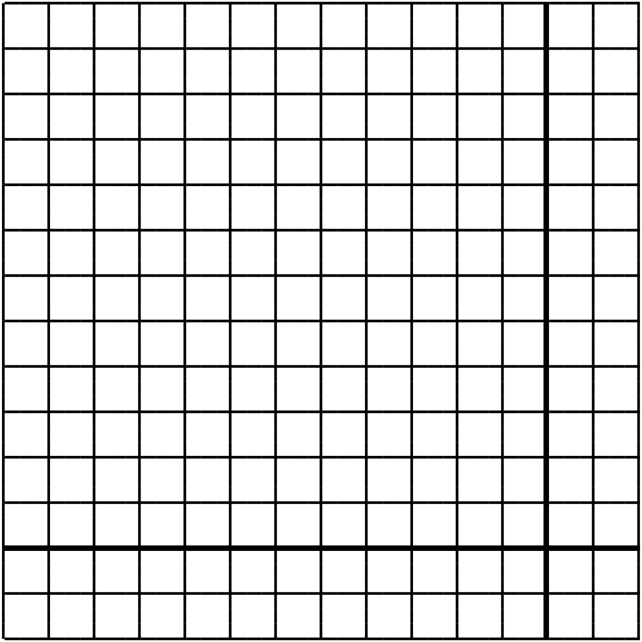
Fill in $I, II, \ldots$ so that all columns have even length.

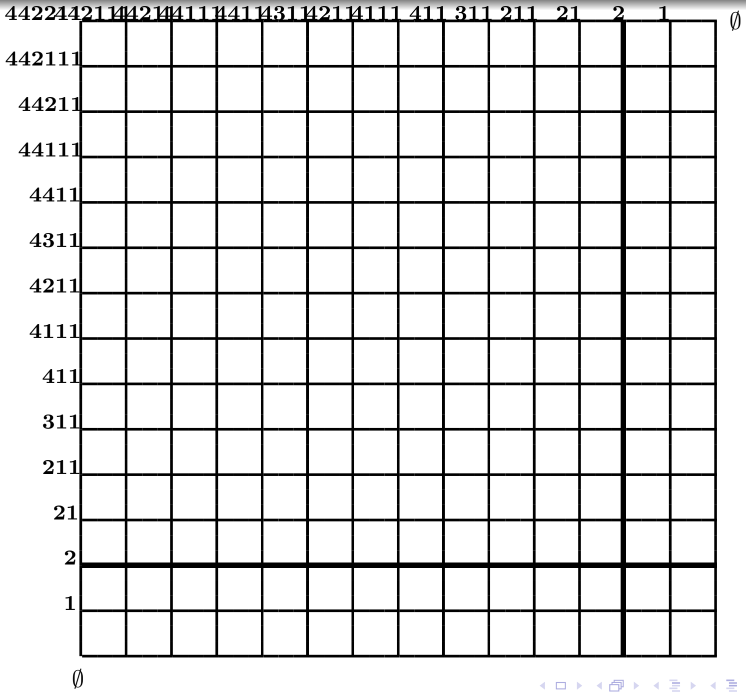
Slide I, II, III, ..., in this order, to the top-left of the tableau. I.e., if s is the entry that is slided to the top-left, s is exchanged with the larger of its top- and left-neighbour, as often as possible.

STEP 2. Put this standard Young tableaux in the alphabet

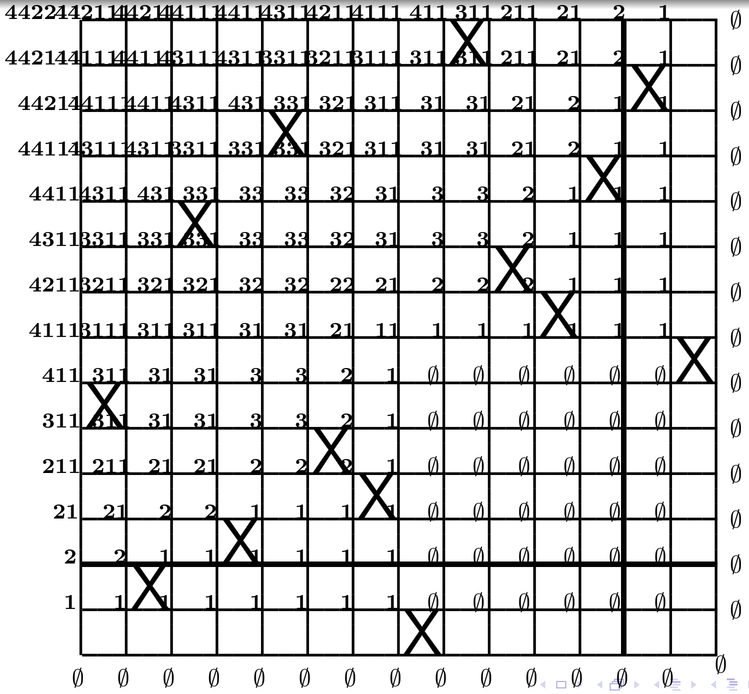
$$I, II, \dots, 1, 2, \dots$$

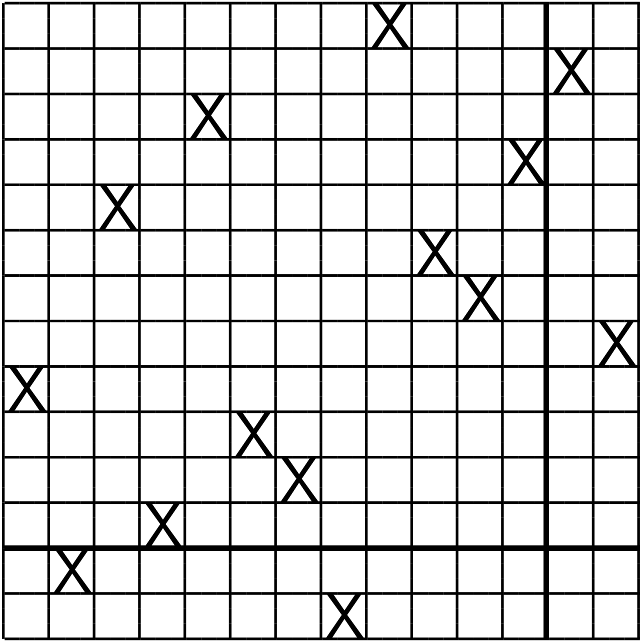
along the left and top side of a (square) growth diagram, and play the (inverse) growth diagram game in direction right/bottom.

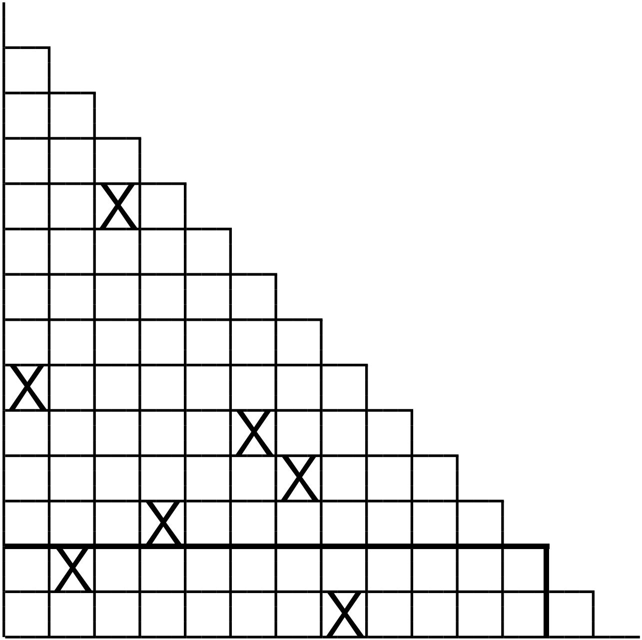




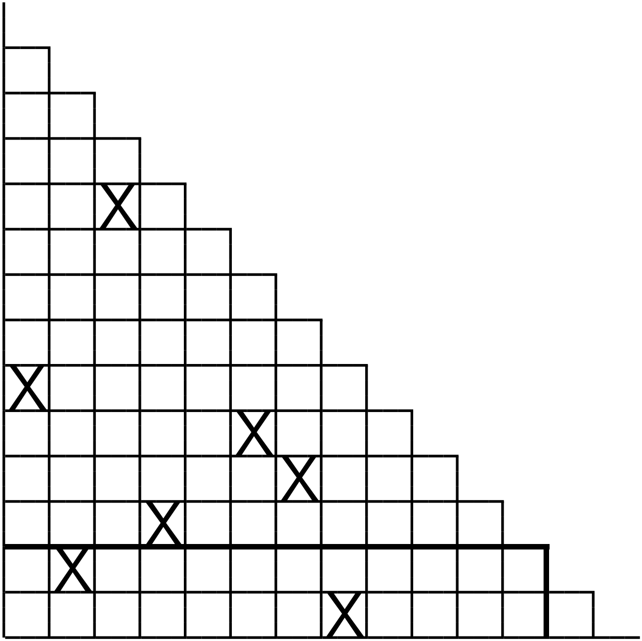
44224421144214411144114311421141114111 411 311 211 21 2 1
∅
442111
44211
44111
4411
4311
4211
4111
411
311
211
21
2
1
∅

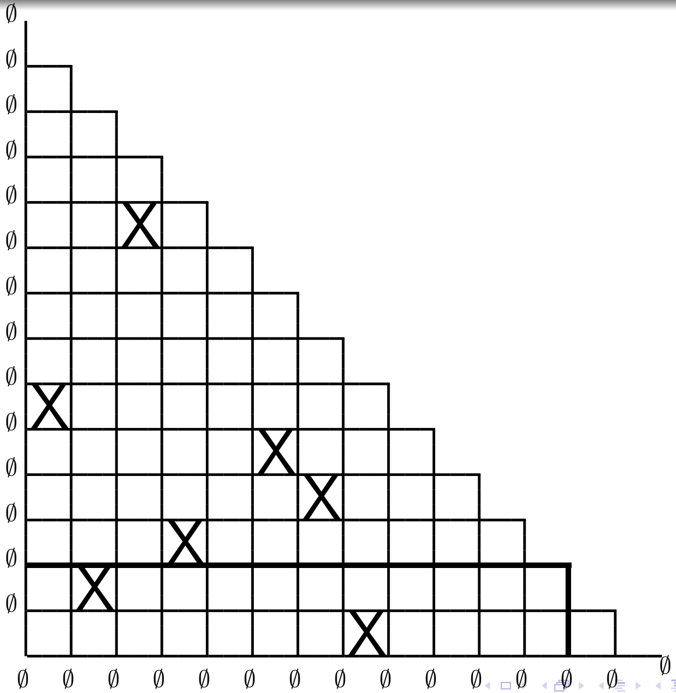


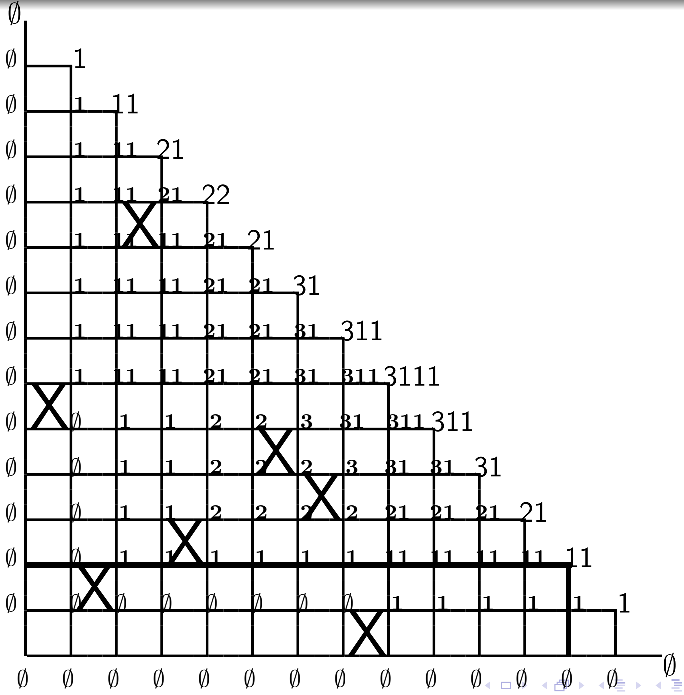


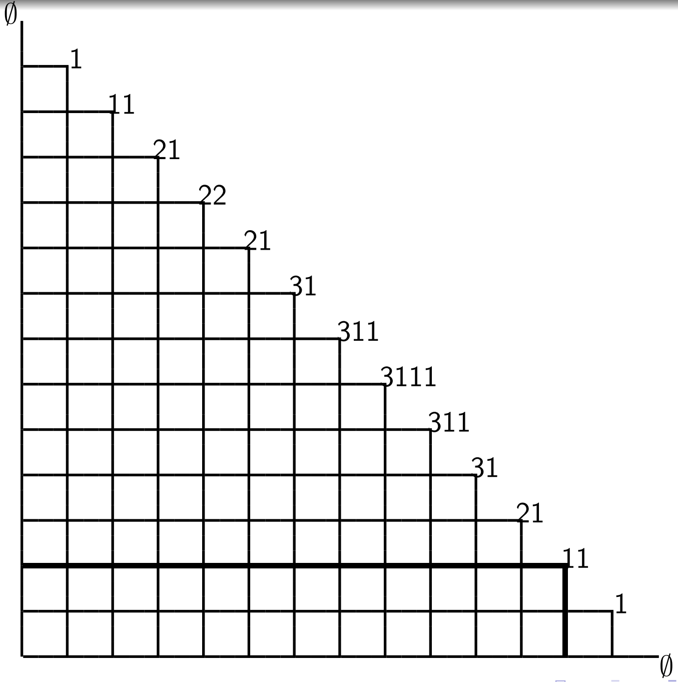

X
X
X
X
X
X
X

STEP 3. Place $\emptyset$'s along the left and bottom side of the (triangular) growth diagram, and play the (forward) growth diagram game in direction right/up.


X
X
X
X
X
X
X




∅
∅ 1
∅ 1 11
∅ 1 11 21
∅ 1 11 21 22
∅ 1 11 11 21 21
∅ 1 11 11 21 21 31
∅ 1 11 11 21 21 31 311
∅ 1 11 11 21 21 31 311 3111
∅ ∅ 1 1 2 2 3 31 311 311
∅ ∅ 1 1 2 2 2 3 31 31 31
∅ ∅ 1 1 2 2 2 2 21 21 21 21
∅ ∅ 1 1 1 1 1 1 11 11 11 11 11
∅ ∅ ∅ ∅ ∅ ∅ ∅ ∅ 1 1 1 1 1 1
∅ ∅ ∅ ∅ ∅ ∅ ∅ ∅ ∅ ∅ ∅ ∅ ∅ ∅ ∅


∅
1
11
21
22
21
31
311
3111
311
31
21
11
1
∅

We read along the diagonal:

$$\emptyset,\ 1,\ 11,\ 21,\ 22,\ 21,\ 31,\ 311,\ 3111,\ 311,\ 31,\ 21,\ 11)$$

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$$\emptyset,\ 1,\ 11,\ 21,\ 22,\ 21,\ 31,\ 311,\ 3111,\ 311,\ 31,\ 21,\ 11)$$

$$\updownarrow$$

1	3	4	8
2	6	7	
5	10		
9	12		
11			

Theorem

Let n, m, k be given non-negative integers. The number of oscillating tableaux of length n with at most k columns, starting at $\emptyset$ and ending at the one-column shape (1^m), is equal to the number of standard Young tableaux of size n with m columns of odd length, all columns of length at most $2k$.

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Using Knuth's generalisation of the Robinson–Schensted correspondence to words and semistandard tableaux, one can generalise the above theorem from standard Young tableaux to semistandard tableaux.