RSK correspondence, jeu de taquin, and growth diagrams

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A *standard Young tableau* is a left-justified array of the integers $1, 2, \ldots, n$, such that entries are increasing along rows and columns.

Given a standard Young tableau $T$, the vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where $\lambda_i$ is the length of the $i$-th row of $T$ is called the *shape* of $T$.

**Example**

$n = 12, \lambda = (4, 3, 2, 2, 1)$.

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The Robinson–Schensted correspondence is a bijection

\[ \pi \leftrightarrow (P, Q) \]

between permutations of \( \{1, 2, \ldots, n\} \) and pairs \((P, Q)\) of standard Young tableaux of the same shape.
The Robinson–Schensted correspondence is a bijection

$$\pi \leftrightarrow (P, Q)$$

between permutations of \(\{1, 2, \ldots, n\}\) and pairs \((P, Q)\) of standard Young tableaux of the same shape.

Hence,

$$n! = \sum_{\lambda \vdash n} f_\lambda^2,$$

where \(f_\lambda\) denotes the number of standard Young tableaux of shape \(\lambda\).
Example

\[ \pi = 5137426: \]

\[ \emptyset \rightarrow 5 \rightarrow \frac{1}{5} \rightarrow \frac{13}{5} \rightarrow \frac{137}{5} \rightarrow 134 \rightarrow \frac{124}{37} \rightarrow \frac{1246}{5} \rightarrow \frac{134}{57} \rightarrow \frac{124}{37} \rightarrow \frac{1246}{5} \]
Robinson–Schensted correspondence

Example

\( \pi = 5137426: \)

\[
(\emptyset, \emptyset) \rightarrow (5, 1) \rightarrow \left( \begin{array}{c} 1 \\ 5 \end{array} , \begin{array}{c} 1 \\ 2 \end{array} \right) \rightarrow \left( \begin{array}{c} 13 \\ 5 \end{array} , \begin{array}{c} 13 \\ 2 \end{array} \right) \\
\rightarrow \left( \begin{array}{c} 137 \\ 5 \end{array} , \begin{array}{c} 134 \\ 2 \end{array} \right) \rightarrow \left( \begin{array}{c} 134 \\ 57 \end{array} , \begin{array}{c} 134 \\ 25 \end{array} \right) \\
\rightarrow \left( \begin{array}{c} 124 \\ 37 \end{array} , \begin{array}{c} 134 \\ 25 \end{array} \right) \rightarrow \left( \begin{array}{c} 1246 \\ 5 \end{array} , \begin{array}{c} 1347 \\ 6 \end{array} \right)
\]
## Growth diagrams

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Informally:

Growth diagrams are diagrams consisting of square cells.
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Inside some cells, there are X’s. In a row, or in a column, there can be at most one X.
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The corners of cells are labelled by partitions.
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The corners of cells are labelled by partitions.

Along each row or column of a diagram, partitions are contained in each other, growing by at most one at a time. Growth by one can only happen if in the row/column to the left/below there is an X.
Growth diagrams

(a) A cell without cross  
(1) If $\rho = \mu = \nu$, and if there is no cross in the cell, then $\lambda = \rho$.
(2) If $\rho = \mu \neq \nu$, then $\lambda = \nu$.
(3) If $\rho = \nu \neq \mu$, then $\lambda = \mu$.
(4) If $\rho, \mu, \nu$ are pairwise different, then $\lambda = \mu \cup \nu$.
(5) If $\rho \neq \mu = \nu$, then $\lambda$ is formed by adding a square to the $(k + 1)$-st row of $\mu = \nu$, given that $\mu = \nu$ and $\rho$ differ in the $k$-th row.
(6) If $\rho = \mu = \nu$, and if there is a cross in the cell, then $\lambda$ is formed by adding a square to the first row of $\rho = \mu = \nu$. 

(b) A cell with cross
\[ \pi = 5137426 \]
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Growth diagrams

a. A cell without cross

b. A cell with cross

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$\pi = 5137426$
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| \( \emptyset \) | 1 |
| \( \emptyset \) | 1 | \( X \) |
| \( \emptyset \) | 1 | 11 | 21 | \( X \) |
| \( X \) | \( \emptyset \) | 1 | 2 |
| \( \emptyset \) | \( \emptyset \) | 1 | 2 | \( X \) |
| \( \emptyset \) | \( \emptyset \) | \( X \) | 1 | 1 | 1 | 1 | 2 |
| \( \emptyset \) | \( \emptyset \) | 1 | 1 | 1 | 1 | \( X \) | 1 |
| \( X \) | \( \emptyset \) | \( \emptyset \) | \( \emptyset \) | \( \emptyset \) | \( \emptyset \) | \( \emptyset \) | \( \emptyset \) | \( \emptyset \) | \( \emptyset \) | 1 |
\[ \pi = 5137426 \]
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Along the right border of the diagram we read:

\[ \emptyset, 1, 2, 21, 31, 311, 411, 421. \]
\[ \pi = 5137426 \]

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\[ \emptyset, 1, 2, 21, 31, 311, 411, 421. \]

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\[ \emptyset, 1, 2, 21, 31, 311, 411, 421. \]

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\[ \emptyset, 1, 11, 21, 31, 32, 321, 421. \]

These two sequences of shapes correspond to the pair

\[
\begin{pmatrix}
1246 & 1347 \\
37 & 25 \\
5 & 6
\end{pmatrix}
\]
Theorem (GREENE)

Given a growth diagram with empty partitions labelling all the corners along the left side and the bottom side of the Ferrers shape, the partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \) labelling corner \( c \) satisfies the following two properties:

\( (G1) \) For any \( k \), the maximal cardinality of the union of \( k \) increasing chains situated in the rectangular region to the left and below of \( c \) is equal to \( \lambda_1 + \lambda_2 + \cdots + \lambda_k \).

\( (G2) \) For any \( k \), the maximal cardinality of the union of \( k \) decreasing chains situated in the rectangular region to the left and below of \( c \) is equal to \( \lambda'_1 + \lambda'_2 + \cdots + \lambda'_k \), where \( \lambda' \) denotes the partition conjugate to \( \lambda \).

In particular, \( \lambda_1 \) is the length of the longest increasing chain in the rectangular region to the left and below of \( c \), and \( \lambda'_1 \) is the length of the longest decreasing chain in the same rectangular region.
Theorem (GREENE)

Given a growth diagram with empty partitions labelling all the corners along the left side and the bottom side of the Ferrers shape, the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ labelling corner $c$ satisfies the following two properties:

(G1) For any $k$, the maximal cardinality of the union of $k$ increasing chains situated in the rectangular region to the left and below of $c$ is equal to $\lambda_1 + \lambda_2 + \cdots + \lambda_k$.

(G2) For any $k$, the maximal cardinality of the union of $k$ decreasing chains situated in the rectangular region to the left and below of $c$ is equal to $\lambda'_1 + \lambda'_2 + \cdots + \lambda'_k$, where $\lambda'$ denotes the partition conjugate to $\lambda$.

In particular, $\lambda_1$ is the length of the longest increasing chain in the rectangular region to the left and below of $c$, and $\lambda'_1$ is the length of the longest decreasing chain in the same rectangular region.
Growth diagrams — global view

\[
\begin{array}{cccccccc}
\emptyset & 1 & 11 & 21 & 31 & 32 & 321 & 421 \\
\emptyset & 1 & 11 & 21 & 21 & 31 & 311 & 411 \\
\emptyset & 1 & 11 & 21 & 21 & 31 & 311 & 311 \\
X & \emptyset & 1 & 2 & 2 & 3 & 31 & 31 \\
X & \emptyset & 1 & 2 & 2 & 2 & 21 & 21 \\
X & \emptyset & 1 & 1 & 1 & 1 & 1 & 2 \\
X & \emptyset & 1 & 1 & 1 & 1 & 1 & 1 \\
X & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\
\emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\
\end{array}
\]
Conjecture (Burrill)

Let $n$ and $k$ be given non-negative integers. The number of oscillating tableaux of length $n$ with at most $k$ columns, starting at $\emptyset$ and ending at some one-column shape is the same as the number of standard Young tableaux of size $n$ with all columns of length at most $2k$.

An oscillating tableau is a sequence $\emptyset = \lambda_0, \lambda_1, \ldots, \lambda_n = \lambda_{n+1}$ of shapes such that $\lambda_i$ and $\lambda_{i+1}$ differ by exactly one cell. (A standard Young tableau is an oscillating tableau where $\lambda_{i+1}$ is always by one cell larger than $\lambda_i$.)

Christian Krattenthaler
RSK, JdT, growth diagrams
Conjecture (Burrill)

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Let $n$ and $k$ be given non-negative integers. The number of oscillating tableaux of length $n$ with at most $k$ columns, starting at $\emptyset$ and ending at some one-column shape is the same as the number of standard Young tableaux of size $n$ with all columns of length at most $2k$.

An oscillating tableau is a sequence $\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^n = \lambda$ of shapes such that $\lambda^i$ and $\lambda^{i+1}$ differ by exactly one cell.

(A standard Young tableau is an oscillating tableau where $\lambda^{i+1}$ is always by one cell larger than $\lambda^i$.)

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RSK, JdT, growth diagrams
Conjecture (Burrill)

Let $n$ and $k$ be given non-negative integers. The number of oscillating tableaux of length $n$ with at most $k$ columns, starting at $∅$ and ending at some one-column shape is the same as the number of standard Young tableaux of size $n$ with all columns of length at most $2k$. 
Conjecture (Burrill)

Let $n$ and $k$ be given non-negative integers. The number of oscillating tableaux of length $n$ with at most $k$ columns, starting at $\emptyset$ and ending at some one-column shape is the same as the number of standard Young tableaux of size $n$ with all columns of length at most $2k$.

Theorem

Let $n, m, k$ be given non-negative integers. The number of oscillating tableaux of length $n$ with at most $k$ columns, starting at $\emptyset$ and ending at the one-column shape $(1^m)$, is equal to the number of standard Young tableaux of size $n$ with $m$ columns of odd length, all columns of length at most $2k$. 
Example

\( n = 12, \ k = 3, \ m = 2. \)

\[
\begin{array}{cccc}
1 & 3 & 4 & 8 \\
2 & 6 & 7 \\
5 & 10 \\
9 & 12 \\
11 \\
\end{array}
\]

\[\emptyset, \ 1, \ 11, \ 21, \ 22, \ 21, \ 31, \ 311, \ 3111, \ 311, \ 31, \ 21, \ 11. \]
The bijection
The bijection

Step 1. *Jeu de taquin.*

```
1  3  4  8  
2  6  7  
5  10  
9  12  
11
```
The bijection

**Step 1. Jeu de taquin.**

```
1  3  4  8
2  6  7  II
5 10
9 12
11
I
```

Fill in $I, II, \ldots$ so that all columns have even length.
The bijection

**Step 1.** *Jeu de taquin.*

\[
\begin{array}{cccc}
1 & 3 & 4 & 8 \\
2 & 6 & 7 & II \\
5 & 10 \\
9 & 12 \\
11 \\
\end{array}
\]

Fill in I, II, . . . so that all columns have even length.

Slide I, II, III, . . . , in this order, to the top-left of the tableau.
I.e., if \( s \) is the entry that is slid to the top-left, \( s \) is exchanged with the larger of its top- and left-neighbour, as often as possible.
The bijection

Step 1. *Jeu de taquin.*

\[
\begin{array}{cccc}
1 & 3 & 4 & 8 \\
2 & 6 & 7 & \text{II} \\
5 & 10 \\
9 & 12 \\
11 \\
\end{array}
\]

Fill in I, II, \ldots so that all columns have even length.

Slide I, II, III, \ldots, in this order, to the top-left of the tableau.

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The bijection

**Step 1.** *Jeu de taquin.*

1 3 4 8
2 6 7 II
5 10
I 12
9
11

Fill in I, II, . . . so that all columns have even length.

Slide I, II, III, . . . , in this order, to the top-left of the tableau.

I.e., if s is the entry that is slided to the top-left, s is exchanged with the larger of its top- and left-neighbour, as often as possible.
The bijection

**Step 1.** *Jeu de taquin.*

```
  1  3  4  8  
  2  6  7  II  
    I 10  
  5  12  
    9  
    11  
```

Fill in *I, II, ...* so that all columns have even length.

Slide *I, II, III, ...*, in this order, to the top-left of the tableau.
I.e., if *s* is the entry that is slid to the top-left, *s* is exchanged with the larger of its top- and left-neighbour, as often as possible.
**The bijection**

**Step 1. *Jeu de taquin.***

1  3  4  8  
I  6  7  II  
2  10  
5  12  
9  
11  

Fill in *I, II, III, ...* so that all columns have even length. Slide *I, II, III, ...*, in this order, to the top-left of the tableau. I.e., if *s* is the entry that is slided to the top-left, *s* is exchanged with the larger of its top- and left-neighbour, as often as possible.
Step 1. *Jeu de taquin.*

\[
\begin{array}{cccc}
I & 3 & 4 & 8 \\
1 & 6 & 7 & II \\
2 & 10 \\
5 & 12 \\
9 \\
11 \\
\end{array}
\]

Fill in $I, II, \ldots$ so that all columns have even length.

Slide $I, II, III, \ldots,$ in this order, to the top-left of the tableau.

I.e., if $s$ is the entry that is slided to the top-left, $s$ is exchanged with the larger of its top- and left-neighbour, as often as possible.
The bijection

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Slide I, II, III, . . . , in this order, to the top-left of the tableau. I.e., if $s$ is the entry that is slided to the top-left, $s$ is exchanged with the larger of its top- and left-neighbour, as often as possible.
The bijection

Step 1. *Jeu de taquin.*

\[
\begin{array}{cccc}
I & 3 & II & 4 \\
1 & 6 & 7 & 8 \\
2 & 10 \\
5 & 12 \\
9 \\
11 \\
\end{array}
\]

Fill in $I, II, \ldots$ so that all columns have even length.

Slide $I, II, III, \ldots$, in this order, to the top-left of the tableau.

I.e., if $s$ is the entry that is slid to the top-left, $s$ is exchanged with the larger of its top- and left-neighbour, as often as possible.
The bijection

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Slide $I, II, III, \ldots$, in this order, to the top-left of the tableau.

I.e., if $s$ is the entry that is slid to the top-left, $s$ is exchanged with the larger of its top- and left-neighbour, as often as possible.
Step 2. Put this standard Young tableaux in the alphabet

$I, II, \ldots, 1, 2, \ldots$

along the left and top side of a (square) growth diagram, and play
the (inverse) growth diagram game in direction right/bottom.
Step 3. Place $\emptyset$’s along the left and bottom side of the (triangular) growth diagram, and play the (forward) growth diagram game in direction right/up.
We read along the diagonal:

\( \emptyset, 1, 11, 21, 22, 21, 31, 311, 3111, 311, 31, 21, 11 \)
We read along the diagonal:

$$\emptyset, 1, 11, 21, 22, 21, 31, 311, 3111, 31111, 3111, 31, 21, 11)$$

$$\uparrow$$

$$\downarrow$$

1 3 4 8
2 6 7
5 10
9 12
11