Bounded Dyck paths, bounded alternating sequences, and reciprocity laws

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Overview

- Classical Enumerative Combinatorics
- Bounded Dyck Paths and Bounded Alternating Sequences
- Reciprocity Laws
- Non-Intersecting Lattice Paths
- Theory of Heaps

Reciprocity Laws

Informally speaking, a (combinatorial) reciprocity law[©] refers to the following:
We are given a sequence $(a_n)_{n\geq 0}$, where a_n is the number of certain objects of "size" n. If it is somehow possible to make sense of a_n for *negative* n *and* it should happen that a_n for negative n has also a combinatorial meaning, then we speak of a (combinatorial) *reciprocity law*.

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Example: Ehrhart Reciprocity. Given a polytope $\mathcal{P}$ in $\mathbb{R}^d$ all of whose vertices have integer coordinates, let $i(\mathcal{P},n) := |n\mathcal{P}\cap\mathbb{Z}^d|$ and $\bar{i}(\mathcal{P},n) := |n\mathcal{P}^o\cap\mathbb{Z}^d|$. Then $i(\mathcal{P},n)$ is a polynomial in n and

$$\bar{i}(\mathcal{P},n) = (-1)^d i(\mathcal{P},-n).$$

[©]Richard Stanley

Reciprocity Laws

Let the generating function of the sequence $(a_n)_{n\ge0}$ be a rational function of the form

$$f(x) = \frac{p(x)}{q(x)} = \sum_{n\ge0} a_n x^n,$$

where $p(x)$ and $q(x)$ are polynomials with $\deg(p(x)) < \deg(q(x))$. If $q(x) = \sum_{i=0}^{d} q_i x^i$, then the sequence $(a_n)_{n\ge0}$ satisfies the rcurrence

$$q_0 a_n + q_1 a_{n-1} + \cdots + q_d a_{n-d} = 0.$$

Hence, $(a_n)_{n\ge0}$ can be extended to negative n.
It is not difficult to see that

$$\sum_{n\ge1} a_{-n} x^n = -f(1/x).$$

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Number of bounded Dyck paths with negative length as Hankel determinants

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This is a continuation of my post Number of bounded Dyck paths with "negative length".

Let $C_n^{(2k+1)}$ be the number of Dyck paths of semilength n bounded by $2k+1$. They satisfy a recursion of order $2k+1$.

Let $C_{-n}^{(2k+1)}$ be the numbers obtained by extending the sequence $C_n^{(2k+1)}$ to negative n using this recursion.

Computations suggest that this extension can also be obtained via Hankel determinants:

$$C_{-n}^{(2k+1)} = \det\left(C_{n+1+i+j}^{(2k+1)}\right)_{i,j=0}^{k-1}.$$

For $k=1$ this reduces to $C_{-n}^{(3)} = C_{n+1}^{(3)}$. This can easily be verified since the sequence $(\ldots 34, 13, 5, 2 | 1, 1, 2, 5, 13, 34, \ldots$ satisfies $a(n) - 3a(n-1) + a(n-2) = 0$.

For $k=2$ we get the sequence $(\cdots, 70, 14, 3, |1, 1, 2, 5, 14, 42, 131, \cdots)$. For example

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For $k=2$ we get the sequence

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If $f(n)$ satisfies a linear recurrence with constant coefficients for all $n \in \mathbb{Z}$ and we set $F(x) = \sum_{n \geq 0} f(n)x^n$, then $\sum_{n \geq 1} f(-n)x^n = -F(1/x)$ (as rational functions). See *Enumerative Combinatorics*, vol. 1, second ed., Prop. 4.2.3.

Addendum. Using Exercise 3.66(d) in *Enumerative Combinatorics*, vol. 1, second ed., it is not hard to show that $c(-n, k)$ is equal to the number of sequences $(a_1, a_2, \ldots, a_{2n-1})$ of positive integers satisfying $1 \leq a_i \leq k+1$ and $a_1 \leq a_2 \geq a_3 \leq a_4 \geq \cdots \geq a_{2n-1}$.

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edited Sep 29 at 23:07

answered Sep 26 at 14:21

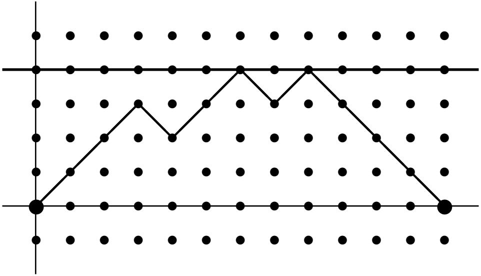
Richard Stanley

37.4k ● 9 ● 130 ● 214

Thank you very much. Is there a combinatorial interpretation of the numbers $c(-n, k)$ related to Dyck paths? – Johann Cigler Sep 26 at 15:08

In fine print:

Let $C_{2n}^{(k)}$ denote the number of paths with steps $(1,1)$ and $(1,-1)$ starting at $(0,0)$ and ending at $(2n,0)$ never passing below the x-axis and never passing above the horizontal line $y=k$. The figure shows one of the 122 such paths for $n=6$ and $k=4$.



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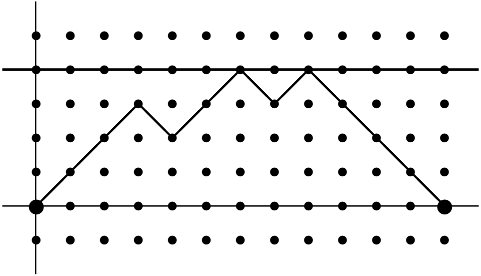

Let $\mathcal{A}_n^{(k)}$ denote the *set* of alternating sequences

$$a_1 \le a_2 \ge a_3 \le a_4 \ge \cdots \diamond\, a_{n-1} \square\, a_n,$$

where $\diamond = \ge$ and $\square = \le$ if n is even and $\diamond = \le$ and $\square = \ge$ if n is odd, in which all a_i's are integers between 1 and k.

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Cigler conjectures

$$\det\left(C_{2n+2i+2j+2}^{(2k+1)}\right)_{0\le i,j\le k-1} = C_{-2n}^{(2k+1)}.$$

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Stanley observes and provides an argument for the relation

$$|\mathcal{A}_{2n-1}^{(k+1)}| = C_{-2n}^{(2k+1)}.$$

$$C_{-2n}^{(2k+1)} = \det\left(C_{2n+2+2i+2j}^{(2k+1)}\right)_{0\le i,j\le k-1}.$$

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I asked Cigler: “What if we lift the upper bound on the paths? Do we then also get determinants on the left-hand side?”

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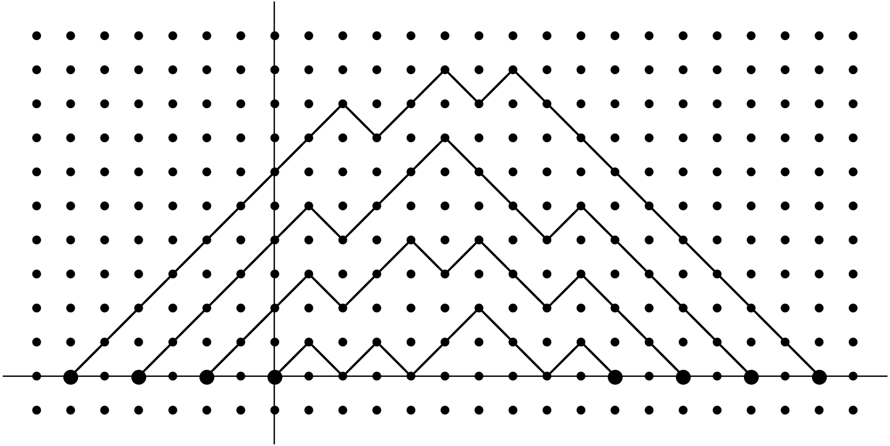
Cigler (next day): "Yes. Here is the — conjectured — formula:

$$\det\left(C_{-2n-2i-2j}^{(2k+2m-1)}\right)_{0\le i,j\le m-1} = \det\left(C_{2n+2i+2j+4m-2}^{(2k+2m-1)}\right)_{0\le i,j\le k-1}.$$

Non-intersecting Lattice Paths

A family $(P_1, P_2, \ldots, P_k)$ of paths P_i is called non-intersecting if no two paths of the family have a vertex in common.



Theorem (Karlin–McGregor, Lindström, Gessel–Viennot, Fisher, John–Sachs, Gronau–Just–Schade–Scheffler–Wojciechowski)

Let G be an acyclic, directed graph, and let $A_1, A_2, \dots, A_n$ and $E_1, E_2, \dots, E_n$ be vertices in the graph with the property that, for $i < j$ and $k < l$, any (directed) path from A_i to E_l intersects with any path from A_j to E_k. Then the number of families $(P_1, P_2, \dots, P_n)$ of non-intersecting (directed) paths, where the i-th path P_i runs from A_i to E_i, $i = 1, 2, \dots, n$, is given by

$$\det_{1 \le i,j \le n} \left(|\mathcal{P}(A_j \to E_i)| \right),$$

where $\mathcal{P}(A \to E)$ denotes the set of paths from A to E.

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where $\mathcal{P}(A \to E)$ denotes the set of paths from A to E.

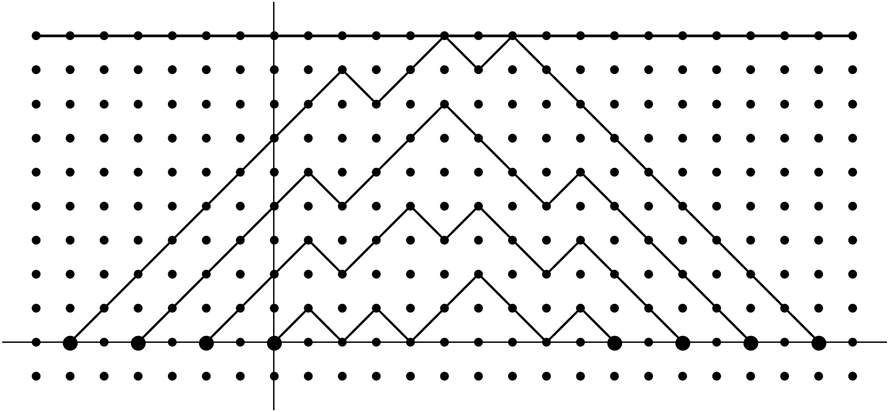
Remark

There holds as well a *weighted* version, in which every edge e is assigned a *weight* $w(e)$, and where the weight of a path (family) $\mathcal{P}$ is defined as the product $\prod_{e\in\mathcal{P}} w(e)$, with the product running over all edges in the path (family).

Combinatorial interpretation of the determinant

$$\det\left(C^{(2k+1)}_{2n+2+2i+2j}\right)_{0\le i,j\le k-1}$$

Example for $n = 4,\ k = 4$:



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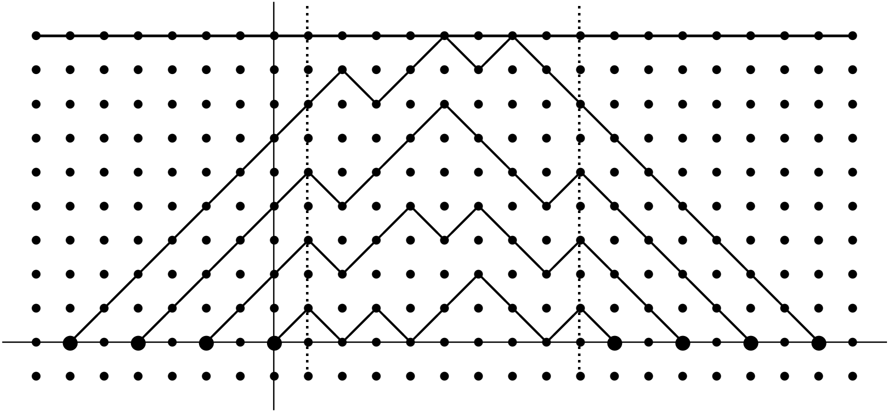

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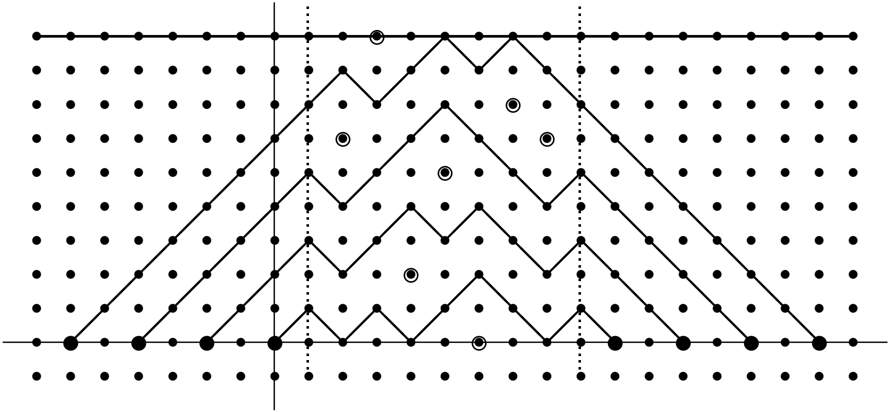

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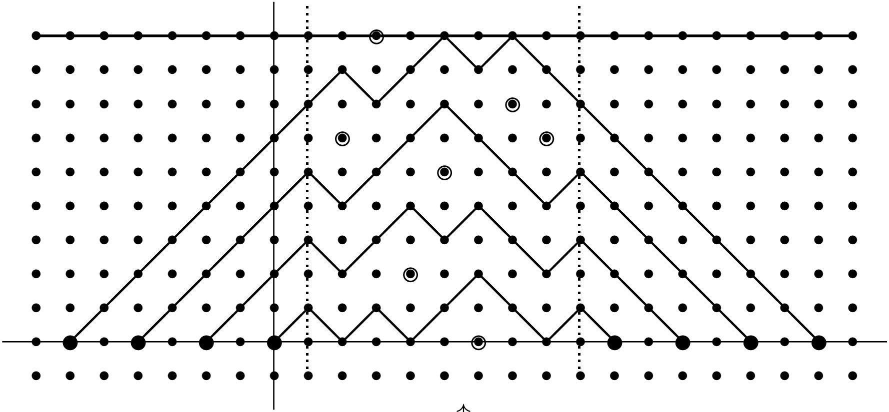

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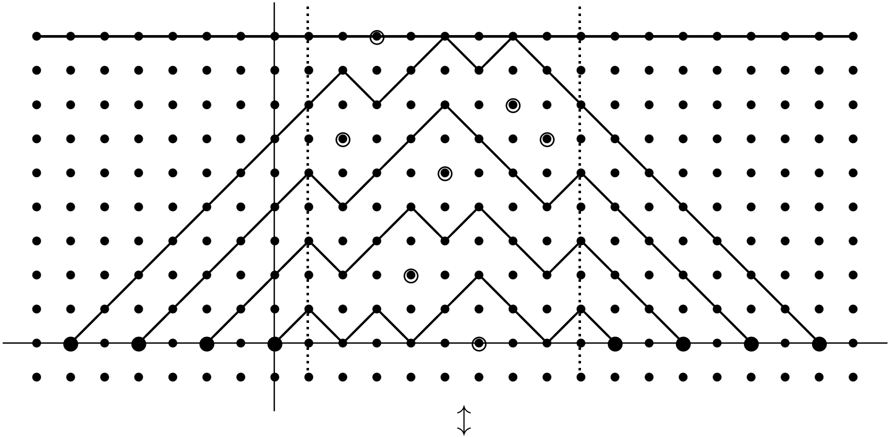

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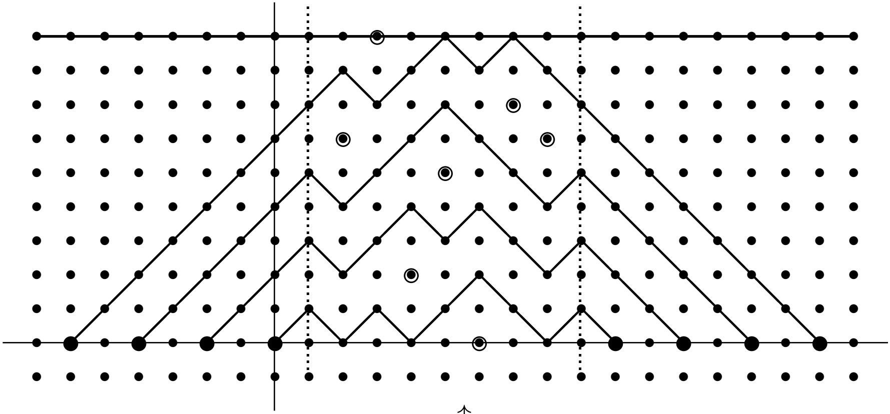

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Let $C_n^{(k)}(r \to s)$ denote the *number* of up-down paths from $(0, r)$ to (n, s) that do not pass below the x-axis and do not pass above the horizontal line $y = k$.

Generating Functions for Bounded Up-Down Paths

Let $C_n^{(k)}(r \to s)$ denote the *number* of up-down paths from $(0, r)$ to (n, s) that do not pass below the x-axis and do not pass above the horizontal line $y = k$.

Theorem (Folklore/Viennot 1983)

For all non-negative integers r, s, k with $0 \le r, s \le k$, we have

$$\sum_{n \ge 0} C_n^{(k)}(r \to s)\, x^n = \begin{cases} \dfrac{U_r(1/2x)\, U_{k-s}(1/2x)}{x\, U_{k+1}(1/2x)}, & \text{if } r \le s, \\ \dfrac{U_s(1/2x)\, U_{k-r}(1/2x)}{x\, U_{k+1}(1/2x)}, & \text{if } r \ge s, \end{cases}$$

where $U_m(x)$ denotes the m-th Chebyshev polynomial of the second kind.

The *m-th Chebyshev polynomial of the second kind*, $U_m(x)$, is given by

$$U_m(\cos\theta) = \frac{\sin((m+1)\theta)}{\sin\theta},$$
$$U_m(x) = \sum_{j\ge 0}(-1)^j\binom{m-j}{j}(2x)^{m-2j}.$$

They satisfy the two-term recurrence

$$2xU_m(x) = U_{m+1}(x) + U_{m-1}(x),$$

with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$.

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This can be proved by means of the transfer-matrix method or by means of the theory of heaps.

Generating Functions for Alternating Sequences

Let $\mathcal{A}_n^{(k)}(r \to s)$ denote the *set* of alternating sequences

$$r \le a_2 \ge a_3 \le a_4 \ge \cdots \diamond a_{n-1} \square s,$$

where $\diamond = \ge$ and $\square = \le$ if n is even and $\diamond = \le$ and $\square = \ge$ if n is odd, in which all a_i's are integers between 1 and k.

Theorem (C., K.)

For all positive integers r, s, k with $1 \le r, s \le k$, we have

$$\sum_{n\ge0} \left|\mathcal{A}_{2n+1}^{(k)}(r \to s)\right| x^{2n}$$

$$= \begin{cases} (-1)^{r+s+1} \dfrac{xU_{2r-2}(x/2)\,U_{2k+1-2s}(x/2)}{U_{2k}(x/2)}, & \text{if } r < s, \\ 1 - \dfrac{xU_{2r-2}(x/2)\,U_{2k+1-2r}(x/2)}{U_{2k}(x/2)}, & \text{if } r = s, \\ (-1)^{r+s+1} \dfrac{xU_{2s-2}(x/2)\,U_{2k+1-2r}(x/2)}{U_{2k}(x/2)}, & \text{if } r > s. \end{cases}$$

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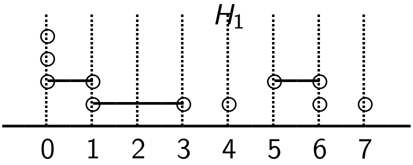
Theorem (C., K.)

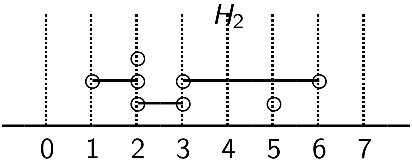
For all positive integers r, s, k with $1 \le r, s \le k$, we have

$$\sum_{n\ge 0} \left|\mathcal{A}_{2n+2}^{(k)}(r \to s)\right| x^{2n+1} = \begin{cases} (-1)^{r+s+1} \dfrac{x U_{2r-2}(x/2) U_{2k-2s}(x/2)}{U_{2k}(x/2)}, & \text{if } r \le s, \\ (-1)^{r+s+1} \dfrac{x U_{2s-1}(x/2) U_{2k-2r+1}(x/2)}{U_{2k}(x/2)}, & \text{if } r > s. \end{cases}$$

Theory of Heaps

A heap of segments on $[1, k]$ is a pile of segments $i{-}j$, with $1 \le i \le j \le k$, allowing multiple pieces of the same kind.





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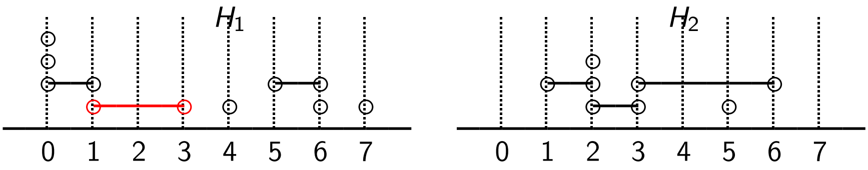

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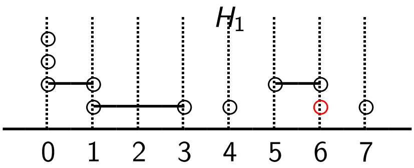

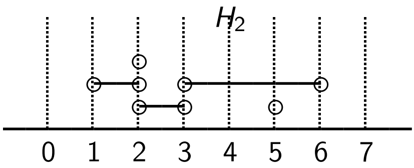

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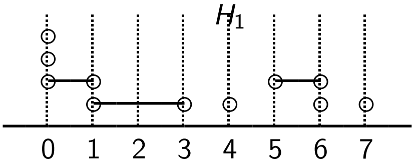
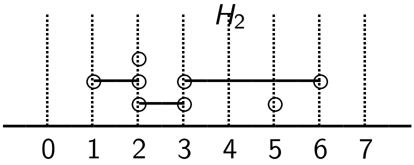




Every segment i–j is assigned the *weight* $w_{i,j}$. The *weight* $w(H)$ of a heap H is by definition the product of all the weights of its segments.

Thus,

$$w(H_1) = w_{0,1} w_{0,0}^2 w_{1,3} w_{4,4} w_{5,6} w_{6,6} w_{7,7}$$

and

$$w(H_2) = w_{1,2} w_{2,3} w_{2,2} w_{3,6} w_{5,5}.$$

Let $\mathcal{S}_k$ denote the set of segments i–j with $1 \le i \le j \le k$.

Theorem (Cartier, Foata 1969/Viennot 1986)

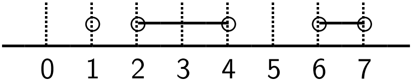
We have

$$\sum_{H \text{ heap of segments on } [1,k]} w(H) = \frac{1}{\sum_{\substack{T \text{ trivial} \\ \text{segments} \subseteq \mathcal{S}_k}} (-1)^{|T|} w(T)},$$

where $|T|$ denotes the number of segments of T.

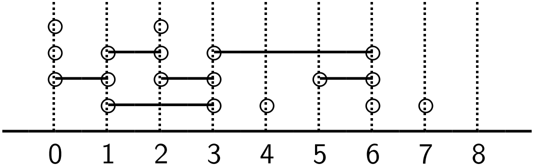
Here, a *trivial heap* is one in which any two of its segments "commute" with each other.

Example.



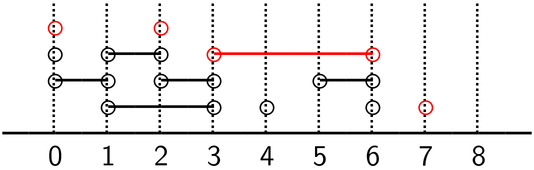
A segment in a heap H is called maximal if it “lies” on top of H and could be moved up vertically without being blocked by any other piece.

Example.



A segment in a heap H is called maximal if it “lies” on top of H and could be moved up vertically without being blocked by any other piece.

Example.



Theory of Heaps

Let $\mathcal{S}_k$ denote the set of segments i–j with $1 \le i \le j \le k$.

Theorem (Viennot 1986)

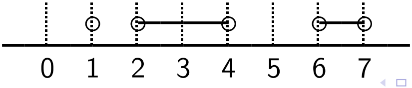
We have

$$\sum_{\substack{H \text{ heap of segments on } [1,k] \\ \text{maximal segments} \subseteq \mathcal{M}}} w(H) = \frac{\displaystyle\sum_{\substack{T \text{ trivial} \\ \text{segments} \subseteq \mathcal{S}_k \backslash \mathcal{M}}} (-1)^{|T|} w(T)}{\displaystyle\sum_{\substack{T \text{ trivial} \\ \text{segments} \subseteq \mathcal{S}_k}} (-1)^{|T|} w(T)},$$

where $|T|$ denotes the number of segments of T.

Here, a *trivial heap* is one in which any two of its segments "commute" with each other.

Example.



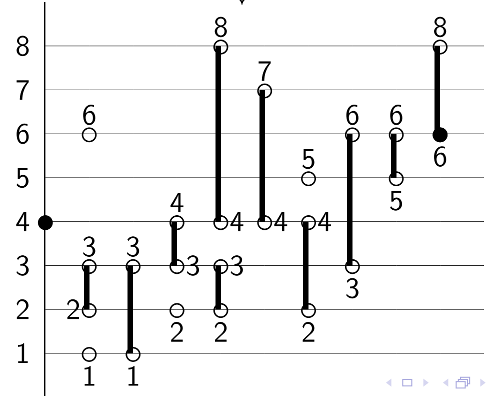
A Bijection between Alternating Sequences and Heaps

$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$$
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$$

$$\downarrow$$

$$3\text{—}2,\ 1\text{—}1,\ 3\text{—}1,\ 4\text{—}3,\ \big|\ 6\text{—}6,\ \big|\ 8\text{—}4,\ \big|\ 7\text{—}4,\ \big|$$
$$2\text{—}2,\ 3\text{—}2,\ 4\text{—}2,\ \big|\ 5\text{—}5,\ \big|\ 6\text{—}3,\ \big|\ 6\text{—}5,\ \big|\ 8\text{—}6$$

$$\downarrow$$

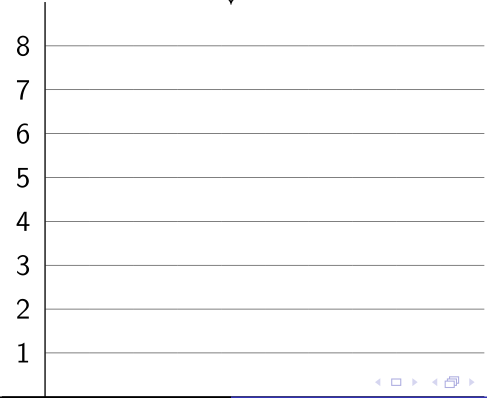


A Bijection between Alternating Sequences and Heaps

$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$$
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$$

$$\downarrow$$

$$\downarrow$$

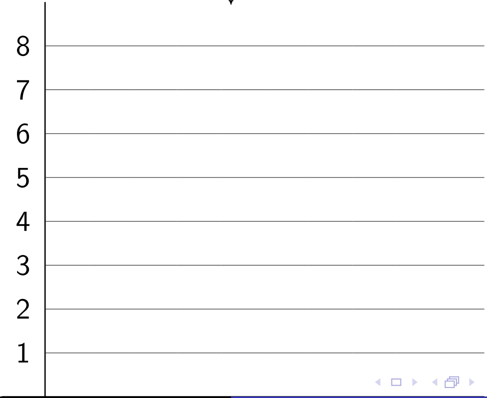


A Bijection between Alternating Sequences and Heaps

$$4 \leq 4 \geq 3 \leq 3 \geq 1 \leq 1 \geq 1 \leq 3 \geq 2 \leq 6 \geq 6 \leq 8 \geq 4 \leq 7 \geq 4 \leq 4 \geq 2$$
$$\leq 3 \geq 2 \leq 2 \geq 2 \leq 5 \geq 5 \leq 6 \geq 3 \leq 6 \geq 5 \leq 8 \geq 6$$

$$\downarrow$$

$$\downarrow$$

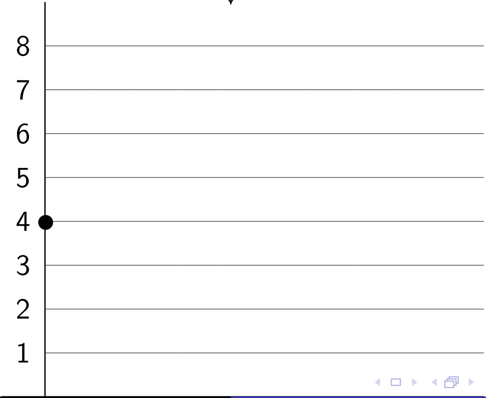


A Bijection between Alternating Sequences and Heaps

$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$$
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$$

$$\downarrow$$

$$\downarrow$$

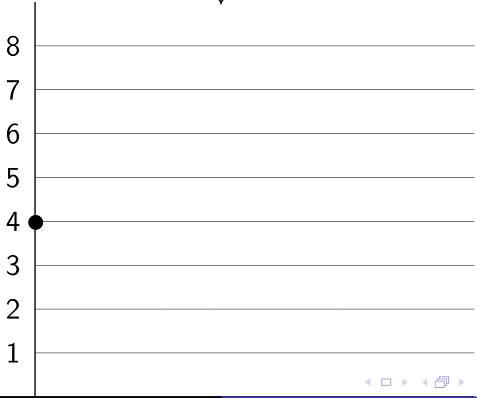


A Bijection between Alternating Sequences and Heaps

$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$
$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$

$\downarrow$

$\downarrow$



A Bijection between Alternating Sequences and Heaps

$4 \le {\color{red}4 \ge 3 \le 3 \ge 1} \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$
$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$

$\downarrow$

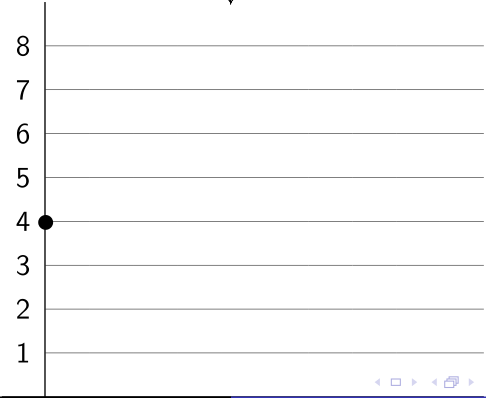
$\downarrow$

A Bijection between Alternating Sequences and Heaps

$4 \leq 4 \geq 3 \leq 3 \geq 1 \leq 1 \geq 1 \leq 3 \geq 2 \leq 6 \geq 6 \leq 8 \geq 4 \leq 7 \geq 4 \leq 4 \geq 2$
$\leq 3 \geq 2 \leq 2 \geq 2 \leq 5 \geq 5 \leq 6 \geq 3 \leq 6 \geq 5 \leq 8 \geq 6$

$\downarrow$

$\downarrow$



A Bijection between Alternating Sequences and Heaps

$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$

$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$

$\downarrow$

$\downarrow$

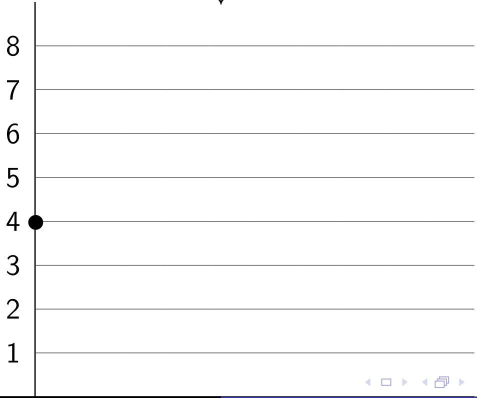
A Bijection between Alternating Sequences and Heaps

$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$
$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$

$\downarrow$

3–2, 1–1, 3–1, 4–3, |

$\downarrow$



A Bijection between Alternating Sequences and Heaps

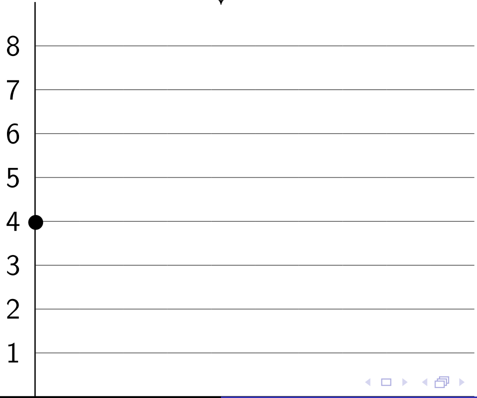
$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$

$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$

$\downarrow$

3—2, 1—1, 3—1, 4—3, |

$\downarrow$



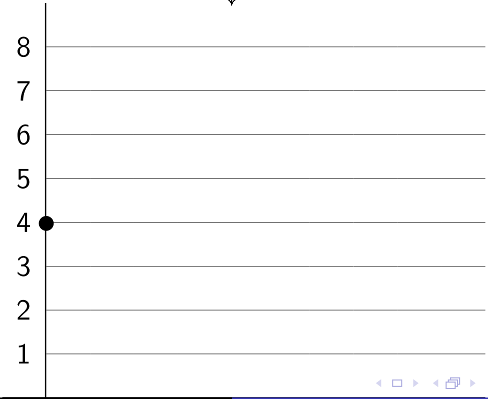
A Bijection between Alternating Sequences and Heaps

$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le \color{red}{6 \ge 6} \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$$
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$$

$$\downarrow$$

$$3\text{—}2,\ 1\text{—}1,\ 3\text{—}1,\ 4\text{—}3,\ \big|\ 6\text{—}6,\ \big|$$

$$\downarrow$$



A Bijection between Alternating Sequences and Heaps

$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le {\color{red}8 \ge 4} \le 7 \ge 4 \le 4 \ge 2$$
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$$

$$\downarrow$$

$$3\text{—}2,\ 1\text{—}1,\ 3\text{—}1,\ 4\text{—}3,\ \big|\ 6\text{—}6,\ \big|$$

$$\downarrow$$

A Bijection between Alternating Sequences and Heaps

$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$
$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$

$\downarrow$

3—$2,\ 1$—$1,\ 3$—$1,\ 4$—$3,\ \mid 6$—$6,\ \mid 8$—$4,\ \mid$

$\downarrow$

A Bijection between Alternating Sequences and Heaps

$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le {\color{red}7 \ge 4} \le 4 \ge 2$$
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$$

$$\downarrow$$

$$3\text{—}2,\ 1\text{—}1,\ 3\text{—}1,\ 4\text{—}3,\ \big|\ 6\text{—}6,\ \big|\ 8\text{—}4,\ \big|$$

$$\downarrow$$

A Bijection between Alternating Sequences and Heaps

$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$$
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$$

$$\downarrow$$

$$3\text{—}2,\ 1\text{—}1,\ 3\text{—}1,\ 4\text{—}3,\ \mid 6\text{—}6,\ \mid 8\text{—}4,\ \mid 7\text{—}4,\ \mid$$

$$\downarrow$$

A Bijection between Alternating Sequences and Heaps

$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$
$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$

$$\downarrow$$

$$3\text{—}2,\ 1\text{—}1,\ 3\text{—}1,\ 4\text{—}3,\ \big|\ 6\text{—}6,\ \big|\ 8\text{—}4,\ \big|\ 7\text{—}4,\ \big|$$

$$\downarrow$$

A Bijection between Alternating Sequences and Heaps

$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$

$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$

$\downarrow$

$3\text{—}2,\ 1\text{—}1,\ 3\text{—}1,\ 4\text{—}3,\ \big|\ 6\text{—}6,\ \big|\ 8\text{—}4,\ \big|\ 7\text{—}4,\ \big|$

$\downarrow$

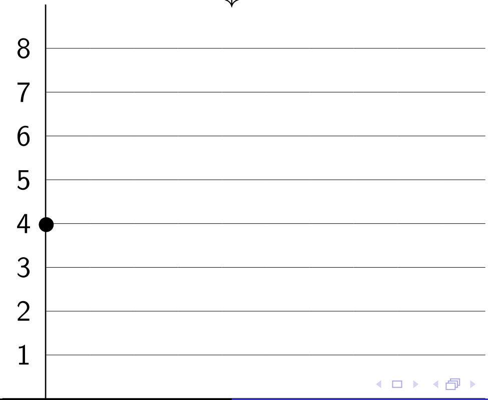
A Bijection between Alternating Sequences and Heaps

$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$
$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$

$$\downarrow$$

3—2, 1—1, 3—1, 4—3, | 6—6, | 8—4, | 7—4, |

$$\downarrow$$



A Bijection between Alternating Sequences and Heaps

$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$$
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$$

$$\downarrow$$

$$3–2,\ 1–1,\ 3–1,\ 4–3,\ \big|\ 6–6,\ \big|\ 8–4,\ \big|\ 7–4,\ \big|$$
$$2–2,\ 3–2,\ 4–2,\ \big|$$

$$\downarrow$$

A Bijection between Alternating Sequences and Heaps

$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$
$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$

$\downarrow$

3–2, 1–1, 3–1, 4–3, | 6–6, | 8–4, | 7–4, |
2–2, 3–2, 4–2, |

$\downarrow$

A Bijection between Alternating Sequences and Heaps

$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$$
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$$

$$\downarrow$$

$$3\text{—}2,\ 1\text{—}1,\ 3\text{—}1,\ 4\text{—}3,\ \big|\ 6\text{—}6,\ \big|\ 8\text{—}4,\ \big|\ 7\text{—}4,\ \big|$$
$$2\text{—}2,\ 3\text{—}2,\ 4\text{—}2,\ \big|\ 5\text{—}5,\ \big|$$

$$\downarrow$$

A Bijection between Alternating Sequences and Heaps

$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$$
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$$

$$\downarrow$$

$$3\text{—}2,\ 1\text{—}1,\ 3\text{—}1,\ 4\text{—}3,\ \big|\ 6\text{—}6,\ \big|\ 8\text{—}4,\ \big|\ 7\text{—}4,\ \big|$$
$$2\text{—}2,\ 3\text{—}2,\ 4\text{—}2,\ \big|\ 5\text{—}5,\ \big|$$

$$\downarrow$$

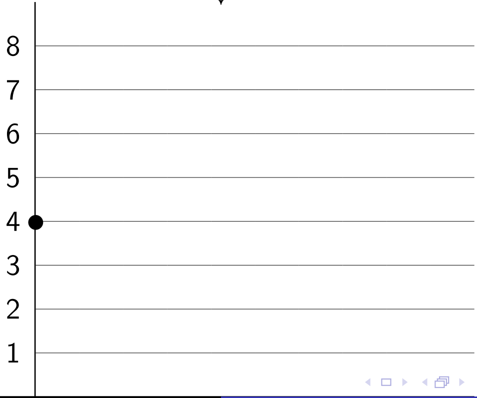
A Bijection between Alternating Sequences and Heaps

$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$$
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$$

$$\downarrow$$

$$3\text{—}2,\ 1\text{—}1,\ 3\text{—}1,\ 4\text{—}3,\ \big|\ 6\text{—}6,\ \big|\ 8\text{—}4,\ \big|\ 7\text{—}4,\ \big|$$
$$2\text{—}2,\ 3\text{—}2,\ 4\text{—}2,\ \big|\ 5\text{—}5,\ \big|\ 6\text{—}3,\ \big|$$

$$\downarrow$$



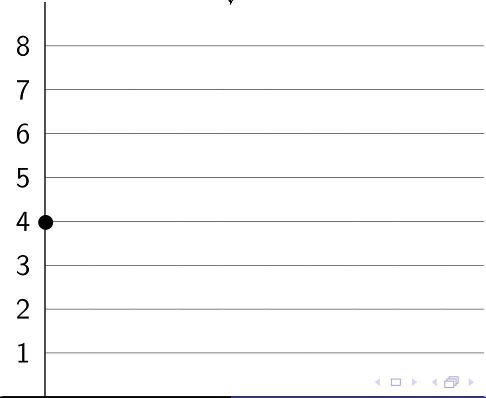
A Bijection between Alternating Sequences and Heaps

$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$$
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$$

$$\downarrow$$

$$3–2,\ 1–1,\ 3–1,\ 4–3,\ \big|\ 6–6,\ \big|\ 8–4,\ \big|\ 7–4,\ \big|$$
$$2–2,\ 3–2,\ 4–2,\ \big|\ 5–5,\ \big|\ 6–3,\ \big|$$

$$\downarrow$$



A Bijection between Alternating Sequences and Heaps

$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$
$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$

$\downarrow$

3—2, 1—1, 3—1, 4—3, | 6—6, | 8—4, | 7—4, |
2—2, 3—2, 4—2, | 5—5, | 6—3, | 6—5, |

$\downarrow$

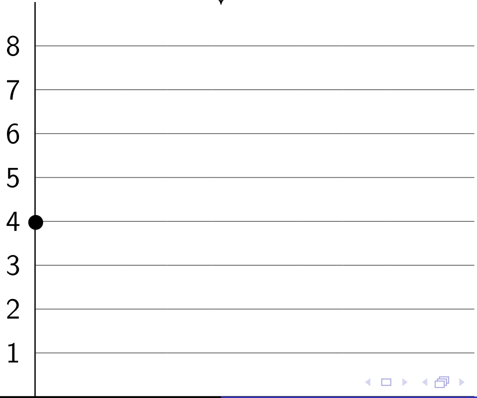
A Bijection between Alternating Sequences and Heaps

$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$
$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$

$\downarrow$

3–2, 1–1, 3–1, 4–3, | 6–6, | 8–4, | 7–4, |
2–2, 3–2, 4–2, | 5–5, | 6–3, | 6–5, |

$\downarrow$



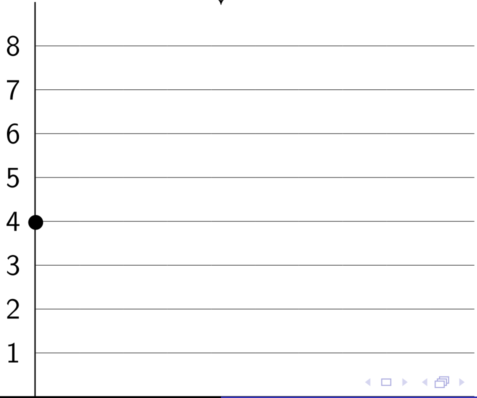
A Bijection between Alternating Sequences and Heaps

$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$$
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le {\color{red}8 \ge 6}$$

$$\downarrow$$

$$3\text{—}2,\ 1\text{—}1,\ 3\text{—}1,\ 4\text{—}3,\ \big|\ 6\text{—}6,\ \big|\ 8\text{—}4,\ \big|\ 7\text{—}4,\ \big|$$
$$2\text{—}2,\ 3\text{—}2,\ 4\text{—}2,\ \big|\ 5\text{—}5,\ \big|\ 6\text{—}3,\ \big|\ 6\text{—}5,\ \big|\ 8\text{—}6$$

$$\downarrow$$



A Bijection between Alternating Sequences and Heaps

$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$$
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$$

$$\downarrow$$

$$3{-}2,\ 1{-}1,\ 3{-}1,\ 4{-}3,\ \big|\ 6{-}6,\ \big|\ 8{-}4,\ \big|\ 7{-}4,\ \big|$$
$$2{-}2,\ 3{-}2,\ 4{-}2,\ \big|\ 5{-}5,\ \big|\ 6{-}3,\ \big|\ 6{-}5,\ \big|\ 8{-}6$$

$$\downarrow$$

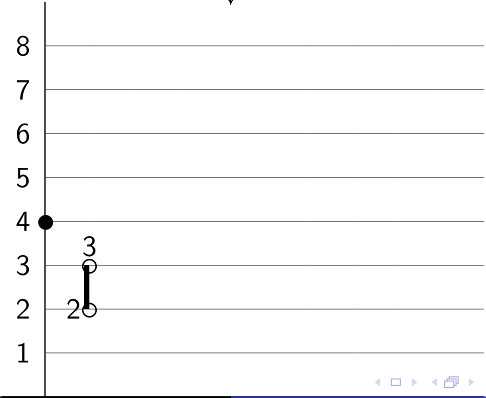
A Bijection between Alternating Sequences and Heaps

$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$$
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$$

$$\downarrow$$

$$\color{red}{3\text{—}2},\ 1\text{—}1,\ 3\text{—}1,\ 4\text{—}3,\ \mid 6\text{—}6,\ \mid 8\text{—}4,\ \mid 7\text{—}4,\ \mid$$
$$2\text{—}2,\ 3\text{—}2,\ 4\text{—}2,\ \mid 5\text{—}5,\ \mid 6\text{—}3,\ \mid 6\text{—}5,\ \mid 8\text{—}6$$

$$\downarrow$$



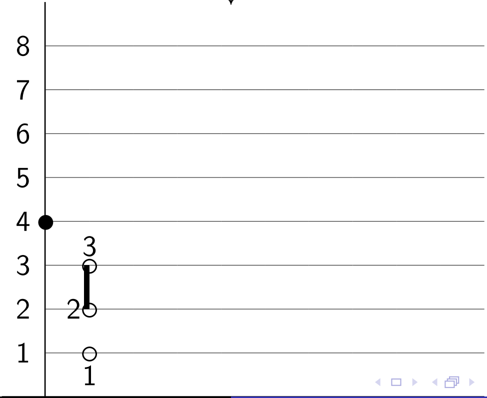
A Bijection between Alternating Sequences and Heaps

$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$$
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$$

$$\downarrow$$

$$3\text{—}2,\ \color{red}{1\text{—}1},\ 3\text{—}1,\ 4\text{—}3,\ \mid 6\text{—}6,\ \mid 8\text{—}4,\ \mid 7\text{—}4,\ \mid$$
$$2\text{—}2,\ 3\text{—}2,\ 4\text{—}2,\ \mid 5\text{—}5,\ \mid 6\text{—}3,\ \mid 6\text{—}5,\ \mid 8\text{—}6$$

$$\downarrow$$



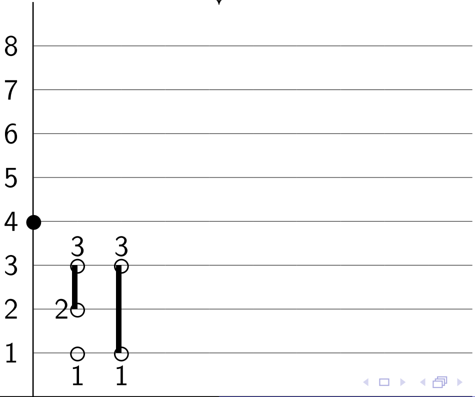
A Bijection between Alternating Sequences and Heaps

$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$
$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$

$\downarrow$

3—2, 1—1, 3—1, 4—3, | 6—6, | 8—4, | 7—4, |
2—2, 3—2, 4—2, | 5—5, | 6—3, | 6—5, | 8—6

$\downarrow$



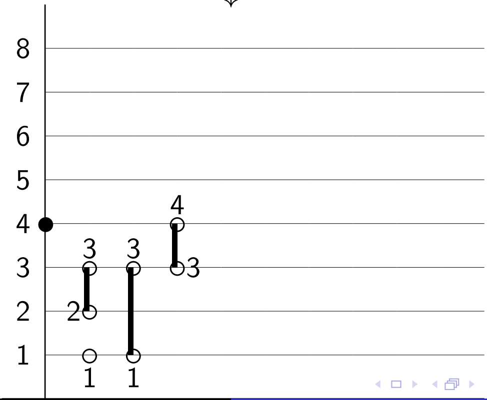
A Bijection between Alternating Sequences and Heaps

$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$
$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$

$$\downarrow$$

3—2, 1—1, 3—1, 4—3, | 6—6, | 8—4, | 7—4, |
2—2, 3—2, 4—2, | 5—5, | 6—3, | 6—5, | 8—6

$$\downarrow$$



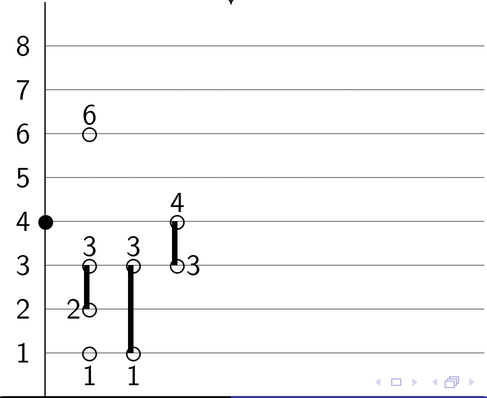
A Bijection between Alternating Sequences and Heaps

$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$$
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$$

$$\downarrow$$

$$3\text{—}2,\ 1\text{—}1,\ 3\text{—}1,\ 4\text{—}3,\ \mid \color{red}{6\text{—}6},\ \mid 8\text{—}4,\ \mid 7\text{—}4,\ \mid$$
$$2\text{—}2,\ 3\text{—}2,\ 4\text{—}2,\ \mid 5\text{—}5,\ \mid 6\text{—}3,\ \mid 6\text{—}5,\ \mid 8\text{—}6$$

$$\downarrow$$



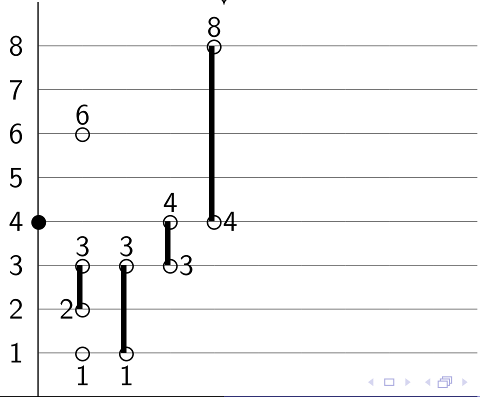
A Bijection between Alternating Sequences and Heaps

$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$$
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$$

$$\downarrow$$

3–2, 1–1, 3–1, 4–3, | 6–6, | 8–4, | 7–4, |
2–2, 3–2, 4–2, | 5–5, | 6–3, | 6–5, | 8–6

$$\downarrow$$



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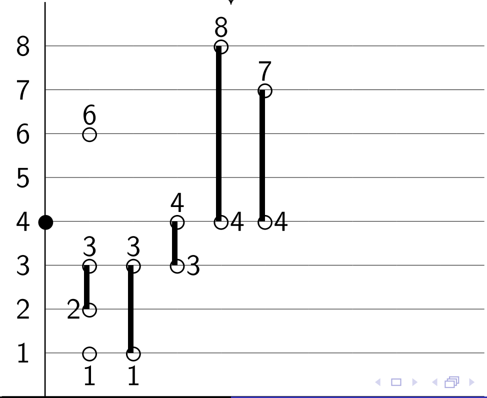
$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$$
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$$

$$\downarrow$$

$$3\!-\!2,\ 1\!-\!1,\ 3\!-\!1,\ 4\!-\!3,\ \mid 6\!-\!6,\ \mid 8\!-\!4,\ \mid {\color{red}7\!-\!4},\ \mid$$
$$2\!-\!2,\ 3\!-\!2,\ 4\!-\!2,\ \mid 5\!-\!5,\ \mid 6\!-\!3,\ \mid 6\!-\!5,\ \mid 8\!-\!6$$

$$\downarrow$$



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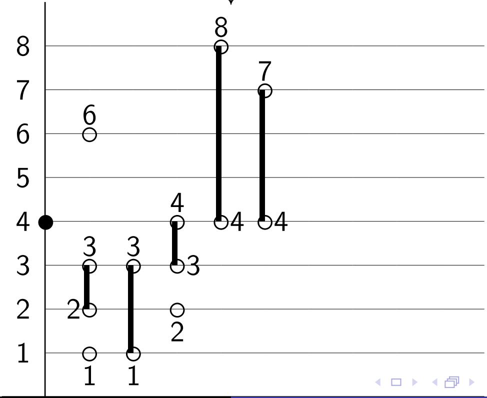
$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$$
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$$

$$\downarrow$$

$$3\text{—}2,\ 1\text{—}1,\ 3\text{—}1,\ 4\text{—}3,\ \big|\ 6\text{—}6,\ \big|\ 8\text{—}4,\ \big|\ 7\text{—}4,\ \big|$$
$$\color{red}{2\text{—}2},\ 3\text{—}2,\ 4\text{—}2,\ \big|\ 5\text{—}5,\ \big|\ 6\text{—}3,\ \big|\ 6\text{—}5,\ \big|\ 8\text{—}6$$

$$\downarrow$$



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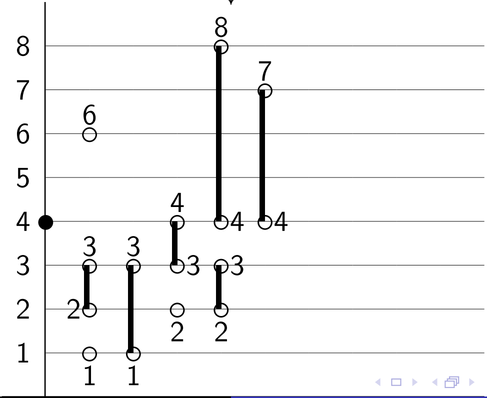
$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$$
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$$

$$\downarrow$$

$$3{-}2,\ 1{-}1,\ 3{-}1,\ 4{-}3,\ \mid 6{-}6,\ \mid 8{-}4,\ \mid 7{-}4,\ \mid$$
$$2{-}2,\ {\color{red}3{-}2},\ 4{-}2,\ \mid 5{-}5,\ \mid 6{-}3,\ \mid 6{-}5,\ \mid 8{-}6$$

$$\downarrow$$



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$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$$
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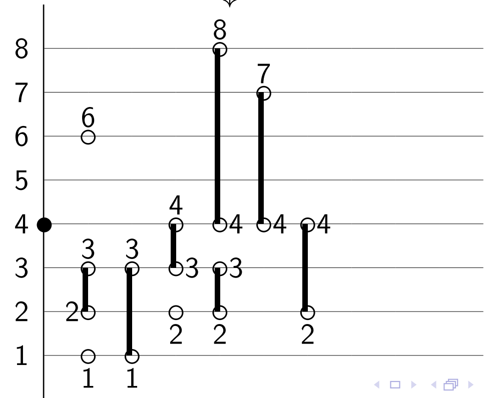
$$\downarrow$$

$$3–2,\ 1–1,\ 3–1,\ 4–3,\ \big|\ 6–6,\ \big|\ 8–4,\ \big|\ 7–4,\ \big|$$
$$2–2,\ 3–2,\ {\color{red}4–2},\ \big|\ 5–5,\ \big|\ 6–3,\ \big|\ 6–5,\ \big|\ 8–6$$

$$\downarrow$$



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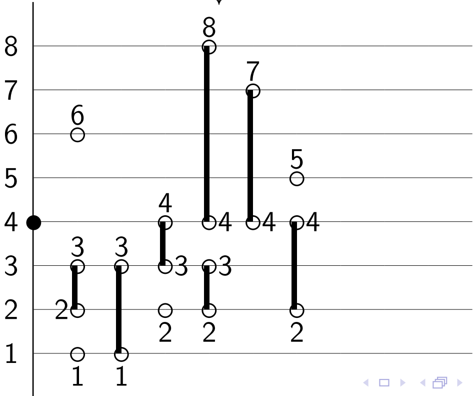
$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$$
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$$

$$\downarrow$$

3—2, 1—1, 3—1, 4—3, | 6—6, | 8—4, | 7—4, |
2—2, 3—2, 4—2, | 5—5, | 6—3, | 6—5, | 8—6

$$\downarrow$$



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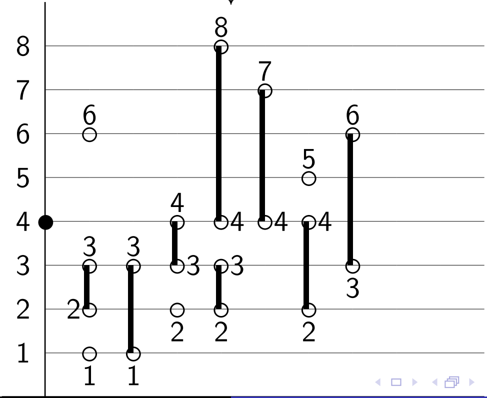
$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$$
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$$

$$\downarrow$$

$$3{-}2,\ 1{-}1,\ 3{-}1,\ 4{-}3,\ \big|\ 6{-}6,\ \big|\ 8{-}4,\ \big|\ 7{-}4,\ \big|$$
$$2{-}2,\ 3{-}2,\ 4{-}2,\ \big|\ 5{-}5,\ \big|\ 6{-}3,\ \big|\ 6{-}5,\ \big|\ 8{-}6$$

$$\downarrow$$



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$$4 \le 4 \ge 3 \le 3 \ge 1 \le 1 \ge 1 \le 3 \ge 2 \le 6 \ge 6 \le 8 \ge 4 \le 7 \ge 4 \le 4 \ge 2$$
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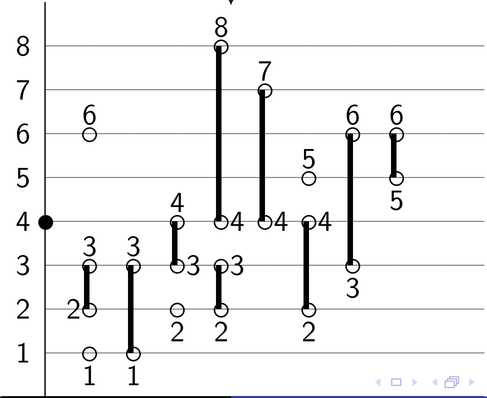
$$\downarrow$$

$$3–2,\ 1–1,\ 3–1,\ 4–3,\ \mid 6–6,\ \mid 8–4,\ \mid 7–4,\ \mid$$
$$2–2,\ 3–2,\ 4–2,\ \mid 5–5,\ \mid 6–3,\ \mid 6–5,\ \mid 8–6$$

$$\downarrow$$



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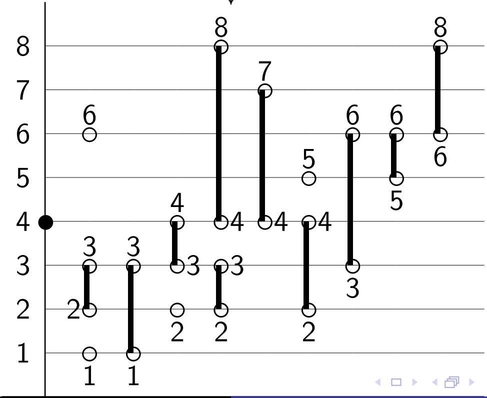
$$\downarrow$$

$$3{-}2,\ 1{-}1,\ 3{-}1,\ 4{-}3,\ \big|\ 6{-}6,\ \big|\ 8{-}4,\ \big|\ 7{-}4,\ \big|$$
$$2{-}2,\ 3{-}2,\ 4{-}2,\ \big|\ 5{-}5,\ \big|\ 6{-}3,\ \big|\ 6{-}5,\ \big|\ 8{-}6$$

$$\downarrow$$



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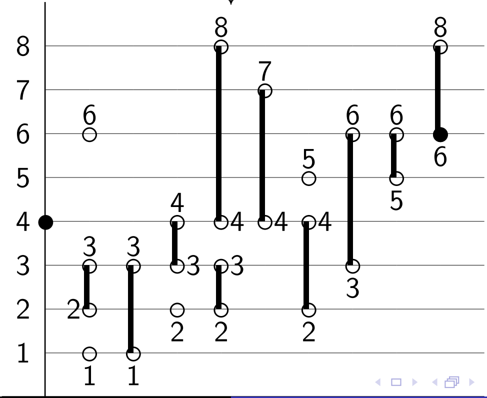
$$\le 3 \ge 2 \le 2 \ge 2 \le 5 \ge 5 \le 6 \ge 3 \le 6 \ge 5 \le 8 \ge 6$$

$$\downarrow$$

$$3\text{—}2,\ 1\text{—}1,\ 3\text{—}1,\ 4\text{—}3,\ \big|\ 6\text{—}6,\ \big|\ 8\text{—}4,\ \big|\ 7\text{—}4,\ \big|$$
$$2\text{—}2,\ 3\text{—}2,\ 4\text{—}2,\ \big|\ 5\text{—}5,\ \big|\ 6\text{—}3,\ \big|\ 6\text{—}5,\ \big|\ 8\text{—}6$$

$$\downarrow$$



A Bijection between Alternating Sequences and Heaps

Lemma

Let n, k, r, s be non-negative integers with $1 \le r \le s \le k$. There is a bijection between $\mathcal{A}_{2n+1}^{(k)}(r \to s)$ and heaps H of n segments on $[1, k]$ with the following two properties:

1. *H has a maximal segment of the form j—s.*
2. *H does not have any maximal segments that are contained in $[1, r-1]$ or $[s+1, k]$.*

A Bijection between Alternating Sequences and Heaps

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Define the weight of a segment to be x^2 (i.e., $w_{i,j} := x^2$), and define the weight of an alternating sequence of length n to be x^n. Then the above bijection is weight-preserving in the sense that the weight of an alternating sequence equals x times the weight of the corresponding heap.

Recall:

Theorem (Viennot 1986)

We have

$$\sum_{\substack{H \text{ heap of segments on } [1,k] \\ \text{maximal segments} \subseteq \mathcal{M}}} w(H) = \frac{\displaystyle\sum_{\substack{T \text{ trivial} \\ \text{segments} \subseteq \mathcal{S}_k \backslash \mathcal{M}}} (-1)^{|T|} w(T)}{\displaystyle\sum_{\substack{T \text{ trivial} \\ \text{segments} \subseteq \mathcal{S}_k}} (-1)^{|T|} w(T)},$$

where $|T|$ *denotes the number of segments of* T*.*

Lemma

Let k be a non-negative integer. The generating function $\sum_T (-1)^{|T|} w(T)$, where the sum is over all trivial heaps T of segments on $[1, k]$, is given by $(-1)^k U_{2k}(x/2)$.

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Lemma

Let r,s,k be positive integers with $1 \le r \le s \le k$. The sum of generating functions

$$\sum_{j=s}^{k} x^2 \sum_{\substack{T \text{ trivial} \\ T \subseteq [1,r-1] \cup [j+1,k]}} (-1)^{|T|} w(T)$$

is given by

$$(-1)^{k+r+s+1} x U_{2r-2}(x/2) U_{2k+1-2s}(x/2).$$

Theorem (C., K.)

For all positive integers r, s, k with $1 \le r, s \le k$, we have

$$\sum_{n\ge0} \left|\mathcal{A}_{2n+1}^{(k)}(r\to s)\right| x^{2n} = \begin{cases} (-1)^{r+s+1}\dfrac{xU_{2r-2}(x/2)\,U_{2k+1-2s}(x/2)}{U_{2k}(x/2)}, & r<s,\\ 1-\dfrac{xU_{2r-2}(x/2)\,U_{2k+1-2r}(x/2)}{U_{2k}(x/2)}, & r=s,\\ (-1)^{r+s+1}\dfrac{xU_{2s-2}(x/2)\,U_{2k+1-2r}(x/2)}{U_{2k}(x/2)}, & r>s. \end{cases}$$

and

$$\sum_{n\ge0} \left|\mathcal{A}_{2n+2}^{(k)}(r\to s)\right| x^{2n+1} = \begin{cases} (-1)^{r+s+1}\dfrac{xU_{2r-2}(x/2)\,U_{2k-2s}(x/2)}{U_{2k}(x/2)}, & r\le s,\\ (-1)^{r+s+1}\dfrac{xU_{2s-1}(x/2)\,U_{2k-2r+1}(x/2)}{U_{2k}(x/2)}, & r>s. \end{cases}$$

Reciprocity

Recall:

Theorem (Folklore/Viennot 1983)

For all non-negative integers r, s, k with $0 \le r, s \le k$, we have

$$\sum_{n\ge0} C_n^{(k)}(r\to s)\,x^n = \begin{cases} \dfrac{U_r(1/2x)\,U_{k-s}(1/2x)}{x\,U_{k+1}(1/2x)}, & \text{if } r\le s,\\[2mm] \dfrac{U_s(1/2x)\,U_{k-r}(1/2x)}{x\,U_{k+1}(1/2x)}, & \text{if } r\ge s,\end{cases}$$

where $U_m(x)$ denotes the m-th Chebyshev polynomial of the second kind.

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Recall:

Theorem (Folklore/Viennot 1983)

For all non-negative integers r, s, k with $0 \le r, s \le k$, we have

$$\sum_{n\ge0} C_n^{(k)}(r \to s)\, x^n = \begin{cases} \dfrac{U_r(1/2x)\,U_{k-s}(1/2x)}{x\,U_{k+1}(1/2x)}, & \text{if } r \le s, \\ \dfrac{U_s(1/2x)\,U_{k-r}(1/2x)}{x\,U_{k+1}(1/2x)}, & \text{if } r \ge s, \end{cases}$$

where $U_m(x)$ denotes the m-th Chebyshev polynomial of the second kind.

Furthermore, if

$$f(x) = \frac{p(x)}{q(x)} = \sum_{n\ge0} a_n x^n$$

is rational with $\deg(p(x)) < \deg(q(x))$, then

$$\sum_{n\ge1} a_{-n} x^n = -f(1/x).$$

Reciprocity

Corollary

Let n, k, r, s be positive integers with $1 \le r, s \le k$. The number $(-1)^{r+s} C_{-2n}^{(2k-1)}(2r-2 \to 2s-2)$ equals $|\mathcal{A}_{2n+1}^{(k)}(r \to s)|$ Furthermore, the number $(-1)^{r+s} C_{-2n+1}^{(2k-1)}(2r-2 \to 2s-1)$ equals $|\mathcal{A}_{2n}^{(k)}(r \to s)|$.

Reciprocity

Corollary

Let n, k, r, s *be positive integers with* $1 \le r, s \le k$. *The number* $(-1)^{r+s} C_{-2n}^{(2k-1)}(2r-2 \to 2s-2)$ *equals* $|\mathcal{A}_{2n+1}^{(k)}(r \to s)|$ *Furthermore, the number* $(-1)^{r+s} C_{-2n+1}^{(2k-1)}(2r-2 \to 2s-1)$ *equals* $|\mathcal{A}_{2n}^{(k)}(r \to s)|$.

Corollary

For positive integers n *and* k, *the number* $C_{-2n}^{(2k-1)}$ *equals* $|\mathcal{A}_{2n-1}^{(k)}|$.

Reciprocity

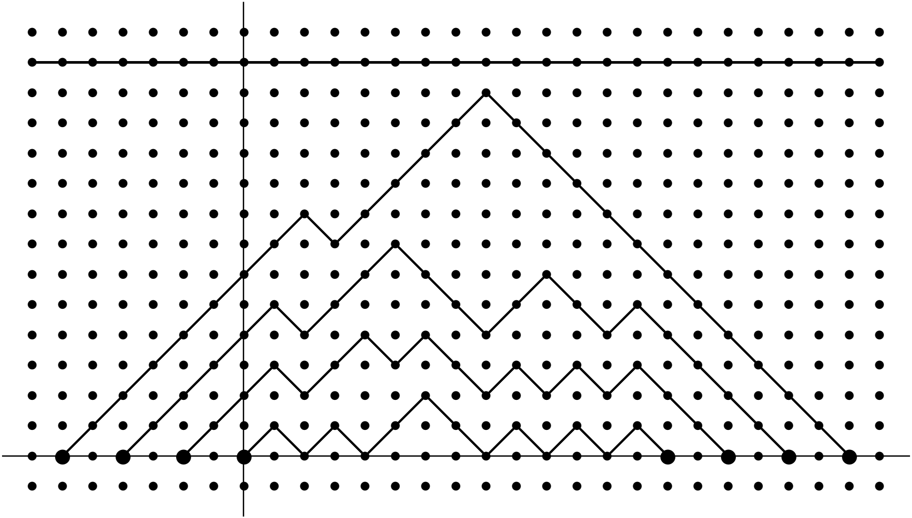
Remember:

I asked Cigler: "What if we lift the upper bound on the paths? Do we then also get determinants on the left-hand side?"

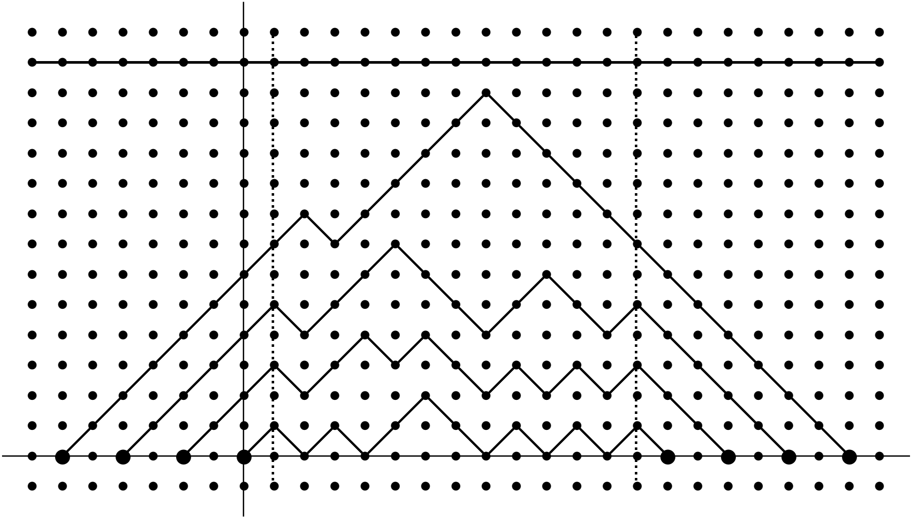
Cigler (next day): "Yes. Here is the — conjectured — formula:

$$\det\left(C^{(2k+2m-1)}_{-2n-2i-2j}\right)_{0\le i,j\le m-1}=\det\left(C^{(2k+2m-1)}_{2n+2i+2j+4m-2}\right)_{0\le i,j\le k-1}.$$

Example for $n = 2$, $k = 4$, $m = 3$:



Example for $n = 2$, $k = 4$, $m = 3$:



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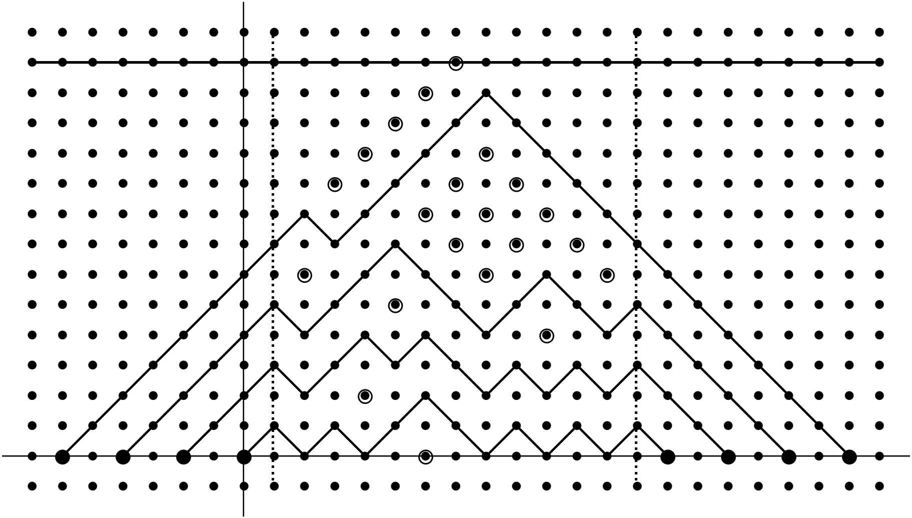

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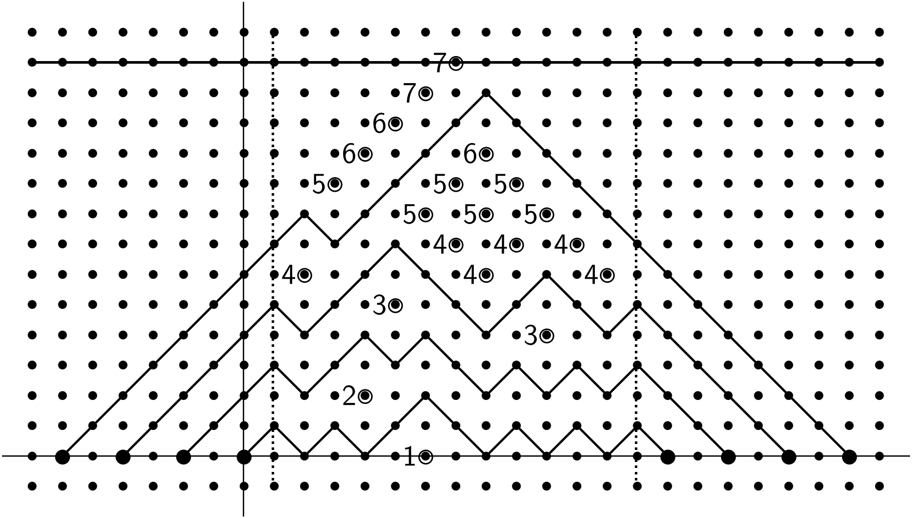
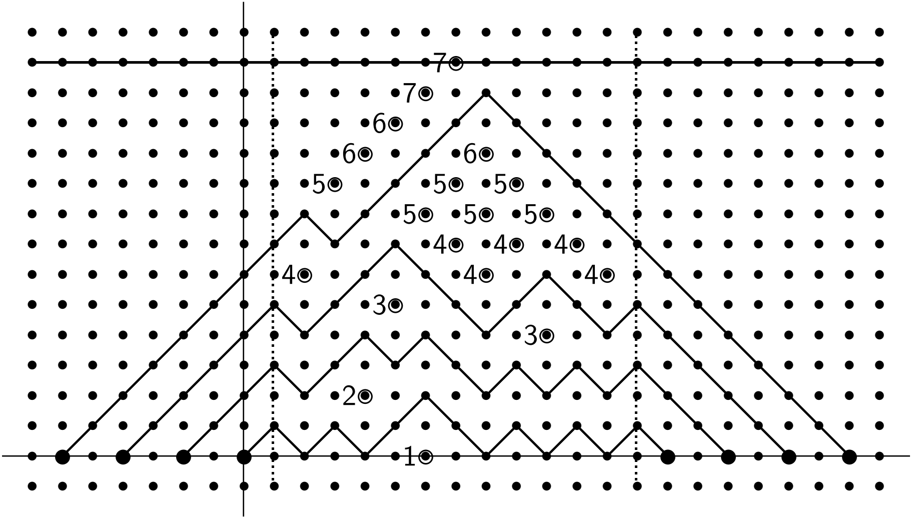
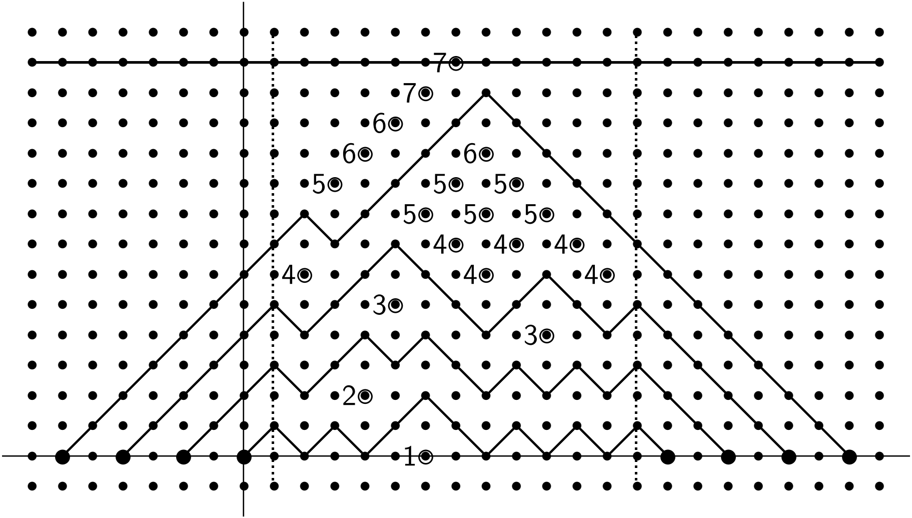


Example for $n = 2$, $k = 4$, $m = 3$:



$$\begin{array}{ccccccccccc} & & & & 7 & 7 & 6 & & & & \\ & & 6 & 6 & 5 & 5 & 5 & 5 & 5 & & \\ 4 & 5 & 2 & 3 & 1 & 4 & 4 & 4 & 3 & 4 & 4 \end{array}$$

Example for $n = 2$, $k = 4$, $m = 3$:

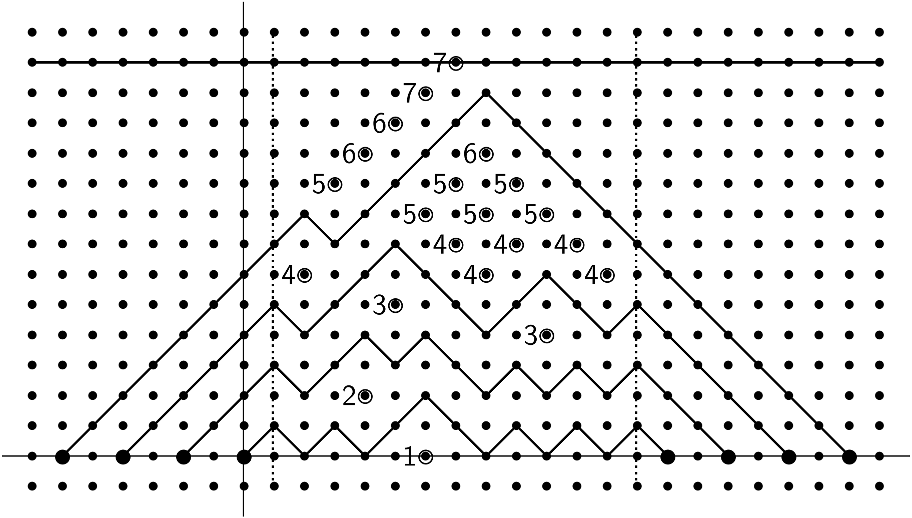


7 7 6

6 6 5 5 5 5 5

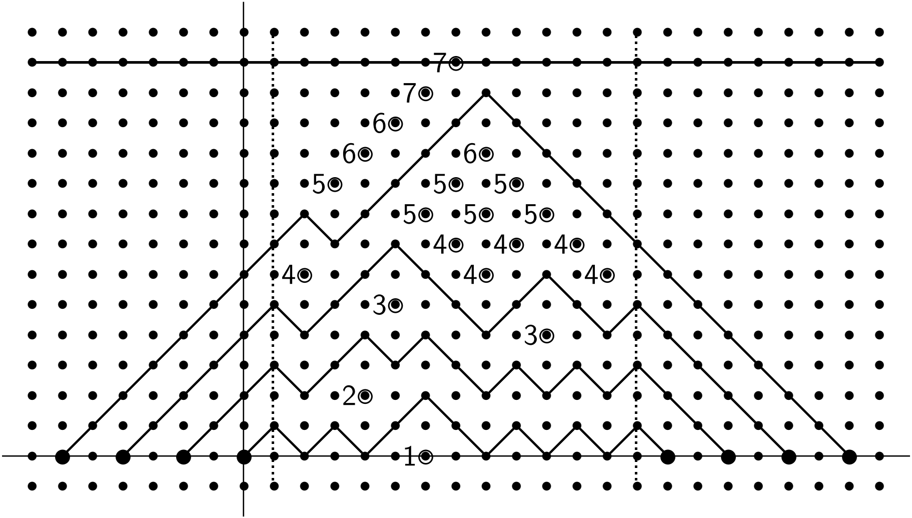
4 5 2 3 1 4 4 4 3 4 4

Example for $n = 2$, $k = 4$, $m = 3$:



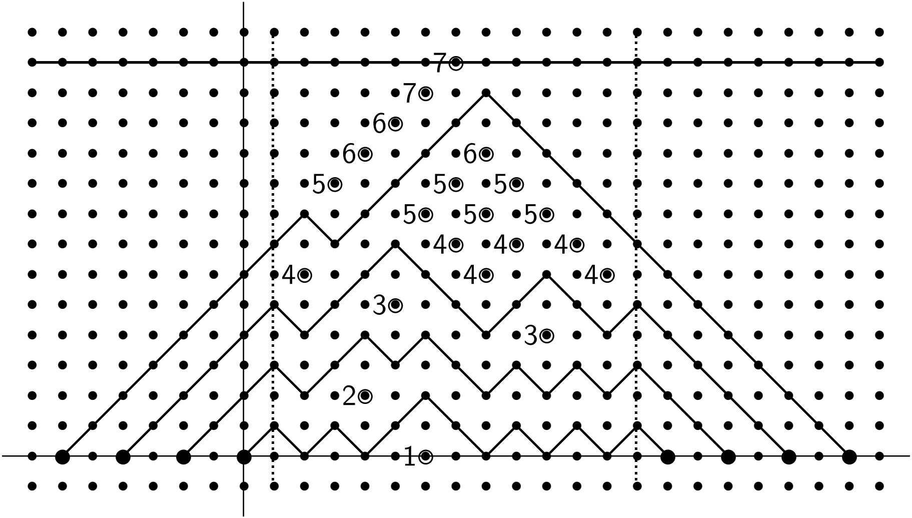
$$
\begin{array}{ccccccccccc}
 & & & & 7 & 7 & 6 & & & & \\
 & & 6 & 6 & 5 & 5 & 5 & 5 & 5 & & \\
4 & 5 & 2 & 3 & 1 & 4 & 4 & 4 & 3 & 4 & 4
\end{array}
$$

Example for $n = 2,\ k = 4,\ m = 3$:



				7	7	6				
		6	6	5	5	5	5	5		
4	5	2	3	1	4	4	4	3	4	4

Example for $n = 2$, $k = 4$, $m = 3$:

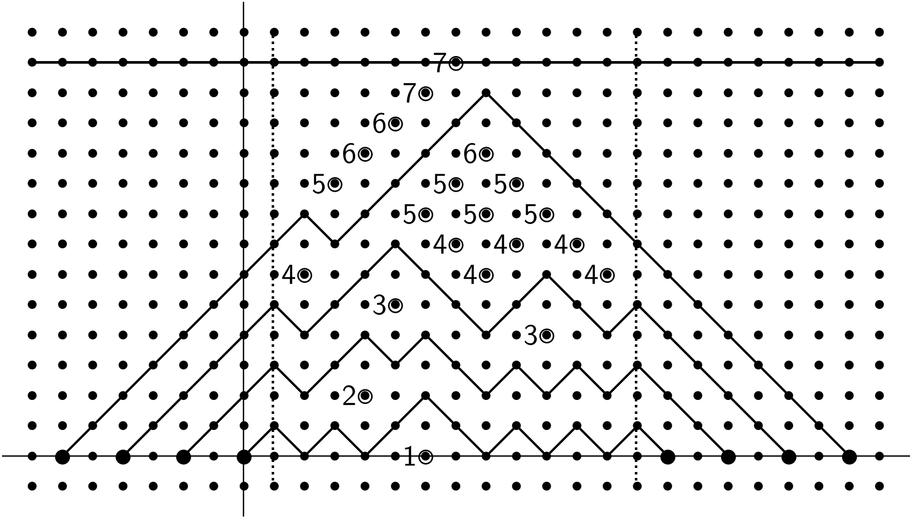


7 7 6

6 6 5 5 5 5 5

4 5 2 3 1 4 4 4 3 4 4

Example for $n = 2,\ k = 4,\ m = 3$:

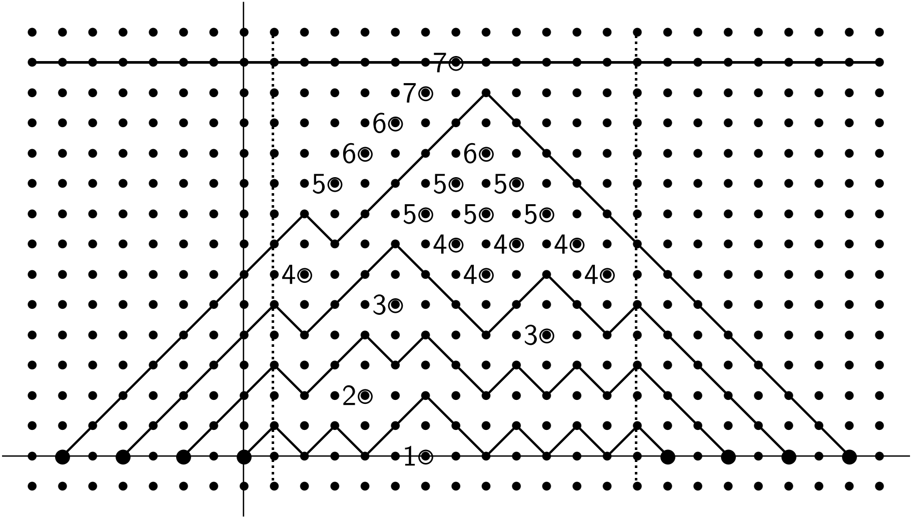


7 7 6

6 6 5 5 5 5 5

4 5 2 3 1 4 4 4 3 4 4

Example for $n = 2$, $k = 4$, $m = 3$:



$$\begin{array}{ccccccccccc} & & & & 7 & 7 & 6 & & & & \\ & & 6 & 6 & 5 & 5 & 5 & 5 & \color{red}{5} & & \\ 4 & 5 & 2 & 3 & 1 & 4 & 4 & \color{red}{4} & 3 & \color{red}{4} & 4 \end{array}$$

Apparently:

The number of trapezoidal arrays of integers of the form

$$\begin{array}{ccccccccccccc}
 & & & & & a_{1,2m-1} & \cdots & a_{1,2n+2m-3} & & & & & \\
 & & & & \iddots & & & & \ddots & & & & \\
 & & & a_{m-2,5} & & & \cdots\cdots\cdots & & a_{m-2,M-5} & & & & \\
 & a_{m-1,3} & a_{m-1,4} & a_{m-1,5} & & & \cdots\cdots\cdots & & a_{m-1,M-5} & a_{m-1,M-4} & a_{m-1,M-3} & & \\
a_{m,1}\ a_{m,2} & a_{m,3} & a_{m,4} & a_{m,5} & & & \cdots\cdots\cdots & & a_{m,M-5} & a_{m,M-4} & a_{m,M-3} & a_{m,M-2} & a_{m,M-1}
\end{array}$$

where $M = 2n + 4m - 4$*, in which each row is alternating, that satisfy*

$$1 \le a_{i,j} \le k + m$$

and

$$a_{i+1,2j} < a_{i,2j+1} > a_{i+1,2j+2}$$

is equal to

$$\det \left(C^{(2k+2m-1)}_{-2n-2i-2j} \right)_{0 \le i,j \le m-1} .$$

Theorem (C., K.)

The number of trapezoidal arrays of integers of the form

$$\begin{array}{cccccccccccccc}
 & & & & & a_{1,2m-1} & \cdots & a_{1,2n+2m-3} & & & & & & \\
 & & & & \iddots & & & & \ddots & & & & & \\
 & & & a_{m-2,5} & & & \cdots\cdots\cdots & & & a_{m-2,M-5} & & & & \\
 & a_{m-1,3} & a_{m-1,4} & a_{m-1,5} & & & \cdots\cdots\cdots & & & a_{m-1,M-5} & a_{m-1,M-4} & a_{m-1,M-3} & & \\
a_{m,1} \; a_{m,2} & a_{m,3} & a_{m,4} & a_{m,5} & & & \cdots\cdots\cdots & & & a_{m,M-5} & a_{m,M-4} & a_{m,M-3} & a_{m,M-2} & a_{m,M-1}
\end{array}$$

where $M = 2n + 4m - 4$, *in which each row is alternating, that satisfy*

$$1 \le a_{i,j} \le k + m$$

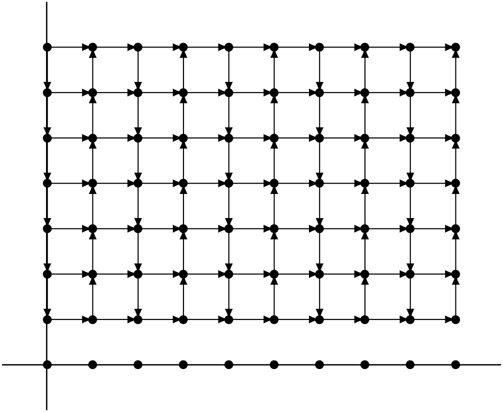
and

$$a_{i+1,2j} < a_{i,2j+1} > a_{i+1,2j+2}$$

is equal to

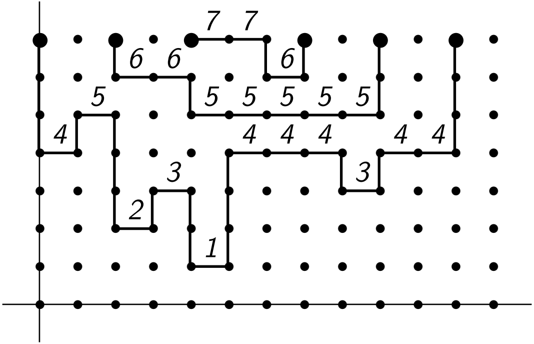
$$\det \left(\left| \mathcal{A}^{(k+m)}_{2n+2i+2j-1} \right| \right)_{0 \le i,j \le m-1} .$$

Sketch of proof. We use again non-intersecting lattice paths, however in a different manner. The directed graph that we need here is of the following form:



				7	7	6				
		6	6	5	5	5	5	5		
4	5	2	3	1	4	4	4	3	4	4

$$\updownarrow$$



More reciprocity laws

Theorem (C., K.)

For all non-negative integers n, k, m, we have

$$\det\left(C^{(2k+2m-1)}_{2n+2i+2j+4m-2}\right)_{0\le i,j\le k-1} = \det\left(C^{(2k+2m-1)}_{-2n-2i-2j}\right)_{0\le i,j\le m-1}.$$

More reciprocity laws

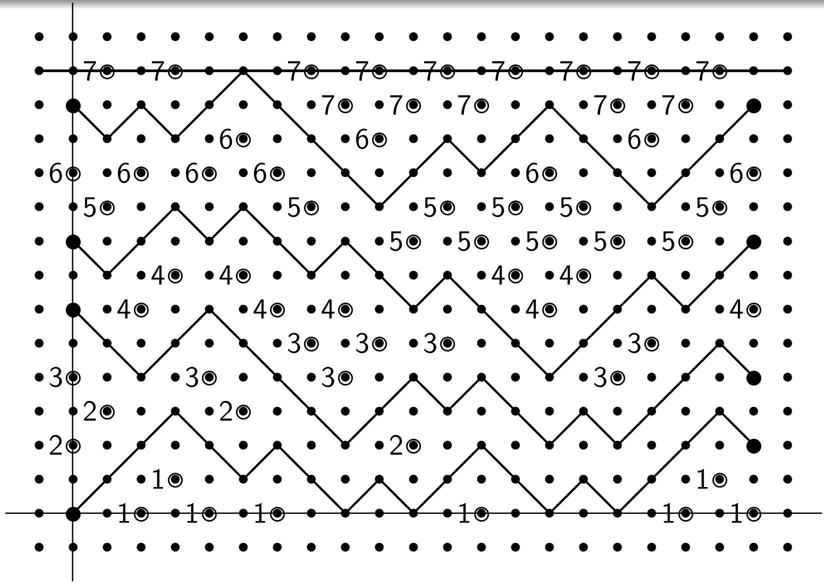
Theorem

Let n, k, m be positive integers, and let $r_0 < r_1 < \cdots < r_{k-1}$ and $s_0 < s_1 < \cdots < s_{k-1}$ be sequences of positive integers with $1 \le r_i, s_i \le k+m$ for all i. Then

$$\det\left(C_{2n}^{(2k+2m-1)}(2r_i - 2 \to 2s_j - 2)\right)_{0\le i,j\le k-1}$$
$$= (-1)^{\sum_{i=0}^{m-1}(\bar{r}_i+\bar{s}_i)} \det\left(C_{-2n}^{(2k+2m-1)}(\bar{r}_i \to \bar{s}_j)\right)_{0\le i,j\le m-1},$$

where

$$\{\bar{r}_0, \bar{r}_1, \ldots, \bar{r}_{m-1}\} = \{1, 2, \ldots, k\} \setminus \{r_0, r_1, \ldots, r_{k-1}\},$$
$$\{\bar{s}_0, \bar{s}_1, \ldots, \bar{s}_{m-1}\} = \{1, 2, \ldots, k\} \setminus \{s_0, s_1, \ldots, r_{k-1}\}.$$



6	7	6	7	6	6	6	7	7	7	7	7	7	7	6	7	7	7	7	7	6
3	5	4	4	3	4	4	5	4	6	5	5	5	5	5	5	5	6	5	5	4
2	2	1	1	1	2	1	3	3	3	2	3	1	4	4	4	3	3	1	1	1

Theorem

Let n, k, m be positive integers, and let $r_0 < r_1 < \cdots < r_{k-1}$ and $s_0 < s_1 < \cdots < s_{k-1}$ be sequences of positive integers with $1 \le r_i, s_i \le k+m$ for all i. The number of rectangular arrays of integers of the form

$$\begin{array}{cccc} a_{1,1} & a_{1,2} & \dots & a_{1,2n+1} \\ a_{2,1} & a_{2,2} & \dots & a_{2,2n+1} \\ & & & \\ a_{m,1} & a_{m,2} & \dots & a_{m,2n+1} \end{array}$$

where $a_{i,1} = \bar{r}_i$ and $a_{i,2n+1} = \bar{s}_i$ for all i, in which each row is alternating, and in which we have

$$1 \le a_{i,j} \le k+m$$

and

$$a_{i+1,2j} < a_{i,2j+1} > a_{i+1,2j+2},$$

is equal to

$$\det\left(\left|\mathcal{A}_{2n+1}^{(k+m)}(r_i \to s_j)\right|\right)_{0 \le i,j \le m-1}.$$

Here,

$$\{\bar{r}_0, \bar{r}_1, \dots, \bar{r}_{m-1}\} = \{1, 2, \dots, k+m\} \setminus \{r_0, r_1, \dots, r_{k-1}\},$$
$$\{\bar{s}_0, \bar{s}_1, \dots, \bar{s}_{m-1}\} = \{1, 2, \dots, k+m\} \setminus \{s_0, s_1, \dots, r_{k-1}\}.$$

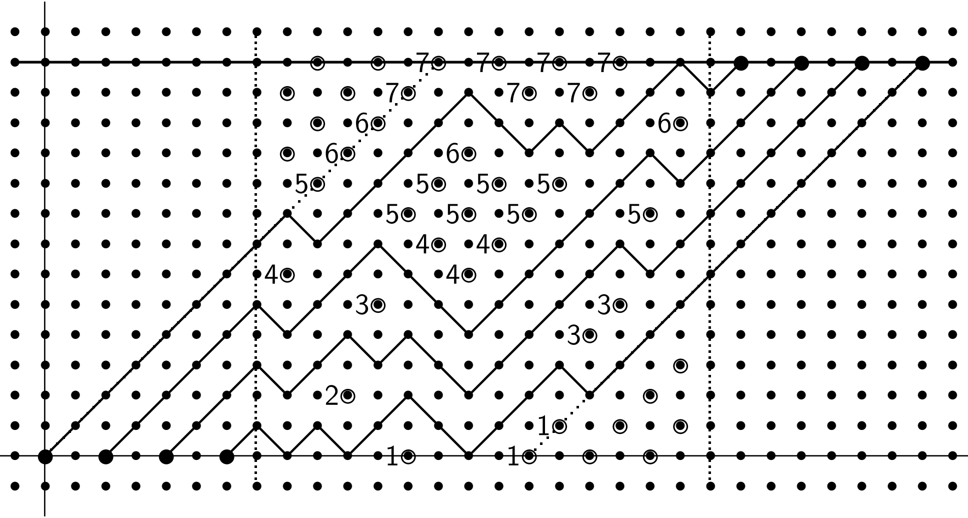
More reciprocity laws

Theorem

For all non-negative integers n, k, m, we have

$$\det\left(D^{(2k+2m-1)}_{2n+2j-2i}\right)_{0\le i,j\le k-1} = (-1)^{km}\det\left(D^{(2k+2m-1)}_{-2n-2j+2i-2k-2m}\right)_{0\le i,j\le m-1}$$

Here, $D^{(k)}_{2n}$ is short for $C^{(k)}_{2n+k}(0\to k)$.



7 7 6 7 7 7 7 7 5 6

6 6 5 5 5 5 5 5 3 3

4 5 2 3 1 4 4 4 1 1

Theorem

Let n, k, m be non-negative integers. The number of rhomboidal arrays of integers of the form

$$
\begin{array}{ccccccccccccc}
 & & & & & & a_{1,2m-1} & & \cdots\cdots\cdots & & a_{1,M-4} & a_{1,M-3} & a_{1,M-2} \\
 & & & & a_{2,2m-3} & a_{2,2m-2} & a_{2,2m-1} & \cdots & a_{2,M-6} & a_{2,M-5} & a_{2,M-4} & & \\
 & & a_{3,2m-5} & a_{3,2m-4} & a_{3,2m-3} & & \cdots\cdots\cdots & & a_{3,M-6} & & & & \\
 & \iddots & & & & & & \iddots & & & & & \\
a_{m,1} & a_{m,2} & a_{m,3} & & \cdots\cdots\cdots\cdots & & & a_{m,2n} & & & & &
\end{array}
$$

where $M = 2m + 2n$, in which each row is alternating, that satisfy

$$1 \leq a_{i,j} \leq k + m$$

and

$$a_{i+1,2j} < a_{i,2j+1} > a_{i+1,2j+2}$$

is equal to

$$(-1)^{km} \det \left(D^{(2k+2m-1)}_{-2n-2i+2j-2k-2m} \right)_{0 \leq i,j \leq m-1} = \det \left(\left| \mathcal{A}^{(k+m)}_{2n+2i-2j} \right| \right)_{0 \leq i,j \leq m-1}$$

What else?

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- Weighted enumeration

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- Enumeration results for “alternating tableaux”

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- Refinement of a generating function result of Bousquet-Mélou and Viennot for parallelogram polyominoes
- Several further conjectures . . .