Determinant identities for moments of orthogonal polynomials

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Prelude

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Cvetković, Rajković and Ivković proved

$$\det\left(C_{i+j} + C_{i+j+1}\right)_{i,j=0}^{n-1} = F_{2n+1}$$

and

$$\det\left(C_{i+j+1} + C_{i+j+2}\right)_{i,j=0}^{n-1} = F_{2n+2}.$$

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Johann Cigler saw

$$\frac{\det\left(\binom{2i+2j+2}{i+j+1}+2\binom{2i+2j+4}{i+j+2}+\binom{2i+2j+6}{i+j+3}\right)_{i,j=0}^{n-1}}{2^n}=\sum_{j=0}^{n}L_{2j+1}^2$$

on a Facebook group (without proof).

Prelude

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Dougherty, French, Saderholm and Qian proved that

$$\det\left(\lambda C_{i+j} + C_{i+j+1}\right)_{i,j=0}^{n-1}$$

satisfies a linear recurrence with constant coefficients of order 2, that

$$\det\left(\lambda C_{i+j} + \mu C_{i+j+1} + C_{i+j+2}\right)_{i,j=0}^{n-1}$$

satisfies a linear recurrence with constant coefficients of order 4, that

$$\det\left(\lambda C_{i+j} + \mu C_{i+j+1} + \nu C_{i+j+2} + C_{i+j+3}\right)_{i,j=0}^{n-1}$$

satisfies a linear recurrence with constant coefficients of order 8.

Prelude

More generally, Dougherty, French, Saderholm and Qian conjectured that

$$\det\left(\lambda_0 C_{i+j} + \lambda_1 C_{i+j+1} + \cdots + \lambda_{d-1} C_{i+j+d-1} + C_{i+j+d}\right)_{i,j=0}^{n-1}$$

satisfies a linear recurrence with constant coefficients of order 2^d.

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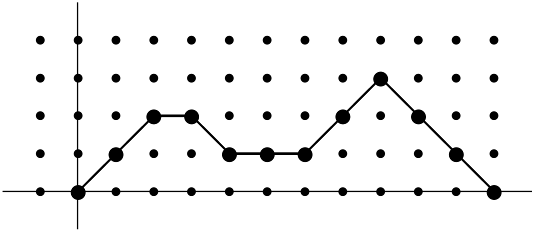
satisfies a linear recurrence with constant coefficients of order 2^d.

Cigler decided to search for the general background of this kind of determinant evaluations.

A conjecture

He considered *Motzkin paths*. These are lattice paths consisting of up-steps $(1,1)$, horizontal steps $(1,0)$ and down-steps $(1,-1)$ that start and end on the x-axis and never run below the x-axis.



The weight of an up-step is defined to be 1, the weight of a horizontal step at height h is s_h, and the weight of a down-step which ends at height h is t_h. By definition, the weight of a Motzkin path is the product of the weights of its steps. Thus, the weight of the above Motzkin path is $s_2 t_1 s_1 s_1 t_2 t_1 t_0 = s_1^2 s_2 t_0 t_1^2 t_2$. Let m_n denote the generating function for Motzkin paths of length n.

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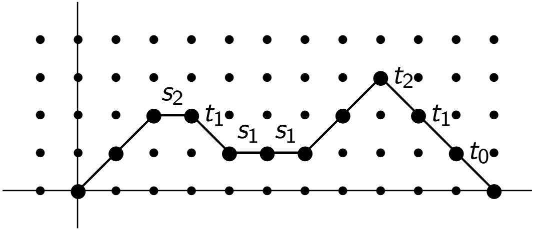

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Cigler found (experimentally) that

$$\frac{\det\left(\alpha\beta m_{i+j} + (\alpha+\beta)m_{i+j+1} + m_{i+j+2}\right)_{i,j=0}^{n-1}}{\det\left(m_{i+j}\right)_{i,j=0}^{n-1}} = \sum_{j=0}^{n} f_j(\alpha) f_j(\beta) \prod_{\ell=j}^{n-1} t_\ell,$$

where

$$f_n(\alpha) = (\alpha + s_{n-1}) f_{n-1}(\alpha) - t_{n-2} f_{n-2}(\alpha),$$

with $f_0(\alpha) = 1$ and $f_{-1}(\alpha) = 0$.

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Facts.

If $s_0 = 1$, $s_i = 2$ for $i \ge 1$, and $t_i = 1$ for all i, then $m_n = C_n$.

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It is classical that the polynomials $p_n(x)$ defined recursively by

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with initial values $p_{-1}(x) = 0$ and $p_0(x) = 1$ are orthogonal with respect to the linear functional L defined by $L(p_n(x)) = \delta_{n,0}$. Their moments are $L(x^n)$, $n = 0, 1, \dots$. Viennot showed that $L(x^n)$ equals the generating function for Motzkin paths denoted here by m_n.

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Remark. (1) $p_n(x)$ and $f_n(x)$ are related by

$$f_n(x) = (-1)^n p_n(-x).$$

(2) It is well-known that

$$\det\left(m_{i+j}\right)_{i,j=0}^{n-1} = \prod_{i=0}^{n-1} t_i^{n-i-1}.$$

The formula: proof by non-intersecting lattice paths

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$$\frac{\det\left(\alpha\beta m_{i+j} + (\alpha+\beta) m_{i+j+1} + m_{i+j+2}\right)_{i,j=0}^{n-1}}{\det\left(m_{i+j}\right)_{i,j=0}^{n-1}} = \sum_{j=0}^{n} f_j(\alpha) f_j(\beta) \prod_{\ell=j}^{n-1} t_\ell,$$

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We figured out that the above identity can be proved "in one picture" by using non-intersecting lattice paths.

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An important special case

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More generally, if $s_i \equiv s$ and $t_i \equiv t$ for $i \ge 1$, then

$$f_n(\alpha) = t^{n/2} U_n\left(\tfrac{\alpha+s}{2\sqrt{t}}\right) - t^{(n-1)/2}(s - s_0) U_{n-1}\left(\tfrac{\alpha+s}{2\sqrt{t}}\right) \\ + t^{(n-2)/2}(t - t_0) U_{n-2}\left(\tfrac{\alpha+s}{2\sqrt{t}}\right), \quad \text{for } n \ge 1.$$

where $U_n(x)$ is the n-th *Chebyshev polynomial of the second kind*

$$U_n(x) = \sum_{k \ge 0} (-1)^k \binom{n-k}{k} (2x)^{n-2k}.$$

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Recall:

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta} = \frac{e^{\mathbf{i}(n+1)\theta} - e^{-\mathbf{i}(n+1)\theta}}{e^{\mathbf{i}\theta} - e^{-\mathbf{i}\theta}} = \frac{Z^{n+1} - Z^{-(n+1)}}{Z - Z^{-1}}.$$

The formula again

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An alternative formula in this special case

Moreover, in that case we have

$$\frac{\det\left(\alpha\beta m_{i+j}+(\alpha+\beta)m_{i+j+1}+m_{i+j+2}\right)_{i,j=0}^{n-1}}{\det\left(m_{i+j}\right)_{i,j=0}^{n-1}}=\frac{\mathrm{Num}(\alpha,\beta)}{\alpha-\beta},$$

where

$$\begin{aligned}\mathrm{Num}(\alpha,\beta)&=t^{\frac{1}{2}(2n+1)}(U_\alpha-U_\beta)\\&\quad\times\left(1-t^{-1/2}(s-s_0)U_\alpha^{-1}+t^{-1}(t-t_0)U_\alpha^{-2}\right)\\&\quad\quad\times\left(1-t^{-1/2}(s-s_0)U_\beta^{-1}+t^{-1}(t-t_0)U_\beta^{-2}\right)U_\beta^n,\end{aligned}$$

with

$$U_\alpha^n\equiv U_n\left(\tfrac{\alpha+s}{2\sqrt{t}}\right),$$

$U_n(x)$ being the n-th *Chebyshev polynomial of the second kind*,

$$U_n(x)=\sum_{k\ge0}(-1)^k\binom{n-k}{k}(2x)^{n-2k}.$$

Interruption: umbral notation

Umbral notation is an elegant short notation that is convenient in certain situations. Given a sequence $(c_n)_{n\ge0}$, one identifies c^n with c_n in polynomial expressions in c.
For example,

$$c^2(\alpha+c)(\beta+c) = c^2\alpha\beta + c^3(\alpha+\beta) + c^4 = c_2\alpha\beta + c_3(\alpha+\beta) + c_4.$$

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Another example: let B_n denote the n-th *Bernoulli number*. The Bernoulli numbers satisfy the recursion

$$B_n=\sum_{k=0}^{n}\binom{n}{k}B_k,$$

or, in umbral notation,

$$B^n=(B+1)^n.$$

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Moreover, in that case we have

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Generalisation of that formula, still in that special case

For the case where $s_i \equiv s$ and $t_i \equiv t$ for $i \geq 1$, we have

$$\frac{\det\left(m^{i+j}\prod_{\ell=1}^{d}(\alpha_\ell+m)\right)_{i,j=0}^{n-1}}{\det\left(m_{i+j}\right)_{i,j=0}^{n-1}} = \frac{\mathrm{Num}(\alpha_1,\dots,\alpha_d)}{\prod_{1\le i<j\le d}(\alpha_i-\alpha_j)},$$

where

$$\mathrm{Num}(\alpha_1,\dots,\alpha_d) = t^{\frac{1}{2}\left(dn+\binom{d}{2}\right)} \prod_{1\le i<j\le d}\left(U_{\alpha_i}-U_{\alpha_j}\right)$$
$$\times \prod_{i=1}^{d}\left(1-t^{-1/2}(s-s_0)U_{\alpha_i}^{-1}+t^{-1}(t-t_0)U_{\alpha_i}^{-2}\right)U_{\alpha_i}^{n},$$

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Maybe this holds without the restriction $s_i \equiv s$ and $t_i \equiv t$ for $i \ge 1$?

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The computer says "yes".

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This should be known.

$\longrightarrow$ Gábor Szegő: *Orthogonal Polynomials* (1939)

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ORTHOGONAL POLYNOMIALS

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Gábor Szegő: *Orthogonal Polynomials* (1939)

and, for $n \geqq 0$,

$$D_n = [(f_\nu, f_\mu)]_{\nu,\mu=0,1,2,\dots,n} > 0. \tag{2.1.7}$$

We write $D_{-1} = 1$ and $D_0(x) = f_0(x)$. The determinant (2.1.7) corresponds to the positive definite quadratic form

$$\begin{aligned} &\| u_0 f_0 + u_1 f_1 + \cdots + u_n f_n \|^2 \\ &\qquad = \int_a^b \{u_0 f_0(x) + u_1 f_1(x) + \cdots + u_n f_n(x)\}^2 \, d\alpha(x), \end{aligned} \tag{2.1.8}$$

so that $D_n > 0$ for each n.

Furthermore, the following integral representations can be established:

$$\begin{aligned} D_n(x) = \frac{1}{n!} \underbrace{\int_a^b \int_a^b \cdots \int_a^b}_{n} &\begin{vmatrix} f_0(x_0) & f_1(x_0) & \cdots & f_n(x_0) \\ f_0(x_1) & f_1(x_1) & \cdots & f_n(x_1) \\ \cdots & \cdots & \cdots & \cdots \\ f_0(x_{n-1}) & f_1(x_{n-1}) & \cdots & f_n(x_{n-1}) \\ f_0(x) & f_1(x) & \cdots & f_n(x) \end{vmatrix} \\ \cdot &\begin{vmatrix} f_0(x_0) & f_1(x_0) & \cdots & f_{n-1}(x_0) \\ f_0(x_1) & f_1(x_1) & \cdots & f_{n-1}(x_1) \\ \cdots & \cdots & \cdots & \cdots \\ f_0(x_{n-1}) & f_1(x_{n-1}) & \cdots & f_{n-1}(x_{n-1}) \end{vmatrix} d\alpha(x_0)\, d\alpha(x_1) \cdots d\alpha(x_{n-1}), \quad n \geqq 1, \end{aligned} \tag{2.1.9}$$

$$D_n = \frac{1}{(n+1)!} \tag{2.1.10}$$

$$\int_a^b \int_a^b \cdots \int_a^b \begin{vmatrix} f_0(x_0) & f_1(x_0) & \cdots & f_n(x_0) \\ f_0(x_1) & f_1(x_1) & \cdots & f_n(x_1) \end{vmatrix}^2 d\alpha(x_0)\,d\alpha(x_1) \cdots d\alpha(x_n)$$

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(2.2.6)
$$p_n(x) = (D_{n-1}D_n)^{-\frac{1}{2}} \begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n-1} & c_n & c_{n+1} & \cdots & c_{2n-1} \\ 1 & x & x^2 & \cdots & x^n \end{vmatrix},$$

where for $n \geqq 0$

(2.2.7)
$$D_n = [c_{\nu+\mu}]_{\nu,\mu=0,1,2,\cdots,n} > 0.$$

In addition to (2.2.6) we have $p_0(x) = D_0^{-\frac{1}{2}} = c_0^{-\frac{1}{2}}$. The determinant (2.2.7) is associated with the positive definite quadratic form

(2.2.8)
$$\sum_{\nu=0}^{n}\sum_{\mu=0}^{n} c_{\nu+\mu}u_\nu u_\mu = \int_a^b (u_0 + u_1x + u_2x^2 + \cdots + u_nx^n)^2\,d\alpha(x),$$

which is called a form of *Hankel* or of *recurrent* type. (See Szegö **1**.)

The determinant in (2.2.6) can be transformed by multiplying the next to the last column by x, subtracting it from the last column, and repeating this operation for each of the preceding columns. In this way we obtain, $n \geqq 1$,

(2.2.9)
$$p_n(x) = (D_{n-1}D_n)^{-\frac{1}{2}} \begin{vmatrix} c_0x - c_1 & c_1x - c_2 & \cdots & c_{n-1}x - c_n \\ c_1x - c_2 & c_2x - c_3 & \cdots & c_nx - c_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n-1}x - c_n & c_nx - c_{n+1} & \cdots & c_{2n-2}x - c_{2n-1} \end{vmatrix}.$$

Furthermore, according to (2.1.9) and (2.1.10), we have the following integral representations:

$$p_n(x) = \frac{(D_{n-1}D_n)^{-\frac{1}{2}}}{n!} \int^b \int^b \cdots \int^b (x - x_0)(x - x_1) \cdots (x - x_{n-1})$$

Gábor Szegő: *Orthogonal Polynomials* (1939)

$$\rho(x) = c(x - x_1)(x - x_2) \cdots (x - x_l), \qquad c \neq 0, \tag{2.5.1}$$

be a π_l which is non-negative in this interval. Then the orthogonal polynomials $\{q_n(x)\}$, associated with the distribution $\rho(x)\,d\alpha(x)$, can be represented in terms of the polynomials $p_n(x)$ as follows:

$$\rho(x)q_n(x) = \begin{vmatrix} p_n(x) & p_{n+1}(x) & \cdots & p_{n+l}(x) \\ p_n(x_1) & p_{n+1}(x_1) & \cdots & p_{n+l}(x_1) \\ \cdots & \cdots & \cdots & \cdots \\ p_n(x_l) & p_{n+1}(x_l) & \cdots & p_{n+l}(x_l) \end{vmatrix}. \tag{2.5.2}$$

In case of a zero x_k, of multiplicity m, $m > 1$, we replace the corresponding rows of (2.5.2) by the derivatives of order $0, 1, 2, \cdots, m - 1$ of the polynomials $p_n(x), p_{n+1}(x), \cdots, p_{n+l}(x)$ at $x = x_k$.

This important result is due to Christoffel (see **1**, actually only in the special case $\alpha(x) = x$). The polynomials $q_n(x)$ are in general not normalized.

The proof is almost obvious. The right-hand member of (2.5.2) is a π_{n+l} which is evidently divisible by $\rho(x)$. Hence it has the form $\rho(x)q_n(x)$, where $q_n(x)$ is a π_n. Moreover, it is a linear combination of the polynomials $p_n(x)$, $p_{n+1}(x), \cdots, p_{n+l}(x)$, so that if $q(x)$ is an arbitrary π_{n-1}, then

$$\int_a^b \rho(x)q_n(x)q(x)\,d\alpha(x) = \int_a^b q_n(x)q(x)\rho(x)\,d\alpha(x) = 0. \tag{2.5.3}$$

Finally, the right side of (2.5.2) is not identically zero. To show this, it suffices to prove that the coefficient of $p_{n+l}(x)$, that is, the determinant $[p_{n+\nu}(x_{\mu+1})]$, $\nu, \mu = 0, 1, 2, \cdots, l - 1$, does not vanish. Suppose it to vanish; then certain real constants $\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_{l-1}$ exist, not all zero, such that

$$\lambda_0 p_n(x) + \lambda_1 p_{n+1}(x) + \cdots + \lambda_{l-1} p_{n+l-1}(x) \tag{2.5.4}$$

tively. (Cf. §3.4 (3).) If $\hat{p}$ denotes the greatest zero of $p(x)$, it is seen from (7.72.3) that the maximum of (7.72.2) is, in this special case,

$$\max(\hat{p}_{m+1}, \hat{q}_m) \quad \text{if} \quad n = 2m, \qquad \max(\hat{r}_{m+1}, \hat{s}_{m+1}) \quad \text{if} \quad n = 2m+1. \tag{7.72.8}$$

The result for the minimum is similar.

(3) Here the general discussion of Tchebichef ends (cf. **7**, p. 395). We can prove, however, that the expressions (7.72.8) are $\hat{p}_{m+1}$ and $\hat{r}_{m+1}$, respectively, so that the following theorem holds:

THEOREM 7.72.1. *Let $w(x)$ be a weight function on the interval $[-1, +1]$. Let $f(x)$ be an arbitrary π_n, not identically zero, and non-negative in $[-1, +1]$. Then the maximum of*

$$\int_{-1}^{+1} f(x)xw(x)\,dx : \int_{-1}^{+1} f(x)w(x)\,dx \tag{7.72.9}$$

is the greatest zero of $p_{m+1}(x)$ if $n = 2m$, and the greatest zero of $p_{m+2}(-1)p_{m+1}(x) - p_{m+1}(-1)p_{m+2}(x)$ if $n = 2m + 1$. Here $\{p_n(x)\}$ is the set of the orthonormal polynomials associated with $w(x)$ in the interval $[-1, +1]$.

According to Theorem 2.5,

$$(1 - x^2)q_m(x) = \text{const.} \begin{vmatrix} p_m(x) & p_{m+1}(x) & p_{m+2}(x) \\ p_m(-1) & p_{m+1}(-1) & p_{m+2}(-1) \\ p_m(1) & p_{m+1}(1) & p_{m+2}(1) \end{vmatrix},$$

$$(1 + x)r_m(x) = \text{const.} \begin{vmatrix} p_m(x) & p_{m+1}(x) \\ p_m(-1) & p_{m+1}(-1) \end{vmatrix}, \tag{7.72.10}$$

An equivalent statement

Experimentally, we found

$$\frac{\det\left(m^{i+j}\prod_{\ell=1}^{d}(\alpha_\ell+m)\right)_{i,j=0}^{n-1}}{\det\left(m_{i+j}\right)_{i,j=0}^{n-1}}=\frac{\det\limits_{1\le i,j\le d}\left(f_{n+i-1}(\alpha_j)\right)}{\prod\limits_{1\le i<j\le d}(\alpha_j-\alpha_i)}.$$

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Equivalently,

$$\det_{1\le i,j\le d}\left(f_{n+i-1}(\alpha_j)\right)=\left(\prod_{1\le i<j\le d}(\alpha_j-\alpha_i)\right)\frac{\det\limits_{0\le i,j\le n-1}\left(m^{i+j}\prod_{\ell=1}^{d}(\alpha_\ell+m)\right)}{\det\limits_{0\le i,j\le n-1}\left(m_{i+j}\right)}.$$

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Proposition (Jacobi)

Let A be an $N\times N$ matrix. Denote the submatrix of A in which rows $i_1,i_2,\dots,i_k$ and columns $j_1,j_2,\dots,j_k$ are omitted by $A^{j_1,j_2,\dots,j_k}_{i_1,i_2,\dots,i_k}$. Then we have

$$\det A\cdot\det A^{1,N}_{1,N}=\det A^1_1\cdot\det A^N_N-\det A^N_1\cdot\det A^1_N.$$

Equivalently,

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$$\det A\cdot\det A_{1,N}^{1,N}=\det A_1^1\cdot\det A_N^N-\det A_1^N\cdot\det A_N^1.$$

Jacobi's *condensation formula* allows (sometimes) for inductive proofs of conjectured determinant identities.

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By Jacobi's condensation formula, the left-hand side satisfies a certain recurrence formula. If we manage to prove that the right-hand side satisfies the same recurrence, then we are done.

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If one works it out, then one sees that we need to prove

$$
\begin{aligned}
&(\alpha_d-\alpha_1)\det_{0\le i,j\le n-1}\left(m^{i+j}\prod_{\ell=1}^{d}(\alpha_\ell+m)\right)\det_{0\le i,j\le n}\left(m^{i+j}\prod_{\ell=2}^{d-1}(\alpha_\ell+m)\right)\\
&=\det_{0\le i,j\le n-1}\left(m^{i+j}\prod_{\ell=1}^{d-1}(\alpha_\ell+m)\right)\det_{0\le i,j\le n}\left(m^{i+j}\prod_{\ell=2}^{d}(\alpha_\ell+m)\right)\\
&-\det_{0\le i,j\le n-1}\left(m^{i+j}\prod_{\ell=2}^{d}(\alpha_\ell+m)\right)\det_{0\le i,j\le n}\left(m^{i+j}\prod_{\ell=1}^{d-1}(\alpha_\ell+m)\right).
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$$=\det_{0\le i,j\le n-1}\left(m^{i+j}\prod_{\ell=1}^{d-1}(\alpha_\ell+m)\right)\det_{0\le i,j\le n}\left(m^{i+j}\prod_{\ell=2}^{d}(\alpha_\ell+m)\right)$$
$$-\det_{0\le i,j\le n-1}\left(m^{i+j}\prod_{\ell=2}^{d}(\alpha_\ell+m)\right)\det_{0\le i,j\le n}\left(m^{i+j}\prod_{\ell=1}^{d-1}(\alpha_\ell+m)\right).$$

If one looks at this properly, then it turns out that this is another instance of Jacobi's condensation formula. □

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We have

$$\frac{\det\left(m^{i+j}\prod_{\ell=1}^{d}(\alpha_\ell+m)\right)_{i,j=0}^{n-1}}{\det\left(m_{i+j}\right)_{i,j=0}^{n-1}}=\frac{\det\limits_{1\le i,j\le d}\left(f_{n+i-1}(\alpha_j)\right)}{\prod\limits_{1\le i<j\le d}(\alpha_j-\alpha_i)}.$$

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What about "non-classical" sources?

$\longrightarrow$ Alain Lascoux:
Symmetric functions & combinatorial operators on polynomials (2003)

SYMMETRIC FUNCTIONS & COMBINATORIAL OPERATORS ON POLYNOMIALS

Alain Lascoux

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CHAPTER 1

Symmetric functions

1.1. Alphabets

We shall handle functions on different sets of indeterminates (called *alphabets*, though we shall mostly use commutative indeterminates for the moment).

A symmetric function of an alphabet $\mathbb{A}$ is a function of the letters which is invariant under permutation of the letters of $\mathbb{A}$.

The simpler symmetric functions are best defined through generating functions. We shall not use the classical notations for symmetric functions (as they can be found in Macdonald's book [**135**]), because it will become clear in the course of these lectures that we need to consider symmetric functions as *functors*, and connect them with operations on vector spaces and representations. It is a small burden imposed on the reader, but the compact notations that we propose greatly simplify manipulations of symmetric functions. Notice that exponents are used for products, and that S^J is different from S_J, except when J is of length one (i.e. is an integer).

$$J = [j_1, j_2, \ldots] \Rightarrow \ \Lambda^J = \Lambda^{j_1}\Lambda^{j_2}\cdots \ \& \ \ S^J = S^{j_1}S^{j_2}\cdots \ \& \ \ \Psi^J = \Psi^{j_1}\Psi^{j_2}\cdots$$

are different from S_J, ψ_J etc.

In case of length 1, we shall indifferently write indices or exponents for the same functions :

$$S^j = S_j \ , \ \Lambda^j = \Lambda_j \ , \ \Psi^j = \Psi_j \ .$$

We need operations on alphabets, the first one being the *addition*, that is the disjoint union that we shall denote by a '+'-sign :

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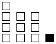
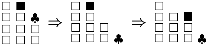
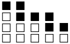
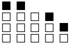
Pushing a box down gives a smaller partition, but it is not true that it gives a pair of consecutive partitions :  and  are not consecutive, because the move of the black box can be performed in two steps:

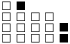


Let J, I be a pair of partitions such that the diagram of J contains the diagram of I. Then the set difference of the two diagrams is called a *skew diagram* and denoted J/I (adding common boxes to I and J does not change J/I. In some problems, one has to consider pairs (J, I) rather than J/I).

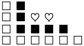
If J/I contains no 2×2 sub-diagram and is connected (resp. J/I contains no two boxes in the same column, res. no two boxes in the same row), then J/I is called a *ribbon* (resp. *horizontal strip*, resp. *vertical strip*). There are strips which are both vertical and horizontal, for example a single box.


ribbon


horizontal strip


vertical strip

A partition of the type $[1^\beta, \alpha{+}1]$ is called a *hook* and is denoted $(\alpha \& \beta)$. The decomposition of the diagram of a partition I into its diagonal hooks (i.e. hooks having their head on the diagonal) is called the *Frobenius code* of I and denoted $\mathfrak{Frob}(I) = (\alpha_1, \alpha_2, \ldots, \alpha_r \,\&\, \beta_1, \beta_2, \ldots, b_r)$ (where r, the number of boxes in the main diagonal, is called the *rank* of the partition).

$I = [2,4,5,6] =$  gives $\mathfrak{Frob}([2,4,5,6]) = (531 \,\&\, 320)$.

$n-r$, then the successive remainders will be divisible by G. This implies that r-th remainder is equal to G (up to a scalar), and that the remainders of higher order are null. Conversely, if $\mathcal{D}_r(\mathbb{A},\mathbb{B})$ is different from 0, and the determinants $\mathcal{D}_p(\mathbb{A},\mathbb{B})$ are all 0 for $p>r$, then the greatest common divisor of $S^m(x-\mathbb{B})$ and $S^n(x-\mathbb{A})$ is of degree $n-r$ and equal to $\mathcal{D}_r(\mathbb{A},\mathbb{B})$.

Let us notice that the determinant $\mathcal{D}_r(\mathbb{A},\mathbb{B})$ also furnishes Euler's multiplicators, i.e. the polynomials $C_{\mathbb{A}}$, $C_{\mathbb{B}}$ such that

$$\mathcal{D}_r(\mathbb{A},\mathbb{B}) = C_{\mathbb{A}}\,R(x,\mathbb{A}) + C_{\mathbb{B}}\,R(x,\mathbb{B}) \ .$$

Indeed, evaluating $\mathcal{D}_r(\mathbb{A},\mathbb{B})$ modulo $R(x,\mathbb{A})$ consists in changing the last column into $[S_{m+r-1}(x-\mathbb{B}),\dots,S_m(x-\mathbb{B}),0,\dots,0]$. Subtracting x to the alphabets in the first $r-1$ rows, one gets, as a last column,

$$[\,S_{m+r-1}(-\mathbb{B}),\dots,S_{m-1}(-\mathbb{B}),\ S_m(x-\mathbb{B}),0,\dots,0\,]$$

that is, $[\,0,\dots,0,R(x-\mathbb{B}),0,\dots,0\,]$, because the $S_k(-\mathbb{B})$ are $\pm$ the elementary symmetric functions of an alphabet of cardinality m, and therefore are null for $k>m$.

Now the cofactor of $R(x-\mathbb{B})$ is the determinant

$$\begin{vmatrix} S_0(-x-\mathbb{B}) & \cdots & S_k(-x-\mathbb{B}) \\ \vdots & & \vdots \\ S_{-r+2}(-x-\mathbb{B}) & \cdots & S_{k-r+2}(-x-\mathbb{B}) \\ S_0(-\mathbb{A}) & \cdots & S_k(-\mathbb{A}) \\ \vdots & & \vdots \\ S_{n+1-m-r}(-\mathbb{A}) & \cdots & S_{r-1}(-\mathbb{A}) \end{vmatrix}$$

Expanding this last determinant according to the first $r-1$ rows, one recognize that it is equal to $S_{\square}(\mathbb{A}-x-\mathbb{B})$, with $\square=(m-n+r)^{r-1}$, $k=m-n+2r-2$.

By symmetry changing $\mathbb{A},\mathbb{B}$, one therefore gets

$$\text{(3.1.5)}\quad \mathcal{D}_r(\mathbb{A},\mathbb{B}) = \pm S_{(m-n+r)^{r-1}}(\mathbb{A}-x-\mathbb{B})R(x,\mathbb{B}) \pm S_{r^{m-n+r-1}}(\mathbb{B}-x-\mathbb{A})R(x,\mathbb{A}) \ ,$$

with signs that specialists will know how to write. This can also be written

$$\text{(3.1.6)}\qquad \mathcal{D}_r(\mathbb{A},\mathbb{B}) = \pm S_{(m-n+r)^{r-1};m}(\mathbb{A}-\mathbb{B};x-\mathbb{B}) \pm S_{r^{m-n+r-1};n}(\mathbb{B}-\mathbb{A};x-\mathbb{A})\,.$$

In particular, when the two polynomials are relatively prime, then the last remainder is equal to the resultant and one has the identity

$[S_{m+r-1}(-\mathbb{B}), \ldots, S_{m-1}(-\mathbb{B}), S_m(x-\mathbb{B}), 0, \ldots, 0]$

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CHAPTER 8

Orthogonal Polynomials

8.1. Orthogonal Polynomials as Symmetric Functions

To any "generic" linear functional $\int$ on $\mathfrak{Pol}(x)$, with $\int 1 = 1$, is associated a (unique) family of orthogonal polynomials:

$$\int P_m(x)P_n(x) = 0 \text{ if } m \neq n \ , \ \int P_n(x)P_n(x) = 1 \ . \tag{8.1.1}$$

We shall treat this subject having only in mind to show algebraic identities. The reader will find a broader point of view in the book of Andrews, Askey, Roy [**5**], and the one of Szegö [**167**].

One can formally suppose that there exists an alphabet $\mathbb{A}$ such that the *moments* $\int x^n$ be the complete functions of $\mathbb{A}$, i.e.

$$\int x^n = S^n(\mathbb{A}) \ , \ n \geq 0 \ .$$

Now $\int$ is a linear functional, that we shall note $\int_{\mathbb{A}}$, with values in symmetric functions:

$$\int_{\mathbb{A}} : \mathfrak{Pol}(x) \mapsto \mathfrak{Sym}(\mathbb{A}) \ .$$

The linear functional can be thought as a quadratic form on the space of polynomials in x, compatible with product :

$$\big(f(x)\,,\,g(x)\big) := \int_{\mathbb{A}} f(x)g(x) = \big(f(x)g(x)\,,\,1\big) \ . \tag{8.1.2}$$

As could be expected, the orthogonal polynomials $P_n(x)$ are Schur functions

This determinant vanishes for $m < n$, having two identical columns. Moreover,

$$\int_{\mathbb{A}} S_{n^n}(\mathbb{A}-x)\, S_{n^n}(\mathbb{A}-x) = \int_{\mathbb{A}} S_{n^n}(\mathbb{A}-x)\, (-x)^n\, S_{(n-1)^n}(\mathbb{A}) = S_{n^n,n}(\mathbb{A})\, S_{(n-1)^n}(\mathbb{A}) \ . \tag{8.1.5}$$

The notation $S_{n^n}(\mathbb{A}-x)$ encodes the classical determinantal expressions of orthogonal polynomials in terms of moments [**23**] :

$$S_{333}(\mathbb{A}-x) = \begin{vmatrix} S^3(\mathbb{A}-x) & S^4(\mathbb{A}-x) & S^5(\mathbb{A}-x) \\ S^2(\mathbb{A}-x) & S^3(\mathbb{A}-x) & S^4(\mathbb{A}-x) \\ S^1(\mathbb{A}-x) & S^2(\mathbb{A}-x) & S^3(\mathbb{A}-x) \end{vmatrix} \ ,$$

$$x^m S_{333}(\mathbb{A}-x) = \begin{vmatrix} S^3(\mathbb{A}) & S^4(\mathbb{A}) & S^5(\mathbb{A}) & x^{m+3} \\ S^2(\mathbb{A}) & S^3(\mathbb{A}) & S^4(\mathbb{A}) & x^{m+2} \\ S^1(\mathbb{A}) & S^2(\mathbb{A}) & S^3(\mathbb{A}) & x^{m+1} \\ S^0(\mathbb{A}) & S^1(\mathbb{A}) & S^2(\mathbb{A}) & x^m \end{vmatrix} \ .$$

Notice that the functional $\int_{\mathbb{A}}$ can also be interpreted as a symmetrizing operator. Indeed, when $\mathbb{A}$ is of finite cardinality n, let ω be the maximal permutation in $\mathfrak{S}_n$. Then

$$a_1^k\, \pi_\omega = S_k(\mathbb{A}) \ , \ k = 0, 1, 2, \ldots \ ,$$

and thus, for any polynomial $f(x)$, one has

$$\int_{\mathbb{A}} f(x) = f(a_1)\, \pi_\omega \ . \tag{8.1.6}$$

Since $a^J\, \pi_\omega = S_J(\mathbb{A})$, $J \in \mathbb{N}$, there is no difficulty in extending the definition of π_ω to an alphabet of infinite cardinality, as is needed in the theory of orthogonal polynomials.

Recursions (8.4.1) can be put into matrix form, as with Euler recursions (4.1.2):

$$(8.4.2)\qquad \begin{bmatrix} \widetilde{Q}_{n-1} & \widetilde{Q}_n \\ \widetilde{P}_{n-1} & \widetilde{P}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & x-\zeta_0 \end{bmatrix} \begin{bmatrix} 0 & \beta_1 \\ 1 & x-\zeta_1 \end{bmatrix} \begin{bmatrix} 0 & \beta_2 \\ 1 & x-\zeta_2 \end{bmatrix} \cdots \begin{bmatrix} 0 & \beta_{n-1} \\ 1 & x-\zeta_{n-1} \end{bmatrix}$$

One can "shift" the linear functional $\int_{\mathbb{A}}$ by a finite alphabet $\mathbb{B}$, defining

$$(8.4.3)\qquad \forall f \in \mathfrak{Pol}(x)\ ,\ \int_{\mathbb{A},\mathbb{B}} f := \int_{\mathbb{A}} f(x)\, R(x,\mathbb{B})\ .$$

Christoffel obtained the associated orthogonal polynomials. A remarkable feature of his result, stated in the following proposition, is that it connects two determinants of different orders (n and $k{+}1$).

Proposition 8.4.1. *Let $\mathbb{B} = \{b_1, \ldots, b_k\}$. Then the orthogonal polynomials relative to $\int_{\mathbb{A},\mathbb{B}}$ are*

$$P_{n,k}(x) = S_{(n+k)^n}(\mathbb{A} - \mathbb{B} - x)\ ,$$

and $P_{n,k}(x)\, R(x,\mathbb{B})$ is proportional, up to a factor independent of x and $\mathbb{B}$, to the Christoffel determinant

$$\left| P_{n-1+j}(b_i) \right|_{1\le i,j\le k+1}\ ,$$

with $b_{k+1} := x$.

Proof. The verification that $P_{n,k}(x)$ is orthogonal to $x^0, \ldots, x^{n-1}$ is the same as in the case of $P_n(x)$ and $\int_{\mathbb{A}}$, apart from changing $\mathbb{A}$ into $\mathbb{A}-\mathbb{B}$, and shifting indices.

The determinant is divisible by the Vandermonde $\Delta(\mathbb{B}+x)$. Evaluating the image of the quotient multiplied by a function of x under $\int_{\mathbb{A},\mathbb{B}}$ is the same as computing the image of the last row, multiplied by the same function, under $\int_{\mathbb{A}}$. Therefore $P_{n,k}(x)$ is orthogonal (with respect to $\int_{\mathbb{A},\mathbb{B}}$) to $x^0, \ldots, x^{n-1}$, while being of degree n in x. It must be proportional to $S_{(n+k)^n}(\mathbb{A}-\mathbb{B}-x)$. The explicit factor is explained by the Bazin formula and is equal to

$$\pm S_{(n-k+1)^{n-k+2}}(\mathbb{A}) \cdots S_{(n-2)^{n-1}}(\mathbb{A}) S_{(n-1)^n}(\mathbb{A})\ .$$

$\square$

It can contain only those powers of x which are congruent to n (mod 2). Indeed, we have for $\nu = 0, 1, 2, \cdots, n-1$

$$\int_{-a}^{a} p_n(-x)x^\nu w(x)\,dx = (-1)^\nu \int_{-a}^{a} p_n(x)x^\nu w(x)\,dx = 0.$$

Consequently, $p_n(-x)$ possesses the same orthogonality property as $p_n(x)$ (in the wider sense). Therefore, comparing the coefficients of x^n, we obtain $p_n(-x) = \text{const.}\, p_n(x) = (-1)^n p_n(x)$.

The linear transformation $x = kx' + l$, $k \neq 0$, carries over the interval $[a, b]$ into an interval $[a', b']$ (or $[b', a']$), and the weight function $w(x)$ into $w(kx' + l)$. Then the polynomials

(2.3.4) $$(\operatorname{sgn} k)^n \,|k|^{\frac{1}{2}}\, p_n(kx' + l)$$

are orthonormal on $[a', b']$ (or $[b', a']$) with the weight function $w(kx' + l)$.

2.4. The classical orthogonal polynomials

1. Let $a = -1$, $b = +1$, $w(x) = (1 - x)^\alpha(1 + x)^\beta$, $\alpha > -1$, $\beta > -1$. Then, except for a constant factor, the orthogonal polynomial $p_n(x)$ is the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ (see §4.1).

2. Let $a = 0$, $b = +\infty$, $w(x) = e^{-x}x^\alpha$, $\alpha > -1$. In this case $p_n(x)$ is, except for a constant factor, the Laguerre polynomial $L_n^{(\alpha)}(x)$ (see §5.1).

3. Let $a = -\infty$, $b = +\infty$, $w(x) = e^{-x^2}$. In this case $p_n(x)$ is, save for a constant factor, the Hermite polynomial $H_n(x)$ (see §5.5).

Some special cases of 1, except for constant factors, are:

The ultraspherical polynomials, for $\alpha = \beta$.

The Tchebichef polynomials of the first kind, $T_n(x) = \cos n\theta$, $x = \cos\theta$, for $\alpha = \beta = -\frac{1}{2}$ (see (1.12.3)).

The Tchebichef polynomials of the second kind, $U_n(x) = \sin(n + 1)\theta/(\sin\theta)$, $x = \cos\theta$, for $\alpha = \beta = +\frac{1}{2}$ (see (1.12.3)).

The polynomials $U_{2n}(\cos(\theta/2)) = \sin(n + \frac{1}{2})\theta/\sin(\theta/2)$ of $\cos\theta = x$, for $\alpha = -\beta = \frac{1}{2}$ (see §1.12).

The Legendre polynomials $P_n(x)$, for $\alpha = \beta = 0$.

A detailed investigation of these polynomials will be given in later chapters.

2.5. A formula of Christoffel

(1) Theorem 2.5. *Let $\{p_n(x)\}$ be the orthonormal polynomials associated with the distribution $d\alpha(x)$ on the interval $[a, b]$. Also let*

Gábor Szegő: *Orthogonal Polynomials* (1939)

$$\rho(x) = c(x - x_1)(x - x_2) \cdots (x - x_l), \qquad c \neq 0, \tag{2.5.1}$$

be a π_l *which is non-negative in this interval. Then the orthogonal polynomials* $\{q_n(x)\}$, *associated with the distribution* $\rho(x)\,d\alpha(x)$, *can be represented in terms of the polynomials* $p_n(x)$ *as follows*:

$$\rho(x)q_n(x) = \begin{vmatrix} p_n(x) & p_{n+1}(x) & \cdots & p_{n+l}(x) \\ p_n(x_1) & p_{n+1}(x_1) & \cdots & p_{n+l}(x_1) \\ \cdots & \cdots & \cdots & \cdots \\ p_n(x_l) & p_{n+1}(x_l) & \cdots & p_{n+l}(x_l) \end{vmatrix}. \tag{2.5.2}$$

In case of a zero x_k, *of multiplicity* m, $m > 1$, *we replace the corresponding rows of* (2.5.2) *by the derivatives of order* $0, 1, 2, \cdots, m - 1$ *of the polynomials* $p_n(x), p_{n+1}(x), \cdots, p_{n+l}(x)$ *at* $x = x_k$.

This important result is due to Christoffel (see **1**, actually only in the special case $\alpha(x) = x$). The polynomials $q_n(x)$ are in general not normalized.

The proof is almost obvious. The right-hand member of (2.5.2) is a π_{n+l} which is evidently divisible by $\rho(x)$. Hence it has the form $\rho(x)q_n(x)$, where $q_n(x)$ is a π_n. Moreover, it is a linear combination of the polynomials $p_n(x)$, $p_{n+1}(x), \cdots, p_{n+l}(x)$, so that if $q(x)$ is an arbitrary π_{n-1}, then

$$\int_a^b \rho(x)q_n(x)q(x)\,d\alpha(x) = \int_a^b q_n(x)q(x)\rho(x)\,d\alpha(x) = 0. \tag{2.5.3}$$

Finally, the right side of (2.5.2) is not identically zero. To show this, it suffices to prove that the coefficient of $p_{n+l}(x)$, that is, the determinant $[p_{n+\nu}(x_{\mu+1})]$, $\nu, \mu = 0, 1, 2, \cdots, l - 1$, does not vanish. Suppose it to vanish; then certain real constants $\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_{l-1}$ exist, not all zero, such that

$$\lambda_0 p_n(x) + \lambda_1 p_{n+1}(x) + \cdots + \lambda_{l-1} p_{n+l-1}(x) \tag{2.5.4}$$

“Christoffel’s formula”

Theorem

Let $\big(p_n(x)\big)_{n\ge0}$ be orthogonal with respect to the linear functional given by $p(x)\mapsto\int p(u)\,d\mu(u)$. Then, as polynomials in $x=\alpha_d$, the polynomials

$$\frac{\det\limits_{1\le i,j\le d}\left(p_{n+i-1}(\alpha_j)\right)}{\prod\limits_{i=1}^{d-1}(\alpha_i-\alpha_d)}$$

are orthogonal with respect to the linear functional

$$p(x)\mapsto\int p(u)\prod_{\ell=1}^{d-1}(u-\alpha_\ell)\,d\mu(u).$$

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are orthogonal with respect to the linear functional

$$p(x)\mapsto\int p(u)\prod_{\ell=1}^{d-1}(u-\alpha_\ell)\,d\mu(u).$$

If $m_n=\int u^n\,d\mu(u)$, then the moments of the above functional are

$$\int u^n\prod_{\ell=1}^{d-1}(u-\alpha_\ell)\,d\mu(u)=m^n\prod_{\ell=1}^{d-1}(m-\alpha_\ell),\qquad n=0,1,\dots.$$

Second proof by theory of orthogonal polynomials

Second proof by theory of orthogonal polynomials

We prove

$$\frac{\det\limits_{0\le i,j\le n-1}\left(m^{i+j}\prod_{\ell=1}^{d}(m-\alpha_\ell)\right)}{\det\limits_{0\le i,j\le n-1}\left(m_{i+j}\right)}=(-1)^{nd}\frac{\det\limits_{1\le i,j\le d}\left(p_{n+i-1}(\alpha_j)\right)}{\prod\limits_{1\le i<j\le d}(\alpha_j-\alpha_i)}.$$

We prove

$$\frac{\det\limits_{0\le i,j\le n-1}\left(m^{i+j}\prod_{\ell=1}^{d}(m-\alpha_\ell)\right)}{\det\limits_{0\le i,j\le n-1}(m_{i+j})}=(-1)^{nd}\frac{\det\limits_{1\le i,j\le d}(p_{n+i-1}(\alpha_j))}{\prod\limits_{1\le i<j\le d}(\alpha_j-\alpha_i)}.$$

Lemma

Let M be a linear functional on polynomials in x with moments ν_n, $n=0,1,\dots$. Then the determinants

$$\det_{0\le i,j\le n-1}(\nu_{i+j+1}-\nu_{i+j}x)$$

are a sequence of orthogonal polynomials with respect to M.

Proof of the lemma.

$$\det_{0\le i,j\le n-1}\left(\nu_{i+j+1}-\nu_{i+j}x\right)$$

$$=\det\begin{pmatrix} 1 & 0 & 0 & \dots & 0\\ \nu_0 & \nu_1-\nu_0 x & \nu_2-\nu_1 x & \dots & \nu_n-\nu_{n-1}x\\ \nu_1 & \nu_2-\nu_1 x & \nu_3-\nu_2 x & \dots & \nu_{n+1}-\nu_n x\\ \multicolumn{5}{c}{\dotfill}\\ \nu_{n-1} & \nu_n-\nu_{n-1}x & \nu_{n+1}-\nu_n x & \dots & \nu_{2n-1}-\nu_{2n-2}x \end{pmatrix}$$

$$=\det\begin{pmatrix} 1 & x & x^2 & \dots & x^n\\ \nu_0 & \nu_1 & \nu_2 & \dots & \nu_n\\ \nu_1 & \nu_2 & \nu_3 & \dots & \nu_{n+1}\\ \multicolumn{5}{c}{\dotfill}\\ \nu_{n-1} & \nu_n & \nu_{n+1} & \dots & \nu_{2n-1} \end{pmatrix}.$$

Second proof by theory of orthogonal polynomials

Using the lemma with $\nu_n = m^n \prod_{\ell=1}^{d-1}(m - \alpha_\ell)$, we see that the determinants in the numerator of the left-hand side of our identity to be proven,

$$\det_{0 \le i,j \le n-1} \left(m^{i+j} \prod_{\ell=1}^{d} (m - \alpha_\ell) \right),$$

seen as polynomials in α_d, are a sequence of orthogonal polynomials for the linear functional with moments

$$m^n \prod_{\ell=1}^{d-1} (m - \alpha_\ell), \quad n = 0, 1, \dots .$$

We are considering a functional with moments

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Second proof by theory of orthogonal polynomials

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$$m^n \prod_{\ell=1}^{d-1} (m - \alpha_\ell), \quad n = 0, 1, \dots .$$

In terms of the functional L of orthogonality for the polynomials $\big(p_n(\alpha_d)\big)_{n\ge 0}$, this linear functional can be expressed as

$$p(\alpha_d) \mapsto L\left(p(\alpha_d) \cdot \prod_{\ell=1}^{d-1} (\alpha_d - \alpha_\ell)\right).$$

Second proof by theory of orthogonal polynomials

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$$p(\alpha_d) \mapsto L\left(p(\alpha_d)\cdot \prod_{\ell=1}^{d-1}(\alpha_d-\alpha_\ell)\right).$$

We claim that also the right-hand side,

$$\frac{\det\limits_{1\le i,j\le d}\left(p_{n+i-1}(\alpha_j)\right)}{\prod\limits_{1\le i<j\le d}(\alpha_j-\alpha_i)},$$

gives a sequence of orthogonal polynomials (in α_d) with respect to this linear functional.

We consider the linear functional

$$p(\alpha_d) \mapsto L\left(p(\alpha_d) \cdot \prod_{\ell=1}^{d-1} (\alpha_d - \alpha_\ell)\right).$$

We claim that also the right-hand side,

$$q_n(\alpha_d) := \frac{\det\limits_{1 \le i,j \le d} \left(p_{n+i-1}(\alpha_j)\right)}{\prod\limits_{1 \le i < j \le d} (\alpha_j - \alpha_i)},$$

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gives a sequence of orthogonal polynomials (in α_d) with respect to this linear functional.

Application of the functional to $\alpha_d^s q_n(\alpha_d)$ is proportional (up to factors that are independent of α_d) to

$$L\left(\alpha_d^s \det_{1\le i,j\le d} \left(p_{n+i-1}(\alpha_j)\right)\right).$$

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Second proof by theory of orthogonal polynomials

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For $0 \le s \le n-1$, this vanishes.

By symmetry, the same argument can also be made for any α_ℓ with $1 \le \alpha_\ell \le d-1$.

Second proof by theory of orthogonal polynomials

Uniqueness of orthogonal polynomials up to scalar factors then implies

$$\det_{0\le i,j\le n-1}\left(m^{i+j}\prod_{\ell=1}^{d}(m-\alpha_\ell)\right)=C\,\frac{\det\limits_{1\le i,j\le d}\left(p_{n+i-1}(\alpha_j)\right)}{\prod\limits_{1\le i<j\le d}(\alpha_j-\alpha_i)},$$

where C is independent of the variables $\alpha_1,\alpha_2,\dots,\alpha_d$.

Uniqueness of orthogonal polynomials up to scalar factors then implies

$$\det_{0\le i,j\le n-1}\left(m^{i+j}\prod_{\ell=1}^{d}(m-\alpha_\ell)\right)=C\frac{\det_{1\le i,j\le d}\left(p_{n+i-1}(\alpha_j)\right)}{\prod_{1\le i<j\le d}(\alpha_j-\alpha_i)},$$

where C is independent of the variables $\alpha_1,\alpha_2,\dots,\alpha_d$.

We look at the highest degree terms (as polynomials in the α_ℓ's) on both sides. On the left-hand side, this is

$$\det_{0\le i,j\le n-1}\left((-1)^d m_{i+j}\prod_{\ell=1}^{d}\alpha_\ell\right)=(-1)^{nd}\prod_{\ell=1}^{d}\alpha_\ell^n\det_{0\le i,j\le n-1}(m_{i+j}).$$

On the right-hand side, this is

$$C\frac{\det_{1\le i,j\le d}\left(\alpha_j^{n+i-1}\right)}{\prod_{1\le i<j\le d}(\alpha_j-\alpha_i)}=C\prod_{\ell=1}^{d}\alpha_\ell^n.$$

Fine, but still ...

Our identity:

$$\frac{\det\left(m^{i+j}\prod_{\ell=1}^{d}(m-\alpha_\ell)\right)_{i,j=0}^{n-1}}{\det\left(m_{i+j}\right)_{i,j=0}^{n-1}} = (-1)^{nd}\frac{\det\limits_{1\le i,j\le d}\left(p_{n+i-1}(\alpha_j)\right)}{\prod\limits_{1\le i<j\le d}(\alpha_j-\alpha_i)}.$$

Fine, but still . . .

Our identity:

$$\frac{\det\left(m^{i+j}\prod_{\ell=1}^{d}(m-\alpha_\ell)\right)_{i,j=0}^{n-1}}{\det\left(m_{i+j}\right)_{i,j=0}^{n-1}}=(-1)^{nd}\frac{\det\limits_{1\le i,j\le d}\left(p_{n+i-1}(\alpha_j)\right)}{\prod\limits_{1\le i<j\le d}(\alpha_j-\alpha_i)}.$$

I asked Mourad Ismail. His replies seem to indicate that he was not aware of *any* source where the identity is stated explicitly.

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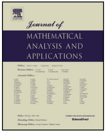

A unified approach for the Hankel determinants of classical combinatorial numbers

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A R T I C L E   I N F O

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We give a general formula for the determinants of a class of Hankel matrices which arise in combinatorics theory. We revisit and extend existent results on Hankel determinants involving the sum of consecutive Catalan, Motzkin and Schröder numbers and we prove a conjecture of [10] about the recurrence relations satisfied by the Hankel transform of linear combinations of Catalans numbers.

of degree n and

$$\mathcal{L}\left(p_n p_m\right) = 0 \quad \text{for } n \neq m.$$

We remark that $b_n = \sum_{k=0}^{r} \lambda_k a_{n+k} = \mathcal{L}\left(x^n q\right)$, where

$$q\left(x\right) = x^r + \lambda_{r-1} x^{r-1} + \ldots + \lambda_0.$$

The r-kernel $\mathcal{K}_{n,P}^{(r)}$ of $P = \left\{p_n\right\}_{n\in\mathbb{N}}$ is defined by

$$\mathcal{K}_{n,P}^{(r)}\left(x_1, x_2, \ldots, x_r\right) = \frac{\det\left(\left(p_{n+i-1}\left(x_j\right)\right)_{1\leq i,j\leq r}\right)}{\prod_{1\leq i<j\leq r}\left(x_j - x_i\right)}$$

for $r \geq 2$ and $\mathcal{K}_{n,P}^{(1)}\left(x\right) = p_n\left(x\right)$. As it will be shown latter, $\mathcal{K}_{n,P}^{(r)}\left(x_1, x_2, \ldots, x_r\right)$ is a polynomial of the variables $x_1, x_2, \ldots$ and x_r.

The following theorem constitutes our main result:

Theorem 1. *We have*

$$\det\left(\mathcal{H}_n\left(b\right)\right) = \left(-1\right)^{nr} \det\left(\mathcal{H}_n\left(a\right)\right) \mathcal{K}_{n,P}^{(r)}\left(\alpha_1, \alpha_2, \ldots, \alpha_r\right), \tag{1.1}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_r$ are the zeros of q.

In most examples considered in the existing literature, b_n has a specific pattern. Namely

$$b_n = a_{n+r} - c a_{n+r-1}, \text{ with } c \in \mathbb{C}.$$

Lemma 4. *If* $\mathcal{K}_{n,P}^{(r)}(\alpha_1, \alpha_2, \dots, \alpha_r) = 0$ *then* $\det\left((b_{i+j})_{0 \le i,j \le n-1}\right) = 0$. *The converse is true.*

Proof. Assume that $\mathcal{K}_{n,P}^{(r)}(\alpha_1, \alpha_2, \dots, \alpha_r) = 0$. Then, the row vectors of the matrix

$$\begin{pmatrix} p_n(\alpha_1) & p_n(\alpha_2) & \cdots & p_n(\alpha_r) \\ p_{n+1}(\alpha_1) & p_{n+1}(\alpha_2) & \ddots & p_{n+1}(\alpha_r) \\ \vdots & & \ddots & \vdots \\ p_{n+r-1}(\alpha_1) & p_{n+r-1}(\alpha_2) & \cdots & p_{n+r-1}(\alpha_r) \end{pmatrix},$$

are linearly dependent, i.e., there exist scalar $c_1, c_2, \dots, c_r$, not all zero such that

$$\sum_{i=0}^{r-1} c_i p_{n+i}(\alpha_k) = 0, \quad \text{for } k = 1, 2, \dots, r.$$

It follows that $\alpha_1, \alpha_2, \dots, \alpha_r$ are zeros of the polynomial $g(x) = \sum_{i=0}^{r-1} c_i p_{n+i}(x)$ and consequently q divides g:

$$g = qh.$$

Obviously h is of degree at most $r-1$, we can write

Interim summary

Interim summary

Theorem

We have

$$\frac{\det\left(m^{i+j}\prod_{\ell=1}^{d}(m-\alpha_\ell)\right)_{i,j=0}^{n-1}}{\det\left(m_{i+j}\right)_{i,j=0}^{n-1}}=(-1)^{nd}\frac{\det\limits_{1\le i,j\le d}\left(p_{n+i-1}(\alpha_j)\right)}{\prod\limits_{1\le i<j\le d}(\alpha_j-\alpha_i)}.$$

This formula exists somehow hidden in the folklore of the theory of orthogonal polynomials.
It was about time that the formula was stated *explicitly and correctly*, and that a *complete proof* (actually, three different proofs) was given.

What can one do with this formula?

Remember:
Dougherty, French, Saderholm and Qian conjectured that

$$\det \left(\lambda_0 C_{i+j} + \lambda_1 C_{i+j+1} + \cdots + \lambda_{d-1} C_{i+j+d-1} + C_{i+j+d} \right)_{i,j=0}^{n-1}$$

satisfies a linear recurrence with constant coefficients of order 2^d.

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The above conjecture now becomes trivial.

What can one do with this formula?

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Recall that the polynomials $p_n(x)$ defined recursively by

$$p_n(x) = (x - s_{n-1})p_{n-1}(x) - t_{n-2}p_{n-2}(x).$$

Let us now again assume that $s_i \equiv s$ and $t_i \equiv t$ for $i \ge 1$.

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Let us now again assume that $s_i \equiv s$ and $t_i \equiv t$ for $i \geq 1$.
The expansion of the determinant on the right-hand side of the formula reads

$$\det_{1 \leq i,j \leq d} \left(p_{n+i-1}(\alpha_j) \right) = \sum_{\sigma \in \mathfrak{S}_d} \operatorname{sgn} \sigma \prod_{j=1}^{d} p_{n+\sigma(j)-1}(\alpha_j).$$

Each product in the sum satisfies the same linear recurrence with constant coefficients, namely the d-fold product recurrence of the recurrence $(*)$.
Since $(*)$ has order 2, the d-fold product recurrence has order 2^d.

What can one do with this formula?

We can even be more precise:

Theorem

Within the above setup, let $s_i \equiv s$ and $t_i \equiv t$ for $i \geq 1$. Furthermore, let

$$H_n = t_0^{-(n-1)} t^{-\binom{n-1}{2}} \det_{0 \leq i,j \leq n-1} \left(\sum_{k=0}^{d} \lambda_k m_{i+j+k} \right),$$

where the λ_k's are some constants and $\lambda_d = 1$. Then the sequence $(H_n)_{n \geq 0}$ of (scaled) Hankel determinants of linear combinations of moments satisfies a linear recurrence of the form

$$\sum_{i=0}^{2^d} c_i H_{n-i} = 0, \quad \text{for } n > 2^d,$$

for some constants c_i, normalised by $c_0 = 1$.

What can one do with this formula?

Explicitly, these constants can be computed as the coefficients of the characteristic polynomial (in x) $\sum_{i=0}^{2^d} c_i x^{2^d - i}$ of the tensor product of 2×2 matrices

$$\begin{pmatrix} x_1 + s & t \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} x_2 + s & t \\ -1 & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} x_d + s & t \\ -1 & 0 \end{pmatrix},$$

where $\lambda_k = e_{d-k}(x_1, x_2, \dots, x_d)$.

Here, the *elementary symmetric functions* are defined by

$$e_k(x_1, x_2, \dots, x_d) := \sum_{1 \le i_1 < i_2 < \cdots < i_k \le d} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

A further generalisation

Ismail's original message:

Dear Christian
I did not do the details but it seems to me that the right side is what you get from a formula due to Christoffel. It is in Szego and also in my book, see Theorem 2.7.1. It is the coefficient of a polynomial in the expansion of $\prod_{1\le j\le d}(x-x_j)$ times another set of orthogonal polynomials. The reason I mention this is because there is a more general formula due to Ouvarouv, also in my book, which replaces $\prod_{1\le j\le d}(x-x_j)$ by a rational function. This is harder and more recent (1960's). It will be really nice if it can be done combinatorially because it involves the function of the second kind.

Uvarov's formula

Theorem

Let $\big(p_n(x)\big)_{n\ge0}$ be orthogonal with respect to the linear functional given by $p(x)\mapsto\int p(u)\,d\mu(u)$. Then, as polynomials in $x=\alpha_1$, the polynomials

$$\frac{\det\limits_{1\le i,j\le k+h}\begin{pmatrix} p_{n-k+i-1}(\alpha_j) & 1\le i\le h\\ q_{n-k+i-h-1}(\beta_j) & h+1\le i\le k+h\end{pmatrix}}{\prod\limits_{\ell=2}^{h}(\alpha_1-\alpha_\ell)}$$

are orthogonal with respect to the linear functional

$$p(x)\mapsto\int p(u)\frac{\prod_{\ell=2}^{h}(u-\alpha_\ell)}{\prod_{\ell=1}^{k}(u-\beta_\ell)}\,d\mu(u).$$

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Here, $q_n(y)=\displaystyle\int\frac{p_n(u)}{y-u}\,d\mu(u).$

The "function of the second kind" $q_n(y)$

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$$\begin{aligned} q_n(y) &= \int \frac{p_n(u)}{y-u}\, d\mu(u) = \sum_{i=0}^{\infty} \int p_n(u) u^i y^{-i-1}\, d\mu(u) \\ &= \sum_{i=n}^{\infty} \int p_n(u) u^i y^{-i-1}\, d\mu(u) \\ &= y^{-n-1} \int p_n(u) u^n\, d\mu(u) + O\left(y^{-n-2}\right) \\ &= \frac{H(n+1)}{H(n)} y^{-n-1} + O\left(y^{-n-2}\right), \end{aligned}$$

where $H(n) := \det\limits_{0 \le i,j \le n-1} (m_{i+j})$.

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Uvarov's formula

Some more smallprint:

We have the expression

$$\frac{\det\limits_{1\le i,j\le k+h}\begin{pmatrix} p_{n-k+i-1}(\alpha_j) & 1\le i\le h \\ q_{n-k+i-h-1}(\beta_j) & h+1\le i\le k+h \end{pmatrix}}{\prod\limits_{\ell=2}^{h}(\alpha_1-\alpha_\ell)}.$$

If $a<0$ then $p_a(x)=0$ and $q_a(y)=y^{-a-1}$.

Uvarov's formula

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$$p(x) \mapsto \int p(u)\prod_{\ell=2}^{h}(u - \alpha_\ell)\,d\mu(u).$$

We have $\int u^n \prod_{\ell=2}^{h}(u-\alpha_\ell)\,d\mu(u) = m^n \prod_{\ell=2}^{h}(m-\alpha_\ell)$, using again umbral notation.

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Here, $q_n(y)=\displaystyle\int\frac{p_n(u)}{y-u}\,d\mu(u)$.

The determinant identity behind Uvarov's formula

Theorem

We have

$$\frac{\det\limits_{0\le i,j\le n-1}\left(\int u^{i+j}\frac{\prod_{\ell=1}^{h}(u-\alpha_\ell)}{\prod_{\ell=1}^{k}(u-\beta_\ell)}\,d\mu(u)\right)}{\det\limits_{0\le i,j\le n-k-1}(m_{i+j})}$$

$$=(-1)^{n(h-k)+kh}\frac{\det_{1\le i,j\le k+h}\begin{pmatrix} p_{n+i-1}(\alpha_j) & 1\le i\le h\\ q_{n+i-h-1}(\beta_j) & h+1\le i\le k+h\end{pmatrix}}{\left(\prod\limits_{1\le i<j\le h}(\alpha_j-\alpha_i)\right)\left(\prod\limits_{1\le i<j\le k}(\beta_i-\beta_j)\right)}.$$

If $n<k$ the previous conventions for negatively indexed $p_a(\alpha)$ and $q_a(\beta)$ apply, and in that case the Hankel determinant in the denominator on the left-hand side has to be interpreted as 1.

The proof

We had three different proofs for the determinant identity behind "Christoffel's formula":

1. Proof by condensation
2. Proof by the use of classical facts from the theory of orthogonal polynomials
3. Proof by a vanishing argument

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For example, if we choose $d\mu(u) = \sqrt{1-u^2}\,du$, the density corresponding to the Chebyshev polynomials $U_n(x)$, then from the special case $k=1$ and $h=0$ one can derive

$$\det_{0\le i,j\le n-1}\left(Y + 2^{-2\lceil (i+j)/2\rceil}\binom{2\lceil (i+j)/2\rceil}{\lceil (i+j)/2\rceil}\right) = 2^{-n(n-1)}(Yn+1),$$

and from the special case $k=h=1$ one can derive

$$\det_{0\le i,j\le n-1}\left(Y + 2^{-2\lceil (i+j+1)/2\rceil}\binom{2\lceil (i+j+1)/2\rceil}{\lceil (i+j+1)/2\rceil}\right) = (-1)^{\binom{n}{2}}2^{-n^2}\left(2\lceil n/2\rceil Y+1\right)$$

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and

$$\det_{0\le i,j\le n-1}\left(Y + 2^{-2\lceil (i+j)/2\rceil}\binom{2\lceil (i+j)/2\rceil}{\lceil (i+j+1)/2\rceil}\right) = \begin{cases} 2^{-n(n-1)}Y, & \text{if } n \equiv 0 \pmod 3,\\ -2^{-n(n-1)}(Y(n+1)+1), & \text{if } n \equiv 1 \pmod 3,\\ 2^{-n(n-1)}(Yn+1), & \text{if } n \equiv 2 \pmod 3.\end{cases}$$

The determinant identity behind Uvarov's formula

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We have

$$\frac{\det\limits_{0\le i,j\le n-1}\left(\int u^{i+j}\,\frac{\prod_{\ell=1}^{h}(u-\alpha_\ell)}{\prod_{\ell=1}^{k}(u-\beta_\ell)}\,d\mu(u)\right)}{\det\limits_{0\le i,j\le n-k-1}(m_{i+j})}$$

$$=(-1)^{n(h-k)+kh}\frac{\det_{1\le i,j\le k+h}\begin{pmatrix} p_{n+i-1}(\alpha_j) & 1\le i\le h\\ q_{n+i-h-1}(\beta_j) & h+1\le i\le k+h\end{pmatrix}}{\left(\prod\limits_{1\le i<j\le h}(\alpha_j-\alpha_i)\right)\left(\prod\limits_{1\le i<j\le k}(\beta_i-\beta_j)\right)}.$$

If $n<k$ the previous conventions for negatively indexed $p_a(\alpha)$ and $q_a(\beta)$ apply, and in that case the Hankel determinant in the denominator on the left-hand side has to be interpreted as 1.