Combinatorial Determinants

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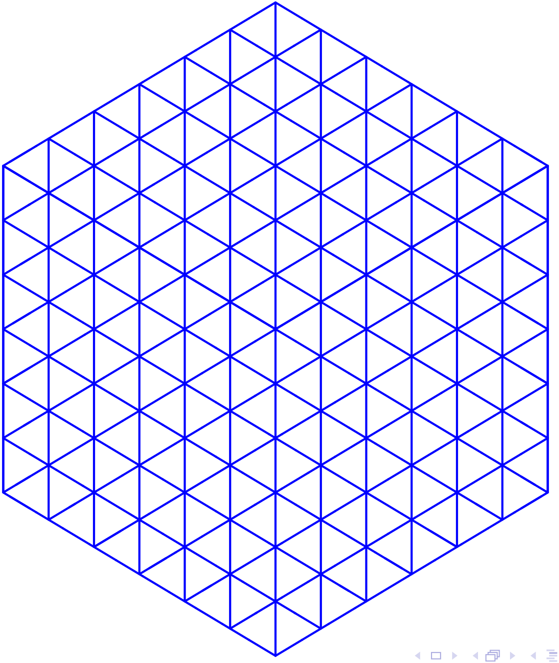
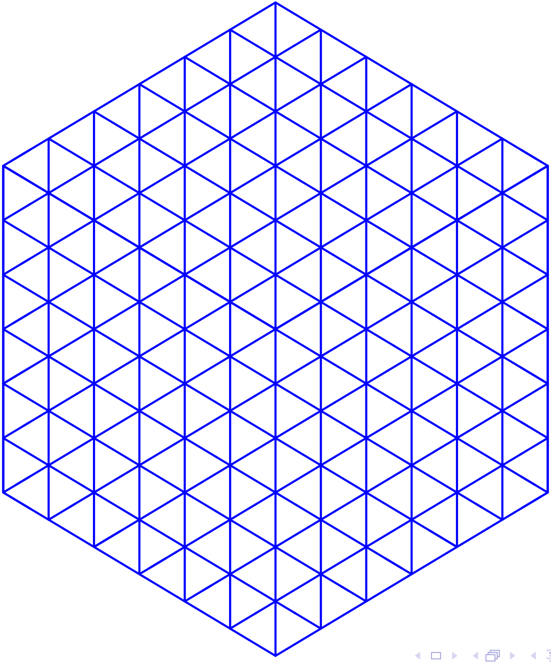
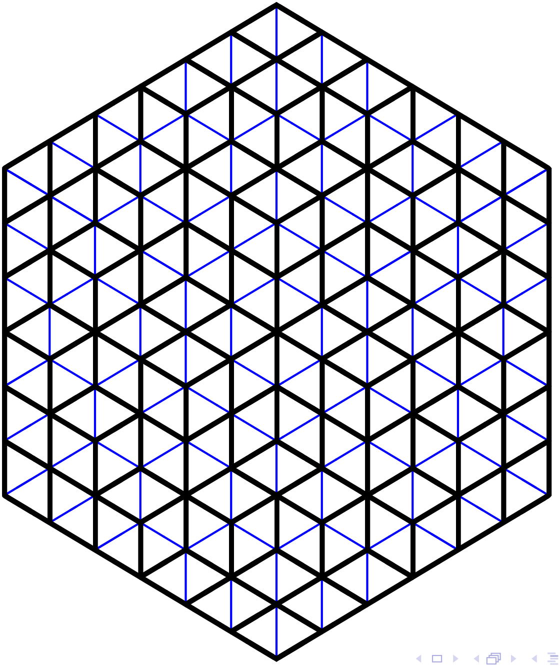
$\longrightarrow$ **non-intersecting lattice paths**

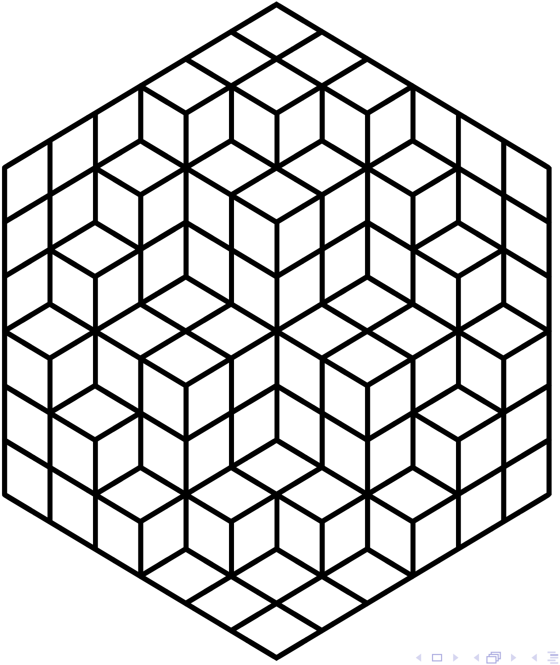
A combinatorial problem



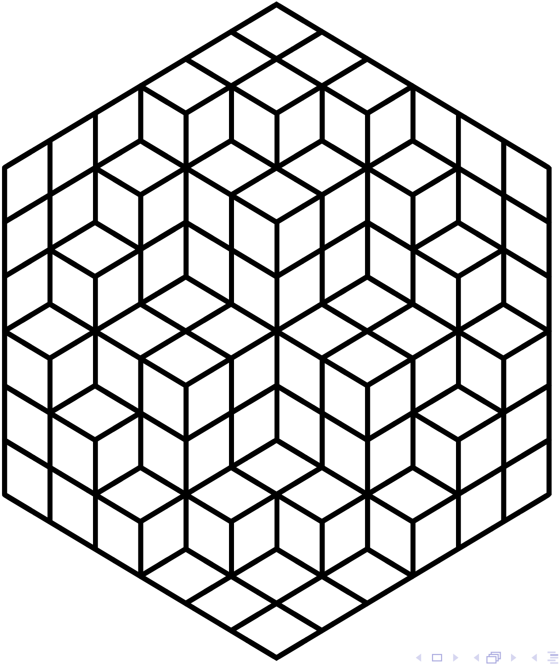
How many rhombus tilings of a hexagon with side lengths a, b, c, a, b, c are there?

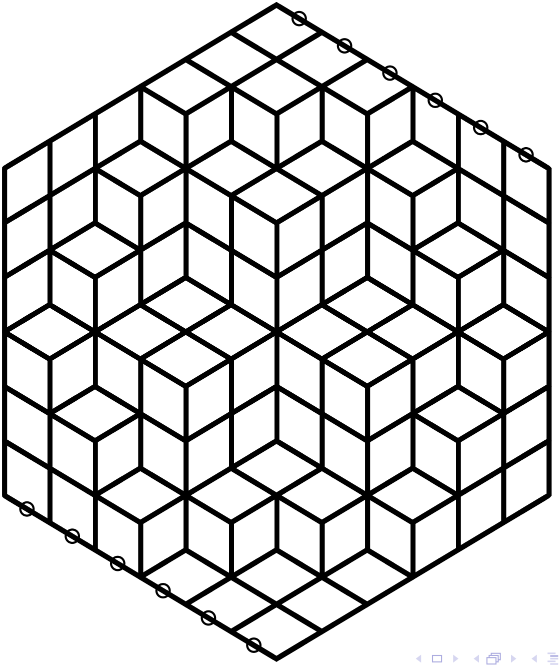


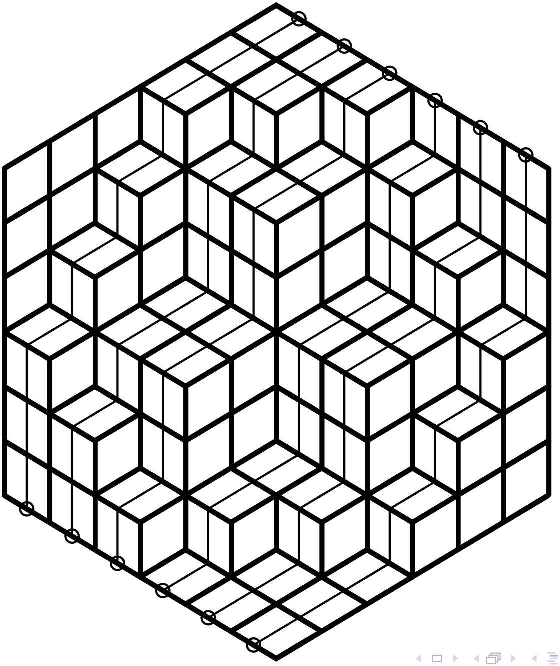


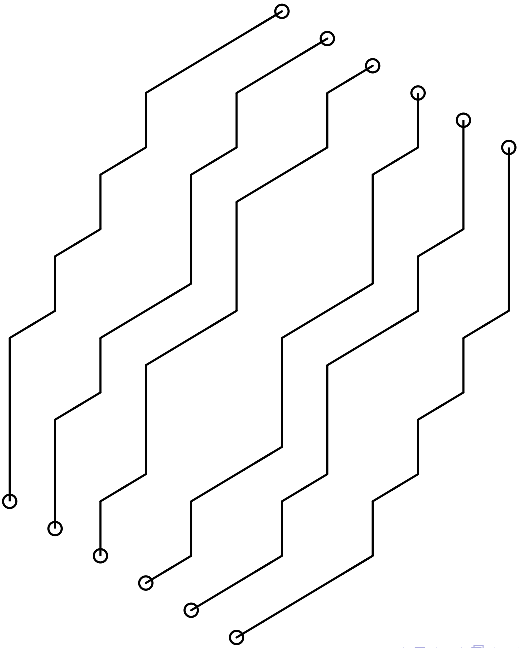


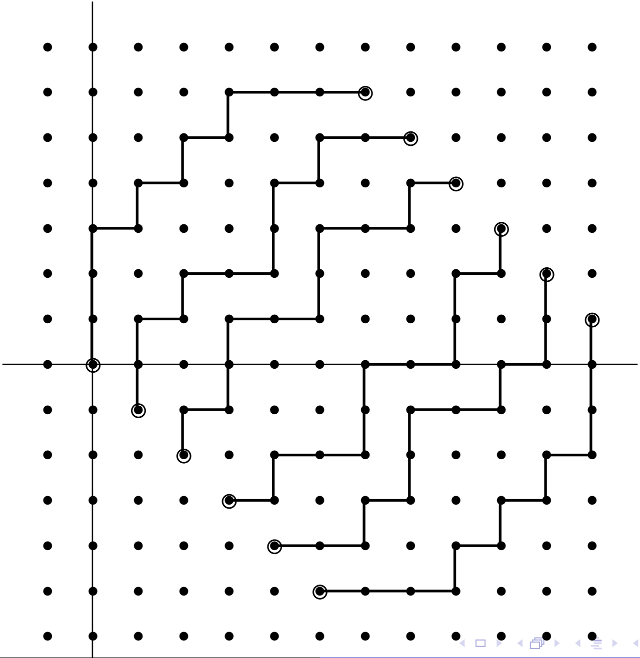
The bijection with non-intersecting lattice paths











We have now converted the original enumeration problem to the following problem:

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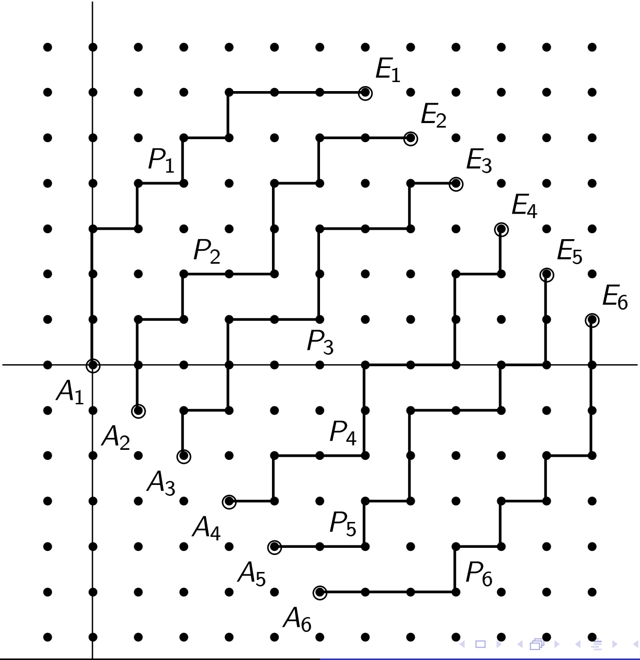
Given starting points $A_1, A_2, \ldots, A_n$ and end points $E_1, E_2, \ldots, E_n$, count families $(P_1, P_2, \ldots, P_n)$ of non-intersecting lattice paths, where the i-th path P_i runs from A_i to E_i, $i = 1, 2, \ldots, n$.


E_1
E_2
E_3
E_4
E_5
E_6
P_1
P_2
P_3
P_4
P_5
P_6
A_1
A_2
A_3
A_4
A_5
A_6

Theorem (Karlin–McGregor, Lindström, Gessel–Viennot, Fisher, John–Sachs, Gronau–Just–Schade–Scheffler–Wojciechowski)

Let G be an acyclic, directed graph, and let $A_1, A_2, \ldots, A_n$ and $E_1, E_2, \ldots, E_n$ be vertices in the graph with the property that, for $i < j$ and $k < l$, any (directed) path from A_i to E_l intersects with any path from A_j to E_k. Then the number of families $(P_1, P_2, \ldots, P_n)$ of non-intersecting (directed) paths, where the i-th path P_i runs from A_i to E_i, $i = 1, 2, \ldots, n$, is given by

$$\det_{1 \le i,j \le n} \left(|\mathcal{P}(A_j \to E_i)| \right),$$

where $\mathcal{P}(A \to E)$ denotes the set of paths from A to E.

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$$\det_{1 \le i,j \le n} (|\mathcal{P}(A_j \to E_i)|),$$

where $\mathcal{P}(A \to E)$ denotes the set of paths from A to E.

Remark

There holds as well a *weighted* version, in which every edge e is assigned a *weight* $w(e)$, and where the weight of a path (family) $\mathcal{P}$ is defined as the product $\prod_{e \in \mathcal{P}} w(e)$, with the product running over all edges in the path (family).

Sketch of Proof

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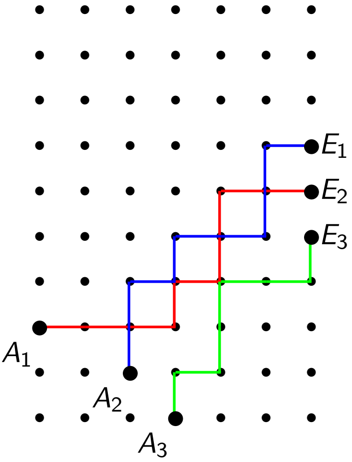
$$\det_{1\le i,j\le n}\left(|\mathcal{P}(A_j \to E_i)|\right) = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}\sigma \prod_{i=1}^{n} |\mathcal{P}(A_{\sigma(i)} \to E_i)|$$

$$= \sum_{\substack{(\sigma,P_1,\dots,P_n)\\ \sigma\in\mathfrak{S}_n\\ P_i:A_{\sigma(i)}\to E_i}} \operatorname{sgn}\sigma.$$

A sign-reversing involution



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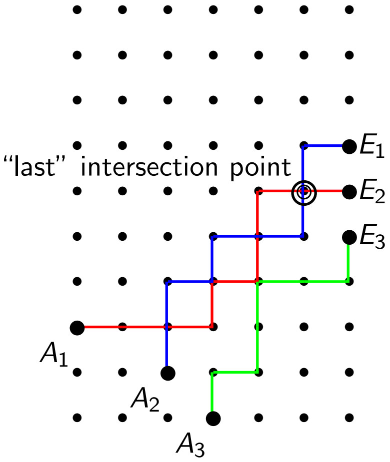

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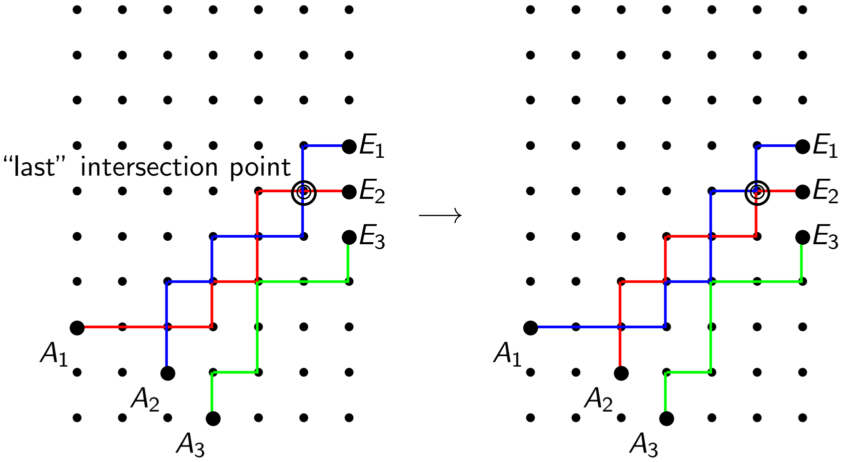

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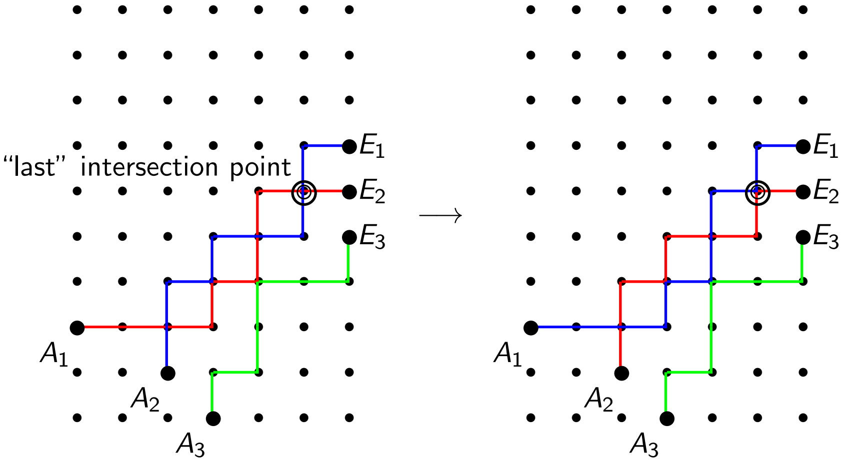

$\sigma = 213$ $\qquad$ $\sigma = 123$

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Hence: the number of rhombus tilings of a hexagon with side lengths a, b, c, a, b, c is given by

$$\det_{1 \le i,j \le b} \left(\binom{a+c}{a+i-j} \right).$$

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It is not very difficult to evaluate this determinant (but this is a different story ...):

$$\det_{1 \le i,j \le b} \left(\binom{a+c}{a+i-j} \right) = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}.$$

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Theorem (MacMahon)

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EXAMPLE: $n=3$:

$$\det\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 3 & 6\end{pmatrix}=\cdots=1.$$

Claim

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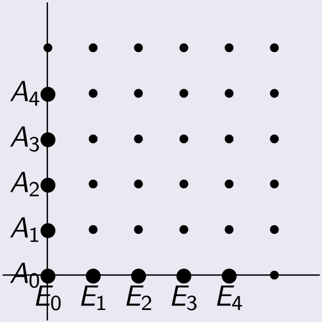

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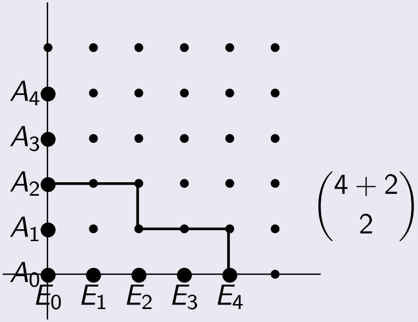

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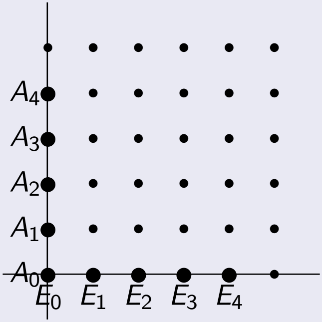

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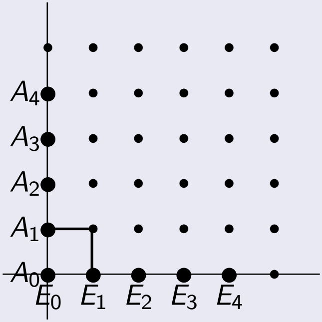

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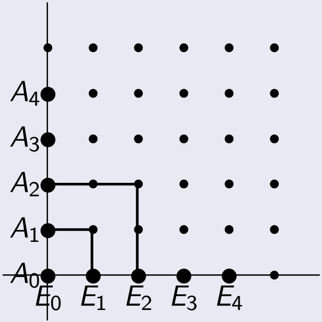

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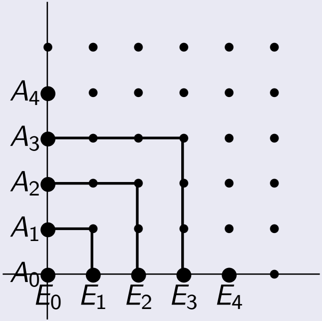

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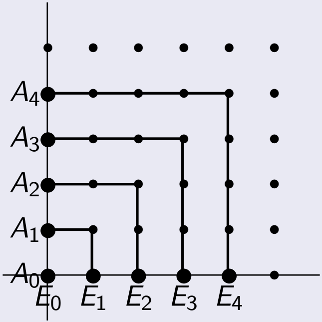
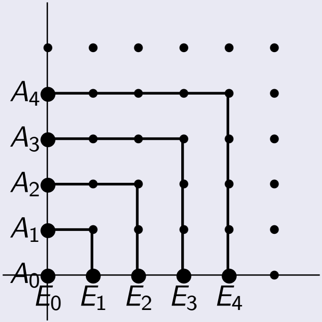


□

Claim

$$\det_{0\le i,j\le n-1}\left(\binom{i+j}{i}\right)\overset{!}{=}1.$$

Proof.



□

Warning:

Warning: This is very beautiful, but normally it is the other way round.

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(I.e., one wants to solve a combinatorial problem, obtains a determinant, and then one has a hard time to evaluate the determinant . . .)

The general form of the non-intersecting lattice path theorem

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Theorem (Lindström, Gessel–Viennot)

Let G be an acyclic, directed graph, and let $A_1, A_2, \dots, A_n$ and $E_1, E_2, \dots, E_n$ be vertices in the graph. Denote by $\mathcal{P}^+(\mathbf{A} \to \mathbf{E})$ the set of all families $(P_1, P_2, \dots, P_n)$ of non-intersecting (directed) paths, such that the i-th path P_i runs from A_i to E_i, $i = 1, 2, \dots, n$. Then

$$\sum_{\sigma \in \mathfrak{S}_n} (\operatorname{sgn} \sigma) \cdot |\mathcal{P}^+(\mathbf{A}_\sigma \to \mathbf{E})| = \det_{1 \le i,j \le n} \left(|\mathcal{P}(A_j \to E_i)| \right),$$

where $\mathbf{A}_\sigma = (A_{\sigma(1)}, A_{\sigma(2)}, \dots, A_{\sigma(n)})$.

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Remark

Again, there holds as well a *weighted* version.

Is this good for anything?

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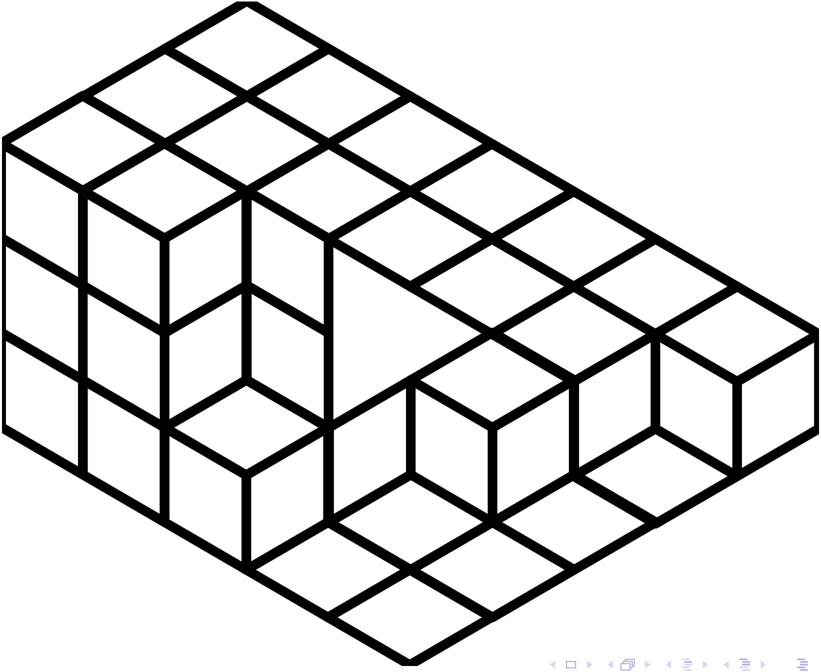
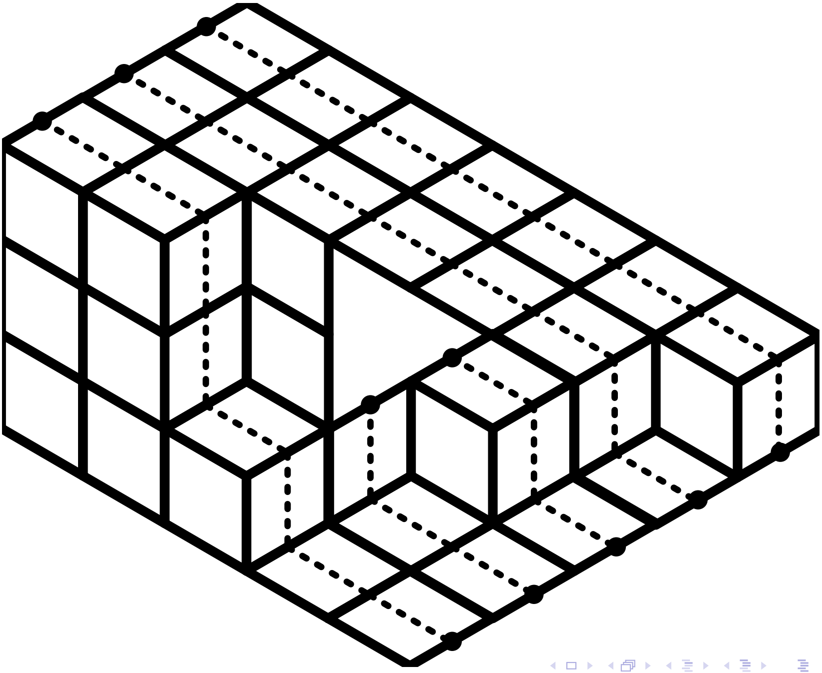
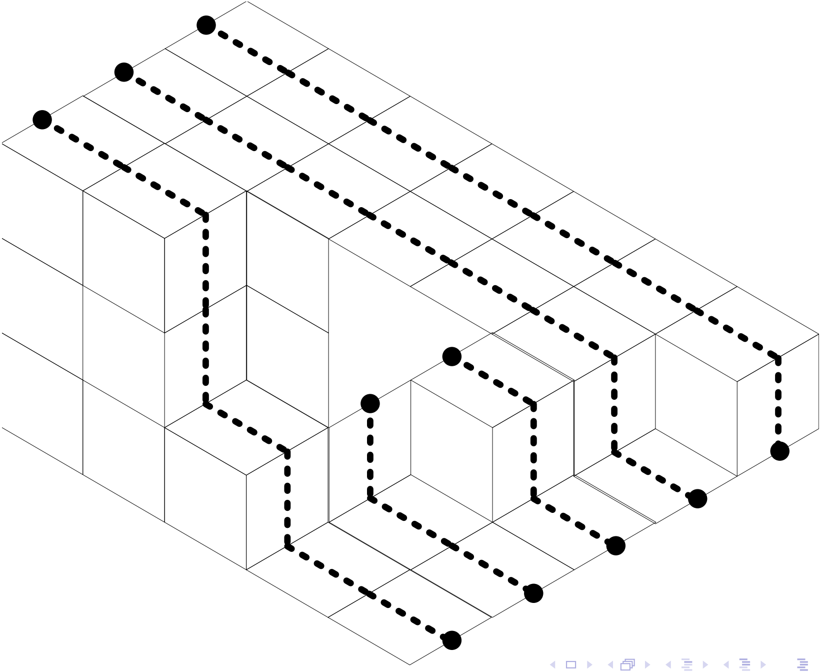
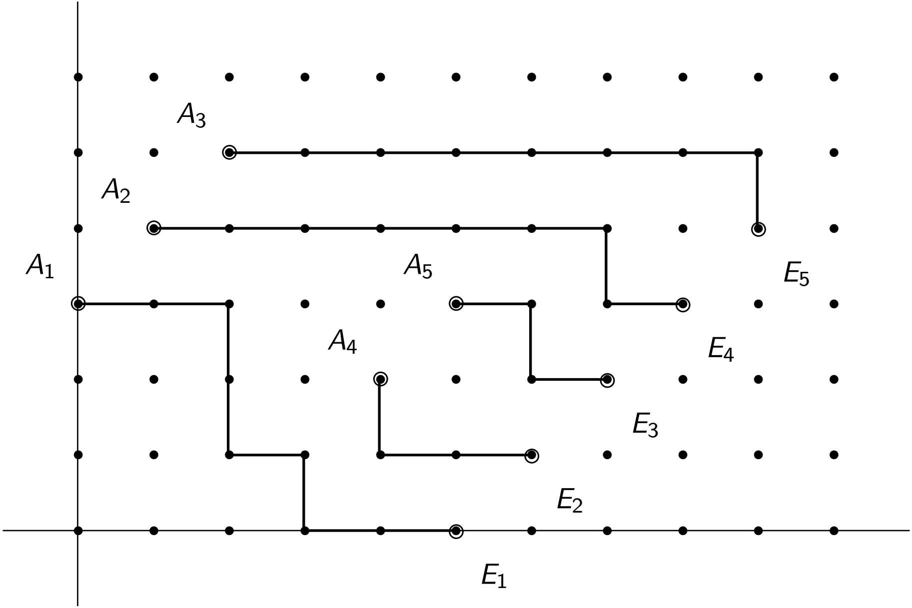
Yes!








A_3
A_2
A_1
A_5
E_5
A_4
E_4
E_3
E_2
E_1

So, at least for even m, the number of rhombus tilings of a hexagon with sides $a, b+m, c, a+m, b, c+m$, with an equilateral triangle of side m removed from its center is given by

$$\det_{1\le i,j\le a+m}\begin{pmatrix} \dbinom{b+c+m}{b-i+j} & 1\le i\le a \\ \dbinom{\frac{b+c}{2}}{\frac{b+a}{2}-i+j} & a+1\le i\le a+m \end{pmatrix}.$$

Theorem (with Ciucu, Eisenkölbl, Zare)

The number of rhombus tilings of the hexagon minus triangle is given by

$$\frac{\mathrm{H}(a+m)\,\mathrm{H}(b+m)\,\mathrm{H}(c+m)\,\mathrm{H}(a+b+c+m)}{\mathrm{H}(a+b+m)\,\mathrm{H}(a+c+m)\,\mathrm{H}(b+c+m)}$$

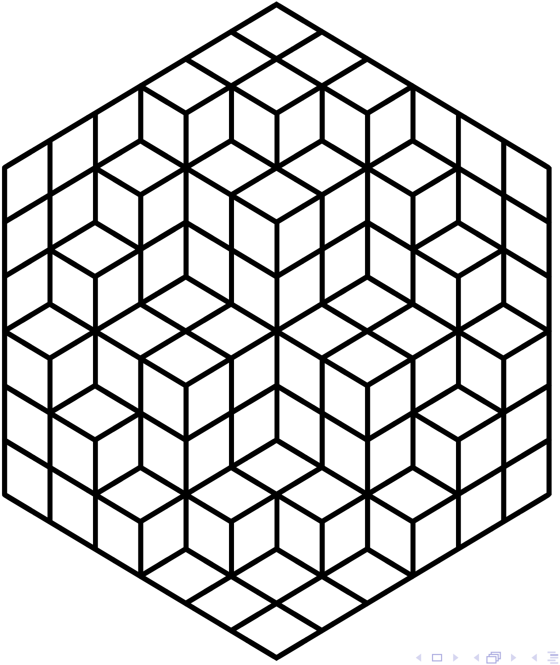
$$\times\frac{\mathrm{H}(m+\lceil\frac{a+b+c}{2}\rceil)\,\mathrm{H}(m+\lfloor\frac{a+b+c}{2}\rfloor)}{\mathrm{H}(\frac{a+b}{2}+m)\,\mathrm{H}(\frac{a+c}{2}+m)\,\mathrm{H}(\frac{b+c}{2}+m)}$$

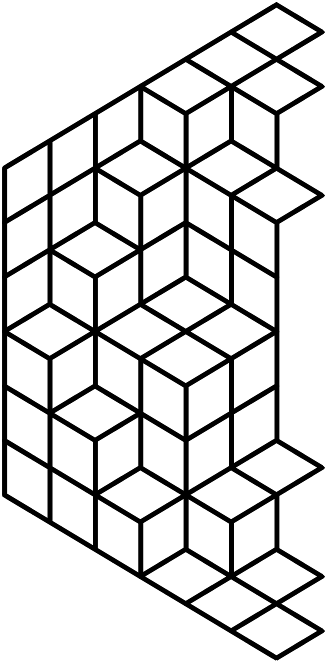
$$\times\frac{\mathrm{H}(\lceil\frac{a}{2}\rceil)\,\mathrm{H}(\lceil\frac{b}{2}\rceil)\,\mathrm{H}(\lceil\frac{c}{2}\rceil)\,\mathrm{H}(\lfloor\frac{a}{2}\rfloor)\,\mathrm{H}(\lfloor\frac{b}{2}\rfloor)\,\mathrm{H}(\lfloor\frac{c}{2}\rfloor)}{\mathrm{H}(\frac{m}{2}+\lceil\frac{a}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lceil\frac{b}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lceil\frac{c}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{a}{2}\rfloor)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{b}{2}\rfloor)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{c}{2}\rfloor)}$$

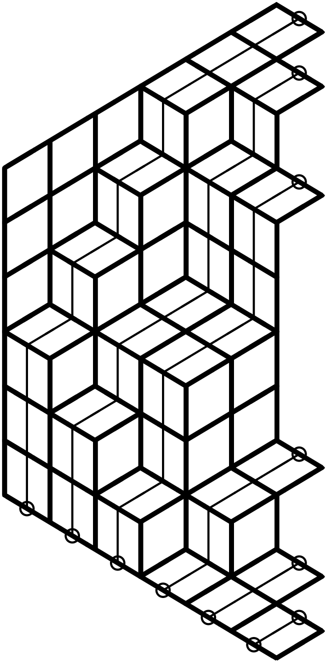
$$\times\frac{\mathrm{H}(\frac{m}{2})^2\,\mathrm{H}(\frac{a+b+m}{2})^2\,\mathrm{H}(\frac{a+c+m}{2})^2\,\mathrm{H}(\frac{b+c+m}{2})^2}{\mathrm{H}(\frac{m}{2}+\lceil\frac{a+b+c}{2}\rceil)\,\mathrm{H}(\frac{m}{2}+\lfloor\frac{a+b+c}{2}\rfloor)\,\mathrm{H}(\frac{a+b}{2})\,\mathrm{H}(\frac{a+c}{2})\,\mathrm{H}(\frac{b+c}{2})},$$

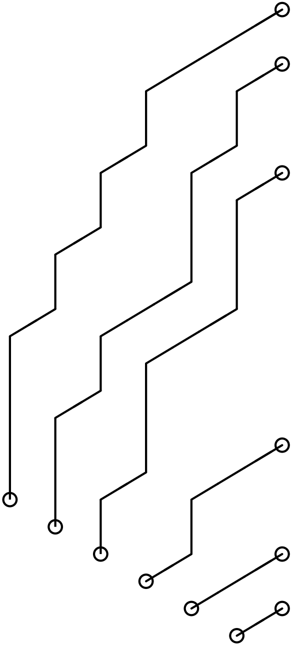
where $\mathrm{H}(n) := \begin{cases}\prod_{k=0}^{n-1}\Gamma(k+1) & \textit{for } n \textit{ an integer},\\ \prod_{k=0}^{n-\frac{1}{2}}\Gamma(k+\frac{1}{2}) & \textit{for } n \textit{ a half-integer}.\end{cases}$

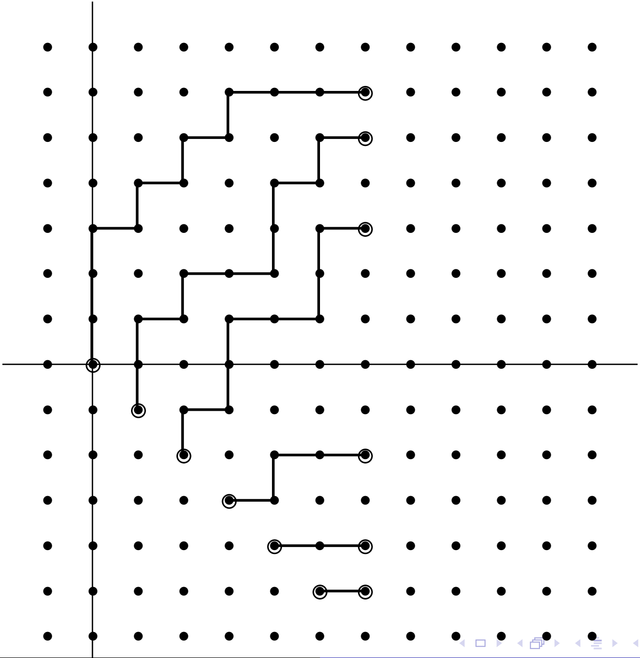
Non-intersecting Lattice Paths and Pfaffians











We encounter a new problem:
Given starting points $A_1, A_2, \dots, A_n$ *and end points* $E_1, E_2, \dots$, *count families* $(P_1, P_2, \dots, P_n)$ *of non-intersecting lattice paths, where the i-th path* P_i *runs from* A_i *to one of the* E_ℓ*'s,* $i = 1, 2, \dots, n$.

The non-intersecting lattice path theorem with variable end points

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Theorem (Okada, Stembridge)

Let G be an acyclic, directed graph, and let $A_1, A_2, \dots, A_{2n}$ and $E_1, E_2, \dots$ be vertices in the graph with the property that, for $i < j$ and $k < l$, any (directed) path from A_i to E_l intersects with any path from A_j to E_k. Then the number of families $(P_1, P_2, \dots, P_{2n})$ of non-intersecting (directed) paths, where the i-th path P_i runs from A_i to one of the E_ℓ's, $i = 1, 2, \dots, 2n$, is given by

$$\mathrm{Pf}_{1 \le i,j \le 2n}(Q_{i,j}),$$

where

$$Q_{i,j} = \sum_{1 \le s < t} \big(|\mathcal{P}(A_i \to E_s)| \cdot |\mathcal{P}(A_j \to E_t)| - |\mathcal{P}(A_j \to E_s)| \cdot |\mathcal{P}(A_i \to E_t)| \big).$$

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Again, there holds as well a *weighted* version.

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COMBINATORIALLY:

$$\mathrm{Pf}(A) = \sum_{\pi \text{ perfect matching of } \{1,\dots,2n\}} (-1)^{\#(\text{crossings of } \pi)} \prod_{(i,j)\in\pi,\, i<j} A_{i,j}.$$

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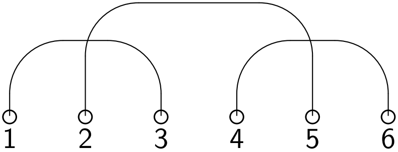
A perfect matching of $\{1, 2, \dots, 6\}$ with 2 crossings:



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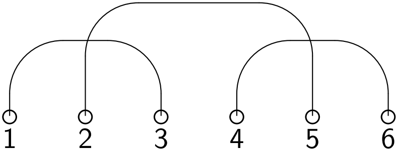
$$\mathrm{Pf}(A) = \sum_{\pi \text{ perfect matching of } \{1,\dots,2n\}} (-1)^{\#(\text{crossings of } \pi)} \prod_{(i,j)\in\pi,\, i<j} A_{i,j}.$$

A perfect matching of $\{1,2,\dots,6\}$ with 2 crossings:



ALGEBRAICALLY: Given a $(2n)\times(2n)$ skew symmetric matrix A,

$$(\mathrm{Pf}(A))^2 = \det A.$$

In our problem, this leads to the Pfaffian

$$\operatorname{Pf}_{1\le i,j\le 2n}\left(\sum_{-n\le s<t\le 2x}\left(\binom{2n+s}{s+j-1}\binom{2n+t}{t+i-1}\right.\right.$$
$$\left.\left.-\binom{2n+s}{s+i-1}\binom{2n+t}{t+j-1}\right)\right).$$

In our problem, this leads to the Pfaffian

$$\operatorname{Pf}_{1\le i,j\le 2n}\left(\sum_{-n\le s<t\le 2x}\left(\binom{2n+s}{s+j-1}\binom{2n+t}{t+i-1} - \binom{2n+s}{s+i-1}\binom{2n+t}{t+j-1}\right)\right).$$

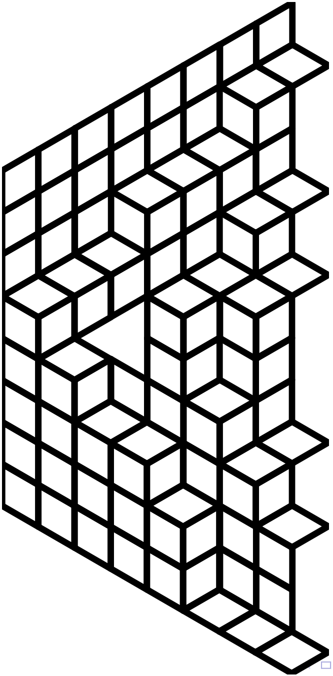
Theorem (Andrews)

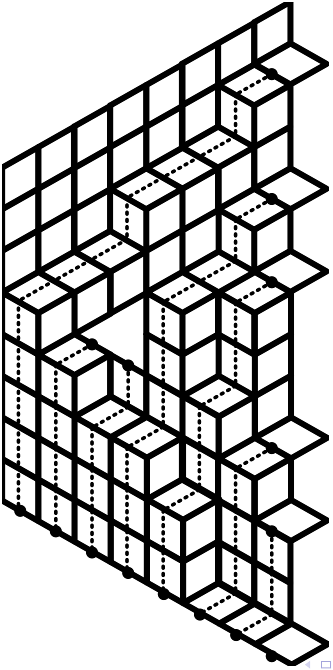
The number of vertically symmetric rhombus tilings of a hexagon with side lengths $2n, 2n, 2x, 2n, 2n, 2x$ is equal to

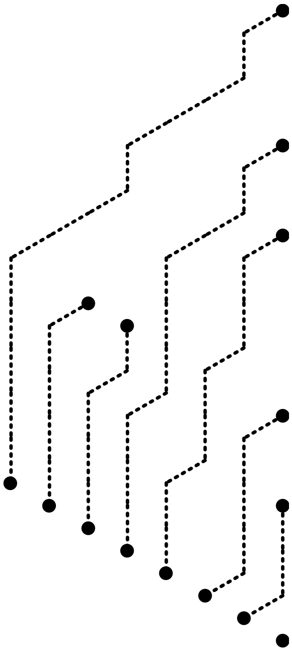
$$\frac{\left(x+\frac{1}{2}\right)_{2n}}{\left(\frac{1}{2}\right)_{2n}} \prod_{s=1}^{n} \frac{(2x+2s)_{4n-4s+1}}{(2s)_{4n-4s+1}},$$

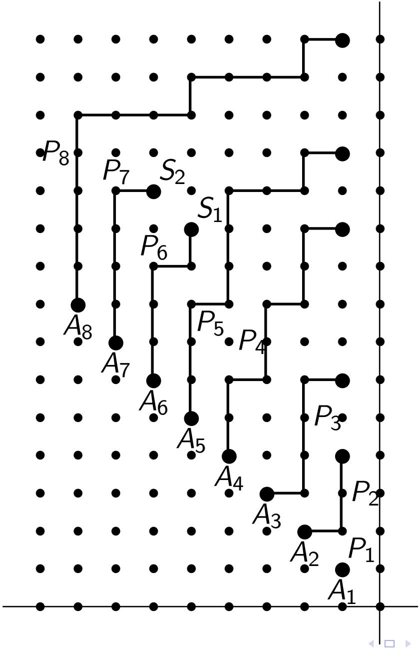
where $(\alpha)_m := \alpha(\alpha+1)\cdots(\alpha+m-1)$ *for* $m \ge 1$, *and* $(\alpha)_0 := 1$.

What about some fixed and some variable end points, at the same time?









P_8
P_7
S_2
S_1
P_6
P_5
P_4
P_3
P_2
P_1
A_8
A_7
A_6
A_5
A_4
A_3
A_2
A_1

Theorem (Stembridge)

Let $\{A_1, A_2, \dots, A_p, S_1, S_2, \dots, S_q\}$ *and* $E = \{E_1, E_2, \dots\}$ *be finite sets of lattice points in the integer lattice* $\mathbb{Z}^2$*. Then the number of families* $(P_1, P_2, \dots, P_p)$ *of non-intersecting lattice paths, with* P_k *running from* A_k *to* S_k*, for* $k = 1, 2, \dots, q$*, and to* E_{j_k}*, for* $k = q+1, q+2, \dots, p$*, the indices being required to satisfy* $j_{q+1} < j_{q+2} < \cdots < j_p$*, is given by*

$$\operatorname{Pf}\begin{pmatrix} Q & H \\ -H^t & 0 \end{pmatrix},$$

where $Q_{i,j}$ *is defined as before, and where the matrix* $H = (H_{i,j})_{1 \le i \le p,\ 1 \le j \le q}$ *is defined by*

$$H_{i,j} = |\mathcal{P}(A_i \to S_j)|.$$

Theorem (with Ciucu)

The number of rhombus tilings of the half hexagon minus triangle of size 2 at distance 2k from the free boundary is equal to

$$\binom{4k+1}{2k}\frac{(n+k)!}{(x+n-k)_{2k+1}}\prod_{s=1}^{n}\frac{(2x+2s)_{4n-4s+1}}{(2s)_{4n-4s+1}}$$
$$\times\sum_{i=0}^{n-k-1}\frac{(\frac{1}{2})_i}{i!\,(n-k-i-1)!^2\,(n+k-i+1)_{n-k}}$$
$$\cdot\frac{1}{(n+k-i+1)_i\,(2n-i+\frac{1}{2})_i}$$
$$\cdot\Big((x)_i\,(x+i+1)_{n-k-i-1}\,(x+n+k+1)_{n-k}$$
$$-(x)_{n-k}\,(x+n+k+1)_{n-k-i-1}\,(x+2n-i+1)_i\Big).$$

The Minor Summation Formula of Ishikawa and Wakayama

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Theorem (Ishikawa–Wakayama)

Let n, p, q be integers such that $n+q$ is even and $0 \le n-q \le p$. Let G be any $n \times p$ matrix, H be any $n \times q$ matrix, and $A = (a_{ij})_{1 \le i,j \le p}$ be any skew-symmetric matrix. Then we have

$$\sum_K \operatorname{Pf}\left(A_K^K\right) \det\left(G_K \vdots H\right) = (-1)^{q(q-1)/2} \operatorname{Pf}\begin{pmatrix} G\,A\,{}^tG & H \\ -{}^tH & 0 \end{pmatrix},$$

where K runs over all $(n-q)$-element subsets of $[1,p]$, A_K^K is the skew-symmetric matrix obtained by picking the rows and columns indexed by K and G_K is the sub-matrix of G consisting of the columns corresponding to K.

Hankel Determinants and Orthogonal Polynomials

Definition

A sequence $(p_n(x))_{n\ge0}$ of polynomials is called (formally) *orthogonal* if $p_n(x)$ has degree n, $n = 0, 1, \dots,$ and if there exists a linear functional L such that $L(p_n(x)p_m(x)) = \delta_{mn}c_n$ for some sequence $(c_n)_{n\ge0}$ of nonzero numbers, with $\delta_{m,n}$ denoting the Kronecker delta (i.e., $\delta_{m,n} = 1$ if $m = n$ and $\delta_{m,n} = 0$ otherwise).

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Theorem (Favard)

Let $(p_n(x))_{n\ge0}$ be a sequence of monic polynomials, the polynomial $p_n(x)$ having degree n, $n=0,1,\dots$. Then the sequence $(p_n(x))$ is (formally) orthogonal if and only if there exist sequences $(a_n)_{n\ge1}$ and $(b_n)_{n\ge1}$, with $b_n\ne0$ for all $n\ge1$, such that the three-term recurrence

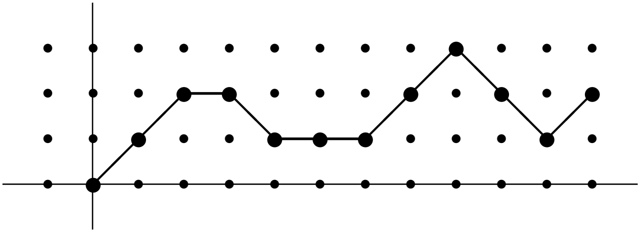
$$p_{n+1}(x)=(x-a_n)p_n(x)-b_np_{n-1}(x),\qquad \text{for } n\ge1, \tag{1}$$

holds, with initial conditions $p_0(x)=1$ and $p_1(x)=x-a_0$. □

Moments of Orthogonal Polynomials and Motzkin Paths

Moments of Orthogonal Polynomials and Motzkin Paths

A (generalised) *Motzkin path* is a lattice path consisting of steps from $\{(1,1),(1,0),(1,-1)\}$ (up-steps, level-steps, down-steps) that does not pass below the x-axis.



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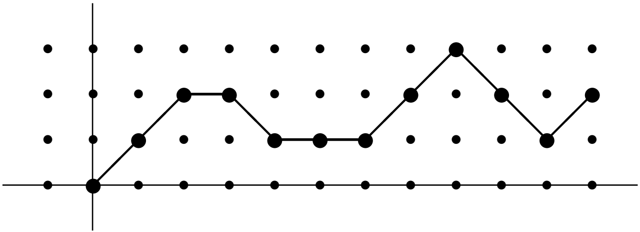

For each of the three kinds of steps, we define a weight:

$$w((1,1)) = 1,\ w((1,0)) = a_h,\ w((1,-1)) = b_h,$$

where h is the height of the starting point of the step. The weight $w(P)$ of a Motzkin path P is the product of all the weights of its steps.

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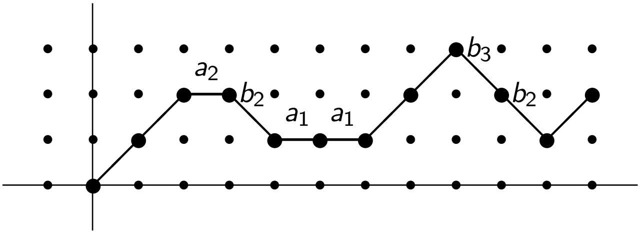

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$$w(.)=a_2 b_2 a_1 a_1 b_3 b_2.$$

Proposition (Viennot)

The moments $\mu_n := L(x^n)$ *of orthogonal polynomials are given by*

$$\sum_P w(P),$$

where the sum is over all Motzkin paths from $(0,0)$ *to* $(n,0)$*.*

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Theorem (Viennot)

Given a sequence of monic orthogonal polynomials $(p_n(x))$, we have

$$L\big(x^n\, p_k(x)\, p_l(x)\big) = b_1 \cdots b_l \cdot \sum_P w(P),$$

where the sum is over all Motzkin paths from $(0,k)$ to (n,l).

Theorem (Heilermann)

Let $(\mu_k)_{k\ge0}$ *be the moments for a monic sequence of orthogonal polynomials. Then the Hankel determinant* $\det_{0\le i,j\le n-1}(\mu_{i+j})$ *equals* $\mu_0^n b_1^{n-1} b_2^{n-2}\cdots b_{n-2}^2 b_{n-1}$.

Another one picture proof

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Without loss of generality, we assume that $\mu_0 = 1$. We must prove:

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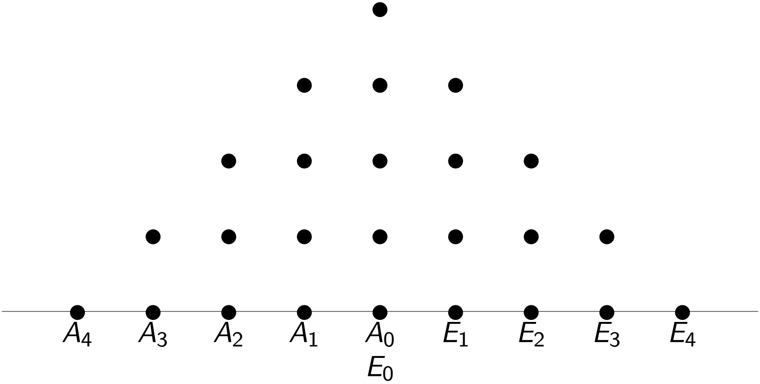

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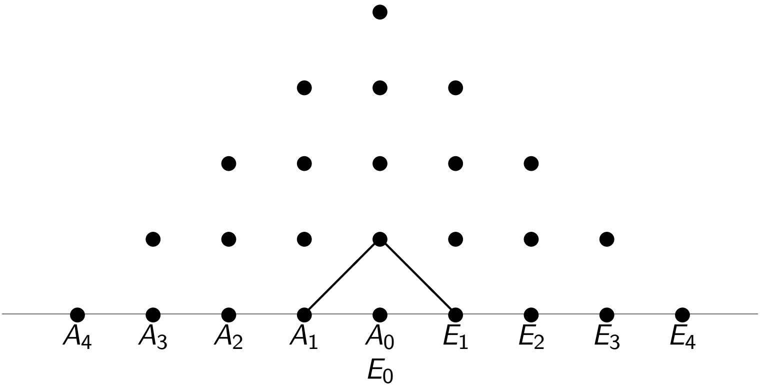

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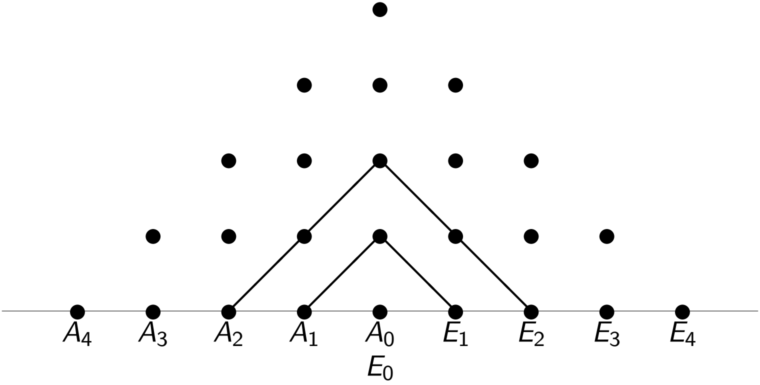

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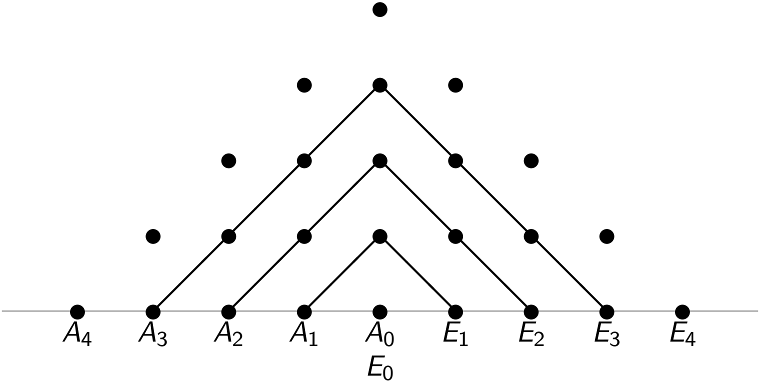

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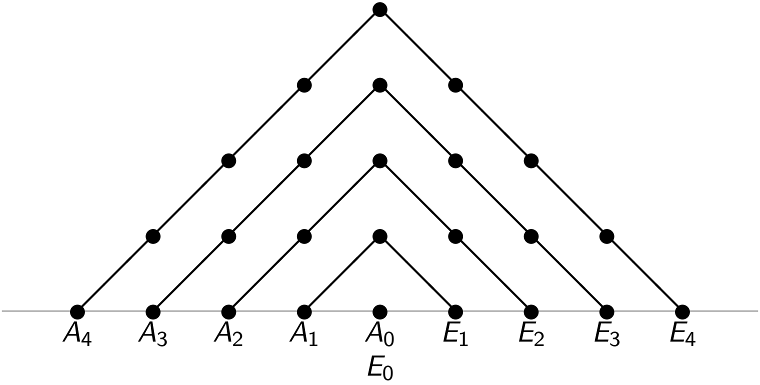

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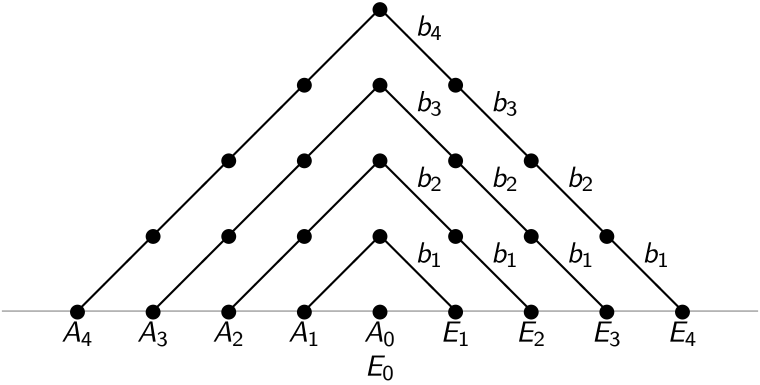

An Application:

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In a certain problem of rhombus tiling enumeration, Markus Fulmek and myself needed to compute the determinant (among others)

$$\det_{0\le i,j\le n-1}\left(B_{i+j+2}\right),$$

where B_k denotes the k-th *Bernoulli number*. (The Bernoulli numbers are defined via their generating function, $\sum_{k=0}^{\infty} B_k z^k/k! = z/(e^z-1)$.)

Solution:

Solution:

ASKEY SCHEME OF HYPERGEOMETRIC ORTHOGONAL POLYNOMIALS

http://aw.twi.tudelft.nl/~koekoek/research.html

${}_4F_3$		Wilson	Racah	
${}_3F_2$	Continuous dual Hahn	Continuous Hahn	Hahn	dual Hahn
${}_2F_1$	Meixner – Pollaczek	Jacobi	Meixner	Kravchouk
${}_2F_1/{}_2F_0$		Laguerre	Charlier	

We need the *continuous Hahn polynomials* (they have the "right" moments).

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If one works everything out, then, according to Heilermann's theorem, we obtain:

$$\det_{0\le i,j,\le n-1}(B_{i+j+2}) = (-1)^{\binom{n}{2}}\left(\frac{1}{6}\right)^n \prod_{i=1}^{n-1}\left(\frac{i(i+1)^2(i+2)}{4(2i+1)(2i+3)}\right)^{n-i}$$
$$= (-1)^{\binom{n}{2}}\frac{1}{6}\prod_{i=1}^{n-1}\frac{i!\,(i+1)!^4\,(i+2)!}{(2i+2)!\,(2i+3)!}.$$

Of course, there is much more ...
You may criticise that I have entirely left out another important application of determinants in combinatorics:

the *Kasteleyn determinants and Pfaffians*

for the enumeration of perfect matchings in planar graphs. In fact, these stand at the beginning of the beautiful asymptotic analysis of planar tiling and matching models due to Cohn–Kenyon–Propp, respectively, in greater generality, to Kenyon–Okounkov–Sheffield. However, that would again be a story by itself ...