Advanced Determinant Calculus

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Advanced Determinant Calculus (Sémimaire Lotharingien de Combinatoire **42** (1999), Art. B42q)

Advanced Determinant Calculus: a Complement (Linear Algebra and Its Applications **411** (2005), 64–166)

“Method” 0. row/column operations

Method 1. take out as many factors as possible until something polynomial remains; match with one of the lemmas in ADC I

Method 2. LU-factorisation

Method 3. condensation

Method 4. identification of factors

Our “demonstration example”

Our "demonstration example"

GERT ALMKVIST:

`I have a sequence of determinants. I must show that they are non-zero. Here are the first few:`

k	$\det(A(k))$
1	$2^{91}3^{8}5^{7}7^{2}$
2	$-2^{523}3^{52}5^{17}7^{14}11^{4}13^{3}$
3	$2^{1367}3^{177}5^{41}7^{25}11^{20}13^{19}17^{5}19^{4}23^{2}$
4	$-2^{3231}3^{167}5^{83}7^{53}11^{28}13^{27}17^{25}19^{8}23^{6}29^{3}31^{2}$
5	$2^{5399}3^{290}5^{345}7^{93}11^{41}13^{37}17^{33}19^{32}23^{10}29^{7}31^{6}37^{3}$

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`Can you help?`

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An expansion due to Bill Gosper:

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An expansion due to Fabrice Bellard:

$$\pi = \frac{1}{740025} \left(\sum_{n=1}^{\infty} \frac{3P(n)}{\binom{7n}{2n} 2^{n-1}} - 20379280 \right),$$

where

$$P(n) = -885673181n^5 + 3125347237n^4 - 2942969225n^3 + 1031962795n^2 - 196882274n + 10996648.$$

Our "demonstration example"

Gert Almkvist and Joakim Petersson:

Are there more expansions of the type

$$\pi = \sum_{n=0}^{\infty} \frac{S(n)}{\binom{mn}{pn} a^n},$$

where $S(n)$ is some polynomial in n (depending on m, p, a)?

Our "demonstration example"

Choose some m, p, a, go to the computer, compute

$$s(k) = \sum_{n=0}^{\infty} \frac{n^k}{\binom{mn}{pn} a^n}$$

to many, many digits for $k = 0, 1, 2, \dots$, put

$$\pi, s(0), s(1), s(2), \dots$$

into the LLL-algorithm, and see if you get an integral linear combination of $\pi, s(0), s(1), s(2), \dots$.

Our "demonstration example"

m	p	a	$\deg(S)$	
3	1	2	1	Gosper
7	2	2	5	Bellard
8	4	-4	4	
10	4	4	8	
12	4	-4	8	
16	8	16	8	
24	12	-64	12	
32	16	256	16	
40	20	-4^5	20	
48	24	4^6	24	
56	28	-4^7	28	
64	32	4^8	32	
72	36	-4^9	36	
80	40	4^{10}	40	

Our "demonstration example"

A few example expansions:

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$$\pi = \frac{1}{r} \sum_{n=0}^{\infty} \frac{S(n)}{\binom{16n}{8n} 16^n},$$

where

$$r = 3^6 5^3 7^2 11^2 13^2$$

and

$$\begin{aligned} S(n) = &-869897157255 - 3524219363487888n \\ &+ 112466777263118189n^2 - 1242789726208374386n^3 \\ &+ 6693196178751930680n^4 - 19768094496651298112n^5 \\ &+ 32808347163463348736n^6 - 28892659596072587264n^7 \\ &+ 10530503748472012800n^8. \end{aligned}$$

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$$\pi = \frac{1}{r} \sum_{n=0}^{\infty} \frac{S(n)}{\binom{32n}{16n} 256^n},$$

where

$$r = 2^3 3^{10} 5^6 7^3 11 \cdot 13^2 17^2 19^2 23^2 29^2 31^2$$

Our "demonstration example"

and

$$
\begin{aligned}
S(n) = &- 20621118847563474790857092808 75\\
&+ 150549174030283902375356971726188209190 0n\\
&- 1124011494040876582138393867162119752919 75n^2\\
&+ 325788165194268289181855772622584067411000 2n^3\\
&- 516773095108906305006078985994630362679612 80n^4\\
&+ 517337977987354819322786909541179043148522 720n^5\\
&- 352639649432956071875808639284125815239024 5120n^6\\
&+ 171145766235995166227501216110074805943799 363584n^7\\
&- 607394166132282199408865396581459044020680 29440n^8\\
&+ 159935882563435860391195903248596461569183 580160n^9\\
&- 313951952615028230229958218839819183812205 608960n^{10}\\
&+ 457341091673257198565533286493831205566468 325376n^{11}\\
&- 486846784774707448105420279985074159657397 780480n^{12}\\
&+ 367314505118245777241612044490633887668208 926720n^{13}\\
&- 185647326591648164598342857319777582801297 080320n^{14}\\
&+ 562246880357070156879991289943246904184673 40288n^{15}\\
&- 768725577881655778607397779514936040861204 4800n^{16}.
\end{aligned}
$$

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Our "demonstration example"

This suggests that *there is a formula*

$$\pi = \sum_{n=0}^{\infty} \frac{S_k(n)}{\binom{8kn}{4kn}(-4)^{kn}},$$

for any $k = 1, 2, \dots$.

How does one prove such identities?

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Let us consider Gosper's formula

$$\pi = \sum_{n=0}^{\infty} \frac{50n-6}{\binom{3n}{n} 2^n}.$$

The beta integral evaluation gives

$$\frac{1}{\binom{3n}{n}} = (3n+1) \int_0^1 x^{2n}(1-x)^n dx.$$

Hence

$$\sum_{n=0}^{\infty} \frac{50n-6}{\binom{3n}{n} 2^n} = \int_0^1 \sum_{n=0}^{\infty} (50n-6)(3n+1) \left(\frac{x^2(1-x)}{2} \right)^n dx.$$

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We have

$$\sum_{n=0}^{\infty}(50n-6)(3n+1)y^n = \frac{2(56y^2+97y-3)}{(1-y)^3}.$$

Thus, if this is substituted, we obtain

$$RHS = 8\int_0^1 \frac{28x^6-56x^5+28x^4-97x^3+97x^2-6}{(x^3-x^2+2)^3}dx$$

$$= \left[\frac{4x(x-1)(x^3-28x^2+9x+8)}{(x^3-x^2+2)^2} + 4\arctan(x-1)\right]_0^1 = \pi. \qquad \square$$

The sums $\sum_{n=0}^{\infty} S(n)/\binom{8kn}{4kn}(-4)^{kn}$, $k = 1, 2, \ldots$

The beta integral evaluation gives

$$\frac{1}{\binom{8kn}{4kn}} = (8kn+1)\int_0^1 x^{4kn}(1-x)^{4kn}dx.$$

Hence, if $S(n)$ has degree d,

$$\sum_{n=0}^{\infty} \frac{S(n)}{\binom{8kn}{4kn}(-4)^{kn}} = \int_0^1 \sum_{n=0}^{\infty}(8kn+1)S(n)\left(\frac{x^{4k}(1-x)^{4k}}{(-4)^k}\right)^n dx$$

$$= \int_0^1 \frac{P(x)}{\left(x^{4k}(1-x)^{4k}-(-4)^k\right)^{d+2}}\,dx.$$

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Let $Q(x) := x^{4k}(1-x)^{4k}-(-4)^k$. Perhaps

$$\int \frac{P(x)}{Q(x)^{d+2}}\,dx = \frac{R(x)}{Q(x)^{d+1}} + 2\arctan(x) + 2\arctan(x-1),$$

for some polynomial $R(x)$ with $R(0)=R(1)=0$.
Then the original sum would indeed be equal to π.

Perhaps

$$\int \frac{P(x)}{Q(x)^{d+2}}\,dx = \frac{R(x)}{Q(x)^{d+1}} + 2\arctan(x) + 2\arctan(x-1),$$

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This is equivalent to

$$\frac{P}{Q^{d+2}} = \frac{R'}{Q^{d+1}} - (d+1)\frac{Q'R}{Q^{d+2}} + 2\left(\frac{1}{x^2+1} + \frac{1}{x^2-2x+2}\right),$$

or

$$QR' - (d+1)Q'R = P - 2Q^{d+2}\left(\frac{1}{x^2+1} + \frac{1}{x^2-2x+2}\right).$$

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$$QR' - (d+1)Q'R = P - 2Q^{d+2}\left(\frac{1}{x^2+1} + \frac{1}{x^2-2x+2}\right).$$

In our examples, we observed that

$$R(x) = (2x-1)\check{R}\big(x(1-x)\big)$$

for a polynomial $\check{R}$. So, let us make the substitution

$$t = x(1-x).$$

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Then, after some simplification, the earlier differential equation becomes

$$-(1-4t)Q\frac{d\check{R}}{dt} + (2Q + 4k(4k+1)(1-4t)t^{4k-1})\check{R} - P + 2(3-2t)\frac{Q^{4k+2}}{t^2 - 2t + 2} = 0,$$

where $Q(t) = t^{4k} - (-4)^k$.

Now, writing $N(k) = 4k(4k+1)$, we make the Ansatz

$$\check{R}(t) = \sum_{j=1}^{N(k)-1} a(j)t^j,$$

$$S(n) = \sum_{j=0}^{4k} a(N(k)+j)n^j$$

(recall: $S(n)$ defines $P(t)$).

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(recall: $S(n)$ defines $P(t)$).

Comparing coefficients of powers of t on both sides of the last equation, we get *a system of $N(k)+4k$ linear equations* for the unknowns $a(1), a(2), \dots, a(N(k)+4k)$.

Hence: *If the determinant of this system of linear equations is non-zero, then there does indeed exist a representation*

$$\pi = \sum_{n=0}^{\infty} \frac{S_k(n)}{\binom{8kn}{4kn}(-4)^n}.$$

The sums $\sum_{n=0}^{\infty} S(n) / \binom{8kn}{4kn} (-4)^{kn}$, $k = 1, 2, \dots$

Some simplification is possible (there are some trivial rows and columns). In the end, one remains with the determinant of a $16k^2 \times 16k^2$ matrix.

Let me now introduce you to this determinant.

The $16k^2 \times 16k^2$ determinant

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$$A(k) := \begin{pmatrix} 0\dots 0\, * & 0\dots 0\, * & 0\dots 0\, * & \dots & \dots & \dots & 0\dots 0\, * \\ F_1 & 0 & 0 & \dots & \dots & \dots & 0 \\ G_1 & F_2 & 0 & \dots & \dots & \dots & 0 \\ 0 & G_2 & F_3 & & & & \vdots \\ 0 & 0 & G_3 & & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & F_{4k-1} & 0 \\ \vdots & & & 0 & 0 & G_{4k-1} & F_{4k} \\ 0 & \dots & \dots & \dots & 0 & 0 & G_{4k} \end{pmatrix}.$$

Here, the F_i's and G_i's are $(4k-1) \times (4k)$ blocks.

The $16k^2 \times 16k^2$ determinant

The ℓ-th non-zero entry in the first row (these entries are marked by $*$) is

$$(-1)^{\ell-1}(-4)^{(\ell+1)k}8k(4k+1)\left(\prod_{i=1}^{4k-\ell}(4ik-1)\right)\left(\prod_{i=1}^{\ell-1}(4ik+1)\right).$$

The $16k^2 \times 16k^2$ determinant

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Here, the F_i's and G_i's are $(4k-1) \times (4k)$ blocks.

Here,

$$F_t = \begin{pmatrix} f_1(4(t-1)k+1) & f_0(4(t-1)k+2) & 0 & \dots \\ 0 & f_1(4(t-1)k+2) & f_0(4(t-1)k+3) & 0 & \dots \\ & \ddots & \ddots \\ & & \ddots & \ddots \\ & & 0 & f_1(4tk-2) & f_0(4tk-1) & 0 \\ & & & 0 & f_1(4tk-1) & f_0(4tk) \end{pmatrix}$$

and

$$G_t = \begin{pmatrix} g_1(4(t-1)k+1) & g_0(4(t-1)k+2) & 0 & \dots \\ 0 & g_1(4(t-1)k+2) & g_0(4(t-1)k+3) & 0 & \dots \\ & \ddots & \ddots \\ & & \ddots & \ddots \\ & & 0 & g_1(4tk-2) & g_0(4tk-1) & 0 \\ & & & 0 & g_1(4tk-1) & g_0(4tk) \end{pmatrix}.$$

The $16k^2 \times 16k^2$ determinant

with

$$\begin{aligned} f_0(j) &= j(-4)^k, \\ f_1(j) &= -(4j+2)(-4)^k, \\ g_0(j) &= (N(k)-j), \\ g_1(j) &= -(4N(k)-4j-2), \end{aligned}$$

where, as before, $N(k) = 4k(4k+1)$.

The $16k^2 \times 16k^2$ determinant

$$A(k) := \begin{pmatrix}
0\dots0\,* & 0\dots0\,* & 0\dots0\,* & \dots & \dots & \dots & 0\dots0\,* \\
F_1 & 0 & 0 & \dots & \dots & \dots & 0 \\
G_1 & F_2 & 0 & \dots & \dots & \dots & 0 \\
0 & G_2 & F_3 & & & & \vdots \\
0 & 0 & G_3 & & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \ddots & F_{4k-1} & 0 \\
\vdots & & & 0 & 0 & G_{4k-1} & F_{4k} \\
0 & \dots & \dots & \dots & 0 & 0 & G_{4k}
\end{pmatrix}.$$

Here, the F_i's and G_i's are $(4k-1)\times(4k)$ blocks.

The $16k^2 \times 16k^2$ determinant

```
In[1]:= A[k_,i_,j_]:=Module[{Var},
Var={Floor[(i-2)/(4k-1)],Floor[(j-1)/(4k)],
Mod[i-2,4k-1],Mod[j-1,4k]};
If[i==1,If[Mod[j,4k]===0,a[k,j],0],
If[Var[[1]]-Var[[2]]==0,
Switch[Var[[3]]-Var[[4]],0,f1[k,i,j],-1,f0[k,i,j],_
,0], If[Var[[1]]-Var[[2]]==1,
Switch[Var[[3]]-Var[[4]],0,g1[k,i,j],-1,g0[k,i,j],_
,0], 0 ] ] ] ]

A[k_]:=Table[A[k,i,j],{i,1,16k^2},{j,1,16k^2}]

f0[k_,i_,j_]:=j(-4)^k
f1[k_,i_,j_]:=-(2+4j)(-4)^k
g0[k_,i_,j_]:=(4k(4k+1)-j)
g1[k_,i_,j_]:=(-4*4k(4k+1)+2+4j)
```

The $16k^2 \times 16k^2$ determinant

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```
In[2]:= Det[A[2]]
```

The $16k^2 \times 16k^2$ determinant

```
In[2]:= Det[A[2]]

Out[2]=
> -601576375580370166777074138698518196031142518971 5\
> 689467122204136674781038302774231725971306459064 07\
> 5121023092662279814015195545600000000000000
```

```
In[2]:= Det[A[2]]

Out[2]=
> -60157637558037016677707413869851819603114251897l5\
> 68946712220413667478103830277423172597130645906407\
> 5121023092662279814015195545600000000000
> 

In[3]:= FactorInteger[%]
```

The $16k^2 \times 16k^2$ determinant

```
In[2]:= Det[A[2]]

Out[2]=
> -60157637558037016677707413869851819603114251897\
> 6894671222041366747810383027742317259713064590640\
> 512102309266227981401519554560000000000
```

```
In[3]:= FactorInteger[%]

Out[3]= {{-1, 1}, {2, 325}, {3, 39}, {5, 11},
> {7, 11}, {11, 3}, {13, 2}}
```

The $16k^2 \times 16k^2$ determinant

In fact, we can prove:

$$\det(A(k)) = (-1)^{k-1} 2^{32k^3+24k^2+2k-1} k^{8k^2+2k} ((4k+1)!)^{4k} \frac{(8k)!}{(4k)!} \prod_{j=1}^{4k} \frac{(2j)!}{j!}.$$

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In fact, we can prove:

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Thus:

Theorem

For all $k \ge 1$ there is a formula

$$\pi = \sum_{n=0}^{\infty} \frac{S_k(n)}{\binom{8kn}{4kn}(-4)^{kn}},$$

where $S_k(n)$ is a polynomial in n of degree $4k$ with rational coefficients. The polynomial $S_k(n)$ can be found by solving the previously described system of linear equations.

The $16k^2 \times 16k^2$ determinant

How to evaluate this determinant?

“Method” 0. Do row and column operations until the determinant reduces to something manageable.

Determinant evaluations

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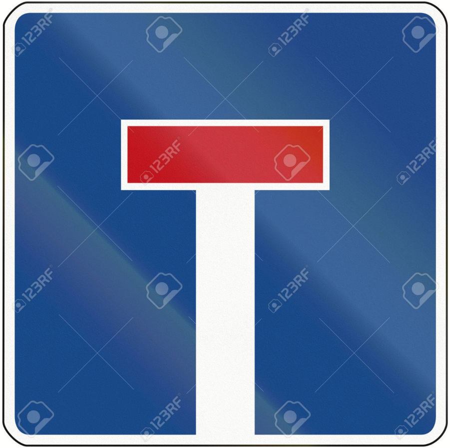

Method 1. Take out as many factors as possible until something polynomial remains. Then match with lemmas in Section 2 of ADC I.

EXAMPLE.

$$\det_{1\le i,j\le n}\left(\binom{s+i-1}{t+j-1}\right)$$

Determinant evaluations

EXAMPLE.

$$\det_{1\le i,j\le n}\left(\binom{s+i-1}{t+j-1}\right)=\det_{1\le i,j\le n}\left(\frac{(s+i-1)!}{(t+j-1)!\,(s-t+i-j)!}\right)$$

$$=\prod_{i=1}^{n}\frac{(s+i-1)!}{(t+i-1)!\,(s-t+i-1)!}$$

$$\times\det_{1\le i,j\le n}\left(\underbrace{(s-t+i-j+1)(s-t+i-j+2)\cdots(s-t+i-1)}_{p_j(i)}\right).$$

Determinant evaluations

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$$=\prod_{i=1}^{n}\frac{(s+i-1)!}{(t+i-1)!\,(s-t+i-1)!}$$

$$\times\det_{1\le i,j\le n}\left(\underbrace{(s-t+i-j+1)(s-t+i-j+2)\cdots(s-t+i-1)}_{p_j(i)}\right).$$

Proposition 1 in ADC I

Let $X_1, X_2, \dots, X_n$ be indeterminates. If $p_1, p_2, \dots, p_n$ are polynomials of the form $p_j(x) = a_j x^{j-1}$ + lower terms, then

$$\det_{1\le i,j\le n}\left(p_j(X_i)\right)=a_1a_2\cdots a_n\prod_{1\le i<j\le n}(X_j-X_i).$$

What about our determinant?

Determinant evaluations

What about our determinant?

Method 2. LU-FACTORISATION.

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Suppose we are given a family of matrices $A(1), A(2), A(3), \ldots$ of which we want to compute the determinants.
Suppose further that we can write

$$A(k) \cdot U(k) = L(k),$$

where $U(k)$ is an upper triangular matrix with 1s on the diagonal, and where $L(k)$ is a lower triangular matrix.
Then

$$\det(A(k)) = \text{product of the diagonal entries of } L(k).$$

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Then

$$\det(A(k)) = \text{product of the diagonal entries of } L(k).$$

But how do we find $U(k)$ and $L(k)$?
We go to the computer, crank out $U(k)$ and $L(k)$ for $k = 1, 2, 3, \dots$, until we are able to make a guess. Afterwards we prove the guess by proving the corresponding identities.

Determinant evaluations

?

Method 3. CONDENSATION.

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This is based on a determinant formula due to Jacobi.

Let A be an $n \times n$ matrix. Let $A^{j_1,j_2,\dots,j_\ell}_{i_1,i_2,\dots,i_\ell}$ denote the submatrix of A in which rows $i_1, i_2, \dots, i_\ell$ and columns $j_1, j_2, \dots, j_\ell$ are omitted. Then

$$\det A \cdot \det A^{1,n}_{1,n} = \det A^1_1 \cdot \det A^n_n - \det A^n_1 \cdot \det A^1_n.$$

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$$\det A \cdot \det A_{1,n}^{1,n} = \det A_1^1 \cdot \det A_n^n - \det A_1^n \cdot \det A_n^1.$$

If we consider a family of matrices $A(1), A(2), \dots,$ and if all the consecutive minors of $A(n)$ belong to the same family, then this allows one to give an inductive proof of a conjectured determinant evaluation for $A(n)$.

EXAMPLE. Consider

$$M_n(b,c) := \det_{1\le i,j\le n}\left(\binom{b+c}{b-i+j}\right) \overset{?}{=} \prod_{i=1}^{n}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}.$$

Determinant evaluations

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Then we have

$$\begin{aligned}\left(M_n(b,c)\right)_n^n &= M_{n-1}(b,c),\\ \left(M_n(b,c)\right)_1^1 &= M_{n-1}(b,c),\\ \left(M_n(b,c)\right)_n^1 &= M_{n-1}(b+1,c-1),\\ \left(M_n(b,c)\right)_1^n &= M_{n-1}(b-1,c+1),\\ \left(M_n(b,c)\right)_{1,n}^{1,n} &= M_{n-2}(b,c).\end{aligned}$$

Method 4. IDENTIFICATION OF FACTORS.

Determinant evaluations

A short proof of the Vandermonde determinant evaluation

$$\det_{1\le i,j\le n}\left(X_i^{j-1}\right)=\prod_{1\le i<j\le n}(X_j-X_i).$$

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PROOF.

- If $X_{i_1} = X_{i_2}$ with $i_1 \neq i_2$, then the determinant vanishes. Hence,

$$\prod_{1\le i<j\le n}(X_j - X_i) \text{ divides } \det_{1\le i,j\le n}\left(X_i^{j-1}\right)$$

as a polynomial in $X_1, X_2, \dots, X_n$.

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  as a polynomial in $X_1,X_2,\dots,X_n$.

- The degree of the product is $\binom{n}{2}$.
  The degree of the determinant is at most $\binom{n}{2}$.
  Consequently,

$$\det_{1\le i,j\le n}\left(X_i^{j-1}\right)=\text{const.}\times\prod_{1\le i<j\le n}(X_j-X_i).$$

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  The degree of the determinant is at most $\binom{n}{2}$.
  Consequently,

$$\det_{1\le i,j\le n}\left(X_i^{j-1}\right)=\text{const.}\times\prod_{1\le i<j\le n}(X_j-X_i).$$

- One can compute the constant by comparing coefficients of $X_1^0X_2^1\cdots X_n^{n-1}$ on both sides. □

Determinant evaluations

What are the essential steps?

Determinant evaluations

What are the essential steps?

(1) Identification of factors
(2) Comparison of degrees
(3) Evaluation of the constant

Objection: This works because there are so many (to be precise: n) variables at our disposal.

What, if there is, say, only one *variable* μ*, and you want to prove that* $(\mu + n)^E$ *divides the determinant?*

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What, if there is, say, only one *variable μ, and you want to prove that $(\mu+n)^E$ divides the determinant?*

Example.

$$\det_{0\le i,j\le n-1}\left(\binom{\mu+i+j}{2i-j}\right)$$
$$=(-1)^{\chi(n\equiv 3 \mod 4)}2^{\binom{n-1}{2}}\prod_{i=1}^{n-1}\frac{(\mu+i+1)_{\lfloor (i+1)/2\rfloor}\left(-\mu-3n+i+\frac{3}{2}\right)_{\lfloor i/2\rfloor}}{(i)_i},$$

where $\chi(\mathcal{A})=1$ if $\mathcal{A}$ is true and $\chi(\mathcal{A})=0$ otherwise, and where the *shifted factorial* $(a)_k$ is defined by
$(a)_k:=a(a+1)\cdots(a+k-1)$, $k\ge 1$, and $(a)_0:=1$.

Important fact:

For proving that $(\mu + n)^E$ *divides the determinant, we put* $\mu = -n$ *in the matrix and find* E *linearly independent vectors in the kernel of the matrix.*

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So, put $\mu = -n$ in the matrix, and compute the kernel for $n = 2, 3, \ldots$:

Determinant evaluations

Important fact:

> *For proving that $(\mu + n)^E$ divides the determinant, we put $\mu = -n$ in the matrix and find E linearly independent vectors in the kernel of the matrix.*

So, put $\mu = -n$ in the matrix, and compute the kernel for $n = 2, 3, \dots$:

```
In[1]:= V[2]
Out[1]= {0, c[1]}
In[2]:= V[3]
Out[2]= {0, c[2], c[2]}
In[3]:= V[4]
Out[3]= {0, c[1], 2 c[1], c[1]}
In[4]:= V[5]
Out[4]= {0, c[1], 3 c[1], c[3], c[1]}
In[5]:= V[6]
Out[5]= {0, c[1], 4 c[1], 2 c[1] + c[4], c[4], c[1]}
In[6]:= V[7]
```

Determinant evaluations

```
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In[6]:= V[7]
Out[6]= {0, c[1], 5 c[1], c[3], -10 c[1] + 2 c[3], -5 c[1]
+ c[3], c[1]}
In[7]:= V[8]
Out[7]= {0, c[1], 6 c[1], c[3], -25 c[1] + 3 c[3], c[5],
-9 c[1] + c[3], c[1]}
In[8]:= V[9]
Out[8]= {0, c[1], 7 c[1], c[3], -49 c[1] + 4 c[3],
-28 c[1] + 2 c[3] + c[6], c[6], -14 c[1] + c[3], c[1]}
```

Apparently, if we put $\mu = -n$ in the n-th matrix, M_n say, then

- the vector $(0,1)$ is in the kernel of M_2,
- the vector $(0,1,1)$ is in the kernel of M_3,
- the vector $(0,1,2,1)$ is in the kernel of M_4,
- the vector $(0,1,3,3,1)$ is in the kernel of M_5 (set $c[1]=1$ and $c[3]=3$),
- the vector $(0,1,4,6,4,1)$ is in the kernel of M_6 (set $c[1]=1$ and $c[4]=4$), etc.

Apparently,

$$\left(0, \binom{n-2}{0}, \binom{n-2}{1}, \binom{n-2}{2}, \dots, \binom{n-2}{n-2}\right)$$

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the vector $(0, 1, 2, 1)$ is in the kernel of M_4,

the vector $(0, 1, 3, 3, 1)$ is in the kernel of M_5 (set $c[1] = 1$ and $c[3] = 3$),

the vector $(0, 1, 4, 6, 4, 1)$ is in the kernel of M_6 (set $c[1] = 1$ and $c[4] = 4$), etc.

Apparently,

$$\left(0, \binom{n-2}{0}, \binom{n-2}{1}, \binom{n-2}{2}, \dots, \binom{n-2}{n-2}\right)$$

is in the kernel of M_n.

And the pattern persists!

Determinant evaluations

```
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```

Something remains to be proved: we must verify that these vectors are indeed in the kernel!

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In our case this amounts to the verification of binomial sums: we need to verify that

$$\sum_{j=1}^{n-1} \binom{n-2}{j-1} \binom{-n+i+j}{2i-j} = 0$$

for $i = 0, 1, \dots, n-1$.

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for $i = 0, 1, \ldots, n-1$.

As we know, nowadays this can be routinely done by using the algorithmic tools that are available (here: the *Gosper–Zeilberger algorithm*).

Back to our determinant

$$A(k) := \begin{pmatrix} 0\dots0* & 0\dots0* & 0\dots0* & \dots & \dots & \dots & 0\dots0* \\ F_1 & 0 & 0 & \dots & \dots & \dots & 0 \\ G_1 & F_2 & 0 & \dots & \dots & \dots & 0 \\ 0 & G_2 & F_3 & & & & \vdots \\ 0 & 0 & G_3 & & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & F_{4k-1} & 0 \\ \vdots & & & 0 & 0 & G_{4k-1} & F_{4k} \\ 0 & \dots & \dots & \dots & 0 & 0 & G_{4k} \end{pmatrix}.$$

Here, the F_i's and G_i's are $(4k-1) \times (4k)$ blocks.

Back to our determinant

with

$$\begin{aligned} f_0(j) &= j(-4)^k, \\ f_1(j) &= -(4j+2)(-4)^k, \\ g_0(j) &= (N(k)-j), \\ g_1(j) &= -(4N(k)-4j-2), \end{aligned}$$

where, as before, $N(k) = 4k(4k+1)$.

Back to our determinant

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$$f_1(j) = -(4j+2)(-4)^k,$$
$$g_0(j) = (N(k) - j),$$
$$g_1(j) = -(4N(k) - 4j - 2),$$

where, as before, $N(k) = 4k(4k+1)$.

with

$$\begin{aligned} f_0(j) &= j(-4)^k, \\ f_1(j) &= -(4j+2)(-4)^k, \\ g_0(j) &= (\quad X \quad - j), \\ g_1(j) &= -(4 \quad X \quad - 4j - 2). \end{aligned}$$

Back to our determinant

```
In[4]:=
f0[k_,i_,j_]:=j(-4)^k
f1[k_,i_,j_]:=-(2+4j)(-4)^k
g0[k_,i_,j_]:=(X-j)
g1[k_,i_,j_]:=(-4*X+2+4j)
```

Back to our determinant

```
In[4]:=
f0[k_,i_,j_]:=j(-4)^k
f1[k_,i_,j_]:=-(2+4j)(-4)^k
g0[k_,i_,j_]:=(X-j)
g1[k_,i_,j_]:=(-4*X+2+4j)
```

```
In[5]:= Factor[Det[A[2]]]
```

Back to our determinant

```
In[4]:=
f0[k_,i_,j_]:=j(-4)^k
f1[k_,i_,j_]:=-(2+4j)(-4)^k
g0[k_,i_,j_]:=(X-j)
g1[k_,i_,j_]:=(-4*X+2+4j)

In[5]:= Factor[Det[A[2]]]

Out[5]= -140639960847488232315491052598657851591838\
> 6968104151763611783762359972003840000000 (-64 + X)
                               2           3           4           5
> (-48 + X) (-40 + X) (-32 + X) (-24 + X) (-16 + X)
             6  7
> (-8 + X) X (9653078694297600 - 916000657637376 X +
                          2                   3                  4
> 36130368757760 X - 758218948608 X + 8928558848 X -
              5            6
> 55938432 X + 145673 X )
```

Back to our determinant

with

$$f_0(j) = j(-4)^k,$$
$$f_1(j) = -(4j+2)(-4)^k,$$
$$g_0(j) = (X - j),$$
$$g_1(j) = -(4X - 4j - 2).$$

with

$$f_0(j) = (\qquad\qquad + j)(-4)^k,$$
$$f_1(j) = -(\qquad\qquad + 4j + 2)(-4)^k,$$
$$g_0(j) = (X - j),$$
$$g_1(j) = -(4X - 4j - 2).$$

with

$$\begin{aligned}
f_0(j) &= (N(k) - X + j)(-4)^k,\\
f_1(j) &= -(4N(k) - 4X + 4j + 2)(-4)^k,\\
g_0(j) &= (X - j),\\
g_1(j) &= -(4X - 4j - 2).
\end{aligned}$$

Back to our determinant

```
In[6]:=
f0[k_,i_,j_]:=(4*k*(4*k+1)-X+j)(-4)^k
f1[k_,i_,j_]:=-(4*4*k*(4*k+1)-4*X+2+4j)(-4)^k
g0[k_,i_,j_]:=(X-j)
g1[k_,i_,j_]:=(-4*X+2+4j)
```

Back to our determinant

```
In[6]:=
f0[k_,i_,j_]:=(4*k*(4*k+1)-X+j)(-4)^k
f1[k_,i_,j_]:=-(4*4*k*(4*k+1)-4*X+2+4j)(-4)^k
g0[k_,i_,j_]:=(X-j)
g1[k_,i_,j_]:=(-4*X+2+4j)

In[7]:= Factor[Det[A[2]]]
```

Back to our determinant

```
In[6]:=
f0[k_,i_,j_]:=(4*k*(4*k+1)-X+j)(-4)^k
f1[k_,i_,j_]:=-(4*4*k*(4*k+1)-4*X+2+4j)(-4)^k
g0[k_,i_,j_]:=(X-j)
g1[k_,i_,j_]:=(-4*X+2+4j)

In[7]:= Factor[Det[A[2]]]

Out[7]= -2967779753976246799013698097944121044541 34\
> 7634940708411155365196124754770317472271790417634\
> 937439881166252558632616674197504000000000
> (-141 + 2 X) (-139 + 2 X) (-137 + 2 X) (-135 + 2 X)
> (-133 + 2 X) (-131 + 2 X) (-129 + 2 X)
```

with

$$
\begin{aligned}
f_0(j) &= (N(k) - X \quad + j)(-4)^k,\\
f_1(j) &= -(4N(k) - 4X \quad + 4j + 2)(-4)^k,\\
g_0(j) &= (X \quad - j),\\
g_1(j) &= -(4X \quad - 4j - 2).
\end{aligned}
$$

Back to our determinant

with

$$f_0(j) = (N(k) - X_2 + j)(-4)^k,$$
$$f_1(j) = -(4N(k) - 4X_1 + 4j + 2)(-4)^k,$$
$$g_0(j) = (X_2 - j),$$
$$g_1(j) = -(4X_1 - 4j - 2).$$

Back to our determinant

```
In[8]:=
f0[k_,i_,j_]:=(4*k*(4*k+1)-X[2]+j)(-4)^k
f1[k_,i_,j_]:=-(4*4*k*(4*k+1)-4*X[1]+2+4j)(-4)^k
g0[k_,i_,j_]:=(X[2]-j)
g1[k_,i_,j_]:=(-4*X[1]+2+4j)
```

Back to our determinant

```
In[8]:=
f0[k_,i_,j_]:=(4*k*(4*k+1)-X[2]+j)(-4)^k
f1[k_,i_,j_]:=-(4*4*k*(4*k+1)-4*X[1]+2+4j)(-4)^k
g0[k_,i_,j_]:=(X[2]-j)
g1[k_,i_,j_]:=(-4*X[1]+2+4j)
```

```
In[9]:= Factor[Det[A[1]]]
```

Back to our determinant

```
In[8]:=
f0[k_,i_,j_]:=(4*k*(4*k+1)-X[2]+j)(-4)^k
f1[k_,i_,j_]:=-(4*4*k*(4*k+1)-4*X[1]+2+4j)(-4)^k
g0[k_,i_,j_]:=(X[2]-j)
g1[k_,i_,j_]:=(-4*X[1]+2+4j)
```

```
In[9]:= Factor[Det[A[1]]]
```

```
Out[9]= 32425917317067571200000 (-37 + 2 X[1])

                                                                 3
> (-35 + 2 X[1]) (-33 + 2 X[1]) (1 + 2 X[1] - 2 X[2])
                            2
> (3 + 2 X[1] - 2 X[2]) (5 + 2 X[1] - 2 X[2])
```

with

$$\begin{aligned}
f_0(j) &= (\ N(k) + j \quad - X_2)(-4)^k,\\
f_1(j) &= -(\ 4N(k) + 4j + 2 \quad - 4X_1)(-4)^k,\\
g_0(j) &= (X_2 - j \quad),\\
g_1(j) &= -(4X_1 - \ 4j - 2 \quad).
\end{aligned}$$

Back to our determinant

with

$$
\begin{aligned}
f_0(j) &= ((N(k)+j)Y - X_2)(-4)^k,\\
f_1(j) &= -((4N(k)+4j+2)Y - 4X_1)(-4)^k,\\
g_0(j) &= (X_2 - j\cdot Y),\\
g_1(j) &= -(4X_1 - (4j+2)Y).
\end{aligned}
$$

Back to our determinant

```
In[10]:=
f0[k_,i_,j_]:=(4*k*(4*k+1)Y-X[2]+j*Y)(-4)^k
f1[k_,i_,j_]:=-(4*4*k*(4*k+1)Y-4*X[1]+(2+4j)Y)(-4)^k
g0[k_,i_,j_]:=(X[2]-j*Y)
g1[k_,i_,j_]:=(-4*X[1]+(2+4j)Y)
```

Back to our determinant

```
In[10]:=
f0[k_,i_,j_]:=(4*k*(4*k+1)Y-X[2]+j*Y)(-4)^k
f1[k_,i_,j_]:=-(4*4*k*(4*k+1)Y-4*X[1]+(2+4j)Y)(-4)^k
g0[k_,i_,j_]:=(X[2]-j*Y)
g1[k_,i_,j_]:=(-4*X[1]+(2+4j)Y)
In[11]:= Factor[Det[A[1]]]
```

Back to our determinant

```
In[10]:=
f0[k_,i_,j_]:=(4*k*(4*k+1)Y-X[2]+j*Y)(-4)^k
f1[k_,i_,j_]:=-(4*4*k*(4*k+1)Y-4*X[1]+(2+4j)Y)(-4)^k
g0[k_,i_,j_]:=(X[2]-j*Y)
g1[k_,i_,j_]:=(-4*X[1]+(2+4j)Y)

In[11]:= Factor[Det[A[1]]]

                                 6
Out[11]= -324259173170675712000 Y  (33 Y - 2 X[1])
> (35 Y - 2 X[1]) (37 Y - 2 X[1])
                          3                        2
> (Y + 2 X[1] - 2 X[2]) (3 Y + 2 X[1] - 2 X[2])
> (5 Y + 2 X[1] - 2 X[2])
```

Back to our determinant

with

$$f_0(j) = ((N(k)+j)Y \quad - X_2 \quad)(-4)^k,$$
$$f_1(j) = -((4N(k)+4j+2)Y \quad - 4X_1 \quad)(-4)^k,$$
$$g_0(j) = (X_2 \quad - j \cdot Y \quad),$$
$$g_1(j) = -(4X_1 \quad - (4j+2)Y \quad).$$

with

$$
\begin{aligned}
f_0(j) &= ((N(k)+j)Y_\ell - X_{2,\ell})(-4)^k,\\
f_1(j) &= -((4N(k)+4j+2)Y_\ell - 4X_{1,\ell})(-4)^k,\\
g_0(j) &= (X_{2,\ell} - j\cdot Y_\ell),\\
g_1(j) &= -(4X_{1,\ell} - (4j+2)Y_\ell).
\end{aligned}
$$

Back to our determinant

```
In[10]:=
f0[k_,i_,j_]:=(4*k*(4*k+1)Y[i]-X[2,i]+j*Y[i])(-4)^k
f1[k_,i_,j_
]:=-(4*4*k*(4*k+1)Y[i]-4*X[1,i]+(2+4j)Y[i])(-4)^k
g0[k_,i_,j_]:=(X[2,i]-j*Y[i])
g1[k_,i_,j_]:=(-4*X[1,i]+(2+4j)Y[i])
```

Back to our determinant

```
In[10]:=
f0[k_,i_,j_]:=(4*k*(4*k+1)Y[i]-X[2,i]+j*Y[i])(-4)^k
f1[k_,i_,j_
]:=-(4*4*k*(4*k+1)Y[i]-4*X[1,i]+(2+4j)Y[i])(-4)^k
g0[k_,i_,j_]:=(X[2,i]-j*Y[i])
g1[k_,i_,j_]:=(-4*X[1,i]+(2+4j)Y[i])
```

```
In[11]:= Factor[Det[A[1]]]
```

Back to our determinant

```
In[10]:=
f0[k_,i_,j_]:=(4*k*(4*k+1)Y[i]-X[2,i]+j*Y[i])(-4)^k
f1[k_,i_,j_
]:=-(4*4*k*(4*k+1)Y[i]-4*X[1,i]+(2+4j)Y[i])(-4)^k
g0[k_,i_,j_]:=(X[2,i]-j*Y[i])
g1[k_,i_,j_]:=(-4*X[1,i]+(2+4j)Y[i])
```

```
In[11]:= Factor[Det[A[1]]]
```

```
Out[11]= 324259173170675712000 (2 X[1, 1] - 33 Y[1])
> Y[1] (2 X[1, 1] - 2 X[2, 1] + Y[1]) (2 X[1, 2] -
> 35 Y[2]) Y[2] (2 X[1, 2] - 2 X[2, 2] + Y[2])
> (-2 X[2, 2] Y[1] + 2 X[1, 1] Y[2] + 3 Y[1] Y[2])
> (2 X[1, 3] - 37 Y[3]) Y[3] (2 X[1, 3] - 2 X[2, 3] +
> Y[3]) (-2 X[2, 3] Y[1] + 2 X[1, 1] Y[3] + 5 Y[1]
> Y[3]) (-2 X[2, 3] Y[2] + 2 X[1, 2] Y[3] + 3 Y[2]
> Y[3])
```

Back to our determinant

Apparently,

$$\det(A^{\text{general}}(k)) = (-1)^{k-1} 4^{2k(4k^2+7k+2)} k^{2k(4k+1)} \prod_{i=1}^{4k} (i+1)_{4k-i+1}$$
$$\times \prod_{a=1}^{4k-1} \left(2X_{1,a} - (32k^2 + 2a - 1) Y_a\right)$$
$$\times \prod_{1 \le a \le b \le 4k-1} (2X_{2,b} Y_a - 2X_{1,a} Y_b - (2b - 2a + 1) Y_a Y_b).$$

The special case that we need in the end to prove our theorem is $X_{1,\ell} = X_{2,\ell} = N(k)$ and $Y_\ell = 1$.

SKETCH OF PROOF.
(1) For each factor of the (conjectured) result, we find a linear combination of the rows which vanishes if the factor vanishes.

For example: the factor $(2X_{1,1} - (32k^2+1)Y_1)$.

If $X_{1,1} = \frac{32k^2+1}{2} Y_1$, then

$$\frac{2(X_{2,4k-1} - (N(k)-1)Y_{4k-1})}{(-4)^{k(4k+1)+1}(16k^2+1)\prod_{\ell=1}^{4k-1}(4\ell k+1)} \cdot (\text{row } 1)$$
$$+ \sum_{s=0}^{4k} \sum_{t=0}^{4k-2} \left(\frac{(-1)^{s(k-1)} 2^t}{4^{sk}} \prod_{\ell=0}^{s-1} \frac{4k-1+4\ell k}{16k^2+1-4\ell k} \right.$$
$$\left. \cdot \prod_{\ell=4k-t}^{4k-1} \frac{2X_{1,\ell} - (32k^2+2\ell-1)Y_\ell}{X_{2,\ell-1} - (16k^2+\ell-1)Y_{\ell-1}} \right)$$
$$\cdot (\text{row } (16k^2 - (4k-1)s - t)) = 0,$$

Back to our determinant

Sketch of proof (continued).
(2) The total degree in the $X_{1,\ell}$'s, $X_{2,\ell}$'s, Y_ℓ's of the product is $16k^2 - 1$.

The degree of the determinant is at most $16k^2 - 1$.

Hence,

$$\det = \text{const.} \times \text{product.}$$

Back to our determinant

Sketch of proof (continued).
(2) The total degree in the $X_{1,\ell}$'s, $X_{2,\ell}$'s, Y_ℓ's of the product is $16k^2 - 1$.

The degree of the determinant is at most $16k^2 - 1$.

Hence,

$$\det = \text{const.} \times \text{product}.$$

(3) Evaluation of the constant.

Compare coefficients of

$$X_{1,1}^{4k} X_{1,2}^{4k-1} \cdots X_{1,4k-1}^{2} Y_1^1 Y_2^2 \cdots Y_{4k-1}^{4k-1}.$$

As it turns out, the constant is equal to a determinant of the same form,

Back to our determinant

but with

$$f_0(j) = (N(k) + j)(-4)^k,$$
$$f_1(j) = 4(-4)^k,$$
$$g_0(j) = -j,$$
$$g_1(j) = -4.$$

Back to our determinant

but with

$$f_0(j) = (N(k) + j)(-4)^k,$$
$$f_1(j) = 4(-4)^k,$$
$$g_0(j) = -j,$$
$$g_1(j) = -4.$$

but with

$$\begin{aligned} f_0(j) &= (Z_\ell + j)(-4)^k, \\ f_1(j) &= 4(-4)^k X_\ell, \\ g_0(j) &= -j, \\ g_1(j) &= -4X_\ell. \end{aligned}$$

Back to our determinant

The computer says that, apparently,

$$\det A^{\text{const.}}(k) = (-1)^{k-1} 2^{16k^3+20k^2+14k-1} k^{4k} (4k+1)! \\ \times \prod_{a=1}^{4k-1} \left(X_a^{4k+1-a} \prod_{b=0}^{a-1} (Z_a - 4bk) \right).$$

Back to our determinant

The computer says that, apparently,

$$\det A^{\text{const.}}(k) = (-1)^{k-1} 2^{16k^3+20k^2+14k-1} k^{4k} (4k+1)! \\ \times \prod_{a=1}^{4k-1} \left(X_a^{4k+1-a} \prod_{b=0}^{a-1} (Z_a - 4bk) \right).$$

Okay. Let us apply the method of identification of factors again.

Back to our determinant

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Okay. Let us apply the method of identification of factors again.

(1) ✓

Back to our determinant

The computer says that, apparently,

$$\det A^{\text{const.}}(k) = (-1)^{k-1} 2^{16k^3+20k^2+14k-1} k^{4k} (4k+1)! \times \prod_{a=1}^{4k-1} \left(X_a^{4k+1-a} \prod_{b=0}^{a-1} (Z_a - 4bk) \right).$$

Okay. Let us apply the method of identification of factors again.

(1) ✓

(2) ✓

Back to our determinant

The computer says that, apparently,

$$\det A^{\text{const.}}(k) = (-1)^{k-1} 2^{16k^3+20k^2+14k-1} k^{4k} (4k+1)! \times \prod_{a=1}^{4k-1} \left(X_a^{4k+1-a} \prod_{b=0}^{a-1} (Z_a - 4bk) \right).$$

Okay. Let us apply the method of identification of factors again.

(1) ✓

(2) ✓

(3) The constant:

is again a determinant of the same form with

$$\begin{aligned} f_0(j) &= (-4)^k, \\ f_1(j) &= 4(-4)^k, \\ g_0(j) &= 0, \\ g_1(j) &= -4. \end{aligned}$$

Back to our determinant

is again a determinant of the same form with

$$f_0(j) = (-4)^k,$$
$$f_1(j) = 4(-4)^k,$$
$$g_0(j) = 0,$$
$$g_1(j) = -4.$$

is again a determinant of the same form with

$$f_0(j) = (-4)^k,$$
$$f_1(j) = 4(-4)^k,$$
$$g_0(j) = 0,$$
$$g_1(j) = -4.$$

For this one "Method" 0 (row and column manipulations) works!

Back to our determinant

Hence:

Theorem

We have

$$\det\left(A^{\text{general}}(k)\right) = (-1)^{k-1} 4^{2k(4k^2+7k+2)} k^{2k(4k+1)} \prod_{i=1}^{4k} (i+1)_{4k-i+1}$$
$$\times \prod_{a=1}^{4k-1} \left(2X_{1,a} - (32k^2 + 2a - 1)Y_a\right)$$
$$\times \prod_{1\le a\le b\le 4k-1} (2X_{2,b}Y_a - 2X_{1,a}Y_b - (2b - 2a + 1)Y_aY_b).$$

Back to our determinant

Corollary

We have

$$\det(A(k)) = (-1)^{k-1} 2^{32k^3+24k^2+2k-1} k^{8k^2+2k} ((4k+1)!)^{4k} \times \frac{(8k)!}{(4k)!} \prod_{j=1}^{4k} \frac{(2j)!}{j!}.$$

Back to our determinant

Corollary

We have

$$\det(A(k)) = (-1)^{k-1} 2^{32k^3+24k^2+2k-1} k^{8k^2+2k} ((4k+1)!)^{4k} \times \frac{(8k)!}{(4k)!} \prod_{j=1}^{4k} \frac{(2j)!}{j!}.$$

This is $\neq 0$!

Back to our determinant

Theorem

For all $k \geq 1$ there is a formula

$$\pi = \sum_{n=0}^{\infty} \frac{S_k(n)}{\binom{8kn}{4kn}(-4)^{kn}},$$

where $S_k(n)$ is a polynomial in n of degree $4k$ with rational coefficients.

Back to our determinant

Theorem

For all $k \geq 1$ there is a formula

$$\pi = \sum_{n=0}^{\infty} \frac{S_k(n)}{\binom{8kn}{4kn}(-4)^{kn}},$$

where $S_k(n)$ is a polynomial in n of degree $4k$ with rational coefficients.