Srinivasa Ramanujan
Life and Mathematics

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Srinivasa Ramanujan
(1887 – 1920)

Ramanujan's mathematical heritage

Ramanujan's interests include *infinite series, integrals, asymptotic expansions and approximations, gamma function, hypergeometric and q-hypergeometric functions, continued fractions, theta functions, class invariants, Diophantine equations, congruences, magic squares*.

- 3 Notebooks
- 37 published mathematical papers
  (J. Indian Math. Soc., Proc. London Math. Soc., Proc. Cambridge Philos. Soc., Proc. Cambridge Philos. Soc., Proc. Royal Soc., Messenger Math., Quart. J. Math.)
- the "Lost Notebook"

Ramanujan reaches his hand from his grave to snatch your theorems from you . . .

(Bill Gosper)

The Rogers–Ramanujan Identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty\,(q^4;q^5)_\infty}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty\,(q^3;q^5)_\infty},$$

where $(\alpha;q)_n := (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1})$, $n \geq 1$, and $(\alpha;q)_0 := 1$.

Partition Congruences

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$n = 1$: 1
$n = 2$: 2, $1 + 1$
$n = 3$: 3, $2 + 1$, $1 + 1 + 1$
$n = 4$: 4, $3 + 1$, $2 + 2$, $2 + 1 + 1$, $1 + 1 + 1 + 1$
$n = 5$: 5, $4 + 1$, $3 + 2$, $3 + 1 + 1$, $2 + 2 + 1$, $2 + 1 + 1 + 1$, $1 + 1 + 1 + 1 + 1$

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$n = 3$: $3,\ 2+1,\ 1+1+1$
$n = 4$: $4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1$
$n = 5$: $5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,$
$1+1+1+1+1$

Let $p(n)$ denote the number of partitions of n.

For example, $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(4) = 5$, $p(5) = 7$.

Partition Congruences

$$\begin{array}{ccccc}
p(1)=1 & p(2)=2, & p(3)=3, & p(4)=5, & p(5)=7 \\
p(6)=11 & p(7)=15, & p(8)=22, & p(9)=30, & p(10)=42 \\
p(11)=56 & p(12)=77, & p(13)=101, & p(14)=135, & p(15)=176 \\
p(16)=231 & p(17)=297, & p(18)=385, & p(19)=490, & p(20)=627
\end{array}$$

By studying MacMahon's (hand-calculated!!) table of the partition numbers $p(n)$ for $n = 1, 2, \ldots, 200$, Ramanujan observed, and then proved, his famous partition congruences.

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Theorem (Ramanujan 1919)

For all non-negative integers n,

$$p(5n+4) \equiv 0 \pmod 5,$$
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Moreover, he conjectured

$$p(5^a 7^b 11^c n + \lambda) \equiv 0 \pmod{5^a 7^b 11^c},$$

where $24\lambda \equiv 1 \pmod{5^a 7^b 11^c}$.

"This theorem is supported by all evidence; but I have not yet been able to find a general proof."

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S. Chowla noticed around 1930 (from the extended tables of H. Gupta) that

$$p(243) = 133978259344888 \not\equiv 0 \pmod{7^3},$$

but $24 \cdot 243 \equiv 1 \pmod{7^3}$.

Partition Congruences

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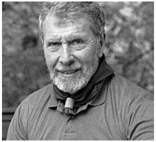
(Arthur Oliver Lonsdale Atkin) 

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Theorem (Euler)

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots}.$$

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Proof:

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\frac{1}{(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots} &= \\
&= \frac{1}{1-q}\cdot\frac{1}{1-q^2}\cdot\frac{1}{1-q^3}\cdot\frac{1}{1-q^4}\cdots \\
&= (1+q^1+q^{1+1}+q^{1+1+1}+\cdots) \\
&\qquad \cdot(1+q^2+q^{2+2}+q^{2+2+2}+\cdots) \\
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Proof of Ramanujan's congruence mod 5:

The following auxiliary results can be derived using *Jacobis triple product formula*

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n-1)/2}x^n=(q;q)_\infty\,(x;q)_\infty\,(q/x;q)_\infty.$$

Lemma

Let $\omega^5=1$, $\omega\neq 1$. *Then*

$$(q;q)_\infty(\omega q;\omega q)_\infty(\omega^2 q;\omega^2 q)_\infty(\omega^3 q;\omega^3 q)_\infty(\omega^4 q;\omega^4 q)_\infty \\ =\frac{(q^5;q^5)_\infty^6}{(q^{25};q^{25})_\infty}.$$

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$$\frac{(q;q)_\infty}{q(q^{25};q^{25})_\infty} = q^{-1}R - 1 - qR^{-1},$$

where R is a power series in q^5.

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Lemma

$$q^{-5}R^5 - 11 - q^5R^{-5} = \frac{(q^5;q^5)_\infty^6}{q^5(q^{25};q^{25})_\infty^6}.$$

Proof of Ramanujan's congruence mod 5:

$$\begin{aligned}
&1+p(1)q+p(2)q^2+p(3)q^3+p(4)q^4+p(5)q^5\\
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&\qquad+p(11)q^{11}+p(12)q^{12}+p(13)q^{13}+p(14)q^{14}+\dots\\
&=q^4\frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6}\cdot\big(q^{-4}R^4+q^{-3}R^3+2q^{-2}R^2+3q^{-1}R+5\\
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5

Proof of Ramanujan's congruence mod 5:

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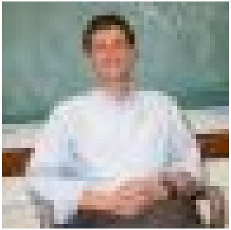
ATKIN found in 1969 that

$$p(206839n + 2623) \equiv 0 \pmod{17}.$$

(We have $206839 = 17 \cdot 23^3$.) More congruences were found over the years.



Scott Ahlgren

Ken Ono

Theorem (Ahlgren and Ono 2001)

If M is coprime to 6, then there are infinitely many non-nested arithmetic progressions $An + B$ for which

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Examples.

$$\begin{aligned} p(48037937n + 1122838) &\equiv 0 \pmod{17}, \\ p(1977147619n + 815655) &\equiv 0 \pmod{19}, \\ p(14375n + 3474) &\equiv 0 \pmod{23}, \\ p(348104768909n + 43819835) &\equiv 0 \pmod{29}, \\ p(4063467631n + 30064597) &\equiv 0 \pmod{31}. \end{aligned}$$

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Matthew Boylan

Theorem (Ahlgren and Boylan 2003)

If ℓ is a prime for which there is a congruence of the form

$$p(\ell n + b) \equiv 0 \pmod{\ell},$$

then $\ell = 5$, 7, or 11.

Freeman Dyson, *Some guesses in the theory of partitions*, Eureka **8** (1944), 10–15.

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Professor Littlewood, when he makes use of an algebraic identity, always saves himself the trouble of proving it; he maintains that an identity, if true, can be verified in a few lines by anybody obtuse enough to feel the need of verification. My object in the following pages is to confute this assertion.

Partition Congruences

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It would be satisfying to have a direct proof of (4). *By this I mean, that although we can prove [...] that the partitions of* $5n+4$ *can be divided into five equally numerous subclasses, it is unsatisfactory to receive from the proofs no concrete idea of how the division to be made. We require a proof which will not appeal to generating functions, but will demonstrate by cross-examination of the partitions themselves the existence of five exclusive, exhaustive and equally numerous subclasses. In the following, I shall not give such a proof, but I shall take the first step towards it.*

Dyson defines the *rank* of a partition as the largest part minus the number of parts of the partition.

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For example,

$$\begin{aligned} \operatorname{rk}(4) &= 4-1 = 3 \\ \operatorname{rk}(3+1) &= 3-2 = 1 \\ \operatorname{rk}(2+2) &= 2-2 = 0 \\ \operatorname{rk}(2+1+1) &= 2-3 = -1 \\ \operatorname{rk}(1+1+1+1) &= 1-4 = -3 \end{aligned}$$

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He conjectured that the rank "explains" Ramanujan's congruences modulo 5 and 7, in the sense that the partitions of $5n+4$ are subdivided into 5 subclasses when one considers the rank of these partitions modulo 5, with an analogous conjecture for the partitions of $7n+5$.

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\mathrm{rk}(2+1+1) &= 2-3 = -1 \equiv 4 \pmod 5,\\
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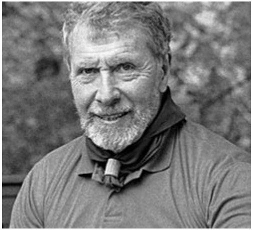
Freeman Dyson, *Some guesses in the theory of partitions*, Eureka **8** (1944), 10–15.

After a few preliminaries, I state certain properties which I am unable to prove: these guesses are then transformed into algebraic identities which are also unproved, although there is conclusive numerical evidence in their support; finally, I indulge in some even vaguer guesses concerning the existence of identities which I am not only unable to prove but also unable to state. I think that this should be enough to disillusion anyone who takes Professor Littlewood's innocent view of the difficulties of algebra. Needless to say, I strongly recommend my readers to supply the missing proofs, or, even better, the missing identities.



Arthur Oliver Lonsdale Atkin

Henry Peter Francis Swinnerton-Dyer

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In the proof, the *rank generating function*

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m,n) w^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(wq;q)_n \, (w^{-1}q;q)_n},$$

where

$$N(m,n) := |\{\text{partitions of } n \text{ with rank } m\}|,$$

and the *theory of modular forms* plays a predominant role.

Dyson's rank conjectures for the partitions of $5n+4$ and $7n+5$ were proved in 1954 by Atkin and Swinnerton-Dyer.

In the proof, the *rank generating function*

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m,n) w^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(wq;q)_n \, (w^{-1}q;q)_n},$$

where

$$N(m,n) := |\{\text{partitions of } n \text{ with rank } m\}|,$$

and the *theory of modular forms* plays a predominant role.

On the other hand, already Dyson noticed that his rank fails to "explain" Ramanujan's congruence modulo 11. In order to remedy this, he proposed that there should be a new, but at this point unknown statistic, the *crank*, which would do the trick.

Freeman Dyson, *Some guesses in the theory of partitions*, Eureka **8** (1944), 10–15.

I hold in fact: That there exists an arithmetical coefficient similar to, but more recondite than, the rank of a partition; I shall call this hypothetical coefficient the "crank" of the partition [...] Whether these guesses are warranted by the evidence, I leave to the reader to decide. Whatever the final verdict of posteriority may be, I believe the "crank" is unique among arithmetical functions in having been named before it was discovered. May it be preserved from the ignominious fate of the planet Vulcan!

George Andrews

Frank Garvan

In 1987, Garvan found the *crank*. He defined it as follows. Let λ be the partition

$$\lambda_1 + \lambda_2 + \cdots + \lambda_s + 1 + \cdots + 1$$

be a partition with r ones. Then

$$\operatorname{crank}(\lambda) := \begin{cases} \lambda_1, & \text{if } r = 0, \\ o(\lambda) - r, & \text{if } r \geq 1, \end{cases}$$

where $o(\lambda)$ is the number of parts of λ that are strictly larger than λ.

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Together with Andrews he proved that the crank "explains" *all* of Ramanujan's congruences modulo 5, 7, 11, in the sense that it divides the corresponding partitions into subclasses of equal sizes.

Let $M(m, N, n)$ be the number of partitions λ of n for which

$$\text{crank}(\lambda) \equiv m \pmod{N}.$$

(Karl Mahlburg)

Theorem (Mahlburg 2005)

Suppose that $\ell \geq 5$ is prime and that τ and j are positive integers. Then there are infinitely many non-nested arithmetic progressions $An + B$ such that

$$M(m, \ell^j, An + B) \equiv 0 \pmod{\ell^\tau}$$

simultaneously for every $0 \leq m \leq \ell^j - 1$.

This theorem is a refinement of the earlier result of Ahlgren and Ono, which it implies.

Mock Theta Functions

"I am extremely sorry for not writing you a single letter up to now . . . I discovered very interesting functions recently which I call Mock θ-functions. Unlike the False θ-functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples."

(Ramanujan in his last letter to Hardy, 1920)

"... Suppose there is a function in the Eulerian form and suppose that all or an infinity of points $q = e^{2i\pi m/n}$ are exponential singularities and also suppose that at these points the asymptotic form of the function closes as neatly as in the cases of (A) and (B). The question is: — is the function taken the sum of two functions one of which is an ordinary theta function and the other a (trivial) function which is $O(1)$ at all the points $e^{2i\pi m/n}$? The answer is it is not necessarily so. When it is not so I call the function Mock ϑ-function. I have not proved rigorously that it is not necessarily so. But I have constructed a number of examples in which it is inconceivable to construct a ϑ-function to cut out the singularities of the original function."

(Ramanujan in his last letter to Hardy, 1920)

Mock theta functions of order 3:

$$f(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n^2},$$

$$\phi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q^2;q^2)_{2n}},$$

$$\psi(q)=\sum_{n=1}^{\infty}\frac{q^{n^2}}{(q;q^2)_{n-1}},$$

$$\chi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}\,(-q;q)_n}{(-q^3;q^3)_n}.$$

Mock theta functions of order 5:

$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n},$$

$$F_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n},$$

$$1 + 2\psi_0(q) = \sum_{n=0}^{\infty} (-1;q)_n q^{\binom{n+1}{2}},$$

$$\phi_0(q) = \sum_{n=0}^{\infty} (-q;q^2)_n q^{n^2},$$

$$f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q)_n},$$

$$F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}},$$

Mock theta functions of order 6:

$$\phi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q;q^2)_n}{(-q;q)_{2n}},$$

$$\psi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2} (q;q^2)_n}{(-q;q)_{2n+1}},$$

$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}} (-q;q)_n}{(q;q^2)_{n+1}},$$

$$\sigma(q) = \sum_{n=0}^{\infty} \frac{q^{\binom{n+2}{2}} (-q;q)_n}{(q;q^2)_{n+1}},$$

$$\lambda(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q;q^2)_n}{(-q;q)_n},$$

$$\mu(q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n}{(-q;q)_n},$$

Mock theta functions of order 7:

$$\mathcal{F}_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n+1};q)_n},$$

$$\mathcal{F}_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{n};q)_n},$$

$$\mathcal{F}_2(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^{n+1};q)_{n+1}}.$$

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$$\mathcal{F}_2(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^{n+1};q)_{n+1}}.$$

He lists several identities and asymptotic properties which these functions satisfy. More are later found in the "Lost Notebook".

Classical (Jacobi) theta functions:

$$j(x;q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n$$
$$= (x;q)_\infty \, (q/x;q)_\infty \, (q;q)_\infty ,$$

where $q = e^{2\pi i\tau}$ and $x = e^{2\pi iz}$ with $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$ (upper half plane).

They transform well under the modular transformations $\tau \to \tau + 1$ and $\tau \to -1/\tau$.

Mock Theta Functions

For example, for the "third order mock theta function"

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2},$$

Ramanujan claimed that, for an explicitly constructed (essentially) modular function $b(q)$,

$$f(q) - (-1)^k b(q) = O(1)$$

as q approaches a primitive $2k$-th root of unity (from the interior of the unit circle).

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Ramanujan's mock theta functions were intensively studied by Watson, Andrews, Dragonette, Selberg, Y.-S. Choi, H. Cohen, F. Dyson, Garvan, B. Gordon, Hickerson, R. McIntosh, M. Wakimoto, and others.

"Ramanujans discovery of the mock theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end. As much as any of his earlier work, the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance. To his students such discoveries will be a source of delight and wonder until the time shall come when we too shall make our journey to that Garden of Proserpine (a.k.a. Persephone)."

(George N. Watson, 1936)

"The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta-functions of Jacobi. This remains a challenge for the future. My dream is that I will live to see the day when our young physicists, struggling to bring the predictions of superstring theory into correspondence with the facts of nature, will be led to enlarge their analytic machinery to include mock theta-functions ... But before this can happen, the purely mathematical exploration of the mock-modular forms and their mock-symmetries must be carried a great deal further.

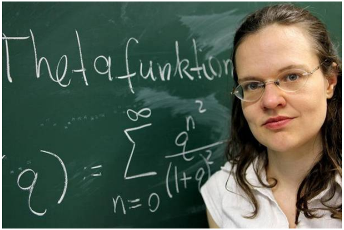
(Freeman Dyson, 1987)

(Sander Zwegers) 

The breakthrough in the understanding of the "mock theta functions" came in 2001 with the Ph.D. thesis of ZWEGERS. For several examples of Ramanujan, he showed that Ramanujan's mock theta functions can be written as a sum of a *harmonic weak Maaß form*[1] of weight 1/2 and an explicitly described non-holomorphic part, its "shadow".

[1]introduced recently by JAN HENDRIK BRUINIER and JENS FUNKE

Mock Theta Functions



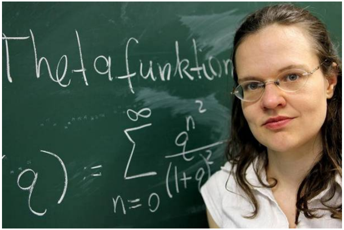
Kathrin Bringmann

Ken Ono

Don Zagier

Further work by BRINGMANN and ONO, and by ZAGIER has developed this into the fascinating theory of *mock modular forms*, which are (essentially) holomorphic parts of harmonic weak Maaß forms. A *mock theta function* is then defined as a mock modular form of weight $1/2$.

Mock Theta Functions



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Evidence that this is in the spirit of Ramanujan's thinking is strong; for example, frequently the shadows of Ramanujan's mock theta functions of a given order are the same!

Epilogue: A Ramanujan Hit Parade

George Andrews

Bruce Berndt

Your Ramanujan Hit Parade: The Top Ten Most Fascinating Formulas in Ramanujans Lost Notebook (*Notices Amer. Math. Soc.* **55** (2013)).

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ome anxiously waiting to learn the
the Top Seven Songs, sung by *Your*
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singer), Snooky Lanson, and Gisele
About twenty years later, that author's
egin to be consumed by Ramanujan's
, but more important than Ramanu-
his evening was how long his parents

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No.3: Cranks

Let

$$G(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n},\quad\text{and}\quad H(q)=\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}.$$

Then

$$\frac{(q;q)_\infty}{(\zeta q;q)_\infty\,(\zeta^{-1}q;q)_\infty}=A(q^5)-q(\zeta+\zeta^{-1})^2B(q^5)\\+q^2(\zeta^2+\zeta^{-2})C(q^5)-q^3(\zeta+\zeta^{-1})D(q^5),$$

where ζ is any primitive fifth root of unity and

$$A(q)=\frac{(q^5;q^5)_\infty\,G^2(q)}{H(q)},\qquad B(q)=(q^5;q^5)_\infty\,G(q),$$

$$C(q)=(q^5;q^5)_\infty\,H(q),\qquad D(q)=\frac{(q^5;q^5)_\infty\,H^2(q)}{G(q)}.$$

No.2: Mock Theta Functions

$$\phi_0(q) = \frac{(-q^2;q^5)_\infty\,(-q^3;q^5)_\infty\,(q^5;q^5)_\infty}{(q^2;q^{10})_\infty\,(q^8;q^{10})_\infty} + 1 - \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q;q^5)_{n+1}\,(q^4;q^5)_n},$$

where

$$\phi_0(q) = \sum_{n=0}^{\infty} q^{n^2}(-q;q^2)_n$$

is one of the fifth order mock theta functions.

No.1: Ranks

Recall the rank generating function

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n)w^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(wq;q)_n\,(w^{-1}q;q)_n},$$

where

$$N(m,n) := |\{\text{partitions of } n \text{ with rank } m\}|.$$

Let ζ be a primitive fifth root of unity and

$$\phi(q) := -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q;q^5)_{n+1}\,(q^4;q^5)_n}$$

$$\psi(q) := -1 + \sum_{n=0}^{\infty} \frac{q^{5n^2}}{(q^2;q^5)_{n+1}\,(q^3;q^5)_n}.$$

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Then

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta q;q)_n\,(\zeta^{-1}q;q)_n} = A(q^5) + (\zeta+\zeta^{-1}-2)\phi(q^5) + qB(q^5) + q^2(\zeta+\zeta^{-1})C(q^5) - q^3(\zeta+\zeta^{-1})\left(D(q^5) - (\zeta^2+\zeta^{-2}-2)\frac{\psi(q^5)}{q^5}\right).$$