Srinivasa Ramanujan
Life and Mathematics

Christian Krattenthaler

Universität Wien

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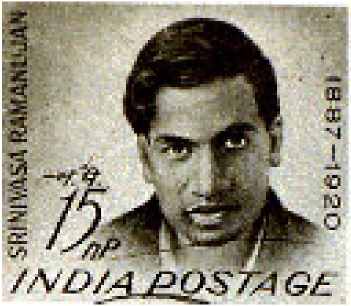

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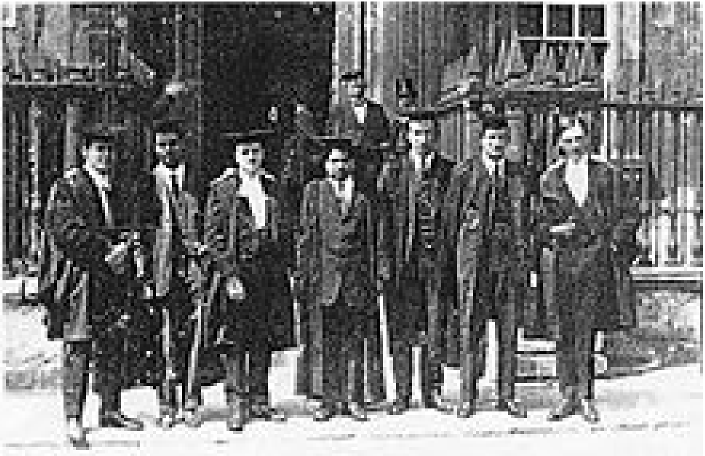

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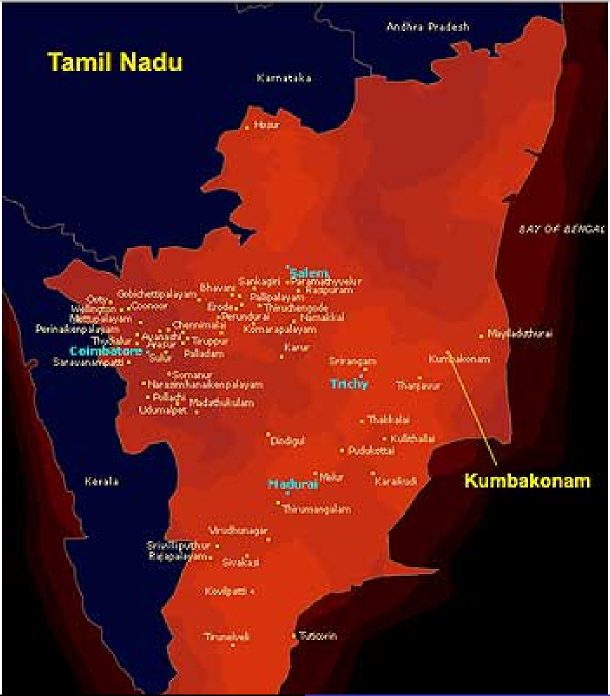
- grew up in Kumbakonam (Madras Presidency = Tamil Nadu)


AFGHANISTAN
PAKISTAN
CHINA
NEPAL
BHUTAN
BANGLADESH
MYANMAR
SRI LANKA
JAMMU & KASHMIR
Srinagar
Jammu
HIMACHAL PRADESH
Shimla
Chandigarh
PUNJAB
HARYANA
DELHI
Dehradun
UTTARAKHAND
RAJASTHAN
Jaipur
UTTAR PRADESH
Lucknow
SIKKIM
Gangtok
ARUNACHAL PRADESH
Itanagar
Dispur
ASOM (ASSAM)
NAGALAND
Kohima
Shillong
MEGHALAYA
Imphal
MANIPUR
Agartala
TRIPURA
Aizawl
MIZORAM
BIHAR
Patna
JHARKHAND
Ranchi
WEST BENGAL
Kolkata
GUJARAT
Gandhinagar
MADHYA PRADESH
Bhopal
CHHATTISGARH
Raipur
ODISHA (ORISSA)
Bhubaneswar
DAMAN & DIU
Daman
DADRA & NAGAR HAVELI
Silvassa
Mumbai
MAHARASHTRA
Hyderabad
ANDHRA PRADESH
Yanam (Puducherry)
ARABIAN SEA
BAY OF BENGAL
Panaji
GOA
KARNATAKA
Bengaluru (Bangalore)
Chennai
PUDUCHERRY (Pondicherry)
Mahe (Puducherry)
Karaikal (Puducherry)
TAMIL NADU
KERALA
Kavaratti
LAKSHADWEEP (INDIA)
Thiruvananthapuram
ANDAMAN & NICOBAR ISLANDS (INDIA)
Port Blair
INDIAN OCEAN


Tamil Nadu
Andhra Pradesh
Karnataka
BAY OF BENGAL
Hosur
Salem
Coimbatore
Trichy
Madurai
Kerala
Kumbakonam
Thanjavur
Erode
Karur
Srirangam
Dindigul
Pudukottai
Thirumangalam
Virudhunagar
Sivakasi
Tirunelveli
Tuticorin

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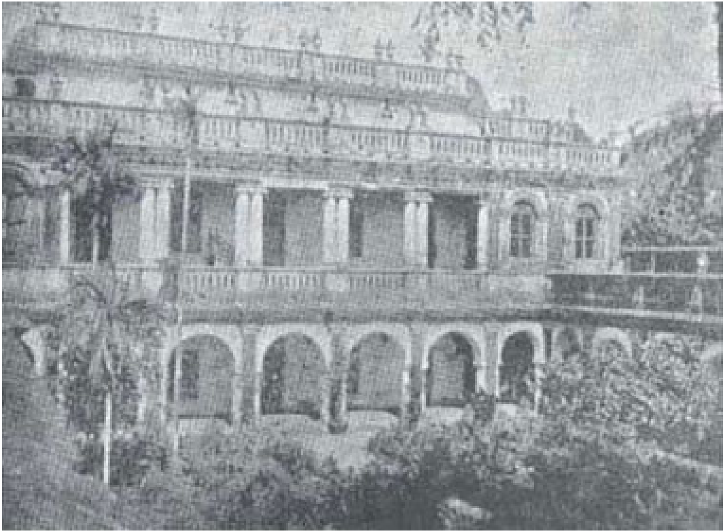

Kumbakonam Town High School

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- 1909: married to *Janaki Ammal* (1899 – 1994)
- 1910: moved to Madras, with the support of *R. Ramachandra Rao*, a tax collector

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- 1913: one of them, *Godfrey Harold Hardy*, professor at Cambridge University, was perplexed, and invited Ramanujan to come to England and to study at Cambridge University

Godfrey Harold Hardy (1877 – 1947)

"[These formulae] defeated me completely. I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them."

What is Hardy talking about?

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$$\cfrac{1}{1+\cfrac{e^{-2\pi}}{1+\cfrac{e^{-4\pi}}{1+\cfrac{e^{-6\pi}}{\ddots}}}} = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right) e^{2\pi/5}$$

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- 1916: Ramanujan is awarded the Bachelor degree ($\sim$ Ph.D.) for a dissertation on "highly composite numbers"
- 1918: Ramanujan is elected Fellow of the Royal Society (F.R.S.), on the proposition of *Hardy* and *Percy Alexander MacMahon*

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- 1919: due to illness, on advice of Hardy he returns to India
- 26 April 1920: Ramanujan dies in his home in Kumbakonam

Entrance to Ramanujan's home in Kumbakonam

Ramanujan's mathematical heritage

Ramanujan's mathematical heritage

Ramanujan's interests include *infinite series, integrals, asymptotic expansions and approximations, gamma function, hypergeometric and q-hypergeometric functions, continued fractions, theta functions, class invariants, Diophantine equations, congruences, magic squares*.

- 3 Notebooks
- 37 published mathematical papers
  (J. Indian Math. Soc., Proc. London Math. Soc., Proc. Cambridge Philos. Soc., Proc. Cambridge Philos. Soc., Proc. Royal Soc., Messenger Math., Quart. J. Math.)
- the "Lost Notebook"

Ramanujan's mathematical heritage

The 3 Notebooks

Ramanujan started to work on them during college time (maybe even earlier). He continued working on them until his departure to England.

The first notebook has 351 pages, organized in 16 chapters.
The second notebook is a revised enlargement of the first. It has 256 pages, organized in 21 chapters.
The third notebook has only 33 pages.

7. $\operatorname{Sec} x = E_1 + \frac{x^2}{\lfloor 2} E_3 + \frac{x^4}{\lfloor 4} E_5 + \frac{x^6}{\lfloor 6} E_7 + \&c$

Cor. $\frac{B_{2n}}{2n} 2^{2n}(2^{2n}-1) = 2 E_1 E_{2n-1} + 2 E_3 E_{2n-3} \frac{(2n-2)(2n-3)}{\lfloor 2} + \&c$
the last term being $2 E_{n-1} E_{n+1} \frac{\lfloor 2n-2}{\lfloor n-2 \, \lfloor n}$ or $(E_n)^2 \frac{\lfloor 2n-2}{(\lfloor n-1)^2}$
according as n is even or odd.

Sol. $\frac{d \tan x}{dx} = \sec^2 x$. Equate the coeff^{ts} of x^{2n-2}.

$E_1 = 1$, $E_3 = 1$, $E_5 = 5$, $E_7 = 61$, $E_9 = 1385$, $E_{11} = 50521$,
$E_{13} = 2702765$, $E_{15} = 199360981$, &c.

8. i. $\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right) \cdots \&c$

Sol. The roots of the equation $\frac{\sin x}{x} = 0$ are $\pm\pi, \pm 2\pi$, &c
and $\frac{\sin x}{x} = 1$ when $x = 0$.

ii. In a similar manner
$\cos x = \left(1 - \frac{4x^2}{\pi^2}\right)\left(1 - \frac{4x^2}{3^2\pi^2}\right)\left(1 - \frac{4x^2}{5^2\pi^2}\right)\left(1 - \frac{4x^2}{7^2\pi^2}\right) \&c$

excerpt from Notebook I

Ramanujan's mathematical heritage

The 3 Notebooks

After the death of Ramanujan, Hardy received copies of the notebooks. He wanted the notebooks to be published and "edited".

In the late 1920s, George Neville Watson (1886 – 1965) and B. M. Wilson (1897 – 1935) began the task of editing the notebooks. They focussed on the second notebook. Wilson was assigned Chapters 2–14, Watson was assigned Chapters 15–21. Wilson died in 1935. Watson wrote over 30 papers inspired by the notebooks before his interest waned in the late 1930s.

1957: The notebooks were published — without editing — by the Tata Institute of Fundamental Research in Bombay.

Ramanujan's mathematical heritage

The 3 Notebooks

Bruce Carl Berndt (*1939)

1974: *Bruce Berndt* (not knowing at the time ...) started the project of systematically proving all entries in the notebooks.

1997: The project is completed by the publication of the fifth volume of *"Ramanujan's Notebooks"*, Part I–V.

Ramanujan's mathematical heritage

The "Lost Notebook"

George Eyre Andrews (*1938)

1976: *George Andrews* visits the Wren Library of Trinity College in Cambridge in order to see whether there is something interesting in G. N. Watson's hand-written notes stored there. He found a pile of papers with Ramanujan's handwriting there.

Ramanujan's mathematical heritage

The "Lost Notebook"

George Eyre Andrews

Bruce Carl Berndt

1998: Bruce Berndt and George Andrews start the project of systematically proving all identities in the "Lost Notebook".

2014: The project is essentially completed. Four volumes (out of five) of *"Ramanujan's Lost Notebook"* are already published.

The Rogers–Ramanujan Identities

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$$\cfrac{1}{1+\cfrac{e^{-2\pi}}{1+\cfrac{e^{-4\pi}}{1+\cfrac{e^{-6\pi}}{\ddots}}}} = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right) e^{2\pi/5}$$

$$\cfrac{1}{1+\cfrac{e^{-2\pi}}{1+\cfrac{e^{-4\pi}}{1+\cfrac{e^{-6\pi}}{\ddots}}}} = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right) e^{2\pi/5}$$

One can show that

$$1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{\ddots}}}} = \frac{\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n}}{\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n}},$$

where $(\alpha;q)_n := (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1})$, $n \geq 1$, and $(\alpha;q)_0 := 1$.

The Rogers–Ramanujan Identities

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Numerator and denominator factor!

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q;q^5)_\infty\,(q^4;q^5)_\infty}$$

and

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n}=\frac{1}{(q^2;q^5)_\infty\,(q^3;q^5)_\infty}.$$

These are the celebrated *Rogers–Ramanujan identities*.

The Rogers–Ramanujan Identities

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The Rogers–Ramanujan Identities

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Combinatorics: the right-hand side rewritten:

$$\begin{aligned}\frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)\cdots} &= \sum_{k_1=0}^{\infty} q^{k_1} \sum_{k_4=0}^{\infty} q^{4k_4} \sum_{k_6=0}^{\infty} q^{6k_6} \sum_{k_9=0}^{\infty} q^{9k_9} \cdots \\ &= \sum q^{k_1\cdot 1+k_4\cdot 4+k_6\cdot 6+k_9\cdot 9+\cdots}.\end{aligned}$$

Hence: the coefficient of q^N on the right-hand side counts all partitions of N into parts (summands) which are congruent to ± 1 modulo 5.

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty \, (q^4;q^5)_\infty}$$

The left-hand side:

$$q^{n^2} = q^1 \cdot q^3 \cdot q^5 \cdots q^{2n-1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty\,(q^4;q^5)_\infty}$$

The left-hand side:

$$q^{n^2} = q^1 \cdot q^3 \cdot q^5 \cdots q^{2n-1}$$

and

$$\frac{1}{(q;q)_n} = \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)\cdots}$$
$$= \sum q^{k_1\cdot 1+k_2\cdot 2+\cdots+k_n\cdot n}.$$

Hence: the coefficient of q^N in this series counts all partitions of N into parts which are at most n.

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty\,(q^4;q^5)_\infty}$$

The left-hand side:

$$q^{n^2} = q^1 \cdot q^3 \cdot q^5 \cdots q^{2n-1}$$

and

$$\frac{1}{(q;q)_n} = \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)\cdots}$$
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Hence: the coefficient of q^N in this series counts all partitions of N into parts which are at most n.

Alternatively: the coefficient of q^N in this series counts all partitions of N into at most n parts.

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty\,(q^4;q^5)_\infty}$$

The left-hand side:

$$q^{n^2} = q^1 \cdot q^3 \cdot q^5 \cdots q^{2n-1}$$

and

$$\frac{1}{(q;q)_n} = \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)\cdots}$$
$$= \sum q^{k_1\cdot 1 + k_2\cdot 2 + \cdots + k_n\cdot n}.$$

The coefficient of q^N in this series counts all partitions of N into at most n parts.

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty\,(q^4;q^5)_\infty}$$

The left-hand side:

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The coefficient of q^N in this series counts all partitions of N into at most n parts.

Altogether: the coefficient of q^N on the left-hand side counts all partitions of N where parts differ by at least 2.

Let $N = 9$.

The coefficient of q^N on the right-hand side counts all partitions of N into parts which are congruent to ± 1 modulo 5:

$$9 = 6+1+1+1 = 4+4+1 = 4+1+1+1+1+1$$
$$= 1+1+1+1+1+1+1+1+1.$$

The coefficient of q^N on the left-hand side counts all partitions of N where parts differ by at least 2:

$$9 = 8+1 = 7+2 = 6+3 = 5+3+1.$$

Ramanujan knew that he had no (rigorous) proof for the two identities.

In 1917 he discovered that the identities had been proved by *Leonard James Rogers* (1862 – 1933) in 1894 [*Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc. **25**, 318–343].

Together, they published another proof of the identities in 1919 [*Proof of certain identities in combinatory analysis*, Proc. Cambridge Philos. Soc. **19**, 214–216].

Independently, *Issai Schur* (1875 – 1941) had also rediscovered the identities in 1917, again by a different approach.

The proof by Ramanujan and Rogers

Let

$$G(x) = \sum_{r=0}^{\infty} (-1)^r q^{\frac{1}{2}r(5r-1)} x^{2r} \frac{(1-xq^{2r})}{(1-x)} \frac{(x;q)_r}{(q;q)_r}.$$

Write $1 - xq^{2r} = 1 - q^r + q^r(1 - xq^r)$, splitting the sum in two parts. Associate the second part of each term with the first part of the succeeding term, to obtain

$$G(x) = \sum_{r=0}^{\infty} (-1)^r q^{\frac{1}{2}r(5r+1)} x^{2r} (1 - xq^{4r+2}) \frac{(xq;q)_r}{(q;q)_r}.$$

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Now consider

$$H(x) = \frac{G(x)}{1-xq} - G(xq).$$

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Substitute the second form for $G(x)$ in the first expression and the first form in the second expression, to get

$$H(x) = xq - \sum_{r=1}^{\infty} (-1)^r q^{\frac{1}{2}r(5r+1)} x^{2r} \frac{1 - q^r + xq^{3r+1}(1 - xq^{r+1})}{1 - xq} \frac{(xq;q)_r}{(q;q)_r}.$$

Associating, as before, the second part of each term with the first part of the succeeding term, one obtains

$$H(x) = xq(1 - xq^2)G(xq^2).$$

Now consider

$$H(x) = \frac{G(x)}{1 - xq} - G(xq).$$

Substitute the second form for $G(x)$ in the first expression and the first form in the second expression, to get

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Associating, as before, the second part of each term with the first part of the succeeding term, one obtains

$$H(x) = xq(1 - xq^2)G(xq^2).$$

Now consider

$$K(x) = \frac{G(x)}{(1 - xq)G(xq)}.$$

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Then, from

$$H(x) = \frac{G(x)}{1-xq} - G(xq)$$

and

$$H(x) = xq(1-xq^2)G(xq^2),$$

one infers

$$\begin{aligned} K(x) &= 1 + \frac{xq}{1+K(xq)} \\ &= 1 + \cfrac{xq}{1+\cfrac{xq^2}{1+\cfrac{xq^3}{\ddots}}}. \end{aligned}$$

In particular, for $x=1$,

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{\ddots}}}} = \frac{1}{K(1)} = \frac{\sum_{r=-\infty}^{\infty}(-1)^r q^{\frac{1}{2}r(5r-1)}}{\sum_{r=-\infty}^{\infty}(-1)^r q^{\frac{1}{2}r(5r+1)}}.$$

By the Jacobi triple product identity,

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n-1)/2}x^n = (q;q)_\infty\,(x;q)_\infty\,(q/x;q)_\infty,$$

this becomes

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{}}}} = \frac{(q;q^5)_\infty\,(q^4;q^5)_\infty}{(q^2;q^5)_\infty\,(q^3;q^5)_\infty}.$$

The Rogers–Ramanujan Identities

If we write

$$F(x) = \frac{G(x)}{(xq;q)_\infty},$$

then

$$F(x) = F(xq) + xqF(xq^2).$$

By induction, one gets

$$F(x) = \sum_{n=0}^{\infty} \frac{x^n q^{n^2}}{(q;q)_n}.$$

In particular,

$$F(1) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{G(1)}{(q;q)_\infty}$$
$$= \frac{(q^2;q^5)_\infty \, (q^3;q^5)_\infty}{(q;q)_\infty} = \frac{1}{(q;q^5)_\infty \, (q^4;q^5)_\infty}.$$

Similarly,

$$F(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n}=\frac{G(q)}{(q^2;q)_\infty}$$
$$=\frac{(q;q^5)_\infty\,(q^4;q^5)_\infty}{(q;q)_\infty}=\frac{1}{(q^2;q^5)_\infty\,(q^3;q^5)_\infty}.$$

Are these identities just curiosities?

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Not at all!

- → Both expressions are modular forms of weight 0
- → Both are denominators in characters of the affine Kac–Moody algebra of type $A_1^{(1)}$
- → The identities were needed in the solution of the so-called *hard hexagon model* by *Rodney James Baxter* (*1940)

....

Are these identities just curiosities?

Not at all!

- → Both expressions are modular forms of weight 0
- → Both are denominators in characters of the affine Kac–Moody algebra of type $A_1^{(1)}$
- → The identities were needed in the solution of the so-called *hard hexagon model* by *Rodney James Baxter* (*1940)

. . . .

- → The identities inspired numerous variations, generalizations, etc.

The Andrews–Gordon identities

For all positive integers k and i,

$$\sum_{r_1,\dots,r_{k-1}\ge 0} \frac{q^{\sum_{j=1}^{k-1} R_j^2+\sum_{j=i}^{k-1} R_j}}{(q;q)_{r_1}\cdots(q;q)_{r_{k-1}}} = \prod_{\substack{n=1\\ n\not\equiv 0,\pm i \pmod{2k+1}}}^{\infty} \frac{1}{1-q^n},$$

where $R_j = r_j + r_{j+1} + \cdots + r_{k-1}$.

The Rogers–Ramanujan identities result for $k = 2$ and $i = 1, 2$.

Srinivasa Ramanujan
(1887 – 1920)