Enumeration of Rhombus Tilings and Determinant Evaluations

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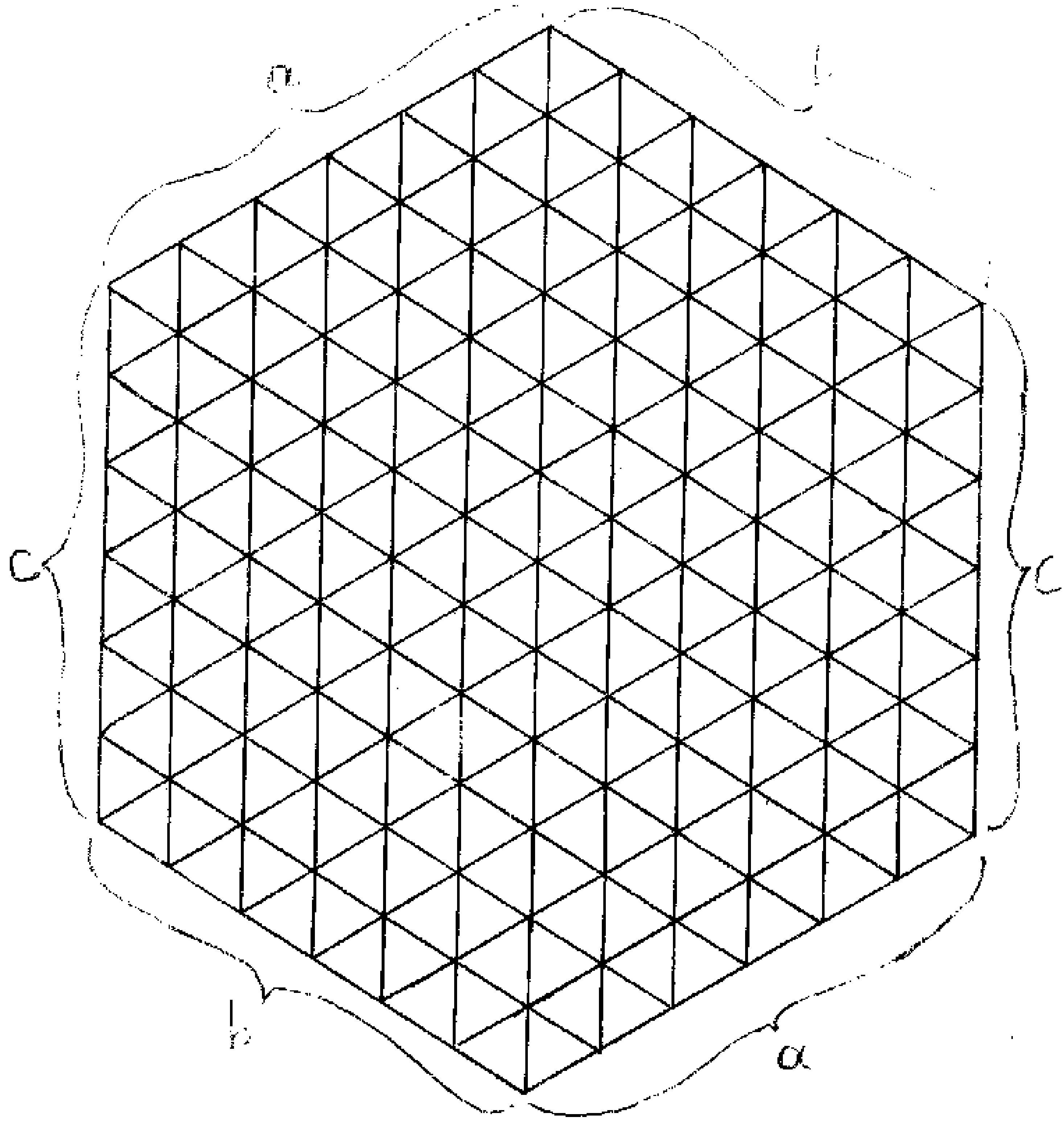
a
b
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b
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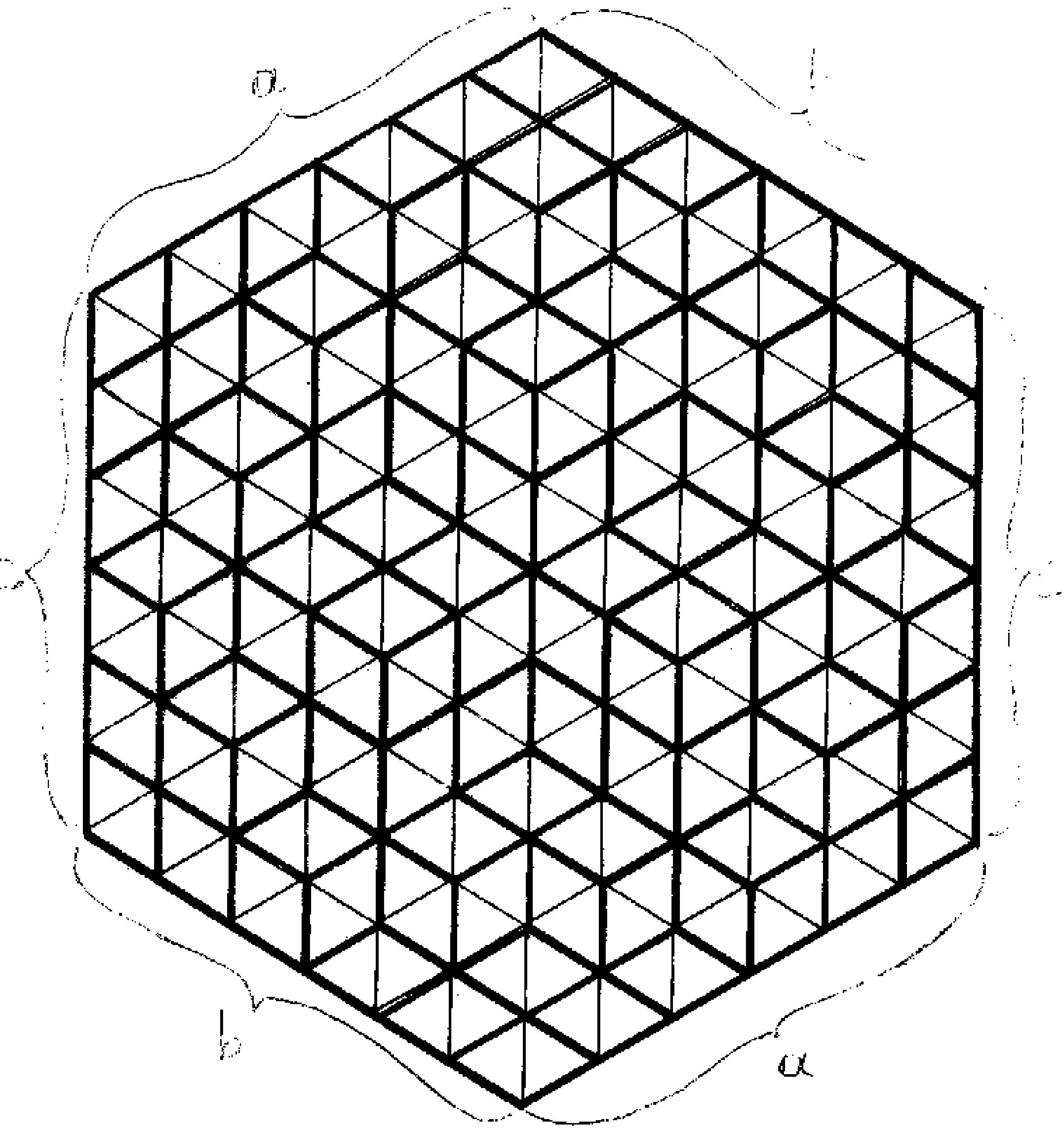
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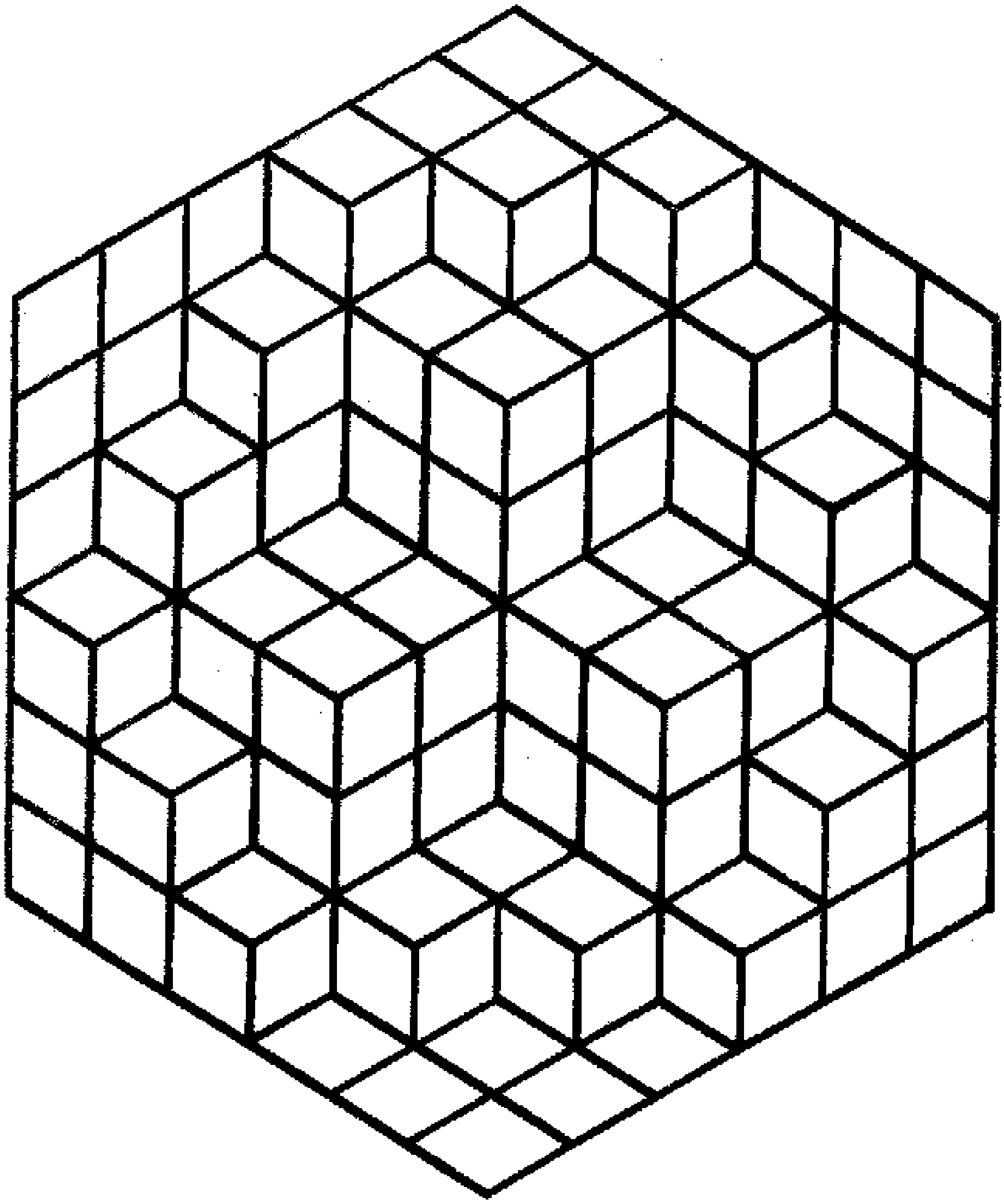
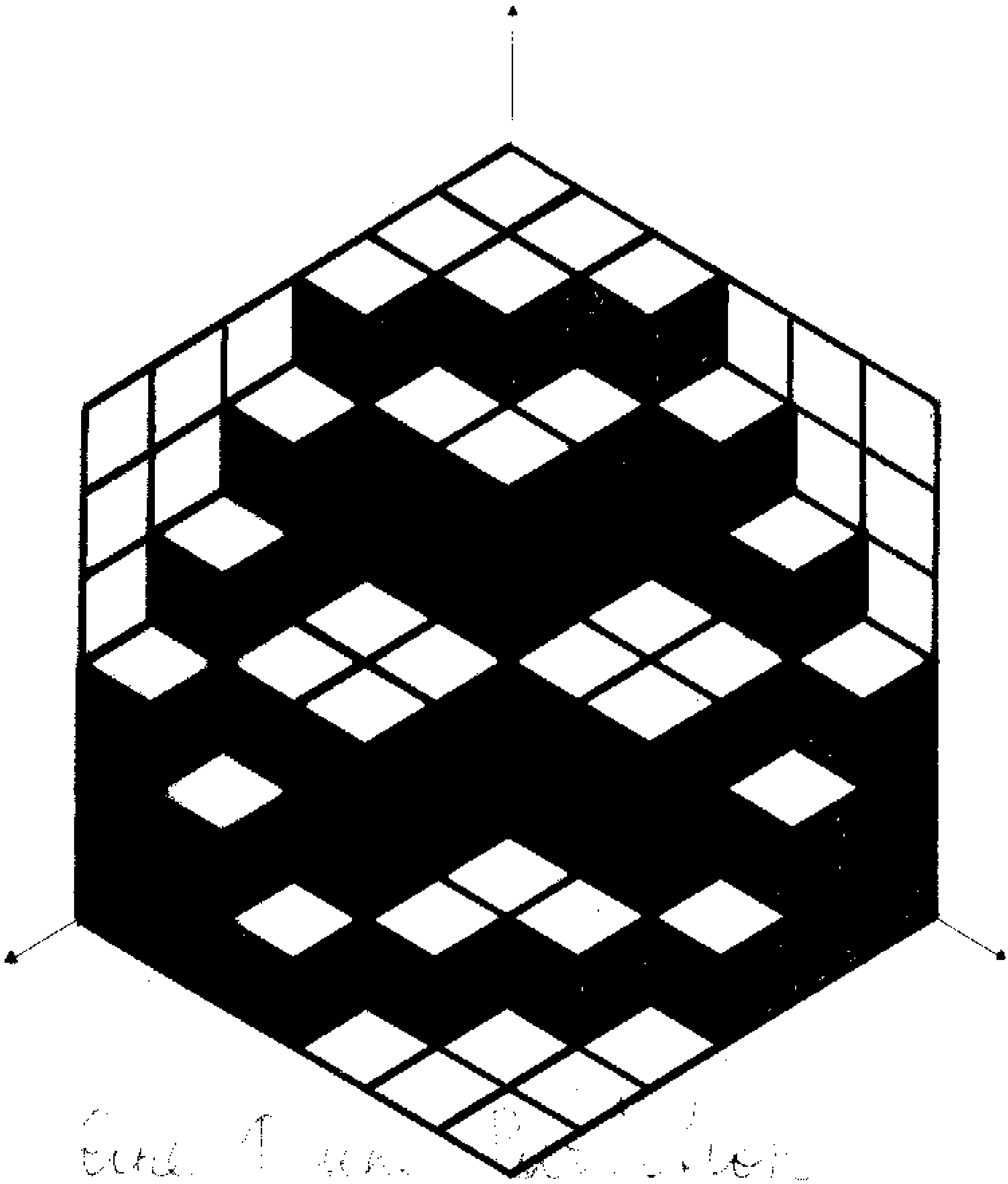
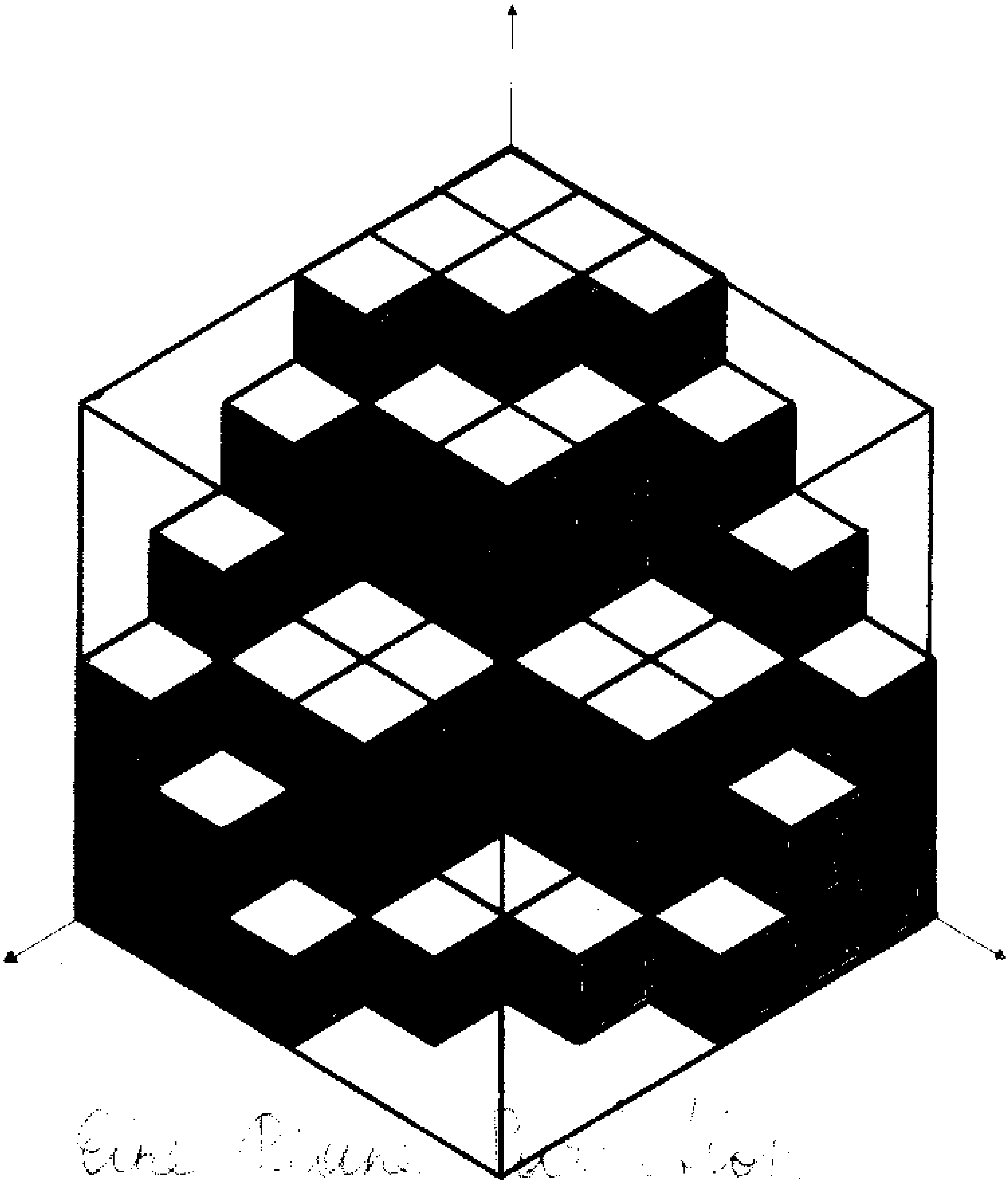
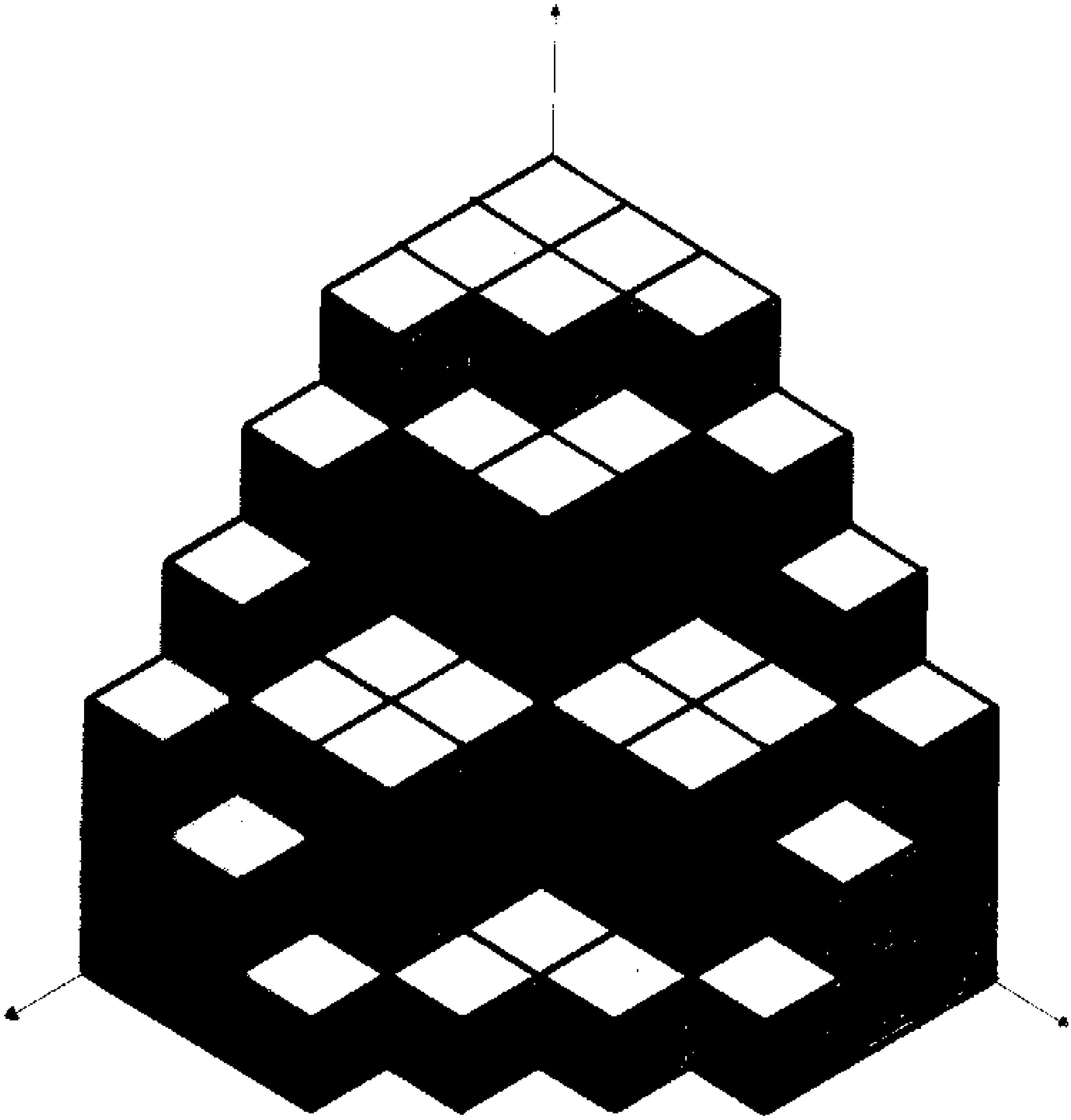
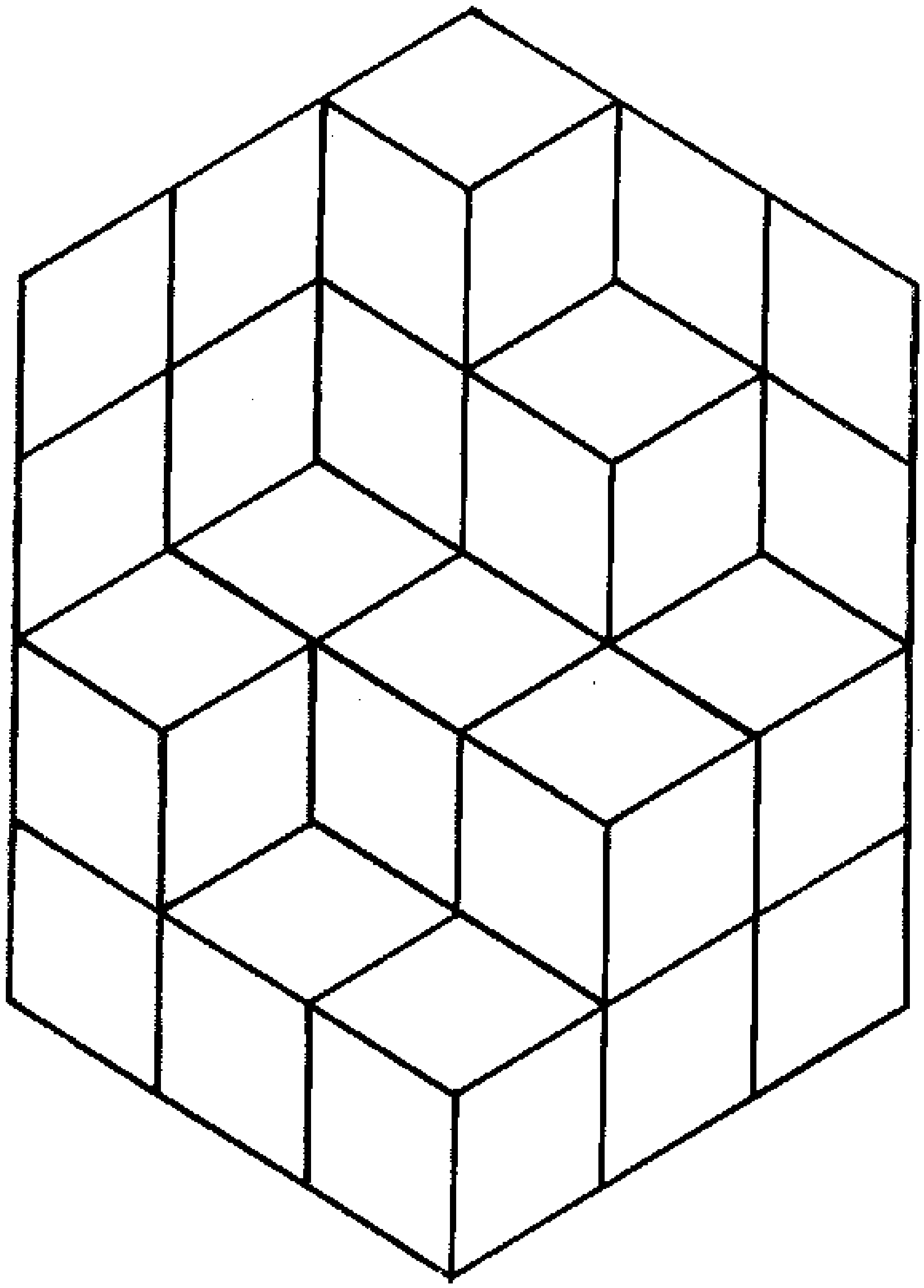


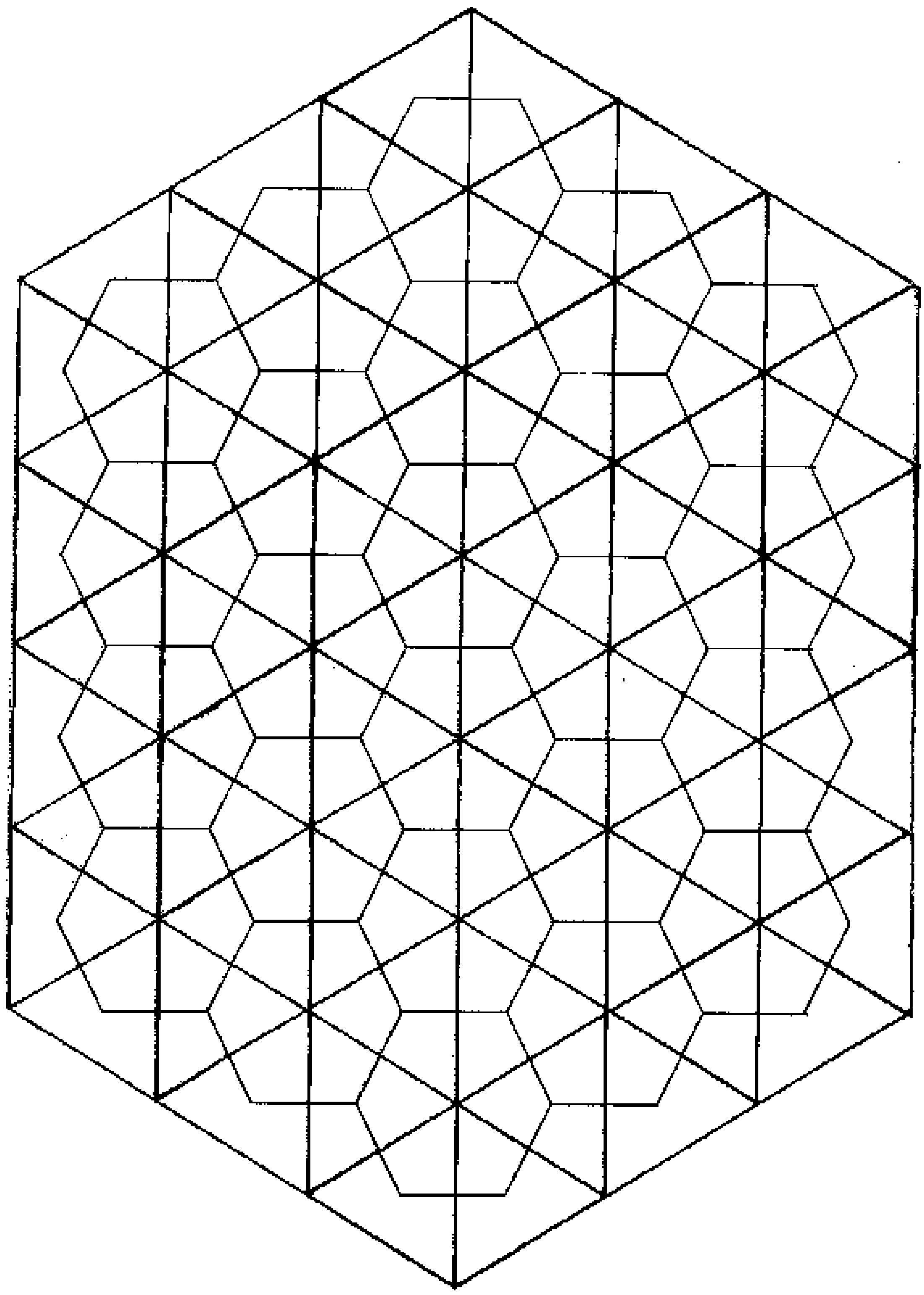


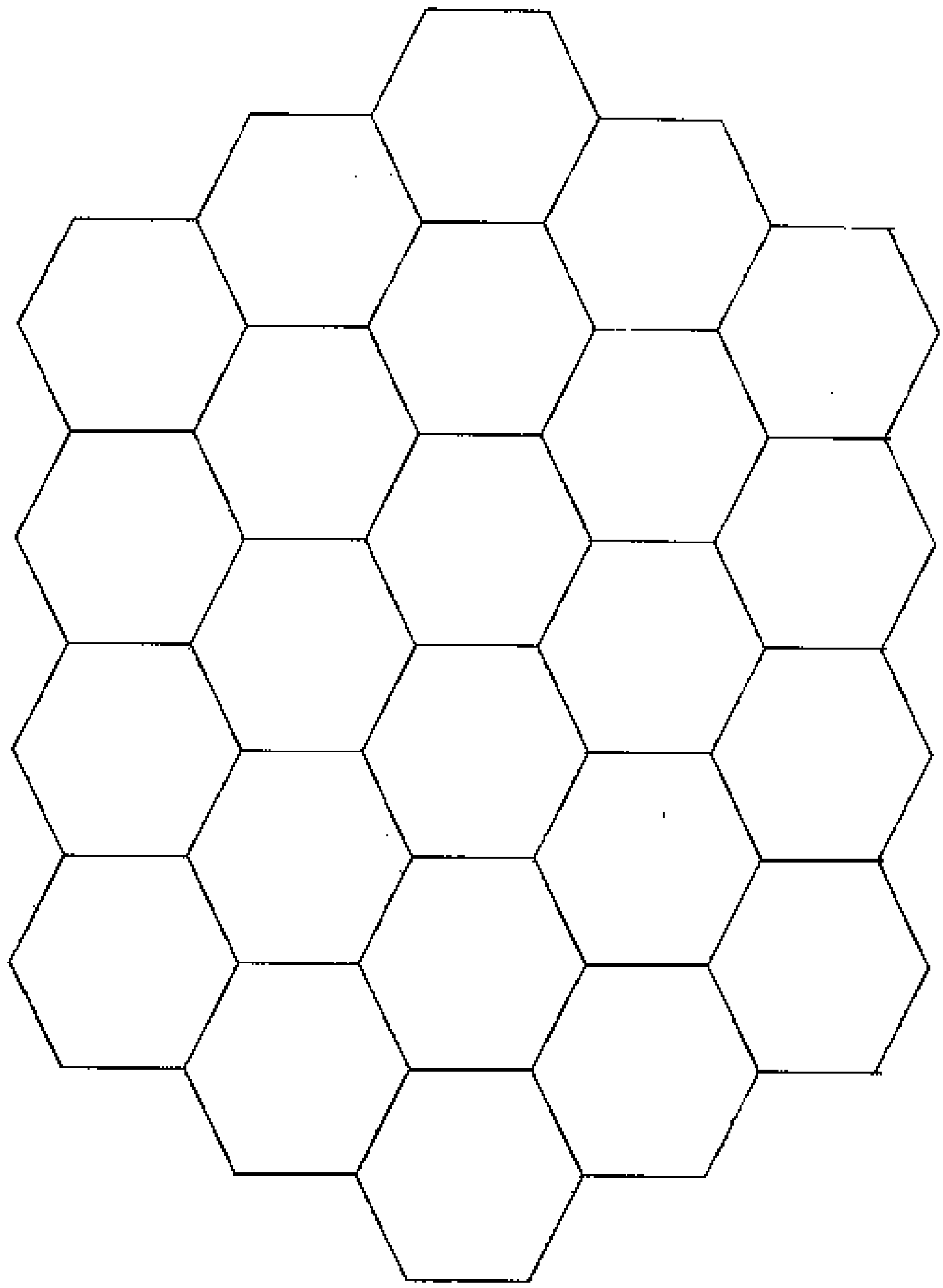


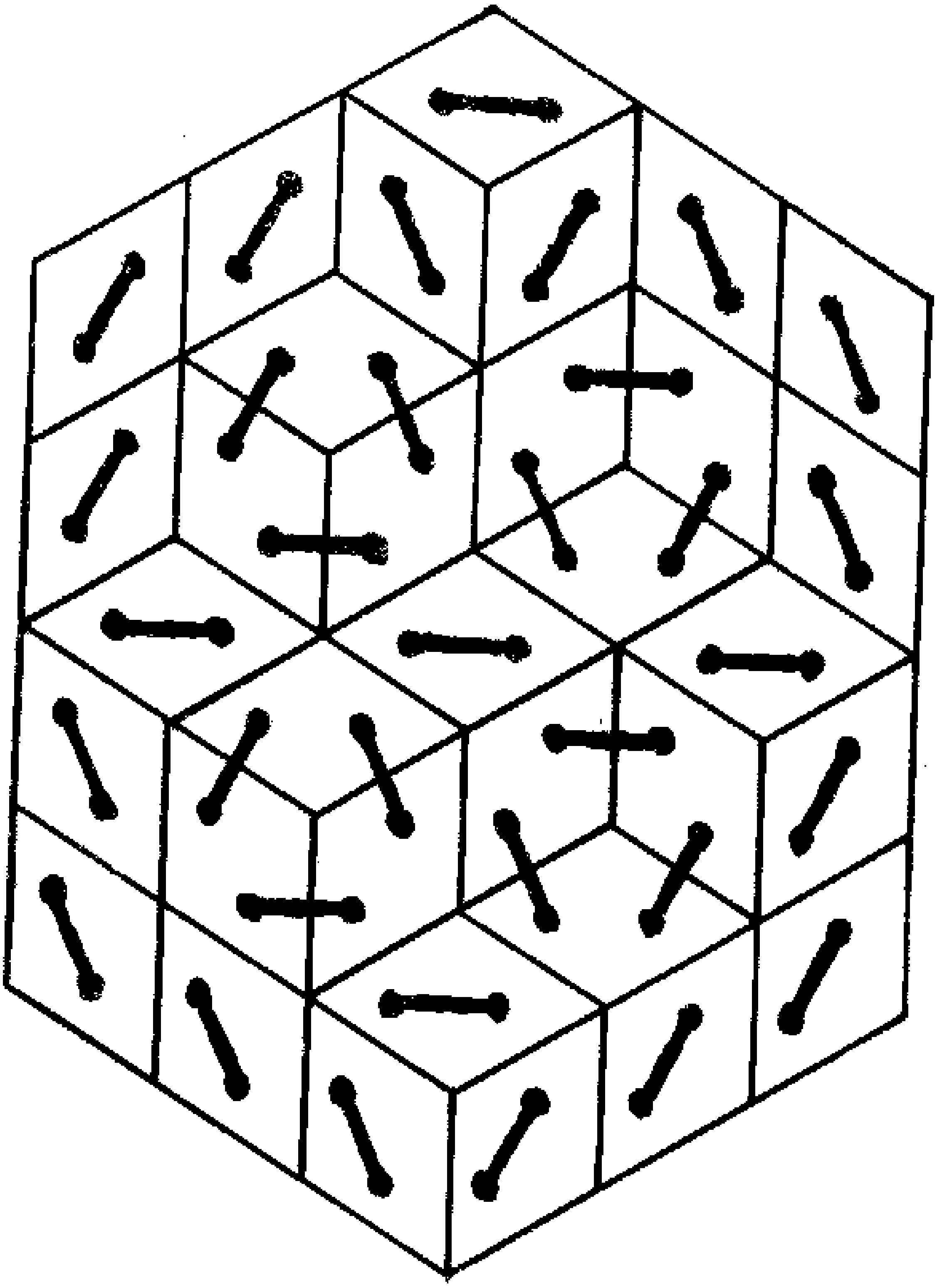
Eine [illegible]
in der [illegible] Box

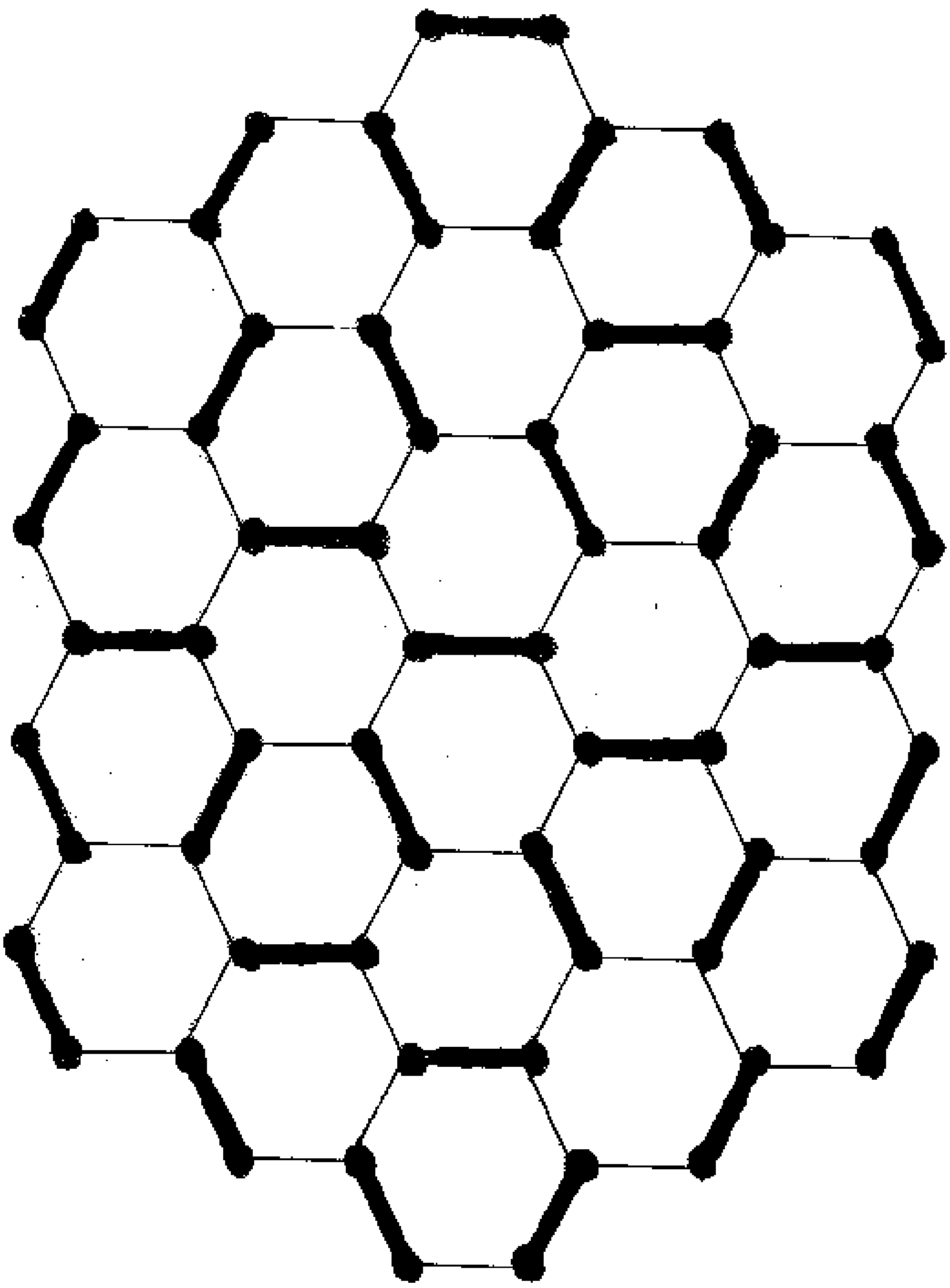








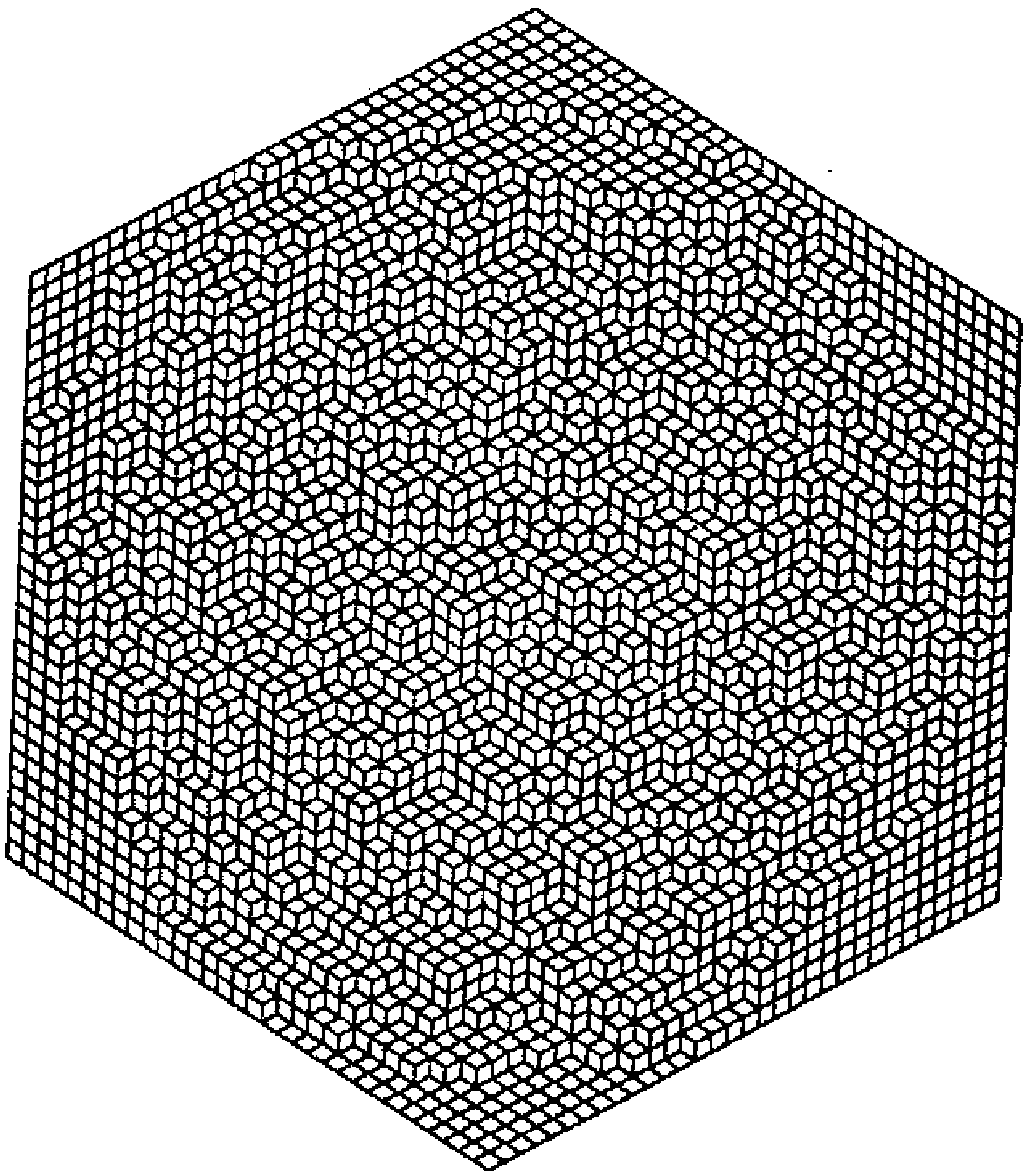




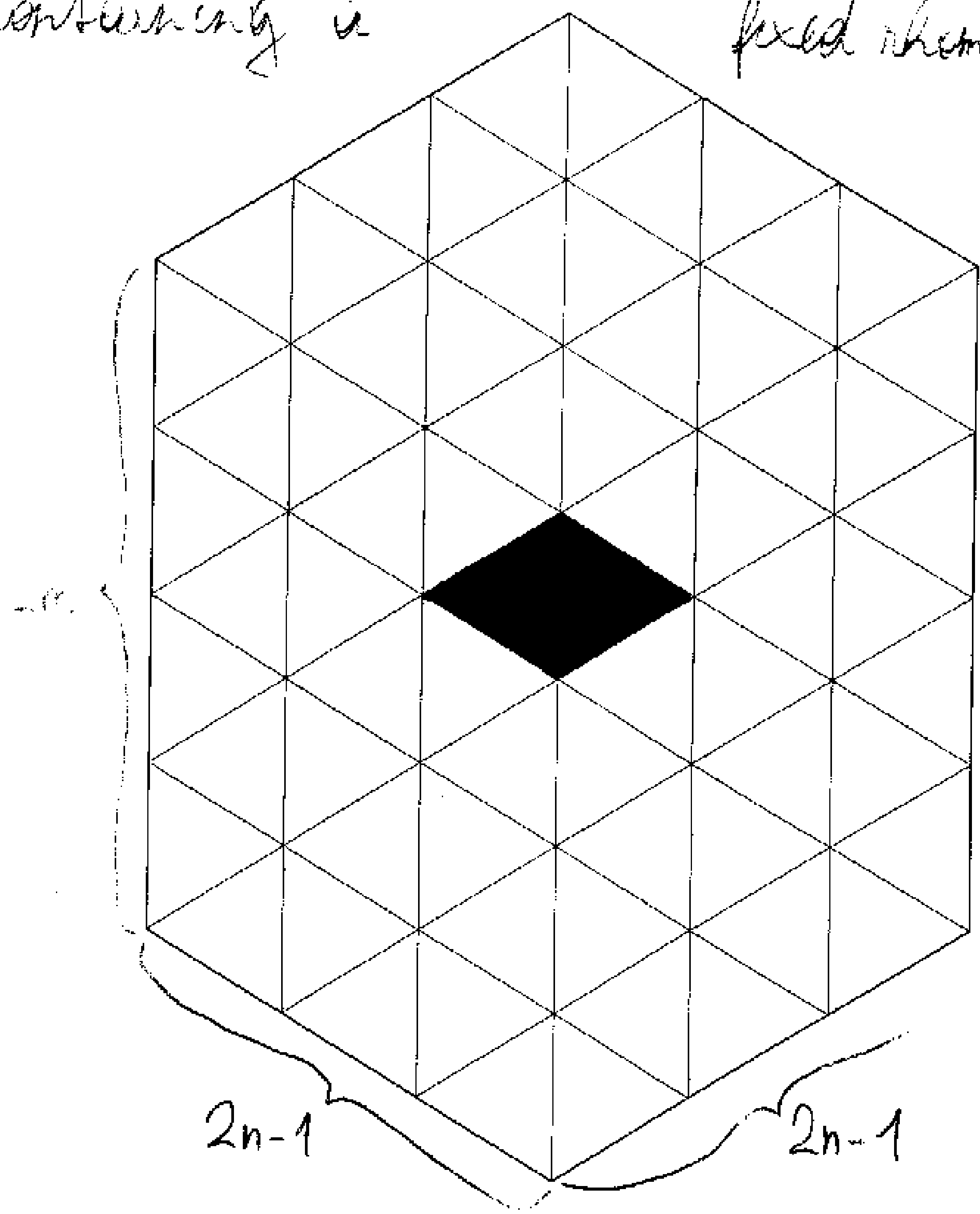
Theorem (MacMahon).

The number of plane partitions contained in an $a \times b \times c$ box, and hence, the number of rhombus tilings of a hexagon with sides a, b, c, a, b, c equals

$$\prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}.$$

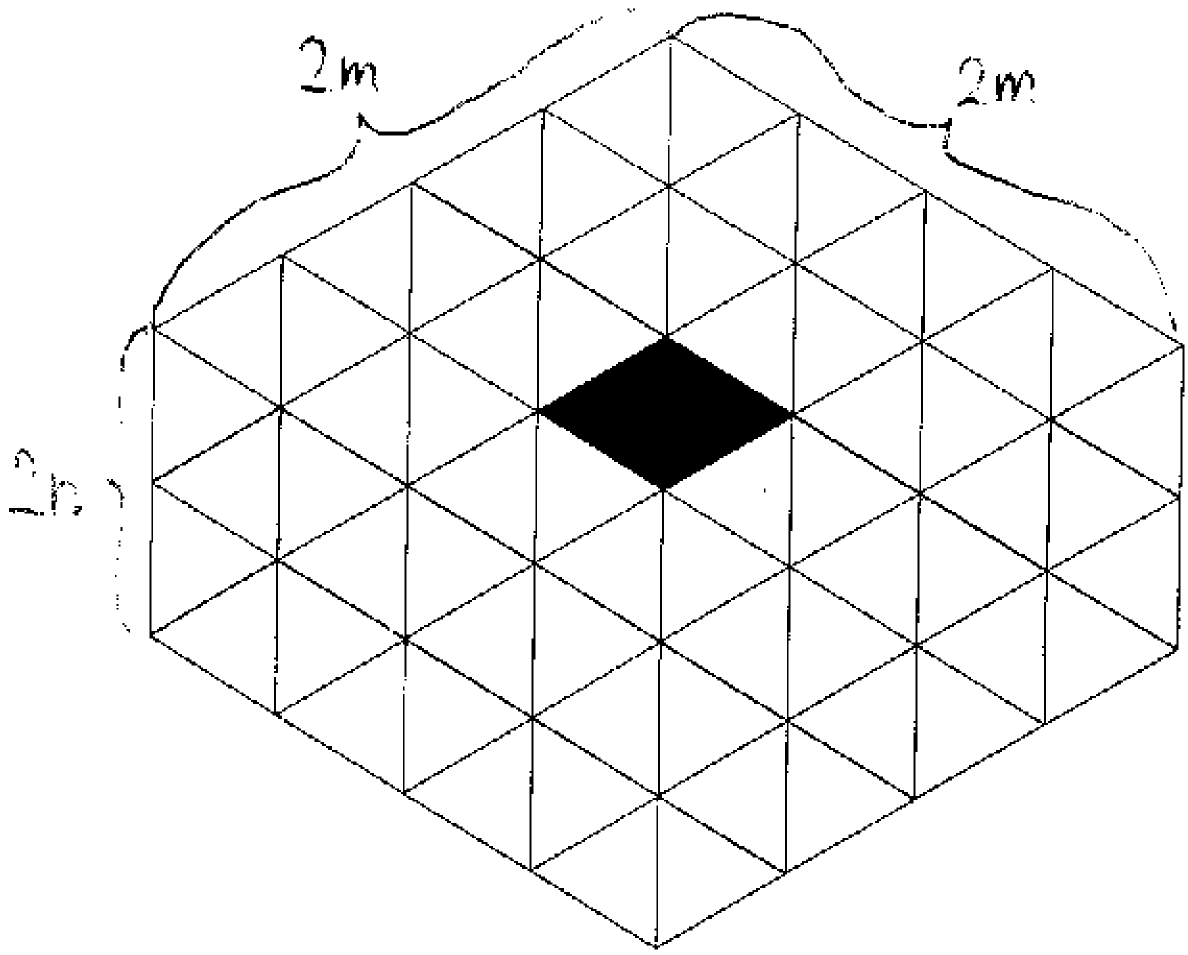


Enumeration of rhombus tilings containing a fixed rhombus



Jim Propp observed empirically: If m=n, then exactly a proportion of $\frac{1}{3}$ in the total number of rhombus tilings contain the central rhombus.

A different problem



Jim Propp observed empirically that for $m=n$ there seems to be a nice formula for the number of rhombus tilings containing this rhombus

Theorem. The number of rhombus tilings of a hexagon with sides $2n$, $2n-1$, $2n-1$, $2m$, $2n-1$, $2n-1$ which contain the central rhombus equals

$$\frac{(2n)!^2\,(2m)!\,(m+2n-1)!}{2\cdot n!^2\,m!\,(2m+4n-2)!}\sum_{i=0}^{n-1}\frac{(-1)^{n-i-1}}{(2n-2i-1)}\frac{(m+n-i)_{2i}}{i!^2}$$

$$\times\prod_{i=1}^{2m}\prod_{j=1}^{2n-1}\prod_{k=1}^{2n-1}\frac{i+j+k-1}{i+j+k-2},$$

(joint with Mihai Ciucu)

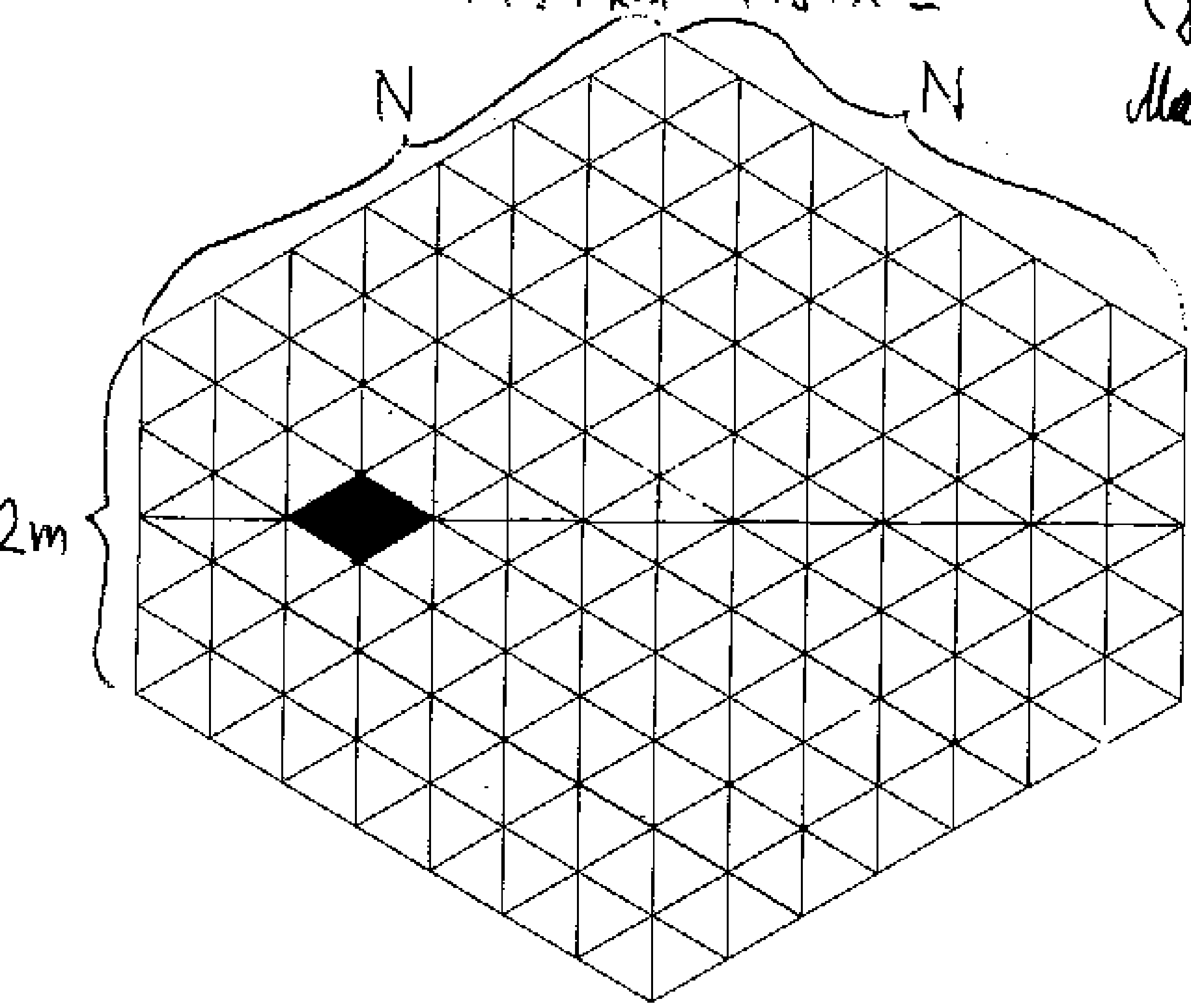
where $(a)_k := a(a+1)\cdots(a+k-1)$.

Theorem. The number of rhombus tilings of a hexagon with sides N, 2m, N, N, 2m, N, which contain the l-th rhombus on the symmetry axis equals

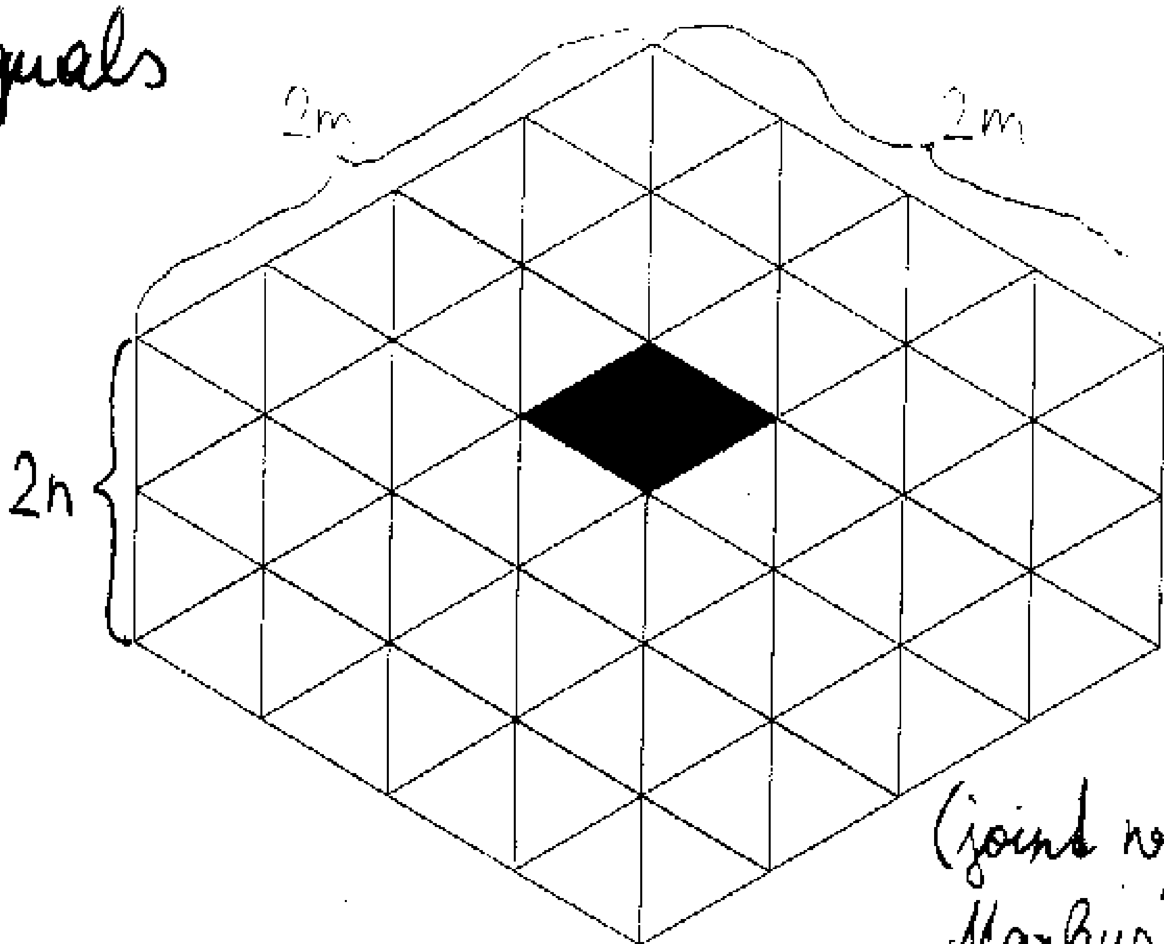
$$\frac{m\binom{m+N}{m}\binom{m+N-1}{m}}{\binom{2m+2N-1}{2m}} \sum_{e=0}^{l-1} (-1)^e \binom{N}{e} \frac{(N-2e)\,(1/2)_e}{(m+e)(m+N-e)(\frac{1}{2}-N)_e}$$

$$\times \prod_{i=1}^{N}\prod_{j=1}^{N}\prod_{k=1}^{2m} \frac{i+j+k-1}{i+j+k-2}.$$

(joint with Markus Fulmek)



Theorem. The number of rhombus tilings of a hexagon with sides 2m, 2n, 2m, 2m, 2n, 2m which contain the "almost central" rhombus equals



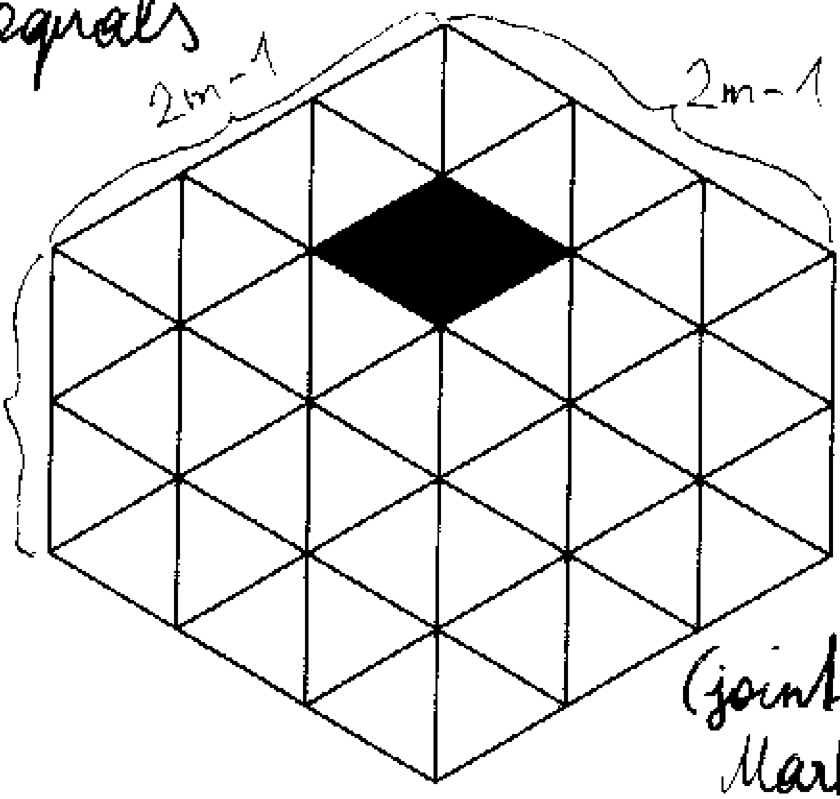
(joint with Markus Fulmek)

$$\frac{mn\binom{2m}{m}^2\binom{2n}{n}}{\binom{4m+2n}{2m+n}}\left(-\frac{1}{(m+n)^2}+\frac{2(2m+1)}{(m+1)(2m-1)(m+n-1)(m+n+1)}\right.$$

$$\left.\times\sum_{h=0}^{m-1}\frac{(2)_h\,(1-m)_h\,(\frac{3}{2}+m)_h\,(1-m-n)_h\,(1+m+n)_h}{(1)_h\,(2+m)_h\,(\frac{3}{2}-m)_h\,(2+m+n)_h\,(2-m-n)_h}\right)$$

$$\times\prod_{i=1}^{2m}\prod_{j=1}^{2n}\prod_{k=1}^{2m}\frac{i+j+k-1}{i+j+k-2}$$

Theorem. The number of rhombus tilings of a hexagon with sides $2m-1, 2n, 2m-1, 2m-1, 2n, 2m-1$ which contain the rhombus above the central rhombus equals



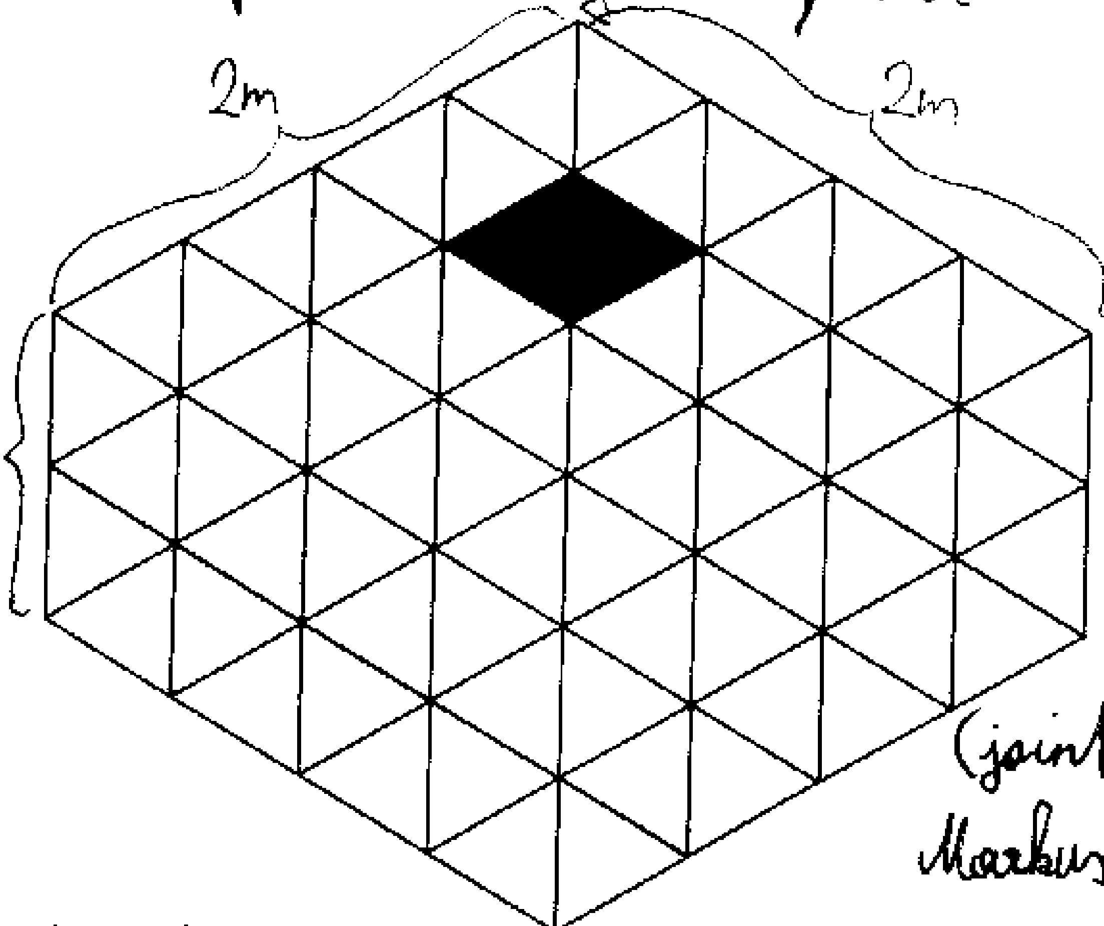
(joint with Markus Fulmek)

$$\frac{(2n-1)\binom{2n-2}{n-1}\binom{2m-1}{m-2}\binom{2m+2}{m+1}}{(m+n-1)(m+n)\binom{4m+2n-3}{2m+n-2}}\left(\frac{m(m+1)(2m-3)(2m-1)(n^2-n-3m+2mn+m^2+2)}{(m-1)(m+n-1)(m+n)(2m+1)}\right.$$

$$\left.+\frac{6}{(m+n-2)(m+n+1)}\sum_{h=0}^{m-2}\frac{(3)_h\left(\frac{5}{2}\right)_h(2-m)_h\left(\frac{3}{2}+m\right)_h(2-m-n)_h(1+m+n)_h}{(1)_h\left(\frac{3}{2}\right)_h(2+m)_h\left(\frac{5}{2}-m\right)_h(2+m+n)_h(3-m-n)_h}\right)$$

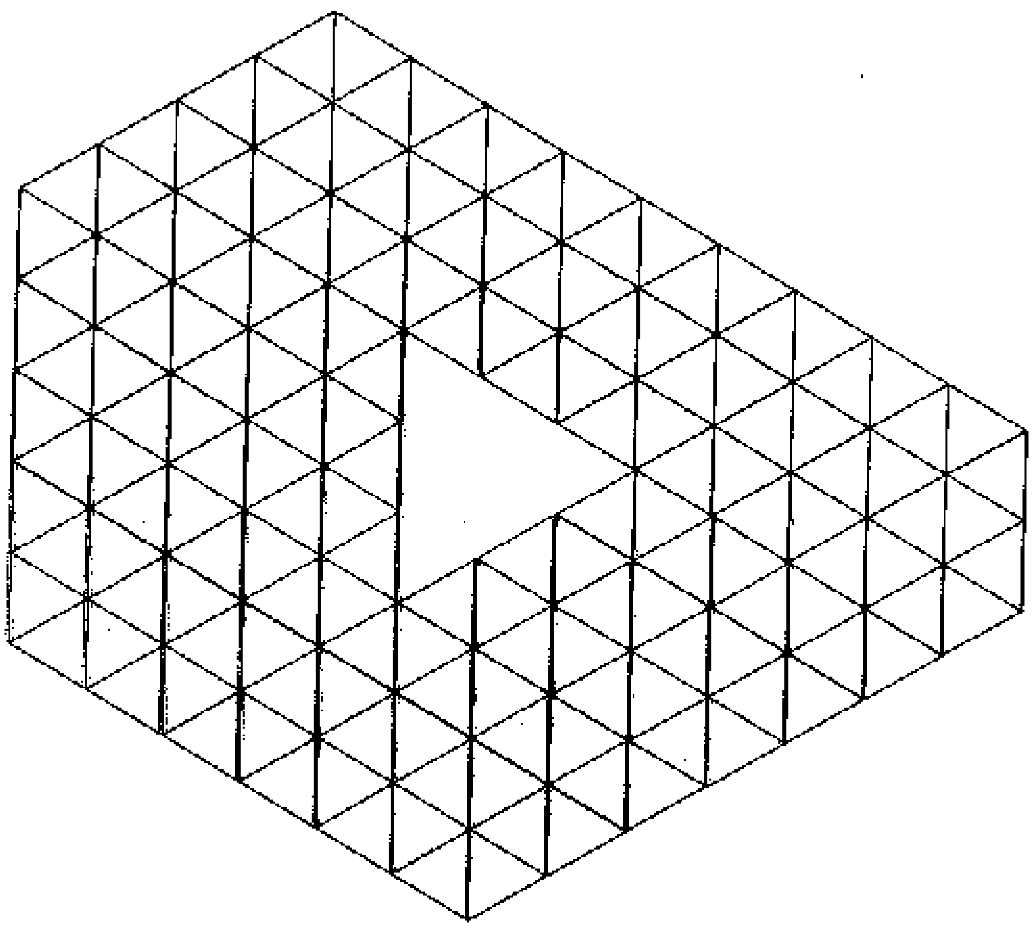
$$\times\prod_{i=1}^{2m-1}\prod_{j=1}^{2n}\prod_{k=1}^{2m-1}\frac{i+j+k-1}{i+j+k-2}.$$

Theorem. The number of rhombus tilings of a hexagon with sides $2m, 2n, 2m, 2m, 2n, 2m$ which contain the rhombus one unit above the center of the hexagon equals



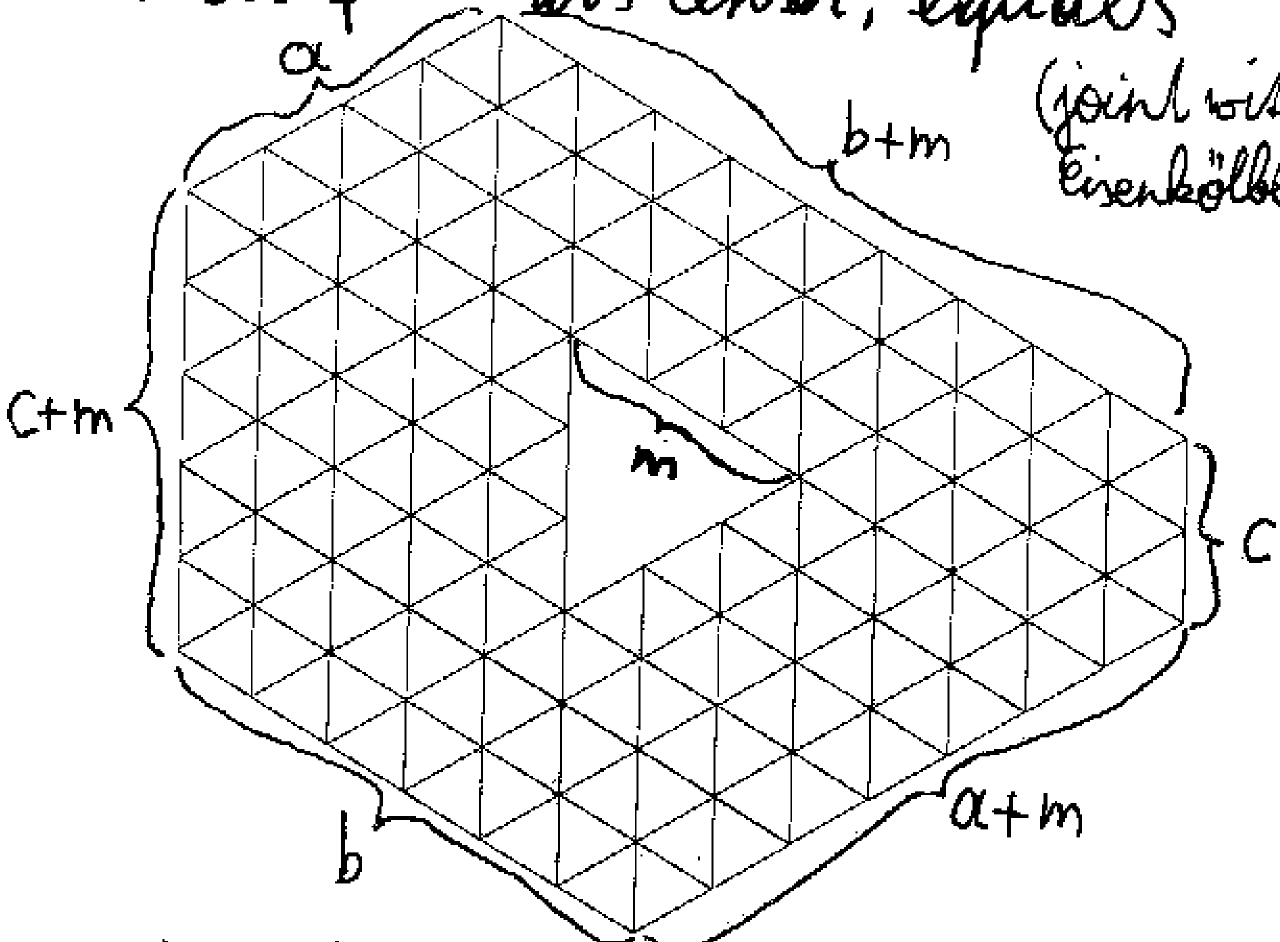
(joint with Markus Fulmek)

$$\frac{\binom{m+n-1}{n}\binom{2m+2}{m-1}\binom{2m+n-1}{m}\binom{2m+n}{2m+1}}{2(2m-3)(2m-1)(2m+2)(m+n-1)(m+n+1)\binom{4m-1}{2m}\binom{4m+2n-1}{2n}}$$
$$\left(\frac{2(m+2)(m+3)(2m-1)(2m-3)X(m,n)}{(m+n-1)(m+n)^2(m+n+1)} - \frac{24(n-1)(2m+n+1)(2m+1)(2m+3)}{(m+n-2)(m+n+2)}\right.$$
$$\left.\cdot\sum_{h=0}^{m-1}\frac{(4)_h(1-m)_h\left(\frac{5}{2}+m\right)_h(2-m-n)_h(2+m+n)_h}{(1)_h(4+m)_h\left(\frac{5}{2}-m\right)_h(3+m+n)_h(3-m-n)_h}\right)\prod_{i=1}^{2m}\prod_{j=1}^{2n}\prod_{k=1}^{2m}\frac{i+j+k-1}{i+j+k-2}.$$



Theorem. If a, b, c have the same parity, then the number of rhombus tilings of a hexagon with side lengths $a, b+m, c, a+m, b, c+m$, with an equilateral triangle of side length m removed from its center, equals

(joint with Ciucu, Eisenkölbl, Zare)



$$\frac{H(a+m)H(b+m)H(c+m)H(a+b+c+m)\,H\left(m+\left\lceil\frac{a+b+c}{2}\right\rceil\right)H\left(m+\left\lfloor\frac{a+b+c}{2}\right\rfloor\right)}{H(a+b+m)H(a+c+m)H(b+c+m)\,H\left(\frac{a+b}{2}+m\right)H\left(\frac{a+c}{2}+m\right)H\left(\frac{b+c}{2}+m\right)}$$

$$\times\frac{H\left(\left\lceil\frac{a}{2}\right\rceil\right)H\left(\left\lceil\frac{b}{2}\right\rceil\right)H\left(\left\lceil\frac{c}{2}\right\rceil\right)H\left(\left\lfloor\frac{a}{2}\right\rfloor\right)H\left(\left\lfloor\frac{b}{2}\right\rfloor\right)H\left(\left\lfloor\frac{c}{2}\right\rfloor\right)}{H\left(\frac{m}{2}+\left\lceil\frac{a}{2}\right\rceil\right)H\left(\frac{m}{2}+\left\lceil\frac{b}{2}\right\rceil\right)H\left(\frac{m}{2}+\left\lceil\frac{c}{2}\right\rceil\right)H\left(\frac{m}{2}+\left\lfloor\frac{a}{2}\right\rfloor\right)H\left(\frac{m}{2}+\left\lfloor\frac{b}{2}\right\rfloor\right)H\left(\frac{m}{2}+\left\lfloor\frac{c}{2}\right\rfloor\right)}$$

$$\times\frac{H\left(\frac{m}{2}\right)^2 H\left(\frac{a+b+m}{2}\right)^2 H\left(\frac{a+c+m}{2}\right)^2 H\left(\frac{b+c+m}{2}\right)^2}{H\left(\frac{m}{2}+\left\lceil\frac{a+b+c}{2}\right\rceil\right)H\left(\frac{m}{2}+\left\lfloor\frac{a+b+c}{2}\right\rfloor\right)H\left(\frac{a+b}{2}\right)H\left(\frac{a+c}{2}\right)H\left(\frac{b+c}{2}\right)}$$

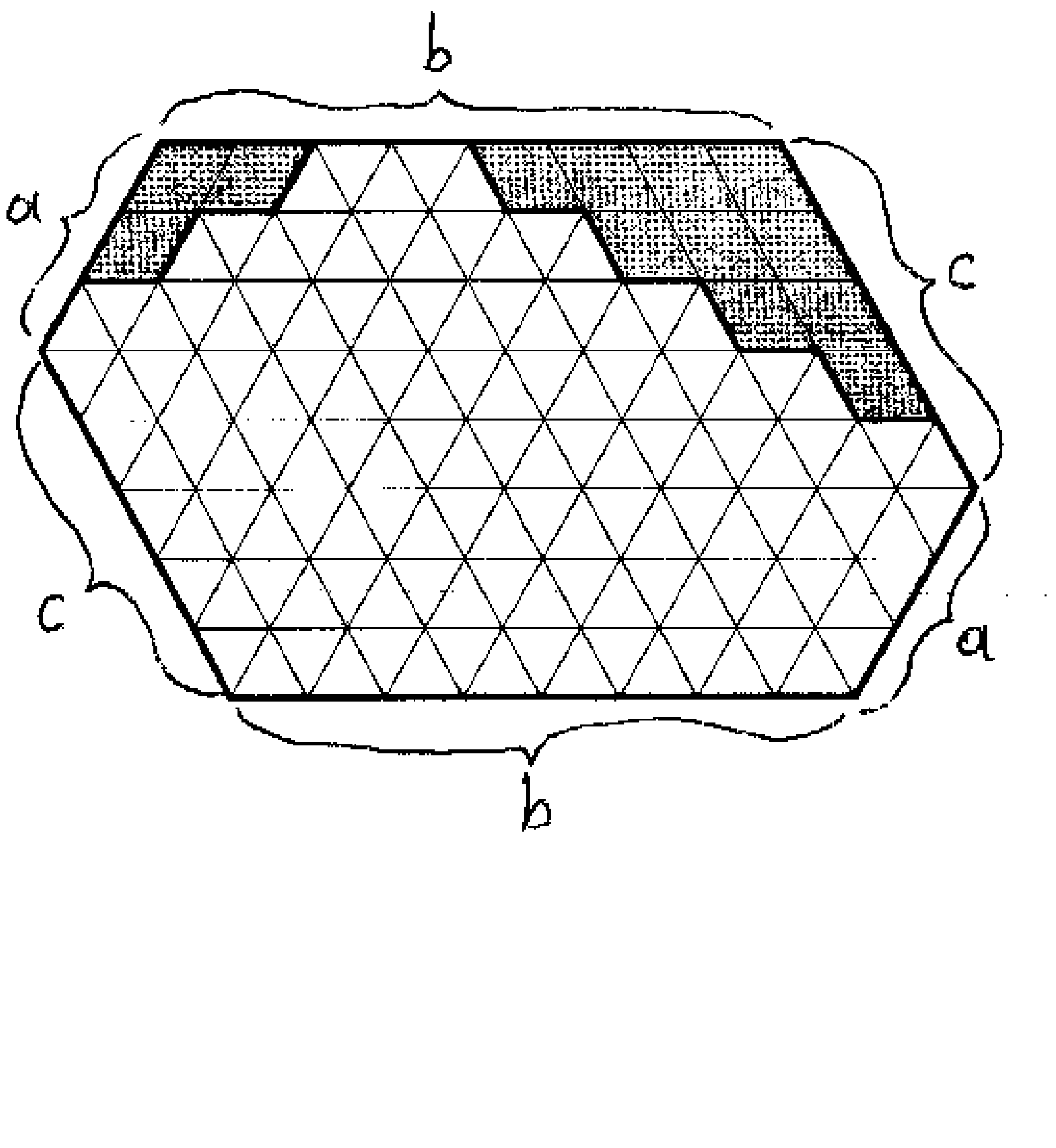
where

$$H(n) := \begin{cases} \prod_{k=0}^{n-1} k! & \text{if } n \text{ is an integer,} \\ \prod_{k=0}^{n-1/2} \Gamma(k+\frac{1}{2}) & \text{if } n \text{ is a half-integer} \end{cases}$$

Theorem. The number of rhombus tilings of a hexagon with side lengths a,b,c,a,b,c with corners cut off as in the figure is equal to

$$\prod_{j=1}^{a} \frac{(j-1)!\,(b+c-a+2j)!\,(b-a-c+2j+1)_j\,(b+2c-a+3j+1)_{a-j}}{(b+2j)!\,(a+c-j)!}$$

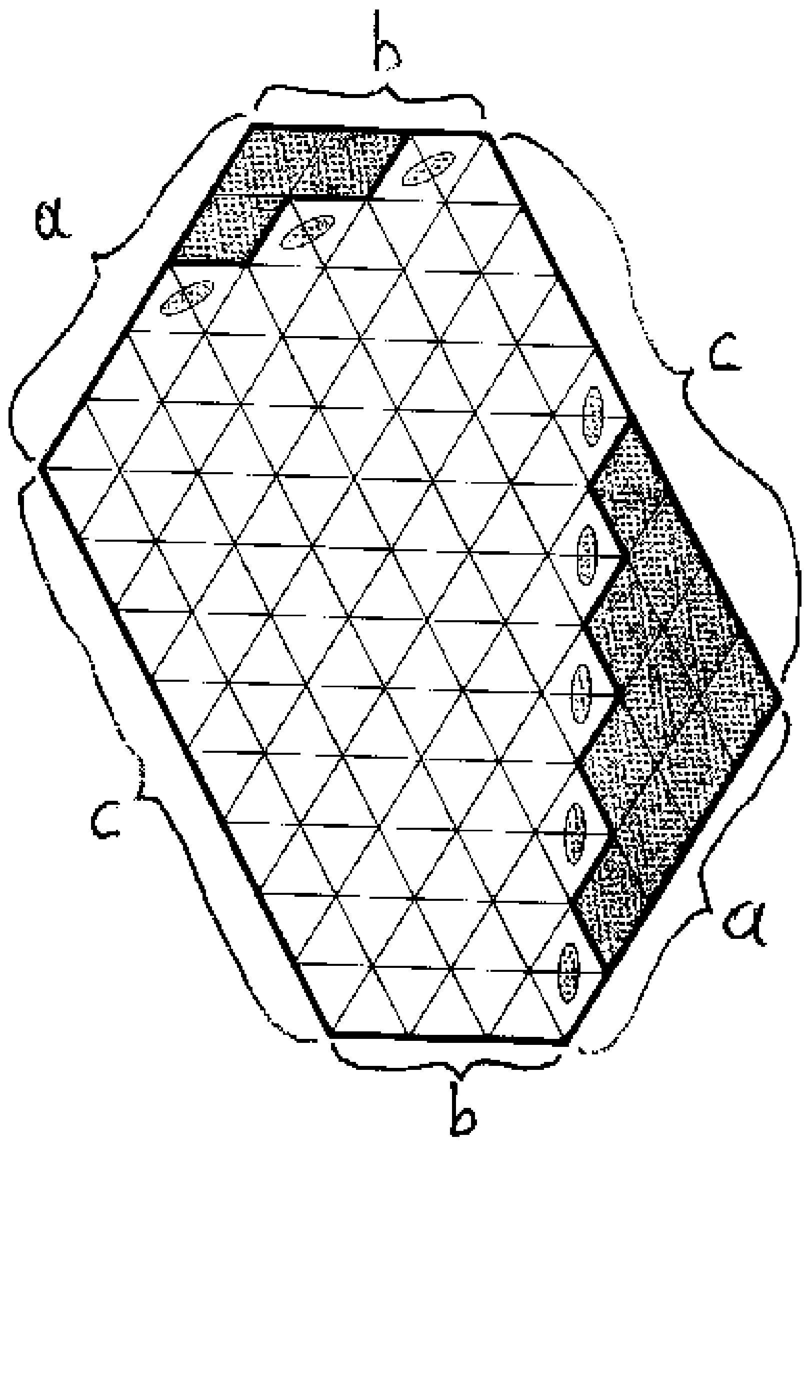
(joint with M. Ciucu)



Theorem. The number of rhombus tilings of a hexagon with side lengths a,b,c,a,b,c with corners cut off as in the figure is equal to

$$\prod_{j=1}^{b} \frac{(j-1)!\,(a+c-b+2j)!\,(3b+c-a-3j+2)_j\,(a+2c+3j+2)_{b-j}}{(a-b+2j-1)!\,(2b+c-j+1)!}.$$

(joint with M. Ciucu)



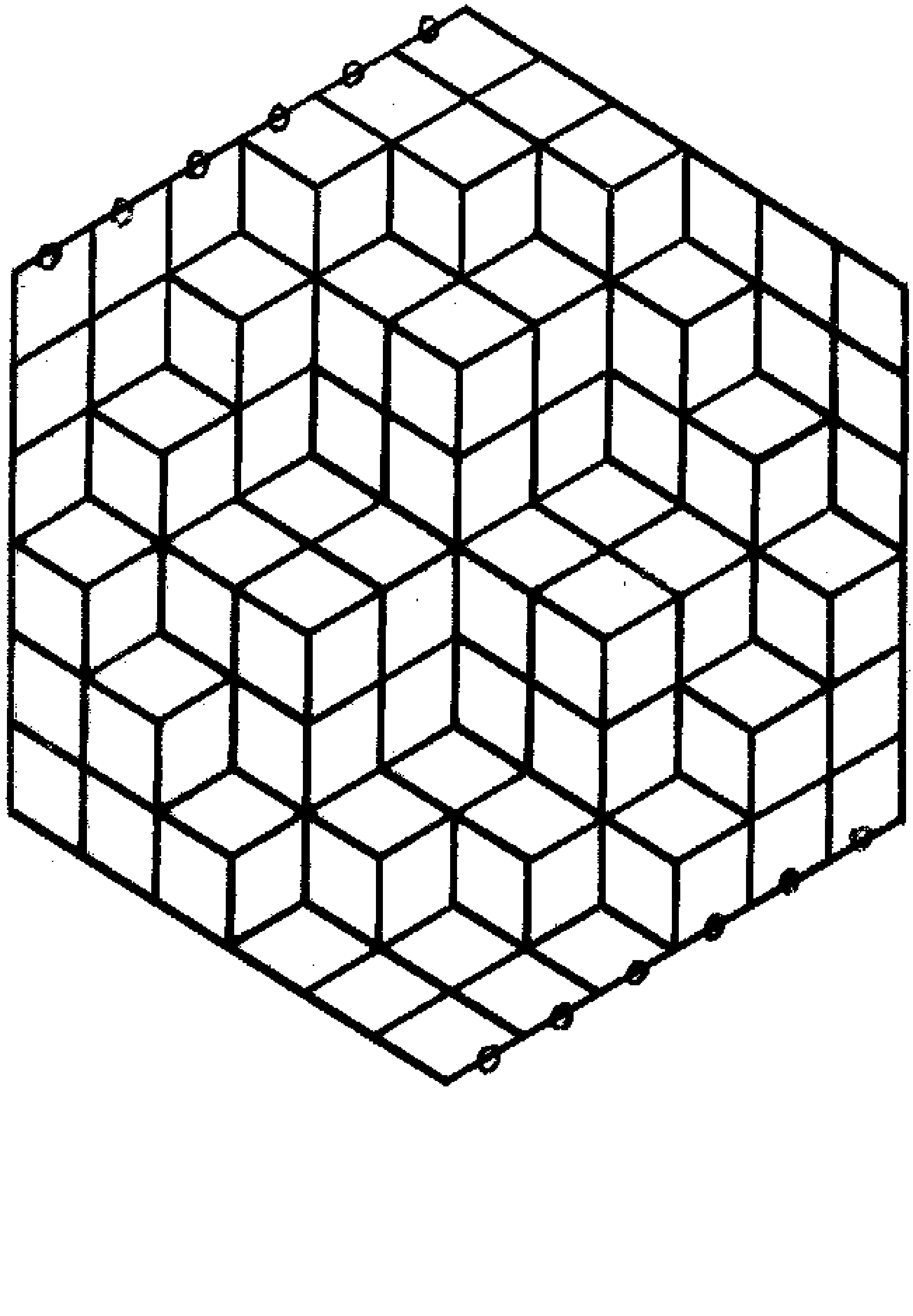
How do we prove these theorems?

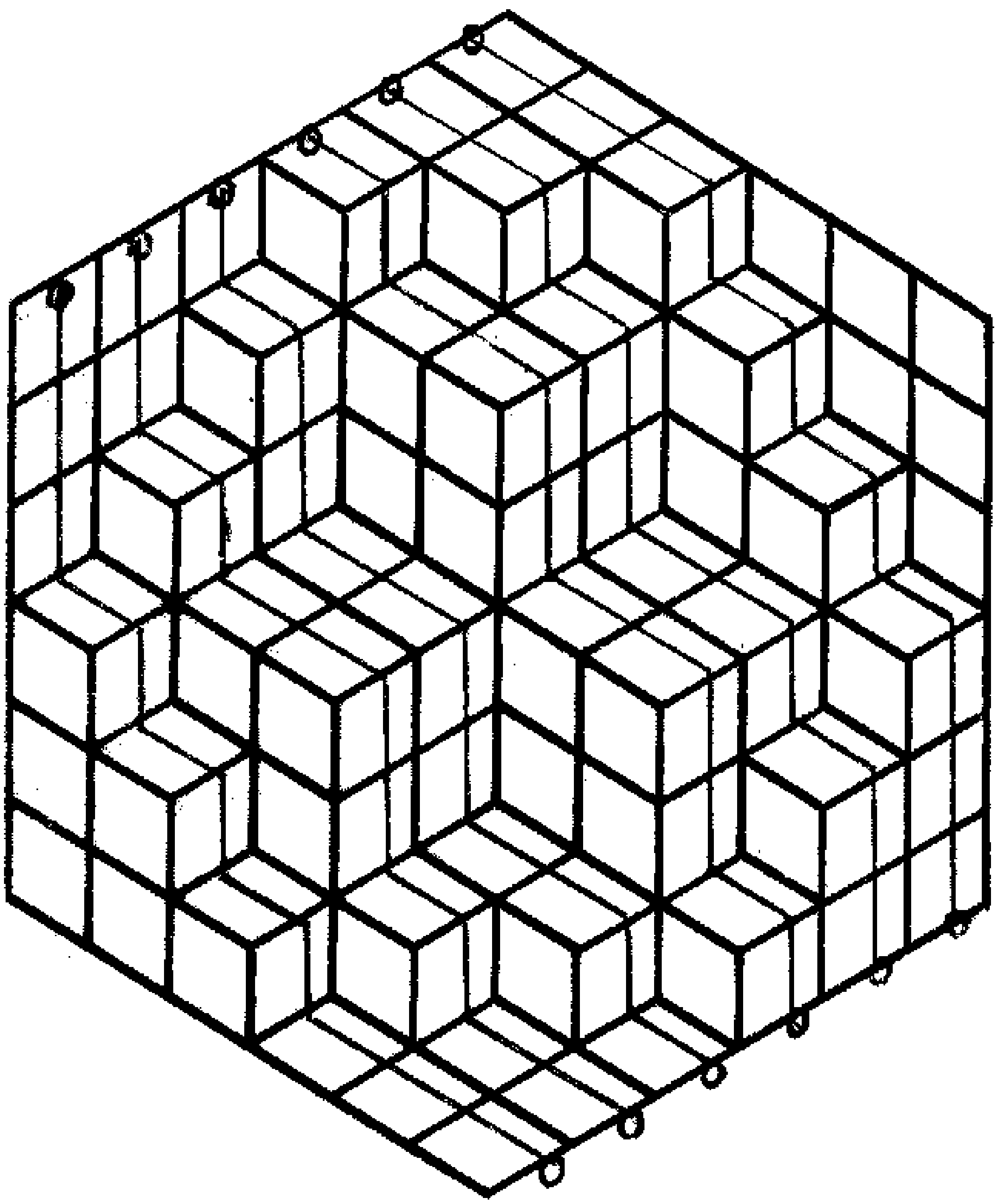
1) Convert the rhombus tilings into non-intersecting lattice paths.

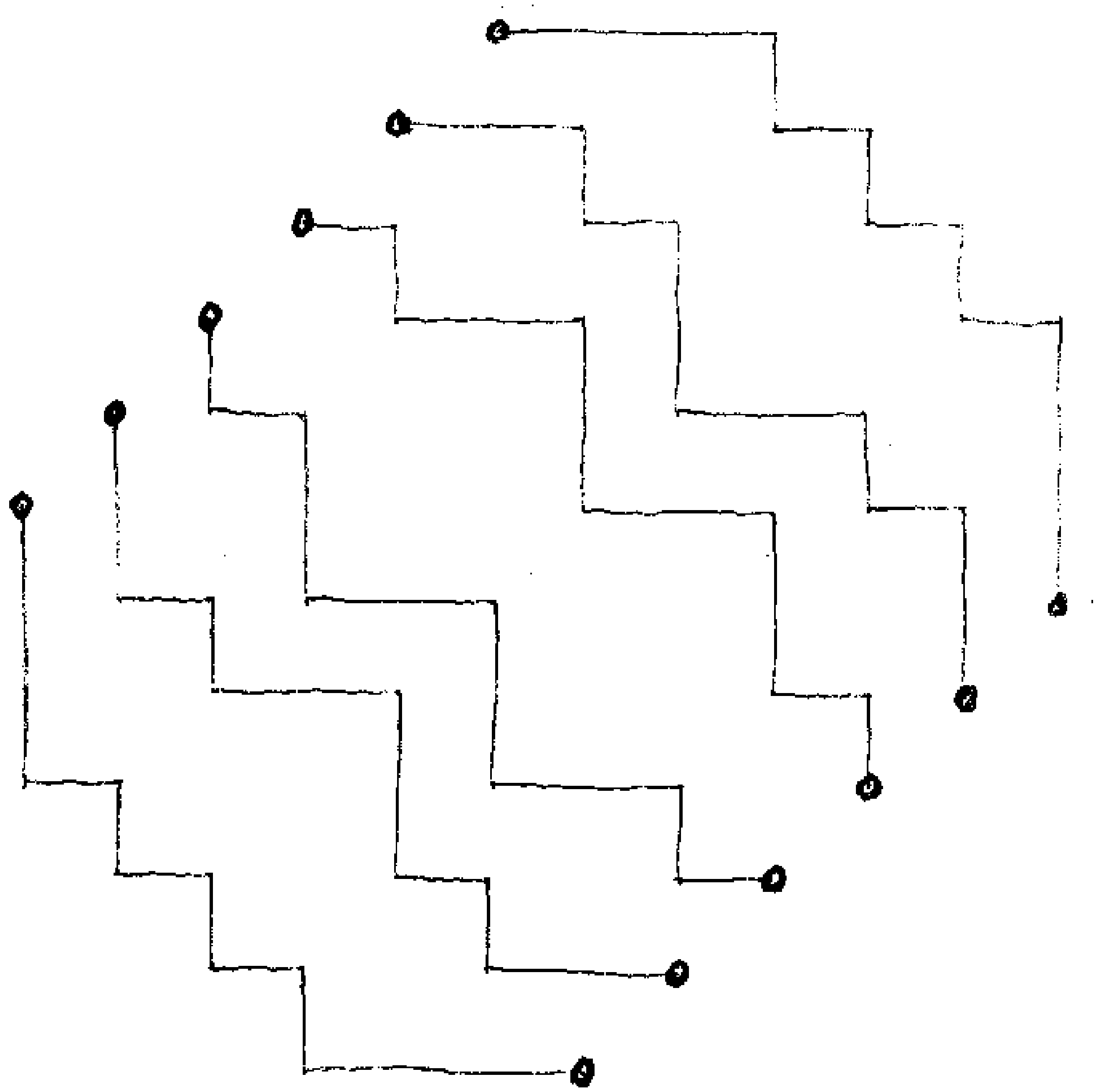
2) Use one of the theorems on non-intersecting lattice paths to find a determinant for the number of tilings.

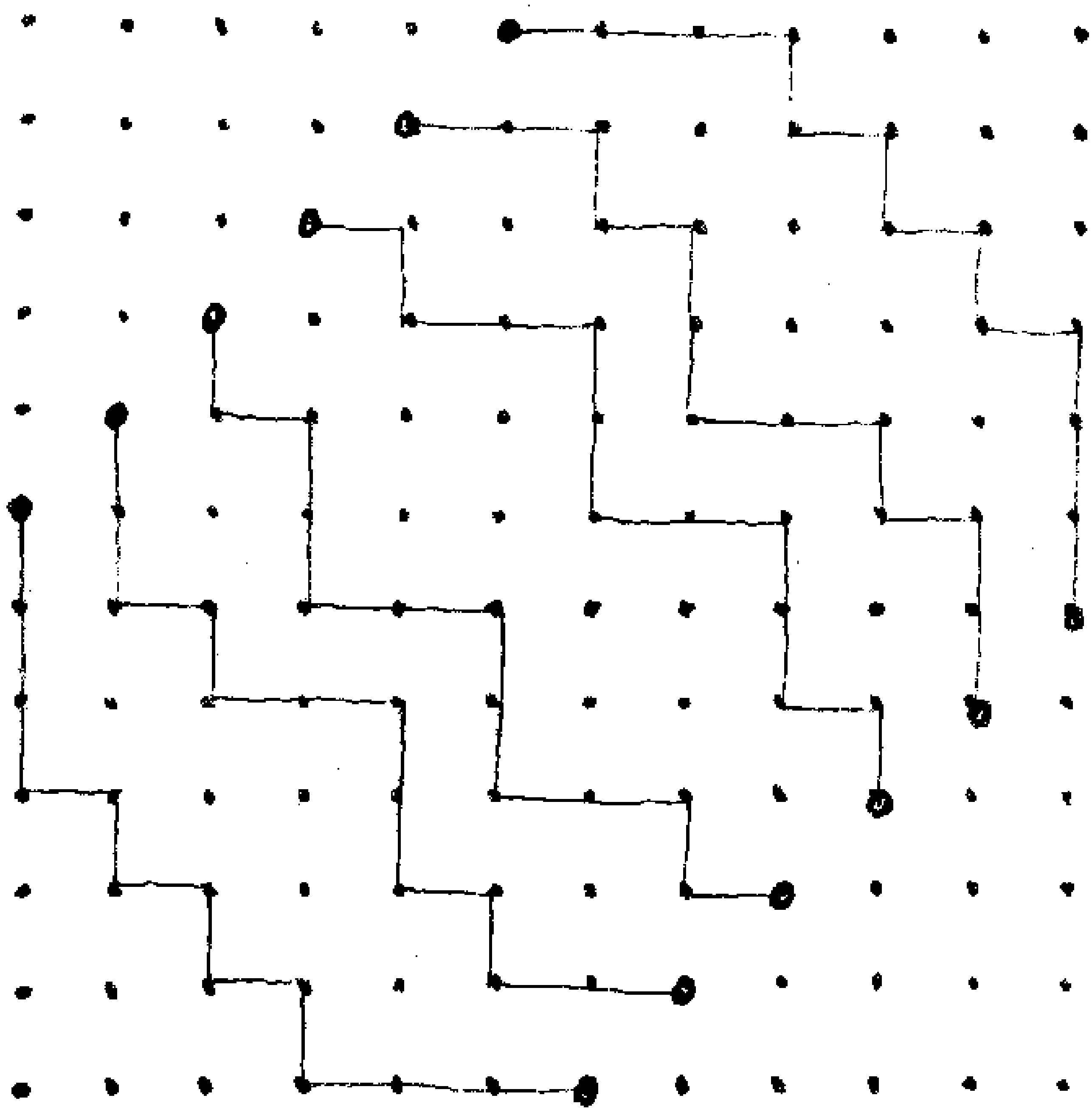
3) Evaluate the determinant.

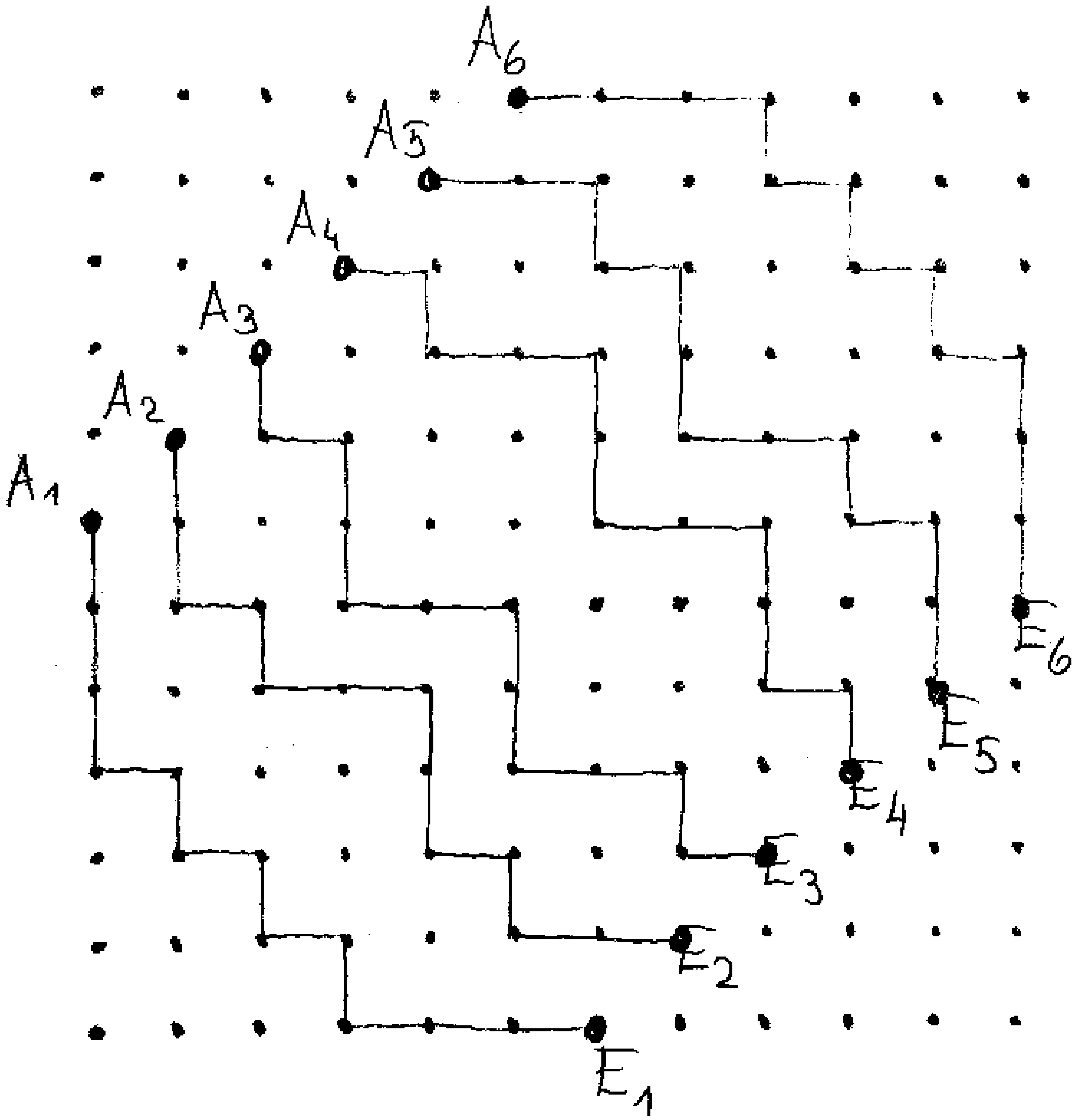
The conversion to non-intersecting lattice paths











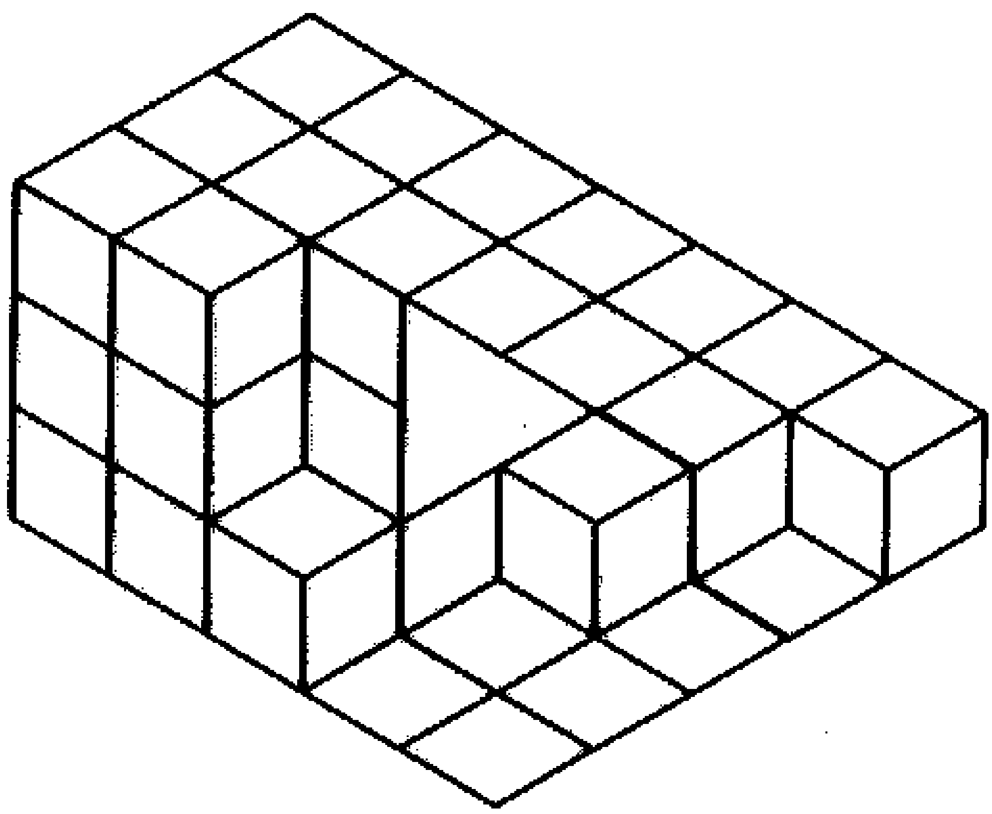
A1
A2
A3
A4
A5
A6
E1
E2
E3
E4
E5
E6

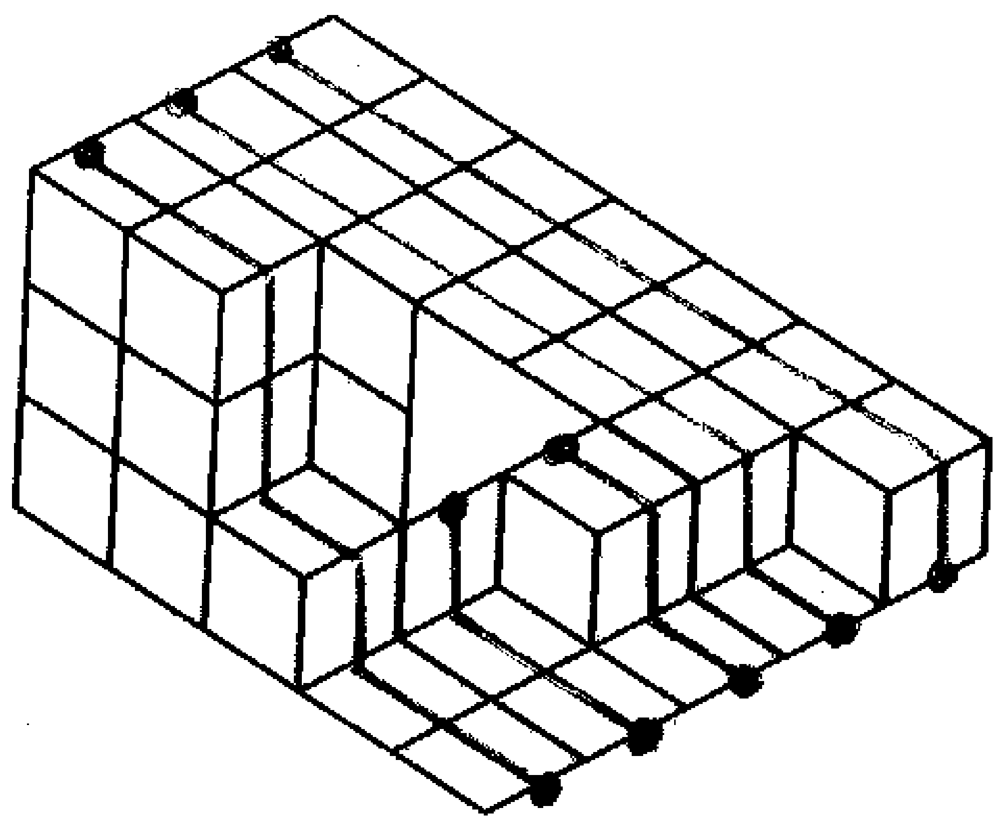
The Karlin-McGregor-Lindström-Gessel-Viennot-Fisher-John-Sachs-Gronau-Just-Schade-Scheffler-Wojciechowski Theorem

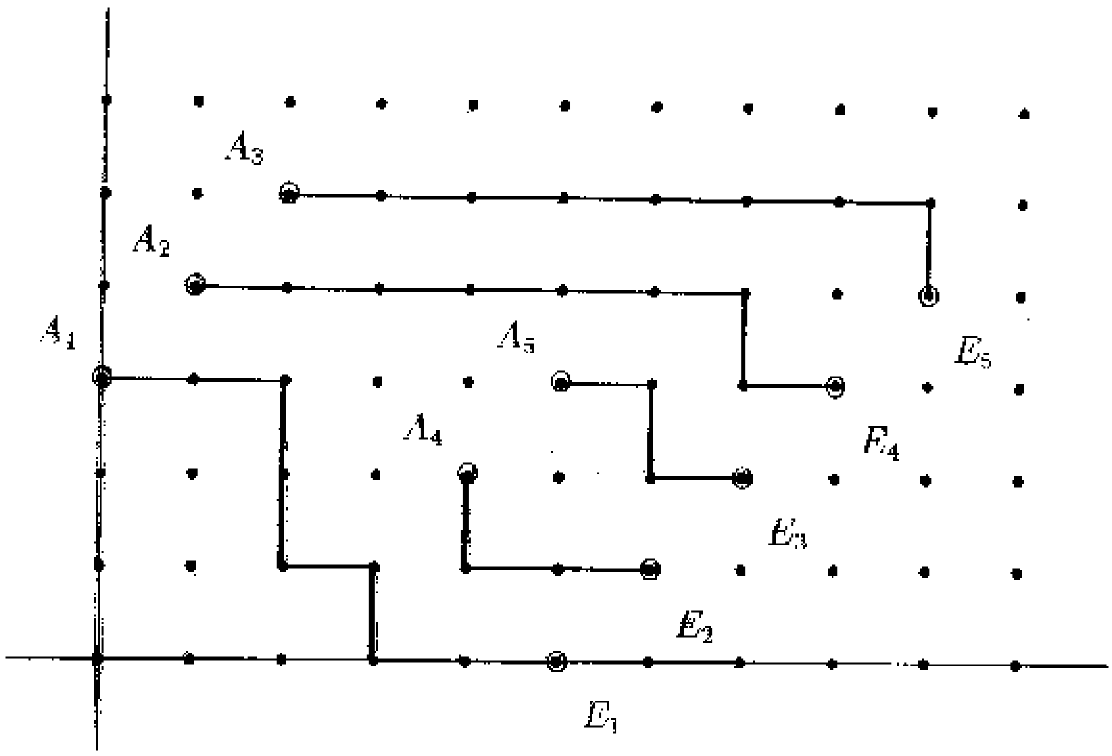
Let $A_1, A_2, \dots, A_n$ and $E_1, E_2, \dots, E_n$ be lattice points. Then the number of all families $(P_1, P_2, \dots, P_n)$ of lattice paths with no common points, P_i running from A_i to E_i, is given by

$$\det_{1 \le i,j \le n} \left(|\mathcal{P}(A_j \to E_i)| \right),$$

where $\mathcal{P}(A \to E)$ denotes the set of all lattice paths from A to E.







A_3
A_2
A_1
A_5
A_4
E_5
E_4
E_3
E_2
E_1

The theorem (as stated) is only true if for all $i<j$ and $k<l$ any path from A_i to E_l intersects any path from A_j to E_k.

The Karlin-McGregor-Lindström-Gessel-Viennot-Fisher-John-Sachs-Gronau-Just-Schade-Scheffler-Wojciechowski

Theorem

Let $A_1, A_2, \ldots, A_n$ and $E_1, E_2, \ldots, E_n$ be lattice points. Then the number of all families $(P_1, P_2, \ldots, P_n)$ of lattice paths with no common points, P_i running from A_i to E_i, is given by

$$\det_{1\le i,j\le n}\left(\left| \mathcal{P}(A_j \to E_i) \right| \right),$$

where $\mathcal{P}(A\to E)$ denotes the set of all lattice paths from A to E.

The Lindström-Gessel-Viennot Theorem (general form)

Let $A_1, A_2, \dots, A_n$ and $E_1, E_2, \dots, E_n$ be arbitrary lattice points. Then

$$\sum_{\mathcal{P}} \operatorname{sgn}(\mathcal{P}) = \det_{1 \le i,j \le n} \left(|\mathcal{P}(A_j \to E_i)| \right),$$

where the sum is over all families $\mathcal{P} = (P_1, P_2, \dots, P_n)$ of nonintersecting lattice paths, P_i running from $A_{\sigma(i)}$ to E_i, for some permutation σ (which may depend on $\mathcal{P}$), and where

$$\operatorname{sgn}(\mathcal{P}) := \operatorname{sgn} \sigma .$$

Proof. $\det_{1\le i,j\le n}\left(|P(A_j \to E_i)|\right)$

$$= \sum_{\sigma\in S_n} \operatorname{sgn}\sigma \underbrace{\prod_{i=1}^{n} |P(A_{\sigma(i)} \to E_i)|}$$

number of n-tuples $(P_1, P_2, \dots, P_n)$

$P_1 : A_{\sigma(1)} \to E_1$

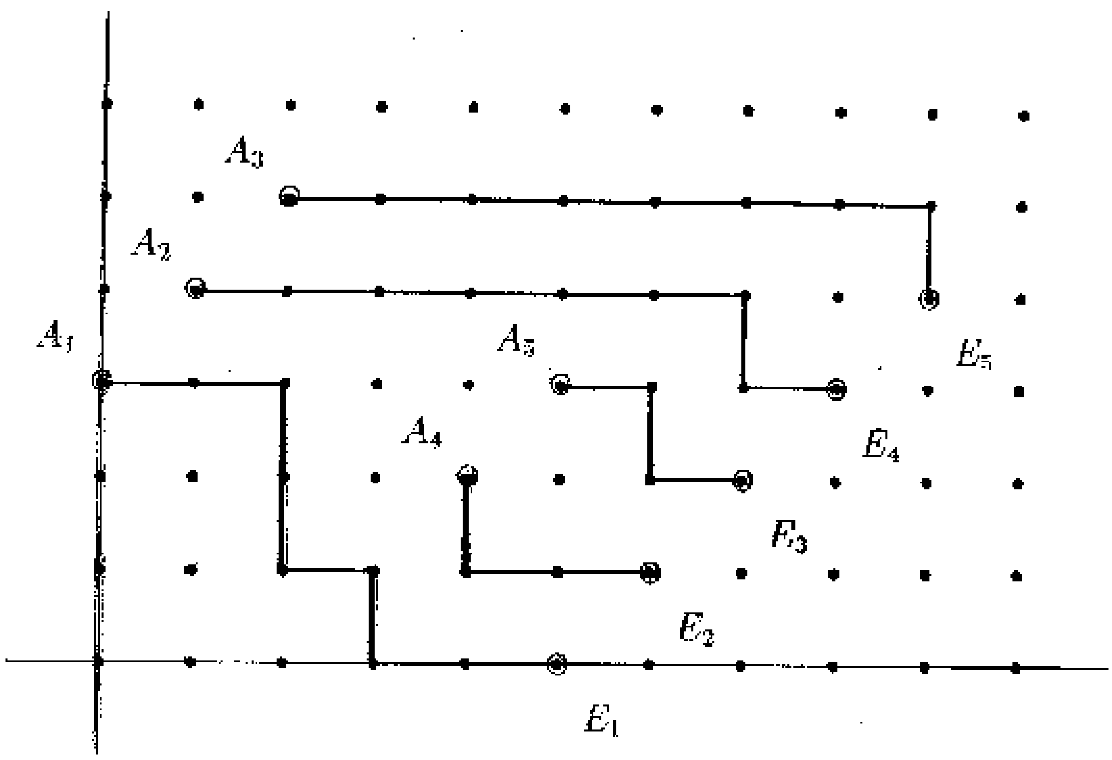
$P_2 : A_{\sigma(2)} \to E_2$

$\dots\dots\dots$

$P_n : A_{\sigma(n)} \to E_n$

$$= \sum_{\substack{(\sigma, P_1, \dots, P_n) \\ P_1 : A_{\sigma(1)} \to E_1 \\ P_2 : A_{\sigma(2)} \to E_2 \\ \dots \\ P_n : A_{\sigma(n)} \to E_n}} \operatorname{sgn}\sigma \ .$$

Now apply the path-portion-switching involutio
Then only the non-intersecting path families survive.



A_3
A_2
A_1
A_5
A_4
E_5
E_4
E_3
E_2
E_1

In the picture:

$$
\begin{array}{llllll}
E_i: & 1 & 2 & 3 & 4 & 5 \\
A_{G(i)}: & 1 & \textcircled{4} & \textcircled{5} & 2 & 3
\end{array}
$$

We could also have:

$$
\begin{array}{llllll}
E_i: & 1 & 2 & 3 & 4 & 5 \\
A_{G(i)}: & 1 & 2 & \textcircled{4} & \textcircled{5} & 3
\end{array}
$$

The permutations have the same sign!

This yields the determinant

$$\det_{1\le i,j\le a+m}\begin{pmatrix} \dbinom{b+c+m}{b-i+j} & 1\le i\le a \\[2ex] \dbinom{\frac{b+c}{2}}{\frac{b+a}{2}-i+j} & a+1\le i\le a+m \end{pmatrix}.$$