How can we evaluate determinants?

After "bare hands" methods (such as playing with row and column operations) have failed, try:

Method 1: Condensation.*

Let M be an $n\times n$ matrix, and let $M^{i_1,\dots,i_k}_{j_1,\dots,j_k}$ denote the submatrix of M, which arose from M by deleting rows $i_1,\dots,i_k$ and columns $j_1,\dots,j_k$. Then

$$|M| = \frac{|M_1^1|\cdot|M_n^n| - |M_n^1|\cdot|M_1^n|}{|M_{1,n}^{1,n}|}.$$

(Desnanot, Jacobi)

This can be used for an inductive proof of a conjectured determinant evaluation.

---

* needs at least 2 parameters.

Example.

$$\det_{1\le i,j\le n}\left(\frac{(x+y+i+j-1)!}{(x+2i-j)!\,(y+2j-i)!}\right)$$

$$=\prod_{i=1}^{n}\frac{(i-1)!\,(x+y+i)!\,(2x+y+2i+1)_{i-1}\,(x+2y+2i+1)_{i-1}}{(x+2i-1)!\,(y+2i-1)!}.$$

Write $M_n(x,y)$ for the matrix

$$\left(\frac{(x+y+i+j-1)!}{(x+2i-j)!\,(y+2j-i)!}\right)_{1\le i,j\le n}.$$

Then:

$$\left(M_n(x,y)\right)_1^1 = M_{n-1}(x+1,y+1)$$

$$\left(M_n(x,y)\right)_n^n = M_{n-1}(x,y)$$

$$\left(M_n(x,y)\right)_n^1 = M_{n-1}(x+2,y-1)$$

$$\left(M_n(x,y)\right)_1^n = M_{n-1}(x-1,y+2)$$

$$\left(M_n(x,y)\right)_{1,n}^{1,n} = M_{n-2}(x+1,y+1).$$

Method 2: The "identification of factors" method *

---

* needs at least 1 parameter.

A short evaluation of the Vandermonde determinant

$$\det [illegible]$$

Proof. If $z(i_1) = z(i_2)$ with $i_1 \neq i_2$, the determinant vanishes. Hence,

$$\prod [illegible]$$

divides the determinant.
On the other hand, the determinant is a polynomial in the $z(i)$'s of degree $\leq \binom{n}{2}$. Therefore the determinant equals this product times a constant.
To compute the constant, compare coefficients of $z(0)^0 z(1)^1 \cdots z(n-1)^{n-1}$. □

S1) Identification of factors
S2) Determination of a degree bound
S3) Computation of the multiplicative constant

The Mills-Robbins-Rumsey determinant

[illegible]

S1) Let us check that [illegible] is a factor of the determinant. In order to do that, we have to show that it vanishes at [illegible]. The latter will be established if we find a nonzero vector in the kernel of the underlying matrix.

More generally:
For proving that $(x+n)^E$ divides the determinant, we find E linearly independent vectors in the kernel of the underlying matrix.

How would we do that?

We go to the computer,
crank out the vectors in the kernel
for [illegible] . . . ,
and try to make a guess what they
are in general.

Once we have worked out a guess,
we prove the corresponding
(hypergeometric) identities.

```
In[1]:= V[2]

Out[1]= {0, c[1]}

In[2]:= V[3]

Out[2]= {0, c[2], c[2]}

In[3]:= V[4]

Out[3]= {0, c[1], 2 c[1], c[1]}

In[4]:= V[5]

Out[4]= {0, c[1], 3 c[1], c[3], c[1]}

In[5]:= V[6]

Out[5]= {0, c[1], 4 c[1], 2 c[1] + c[4], c[4], c[1]}

In[6]:= V[7]

Out[6]= {0, c[1], 5 c[1], c[3], -10 c[1] + 2 c[3], -5 c[1] + c[3], c[1]}

In[7]:= V[8]

Out[7]= {0, c[1], 6 c[1], c[3], -25 c[1] + 3 c[3], c[5], -9 c[1] + c[3], c[1]}

In[8]:= V[9]

Out[8]= {0, c[1], 7 c[1], c[3], -49 c[1] + 4 c[3],
>     -28 c[1] + 2 c[3] + c[6], c[6], -14 c[1] + c[3], c[1]}

In[9]:= V[10]

Out[9]= {0, c[1], 8 c[1], c[3], -84 c[1] + 5 c[3], c[5],
>     196 c[1] - 10 c[3] + 2 c[5], 98 c[1] - 5 c[3] + c[5], -20 c[1] + c[3],
>     c[1]}

In[10]:= V[11]

Out[10]= {0, c[1], 9 c[1], c[3], -132 c[1] + 6 c[3], c[5],
      648 c[1] - 25 c[3] + 3 c[5], c[7], 234 c[1] - 9 c[3] + c[5],
      -27 c[1] + c[3], c[1]}
```

```
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Out[8]= {0, c[1], 7 c[1], c[3], -49 c[1] + 4 c[3],
      -28 c[1] + 2 c[3] + c[6], c[6], -14 c[1] + c[3], c[1]}

n[9]:= V[10]

ut[9]= {0, c[1], 8 c[1], c[3], -84 c[1] + 5 c[3], c[5],
      196 c[1] - 10 c[3] + 2 c[5], 98 c[1] - 5 c[3] + c[5], -20 c[1] + c[3],
      c[1]}

n[10]:= V[11]

ut[10]= {0, c[1], 9 c[1], c[3], -132 c[1] + 6 c[3], c[5],
      648 c[1] - 25 c[3] + 3 c[5], c[7], 234 c[1] - 9 c[3] + c[5],
      -27 c[1] + c[3], c[1]}
```

```
In[1]:= V[2]
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Out[2]= {0, c[2], c[2]}
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Out[3]= {0, c[1], 2 c[1], c[1]}
In[4]:= V[5]
Out[4]= {0, c[1], 3 c[1], c[3], c[1]}
In[5]:= V[6]
Out[5]= {0, c[1], 4 c[1], 2 c[1] + c[4], c[4], c[1]}
In[6]:= V[7]
Out[6]= {0, c[1], 5 c[1], c[3], -10 c[1] + 2 c[3], -5 c[1] + c[3], c[1]}
In[7]:= V[8]
Out[7]= {0, c[1], 6 c[1], c[3], -25 c[1] + 3 c[3], c[5], -9 c[1] + c[3], c[1]}
In[8]:= V[9]
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     c[1]}
In[10]:= V[11]
Out[10]= {0, c[1], 9 c[1], c[3], -132 c[1] + 6 c[3], c[5],
     648 c[1] - 25 c[3] + 3 c[5], c[7], 234 c[1] - 9 c[3] + c[5],
     -27 c[1] + c[3], c[1]}
```

(set $c_1=1$, $c_3=3$)

(set $c_1=1$, $c_4=4$)

Set $c_1 = 0$.
The pattern persists!
(in a shifted sense)

Apparently,

$$\left(0, \binom{n-2}{0}, \binom{n-1}{1}, \binom{n-2}{2}, \ldots, \binom{n-2}{n-2}\right)$$

is in the kernel.

Now we have to prove:

$$\sum_{j=1}^{n-1} \binom{n-2}{j-1} \binom{-n+i+j}{2i-j} = 0 .$$

Zeilberger's algorithm proves binomial identities automatically

Let [illegible], where

$$F(n,j) \; \text{[illegible]}$$

where $P(n,j)$ is a polynomial in n and j, and $a_\ell, b_\ell, u_\ell, v_\ell$ are integers.

Then Zeilberger's algorithm finds $G(n,j)$ and polynomials $d_0(n), d_1(n), \dots, d_k(n)$, for some k, such that

$$d_0(n)\,F(n,j) + d_1(n)\,F(n+1,j) + \cdots + d_k(n)\,F(n+k,j)$$
$$= G(n,j+1) - G(n,j).$$

Summing over j we get

$$d_0(n)\,S(n) + d_1(n)\,S(n+1) + \cdots + d_k(n)\,S(n+k) = 0.$$

To prove a conjectured identity, we just have to check whether the conjectured right-hand side satisfies the same recurrence, plus a few initial values.

```
> read Ekhad;
```

Version of Sept. 13, 2000: adapted to Maple 6
Also works on Maple 5 and below
In the penultimate Version of Feb 25, 1999 a suggestion
of Frederic Chyzak was used, with considerable
speed-up. We thank him SO MUCH!
The penpenultimate version, Feb. 1997,
corrected a subtle bug discovered by Helmut Prodinger
Previous versions benefited from comments by Paula Cohen,
Lyle Ramshaw, and Bob Sulanke.
This is EKHAD, One of the Maple packages
accompanying the book
"A=B"
(published by A.K. Peters, Wellesley, 1996)
by Marko Petkovsek, Herb Wilf, and Doron Zeilberger.
The most current version is available on WWW at:
http://www.math.temple.edu/~zeilberg .
*Information about the book, and how to order it, can be found i *
n
http://www.central.cis.upenn.edu/~wilf/AeqB.html .
Please report all bugs to: zeilberg@math.temple.edu .
*All bugs or other comments used will be acknowledged in futur *
e
versions.

For general help, and a list of the available functions,

type "ezra();". For specific help type "ezra(procedure_name)"

```
> ct(binomial(n-2,j-1)*binomial(-n+i+j,2*i-j),2,j,n,N);
```

$$-3\,n+3+(2\,n-3\,i-1)\,N+(n-i)\,N^2,$$

$$-\frac{(n-1)\,(2\,n-i-j+1)\,(j-1)\,(n+1+i-2\,j)\,(n+i-2\,j)}{(n-j)\,(n-j+1)\,(n-i-j+1)\,(n-i-j)}$$

> [illegible]

[illegible]

where

$$S(n) = \sum_j \binom{n-2}{j-1}\binom{-n+i+j}{2i-j} = \sum_j F(n,j)$$

The term in the last line is the ratio

[illegible] $/ F(n,j)$.

Method 3: LU-factorization

$$\det M(n) = ?$$

$$M(n)\cdot \underbrace{\begin{pmatrix}1 & & & * \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1\end{pmatrix}}_{U(n)} = \underbrace{\begin{pmatrix}* & & & 0\\ & * & & \\ & & \ddots & \\ * & & & *\end{pmatrix}}_{L(n)}$$

Go to the computer, crank out $U(n)$, $L(n)$ for $n=1,2,3,\ldots$, try to make a guess for the entries.

Then we need "only" prove the underlying identities.

Both methods, LU-factorization and "identification of factors" are similar in spirit:

- first, there is something to guess
- then, the guess is proved by proving some (hypergeometric) identity

Both steps can be largely automatised!

Proving hypergeometric identities:

Zeilberger algorithm (single sums)

WZ method, Karr's algorithm
(multiple sums)

Automatic guessing

- "Superseeker", the electronic version of the Sloane-Plouffe Handbook of Integer Sequences
- "gfun", a Maple package developed by Bruno Salvy and Paul Zimmermann

Automatic guessing of
"nice" formulas

```
{1, 2, 5, 14, 42, 132, 429, 1430,
 4862, 16796, 58786, 208012,
 742900, 2674440, 9694845,
 35357670, 129644790, 477638700,
 1767263190, 6564120420}

In[5]:= FactorInteger[477638700]

Out[5]= {{2, 2}, {3, 1},
  {5, 2}, {7, 1}, {11, 1},
  {23, 1}, {29, 1}, {31, 1}}

In[6]:= FactorInteger[1767263190]

Out[6]= {{2, 1}, {3, 1}, {5, 1},
  {7, 1}, {11, 1}, {23, 1},
  {29, 1}, {31, 1}, {37, 1}}

In[7]:= FactorInteger[6564120420]

Out[7]= {{2, 2}, {3, 1}, {5, 1},
  {11, 1}, {13, 1}, {23, 1},
  {29, 1}, {31, 1}, {37, 1}}
```

```
{1, 2, 9, 272, 589185}
In[9]:= FactorInteger[272]
Out[9]= {{2, 4}, {17, 1}}
In[10]:= FactorInteger[589185]
Out[10]= {{3, 2}, {5, 1}, {13093, 1}}
```

(The number of perfect matchings of the n-dimensional hypercube)

"Definition". A "nice" formula is a formula of the type

[illegible]

where [illegible] and [illegible]

In[1]:= << rate.m

In[2]:= Rate[1, 2, 3, 4]

Out[2]= {i0}

In[3]:= Rate[1, 3, 6, 10]

Out[3]= $\left\{\frac{1}{2}\,\text{i0}\,(1+\text{i0})\right\}$

In[4]:= Rate[1, 2, 6, 24]

Out[4]= {Gamma[1 + i0]}

In[5]:= Rate[1, 2, 7, 42, 429, 7436, 218348, 10850216]

Out[5]= $$\left\{\prod_{\text{i1}=1}^{-1+\text{i0}}\left(2^{-2-4\,\text{i1}}\,3^{\frac{3}{2}+3\,\text{i1}}\,\text{Gamma}\left[\frac{2}{3}+\text{i1}\right]\,\text{Gamma}\left[\frac{4}{3}+\text{i1}\right]\right)\Big/\left(\text{Gamma}\left[\frac{1}{2}+\text{i1}\right]\,\text{Gamma}\left[\frac{3}{2}+\text{i1}\right]\right)\right\}$$

Automatic guessing

The sequences which are encountered within the binomial/hypergeometric "paradigm" are of the following form:

$(a_n)_{n\geq 1}$, where a_n equals

rational function of n

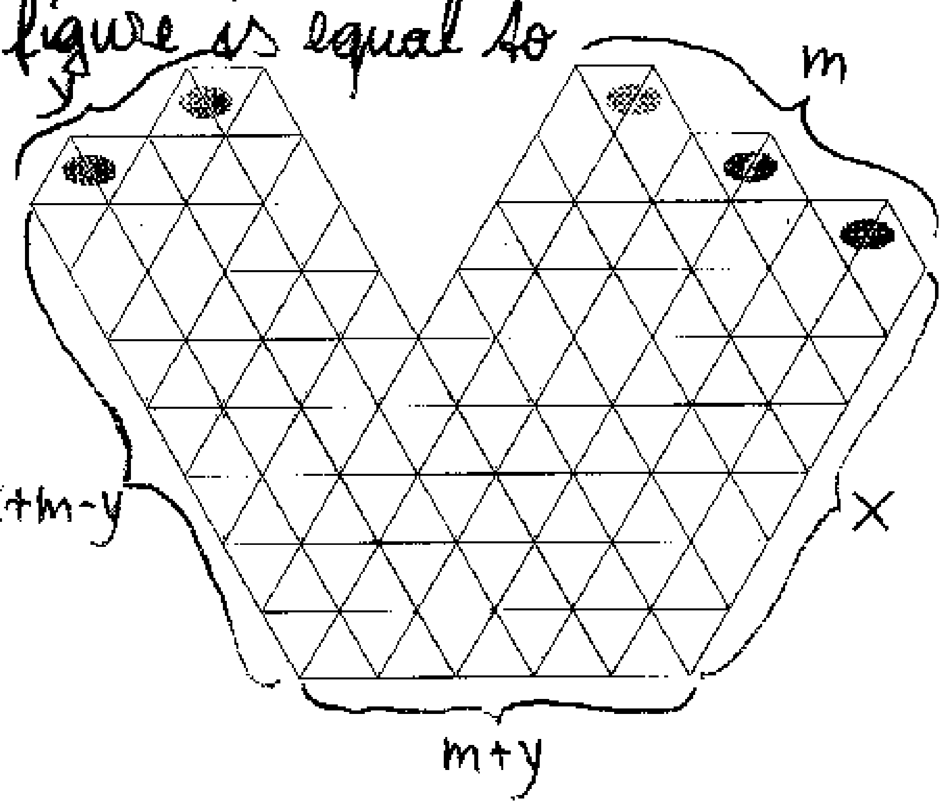
or $\prod_{i=1}^{n}$ (rational function of i)

or $\prod_{i=1}^{n}\prod_{j=1}^{i}$ (rational function of j)

etc.

Given enough values of the sequence, these forms can be found using rational interpolation.

Conjecture. The number of rhombus tilings of the V-shaped part of a hexagon with side lengths $x, m+y, x+m-y, x, m+y, x+m-y$ as shown in the figure is equal to



$$\prod_{i=1}^{[illegible]} (x-i+m+y+1)!\,(i-1)! \prod_{i=m+1}^{m+y} (x+2y-2i+1)!\,(m+2y+i-1)! \quad [illegible]$$

$$\times 2^{\binom{m}{2}+\binom{y}{2}} \prod_{i=1}^{m-1} i! \prod_{i=1}^{y-1} i! \prod_{i\geq 0} \left(x+i+\frac{3}{2}\right)_{m-2i-1}$$

$$\times \prod_{i\geq 0} \left(x-y+\frac{5}{2}+3i\right)_{\left\lfloor \frac{3y}{2}-\frac{9i}{2}\right\rfloor-2} \prod_{i\geq 0} \left(x+\frac{3m}{2}-y+\left\lceil\frac{3i}{2}\right\rceil+\frac{3}{2}\right)_{3\left\lceil\frac{y}{2}\right\rceil-\left\lceil\frac{9i}{2}\right\rceil}$$

$$\times \prod_{i\geq 0} \left(x+\frac{3m}{2}-y+\left\lfloor\frac{3i}{2}\right\rfloor+2\right)_{3\left\lfloor\frac{y}{2}\right\rfloor-\left\lfloor\frac{9i}{2}\right\rfloor-1}$$

$$\times \prod_{i\geq 0} \left(x+m-\left\lfloor\frac{y}{2}\right\rfloor+i+1\right)_{2\left\lfloor\frac{y}{2}\right\rfloor-m-2i} \prod_{i\geq 0} \left(x+\left\lfloor\frac{y}{2}\right\rfloor+i+2\right)_{m-2\left\lfloor\frac{y}{2}\right\rfloor-2i-2}$$

$$\times \frac{\prod_{i=0}^{y} (x-y+3i+1)_{m+2y-4i} \prod_{i=0}^{\left\lceil\frac{y}{2}\right\rceil-1} (x+m-y+i+1)_{3y-m-4i}}{\prod_{i\geq 0} \left(x+\frac{m}{2}-\frac{y}{2}+i+1\right)_{y-2i} \left(x+\frac{m}{2}-\frac{y}{2}+i+\frac{3}{2}\right)_{y-2i-1}}$$

$$\times \frac{\prod_{i=0}^{y} (x+i+2)_{2m-2i-1}}{(x+y+2)_{m-y-1}\,(m+x-y+1)_{m+y}}.$$

In order to prove the conjecture, it "suffices" to evaluate the determinant

$$\det_{1\le i,j\le m+y}\left(\begin{cases}\binom{x+i}{x-i+j} & i=1,\dots,m\\[2ex] \binom{x+2m-i+1}{m+y-2i+j+1} & i=m+1,\dots,m+y\end{cases}\right).$$