Enumeration of standard Young tableaux of shapes of the form “staircase minus rectangle”

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Standard Young tableaux

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Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be two n-tuples of non-negative integers which are in non-increasing order and satisfy $\lambda_i \geq \mu_i$ for all i.

A standard Young tableau of skew shape λ/μ is an arrangement of the numbers $1, 2, \ldots, \sum_{i=1}^{n}(\lambda_i - \mu_i)$ of the form

$$\begin{array}{ccccccc} & & & \pi_{1,\mu_1+1} & \cdots\cdots\cdots & & \pi_{1,\lambda_1} \\ & & \pi_{2,\mu_2+1} \quad \cdots & \pi_{2,\mu_1+1} & \cdots\cdots & \pi_{2,\lambda_2} & \\ & \iddots & & \vdots & \iddots & & \\ \pi_{n,\mu_n+1} & & \cdots\cdots\cdots\cdots\cdots & & \pi_{n,\lambda_n} & & \end{array}$$

such that numbers along rows and columns are increasing.

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such that numbers along rows and columns are increasing.

A standard Young tableau of shape $(6, 5, 4, 3, 2, 1)/(3, 3, 0, 0, 0, 0)$:

$$\begin{array}{cccccc} & & & 2 & 5 & 13 \\ & & & 3 & 9 & \\ 1 & 4 & 8 & 12 & & \\ 6 & 11 & 15 & & & \\ 7 & 14 & & & & \\ 10 & & & & & \end{array}$$

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Classical ballot problem

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In the *n-candidate ballot problem*, there are n candidates C_1, C_2, ..., C_n in an election, C_1 receiving λ_1 votes, C_2 receiving λ_2 votes, ..., C_n receiving λ_n votes. How many ways of counting the votes are there, such that at any stage during the counting C_1 has at least as many votes as C_2, C_2 has at least as many votes as C_3, etc.?

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Example

Let $n = 3$, $\lambda_1 = 5$, $\lambda_2 = 3$, $\lambda_3 = 3$. An "admissible" counting of the votes is

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$\to$ If the i-th letter in this word is C_j, we place i into the j-th row of a tableau, $i = 1, 2, \ldots$ This produces a standard Young tableau.

In general, this defines a bijection between standard Young tableaux of shape λ and "admissible" vote countings for the ballot problem, where candidate C_j receives λ_j votes, $j = 1, 2, \ldots, n$.

We can also accomodate *skew* standard Young tableaux:

Ballot problem with initial bias

We can also accomodate *skew* standard Young tableaux:

Given n candidates C_1, C_2, ..., C_n in an election, C_1 receiving λ_1 votes, C_2 receiving λ_2 votes, ..., C_n receiving λ_n votes. How many ways of counting the votes are there, such that we start at a stage where already μ_1 votes were counted for C_1, μ_2 votes for C_2, ..., and μ_n votes for C_n, and at any stage during the (subsequent) counting C_1 has at least as many votes as C_2, C_2 has at least as many votes as C_3, etc.?

Example

Let $n = 6$, $\mu = (3, 3, 0, 0, 0, 0)$, $\lambda = (6, 5, 4, 3, 2, 1)$. An "admissible" counting of the votes is

$$C_3\, C_1\, C_2\, C_3\, C_1\, C_4\, C_5\, C_3\, C_2\, C_6\, C_4\, C_3\, C_1\, C_5\, C_4.$$

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$\rightarrow$ If the i-th letter in this word is C_j, we place i into the j-th row of a tableau. This produces a *skew* standard Young tableau.

Lattice paths and reflection principle

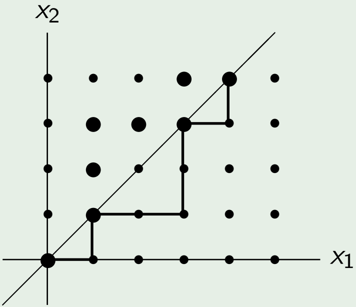
Given a standard Young tableau of shape λ, we form a lattice path from $\mathbf{0} = (0, 0, \dots, 0)$ to $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ by reading through the entries of the tableau (in order) and drawing a step in x_j-direction, if entry i is in the j-th row of the tableaux, $i = 1, 2, \dots$.

This defines a bijection between standard Young tableaux of shape λ and lattice paths from the origin to $(\lambda_1, \lambda_2, \dots, \lambda_n)$ which are staying in the region $x_1 \geq x_2 \geq \cdots \geq x_n$.

Lattice paths and reflection principle

Example

$$\begin{matrix} 1 & 3 & 4 & 7 \\ 2 & 5 & 6 & 8 \end{matrix} \quad \longleftrightarrow$$



More generally, given a standard Young tableau of skew shape λ/μ, we form a lattice path from $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ to $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ by reading through the entries of the tableau (in order) and drawing a step in x_j-direction, if entry i is in the j-th row of the tableaux, $i = 1, 2, \dots$.

This defines a bijection between standard Young tableaux of shape λ/μ and lattice paths from $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ to $(\lambda_1, \lambda_2, \dots, \lambda_n)$ which are staying in the region $x_1 \geq x_2 \geq \cdots \geq x_n$.

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- count the corresponding lattice paths
- by the reflection principle!

The "reflection principle"

For simplicity, let us take $n = 2$, and count lattice paths from (μ_1, μ_2) to (λ_1, λ_2) staying in the region $x_1 \ge x_2$.

Clearly,

$$\begin{aligned} &\left|P\big((\mu_1,\mu_2)\to(\lambda_1,\lambda_2) \mid x_1 \ge x_2\big)\right| \\ &\qquad = \left|P\big((\mu_1,\mu_2)\to(\lambda_1,\lambda_2)\big)\right| \\ &\qquad\qquad - \left|P\big((\mu_1,\mu_2)\to(\lambda_1,\lambda_2) \mid x_1 \not\ge x_2 \text{ at least once}\big)\right|. \end{aligned}$$

The "reflection principle"

$$\begin{aligned}&\left|P\big((\mu_1,\mu_2)\to(\lambda_1,\lambda_2)\mid x_1\ge x_2\big)\right|\\&\qquad=\left|P\big((\mu_1,\mu_2)\to(\lambda_1,\lambda_2)\big)\right|\\&\qquad\qquad-\left|P\big((\mu_1,\mu_2)\to(\lambda_1,\lambda_2)\mid x_1\not\ge x_2\text{ at least once}\big)\right|.\end{aligned}$$

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$\left|P\big((\mu_1,\mu_2)\to(\lambda_1,\lambda_2)\big)\right|$ is given by a binomial coefficient.

We claim:

$$\left|P\big((\mu_1,\mu_2)\to(\lambda_1,\lambda_2)\mid x_1\not\ge x_2 \text{ at least once}\big)\right| \\ = \left|P\big((\mu_2-1,\mu_1+1)\to(\lambda_1,\lambda_2)\big)\right|.$$

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The last expression is again given by a binomial coefficient.

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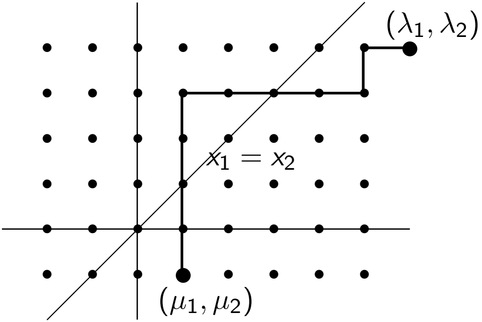

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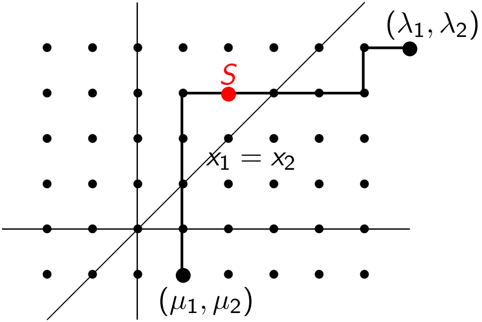

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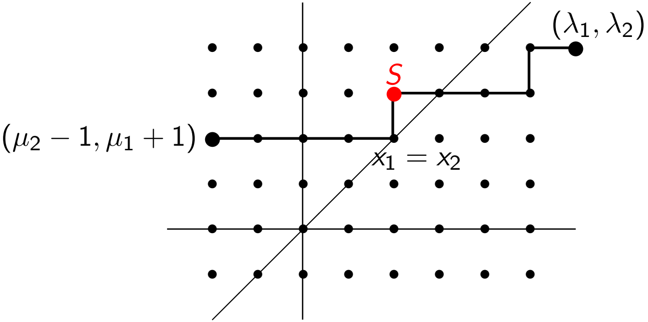

As a consequence, we obtain that the number of standard Young tableaux of shape $(\lambda_1, \lambda_2)/(\mu_1, \mu_2)$ is given by

$$\binom{\lambda_1 - \mu_1 + \lambda_2 - \mu_2}{\lambda_1 - \mu_1} - \binom{\lambda_1 - \mu_1 + \lambda_2 - \mu_2}{\lambda_1 - \mu_2 + 1}$$
$$= \left(\sum_{i=1}^{2} (\lambda_i - \mu_i) \right)! \cdot \det_{1 \le i,j \le 2} \left(\frac{1}{(\lambda_i - i - \mu_j + j)!} \right).$$

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$$=\left(\sum_{i=1}^{2}(\lambda_i-\mu_i)\right)!\cdot\det_{1\le i,j\le 2}\left(\frac{1}{(\lambda_i-i-\mu_j+j)!}\right).$$

More generally:

Aitken's Formula

The number of all standard Young tableaux of shape λ/μ equals

$$\left(\sum_{i=1}^{n}(\lambda_i-\mu_i)\right)!\cdot\det_{1\le i,j\le n}\left(\frac{1}{(\lambda_i-i-\mu_j+j)!}\right).$$

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Are there cases where the determinant can be evaluated?

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Are there cases where the determinant can be evaluated?

Yes! Namely, if $\mu_1=\mu_2=\cdots=\mu_n=0$.

The hook-length formula

Theorem (Frame, Robinson, Thrall)

The number of all standard Young tableaux of shape λ equals

$$\frac{(\lambda_1 + \lambda_2 + \cdots + \lambda_n)!}{\prod_{\rho \in \lambda} h(\rho)},$$

where $h(\rho)$ denotes the hook-length of the cell ρ.

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Example

The hook-lengths of the cells of the shape $(5, 3, 3)$ are

$$\begin{matrix} 7 & 6 & 5 & 2 & 1 \\ 4 & 3 & 2 & & \\ 3 & 2 & 1 & & \end{matrix}$$

Hence, the number of standard Young tableaux of shape $(5, 3, 3)$ is equal to

$$\frac{(5+3+3)!}{7 \cdot 6 \cdot 5 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 1} = 660.$$

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Lattice paths and reflection principle

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Are there cases where the determinant can be evaluated other than $\mu = (0, 0, \ldots, 0)$?

John Stembridge:

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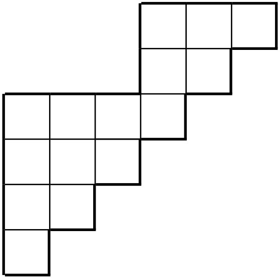
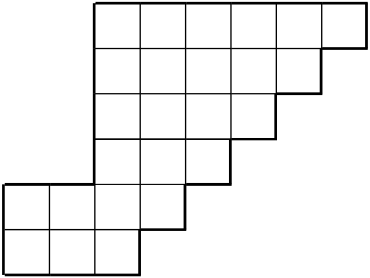
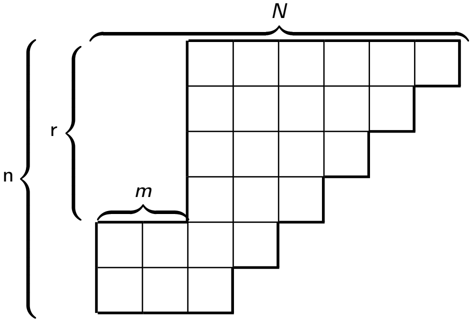


We shall do something more general than DeWitt here: we shall enumerate all standard Young tableaux of a skew shape, where the outer shape is a (possibly incomplete) staircase and the inner shape is a rectangle.



Our goal: Let N, n, m, r be non-negative integers. Compute the number of all standard Young tableaux of shape $(N, N-1, \dots, N-n+1)/(m^r)$, where (m^r) stands for $(m, m, \dots, m, 0, \dots, 0)$ with r components m).



“staircase minus rectnngle”

Recall:

Aitken’s Formula

The number of all standard Young tableaux of shape λ/μ *equals*

$$\left(\sum_{i=1}^{n}(\lambda_i-\mu_i)\right)!\cdot\det_{1\le i,j\le n}\left(\frac{1}{(\lambda_i-i-\mu_j+j)!}\right).$$

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We substitute in Aitken’s formula:

$$\left(\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right)!\det_{1\le i,j\le n}\left(\begin{cases}\dfrac{1}{(N+1-2i-m+j)!} & j\le r\\ \dfrac{1}{(N+1-2i+j)!} & j>r\end{cases}\right).$$

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We now do a Laplace expansion with respect to the first r columns:

$$\left(\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right)!$$
$$\times \sum_{1\le k_1<\dots<k_r\le n}(-1)^{\binom{r+1}{2}+\sum_{i=1}^r k_i}\det_{1\le i,j\le r}\left(\frac{1}{(N+1-2k_i-m+j)!}\right)$$
$$\cdot \det_{\substack{1\le i\le n,\ i\notin\{k_1,\dots,k_r\}\\ r+1\le j\le n}}\left(\frac{1}{(N+1-2i+j)!}\right).$$

“staircase minus rectnngle”

$$\left(\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right)!$$

$$\times \sum_{1\le k_1<\cdots<k_r\le n} (-1)^{\binom{r+1}{2}+\sum_{i=1}^r k_i} \det_{1\le i,j\le r}\left(\frac{1}{(N+1-2k_i-m+j)!}\right)$$

$$\cdot \det_{\substack{1\le i\le n,\, i\notin\{k_1,\dots,k_r\}\\ r+1\le j\le n}}\left(\frac{1}{(N+1-2i+j)!}\right).$$

$$\left(\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right)!$$
$$\times \sum_{1\le k_1<\cdots<k_r\le n}(-1)^{\binom{r+1}{2}+\sum_{i=1}^r k_i}\det_{1\le i,j\le r}\left(\frac{1}{(N+1-2k_i-m+j)!}\right)$$
$$\cdot \det_{\substack{1\le i\le n,\, i\notin\{k_1,\dots,k_r\}\\ r+1\le j\le n}}\left(\frac{1}{(N+1-2i+j)!}\right).$$

Both determinants can be evaluated by means of

$$\det_{1\le i,j\le s}\left(\frac{1}{(X_i+j)!}\right)=\prod_{i=1}^{s}\frac{1}{(X_i+s)!}\prod_{1\le i<j\le s}(X_i-X_j),$$

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After a lot of simplification, one arrives at

$$(-1)^{\binom{r}{2}} 2^{\binom{r}{2}+\binom{n-r}{2}} \left(\binom{N+1}{2} - \binom{N-n+1}{2} - mr \right)!$$

$$\times \prod_{i=1}^{n} \frac{(i-1)!}{(N+n+1-2i)!} \prod_{i=1}^{r} \frac{(N+n-1)!}{(n-1)!\,(N-m+r-1)!}$$

$$\times \sum_{0\le k_1<\dots<k_r\le n-1} \prod_{1\le i<j\le r} (k_j-k_i)^2$$

$$\cdot \prod_{i=1}^{r} \frac{\left(-\frac{N-m+r-1}{2}\right)_{k_i} \left(-\frac{N-m+r-2}{2}\right)_{k_i} (-n+1)_{k_i}}{\left(-\frac{N+n-1}{2}\right)_{k_i} \left(-\frac{N+n-2}{2}\right)_{k_i} k_i!}.$$

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$$\cdot \prod_{i=1}^{r} \frac{\left(-\frac{N-m+r-1}{2}\right)_{k_i} \left(-\frac{N-m+r-2}{2}\right)_{k_i} (-n+1)_{k_i}}{\left(-\frac{N+n-1}{2}\right)_{k_i} \left(-\frac{N+n-2}{2}\right)_{k_i} k_i!}.$$

$\longrightarrow$ multiple hypergeometric series associated to root systems!

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“staircase minus rectnngle”

$\longrightarrow$ multiple hypergeometric series associated to root systems!

MICHAEL SCHLOSSER

$\longrightarrow$ multiple hypergeometric series associated to root systems!

Michael Schlosser (translated into English):

In a multi-dimensional ${}_{12}V_{11}$ transformation formula for elliptic hypergeometric series conjectured by Warnaar, which has subsequently been proven by Rains and, independently, by Coskun and Gustafson, let $p = 0$, $d \to aq/d$, $f \to aq/f$, and then $a \to 0$ and $q \to 1$. If I am not mistaken, this should do the trick.

Elliptic hypergeometric series?

Elliptic hypergeometric series?

Given a complex number p with $|p| < 1$, define a (rescaled) theta function $\theta(x;p)$ by

$$\theta(x;p) = \prod_{j=0}^{\infty} (1 - p^j x)(1 - p^{j+1}/x).$$

Furthermore, fixing another complex parameter, q say, and a non-negative integer m, we set

$$(a;q,p)_m = \theta(a;p)\,\theta(aq;p)\cdots\theta(aq^{m-1};p).$$

For convenience, we also employ the short notation

$$(a_1, a_2, \dots, a_k; q, p)_m = (a_1;q,p)_m\,(a_2;q,p)_m \cdots (a_k;q,p)_m.$$

“staircase minus rectnngle”

An elliptic transformation formula (Rains, Coskun and Gustafson)

Let a, b, c, d, e, f be indeterminates, let m be a nonnegative integer, and $r \geq 1$. Then

$$\sum_{0 \le k_1 < k_2 < \cdots < k_r \le m} q^{\sum_{i=1}^{r}(2i-1)k_i} \prod_{1 \le i < j \le r} \theta(q^{k_i - k_j}; p)^2 \, \theta(aq^{k_i + k_j}; p)^2$$

$$\times \prod_{i=1}^{r} \frac{\theta(aq^{2k_i}; p)(a, b, c, d, e, f; q, p)_{k_i}}{\theta(a; p)(q, aq/b, aq/c, aq/d, aq/e, aq/f; q, p)_{k_i}}$$

$$\times \prod_{i=1}^{r} \frac{(\lambda a q^{2-r+m}/ef, q^{-m}; q, p)_{k_i}}{(efq^{r-1-m}/\lambda, aq^{1+m}; q, p)_{k_i}}$$

$$= \prod_{i=1}^{r} \frac{(b,c,d,ef/a;q,p)_{i-1}}{(\lambda b/a,\lambda c/a,\lambda d/a,ef/\lambda;q,p)_{i-1}}$$

$$\times \prod_{i=1}^{r} \frac{(aq;q,p)_m\,(aq/ef;q,p)_{m+1-r}\,(\lambda q/e,\lambda q/f;q,p)_{m-i+1}}{(\lambda q;q,p)_m\,(\lambda q/ef;q,p)_{m+1-r}\,(aq/e,aq/f;q,p)_{m-i+1}}$$

$$\times \sum_{0\le k_1<k_2<\cdots<k_r\le m} q^{\sum_{i=1}^{r}(2i-1)k_i} \prod_{1\le i<j\le r} \theta(q^{k_i-k_j};p)^2\,\theta(\lambda q^{k_i+k_j};p)^2$$

$$\times \prod_{i=1}^{r} \frac{\theta(\lambda q^{2k_i};p)(\lambda,\lambda b/a,\lambda c/a,\lambda d/a,e,f;q,p)_{k_i}}{\theta(\lambda;p)(q,aq/b,aq/c,aq/d,\lambda q/e,\lambda q/f;q,p)_{k_i}}$$

$$\times \prod_{i=1}^{r} \frac{(\lambda a q^{2-r+m}/ef,q^{-m};q,p)_{k_i}}{(efq^{r-1-m}/\lambda,\lambda q^{1+m};q,p)_{k_i}},$$

where $\lambda = a^2q^{2-r}/bcd$.

In the elliptic transformation formula, we let $p = 0$, $d \to aq/d$, $f \to aq/f$, and then $a \to 0$. Next we perform the substitutions $b \to q^b$, $c \to q^c$, etc., we divide both sides of the identity obtained so far by $(1-q)^{\binom{r}{2}}$, and we let $q \to 1$.

"staircase minus rectnngle"

Corollary

For all non-negative integers m, r and s, we have

$$\sum_{0\le k_1<k_2<\dots<k_r\le m}\ \prod_{1\le i<j\le r}(k_i-k_j)^2\prod_{i=1}^{r}\frac{(d+k_i)_s\,(b)_{k_i}\,(-m)_{k_i}}{k_i!\,(f)_{k_i}}$$

$$=\frac{(-1)^{\binom{r}{2}}}{(r+s-1)!^{s-1}}\prod_{i=1}^{r}\frac{(b)_{i-1}\,(-f+b+s+2r-i-m)_{m-r+1}}{(-f-m+i)_{m-i+1}}$$

$$\times\prod_{i=1}^{r+s-1}\frac{(i-1)!\,m!}{(m-i)!}\prod_{i=r}^{r+s-1}\frac{(d-b+1-r)_i}{(r+s-i-1)!\,(d)_{i-r}\,(f-b-s+1-r)_i}$$

$$\times\sum_{0\le\ell_1<\ell_2<\dots<\ell_s\le r+s-1}\ \prod_{1\le i<j\le s}(\ell_i-\ell_j)^2$$

$$\times\prod_{i=1}^{s}\frac{(d)_{\ell_i}\,(f-b-s+1-r)_{\ell_i}\,(-r-s+1)_{\ell_i}}{\ell_i!\,(d-b+1-r)_{\ell_i}\,(-m)_{\ell_i}}.$$

“staircase minus rectnngle”

Corollary

For all non-negative integers m, r and s, we have

$$\sum_{0\le k_1<k_2<\cdots<k_r\le m}\ \prod_{1\le i<j\le r}(k_i-k_j)^2\prod_{i=1}^{r}\frac{(d+k_i)_s\,(b)_{k_i}\,(-m)_{k_i}}{k_i!\,(f)_{k_i}}$$

$$=\frac{(-1)^{\binom{r}{2}}}{(r+s-1)!^{s-1}}\prod_{i=1}^{r}\frac{(b)_{i-1}\,(-f+b+s+2r-i-m)_{m-r+1}}{(-f-m+i)_{m-i+1}}$$

$$\times\prod_{i=1}^{r+s-1}\frac{(i-1)!\,m!}{(m-i)!}\prod_{i=r}^{r+s-1}\frac{(d-b+1-r)_i}{(r+s-i-1)!\,(d)_{i-r}\,(f-b-s+1-r)_i}$$

$$\times\sum_{0\le \ell_1<\ell_2<\cdots<\ell_s\le r+s-1}\ \prod_{1\le i<j\le s}(\ell_i-\ell_j)^2$$

$$\times\prod_{i=1}^{s}\frac{(d)_{\ell_i}\,(f-b-s+1-r)_{\ell_i}\,(-r-s+1)_{\ell_i}}{\ell_i!\,(d-b+1-r)_{\ell_i}\,(-m)_{\ell_i}}.$$

“staircase minus rectnngle”

Theorem

If $N-n$ is even, the number of standard Young tableaux of shape $(N, N-1, \dots, N-n+1)/(m^r)$ equals

$$(-1)^{\binom{(N-n)/2}{2}+\frac{1}{2}r(N-n)}2^{\binom{n}{2}+(N-n-m)r}\left(\binom{N+1}{2}-\binom{N-n+1}{2}-mr\right)!$$

$$\times\frac{1}{\left(r+\frac{N-n-2}{2}\right)!^{(N-n)/2}\left(\frac{N+n-2}{2}\right)!^{(N-n)/2}}\frac{\prod_{i=1}^{(N+n)/2}(i-1)!}{\prod_{i=1}^{n}(N-n+2i-1)!}$$

$$\times\prod_{i=1}^{r}\frac{\left(\frac{N-n}{2}+i-1\right)!\,(n+m-r+2i-1)!\,\left(\frac{n+m-r}{2}+i\right)_{(N-n)/2}}{(m+i-1)!\,(N-m-r+2i-1)!}$$

$$\times\sum_{0\le\ell_1<\ell_2<\cdots<\ell_{(N-n)/2}\le r+\frac{N-n-2}{2}}(-1)^{\sum_{i=1}^{(N-n)/2}\ell_i}\left(\prod_{1\le i<j\le\frac{N-n}{2}}(\ell_i-\ell_j)^2\right)$$

$$\cdot\prod_{i=1}^{\frac{N-n}{2}}\binom{\frac{N-n-2}{2}+r}{\ell_i}\frac{\left(\frac{N+n}{2}-\ell_i\right)_{\ell_i}\left(\frac{n+m-r+1}{2}-i\right)_{r+i-\ell_i-1}\left(\frac{N-m-r+2}{2}-i\right)_{r+i-\ell_i-1}}{\left(\frac{N+m-r+2}{2}-i\right)_{r+i-\ell_i-1}},$$

and there is a similar statement if $N-n$ is odd.

In the case of a full staircase (i.e., $n = N$), the formula reduces to DeWitt's original result.

Corollary

The number of standard Young tableaux of shape $(n, n-1, \dots, 1)/(m^r)$ *equals*

$$2^{\binom{n}{2}-rm}\left(\binom{n+1}{2}-mr\right)!\prod_{i=1}^{n}\frac{(i-1)!}{(2i-1)!} \times\prod_{i=1}^{r}\frac{(n+m-r+2i-1)!\,(i-1)!}{(m+i-1)!\,(n-m-r+2i-1)!},$$

The “next” case:

Corollary

The number of standard Young tableaux of shape $(n+1, n, \dots, 2)/(m^r)$ *equals*

$$2^{\binom{n}{2}-(m-1)r}\left(\binom{n+2}{2}-mr-1\right)!\prod_{i=1}^{n}\frac{(i-1)!}{(2i)!}$$
$$\times\prod_{i=1}^{r}\frac{(n+m-r+2i-1)!\,(i-1)!}{(m+i-1)!\,(n-m-r+2i)!}$$
$$\times\sum_{\ell=0}^{r}(-1)^{r-\ell}\binom{r}{\ell}\frac{(n-\ell+1)_\ell\left(\frac{n+m-r}{2}\right)_{r-\ell}\left(\frac{n-m-r+1}{2}\right)_{r-\ell}}{\left(\frac{n+m-r+1}{2}\right)_{r-\ell}}.$$

In general:
The number of standard Young tableaux of shape $(N, N-1, \dots, N-n)/(m^r)$ equals an $\lceil (N-n)/2 \rceil$-fold hypergeometric sum.

JOHN STEMBRIDGE:

“staircase minus rectnngle”

John Stembridge:

`I think her approach is much simpler;`

John Stembridge:

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I think her approach is much simpler;
but I don’t think it would extend to the ‘‘next
case’’ you mention.
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