Uvarov's formula — a thorough discussion

Christian Krattenthaler

Universität Wien

“Christoffel’s formula”

Notation

Except when otherwise stated, I consider *formal* orthogonal polynomials, that is, polynomials $p_n(x)$ that satisfy a three-term recurrence of the form

$$p_n(x) = (x - s_{n-1})p_{n-1}(x) - t_{n-2}p_{n-2}(x),$$

with initial values $p_{-1}(x) = 0$ and $p_0(x) = 1$.

They are orthogonal with respect to the linear functional L defined by $L(p_n(x)) = \delta_{n,0}$, that is,

$$L(p_n(x)p_m(x)) = \omega_n\delta_{n,m}$$

for some (non-zero) ω_n, $n = 0, 1, 2, \ldots$.

The moments m_n of the linear functional L are $m_n = L(x^n)$, $n = 0, 1, \ldots$.

The problem

Given a sequence $\big(p_n(x)\big)_{n\ge 0}$ with functional of orthogonality given by

$$L : p(x) \mapsto \int p(u)\,d\mu(u),$$

find the sequence of orthogonal polynomials for the modified linear functional

$$p(x) \mapsto \int p(u) \prod_{\ell=1}^{d-1} (u-\alpha_\ell)\,d\mu(u).$$

Ideally, express these orthogonal polynomials as linear combinations in the basis $\big(p_n(x)\big)_{n\ge 0}$.

"Christoffel's formula"

Theorem

Let $\left(p_n(x)\right)_{n\ge0}$ be orthogonal with respect to the linear functional given by $p(x)\mapsto\int p(u)\,d\mu(u)$. Then, as polynomials in $x=\alpha_d$, the polynomials

$$\frac{\det\limits_{1\le i,j\le d}\left(p_{n+i-1}(\alpha_j)\right)}{\prod\limits_{i=1}^{d-1}(\alpha_i-\alpha_d)}$$

are orthogonal with respect to the linear functional

$$p(x)\mapsto\int p(u)\prod_{\ell=1}^{d-1}(u-\alpha_\ell)\,d\mu(u).$$

Umbral notation

Umbral notation is an elegant short notation that is convenient in certain situations. Given a sequence $(c_n)_{n\geq 0}$, one identifies c^n with c_n in polynomial expressions in c.

For example,

$$c^2(\alpha+c)(\beta+c)=c^2\alpha\beta+c^3(\alpha+\beta)+c^4=c_2\alpha\beta+c_3(\alpha+\beta)+c_4.$$

If $m_n=\int u^n\,d\mu(u)$ are the moments of the original linear functional L, then the moments of the modified functional are

$$\int u^n\prod_{\ell=1}^{d-1}(u-\alpha_\ell)\,d\mu(u)=m^n\prod_{\ell=1}^{d-1}(m-\alpha_\ell),\quad n=0,1,\dots.$$

An identity behind "Christoffel's formula"

Lemma

Let M be a linear functional on polynomials in x with moments ν_n, $n = 0, 1, \dots$. Then the determinants

$$\det_{0 \le i,j \le n-1} \left(\nu_{i+j+1} - \nu_{i+j} x \right)$$

are a sequence of orthogonal polynomials with respect to M.

We use the lemma with $\nu_n = m^n \prod_{\ell=1}^{d-1} (m - \alpha_\ell)$, the moments of our modified functional.

We compute

$$\nu_{i+j+1} - \nu_{i+j}\alpha_d = m^{i+j+1} \prod_{\ell=1}^{d-1} (m - \alpha_\ell) - \alpha_d m^{i+j} \prod_{\ell=1}^{d-1} (m - \alpha_\ell)$$
$$= m^{i+j} \prod_{\ell=1}^{d} (m - \alpha_\ell).$$

An identity behind "Christoffel's formula"

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$$\begin{aligned}\nu_{i+j+1} - \nu_{i+j}\alpha_d &= m^{i+j+1} \prod_{\ell=1}^{d-1}(m - \alpha_\ell) - \alpha_d m^{i+j} \prod_{\ell=1}^{d-1}(m - \alpha_\ell) \\ &= m^{i+j} \prod_{\ell=1}^{d}(m - \alpha_\ell).\end{aligned}$$

Thus, we have

$$\det_{0 \le i,j \le n-1} \left(\nu_{i+j+1} - \nu_{i+j}\alpha_d\right) = \det_{0 \le i,j \le n-1} \left(m^{i+j} \prod_{\ell=1}^{d}(m - \alpha_\ell). \right).$$

Thus, we have

$$\det_{0\le i,j\le n-1}\left(\nu_{i+j+1}-\nu_{i+j}\alpha_d\right)=\det_{0\le i,j\le n-1}\left(m^{i+j}\prod_{\ell=1}^{d}(m-\alpha_\ell).\right).$$

According to the lemma, these are orthogonal polynomials (in the variable $x=\alpha_d$) for the modified linear functional

$$p(x)\mapsto\int p(u)\prod_{\ell=1}^{d-1}(u-\alpha_\ell)\,d\mu(u).$$

On the other hand, "Christoffel's formula" provides us with a formula for these orthogonal polynomials, namely

$$\frac{\det\limits_{1\le i,j\le d}\left(p_{n+i-1}(\alpha_j)\right)}{\prod\limits_{i=1}^{d-1}(\alpha_i-\alpha_d)}.$$

Hence, the two expressions must be related to each other.

An identity behind "Christoffel's formula"

Hence, the two expressions must be related to each other.

Conjecture

We have

$$\frac{\det\left(m^{i+j}\prod_{\ell=1}^{d}(m-\alpha_\ell)\right)_{i,j=0}^{n-1}}{\det\left(m_{i+j}\right)_{i,j=0}^{n-1}} = (-1)^{nd}\frac{\det\limits_{1\le i,j\le d}\left(p_{n+i-1}(\alpha_j)\right)}{\prod\limits_{1\le i<j\le d}(\alpha_j-\alpha_i)}.$$

I did not find this identity in the orthogonal polynomials literature.
So, what about a proof?

An equivalent statement

Experimentally, we found

$$\frac{\det\left(m^{i+j}\prod_{\ell=1}^{d}(m-\alpha_\ell)\right)_{i,j=0}^{n-1}}{\det\left(m_{i+j}\right)_{i,j=0}^{n-1}} = (-1)^{nd}\frac{\det\limits_{1\le i,j\le d}\left(p_{n+i-1}(\alpha_j)\right)}{\prod\limits_{1\le i<j\le d}(\alpha_j-\alpha_i)}.$$

Equivalently,

$$\det_{1\le i,j\le d}\left(p_{n+i-1}(\alpha_j)\right)$$
$$= (-1)^{nd}\left(\prod_{1\le i<j\le d}(\alpha_j-\alpha_i)\right)\frac{\det\limits_{0\le i,j\le n-1}\left(m^{i+j}\prod_{\ell=1}^{d}(m-\alpha_\ell)\right)}{\det\limits_{0\le i,j\le n-1}\left(m_{i+j}\right)}.$$

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We prove this identity by the condensation method.†

Proposition (Jacobi)

Let A be an $N\times N$ matrix. Denote the submatrix of A in which rows $i_1,i_2,\dots,i_k$ and columns $j_1,j_2,\dots,j_k$ are omitted by $A_{i_1,i_2,\dots,i_k}^{j_1,j_2,\dots,j_k}$. Then we have

$$\det A\cdot\det A_{1,N}^{1,N}=\det A_1^1\cdot\det A_N^N-\det A_1^N\cdot\det A_N^1.$$

Jacobi's *condensation formula* allows for inductive proofs.

† This method was a favourite of Charles Lutwidge Dodgson (Lewis Carroll).

Equivalently,

$$\det_{1\le i,j\le d}\left(p_{n+i-1}(\alpha_j)\right)$$

$$=(-1)^{nd}\left(\prod_{1\le i<j\le d}(\alpha_j-\alpha_i)\right)\frac{\det\limits_{0\le i,j\le n-1}\left(m^{i+j}\prod_{\ell=1}^{d}(m-\alpha_\ell)\right)}{\det\limits_{0\le i,j\le n-1}\left(m_{i+j}\right)}.$$

By Jacobi's condensation formula, the left-hand side satisfies a certain recurrence formula. If we manage to prove that the right-hand side satisfies the same recurrence, then we are done.

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If one works it out, then one sees that we need to prove

$$(\alpha_1-\alpha_d)\det_{0\le i,j\le n-1}\left(m^{i+j}\prod_{\ell=1}^{d}(m-\alpha_\ell)\right)\det_{0\le i,j\le n}\left(m^{i+j}\prod_{\ell=2}^{d-1}(m-\alpha_\ell)\right)$$
$$=\det_{0\le i,j\le n-1}\left(m^{i+j}\prod_{\ell=1}^{d-1}(m-\alpha_\ell)\right)\det_{0\le i,j\le n}\left(m^{i+j}\prod_{\ell=2}^{d}(m-\alpha_\ell)\right)$$
$$-\det_{0\le i,j\le n-1}\left(m^{i+j}\prod_{\ell=2}^{d}(m-\alpha_\ell)\right)\det_{0\le i,j\le n}\left(m^{i+j}\prod_{\ell=1}^{d-1}(m-\alpha_\ell)\right).$$

If one looks at this properly, then it turns out that this is another instance of Jacobi's condensation formula. □

Great, but ...

Great! We found a proof, and we have a theorem.

Theorem

We have

$$\frac{\det\left(m^{i+j}\prod_{\ell=1}^{d}(m-\alpha_\ell)\right)_{i,j=0}^{n-1}}{\det\left(m_{i+j}\right)_{i,j=0}^{n-1}} = (-1)^{nd}\frac{\det\limits_{1\le i,j\le d}\left(p_{n+i-1}(\alpha_j)\right)}{\prod\limits_{1\le i<j\le d}(\alpha_j-\alpha_i)}.$$

But still, is this really new? Hard to believe ...

I asked Mourad Ismail. His replies seem to indicate that he was not aware of *any* source where the identity is stated explicitly.

A literature search

I asked Mourad Ismail. His replies seem to indicate that he was not aware of *any* source where the identity is stated explicitly.

If you don't find the identity in the classical orthogonal polynomial literature, then what about "non-classical" sources?

• A. Lascoux, *Symmetric Functions and Combinatorial Operators on Polynomials*, CBMS Series, vol. 99, Amer. Math. Soc., 2003.
Proof by the use of classical facts from the theory of orthogonal polynomials (result not quite right).

• M. Elouafi, *A unified approach for the Hankel determinants of classical combinatorial numbers*, J. Math. Anal. Appl. **431** (2015), 1253–1274.
Proof by a vanishing argument (has a serious gap).

• E. Brézin and S. Hikami, *Characteristic polynomials of random matrices*, Comm. Math. Phys. **214** (2000), 111–135.
Proof by random matrix theory.

Ismail's original message:

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Dear Christian
I did not do the details but it seems to me that the
right side is what you get from a formula due to
Christoffel.  It is in Szego and also in my book, see
Theorem 2.7.1.  It is the coefficient of a polynomial
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in the expansion of $\prod_{1\le j\le d}(x-x_j)$ times another set of orthogonal polynomials. The reason I mention this is because there is a more general formula due to Ouvarouv, also in my book, which replaces $\prod_{1\le j\le d}(x-x_j)$ by a rational function. This is harder and more recent (1960's). It will be really nice if it can be done combinatorially because it involves the function of the second kind.

Theorem

Let $\big(p_n(x)\big)_{n\ge0}$ *be orthogonal with respect to the linear functional given by* $p(x)\mapsto\int p(u)\,d\mu(u)$*. Then, as polynomials in* $x=\alpha_1$*, the polynomials*

$$\frac{\det\limits_{1\le i,j\le k+h}\begin{pmatrix} p_{n-k+i-1}(\alpha_j) & 1\le i\le h\\ q_{n-k+i-h-1}(\beta_j) & h+1\le i\le k+h\end{pmatrix}}{\prod\limits_{\ell=2}^{h}(\alpha_1-\alpha_\ell)}$$

are orthogonal with respect to the linear functional

$$p(x)\mapsto\int p(u)\frac{\prod_{\ell=2}^{h}(u-\alpha_\ell)}{\prod_{\ell=1}^{k}(u-\beta_\ell)}\,d\mu(u).$$

Here, $q_n(y)=\displaystyle\int\frac{p_n(u)}{y-u}\,d\mu(u)$*.*

The "function of the second kind" $q_n(y)$

$$
\begin{aligned}
q_n(y) &= \int \frac{p_n(u)}{y-u}\,d\mu(u) = \sum_{i=0}^{\infty} \int p_n(u)u^i y^{-i-1}\,d\mu(u) \\
&= \sum_{i=n}^{\infty} \int p_n(u)u^i y^{-i-1}\,d\mu(u) \\
&= y^{-n-1} \int p_n(u)u^n\,d\mu(u) + O\left(y^{-n-2}\right) \\
&= \frac{H(n+1)}{H(n)} y^{-n-1} + O\left(y^{-n-2}\right),
\end{aligned}
$$

where $H(n) := \det\limits_{0\le i,j\le n-1} (m_{i+j})$.

Uvarov's formula

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Here, $q_n(y)=\displaystyle\int \frac{p_n(u)}{y-u}\,d\mu(u)$.

Some more smallprint:

We have the expression

$$\frac{\det\limits_{1\le i,j\le k+h}\begin{pmatrix} p_{n-k+i-1}(\alpha_j) & 1\le i\le h \\ q_{n-k+i-h-1}(\beta_j) & h+1\le i\le k+h \end{pmatrix}}{\prod\limits_{\ell=2}^{h}(\alpha_1-\alpha_\ell)}.$$

If $a<0$ then $p_a(x)=0$ and $q_a(y)=y^{-a-1}$.

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Here, $q_n(y)=\displaystyle\int\frac{p_n(u)}{y-u}\,d\mu(u)$.

The determinant identity behind Uvarov's formula

Is there again an *identity* hidden behind this formula?

The determinant identity behind Uvarov's formula

Theorem

We have

$$\frac{\det\limits_{0\le i,j\le n-1}\left(\displaystyle\int u^{i+j}\frac{\prod_{\ell=1}^{h}(u-\alpha_\ell)}{\prod_{\ell=1}^{k}(u-\beta_\ell)}\,d\mu(u)\right)}{\det\limits_{0\le i,j\le n-k-1}\left(m_{i+j}\right)}$$

$$=(-1)^{n(h-k)+kh}\frac{\det_{1\le i,j\le k+h}\begin{pmatrix} p_{n+i-1}(\alpha_j) & 1\le i\le h\\ q_{n+i-h-1}(\beta_j) & h+1\le i\le k+h\end{pmatrix}}{\left(\prod\limits_{1\le i<j\le h}(\alpha_j-\alpha_i)\right)\left(\prod\limits_{1\le i<j\le k}(\beta_i-\beta_j)\right)}.$$

If $n<k$ the previous conventions for negatively indexed $p_a(\alpha)$ and $q_a(\beta)$ apply, and in that case the Hankel determinant in the denominator on the left-hand side has to be interpreted as 1.

The proof

We had four different proofs for the determinant identity behind "Christoffel's formula":

1. Proof by condensation
2. Proof by the use of classical facts from the theory of orthogonal polynomials
3. Proof by a vanishing argument
4. Proof by random matrix theory

Which of these methods can we apply here?

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1. Proof by condensation !

Which of these methods can we apply here?

We do an induction on $h+k$. The previous "condensation lemma" is again needed, as well as the similarly looking one.

Lemma

Let $(c_n)_{n\ge0}$ be a given sequence, and α and β be variables. Then, for all positive integers n, we have

$$\det_{0\le i,j\le n-1}\left(\alpha c_{i+j}+c_{i+j+1}\right)\det_{0\le i,j\le n-1}\left(\beta c_{i+j}+c_{i+j+1}\right)$$
$$=-\det_{0\le i,j\le n}\left(c_{i+j}\right)\det_{0\le i,j\le n-2}\left(\alpha\beta c_{i+j}+(\alpha+\beta)c_{i+j+1}+c_{i+j+2}\right)$$
$$+\det_{0\le i,j\le n-1}\left(c_{i+j}\right)\det_{0\le i,j\le n-1}\left(\alpha\beta c_{i+j}+(\alpha+\beta)c_{i+j+1}+c_{i+j+2}\right).$$

When I placed the preprint on the arχiv, Arno Kuijlaars pointed out:

"Your identity is contained in:

J. Baik, P. Deift and E. Strahov, *Products and ratios of characteristic polynomials of random Hermitian matrices*, J. Math. Phys. **44** (2003), 3657–3670."

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Products and ratios of characteristic polynomials of random Hermitian matrices

Jinho Baik[a)]
Department of Mathematics, Princeton University, Princeton, New Jersey 08544 and Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

Percy Deift[b)]
Courant Institute of Mathematical Sciences, New York University, New York, New York 10012 and School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540

Eugene Strahov[c)]
Department of Mathematical Sciences, Brunel University, Uxbridge, UB8 3PH, United Kingdom

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We present new and streamlined proofs of various formulas for products and ratios of characteristic polynomials of random Hermitian matrices that have appeared recently in the literature.
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I. INTRODUCTION

In random matrix theory, unitary ensembles of $N\times N$ matrices $\{H\}$ play a central role.[15] Such

The determinant identity behind Uvarov's formula

$$\left\langle \prod_{j=1}^{M} D_N^{-1}[\epsilon_j,H] \right\rangle_\alpha = (-1)^{M(M-1)/2} \frac{\prod_{j=N-M}^{N-1} \gamma_j}{\Delta(\epsilon)} \begin{vmatrix} h_{N-M}(\epsilon_1) & \cdots & h_{N-1}(\epsilon_1) \\ \vdots & & \\ h_{N-M}(\epsilon_M) & \cdots & h_{N-1}(\epsilon_M) \end{vmatrix}. \qquad (2.24)$$

Proof: When $M=1$, we use the identity (2.21) together with (2.7) and the relation (see, e.g., Ref. 18)

$$\gamma_{n-1} = -2\pi i n \frac{Z_{n-1}}{Z_n} \qquad (2.25)$$

to obtain

$$\langle D_N^{-1}[\epsilon,H] \rangle_\alpha = \gamma_{N-1} h_{N-1}(\epsilon). \qquad (2.26)$$

We rewrite the average in Eq. (2.24) as follows:

$$\left\langle \prod_{j=1}^{M} D_N^{-1}[\epsilon_j,H] \right\rangle_\alpha = \frac{Z_N^{[0,M]}}{Z_{N-1}^{[0,M-1]}} \frac{Z_{N-1}^{[0,M-1]}}{Z_{N-2}^{[0,M-2]}} \cdots \frac{Z_{N-M}^{[0,0]}}{Z_N^{[0,0]}}, \qquad (2.27)$$

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The determinant identity behind Uvarov's formula

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where

$$Z_N^{[0,M]} = \int \cdots \int \Delta^2(x) d\alpha^{[0,M]}(x), \tag{2.28}$$

The determinant identity behind Uvarov's formula

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where

$$Z_N^{[0,M]} = \int \cdots \int \Delta^2(x) d\alpha^{[0,M]}(x), \quad (2.28)$$

$Z_N^{[0,0]} \equiv Z_N$ and $d\alpha^{[0,0]}(x) = d\alpha(x)$. The following relation can be observed from Eqs. (2.26) and (2.25):

$$\left\langle \prod_{j=1}^{M} D_N^{-1}[\epsilon_j,H] \right\rangle_\alpha = (-1)^{M(M-1)/2} \frac{\prod_{j=N-M}^{N-1} \gamma_j}{\Delta(\epsilon)} \begin{vmatrix} h_{N-M}(\epsilon_1) & \cdots & h_{N-1}(\epsilon_1) \\ \vdots & & \vdots \\ h_{N-M}(\epsilon_M) & \cdots & h_{N-1}(\epsilon_M) \end{vmatrix}. \tag{2.24}$$

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$$\frac{Z_{N-K}^{[0,m]}}{Z_{N-K-1}^{[0,m-1]}} = -2\pi i (N-K) h_{N-K-1}^{[0,m-1]}(\epsilon_m). \tag{2.29}$$

Proof: When $M=1$, we use the identity (2.21) together with (2.7) and the relation (see, e.g., Ref. 18)

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$$Z_N^{[0,M]} = \int \cdots \int \Delta^2(x) \, d\alpha^{[0,M]}(x), \tag{2.28}$$

$Z_N^{[0,0]} \equiv Z_N$ and $d\alpha^{[0,0]}(x) = d\alpha(x)$. The following relation can be observed from Eqs. (2.26) and (2.25):

$$\frac{Z_{N-K}^{[0,m]}}{Z_{N-K-1}^{[0,m-1]}} = -2\pi i (N-K) h_{N-K-1}^{[0,m-1]}(\epsilon_m). \tag{2.29}$$

Inserting this relation in (2.27) we find

$$\gamma_{n-1} = -2\pi i n \frac{Z_{n-1}}{Z_n} \tag{2.25}$$

to obtain

$$\langle D_N^{-1}[\epsilon,H]\rangle_\alpha = \gamma_{N-1} h_{N-1}(\epsilon). \tag{2.26}$$

We rewrite the average in Eq. (2.24) as follows:

$$\left\langle \prod_{j=1}^{M} D_N^{-1}[\epsilon_j,H] \right\rangle_\alpha = \frac{Z_N^{[0,M]}}{Z_{N-1}^{[0,M-1]}} \frac{Z_{N-1}^{[0,M-1]}}{Z_{N-2}^{[0,M-2]}} \cdots \frac{Z_{N-M}^{[0,0]}}{Z_N^{[0,0]}}, \tag{2.27}$$

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Our result (2.24) immediately follows from the above equation and formula (2.23). □

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We now repeat the above considerations for the case

$$d\alpha^{[\ell,m]}(t) = \frac{(\mu_1 - t)\cdots(\mu_\ell - t)}{\cdots}\, d\alpha(t) \tag{2.31}$$

The determinant identity behind Uvarov's formula

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$$d\alpha^{[\ell,m]}(t) = \frac{(\mu_1 - t)\cdots(\mu_\ell - t)}{(\epsilon_1 - t)\cdots(\epsilon_m - t)}\, d\alpha(t). \tag{2.31}$$

The first result is a Christoffel-type formula for the measure (2.31), which is due to Uvarov.[19]

Lemma 2.11: Suppose $0 \le m \le n$. Then the monic orthogonal polynomials $\pi^{[\ell,m]}(t)$'s with

The determinant identity behind Uvarov's formula

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The first result is a Christoffel-type formula for the measure (2.31), which is due to Uvarov.[19]

Lemma 2.11: Suppose $0 \leqslant m \leqslant n$. *Then the monic orthogonal polynomials* $\pi_n^{[\ell,m]}(t)$*'s with respect to the measure* $d\alpha^{\ell,m]}(t)$ *have the following representation:*

$$\pi_n^{[\ell,m]}(t)=\frac{1}{(t-\mu_\ell)\cdots(t-\mu_1)}\frac{\begin{vmatrix} h_{n-m}(\epsilon_1) & \cdots & h_{n+\ell}(\epsilon_1)\\ \vdots & & \\ h_{n-m}(\epsilon_m) & \cdots & h_{n+\ell}(\epsilon_m)\\ \pi_{n-m}(\mu_1) & \cdots & \pi_{n+\ell}(\mu_1)\\ \vdots & & \\ \pi_{n-m}(\mu_\ell) & \cdots & \pi_{n+\ell}(\mu_\ell)\\ \pi_{n-m}(t) & \cdots & \pi_{n+\ell}(t)\end{vmatrix}}{\begin{vmatrix} h_{n-m}(\epsilon_1) & \cdots & h_{n+\ell}(\epsilon_1)\\ \vdots & & \\ h_{n-m}(\epsilon_m) & \cdots & h_{n+\ell}(\epsilon_m)\\ \pi_{n-m}(\mu_1) & \cdots & \pi_{n+\ell}(\mu_1)\\ \vdots & & \\ \pi_{n-m}(\mu_\ell) & \cdots & \pi_{n+\ell}(\mu_\ell)\end{vmatrix}}. \tag{2.32}$$

Proof: As in the previous cases we define $q_n^{[\ell,m]}(t)$ to be the determinant in the numerator of (2.32). Observe that

$$q_n^{[\ell,m]}(\mu_1)=\cdots=q_n^{[\ell,m]}(\mu_\ell)=0 \tag{2.33}$$

The determinant identity behind Uvarov's formula

$$\pi_n^{[\ell,m]}(t) = \frac{1}{(t-\mu_\ell)\cdots(t-\mu_1)} \frac{\begin{vmatrix} h_{n-m}(\epsilon_1) & \cdots & h_{n+\ell}(\epsilon_1) \\ \vdots & & \\ h_{n-m}(\epsilon_m) & \cdots & h_{n+\ell}(\epsilon_m) \\ \pi_{n-m}(\mu_1) & \cdots & \pi_{n+\ell}(\mu_1) \\ \vdots & & \\ \pi_{n-m}(\mu_\ell) & \cdots & \pi_{n+\ell}(\mu_\ell) \\ \pi_{n-m}(t) & \cdots & \pi_{n+\ell}(t) \end{vmatrix}}{\begin{vmatrix} h_{n-m}(\epsilon_1) & \cdots & h_{n+\ell}(\epsilon_1) \\ \vdots & & \\ h_{n-m}(\epsilon_m) & \cdots & h_{n+\ell}(\epsilon_m) \\ \pi_{n-m}(\mu_1) & \cdots & \pi_{n+\ell}(\mu_1) \\ \vdots & & \\ \pi_{n-m}(\mu_\ell) & \cdots & \pi_{n+\ell}(\mu_\ell) \end{vmatrix}}. \tag{2.32}$$

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and that

$$\int q_n^{[\ell,m]}(t)\,d\alpha(t) \quad \int q_n^{[\ell,m]}(t)\,d\alpha(t) \quad 0 \tag{2.34}$$

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The next steps are the same as in the proofs of Lemma 2.1 and Lemma 2.5.

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The next steps are the same as in the proofs of Lemma 2.1 and Lemma 2.5. □

Corollary 2.12:

$$\begin{vmatrix} h_{N-M}(\epsilon_1) & \cdots & h_{N+K-1}(\epsilon_1) \\ \vdots & & \\ h_{N-M}(\epsilon_M) & \cdots & h_{N+K-1}(\epsilon_M) \\ \pi_{N-M}(\mu_1) & \cdots & \pi_{N+K-1}(\mu_1) \\ \vdots & & \end{vmatrix}$$

$$\begin{vmatrix} \vdots & & \\ \pi_{n-m}(\mu_\ell) & \cdots & \pi_{n+\ell}(\mu_\ell) \end{vmatrix}$$

Proof: As in the previous cases we define $q_n^{[\ell,m]}(t)$ to be the determinant in the numerator of (2.32). Observe that

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The next steps are the same as in the proofs of Lemma 2.1 and Lemma 2.5. □

Corollary 2.12:

$$\left\langle \prod_{j=1}^{K} D_N[\mu_j,H] \right\rangle_{\alpha^{[0,M]}} = \frac{1}{\Delta(\mu)} \frac{\begin{vmatrix} h_{N-M}(\epsilon_1) & \cdots & h_{N+K-1}(\epsilon_1) \\ \vdots & & \\ h_{N-M}(\epsilon_M) & \cdots & h_{N+K-1}(\epsilon_M) \\ \pi_{N-M}(\mu_1) & \cdots & \pi_{N+K-1}(\mu_1) \\ \vdots & & \\ \pi_{N-M}(\mu_K) & \cdots & \pi_{N+K-1}(\mu_K) \end{vmatrix}}{\begin{vmatrix} h_{N-M}(\epsilon_1) & \cdots & h_N(\epsilon_1) \\ \vdots & & \\ h_{N-M}(\epsilon_M) & \cdots & h_N(\epsilon_M) \end{vmatrix}}. \tag{2.35}$$

Proof: Identity (2.35) follows from Eqs. (2.10) and (2.32) once we note that Eq. (2.32) can be

ЖУРНАЛ
ВЫЧИСЛИТЕЛЬНОЙ МАТЕМАТИКИ И МАТЕМАТИЧЕСКОЙ ФИЗИКИ

Том 9 Ноябрь 1969 Декабрь № 6

УДК 518:517.864

О СВЯЗИ СИСТЕМ ПОЛИНОМОВ, ОРТОГОНАЛЬНЫХ ОТНОСИТЕЛЬНО РАЗЛИЧНЫХ ФУНКЦИЙ РАСПРЕДЕЛЕНИЯ

В. Б. УВАРОВ

(Москва)

Введение

При вычислении определенных интегралов вида

Так как

$$\int_{-\infty}^{\infty} \frac{P_n^{(1,0)}(x)}{x-\beta_1}\, d\rho(x) = 0,$$

то из (4) получим $C_{n,n}Q_n(\beta_1) + C_{n,n-1}Q_{n-1}(\beta_1) = 0$, т. е. $C_{n,n} = B_nQ_n(\beta_1)$, $C_{n,n-1} = -B_nQ_n(\beta_1)$, где B_n — некоторая постоянная. Подставляя эти выражения в (4), получим (3). Лемма доказана.

С помощью доказанной леммы можно заметить, что в общем случае полиномы $P_n^{(k,l)}(x)$ целесообразно искать в виде

$$P_n^{(k,l)}(x) = \left[\prod_{j=1}^{l}(x-\alpha_j)\right]^{-1} \sum_{s=0}^{n+l} C_s P_s(x).$$

Здесь

$$C_s = \frac{1}{d_s^2} \int_{-\infty} P_n^{(k,l)}(x) \prod_{j=1}^{l}(x-\alpha_j) P_s(x)\, d\rho(x).$$

Преобразуем выражение для C_s следующим образом:

$$C_s = \frac{1}{d_s^2} \int_{-\infty}^{\infty} P_n^{(k,l)}(x) P_s(x) \prod_{j=1}^{k}(x-\beta_j)\, d\rho_{k,l}(x).$$

$$\int\limits_{-\infty} \frac{\cdots (x)}{x-\beta_1}\, d\rho(x) = 0,$$

то из (4) получим $C_{n,\,n} Q_n(\beta_1) + C_{n,\,n-1} Q_{n-1}(\beta_1) = 0$, т. е. $C_{n,\,n} = B_n Q_n(\beta_1)$, $C_{n,\,n-1} = -B_n Q_n(\beta_1)$, где B_n — некоторая постоянная. Подставляя эти выражения в (4), получим (3). Лемма доказана.

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В силу свойств ортогональности полиномов $P_n^{(k,\,l)}(x)$ коэффициенты $C_s = 0$ при $s < n-k$, т. е. для $P_n^{(k,\,l)}(x)$ при $n \geqslant k$ справедливо разложение (случай $n < k$ будет рассмотрен особо)

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Очевидно, что

$$\sum^{n+l} C_s P_s(\alpha_j) = 0 \qquad (j = 1, 2, \ldots, l). \qquad (6)$$

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Недостающие для определения коэффициентов разложения k уравнений можно получить из условий ортогональности

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$$\int_{-\infty}^{\infty} P_n^{(k,\,l)}(x) \prod_{\substack{j=1\\ j\neq i}}^{k} (x - \beta_j)\, d\rho_{k,\,l}(x) = 0 \qquad (i = 1, 2, \ldots, k).$$

Подставляя в эти равенства выражения (5) и (1), получим систему уравнений

$C_s = 0$ при $s < n - k$, т. е. для $P_n^{(k,l)}(x)$ при $n \geqslant k$ справедливо разложение (случай $n < k$ будет рассмотрен особо)

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$$\sum_{s=n-k}^{n+l} C_s Q_s(\beta_i) = 0 \qquad (i = 1, 2, \ldots, k). \qquad (7)$$

В результате исключения C_s из (5) — (7), считая C_{n+l} известным, приходим к формуле

$$P_n^{(k,\,l)}(x) = A_n^{(k,\,l)} \left[\prod_{j=1}^{l} (x - \alpha_j) \right]^{-1} \begin{vmatrix} P_{n-k}(\alpha_1) & \ldots & P_{n+l}(\alpha_1) \\ \ldots & \ldots & \ldots \\ P_{n-k}(\alpha_l) & \ldots & P_{n+l}(\alpha_l) \\ Q_{n-k}(\beta_1) & \ldots & Q_{n+l}(\beta_1) \\ \ldots & \ldots & \ldots \\ Q_{n-k}(\beta_k) & \ldots & Q_{n+l}(\beta_k) \\ P_{n-k}(x) & \ldots & P_{n+l}(x) \end{vmatrix}, \tag{8}$$

где $A_n^{(k,l)} = C_{n+l} / \Delta$ — нормировочный коэффициент,

$$\Delta = \begin{vmatrix} P_{n-k}(\alpha_1) & \ldots & P_{n+l-1}(\alpha_1) \\ \ldots & \ldots & \ldots \\ P_{n-k}(\alpha_l) & \ldots & P_{n+l-1}(\alpha_l) \\ Q_{n-k}(\beta_1) & \ldots & Q_{n+l-1}(\beta_1) \\ \ldots & \ldots & \ldots \\ Q_{n-k}(\beta_k) & \ldots & Q_{n+l-1}(\beta_k) \end{vmatrix}. \tag{9}$$

Нам остается найти связь полиномов $P_n^{(k,\,l)}(x)$ и $P_n(x)$ для случая, когда $n < k$.

Будем искать $P_n^{(k,\,l)}(x)$ при $n < k$ в виде, похожем на (8), а именно

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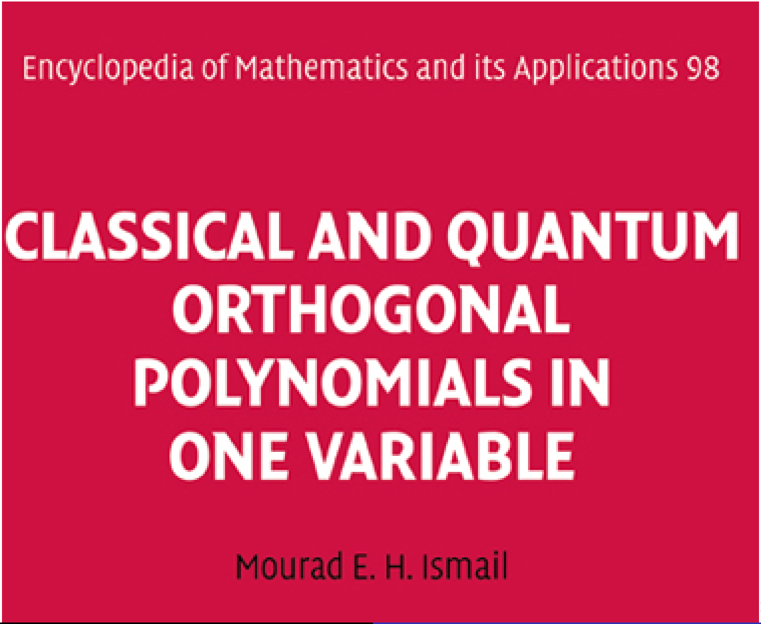

Uvarov's formula in Ismail's book

Theorem 2.7.3 (Uvarov) *Let ν be as in (2.7.4) and assume that $\{P_n(x;m,k)\}$ are orthogonal with respect to ν. Set*

$$\tilde{Q}_n(x) := \int_{\mathbb{R}} \frac{P_n(y)}{x-y}\, d\mu(y). \tag{2.7.7}$$

Then for $n \ge k$ we have

$$\left[\prod_{i=1}^{m}(x-x_i)\right] P_n(x;m,k) = \begin{vmatrix} P_{n-k}(x_1) & P_{n-k+1}(x_1) & \cdots & P_{n+m}(x_1) \\ \vdots & \vdots & \vdots & \vdots \\ P_{n-k}(x_m) & P_{n-k+1}(x_m) & \cdots & P_{n+m}(x_m) \\ \tilde{Q}_{n-k}(y_1) & \tilde{Q}_{n-k+1}(y_1) & \cdots & \tilde{Q}_{n+m}(y_1) \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{Q}_{n-k}(y_k) & \tilde{Q}_{n-k+1}(y_k) & \cdots & \tilde{Q}_{n+m}(y_k) \\ P_{n-k}(x) & P_{n+1}(x) & \cdots & P_{n+m}(x) \end{vmatrix}. \tag{2.7.8}$$

$$\begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ \tilde{Q}_{n-k}(y_k) & \tilde{Q}_{n-k+1}(y_k) & \cdots & \tilde{Q}_{n+m}(y_k) \\ P_{n-k}(x) & P_{n+1}(x) & \cdots & P_{n+m}(x) \end{vmatrix}$$

If $n < k$ *then*

$$\left[\prod_{i=1}^{m}(x-x_i)\right] P_n(x;m,k)$$

$$= \begin{vmatrix} a_{1,1} & \cdots & a_{1,k-n} & P_0(x_1) & \cdots & P_{n+m}(x_1) \\ \vdots & \vdots & \vdots & \vdots & & \\ a_{m,1} & \cdots & a_{m,k-n} & P_0(x_m) & \cdots & P_{n+m}(x_m) \\ b_{1,1} & \cdots & b_{1,k-n} & \hat{Q}_0(y_1) & \cdots & \tilde{Q}_{n+m}(y_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{k,1} & \cdots & b_{k,k-n} & \tilde{Q}_0(y_k) & \cdots & \tilde{Q}_{n+m}(y_k) \\ c_1 & \cdots & c_{k-n} & P_0(x) & \ldots & P_{n+m}(x) \end{vmatrix}, \qquad (2.7.9)$$

where

$$b_{ij} = y_i^{j-1}, \quad 1 \le i \le k, \quad 1 < j \le k-n,$$

$a_{ij} = 0$; $1 \le i \le m$, $1 \le j \le k-n$, $c_j = 0$.

If an x_j *(or* y_l *is repeated* r *times, then the corresponding* r *rows will contain* $P_s(x_j), \ldots, P_s^{(r-1)}(x_j)$ $(\tilde{Q}_s(x_j), \ldots, \tilde{Q}_s^{(r-1)}(x_j))$*, respectively.*

Uvarov proved this result in a brief announcement (Uvarov, 1959) and later gave

Uvarov's formula in Ismail's book

$$
\begin{vmatrix}
a_{m,1} & \cdots & a_{m,k-n} & P_0(x_m) & \cdots & P_{n+m}(x_m) \\
b_{1,1} & \cdots & b_{1,k-n} & \tilde{Q}_0(y_1) & \cdots & \tilde{Q}_{n+m}(y_1) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
b_{k,1} & \cdots & b_{k,k-n} & \tilde{Q}_0(y_k) & \cdots & \tilde{Q}_{n+m}(y_k) \\
c_1 & \cdots & c_{k-n} & P_0(x) & \ldots & P_{n+m}(x)
\end{vmatrix},
$$

where

$$b_{ij} = y_i^{j-1}, \quad 1 \le i \le k, \quad 1 < j \le k-n,$$

$a_{ij} = 0$; $1 \le i \le m$, $1 \le j \le k-n$, $c_j = 0$.

If an x_j *(or* y_l *is repeated* r *times, then the corresponding* r *rows will contain* $P_s(x_j), \ldots, P_s^{(r-1)}(x_j)$ $(\tilde{Q}_s(x_j), \ldots, \tilde{Q}_s^{(r-1)}(x_j))$, *respectively.*

Uvarov proved this result in a brief announcement (Uvarov, 1959) and later gave the details in (Uvarov, 1969). The proof given below is a slight modification of Uvarov's original proof.

Proof of Theorem 2.7.3 Let $\pi_j(x)$ denote a generic polynomials in x of degree at most j and denote the determinant on the right-hand side of (2.7.8) by $\Delta_{k,m,n}(x)$. Clearly $\Delta_{k,m,n}(x)$ vanishes at the points $x = x_j$, with $1 \le j \le m$ so let $\Delta_{k,m,n}(x)$

Uvarov's formula in Ismail's book

be $S_n(x) \prod_{i=1}^{m} (x - x_i)$ with S_n of degree at most n. Moreover, $S_n(x) \not\equiv 0$, so we let

$$S_n(x) = \pi_{n-k}(x) \prod_{i=1}^{k} (x - y_i) + \pi_{k-1}(x),$$

and note the partial fraction decomposition

$$\frac{\pi_{k-1}(x)}{\prod_{i=1}^{k} (x - y_i)} = \sum_{j=1}^{k} \frac{\alpha_j}{x - y_j}.$$

With ν as in (2.7.4) we have

$$\int_{\mathbb{R}} S_n^2(x)\, d\nu(x) = \int_{\mathbb{R}} S_n(x) \prod_{j=1}^{m} (x - x_j) \left\{ \pi_{n-k}(x) + \frac{\pi_{k-1}(x)}{\prod_{i=1}^{k} (x - y_i)} \right\} d\mu(x)$$

be $S_n(x) \prod_{i=1}^{m} (x - x_i)$ with S_n of degree at most n. Moreover, $S_n(x) \not\equiv 0$, so we let

$$S_n(x) = \pi_{n-k}(x) \prod_{i=1}^{k} (x - y_i) + \pi_{k-1}(x),$$

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With ν as in (2.7.4) we have

$$\int_{\mathbb{R}} S_n^2(x)\, d\nu(x) = \int_{\mathbb{R}} S_n(x) \prod_{j=1}^{m} (x - x_j) \left\{ \pi_{n-k}(x) + \frac{\pi_{k-1}(x)}{\prod_{i=1}^{k} (x - y_i)} \right\} d\mu(x)$$

Uvarov's formula in Ismail's book

From my correspondence with Mourad:

```
Date:  Sun, 28 Feb 2021 18:03:36 +0100 (CET)
From:  Christian Krattenthaler
To:  Mourad Ismail <mourad.eh.ismail@gmail.com>
Subject:  Re:  Uvarov

> Dear Christian,In my book I said S_n is not
> identically zero.

Right.

> I do not see why right now but I will check it
> and get back to you.
> Sorry but glad you are interested.

Yes, I am very interested to see why.

With best wishes,

Christian
```

Proof of Uvarov's formula

Let $d\mu(u)$ be the density of a positive measure with infinite support all of whose moments exist. We let $d\nu(u)$ be the modified density

$$d\nu(u) = \frac{\prod_{\ell=2}^{h}(u - \alpha_\ell)}{\prod_{\ell=1}^{k}(u - \beta_\ell)}\, d\mu(u),$$

where $\alpha_2, \alpha_3, \dots, \alpha_h$ and $\beta_1, \beta_2, \dots, \beta_k$ are real numbers chosen so that the modification factor

$$\frac{\prod_{\ell=2}^{h}(u - \alpha_\ell)}{\prod_{\ell=1}^{k}(u - \beta_\ell)}$$

is positive for all u in the support of the measure μ.

Proof of Uvarov's formula

Theorem

We have

$$\frac{\det\limits_{0\le i,j\le n-1}\left(\int u^{i+j}\frac{\prod_{\ell=1}^{h}(u-\alpha_\ell)}{\prod_{\ell=1}^{k}(u-\beta_\ell)}\,d\mu(u)\right)}{\det\limits_{0\le i,j\le n-k-1}(m_{i+j})}$$

$$=(-1)^{n(h-k)+kh}\frac{\det_{1\le i,j\le k+h}\begin{pmatrix}p_{n+i-1}(\alpha_j) & 1\le i\le h\\ q_{n+i-h-1}(\beta_j) & h+1\le i\le k+h\end{pmatrix}}{\left(\prod\limits_{1\le i<j\le h}(\alpha_j-\alpha_i)\right)\left(\prod\limits_{1\le i<j\le k}(\beta_i-\beta_j)\right)}.$$

If $n<k$ the previous conventions for negatively indexed $p_a(\alpha)$ and $q_a(\beta)$ apply, and in that case the Hankel determinant in the denominator on the left-hand side has to be interpreted as 1.

Proof of Uvarov's formula

If we use the lemma from before with

$$\nu_n = \int u^n \frac{\prod_{\ell=2}^{h}(u-\alpha_\ell)}{\prod_{\ell=1}^{k}(u-\beta_\ell)}\, d\mu(u),$$

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$$\nu_n = \int u^n \frac{\prod_{\ell=2}^{h}(u-\alpha_\ell)}{\prod_{\ell=1}^{k}(u-\beta_\ell)}\, d\mu(u),$$

Lemma

Let M be a linear functional on polynomials in x with moments ν_n, $n = 0, 1, \dots$. Then the determinants

$$\det_{0\le i,j\le n-1}\left(\nu_{i+j+1} - \nu_{i+j}x\right)$$

are a sequence of orthogonal polynomials with respect to M.

Proof of Uvarov's formula

If we use of the lemma from before with

$$\nu_n = \int u^n \frac{\prod_{\ell=2}^{h}(u-\alpha_\ell)}{\prod_{\ell=1}^{k}(u-\beta_\ell)}\, d\mu(u),$$

then we see that (as polynomials in $x=\alpha_1$) the polynomials

$$\det_{0\le i,j\le n-1}\left(\int u^{i+j}\frac{\prod_{\ell=1}^{h}(u-\alpha_\ell)}{\prod_{\ell=1}^{k}(u-\beta_\ell)}\, d\mu(u)\right)$$

are orthogonal with respect to the modified density $d\nu(u)$.
These are exactly the determinants on the left-hand side of our identity of the theorem!
By the identity, these determinants are proportional to the right-hand sides. Hence, the right-hand sides are orthogonal polynomials with respect to the modified density $d\nu(u)$, and these are the determinants that appear in Uvarov's formula. □

The determinant identity behind Uvarov's formula

Theorem

We have

$$\frac{\det\limits_{0\le i,j\le n-1}\left(\int u^{i+j}\frac{\prod_{\ell=1}^{h}(u-\alpha_\ell)}{\prod_{\ell=1}^{k}(u-\beta_\ell)}\,d\mu(u)\right)}{\det\limits_{0\le i,j\le n-k-1}(m_{i+j})}$$

$$=(-1)^{n(h-k)+kh}\frac{\det_{1\le i,j\le k+h}\begin{pmatrix}p_{n+i-1}(\alpha_j) & 1\le i\le h\\ q_{n+i-h-1}(\beta_j) & h+1\le i\le k+h\end{pmatrix}}{\left(\prod\limits_{1\le i<j\le h}(\alpha_j-\alpha_i)\right)\left(\prod\limits_{1\le i<j\le k}(\beta_i-\beta_j)\right)}.$$

If $n<k$ the previous conventions for negatively indexed $p_a(\alpha)$ and $q_a(\beta)$ apply, and in that case the Hankel determinant in the denominator on the left-hand side has to be interpreted as 1.