Proof of some conjectures of Jean-Bernard ZUBER on the enumeration of fully packed loop configurations

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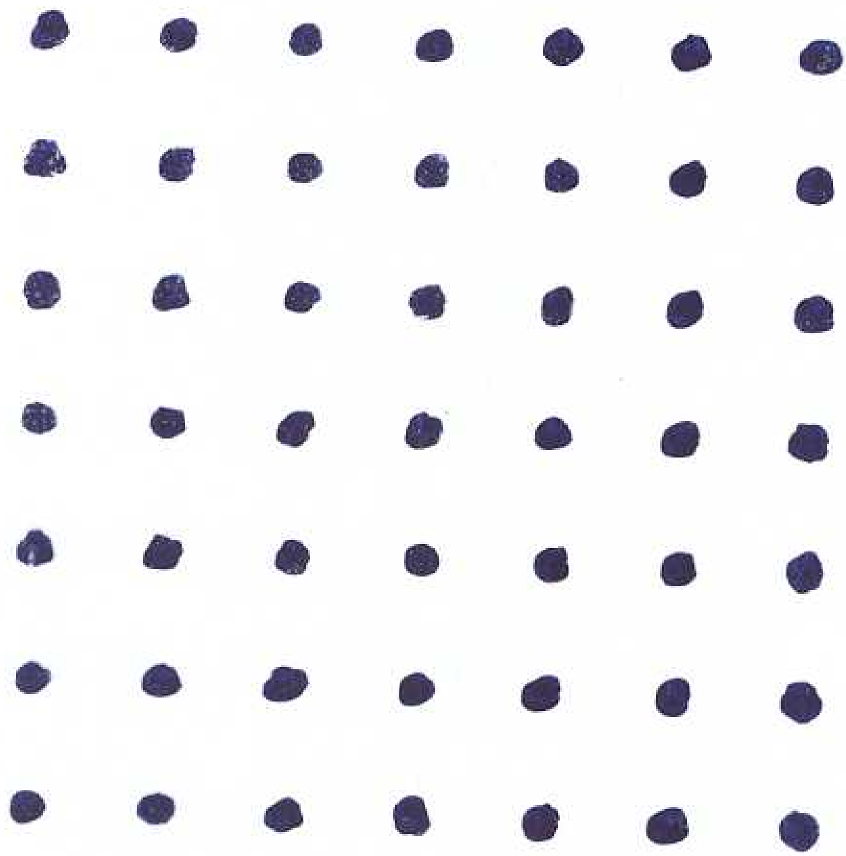
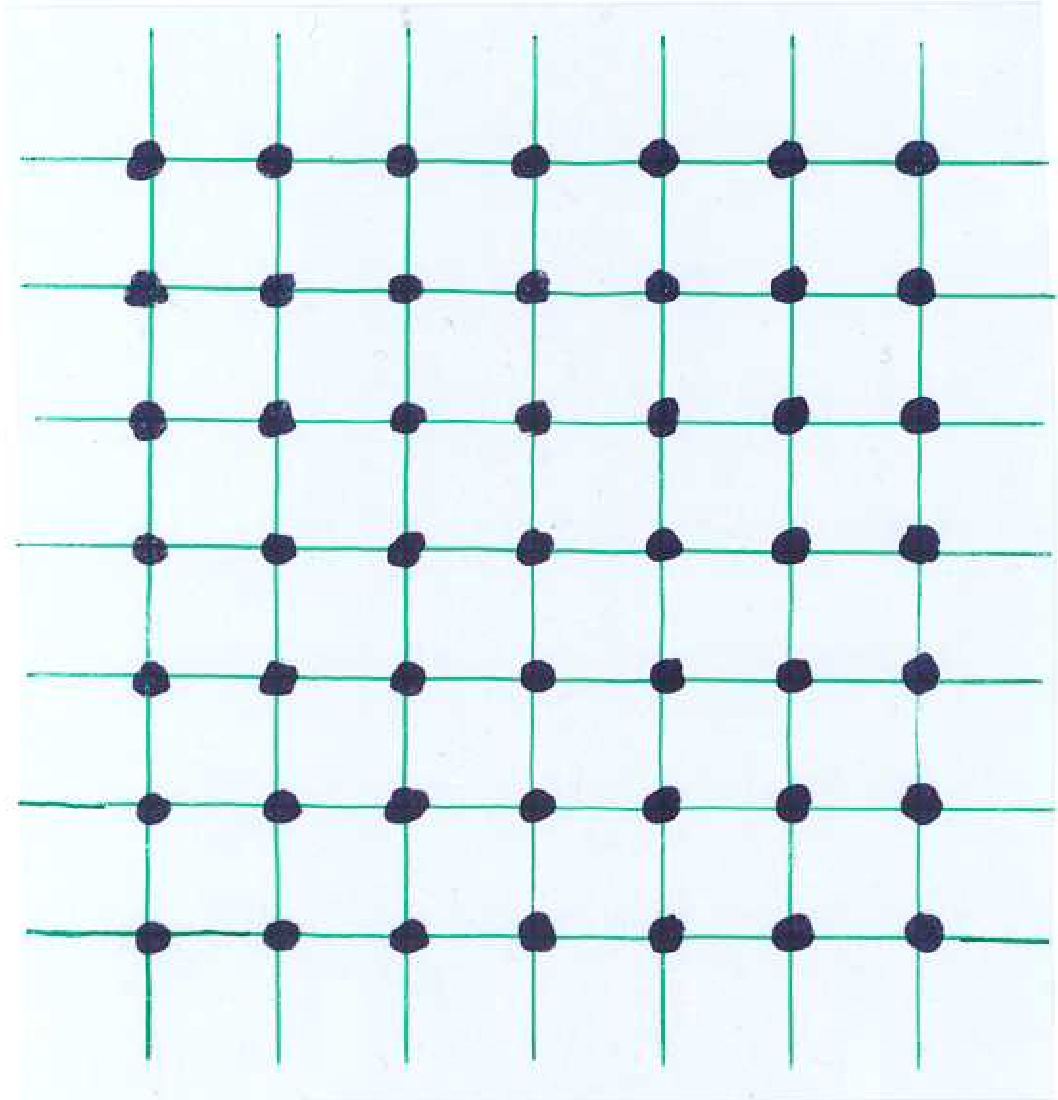
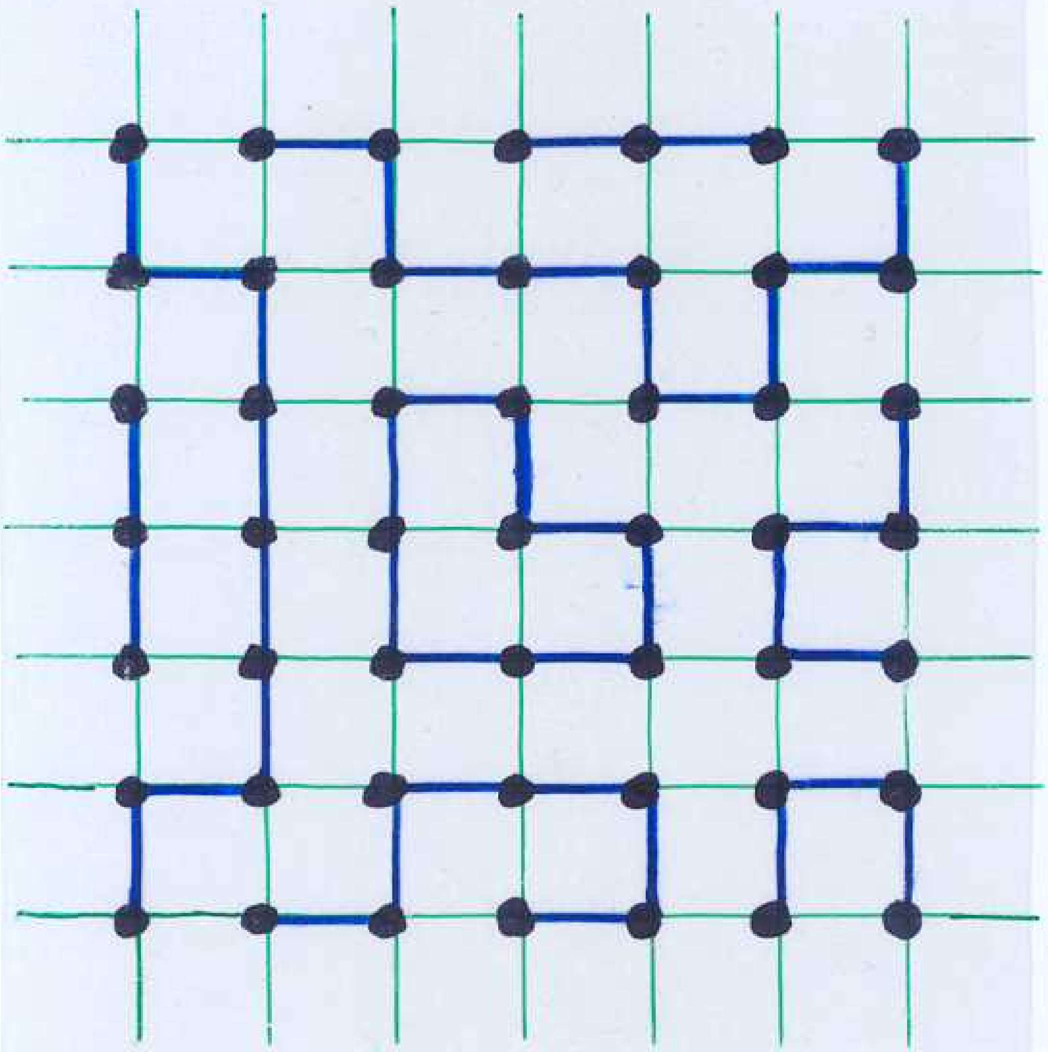
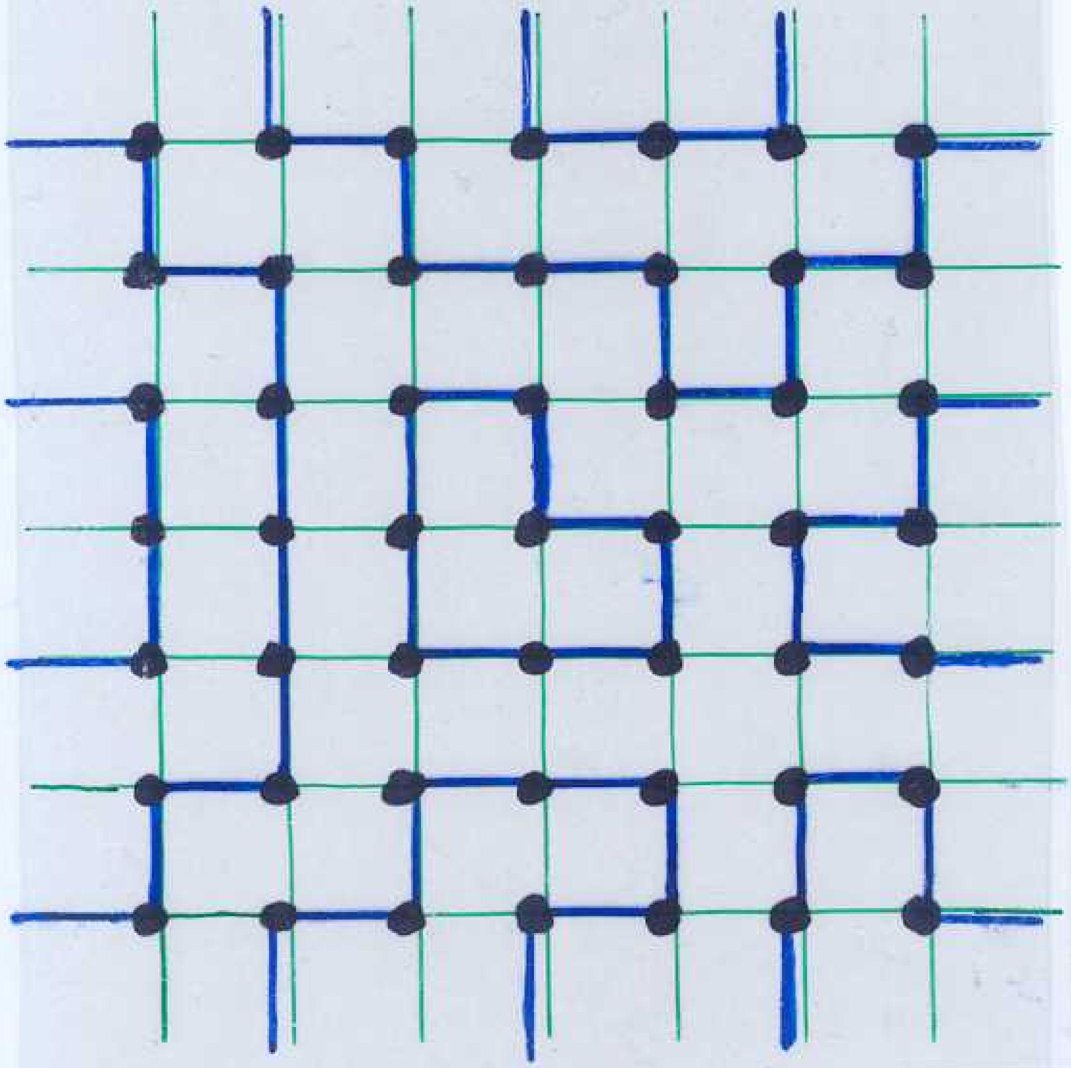
A *fully packed loop configuration* on some graph is a collection of edges such that each vertex lies on exactly two edges.







Border condition: Every other edge along the border belongs to the FPL configurations.



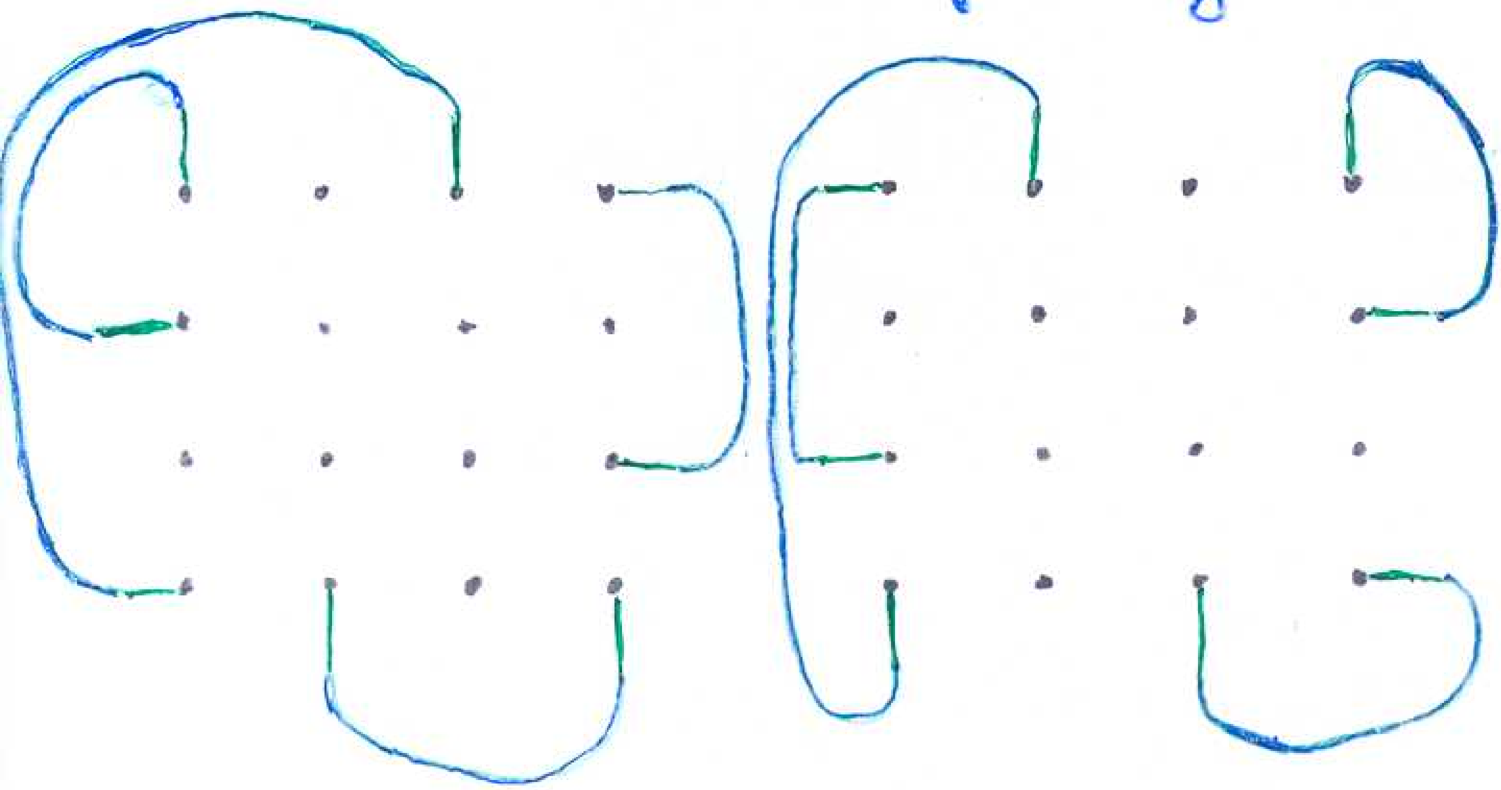
Theorem. The total number of fully packed loop configurations on the $n \times n$ grid with these "alternating" boundary conditions is equal to

$$A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!},$$

the number of $n \times n$ alternating sign matrices.

The enumeration of FPL configurations with a fixed matching pattern is motivated by a conjecture of Razumov and Stroganov which predicts that the arising numbers appear as the coordinates of the groundstate of a Hamiltonian in the dense O(1) loop model.

Theorem (Wieland).
The number of FPL configurations with matching pattern m is equal to the number of FPL configurations with matching pattern m^r, where m^r is the "rotation of m by one unit".



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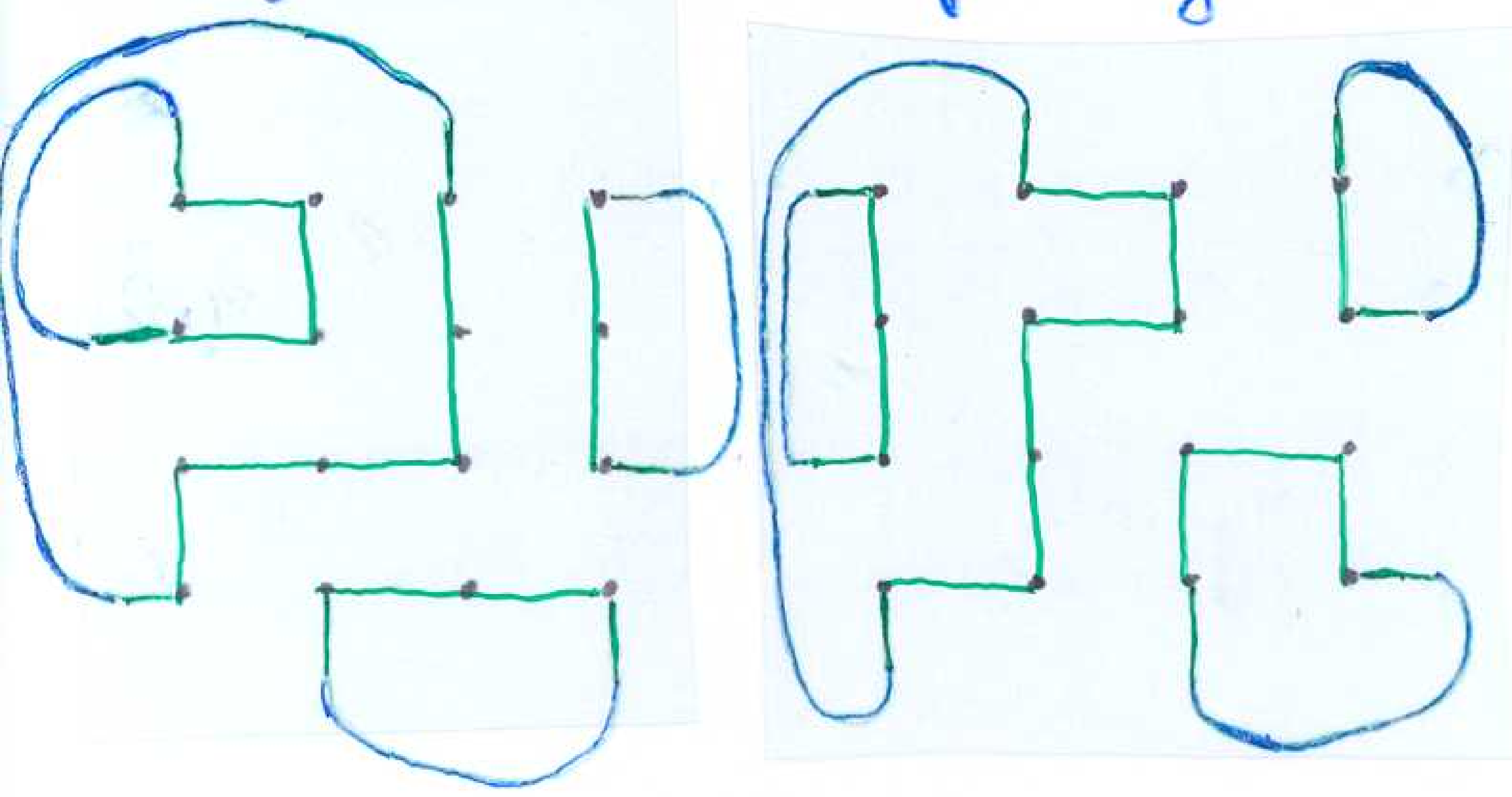
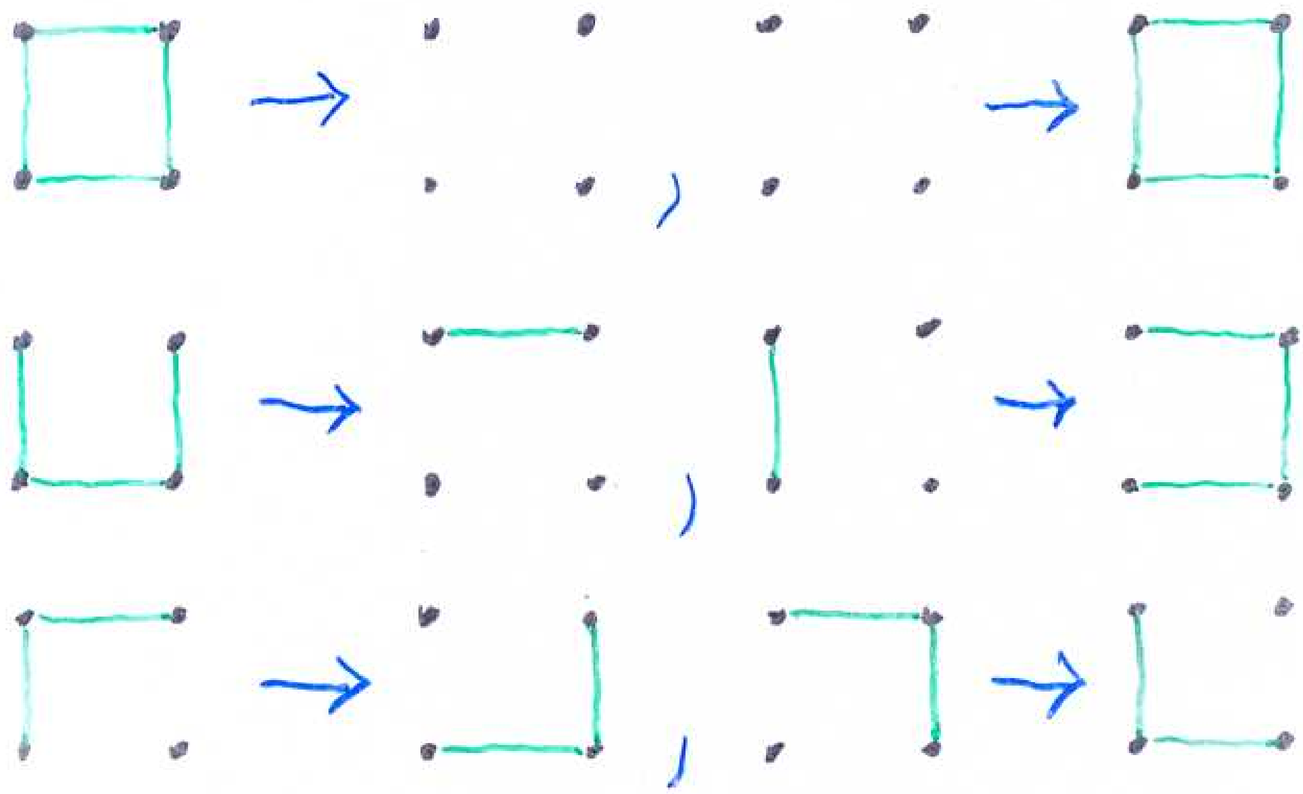
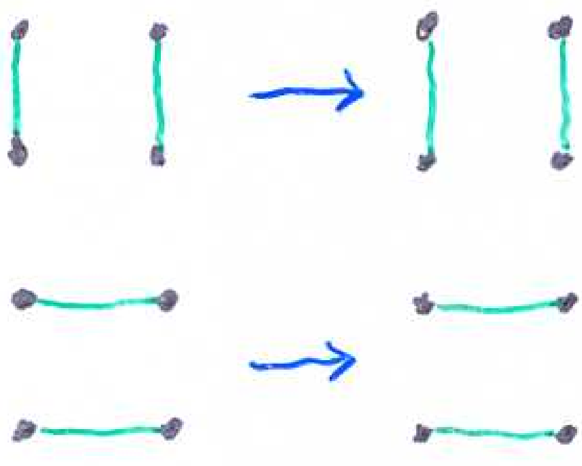


Within every other square do:
If an edge was in the FPL configuration, remove it, and vice versa, i.e.,



etc., except

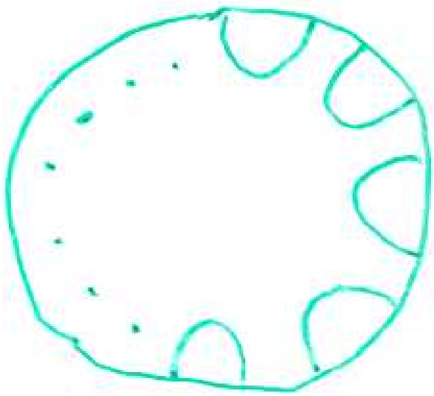


The enumeration of FPL configurations with a fixed matching pattern

Notation:

 = number of FPL configurations with this matching pattern

Conjecture (Propp-Wilson).



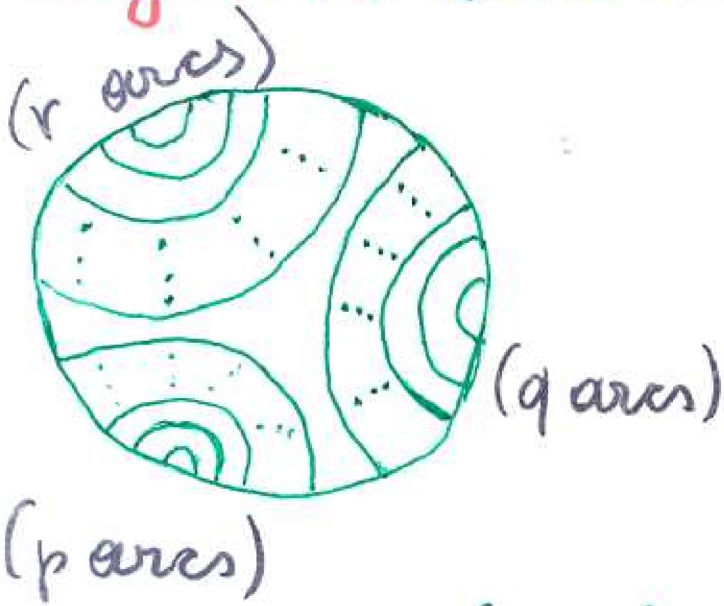
$$= A_{n-1} = \prod_{i=0}^{n-2} \frac{(3i+1)!}{(n+i-1)!}$$

(n arcs)

= number of $(n-1)\times(n-1)$ alternating sign matrices

There is a generalization by Wieland, and several variations by Zuber.

Conjecture (Zuber).

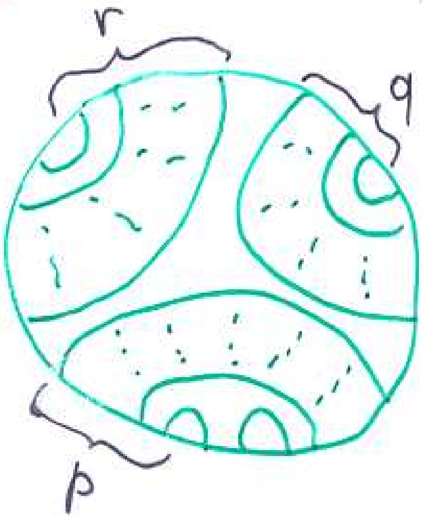


= number of rhombus tilings of a hexagon with side lengths p, q, r, p, q, r

$$= \frac{p^{\blacktriangle}\, q^{\blacktriangle}\, r^{\blacktriangle}\, (p+q+r)^{\blacktriangle}}{(p+q)^{\blacktriangle}\, (p+r)^{\blacktriangle}\, (q+r)^{\blacktriangle}}$$

where $n^{\blacktriangle} := 1!\, 2! \cdots (n-1)!$.

Conjecture (Zuber).

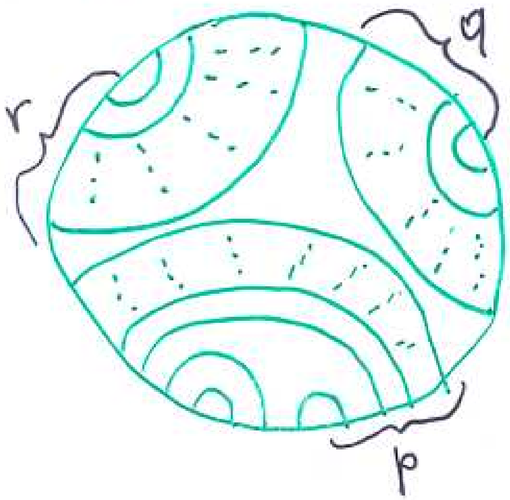


$$= \frac{(p+q+r+1)\text{¡}\,(p+1)\text{¡}\,q\text{¡}\,r\text{¡}}{(p+q+2)\text{¡}\,(p+r+2)\text{¡}\,(q+r)\text{¡}}$$

$$\times (p+q)!\,(p+r)!$$

$$\times \Big((p+1)\big(q(p+q+1)+r(p+r+1)\big)+p(p+q+1)(p+r+1)\Big).$$

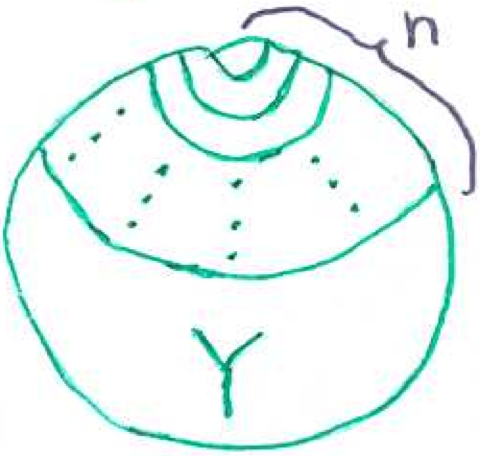
Conjecture (Zuber).



$$= \frac{(p+q+r+2)\blacktriangle\, r\blacktriangle\, q\blacktriangle\, (p+2)\blacktriangle}{2(p+q+4)\blacktriangle\, (p+r+2)\blacktriangle\, (q+r)\blacktriangle} \times (p+q+1)!\,(p+q+2)!\,(p+r+3)!\,(p+r)!\,(p+2)$$

$$\times \Big((p+1)(p+q+3)(p+r+1)\big(p(p+r+2)+pq+4qr\big) + 2p(p+q+r)q(p+r+1)(p+q+2) + 2(p+1)(p+q+3)(p+r+1)(p+r+2)r + (p+3)(p+r+1)(p+r+2)r(r-1) + (p+1)(p+q+2)(p+q+3)q(q+1) \Big).$$

Conjecture (Zuber).



$$= \frac{P_Y(n)}{|Y|!},$$

where $P_Y(n)$ is a polynomial in n with integer coefficients with leading coefficient $\dim Y$.

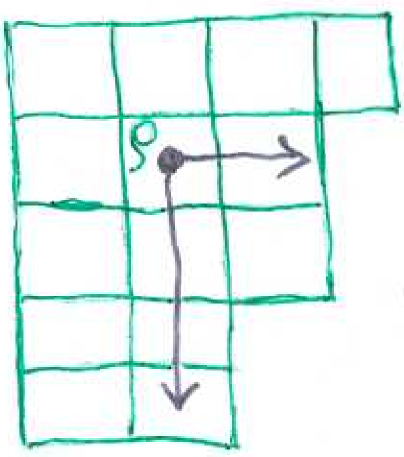
$\dim Y$ is the dimension of the irreducible representation of S_n indexed by Y, where $n = |Y| = \#$cells of Y.

An explicit formula is given by:

Hook formula (Frame, Robinson, Thrall)

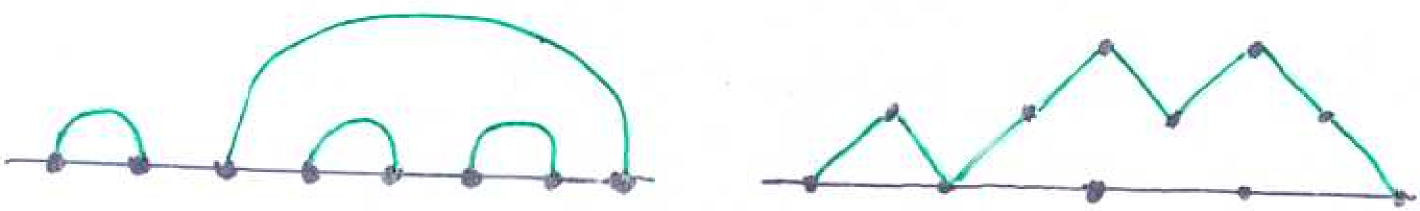
$$\dim Y = \frac{|Y|!}{\prod_{\rho \in Y} h_\rho},$$

where h_ρ is the hook length of the cell ρ = number of cells to the right (in the same row) and below (in the same column) of ρ, including ρ.



$h_\rho = 5$

A correspondence between matchings and Ferrers diagrams



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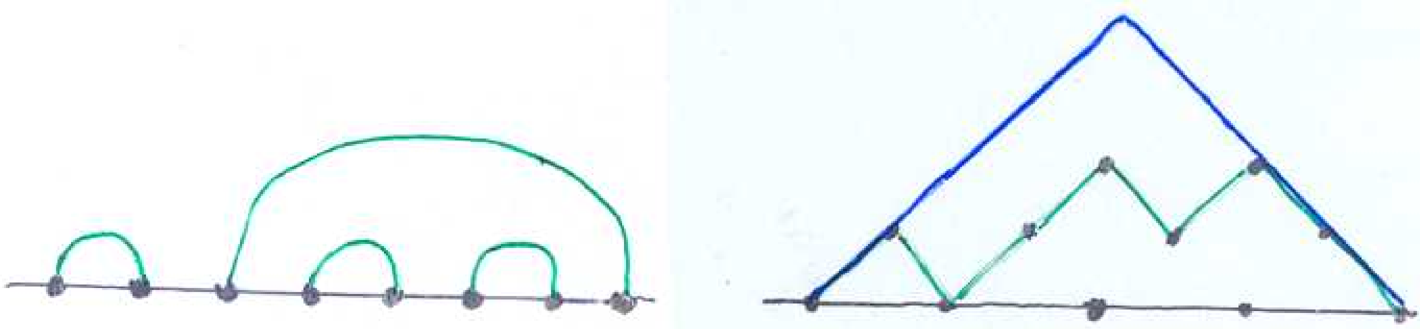

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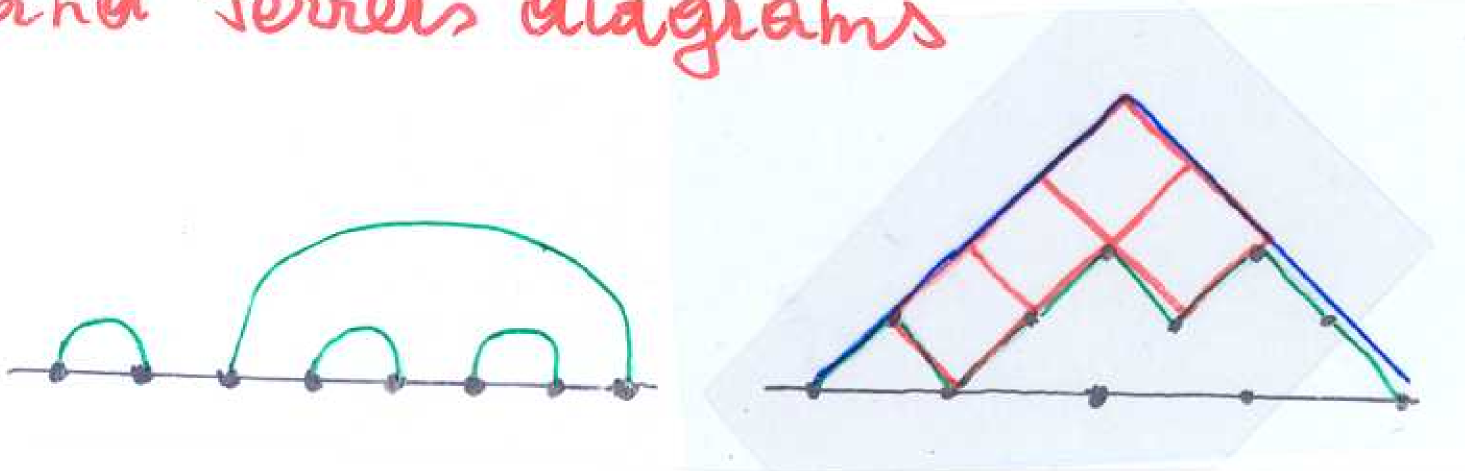

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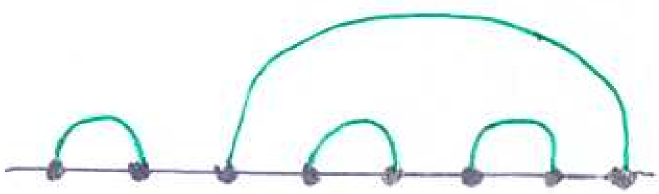

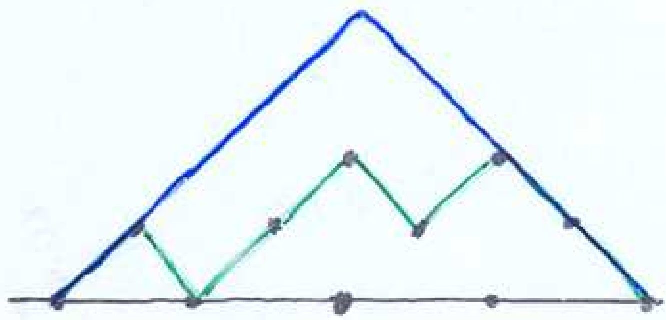

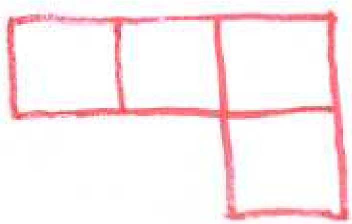

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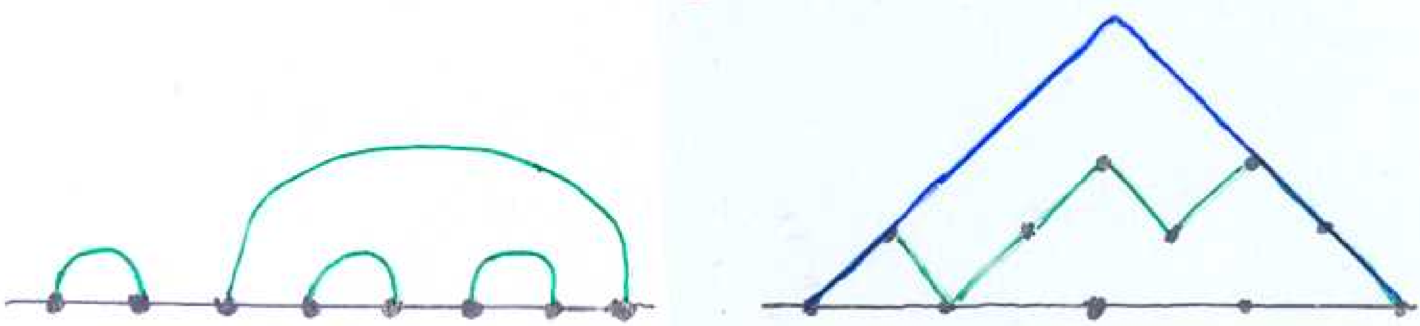
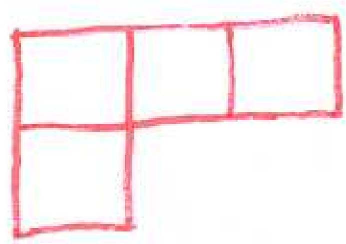
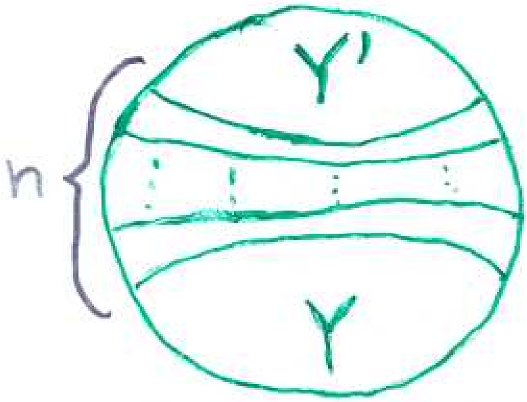




Conjecture (Zuber).

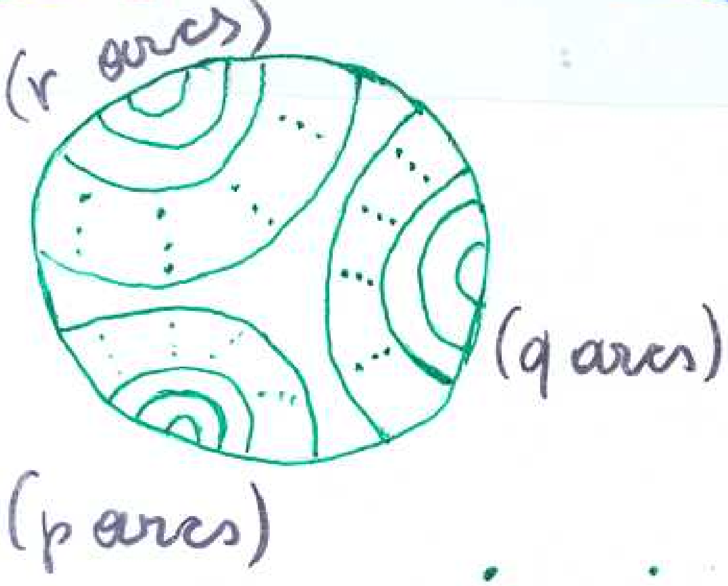


$$= \frac{\tilde{P}_{Y,Y'}(n)}{|Y|!\,|Y'|!},$$

where $\tilde{P}_{Y,Y'}(n)$ is a polynomial in n with integer coefficients and leading term $(\dim Y)\cdot(\dim Y')$.

Theorem (Di Francesco, Zinn-Justin, Zuber).

~~**Conjecture** (Zuber)~~.



= number of rhombus tilings of a hexagon with side lengths p, q, r, p, q, r

$$= \frac{p\blacktriangle\ q\blacktriangle\ r\blacktriangle\ (p+q+r)\blacktriangle}{(p+q)\blacktriangle\ (p+r)\blacktriangle\ (q+r)\blacktriangle}$$

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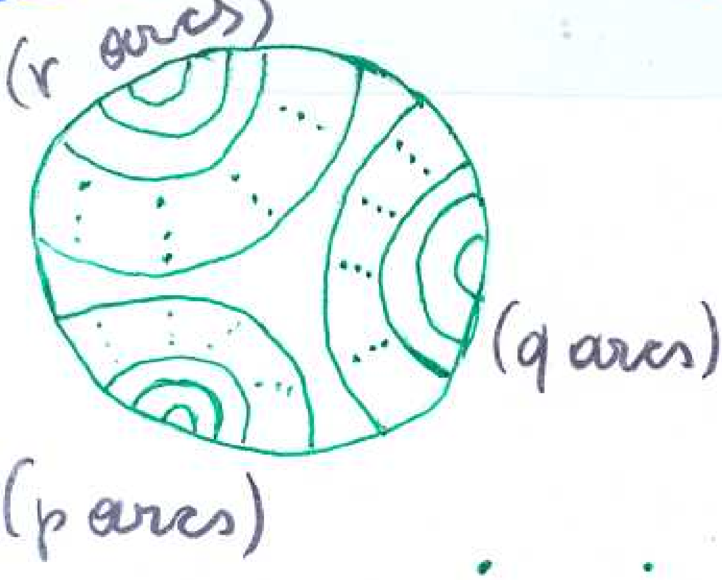

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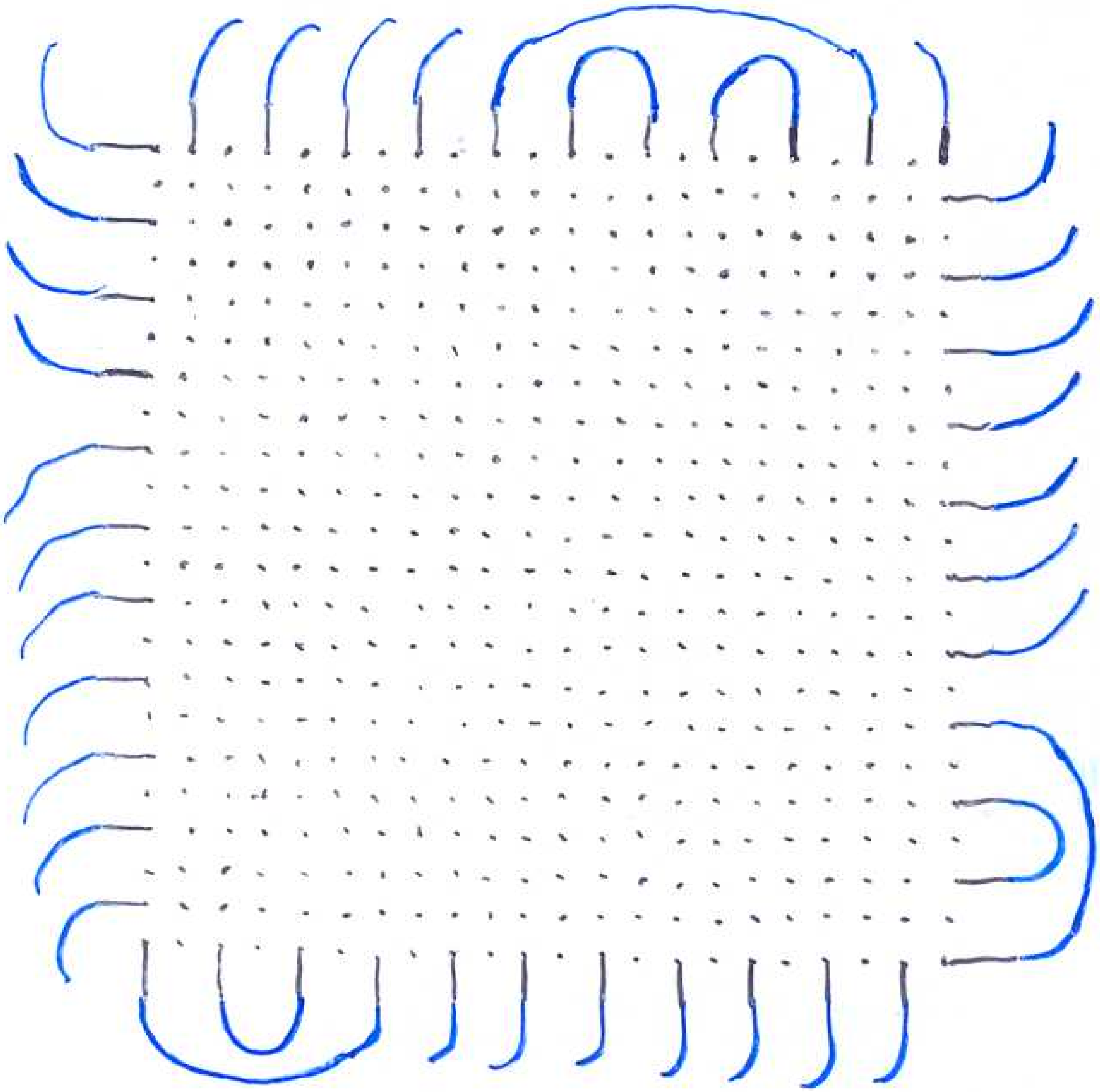
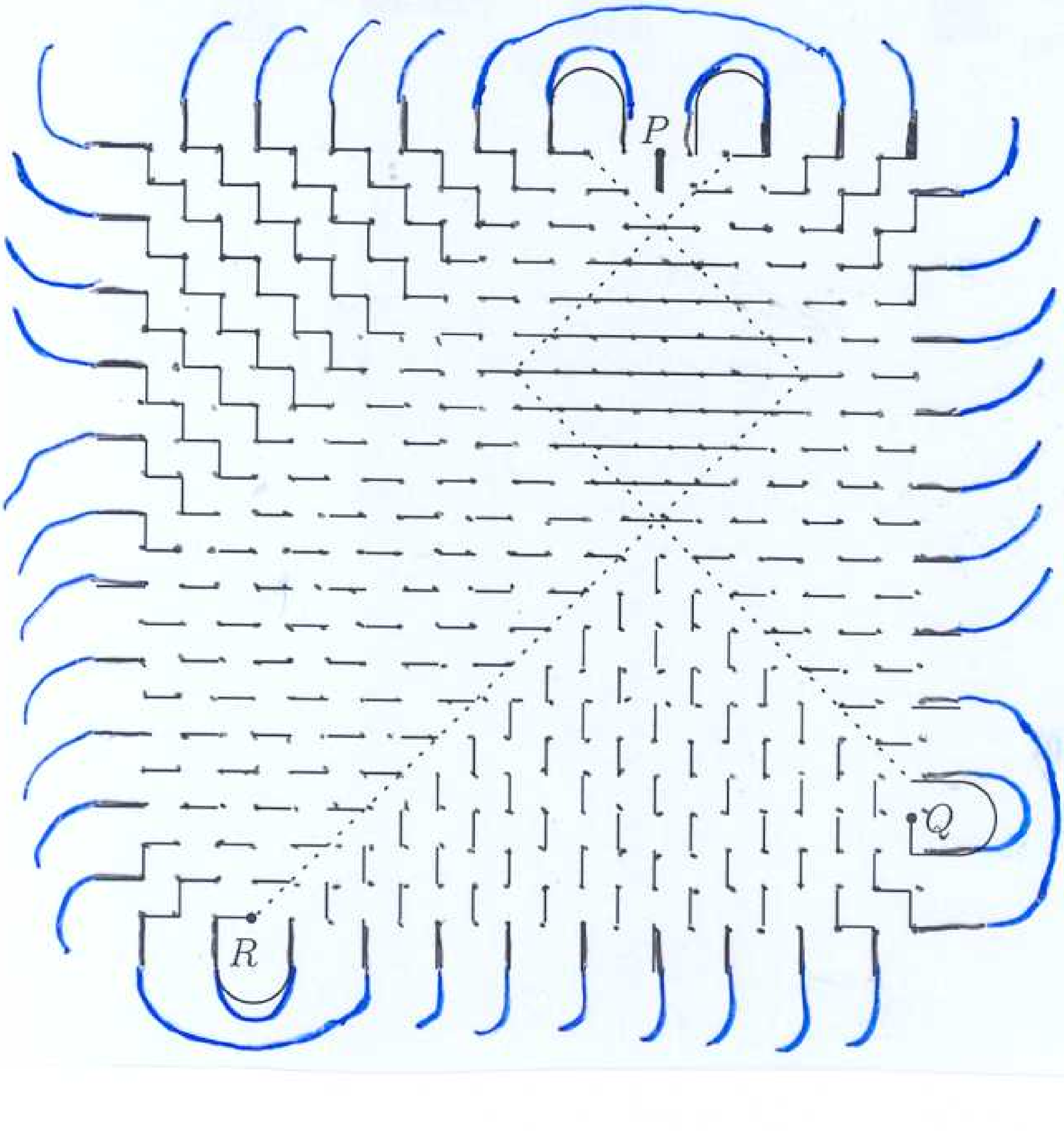
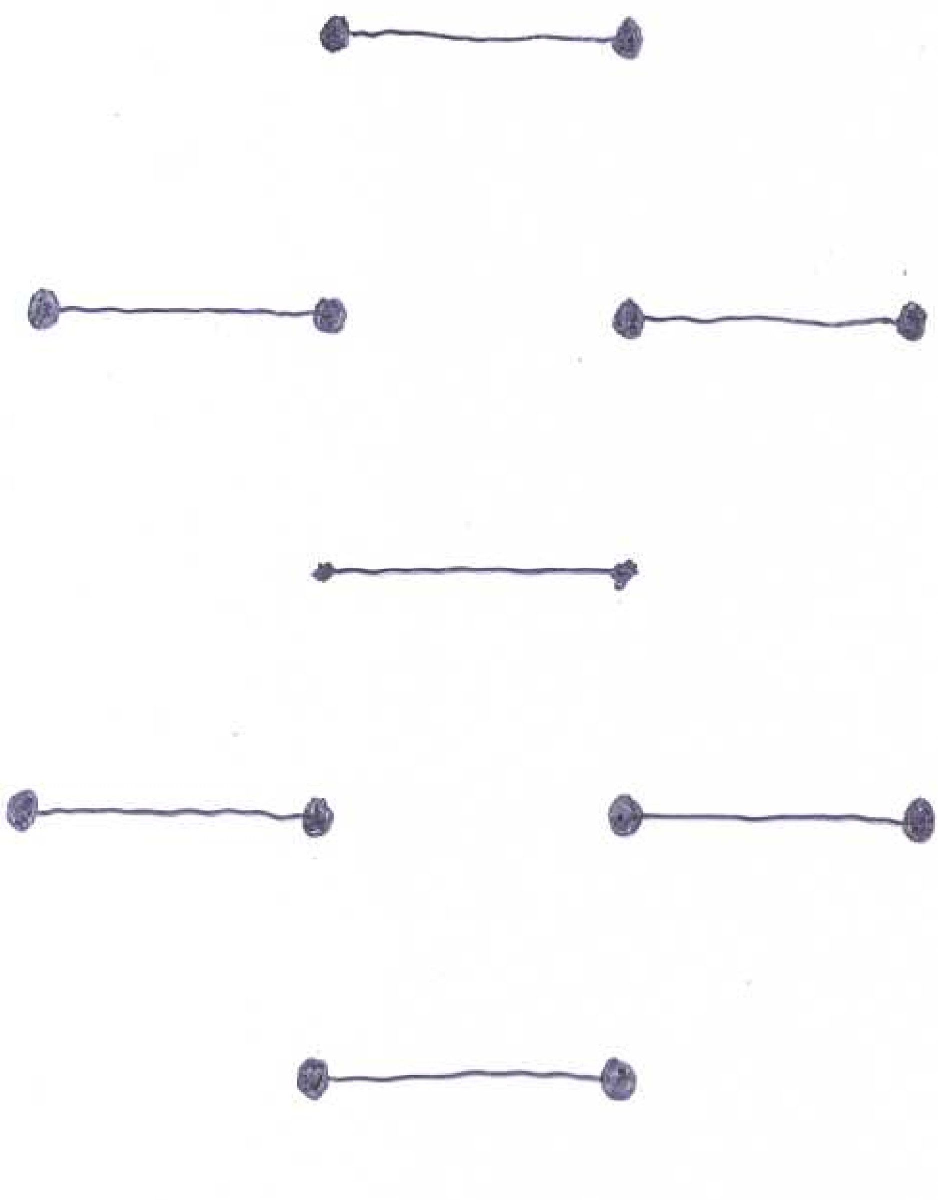
The last four conjectures are theorems (with Caselli, Lass, Nadeau).




P
Q
R

(de Gier)

(de Gier)



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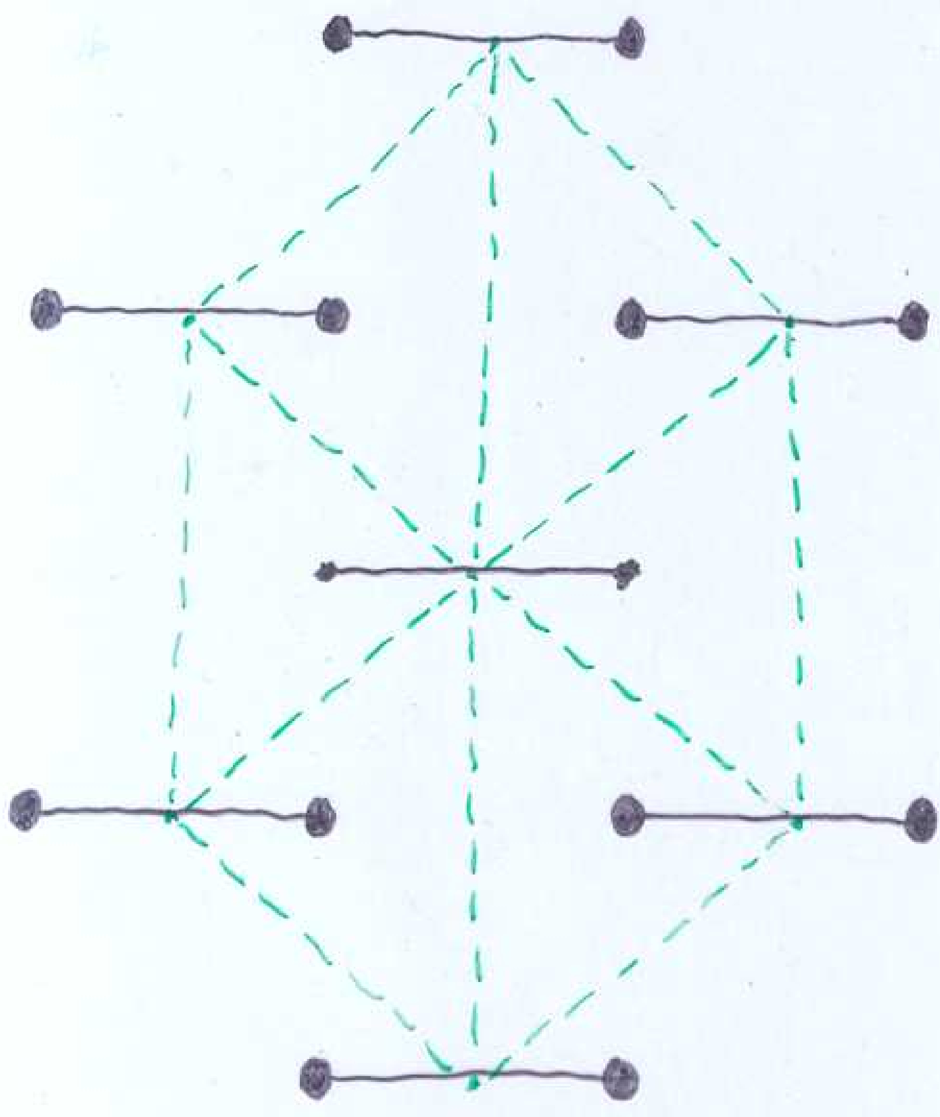

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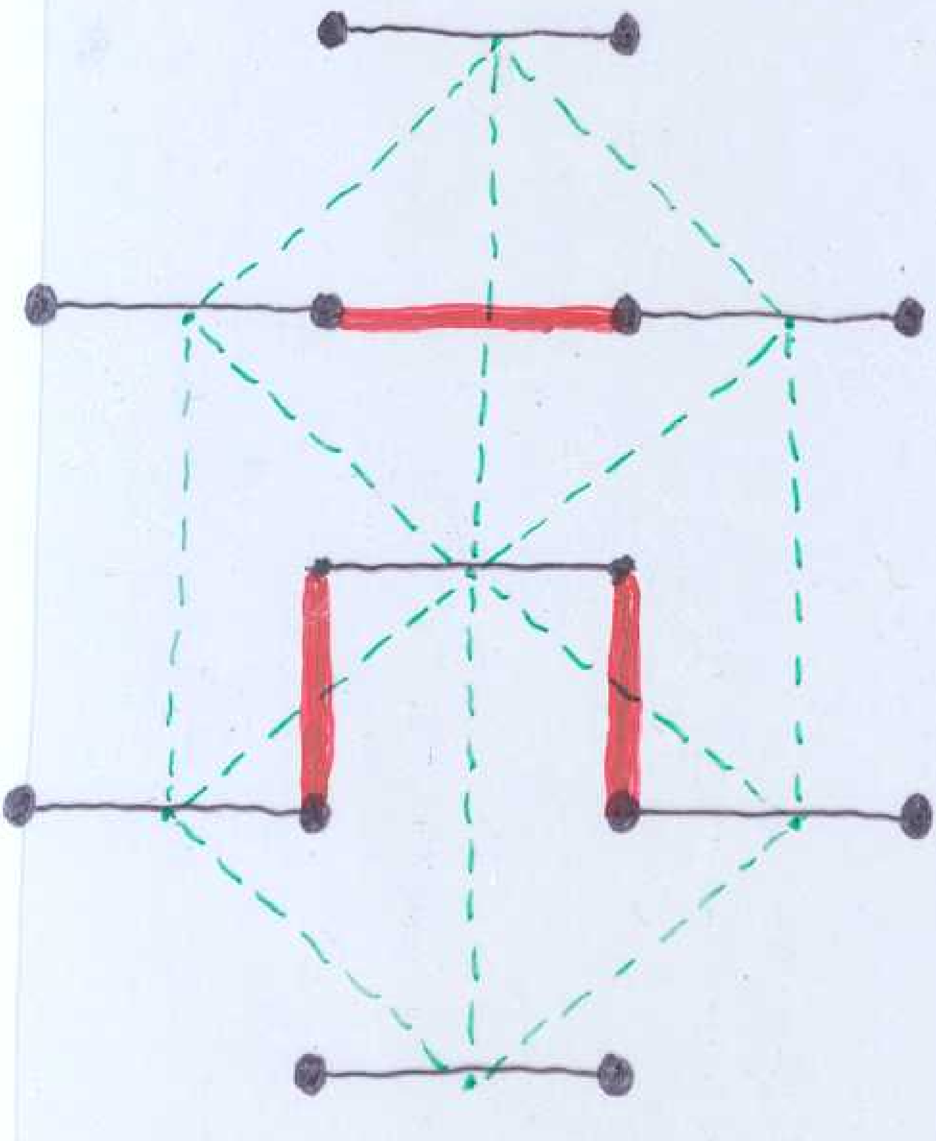

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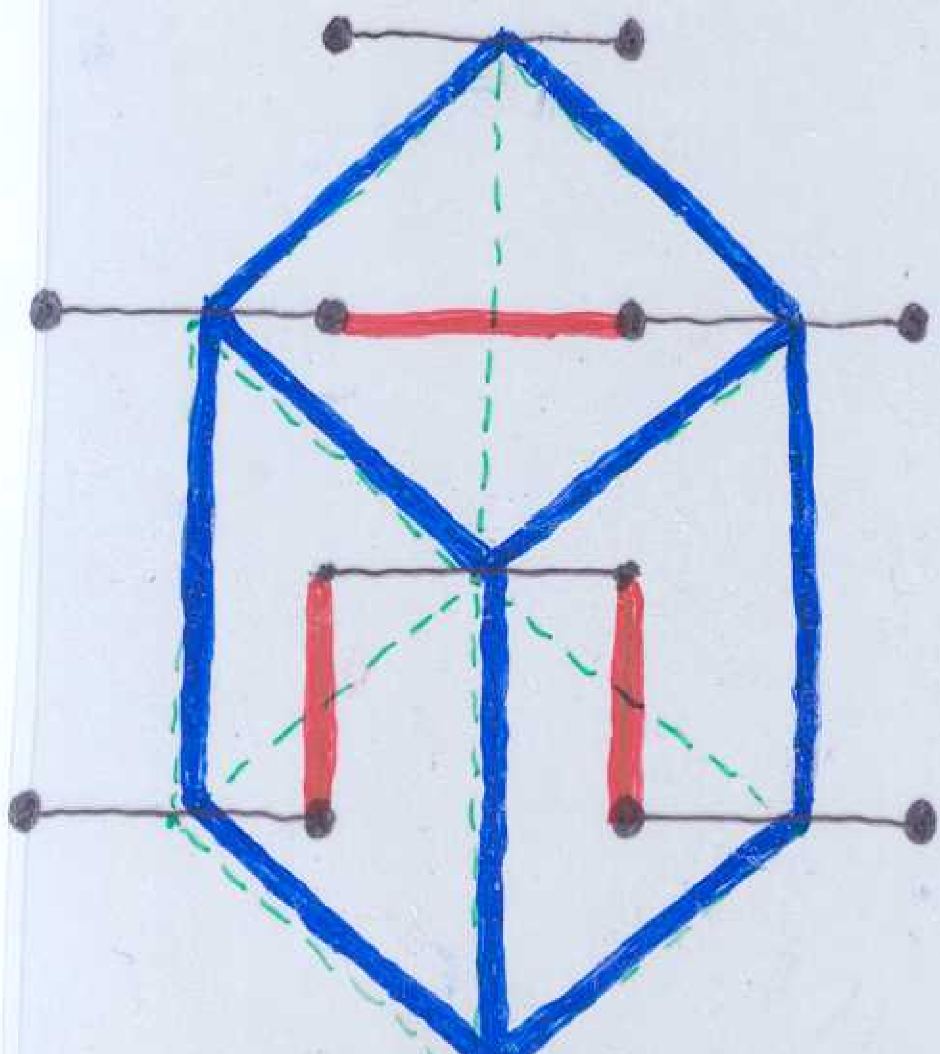
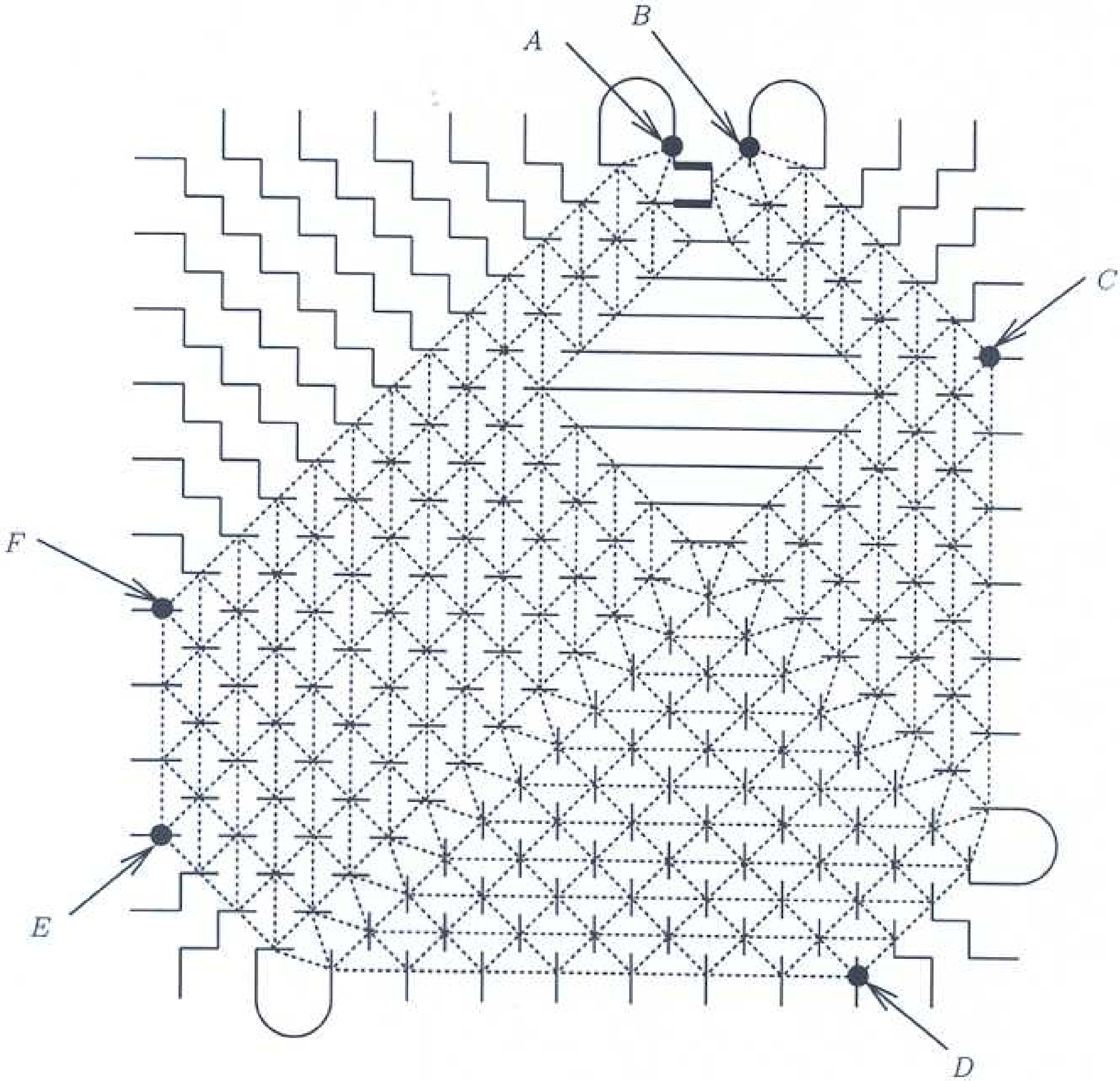
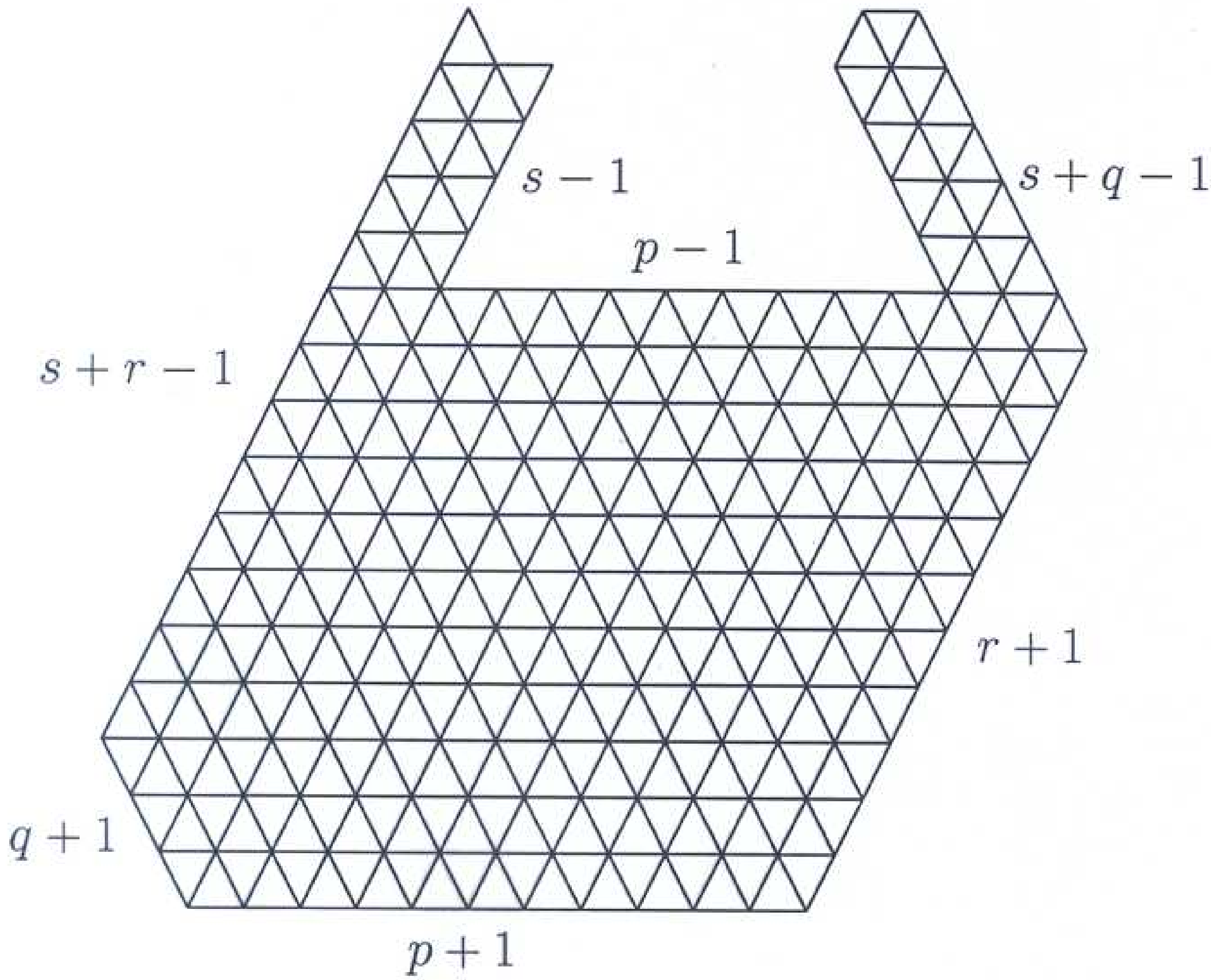
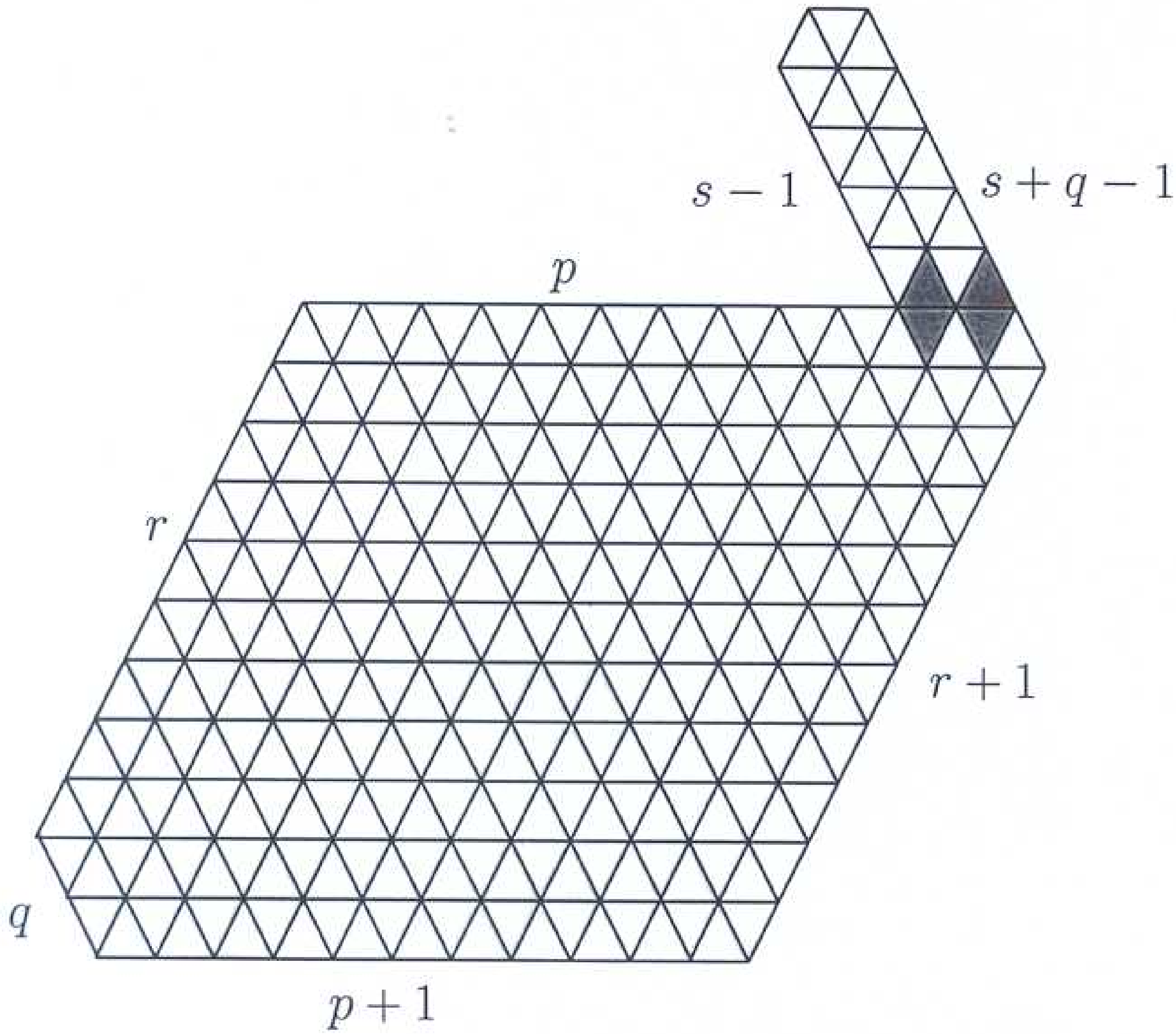
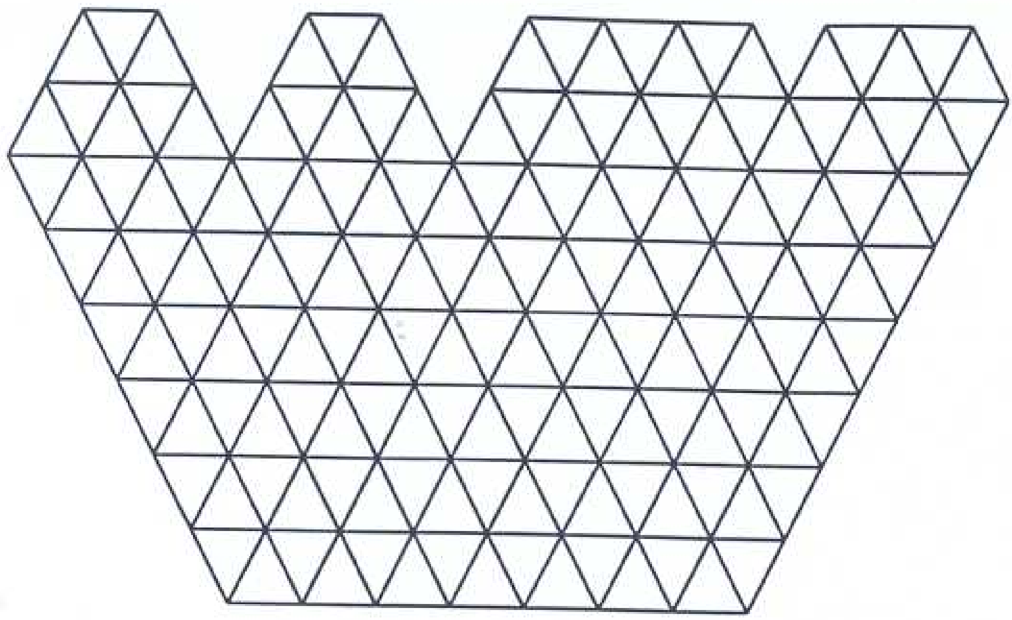


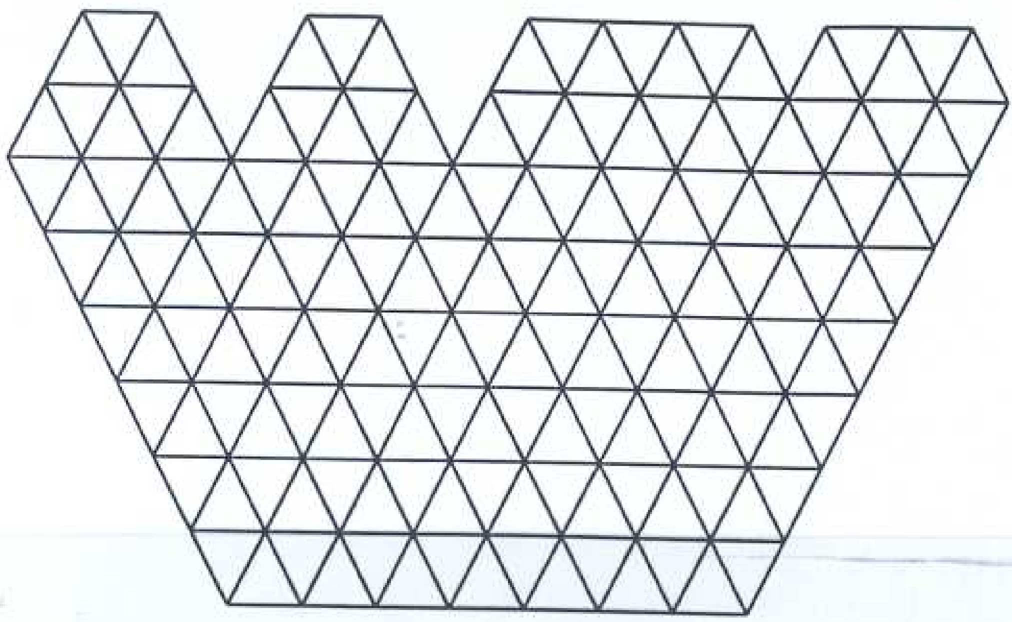

A
B
C
D
E
F

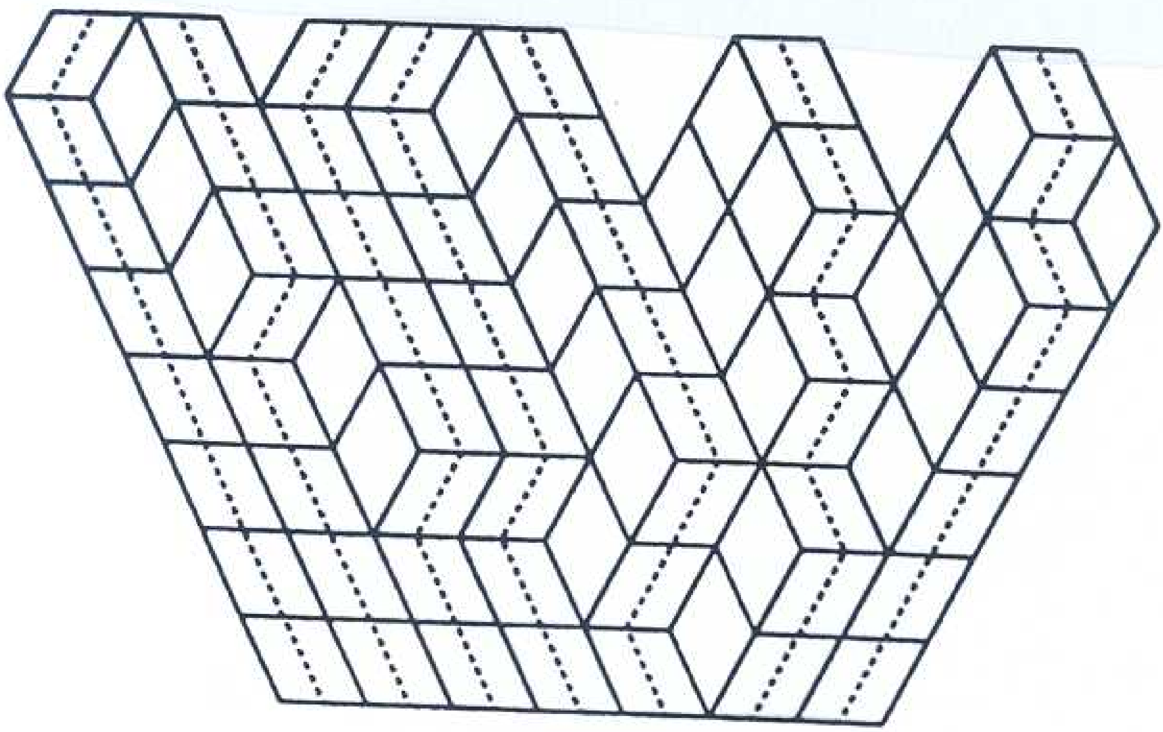

$s-1$
$s+q-1$
$p-1$
$s+r-1$
$r+1$
$q+1$
$p+1$

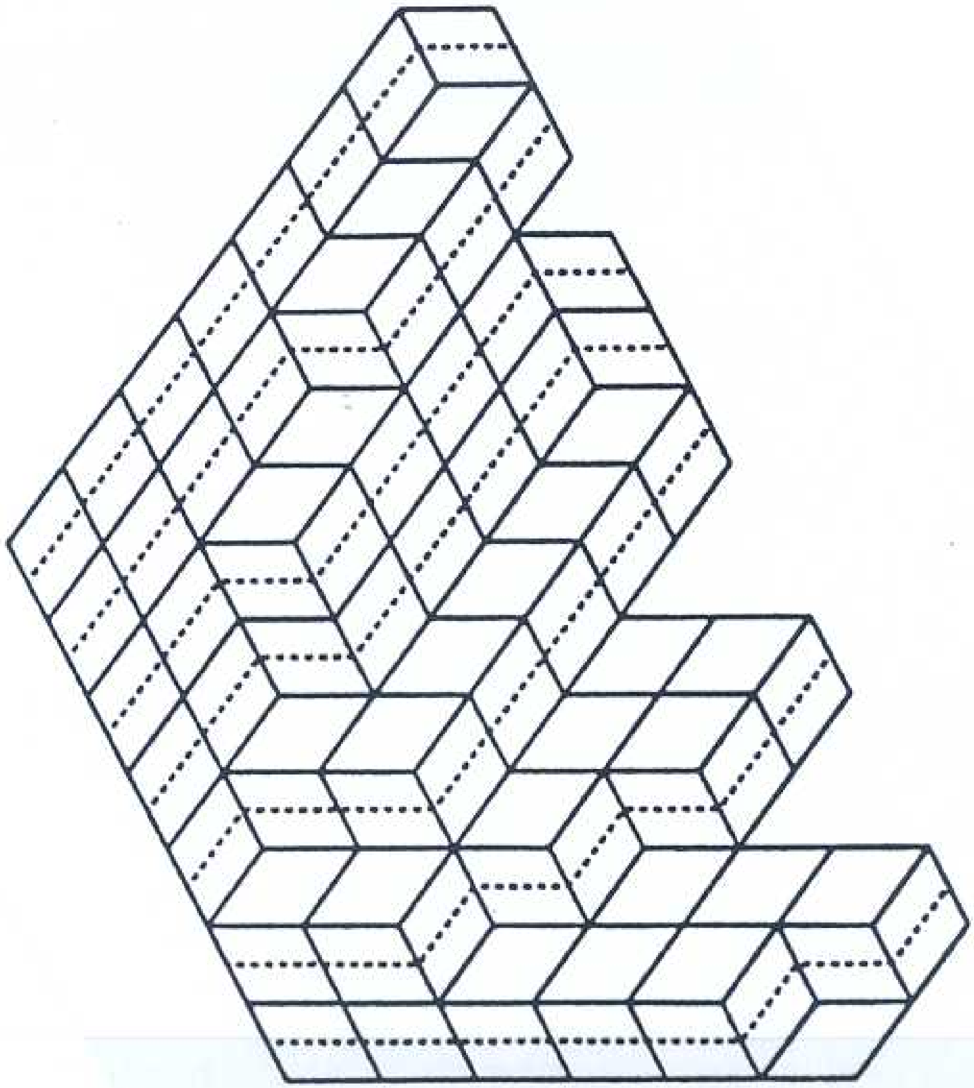


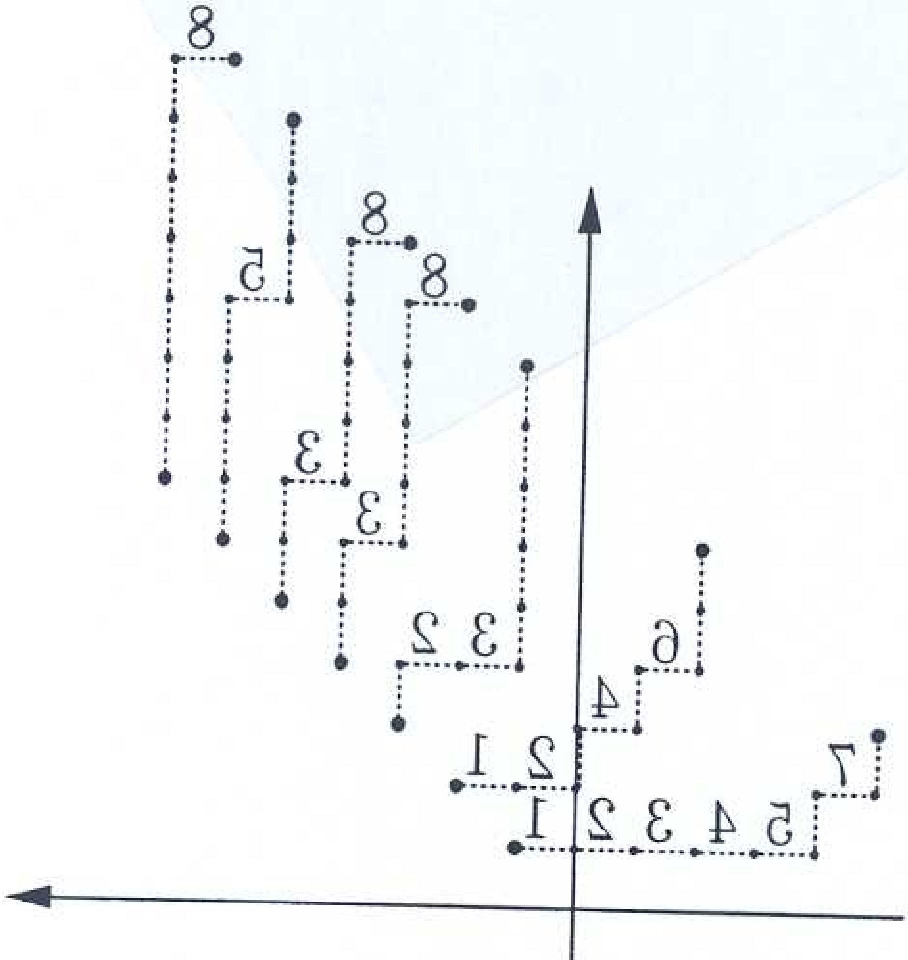
$s-1$
$s+q-1$
p
r
$r+1$
q
$p+1$

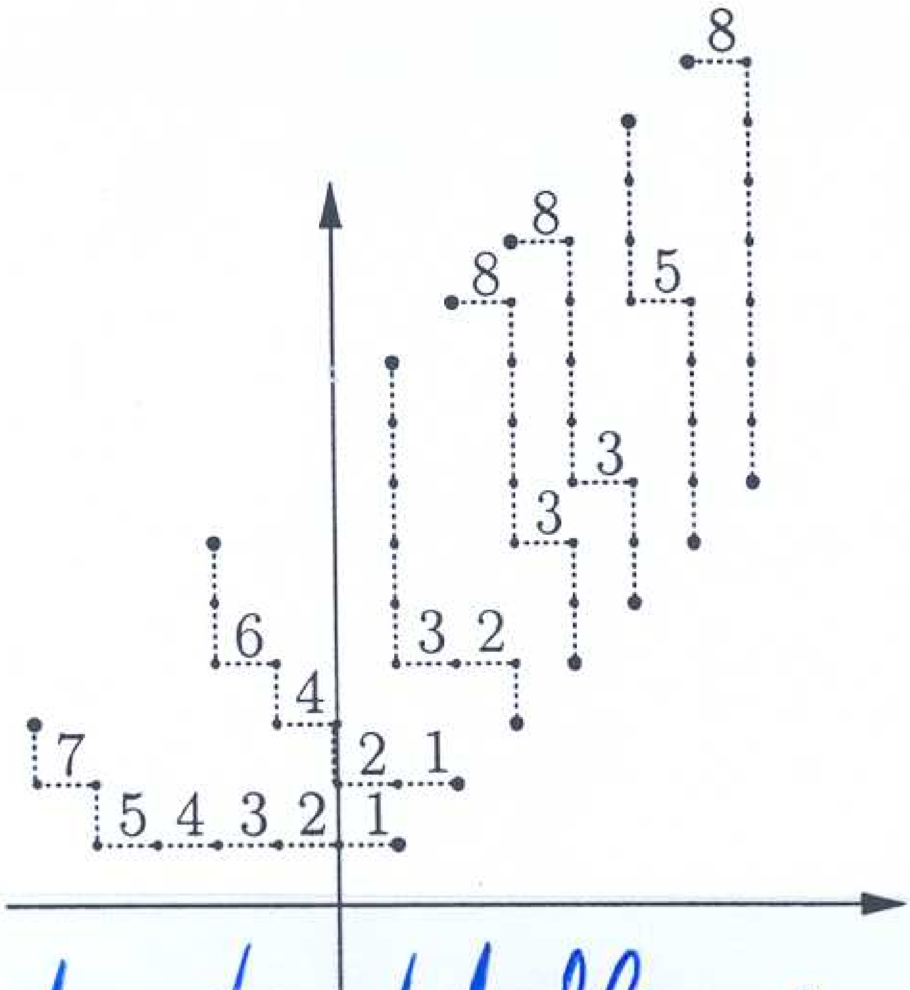












a semistandard tableau:

1	1	2	3	3	5	8
2	2	3	8	8		
3	4					
4	6					
5						
7						

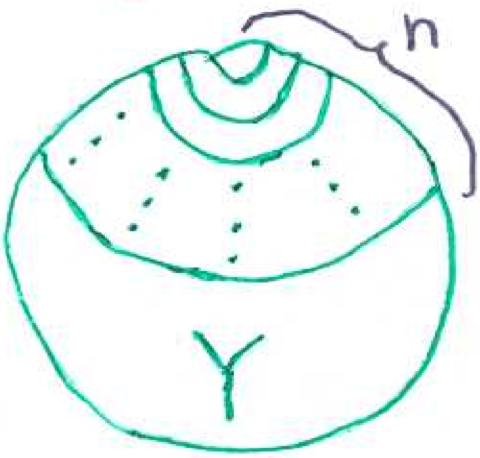
Now we need:

The hook-content formula (Stanley).

The number of semistandard tableaux of shape λ with entries $\leq a$ is

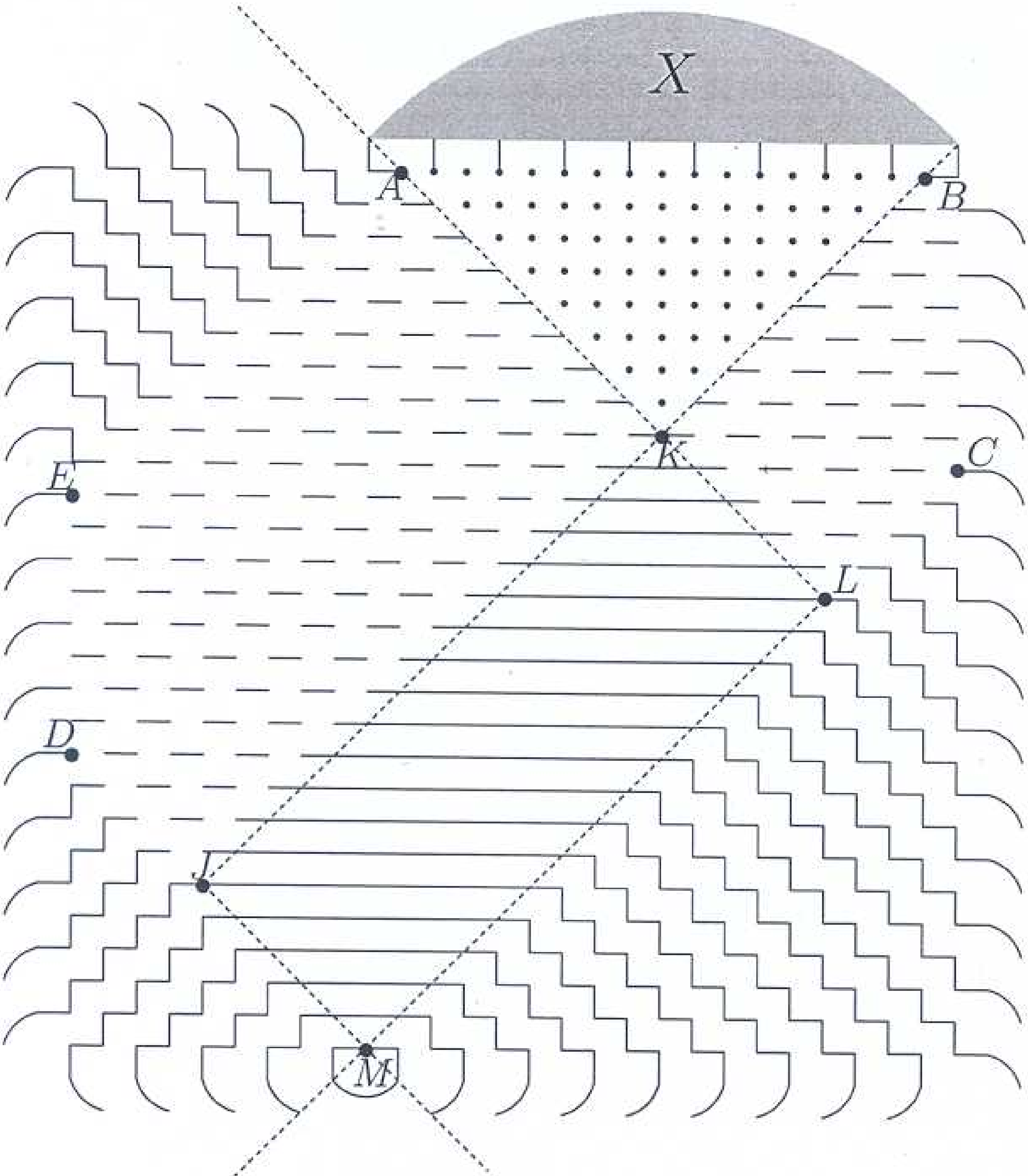
$$\prod_{\rho \in \lambda} \frac{a + c_\rho}{h_\rho}.$$

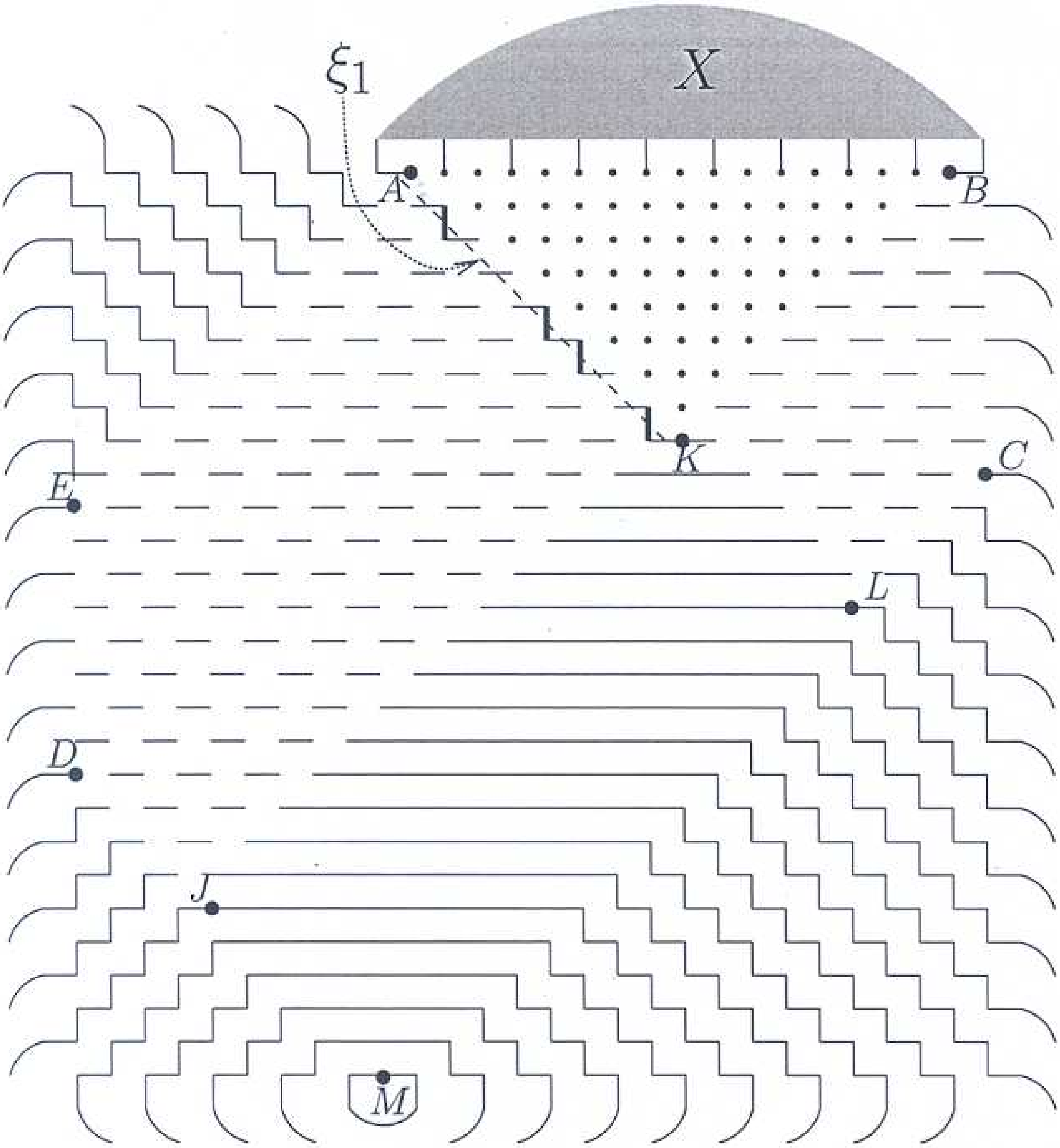
Conjecture (Zuber).



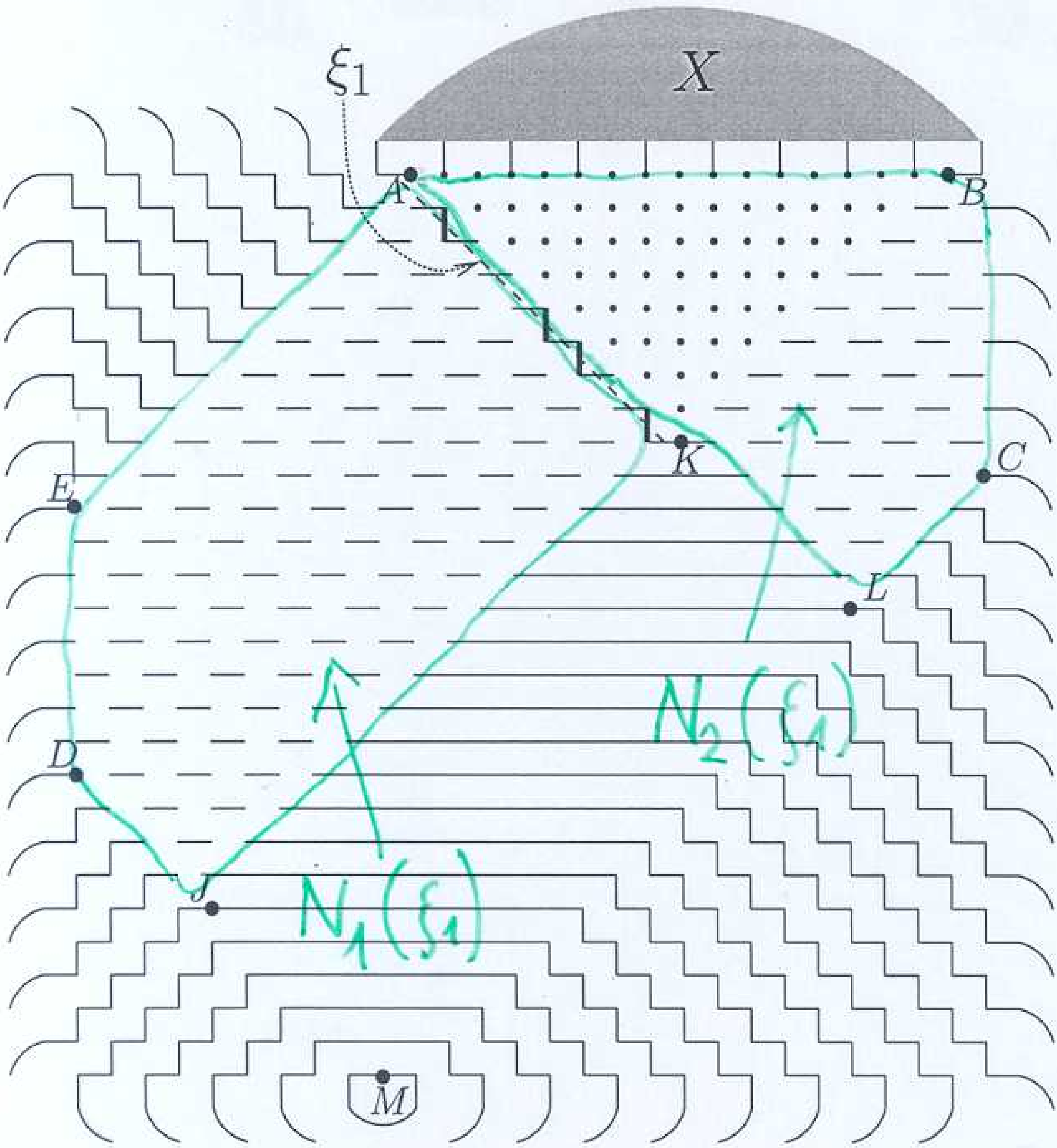
$$= \frac{P_Y(n)}{|Y|!},$$

where $P_Y(n)$ is a polynomial in n with integer coefficients with leading coefficient $\dim Y$.


X
A
B
C
K
E
L
D
J
M



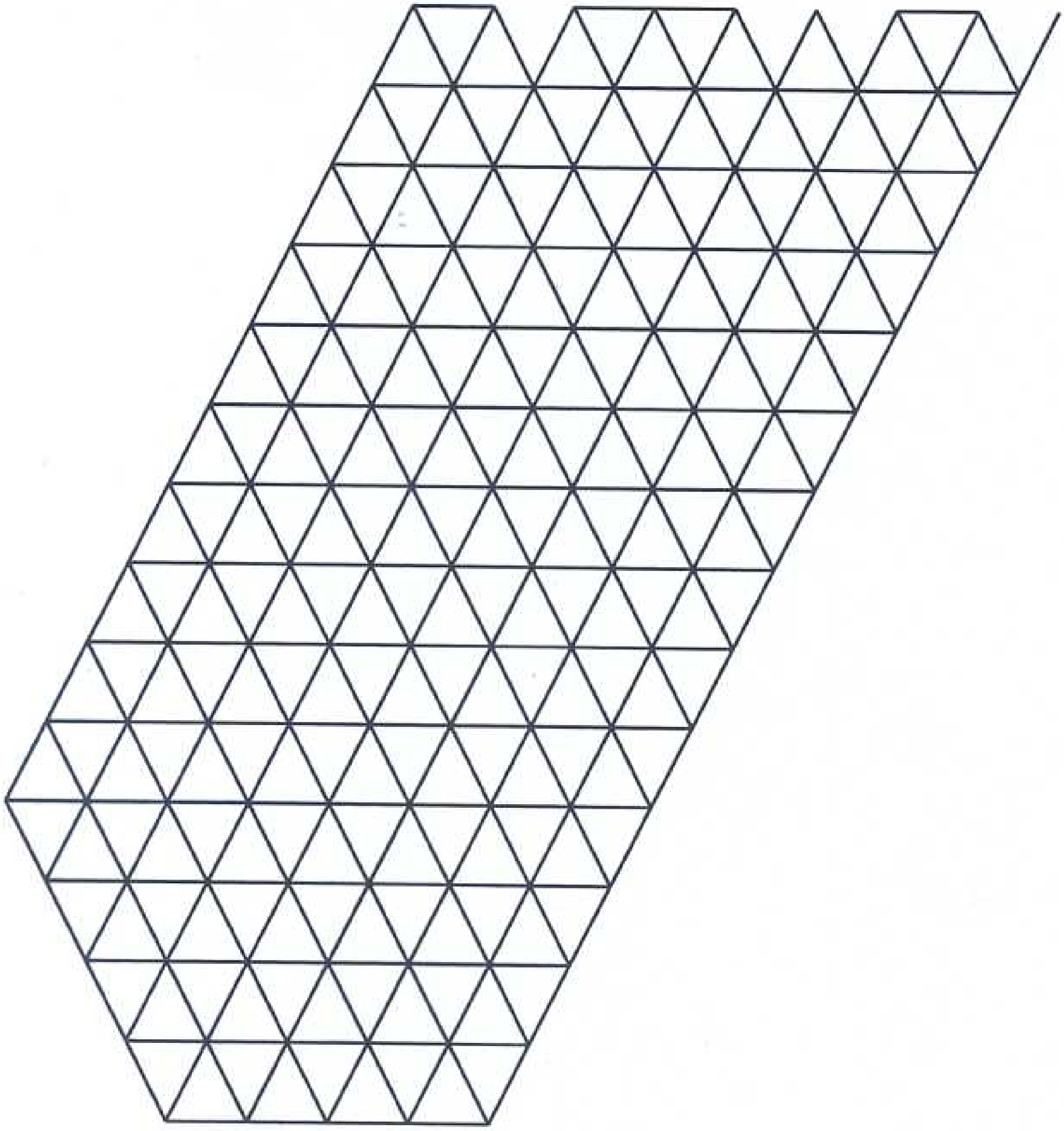
ξ_1
X
A
B
K
C
E
L
D
J
M


X
ξ_1
A
B
C
E
K
L
D
J
M
$N_1(\xi_1)$
$N_2(\xi_1)$

Thus, the number of FPL configurations that we are looking for is given by

$$\sum_{\xi_1} N_1(\xi_1) \cdot N_2(\xi_1),$$

where the sum is over all possible choices ξ_1 of vertical edges along the border between the two regions.



Hence:

For any ξ_1, the number $N_1(\xi_1)$ is equal to the number of semistandard tableaux of some fixed shape.

Application of the hook-content formula then proves the polynomiality in n.

If one looks closer, then also the other claims of the conjecture become evident.

However:

All this holds only if $n \geq 3d$,
given that X is a matching of
$2d$ pending edges.

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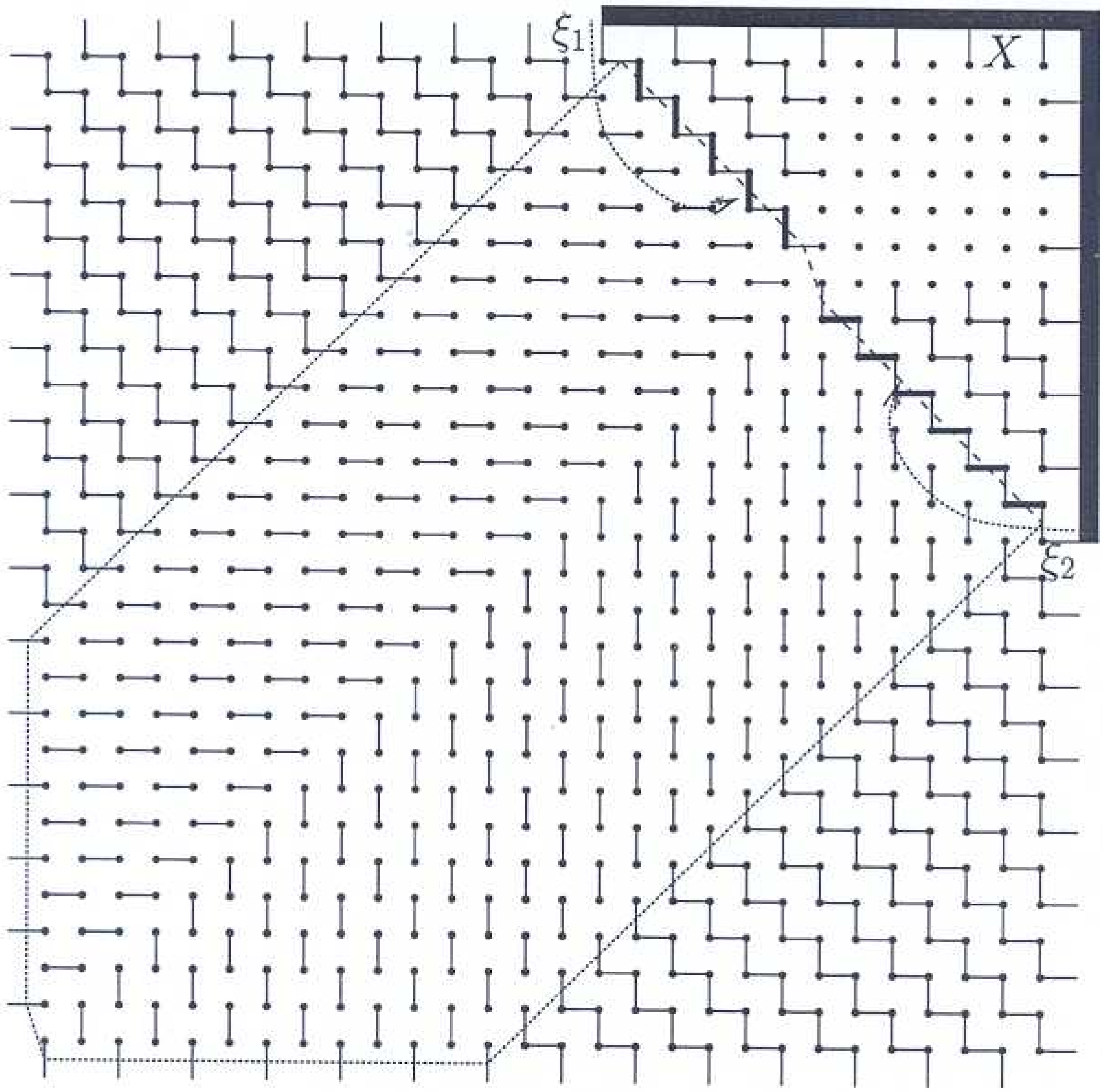
What about small n?

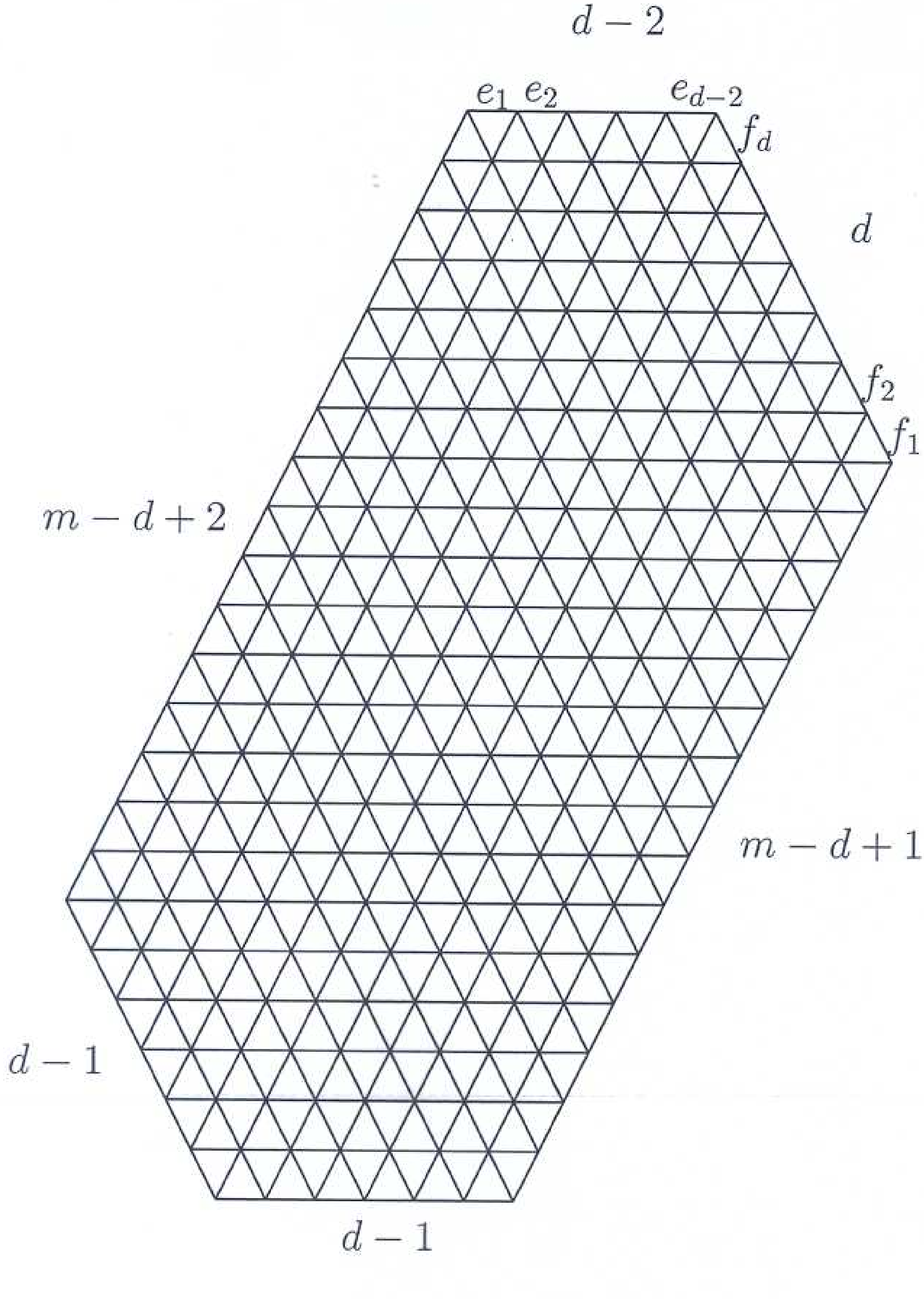


ξ_1
X
ξ_2

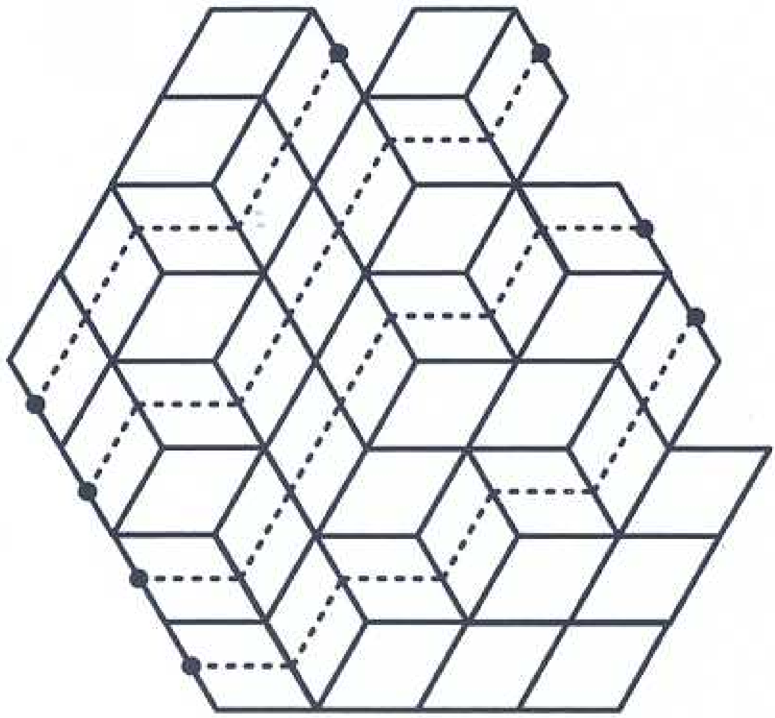
Thus, the number of FPL configurations that we are looking for is given by

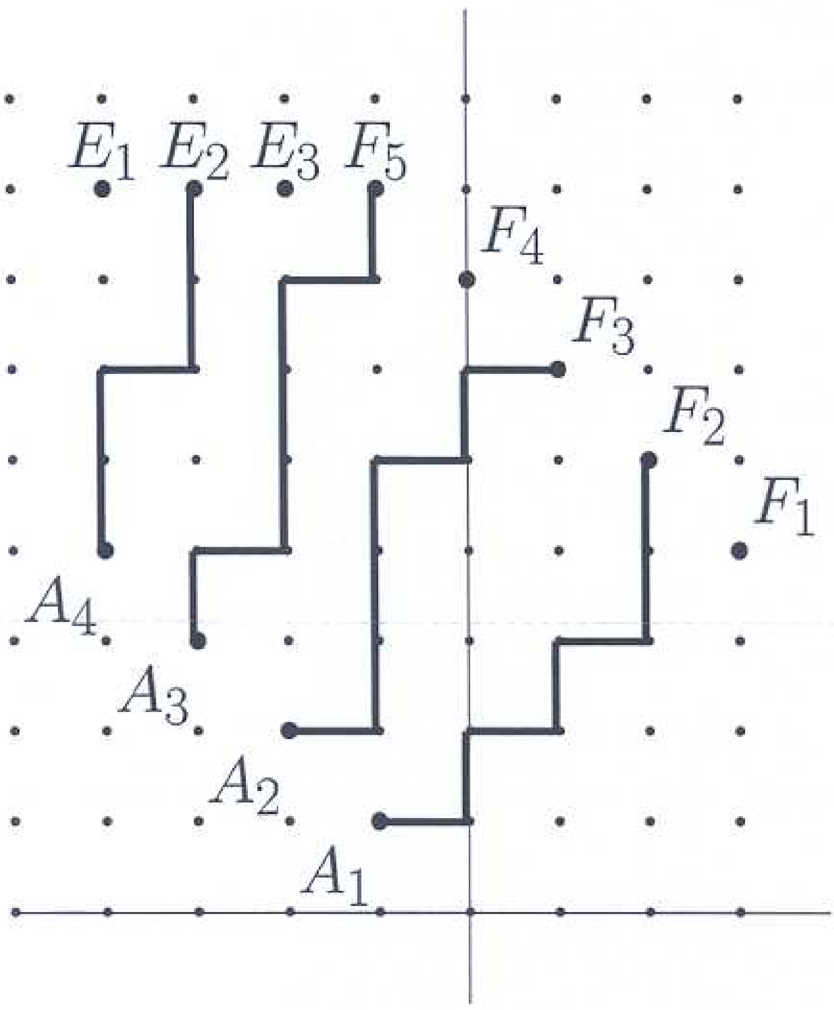
$$\sum_{\xi} N_3(\xi) \cdot N_4(\xi),$$

where the sum is over all possible choices ξ of vertical edges along the border between the two regions.



$d-2$
e_1 e_2 e_{d-2}
f_d
d
f_2
f_1
$m-d+2$
$m-d+1$
$d-1$
$d-1$





E_1 E_2 E_3 F_5
F_4
F_3
F_2
F_1
A_4
A_3
A_2
A_1

By applying the Lindström-Gessel-Viennot theorem for non-intersecting lattice paths, one finds that any $N_3(\xi)$ is given by a determinant of the form

$$\det\left(\begin{pmatrix} n+\ldots \\ \ldots \end{pmatrix}\right).$$

In particular, $N_3(\xi)$ is polynomial in n, while $N_4(\xi)$ is independent of n.

Moreover, one can read off that

$$N_3(\xi)\Big|_{n=0} = \begin{cases} 1 & \xi \text{ is the "barrier" choice in the figure,} \\ 0 & \text{otherwise.} \end{cases}$$

Now let

$$A_X(n) = \text{number of FPL configurations with matching pattern } X \cup n \text{ arcs},$$

$$H_X(n) = \sum_{\xi} N_3(\xi) \cdot N_4(\xi).$$

We have:

$$A_X(n) = H_X(n) \quad \text{for } n \geq 2d$$

$$A_X(0) = H_X(0)$$

$$A_{X \cup s \text{ arcs}}(n) = A_X(n+s).$$

Hence:

$$H_{X \cup s \text{ arcs}}(n) = H_X(n+s)$$

$$\text{for } n \geq 2d.$$

These are infinitely many values of n. Therefore $H_{X \cup s \text{ arcs}}(n)$ and $H_X(n+s)$ are identical as polynomials in n.

So:

$$H_X(s) = H_{X \cup s \text{ arcs}}(0) = A_{X \cup s \text{ arcs}}(0) = A_X(s),$$

for all $s \geq 0$. $\square$