Nonlinear Functional Analysis, SS 2008

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Thank you all in advance,

Andreas Kriegl, August 1993

In the second edition an extensive list of misprints and corrections provided by Eva Adam has been taken gratefully into account.

Andreas Kriegl, September 1994

After some minor corrections I ported to source to LaTeX. Since chapter 1 and 2 have been incorperated into the book [14] they have been replaced by the (slightly modified) sections from there.

Andreas Kriegl, Feber 2008

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0. Motivation

0.1 Equations on function spaces. It should be unnecessary to convince the reader, that differential calculus is an important tool in mathematics. But probably some motivation is necessary why one should extend it to infinite dimensional spaces. One of our main tasks as mathematicians is, like it or not, to solve equations like

$$f(u) = 0.$$

However quite often one has to consider functions f which don't take (real) numbers as arguments u but functions. Let us just mention DIFFERENTIAL EQUATIONS, where f is of the following form

$$f(u)(t) := F(t, u(t), u'(t), \dots, u^{(n)}(t)).$$

Note that this is not the most general form of a differential equation, consider for example the function f given by $f(u) := u' - u \circ u$, which is not treated by the standard theory.

If the arguments t of u are (real) numbers, then this is the general form of an ORDINARY DIFFERENTIAL EQUATION, and in the generic case one can solve this implicit equation $F(t, u(t), u'(t), \dots, u^{(n)}(t)) = 0$ with respect to $u^{(n)}(t)$ and obtains an equation of the form

$$u^{(n)}(t) = g(t, u(t), u'(t), \dots, u^{(n-1)}(t)).$$

By substituting $u_0(t) := u(t), u_1(t) := u^{(1)}(t), \dots, u_{n-1}(t) := u^{(n-1)}(t)$ one obtains a (vector valued) equation

$$\begin{aligned}
u_0'(t) &= u_1(t)\\
u_1'(t) &= u_2(t)\\
&\vdots\\
u_{n-2}'(t) &= u_{n-1}(t)\\
u_{n-1}'(t) &= g(t, u_0(t), \dots, u_{n-1}(t))
\end{aligned}$$

And if we write $\mathbf{u} := (u_0, \dots, u_{n-1})$ and

$$\mathbf{g}(t, \mathbf{u}) := (u_1(t), \dots, u_{n-1}(t), g(t, u_0(t), \dots, u_{n-1}(t))),$$

we arrive at the ORDINARY DIFFERENTIAL EQUATION OF ORDER 1

$$\mathbf{u}'(t) = \mathbf{g}(t, \mathbf{u}(t)).$$

So we are searching for a solution $\mathbf{u}$ of the equation $\mathbf{u}' = G(\mathbf{u})$, where $G(\mathbf{u})(t) := \mathbf{g}(t, \mathbf{u}(t))$. The general existence and uniqueness results for equations usually depend on some fixed-point theorem and so the domain and the range space have to be equal or at least to be isomorphic. So we need that $u \mapsto u' - G(u)$ is a selfmapping. In order to apply it to a function u, we need that u is 1-times differentiable, but in order that the image $u' - G(u)$ is 1-times differentiable, we need that u is twice differentiable. Inductively we come to the conclusion that u should be smooth. So are there spaces of smooth functions, to which we can apply some fixed point theorem?

0.2 Spaces of continuous and differentiable functions. In [5, 3.2.5] we have shown that the space $C(X, \mathbb{R})$ of continuous real-valued functions on X is a Banach-space with respect to the supremum-norm, provided X is compact. Recall that the proof goes as follows: If f_n is a Cauchy-sequence, then it converges pointwise (since the point-evaluations $\mathrm{ev}_x = \delta_x : C(X, \mathbb{R}) \to \mathbb{R}$ are continuous linear functionals), by the triangle inequality the convergence is uniform and by elementary analysis (e.g. see [1, 4.2.8]) a uniform limit of continuous functions is continuous.

If X is not compact, one can nevertheless consider the linear restriction maps $C(X, \mathbb{R}) \to C(K, \mathbb{R})$ for compact subsets $K \subseteq X$ and then use the initial structure on $C(X, \mathbb{R})$, given by the seminorms $f \mapsto \|f|_K\|_\infty$, where K runs through some basis of the compact sets, see [5, 3.2.8]. If X has a countable basis of compact sets, then we obtain a countably seminormed space $C(X, \mathbb{R})$. If we try to show completeness, we get as candidate for the limit a function f, which is on compact sets the uniform limit of the Cauchy-sequence f_n, and hence is continuous on these sets. If X is Kelley (= compactly generated, i.e. a set is open if its trace to all compact subsets is open, or equivalently if X carries the final topology with respect to all the inclusions of compact subsets, see [8, 2.3.1]) then we can conclude that f is continuous and hence $C(X, \mathbb{R})$ is complete. So under these assumptions (and in particular if X is locally compact) the space $C(X, \mathbb{R})$ is a Fréchet-space.

Is it really necessary to use countably many seminorms for non-compact X? – There is no norm which defines an equivalent structure on $C(X, \mathbb{R})$: Otherwise some seminorm $p_K := \|\_|_K\|_\infty$ must dominate it. However, this is not possible, since p_K is not a norm. In fact, since X is not compact there is some point $a \in X \setminus K$ and hence the function f defined by $f|_K = 0$ and $f(a) = 1$ is continuous on $K \cup \{a\}$. By Tietze-Urysohn [8, 1.3.2] it can be extended to a continuous function on X, which is obviously in the kernel of p_K but not zero.
Is there some other reasonable norm turning $C(X, \mathbb{R})$ into a Banach space E? – By reasonable we mean that at least the point-evaluations should be continuous (i.e. the topology should be finer than that of pointwise convergence). Then the identity mapping $E \to C(X, \mathbb{R})$ would be continuous by the application in [5, 5.3.8] of the closed graph theorem. Hence by the open mapping theorem [5, 5.3.5] for Fréchet spaces the identity would be an isomorphism, and thus $E \cong C(X, \mathbb{R})$ is not Banach. Note that this shows that, in a certain sense, the structure of $C(X, \mathbb{R})$ is unique.

Now what can be said about spaces of differentiable functions? – Of course the space $D^1(X, \mathbb{R})$ of differentiable functions on some interval X is contained in $C(X, \mathbb{R})$. However it is not closed in $C(X, \mathbb{R})$ and hence not complete in the supremum-norm, since a uniform limit of differentiable functions need not be differentiable anymore, see the example in [1, 4.2.11]. We need some control on the derivative. So we consider the space $C^1(X, \mathbb{R})$ of continuously differentiable functions with the initial topology induced by the inclusion in $C(X, \mathbb{R})$ and by the map $d : C^1(X, \mathbb{R}) \to C(X, \mathbb{R})$ given by $f \mapsto f'$. If X is compact we can consider instead of the corresponding two seminorms $f \mapsto \|f\|_\infty$ and $f \mapsto \|f'\|_\infty$ equally well their maximum (or sum) and obtain a norm $f \mapsto \max\{\|f\|_\infty, \|f'\|_\infty\}$ on $C^1(X, \mathbb{R})$. Again elementary analysis gives completeness, since for a Cauchy-sequence f_n we have a uniform limit f_∞ of f_n and a uniform limit f^1_∞ of f'_n, and hence (e.g. see [1, 4.2.11] or 2.40) f_∞ is differentiable with derivative f^1_∞. Inductively, we obtain that for compact intervals X and natural numbers n the spaces $C^n(X, \mathbb{R})$ can be made canonically into Banach-spaces, see [5, 4.2.5].

0.3 Spaces of smooth functions. What about the space $C^\infty(X,\mathbb{R})$ of infinite differentiable maps on a compact interval X? – Here we have countably many seminorms $f\mapsto \|f^{(n)}\|_\infty$, and as before we obtain completeness. So we have again a Fréchet space.

Again the question arises: Is it really necessary to use countably many seminorms? Since X is assumed to be compact we have a continuous norm, the supremum norm, and we cannot argue as before. So let us assume that there is some norm on $C^\infty(X,\mathbb{R})$ defining an equivalent structure. In particular it has to be continuous and hence has to be dominated by the maximum of the suprema of finitely many derivatives. Let us take an even higher derivative. Then the supremum of this derivative must be dominated by the norm. However, this is not possible, since there exist smooth functions f, for which all derivatives of order less than n are globally bounded by 1, but which have arbitrarily large n-th derivative at a given point, say 0. In fact, without loss of generality, we may assume assume that n is even and let $b\ge 1$. Take $f(x):=a\cos bx$ with $a:=1/b^{n-1}$. Then $|f^{(k)}(x)|=b^{k+1-n}\le 1$ for $k<n$, but $f^{(n)}(0)=\pm b\cos 0$.
Is there some reasonable (nonequivalent) norm which turns $C^\infty(X,\mathbb{R})$ into a Banach-space? – Well, the same arguments as before show that any reasonable Fréchet-structure on $C^\infty(X,\mathbb{R})$ is identical to the standard one and hence not normable.

0.4 ODE's. By what we have said in 0.1 the straight forward formulation of a fixed point equation for a general ordinary differential equation, does not lead to Banach spaces but to Fréchet spaces. There is however a classical way around this difficulty. The idea can be seen from the simplest differential equation, namely when G doesn't depend on u, i.e. $u'(t)=G(t)$. Then the (initial value) problem can be solved by integration: $u(t)=u(0)+\int_0^t G(s)\,ds$ and in fact similar methods work in the case of separated variables, i.e. $u'(t)=G_1(t)\,G_2(u)$, since then $H_2(u):=\int\frac{1}{G_2(u)}\,du=c+\int G_1(t)\,dt=:H_1(t)$ and hence $u(t)=H_2^{-1}(H_1(t))$.
In [**5**, 1.3.2] of [**2**, 6.2.14] we have seen how to prove an existence and uniqueness result for differential equations $u'(t)=g(t,u(t))$ with initial value conditions $u(0)=a$. Namely, by integration one transforms it into the INTEGRAL EQUATION

$$u(t)=a+\int_0^t g(s,u(s))\,ds.$$

Thus one has to find a fixed point u of $u=G(u)$, where G is the integral operator given by

$$G(u)(t):=a+\int_0^t g(s,u(s))\,ds.$$

As space of possible solutions u one can now take the space $C(I,\mathbb{R})$ for some interval I around 0. If one takes I sufficiently small then it is easily seen that G is a contraction provided g is sufficiently smooth, e.g. locally Lipschitz. Hence the existence of a fixed point follows from Banach's fixed point theorem [**5**, 1.2.2] (or [**8**, 3.1.7], or [**1**, 3.4.12]).

A more natural approach was taken in [**2**, 6.2.10]: The idea there is to solve the equation $0=u'-f\circ u=:(d-f_*)(u)$ on a space of differentiable functions u. However, since we cannot expect global existence of u but only on some interval $[-a,a]$ we transform the $u\in C^1([-a,a],\mathbb{R})$ into $u_a\in C^1([-1,1],\mathbb{R})$, via $u_a(t)=u(ta)$ and the differential equation then becomes $u_a'(t)=au'(ta)=af(u(ta))=a\,f(u_a(t))$, an implicit equation $0=g(a,u_a)$, where $g:\mathbb{R}\times C^1([-a,a],\mathbb{R})\to C([-a,a])$ is given by $g(a,u)(t)=u'(t)-a\,f(u(t))=(d-a\,f_*)(u)(t)$. In order to apply the implicit function theorem we need that g is C^1 and $\partial_2 g(0,0):C^1([-1,1],\mathbb{R})\to C^0([-1,1,\mathbb{R}])$ is invertible. Since $d:C^1([-1,1],\mathbb{R})\to C([-1,1],\mathbb{R})$ is linear and continuous we only have to show that f_* is C^1. Since $\mathrm{ev}_x : C([-1,1],\mathbb{R}) \to \mathbb{R}$ is continuous and linear a possible (directional) derivative $(f_*)'(g)(h)$ should satisfy:

$$\begin{aligned}(f_*)'(g)(h)(x) &= \frac{d}{dt}|_{t=0}(\mathrm{ev}_x \circ f_*)(g + t\,h)(x)\\ &= \frac{d}{dt}|_{t=0} f(g(x) + t\,h(x)) = f'(g(x))(h(x)),\end{aligned}$$

so we need that f is C^1 and then one can show that $f_* : C([-1,1],\mathbb{R}) \to C([-1,1],\mathbb{R})$ is C^1 with derivative $(f_*)' = (f')_*$, see [2, 6.2.10] for the details (in a more general situation). Then $\partial_2 g(0,0) = d$ is an isomorphism if we replace $C^1([-1,1],\mathbb{R})$ by the closed hyperplane $\{u \in C^1([-1,1],\mathbb{R}) : u(0) = 0\}$ involving the initial condition.

In the particular case of LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS $u' = Au$ we have seen in [5, 3.5.1] the (global) solution u with initial condition $u(0) = u_0$ is given by $u(t) := e^{tA}\,u_0$. Furthermore the solution of a general initial value problem of a LINEAR DIFFERENTIAL EQUATION OF ORDER n

$$u^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t)\,u^{(i)}(t) = s(t), \quad u(0) = u_0, \dots, u^{(n-1)}(0) = u_{n-1}.$$

is given by an integral operator $G : f \mapsto u$ defined by $(Gf)(t) := f(t) + \int_0^1 g(t,\tau)\,d\tau$, with a certain continuous integral kernel g. We have also seen in [5, 3.5.5] that a BOUNDARY VALUE PROBLEM OF SECOND ORDER

$$u''(t) + a_1(t)\,u'(t) + a_0(t)\,u(t) = s(t), \quad R_a(u) = 0 = R_b(u),$$

where the boundary conditions are $R_a(u) := r_{a,0}\,u(a) + r_{a,1}\,u'(a)$ and $R_b(u) := r_{b,0}\,u(b) + r_{b,1}\,u'(b)$ is also solved in the generic case by an integral operator

$$u(t) = \int_a^b g(t,\tau)\,f(\tau)\,d\tau,$$

with continuous integral kernel obtained from the solutions of corresponding initial value problems.

0.5 PDE's. Now what happens, if the u in the differential equation are functions of several numerical variables. Then the derivatives $u^{(k)}$ are given by the corresponding Jacobi-matrices of partial derivatives, and our differential equation $F(t, u(t), \dots u^{(n)}) = 0$ of 0.1 is a PARTIAL DIFFERENTIAL EQUATION, see [5, 4.7.1].

Even if we have a PARTIAL LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS as in [5, 7.4.2]

$$F(u)(x) := p(\partial)(u)(x) := \sum_{|\alpha| \le n} a_\alpha \cdot \partial^\alpha u(x) = s(x),$$

where p is the polynomial $p(z) = \sum_{|\alpha|\le n} a_\alpha\, z^\alpha$, we cannot apply the trick from above. The first problem is, that we no longer have a natural candidate, with respect to which we could pass to a explicit equation. In some special cases one can do. An example is the EQUATION OF HEAT-CONDUCTION

$$\frac{\partial}{\partial t} u = \Delta u,$$

where $u : \mathbb{R} \times X \to \mathbb{R}$ is the heat-distribution at the time t in the point x and Δ denotes the LAPLACE-OPERATOR given on $X = \mathbb{R}^n$ by $\Delta := \sum_{k=1}^n \left(\frac{\partial}{\partial x^k}\right)^2$. So this is an ordinary linear differential equation in an infinite dimensional space of functions on X. If we want Δ to be a self-mapping, we need smooth functions. But if we want to solve the equation as $u(t) = e^{t\Delta}u_0$ we need the functional calculus (i.e. applicability of the analytic function $e \mapsto e^t$ to the Operator Δ) and hence a Hilbert space of functions. But then Δ becomes an unbounded (symmetric) operator. This we treated in [6, 12.48].

Another example of such a situation is the Schrödinger equation

$$i\hbar\frac{d}{dt}u = S\,u.$$

where the Schrödinger-operator is given by $S = -\frac{\hbar^2}{2m}\Delta + U(x)$ for some potential U.

A third important equation is the wave-equation $(\frac{\partial}{\partial t})^2 u = \Delta u$, see [3, 9.3.1] or [5, 5.4]. If one makes an Ansatz of separated variables $u(t,x) = u_1(t)\,u_2(x)$ one obtains an Eigen-value equation $\Delta u(x) = \lambda u(x)$ for Δ and after having obtained the Eigen-functions $u_n : X \to \mathbb{R}$, one is lead to the problem of finding coefficients a_k and b_k such that

$$u(t,x) := \sum_k \Big(a_k \cos(\sqrt{\lambda_k}\,t) + b_k \sin(\sqrt{\lambda_k}\,t)\Big)\,u_k(x)$$

solves the initial conditions

$$u(0,x) = \sum_k a_k u_k(x) \qquad \text{and} \qquad \partial_1 u(0,x) = \sum_k \sqrt{\lambda_k}\,b_k\,u_k(x)$$

If we would have an inner-product, for which the u_k are orthonormal, then we could easily calculate the coefficients a_k and b_k. The space $C_{2\pi}$ of 2π-periodic functions is however not a Hilbert space. Otherwise it would be isomorphic to its dual, by the Riesz Representation theorem [5, 6.2.9]: However for $t \neq s$ we have that $\|\operatorname{ev}_t - \operatorname{ev}_s\| = \sup\{|f(t) - f(s) : \|f\|_\infty \leq 1\} = 1$ if we chose $f(t) = 1$ and $f(s) = 0$. Thus $C(X,\mathbb{R})'$ is not separable, since otherwise for every t there would be an ℓ_t in a fixed dense countable subset with $\|\operatorname{ev}_t - \ell_t\| < \frac{1}{2}$. Since the t are uncountable there have to be $t \neq s$ for which $\ell_t = \ell_s$, a contradiction. Another method to see this is to use Krein-Milman [5, 7.5.1]: If $C(X)$ were a dual-space, then its unit-ball would have to be contained in the closed convex hull of its extremal points. A function f in the unit-ball, which is not everywhere of absolute value 1, is is not extremal. In fact, take a t_0 with $|f(t_0)| < 1$ and a function v with support in a neighborhood of t_0. Then $f + s\,v$ lies in the unit ball for all values of s near 0. Hence we have by far too few extremal points, since those real-valued functions have to be constant on connectivity components.

In analogy to the inner product on $\mathbb{R}^n$ we can consider the continuous positive definite hermitian bilinear map $(f,g) \mapsto \int_X f(x)\,\overline{g(x)}\,dx$. By what we said above, it cannot yield a complete norm on $C(X,\mathbb{R})$. But we can take the completion of $C(X,\mathbb{R})$ with respect to this norm and arrive by [5, 4.12.5] at $L^2(X)$, a space not consisting of functions, but equivalence classes thereof. Now for the one-dimensional wave-equation, i.e. the equation of an vibrating string, we can solve the Eigenvalue-problem directly (it is given by an ordinary differential equation). And Fourier-series solves the problem, see [5, 5.4] and [5, 6.3.8].

For general compact oriented manifolds X the Laplace operator will be symmetric with respect to that inner product, see [4, 49.1]. If it were bounded, then it would be selfadjoint and one could apply geometry in order to find Eigen-values and Eigen-vectors by minimizing the angle between x and Tx, or equivalently by maximizing $|\langle Tx, x\rangle|$, see [5, 6.5.3]. It is quite obvious that for a selfadjoint bounded operator the supremum of $|\langle Tx, x\rangle|$ is its norm, and that a point were it is attained is an Eigen-vector with maximal absolute Eigen-value. So one needs compactness to show the existence of such a point. Since Eigen-vectors to different Eigen-values are orthogonal to each other, one can then proceed recursively, provided the operator is compact.

Again the idea is that, although the linear differential-operator F is not bounded, its inverse should be an integral operator G (the Green-operator) with continuous kernel ε and hence compact. And instead of solving $Fu = \lambda u$ we can equally well solve $\frac{1}{\lambda}u = Gu$, see [4, 49.6].

In order to find the Green operator, we have seen in [5, 4.7.7] that a possible solution operator $G : s \mapsto u$ would be given by convolution of s with a Green-function ε, i.e. a solution of $F(\varepsilon) = \delta$, where δ is the neutral element with respect to convolution. In fact, since $u := \varepsilon \star s$ should be a solution of $F(u) = s$, we conclude that $s = F(u) = F(\varepsilon \star s) = F(\varepsilon) \star s$. However such an element doesn't exist in the algebra of smooth functions, and one has to extend the notion of function to include so called generalized functions or distributions. These are the continuous linear functionals on the space $\mathcal{D}$ of smooth functions with compact support.

As we have seen in [5, 4.8.2] the space $\mathcal{D}$ is no longer a Fréchet space, but a strict inductive limit of the Fréchet spaces $C^\infty_K(X) := \{f \in C^\infty : \operatorname{supp} f \subseteq K\}$. Assume that there is some reasonable Fréchet structure on C^∞_c. Then by the same arguments as before the identity from $\mathcal{D}$ to C^∞_c would be continuous, hence closed, and hence the inverse to the webbed space $\mathcal{D}$ would be continuous too, i.e. a homeomorphism. Remains to show that the standard structure is not a Fréchet structure. If it were, then $\mathcal{D}$ would be Baire. However the closed linear subspaces C^∞_K have as union $\mathcal{D}$ and have empty interior, since non-empty open sets are absorbing. A contradiction to the Baire-property.

By passing to the transposed, we have seen in [5, 4.9.1] that every linear partial differential operator F can be extended to a continuous linear map $\tilde{F} : \mathcal{D}' \to \mathcal{D}'$, and so one can consider distributional solutions of such differential equations. In [5, 8.3.1] we have proven the Malgrange Ehrenpreis theorem on the existence of distributional fundamental solutions using the generalization of Fourier-series, namely the Fourier-transform $\mathcal{F}$. The idea is that $1 = \mathcal{F}(\delta) = \mathcal{F}(F(\varepsilon)) = \mathcal{F}(p(\partial)(\varepsilon)) = p \cdot \mathcal{F}(\varepsilon)$ and hence $\varepsilon = \mathcal{F}^{-1}(1/p)$. For this we have to consider the Schwartz-space $\mathcal{S}$ of rapidly decreasing smooth functions, which is a Fréchet space, and its dual $\mathcal{S}'$. In order that the poles of $1/p$ make no trouble we had to show that the Fourier-transform of smooth functions with compact support and even of distributions with compact support are entire functions.

If we want to solve *linear partial differential equations with non-constant coefficients* or even non-linear partial differential equations, we have to consider not only the linear theory of $\mathcal{D}$ but the non-linear one. See [10] for an approach to this.

0.6 Differentiation and integration commute. Let us consider a much more elementary result. In fact, even in the introductory courses in analysis one considers infinite dimensional results, but usually disguised. Recall the result about differentiation under the integral sign. There one considers a function f of two real variables and takes the integral $\int_0^1 f(t,s)\,ds$ with respect to one variable, and then one asks the question: Which assumptions guarantee that the resulting function is differentiable with respect to remaining variable t and what is its derivative? Before we try to remember the correct answer let us reformulate this result without being afraid of infinite dimensions. We are given the function $f : \mathbb{R}\times I \to \mathbb{R}$, $(t,s) \mapsto f(t,s)$. What do we actually mean by writing down $\int_0^1 f(t,s)\,ds$? – Well we keep t fixed and consider the function $f_t : I \to \mathbb{R}$ given by $s \mapsto f(t,s)$ and integrate it, i.e. $\int_0^1 f(t,s)\,ds := \int(f_t)$, where $\int$ denotes the integration operator

$\int : C[0,1] \to \mathbb{R}$, $g \mapsto \int_0^1 g(s)\,ds$. But now we want to vary t, so we have to consider the result as a function $t \mapsto \int(f_t)$, so we have to consider $t \mapsto f_t$ and we denote this function by $\check{f}$. It is given by the formula $\check{f}(t)(s) = f_t(s) = f(t,s)$. Then $\int(f_t) = (\int \circ \check{f})(t)$. Thus what we actually are interested in is, whether the composition $\int \circ \check{f}$ is differentiable and what its derivative is. This problem is usually solved by the CHAIN-RULE, but the situation here is much easier. In fact recall that integration is linear and continuous with respect to the supremum norm (or even the 1-norm) and $\check{f}$ is a curve (into some function space). Now if ℓ is continuous and linear and c is a differentiable curve then $\ell \circ c$ is differentiable with derivative $\ell(c'(t))$ at t: In fact

$$\lim_{s\to 0} \frac{\ell(c(t+s)) - \ell(c(t))}{s} = \lim_{s\to 0} \ell\left(\frac{c(t+s)-c(t)}{s}\right) = \\ = \ell\left(\lim_{s\to 0}\frac{c(t+s)-c(t)}{s}\right) = \ell(c'(t)).$$

So it remains to show that $\check{f} : \mathbb{R} \to C(I,\mathbb{R})$ is differentiable and to find its derivative. Let us assume it is differentiable and try to determine the derivative. On $C(I,\mathbb{R})$ we have nice continuous linear functionals, namely the POINT EVALUATIONS $\mathrm{ev}_s : g \mapsto g(s)$. These are continuous and linear and separate points (they are far from being all continuous linear functionals, see Riesz's Representation theorem [**5**, 7.3.3] and [**5**, 7.3.4]). Applying what we said before to $\ell := \mathrm{ev}_s$ and $c := \check{f}$ we obtain $\mathrm{ev}_s(\check{f}'(t)) = (\mathrm{ev}_s \circ \check{f})'(t)$, and $(\mathrm{ev}_s \circ \check{f})(t) = \mathrm{ev}_s(\check{f}(t)) = \check{f}(t)(s) = f(t,s)$. Hence $\mathrm{ev}_s(\check{f}'(t))$ is nothing else but the first partial derivative $\frac{\partial}{\partial t} f(t,s)$. Conversely, assume that the first partial derivative of f exists on $\mathbb{R}\times I$ and is continuous, then we want to show, that $\check{f}$ is differentiable, and $(\check{f})'(t)(s) = \frac{\partial}{\partial t} f(t,s) = \partial_1 f(t,s)$, or in other words $(\partial_1 f)^\vee = (\check{f})'$.

For this we first consider the corresponding topological problem: Are the continuous mappings $f : \mathbb{R}\times I \to \mathbb{R}$ exactly the continuous maps $\check{f} : \mathbb{R} \to C(I,\mathbb{R})$? This has been solved in the calculus courses. In fact a mapping $\check{f}$ is well-defined iff $f(x, \_)$ is continuous for all x and it is continuous iff $f(\_, y)$ is equi-continuous with respect to y, i.e.

$$\forall x \in \mathbb{R}\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall x' \in \mathbb{R}\ \forall y \in I : |x'-x| < \delta \Rightarrow |f(x,y) - f(x',y)| < \varepsilon.$$

However, these two conditions together are equivalent to the continuity of f, as can be seen for example in [**1**, 3.2.8].

Now to the differentiability question. We assume that $\partial_1 f$ exists and is continuous. Hence $(\partial_1 f)^\vee : \mathbb{R} \to C(I,\mathbb{R})$ is continuous. We want to show that $\check{f} : \mathbb{R} \to C(I,\mathbb{R})$ is differentiable (say at 0) with $(\partial_1 f)^\vee$ (at 0) as derivative. So we have to show that the mapping $t \mapsto \frac{\check{f}(t) - \check{f}(0)}{t}$ is continuously extendable to $\mathbb{R}$ by defining its value at 0 as $(\partial_1 f)^\vee(0)$. Or equivalently, by what we have shown for continuous maps before, that the map

$$(t,s) \mapsto \begin{cases} \frac{f(t,s)-f(0,s)}{t} & \text{for } t \neq 0 \\ \partial_1 f(0,s) & \text{otherwise} \end{cases}$$

is continuous. This follows immediately from the continuity of ∂_1 and that of $\int_0^1 \_\, dr$, since it can be written as $\int_0^1 \partial_1 f(r\,t, s)\,dr$ by the fundamental theorem.

So we arrive under this assumption at the conclusion, that $\int_0^1 f(t,s)\,ds$ is differentiable with derivative

$$\frac{d}{dt}\int_0^1 f(t,s)\,ds = \int((\check{f})'(t)) = \int_0^1 \frac{\partial}{\partial t} f(t,s)\,ds$$

and we have proved the

Proposition. *For a continuous map $f:\mathbb{R}\times I\to\mathbb{R}$ the partial derivative $\partial_1 f$ exists and is continuous iff $\check{f}:\mathbb{R}\to C(I,\mathbb{R})$ is continuously differentiable. And in this situation $\int((\check{f})'(t)) = \frac{d}{dt}\int_0^1 f(t,s)\,ds = \int_0^1 \frac{\partial}{\partial t} f(t,s)\,ds$.* □

And we see, it is much more natural to formulate and prove this result with the help of the infinite dimensional space $C([0,1],\mathbb{R})$. But this not only clarifies the proof, but is of importance for its own sake, as we will see in 0.8.

0.7 Exponential law for continuous mappings. Let us try to generalize this result. We will write Y^X for the function spaces $C(X,Y)$ for reasons of cardinality. So the question is whether the continuous mappings $f: X\times Y\to Z$ correspond exactly to the continuous maps $\check{f}: X\to C(Y,Z)$?

For this we need a topology on $C(X,Y)$. If Y is a locally convex space (or a uniform space) we can use the topology of uniform convergence on compact subsets of X, given by the seminorms $f\mapsto\sup\{q(f(x)):x\in K\}$, where $K\subseteq X$ runs through the compact subsets and q through the seminorms of Y, see [5, 3.2.8]. For general Y we consider the compact-open topology, which has as subbasis the sets $N_{K,U}:=\{f:f(K)\subseteq U\}$ where K runs through (a basis of) the compact subsets of X and Y through (a basis of) the open subsets of Y, see [8, 2.4.2].

Let us show first that for locally convex spaces F and topological spaces X the compact-open topology is the locally convex topology of uniform convergence on compact subsets:

So let $K\subseteq X$ be compact, $V\subseteq F$ be open, and $f\in N_{K,V}$, i.e. $f(K)\subseteq V$. Then for each $x\in K$ there exists a seminorm q on F and an $\varepsilon>0$ such that $V_{f(x)}:=\{y\in F: q(y-f(x))<\varepsilon\}\subseteq V$. The sets $U_x:=\{z'\in X: q(f(z')-f(x))<\frac{\varepsilon}{2}\}$ with $x\in K$ form an open converging, so there are finitely many $x_1,\dots,x_n$ with $K\subseteq\bigcup_{i=1}^n U_i$, where $U_i:=U_{x_i}$. Let q_i be the seminorm and ε_i the radius corresponding to x_i and $K_i:=\{z'\in K: q_i(f(z')-f(x_i))\le\frac{\varepsilon_i}{2}\}$. We claim, that $q_i(g(x)-f(x))<\frac{\varepsilon_i}{2}$ for all i and $x\in K_i$ implies $g\in N_{K,U}$. In fact, let $x\in K$, then there exists an i with $x\in U_i\cap K\subseteq K_i$ and hence $q_i(g(x)-f(x_i))\le q_i(g(x)-f(x))+q_i(f(x)-f(x_i))<\frac{\varepsilon_i}{2}+\frac{\varepsilon_i}{2}=\varepsilon_i$, i.e. $g(x)\in V_{f(x_i)}\subseteq V$.

Conversely, let a compact $K\subseteq X$, a seminorm q on F, an $\varepsilon>0$, and $f\in C(X,F)$ be given. Note that $g\in C(X,F)$ is a subset of $W:=\{(x,y):x\in K\Rightarrow q(y-f(x))<\varepsilon\}$ iff $q(g(x)-f(x))<\varepsilon$ for all $x\in K$. For $x\in K$ let $U_x:=\{x': q(f(x')-f(x))<\frac{\varepsilon}{3}\}$ and take finitely many $x_1,\dots,x_n$ such that the $U_i:=U_{x_i}$ cover K. Let $K_i:=\{x\in K: q(f(x)-f(x_i))\le\frac{\varepsilon}{3}\}$ and $V_i:=\{y: q(y-f(x_i))<\frac{\varepsilon}{2}\}$ then $f(K_i)\subseteq V_i$. If $g\in\bigcap_i N_{K_i,V_i}$ then for each $x\in K$ there exists an i with $x\in U_i\cap K\subseteq K_i$ and thus $q(g(x)-f(x))\le q(g(x)-f(x_i))+q(f(x_i)-f(x))<\frac{\varepsilon}{2}+\frac{\varepsilon}{3}<\varepsilon$, i.e. $g\subseteq W$.

How is $\hat{g}: X\times Y\to Z$ constructed from a continuous $g: X\to Z^Y$. Well, one can consider $g\times Y: X\times Y\to Z^Y\times Y$ and compose it with the EVALUATION MAP $\mathrm{ev}: Z^Y\times Y\to Z$. Since the product of continuous maps is continuous, it remains to show that the evaluation map is continuous in order to obtain that $\hat{g}$ is continuous. So let $f\in Z^Y$ and $y\in Y$ and let U be a neighborhood of $f(y)$. If Y is locally compact, we can find a compact neighborhood W of y and then $f\in N_{W,U}:=\{g: g(W)\subset U\}$ and $\mathrm{ev}(N_{W,U}\times W)\subseteq U$.

Conversely let a continuous $f: X\times Y\to Z$ be given. Then we consider $f_*:=f^Y:(X\times Y)^Y\to Z^Y$ and compose it from the right with the INSERTION MAP $\mathrm{ins}: X\to(X\times Y)^Y$ given by $x\mapsto(y\mapsto(x,y))$. Then we arrive at $\check{f}$. Obviously f_* is continuous since $(f_*)^{-1}N_{K,U}=N_{K,f^{-1}U}$. The insertion map is continuous, since

$\mathrm{ins}^{-1}(N_{K,U\times V}) = U$ if $K \subseteq V$ and is empty otherwise, so $\check{f}$ is continuous. Thus the only difficult part was the continuity of the evaluation map.

Moreover we have the

Proposition. *Let X, Y and Z be topological spaces with Y being locally compact. Then we have a homeomorphism $Z^{X\times Y} \cong (Z^Y)^X$, given by $f \mapsto \check{f}$, where the function spaces carry the compact open topology.*

Proof. We have already proved that we have a bijection. That this gives a homeomorphism follows, since the corresponding subbases $N_{K_1\times K_2,U}$ and $N_{K_1,N_{K_2,U}}$ correspond to each other. □

In general the compact open topology on Z^Y will not be locally compact even for locally compact spaces Y and Z (e.g. $C([0,1],\mathbb{R})$ is an infinite dimensional and hence not locally compact Banach space). So in order to get an intrinsic exponential law, one can modify the notion of continuity and call a mapping $f : X \to Y$ between Hausdorff topological spaces COMPACTLY-CONTINUOUS iff its restriction to every compact subset $K \subseteq X$ is continuous. Thus $f : X \times Y \to Z$ is continuous iff $f|_{K\times L} : K \times L \to Z$ is continuous for all compact subsets $K \subseteq X$ and $L \subseteq Y$. By the exponential law for compact sets this is equivalent to $\check{f} : K \to Z^L$ being continuous. Since Z^Y carries the initial structure with respect to $\mathrm{inkl}^* : Z^Y \to Z^L$, this is furthermore equivalent to the continuity of $\check{f} : K \to Z^Y$, and thus to $\check{f} : X \to Z^Y$ being compactly-continuous, but for this we have to denote with Z^Y the space of compactly continuous maps from $Y \to Z$.

Instead of the category of compactly continuous maps between Hausdorff topological spaces, one can use the EQUIVALENT CATEGORY (see [7, 1.22]) of continuous mappings between compactly generated spaces. Recall that a Hausdorff topological space is called COMPACTLY GENERATED or a KELLEY SPACE iff it carries the final topology with respect to the inclusions of its compact subsets with their trace topology. The equivalence between these two categories is given by the identity functor on one side, and on the other side by the Kelley-fication, i.e. by replacing the topology by the final topology with respect to the compact subsets. Note that the identity is compactly continuous in both directions. However, the natural topology on the products in this category is the Kelley-fication of the product topology and also on the function spaces one has to consider the Kelly-fication of the compact open topology, see [8, 2.4].

0.8 Variational calculus. In physics one is not a priori given an equation $f(x) = 0$, but often some OPTIMIZATION PROBLEM. One is searching for those x, for which the values $f(x)$ of some real-valued function (like the LAGRANGE FUNCTION in classical mechanics, which is given by the difference of kinematic energy and the potential) attain an extremum (i.e. are minimal or maximal), see for example [4, 45]. Again x is often not a finite dimensional vector but functions and then f is often given by some integral (like the action (german: Wirkungsintegral) in classical mechanics)

$$f(x) := \int_0^1 F(t, x(t), x'(t))\, dt.$$

For finite dimensional vectors x one finds solutions of the problem $f(x) \to \min$ by applying differential calculus and searching for solutions of $f'(x) = 0$. In infinite dimensions one proceeds similarly in the calculus of variations (see [3, 9.4.3]), by finding those points x, where the directional derivatives $f'(x)(v)$ vanish for all directions v. Since the boundary values of x are given, the variation v has to vanish on the boundary $\{0,1\}$. One can calculate the directional derivative by what we have shown before as follows:

$$\begin{aligned}
f'(x)(v) &:= \frac{d}{dt}\Big|_{t=0} f(x+tv)\\
&= \frac{d}{dt}\Big|_{t=0} \int_0^1 F\Big(s,(x+tv)(s),(x+tv)'(s)\Big)\,ds\\
&= \int_0^1 \frac{\partial}{\partial t}\Big|_{t=0} F\Big(s,(x+tv)(s),(x+tv)'(s)\Big)\,ds\\
&= \int_0^1 \Big(\partial_2 F(s,x(s),x'(s))\cdot v(s) + \partial_3 F(s,x(s),x'(s))\cdot v'(s)\Big)\,ds\\
&= \int_0^1 \left(\partial_2 F(s,x(s),x'(s)) - \frac{d}{ds}\partial_3 F(s,x(s),x'(s))\right)\cdot v(s)\,ds
\end{aligned}$$

We have used partial integration and that the variation v has to vanish at the boundary points 0 and 1. Since $f'(x)(v)$ has to be 0 for all such v we arrive at the Euler-Lagrange partial differential equation

$$\partial_2 F(s,x(s),x'(s)) = \frac{d}{ds}\partial_3 F(s,x(s),x'(s)),$$

or with slight abuse of notation:

$$\frac{\partial}{\partial x}F = \left(\frac{\partial}{\partial \dot x}F\right)^{\cdot},$$

where $(\_)^{\cdot}$ denotes the derivative with respect to time s.

Warning: abuse may lead to disaster! In physics for example one has the gas-equation $p\cdot V\cdot t = 1$, where p is pressure, V the volume and t the temperature scaled appropriately. So we obtain the following partial derivatives:

$$\begin{aligned}
\frac{\partial p}{\partial V} &= \frac{\partial}{\partial V}\frac{1}{V\,t} = -\frac{1}{t\,V^2}\\
\frac{\partial V}{\partial t} &= \frac{\partial}{\partial t}\frac{1}{t\,p} = -\frac{1}{p\,t^2}\\
\frac{\partial t}{\partial p} &= \frac{\partial}{\partial p}\frac{1}{p\,V} = -\frac{1}{V\,p^2}
\end{aligned}$$

And hence cancellation yields

$$1 = \frac{\partial p}{\partial V}\cdot\frac{\partial V}{\partial t}\cdot\frac{\partial t}{\partial p} = (-1)^3\frac{1}{t\,V^2}\cdot\frac{1}{p\,t^2}\cdot\frac{1}{V\,p^2} = -\frac{1}{(pVt)^3} = -1.$$

Try to find the mistake!

0.9 Flows as 1-parameter subgroups of diffeomorphisms. Another situation, where it is natural to consider differentiable curves into function spaces, are flows. So we are considering ordinary time-independent differential equations, i.e. equations of the form $\dot u = f(u)$. For given initial value $u(0) = a$ we can consider the solution u_a and obtain a mapping $u : \mathbb{R}\times X \to X$ given by $(t,a)\mapsto u_a(t)$. Obviously $u(0,x) = x$ and by uniqueness we have $u(t+s,x) = u(t,u(s,x))$, i.e. u is a flow on X, see [4, 28.3]. Conversely, we can reconstruct the differential equation by differentiating the flow with respect to t at $t=0$, i.e. $\frac{\partial}{\partial t}|_{t=0}u(t,x) = f(u(t,x))|_{t=0} = f(x)$. It would be more natural to consider the associate mapping $\check u$ with values in some space of mappings from $X\to X$. The flow property translates into the assumption that $t\mapsto \check u(t)$ is a group-homomorphism from $\mathbb{R}$ into the group of invertible maps on X. The VECTOR FIELD f can thus be interpreted as the tangent vector $\check{u}'(0)$ at 0 of the curve $\check{u}$. Thus we should have that $\check{u}$ is differentiable into a group $\mathrm{DIFF}(X)$ OF DIFFEOMORPHISMS on X, and this group should carry some smooth structure, analogously to classical Lie-groups. In particular the composition $\mathrm{Diff}(X)\times\mathrm{Diff}(X)\to\mathrm{Diff}(X)$ map should be differentiable. Since $(f,g)\mapsto f\circ g$ is linear in the first variable (if we consider the range space X as submanifold of some $\mathbb{R}^n$), the difficult part is the differentiability in the second variable, i.e. that of the map $f_*: g\mapsto f\circ g$. We have noted at the end of 0.4 that for f_* to be differentiable we need that f is differentiable since $(f_*)'=(f')_*$. Thus in order that the composition map is differentiable, we need that its first variable f is differentiable, hence Diff should mean at least 1-times differentiable. But then in order that the derivative of the composition map has 1-time differentiable values we need that f' is 1-times differentiable, i.e. f is twice differentiable. Inductively we arrive at the smoothness of f, i.e. infinite often differentiability. But as we have mentioned before, even in the simplest case $C^\infty([0,1],\mathbb{R})$ or $C^\infty(S^1,\mathbb{R})$, these function spaces are not Banach-spaces anymore, but Fréchet-spaces.

0.10 Exponential law for differentiable mappings. A similar thing happens when searching for an exponential law for differentiable functions. If we want a nice correspondence between differentiable functions on a product and differentiable functions into a function space, we have seen that a curve $c:\mathbb{R}\to C(\mathbb{R},\mathbb{R})$ is C^1 if and only if $\partial_1\hat{c}:\mathbb{R}^2\to\mathbb{R}$ exists and is continuous. If we want a (differentiability-)property which is invariant under base-change in $\mathbb{R}^2$, then $\partial_2\hat{c}:\mathbb{R}^2\to\mathbb{R}$ should exist and be continuous, and hence $c:\mathbb{R}\to C(\mathbb{R},\mathbb{R})$ should have values in $C^1(\mathbb{R},\mathbb{R})$ and be continuous $\mathbb{R}\to C^1(\mathbb{R},\mathbb{R})$. Thus $\hat{c}:\mathbb{R}^2\to\mathbb{R}$ is C^1 if and only if $c:\mathbb{R}\to C(\mathbb{R},\mathbb{R})$ is C^1 (with derivative $c'(t)^\wedge=\partial_1\hat{c}$) and is C^0 into $C^1(\mathbb{R},\mathbb{R})$ (with $(d\circ c)^\wedge=\partial_2\hat{c}$). So if we want to use just a single functions space (instead of $C^0(\mathbb{R},\mathbb{R})$ and $C^1(\mathbb{R},\mathbb{R})$ at the same time) we should assume $c:\mathbb{R}\to C^1(\mathbb{R},\mathbb{R})$ to be C^1. But then $c':\mathbb{R}\to C^1(\mathbb{R},\mathbb{R})$ has to be continuous, and thus $d\circ c':\mathbb{R}\to C(\mathbb{R},\mathbb{R})$ has to be continuous, i.e. $(d\circ c')^\wedge=\partial_2\partial_1\hat{c}:\mathbb{R}^2\to\mathbb{R}$ should be continuous. Assumed invariance under base-change yields that $\hat{c}:\mathbb{R}^2\to\mathbb{R}$ should be C^2 and then $\hat{c}:\mathbb{R}\to C(\mathbb{R},\mathbb{R})$ has to be C^2, $\hat{c}:\mathbb{R}\to C1(\mathbb{R},\mathbb{R})$ has to be C^1, and $\hat{c}:\mathbb{R}\to C^2(\mathbb{R},\mathbb{R})$ has to be C^0. Inductively we get that the exponential law for differentiable functions can only be valid for C^∞-functions.

0.11 Continuity of the derivative. Well, as has been discovered around 1900, the derivative should be a linear (more precisely, an affine) approximation to the function. Assume we have already defined the concept of DERIVATIVE $f'(x)\in L(E,F)$ for functions $f:E\supseteq U\to F$ at a given point $x\in U$. By collecting for all x in the open domain U of f these derivatives $f'(x)$, we obtain a mapping $x\mapsto f'(x)$, the derivative $f':E\supseteq U\to L(E,F)$ with values in the space of continuous linear mappings. In order to speak about continuous differentiable (short: C^1) mappings, we need some topology on $L(E,F)$ and then this amounts to the assumption, that $f':U\to L(E,F)$ is continuous. For C^1-maps we should have a CHAIN-RULE, which guarantees that the composite $f\circ g$ of C^1-maps is again C^1 and the derivative should be $(f\circ g)'(x)=f'(g(x))\circ g'(x)$. This map is thus given by the following description: Given x then first calculate $g(x)$ and then $f'(g(x))\in L(F,G)$ and $g'(x)\in L(E,F)$, and finally apply the composition map $L(F,G)\times L(E,F)\to L(E,G)$ to obtain $f'(g(x))\circ g'(x)$. Since f and g are assumed to be C^1 the components $f'\circ g$ and g' are continuous. So it remains to show the continuity of the composition mapping. Let us consider the simplified case where $G=E=\mathbb{R}$. Then composition reduces to the evaluation map $\mathrm{ev}:F'\times F\to\mathbb{R}$ and we are looking for a topology on F' such that this map is continuous. Assume we have found such a topology. Then there exists 0-neighborhoods V in F' and U in F such that $\mathrm{ev}(V \times U) \subseteq [-1, 1]$. Since scalar-multiplication on F' should be continuous, we can find for every $\ell \in F'$ a number $K > 0$, such that $\ell \in K\,V$. Thus for $x \in U$ we have $\ell(x) = \mathrm{ev}(K\frac{1}{K}\ell, x) = K\,\mathrm{ev}(\frac{1}{K}\ell, x) \in K\,\mathrm{ev}(V \times U) \subseteq [-K, K]$. This shows that U is scalarly bounded, and hence is bounded by the corollary in [**5**, 5.2.7]. However, a seminormed space, which has a bounded 0-neighborhood has to be normed, by Kolmogoroff's theorem [**5**, 2.6.2].

So it seems that there is no reasonable notion of C^1, which applies to more than just functions between Banach spaces. However, we have assumed that continuity is meant with respect to topologies. In fact, there have been several (more or less successful) attempts in the past to remedy this situation by considering convergence structures on $L(E, F)$. If one defines that a net (or a filter) f_α should converge to f in $L(E, F)$ iff for nets (or filters) x_β converging to some x in E the net (or filter) $f_\alpha(x_\beta)$ should converge to $f(x)$, then the evaluation map, and more generally the composition map becomes continuous. A second way to come around this problem, is to assume for C^1 the continuity of $\widehat{f'} : U \times E \to F$ instead. Then the chain-rule becomes easy. However this notion is bad, since we cannot prove the INVERSE FUNCTION THEOREM for C^1 even for Banach spaces, see [**2**, 6.2.1] and [**2**, 6.3.15]. See [**2**, 6.1.19] for an example of a differentiable function f on a Hilbert space for which $\widehat{f'}$ is continuous, but f' is not. This examples shows in particular that the exponential law is wrong for continuous functions $\ell^2 \times \ell^2 \to \mathbb{R}$ which are linear in the second variable if one uses the operator norm on $L(\ell^2, \mathbb{R}) = (\ell^2)' \cong \ell^2$.

0.12 Derivatives of higher order. If we want to define higher derivatives - as we need them in conditions for local extrema and the like - we would call a function f by recursion $(n+1)$-times differentiable iff f' exists and is n-times differentiable (D^n for short). In order to show that the composite $f \circ g$ of two D^2-maps is again D^2, we have to show that $(f \circ g)' : x \mapsto f'(g(x)) \circ g'(x)$ is again D^1. Now this map is given by the following composition: Given x then first calculate $g(x)$ and then $f'(g(x)) \in L(F, G)$ and $g'(x) \in L(E, F)$, and finally apply the composition map $L(F, G) \times L(E, F) \to L(E, G)$ to obtain $f'(g(x)) \circ g'(x)$. By the chain-rule for D^1-mappings, we would obtain that $f' \circ g \in D^1$ and by assumption $g' \in D^1$. So it remains to differentiate the bilinear composition map. Since it is linear in both entries separately, its partial derivatives should obviously exist and the derivative also. But recall that it is not even continuous.

0.13 Résumé. We have learned a few things from these introductory words:

(1) Problems in finite dimensions often have a more natural formulation (and proof) involving infinite dimensional function-spaces, which are quite often not Banach spaces, but Fréchet spaces like $C(\mathbb{R},\mathbb{R})$ and $C^\infty(I,\mathbb{R})$ or even more general ones like $\mathcal{D}$ and $\mathcal{D}'$.
(2) Mappings of two variables $f : X \times Y \to Z$, should often be considered as mappings $\check{f}$ from X to a space of mappings from Y to Z and properties such as continuity or differentiability should translate nicely. For differentiability this can only be true for C^∞.
(3) It is not clear, how to obtain the basic ingredient to calculus, the chain-rule. For this the composition map, or at least the evaluation map, should be smooth, although it is not continuous in the topological setting.
(4) There is no reasonable notion of C^1 generalizing classical (Fréchet-)calculus to mappings between spaces beyond Banach spaces.

After having found lots of, at first view devastating, difficulties, let's look what can be done easily:

(1) It is obvious what differentiability for a curve c into any locally convex space means, since limits of difference quotients make sens. Hence we have also the notion of continuous differentiable, of n-times differentiable, and of smoothness for such curves.
(2) Continuous (multi-)linear mappings preserve smoothness of curves, and satisfy the chain-rule.
(3) Directional derivatives can be easily defined for mappings f between arbitrary locally convex spaces, since they are just derivatives of the curves $c : t \mapsto f(x + t\,v)$ obtained by composing f with an affine line $t \mapsto x + t\,v$
(4) Candidates for derivatives $f'(x)$ of mappings f can be obtained by reduction to 1-dimensional analysis via affine mappings: $\ell(f'(x)(v)) = \frac{d}{dt}|_{t=0}\ell(f(x + t\,v))$.

Chapter I
Calculus of Smooth Mappings

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This chapter is devoted to calculus of smooth mappings in infinite dimensions. The leading idea of our approach is to base everything on smooth curves in locally convex spaces, which is a notion without problems, and a mapping between locally convex spaces will be called smooth if it maps smooth curves to smooth curves.

We start by looking at the set of smooth curves $C^\infty(\mathbb{R}, E)$ with values in a locally convex space E, and note that it does not depend on the topology of E, only on the underlying system of bounded sets, its bornology. This is due to the fact, that for a smooth curve difference quotients converge to the derivative much better 2.1 than arbitrary converging nets or filters: we may multiply it by some unbounded sequences of scalars without disturbing convergence (or, even better, boundedness).

Then the basic results are proved, like existence, smoothness, and linearity of derivatives, the chain rule 3.18, and also the most important feature, the 'exponential law' 3.12 and 3.13: We have

$$C^\infty(E \times F, G) \cong C^\infty(E, C^\infty(F, G)),$$

without any restriction, for a natural structure on $C^\infty(F, G)$.

Smooth curves have integrals in E if and only if a weak completeness condition is satisfied: it appeared as bornological completeness, Mackey completeness, or local completeness in the literature, we call it c^∞-complete. This is equivalent to the condition that weakly smooth curves are smooth 2.14. All calculus in later chapters in this book will be done on CONVENIENT VECTOR SPACES: These are locally convex vector spaces which are c^∞-complete; note that the locally convex topology on a convenient vector space can vary in some range, only the system of bounded sets must remain the same.

Linear or more generally multilinear mappings are smooth if and only if they are bounded 5.5, and one has corresponding exponential laws 5.2 for them as well. Furthermore, there is an appropriate tensor product, the bornological tensor product 5.7, satisfying

$$L(E \otimes_\beta F, G) \cong L(E, F; G) \cong L(E, L(F, G)).$$

An important tool for convenient vector spaces are uniform boundedness principles as given in 5.18, 5.24 and 5.26.

It is very natural to consider on E the final topology with respect to all smooth curves, which we call the c^∞-topology, since all smooth mappings are continuous for it: the vector space E, equipped with this topology is denoted by $c^\infty E$, with lower case c in analogy to kE for the Kelley-fication and in order to avoid any confusion with any space of smooth functions or sections. The special curve lemma 2.8 shows that the c^∞-topology coincides with the usual Mackey closure topology. The space $c^\infty E$ is not a topological vector space in general. This is related to the fact that the evaluation $E \times E' \to \mathbb{R}$ is jointly continuous only for normable E, but it is always smooth and hence continuous on $c^\infty(E \times E')$. The c^∞-open subsets are the natural domains of definitions of locally defined functions. For nice spaces (e.g. Fréchet and strong duals of Fréchet-Schwartz spaces, see 4.11) the c^∞-topology coincides with the given locally convex topology. In general, the c^∞-topology is finer than any locally convex topology with the same bounded sets.

In the last section of this chapter we discuss the structure of spaces of smooth functions on finite dimensional manifolds and, more generally, of smooth sections of finite dimensional vector bundles. They will become important in chapter IX as modeling spaces for manifolds of mappings. Furthermore, we give a short account of reflexivity of convenient vector spaces and on (various) approximation properties for them.

1. Smooth Curves

1.1. Notation. Since we want to have unique derivatives all locally convex spaces E will be assumed Hausdorff. The family of all bounded sets in E plays an important rôle. It is called the *bornology* of E. A linear mapping is called *bounded*, sometimes also called bornological, if it maps bounded sets to bounded sets. A bounded linear bijection with bounded inverse is called *bornological isomorphism*. The *space of all continuous linear functionals on* E will be denoted by E^* and the *space of all bounded linear functionals on* E by E'. The *adjoint* or *dual mapping* of a linear mapping ℓ, however, will be always denoted by ℓ^*, because of differentiation.

See also the appendix 52 for some background on functional analysis.

1.2. Differentiable curves. The concept of a smooth curve with values in a locally convex vector space is easy and without problems. Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called *differentiable* if the *derivative* $c'(t) := \lim_{s\to 0} \frac{1}{s}(c(t+s) - c(t))$ at t exists for all t. A curve $c : \mathbb{R} \to E$ is called *smooth* or C^∞ if all iterated derivatives exist. It is called C^n for some finite n if its iterated derivatives up to order n exist and are continuous.

A curve $c : \mathbb{R} \to E$ is called *locally Lipschitzian* if every point $r \in \mathbb{R}$ has a neighborhood U such that the *Lipschitz condition* is satisfied on U, i.e., the set

$$\left\{\tfrac{1}{t-s}\Big(c(t) - c(s)\Big) : t \neq s; t, s \in U\right\}$$

is bounded. Note that this implies that the curve satisfies the Lipschitz condition on each bounded interval, since for (t_i) increasing

$$\frac{c(t_n) - c(t_0)}{t_n - t_0} = \sum \frac{t_{i+1} - t_i}{t_n - t_0}\,\frac{c(t_{i+1}) - c(t_i)}{t_{i+1} - t_i}$$

is in the absolutely convex hull of a finite union of bounded sets.

A curve $c : \mathbb{R} \to E$ is called $\mathcal{L}ip^k$ or $C^{(k+1)-}$ if all derivatives up to order k exist and are locally Lipschitzian.

1.3. Lemma. Continuous linear mappings are smooth. *A continuous linear mapping $\ell : E \to F$ between locally convex vector spaces maps $\mathcal{L}\mathrm{ip}^k$-curves in E to $\mathcal{L}\mathrm{ip}^k$-curves in F, for all $0 \le k \le \infty$, and for $k > 0$ one has $(\ell \circ c)'(t) = \ell(c'(t))$.*

Proof. As a linear map ℓ commutes with the formation of difference quotients, hence the image of a Lipschitz curve is Lipschitz since ℓ is bounded.

As a continuous map it commutes with the formation of the respective limits. Hence $(\ell \circ c)'(t) = \ell(c'(t))$.

Now the rest follows by induction. □

Note that a differentiable curve is continuous, and that a continuously differentiable curve is locally Lipschitzian: For $\ell \in E^*$ we have

$$\ell\left(\frac{c(t) - c(s)}{t - s}\right) = \frac{(\ell \circ c)(t) - (\ell \circ c)(s)}{t - s} = \int_0^1 (\ell \circ c)'(s + (t - s)r)dr,$$

which is bounded, since $(\ell \circ c)' = \ell \circ c'$ is locally bounded. Since boundedness can be tested by continuous linear functionals (see [5, 5.2.7]) we conclude that c is locally Lipschitzian.

More general, we have by induction the following implications:

$$C^{n+1} \implies \mathcal{L}\mathrm{ip}^n \implies C^n,$$
$$\text{differentiable} \implies C.$$

1.4. The mean value theorem. In classical analysis the basic tool for using the derivative to get statements on the original curve is the mean value theorem. So we try to generalize it to infinite dimensions. For this let $c : \mathbb{R} \to E$ be a differentiable curve. If $E = \mathbb{R}$ the classical mean value theorem states, that the difference quotient $(c(a)-c(b))/(a-b)$ equals some intermediate value of c'. Already if E is two dimensional this is no longer true. Take for example a parameterization of the circle by arclength. However, we will show that $(c(a) - c(b))/(a - b)$ lies still in the closed convex hull of $\{c'(r) : r\}$. Having weakened the conclusion, we can try to weaken the assumption. And in fact c may be not differentiable in at most countably many points. Recall however, that there exist strictly monotone functions $f : \mathbb{R} \to \mathbb{R}$, which have vanishing derivative outside a Cantor set (which is uncountable, but has still measure 0).

Sometimes one uses in one dimensional analysis a generalized version of the mean value theorem: For an additional differentiable function h with non-vanishing derivative the quotient $(c(a)-c(b))/(h(a)-h(b))$ equals some intermediate value of c'/h'. A version for vector valued c (for real valued h) is that $(c(a) - c(b))/(h(a) - h(b))$ lies in the closed convex hull of $\{c'(r)/h'(r) : r\}$. One can replace the assumption that h' vanishes nowhere by the assumption that h' has constant sign, or, more generally, that h is monotone. But then we cannot form the quotients, so we should assume that $c'(t) \in h'(t) \cdot A$, where A is some closed convex set, and we should be able to conclude that $c(b) - c(a) \in (h(b) - h(a)) \cdot A$. This is the version of the *mean value theorem* that we are going to prove now. However, we will make use of it only in the case where $h = \mathrm{Id}$ and c is everywhere differentiable in the interior.

Proposition. Mean value theorem. *Let $c : [a, b] =: I \to E$ be a continuous curve, which is differentiable except at points in a countable subset $D \subseteq I$. Let h be a continuous monotone function $h : I \to \mathbb{R}$, which is differentiable on $I \setminus D$. Let A be a convex closed subset of E, such that $c'(t) \in h'(t) \cdot A$ for all $t \notin D$.*

Then $c(b) - c(a) \in (h(b) - h(a)) \cdot A$.

Proof. Assume that this is not the case. By the theorem of Hahn Banach [5, 7.2.1] there exists a continuous linear functional ℓ with $\ell(c(b)-c(a)) \notin \overline{\ell((h(b)-h(a))\cdot A)}$. But then $\ell\circ c$ and $\overline{\ell(A)}$ satisfy the same assumptions as c and A, and hence we may assume that c is real valued and A is just a closed interval $[\alpha,\beta]$. We may furthermore assume that h is monotonely increasing. Then $h'(t)\ge 0$, and $h(b)-h(a)\ge 0$. Thus the assumption says that $\alpha h'(t)\le c'(t)\le \beta h'(t)$, and we want to conclude that $\alpha(h(b)-h(a))\le c(b)-c(a)\le \beta(h(b)-h(a))$. If we replace c by $c-\beta h$ or by $\alpha h - c$ it is enough to show that $c'(t)\le 0$ implies that $c(b)-c(a)\le 0$. For given $\varepsilon>0$ we will show that $c(b)-c(a)\le \varepsilon(b-a+1)$. For this let J be the set $\{t\in[a,b] : c(s)-c(a)\le \varepsilon\left((s-a)+\sum_{t_n<s}2^{-n}\right) \text{ for } a\le s<t\}$, where $D =: \{t_n : n\in\mathbb{N}\}$. Obviously, J is a closed interval containing a, say $[a,b']$. By continuity of c we obtain that $c(b')-c(a)\le\varepsilon\left((b'-a)+\sum_{t_n<b'}2^{-n}\right)$. Suppose $b'<b$. If $b'\notin D$, then there exists a subinterval $[b',b'+\delta]$ of $[a,b]$ such that for $b'\le s<b'+\delta$ we have $c(s)-c(b')-c'(b')(s-b')\le\varepsilon(s-b')$. Hence we get

$$c(s)-c(b')\le c'(b')(s-b')+\varepsilon(s-b')\le\varepsilon(s-b'),$$

and consequently

$$\begin{aligned}c(s)-c(a)&\le c(s)-c(b')+c(b')-c(a)\\&\le\varepsilon(s-b')+\varepsilon\Big(b'-a+\sum_{t_n<b'}2^{-n}\Big)\le\varepsilon\Big(s-a+\sum_{t_n<s}2^{-n}\Big).\end{aligned}$$

On the other hand if $b'\in D$, i.e., $b'=t_m$ for some m, then by continuity of c we can find an interval $[b',b'+\delta]$ contained in $[a,b]$ such that for all $b'\le s<b'+\delta$ we have

$$c(s)-c(b')\le\varepsilon 2^{-m}.$$

Again we deduce that

$$c(s)-c(a)\le\varepsilon2^{-m}+\varepsilon\Big(b'-a+\sum_{t_n<b'}2^{-n}\Big)\le\varepsilon\Big(s-a+\sum_{t_n<s}2^{-n}\Big).$$

So we reach in both cases a contradiction to the maximality of b'. □

Warning: One cannot drop the monotonicity assumption. In fact take $h(t):=t^2$, $c(t):=t^3$ and $[a,b]=[-1,1]$. Then $c'(t)\in h'(t)[-2,2]$, but $c(1)-c(-1)=2\notin\{0\}=(h(1)-h(-1))[-2,2]$.

1.5. Testing with functionals. Recall that in classical analysis vector valued curves $c:\mathbb{R}\to\mathbb{R}^n$ are often treated by considering their components $c_k:=\mathrm{pr}_k\circ c$, where $\mathrm{pr}_k:\mathbb{R}^n\to\mathbb{R}$ denotes the canonical projection onto the k-th factor $\mathbb{R}$. Since in general locally convex spaces do not have appropriate bases, we use all continuous linear functionals instead of the projections pr_k. We will say that a property of a curve $c:\mathbb{R}\to E$ is *scalarly true*, if $\ell\circ c:\mathbb{R}\to E\to\mathbb{R}$ has this property for all continuous linear functionals ℓ on E.

We want to compare scalar differentiability with differentiability. For finite dimensional spaces we know the trivial fact that these two notions coincide. For infinite dimensions we first consider $\mathcal{L}$ip-curves $c:\mathbb{R}\to E$. Since by [5, 5.2.7] boundedness can be tested by the continuous linear functionals we see, that c is $\mathcal{L}$ip if and only if $\ell\circ c:\mathbb{R}\to\mathbb{R}$ is $\mathcal{L}$ip for all $\ell\in E^*$. Moreover, if for a bounded interval $J\subset\mathbb{R}$ we take B as the absolutely convex hull of the bounded set $c(J)\cup\{\frac{c(t)-c(s)}{t-s} : t\ne s; t,s\in J\}$, then we see that $c|_J : J\to E_B$ is a well defined $\mathcal{L}$ip-curve into E_B. We denote by E_B the linear span of B in E, equipped with the *Minkowski functional* $p_B(v):=\inf\{\lambda>0 : v\in\lambda.B\}$. This is a normed space. Thus we have the following equivalent characterizations of $\mathcal{L}$ip-curves:

(1) locally c factors over a $\mathcal{L}$ip-curve into some E_B;
(2) c is $\mathcal{L}$ip;
(3) $\ell \circ c$ is $\mathcal{L}$ip for all $\ell \in E^*$.

For continuous instead of Lipschitz curves we obviously have the analogous implications ($1 \Rightarrow 2 \Rightarrow 3$). However, if we take a non-convergent sequence $(x_n)_n$, which converges weakly to 0 (e.g. take an orthonormal base in a separable Hilbert space), and consider an infinite polygon c through these points x_n, say with $c(\frac{1}{n}) = x_n$ and $c(0) = 0$. Then this curve is obviously not continuous but $\ell \circ c$ is continuous for all $\ell \in E^*$.

Furthermore, the "worst" continuous curve - i.e. $c : \mathbb{R} \to \prod_{C(\mathbb{R},\mathbb{R})} \mathbb{R} =: E$ given by $(c(t))_f := f(t)$ for all $t \in \mathbb{R}$ and $f \in C(\mathbb{R}, \mathbb{R})$ - cannot be factored locally as a continuous curve over some E_B. Otherwise, $c(t_n)$ would converge into some E_B to $c(0)$, where t_n is a given sequence converging to 0, say $t_n := \frac{1}{n}$. So $c(t_n)$ would converge *Mackey* to $c(0)$, i.e., there have to be $\mu_n \to \infty$ with $\{\mu_n(c(t_n) - c(0)) : n \in \mathbb{N}\}$ bounded in E. Since a set is bounded in the product if and only if its coordinates are bounded, we conclude that for all $f \in C(\mathbb{R}, \mathbb{R})$ the sequence $\mu_n(f(t_n) - f(0))$ has to be bounded. But we can choose a continuous function f with $f(0) = 0$ and $f(t_n) = \frac{1}{\sqrt{\mu_n}}$ and conclude that $\mu_n(f(t_n) - f(0)) = \sqrt{\mu_n}$ is unbounded.

Similarly, one shows that the reverse implications do not hold for differentiable curves, for C^1-curves and for C^n-curves. However, if we put instead some Lipschitz condition on the derivatives, there should be some chance, since this is a bornological concept. In order to obtain this result, we should study convergence of sequences in E_B.

1.6. Lemma. Mackey-convergence. *Let B be a bounded and absolutely convex subset of E and let $(x_\gamma)_{\gamma\in\Gamma}$ be a net in E_B. Then the following two conditions are equivalent:*

(1) *x_γ converges to 0 in the normed space E_B;*
(2) *There exists a net $\mu_\gamma \to 0$ in $\mathbb{R}$, such that $x_\gamma \in \mu_\gamma \cdot B$.*

In (2) we may assume that $\mu_\gamma \geq 0$ and is decreasing with respect to γ, at least for large γ. In the particular case of a sequence (or where we have a confinal countable subset of Γ) we can choose $\mu_\gamma > 0$ for all large γ and hence we may divide.

A net (x_γ) for which a bounded absolutely convex $B \subseteq E$ exists, such that x_γ converges to x in E_B is called Mackey convergent to x or short *M-convergent.*

Proof. ($\Uparrow$) Let $x_\gamma = \mu_\gamma \cdot b_\gamma$ with $b_\gamma \in B$ and $\mu_\gamma \to 0$. Then $p_B(x_\gamma) = |\mu_\gamma|\, p_B(b_\gamma) \leq |\mu_\gamma| \to 0$, i.e. $x_\gamma \to 0$ in E_B.

($\Downarrow$) Set $\mu_\gamma := 2\, p_B(x_\gamma)$ and $b_\gamma := \frac{x_\gamma}{\mu_\gamma}$ if $\mu_\gamma \neq 0$ and $b_\gamma := 0$ otherwise. Then $p_B(b_\gamma) = \frac{1}{2}$ or $p_B(b_\gamma) = 0$, so $b_\gamma \in B$. By assumption, $\mu_\gamma \to 0$ and $x_\gamma = \mu_\gamma\, b_\gamma$.

For the final assertions, choose γ_1 such that $|\mu_\gamma| \leq 1$ for $\gamma \geq \gamma_1$, and for those γ we replace μ_γ by $\sup\{|\mu_{\gamma'}| : \gamma' \geq \gamma\} \geq |\mu_\gamma| \geq 0$ which is decreasing with respect to γ.

If we have a sequence $(\gamma_n)_{n\in\mathbb{N}}$ which is confinal in Γ, i.e. for every $\gamma \in \Gamma$ there exists an $n \in \mathbb{N}$ with $\gamma \leq \gamma_n$, then $\gamma \mapsto \nu_\gamma := 1/\min\{n : \gamma \leq \gamma_n\} > 0$ converges to 0, and we can replace μ_γ by $\max\{\mu_\gamma, \nu_\gamma\} > 0$. □

If Γ is the ordered set of all countable ordinals, then it is not possible to find a net $(\mu_\gamma)_{\gamma\in\Gamma}$, which is positive everywhere and converges to 0, since a converging net is finally constant.

1.7. The difference quotient converges Mackey. Now we show how to describe the quality of convergence of the difference quotient.

Corollary. *Let $c : \mathbb{R} \to E$ be a $\mathcal{L}\mathrm{ip}^1$-curve. Then the curve*

$$t \mapsto \frac{1}{t}\Big(\frac{c(t)-c(0)}{t} - c'(0)\Big)$$

is bounded on bounded subsets of $\mathbb{R} \setminus \{0\}$.

Proof. We apply 1.4 to c and obtain:

$$\begin{aligned}\frac{c(t)-c(0)}{t} - c'(0) &\in \Big\langle c'(r) : 0 < |r| < |t| \Big\rangle_{\text{closed, convex}} - c'(0)\\ &= \Big\langle c'(r) - c'(0) : 0 < |r| < |t| \Big\rangle_{\text{closed, convex}}\\ &= \Big\langle r\,\frac{c'(r)-c'(0)}{r} : 0 < |r| < |t| \Big\rangle_{\text{closed, convex}}\end{aligned}$$

Let $a > 0$. Since $\{\frac{c'(r)-c'(0)}{r} : 0 < |r| < a\}$ is bounded and hence contained in a closed absolutely convex and bounded set B, we can conclude that

$$\frac{1}{t}\left(\frac{c(t)-c(0)}{t} - c'(0)\right) \in \Big\langle \frac{r}{t}\,\frac{c'(r)-c'(0)}{r} : 0 < |r| < |t| \Big\rangle_{\text{closed, convex}} \subseteq B. \quad \square$$

1.8. Corollary. Smoothness of curves is a bornological concept. *For $0 \le k < \infty$ a curve c in a locally convex vector space E is $\mathcal{L}\mathrm{ip}^k$ if and only if for each bounded open interval $J \subset \mathbb{R}$ there exists an absolutely convex bounded set $B \subseteq E$ such that $c|_J$ is a $\mathcal{L}\mathrm{ip}^k$-curve in the normed space E_B.*

Attention: A smooth curve factors locally into some E_B as a $\mathcal{L}\mathrm{ip}^k$-curve for each finite k only, in general. Take the "worst" smooth curve $c : \mathbb{R} \to \prod_{C^\infty(\mathbb{R},\mathbb{R})} \mathbb{R}$, analogously to 1.5, and, using Borel's theorem, deduce from $c^{(k)}(0) \in E_B$ for all
Details! $k \in \mathbb{N}$ a contradiction.

Proof. ($\Uparrow$) This follows from lemma 1.3, since the inclusion $E_B \to E$ is continuous.

($\Downarrow$) For $k = 0$ this was shown in 1.5. For $k \ge 1$ take a closed absolutely convex bounded set $B \subseteq E$ containing all derivatives $c^{(i)}$ on J up to order k as well as their difference quotients on $\{(t,s) : t \ne s, t, s \in J\}$. We show first that c is differentiable in E_B, say at 0, with derivative $c'(0)$. By the proof of the previous corollary 1.7 we have that the expression $\frac{1}{t}(\frac{c(t)-c(0)}{t} - c'(0))$ lies in B. So $\frac{c(t)-c(0)}{t} - c'(0)$ converges to 0 in E_B. For the higher order derivatives we can now proceed by induction. $\square$

A consequence of this is, that smoothness does not depend on the topology but only on the dual (so all topologies with the same dual have the same smooth curves), and in fact it depends only on the bounded sets, i.e. the bornology. Since on $L(E,F)$ there is essentially only one bornology (by the uniform boundedness principle, see [5, 5.2.2]) there is only one notion of $\mathcal{L}\mathrm{ip}^n$-curves into $L(E,F)$. Furthermore, the class of $\mathcal{L}\mathrm{ip}^n$-curves doesn't change if we pass from a given locally convex topology to its *bornologification*, see 4.2, which by definition is the finest locally convex topology having the same bounded sets.

Let us now return to scalar differentiability. Corollary 1.7 gives us $\mathcal{L}\mathrm{ip}^n$-ness provided we have appropriate candidates for the derivatives.

1.9. Corollary. Scalar testing of curves. *Let $c^k : \mathbb{R} \to E$ for $k < n+1$ be curves such that $\ell \circ c^0$ is $\mathcal{L}\mathrm{ip}^n$ and $(\ell \circ c^0)^{(k)} = \ell \circ c^k$ for all $k < n+1$ and all $\ell \in E^*$. Then c^0 is $\mathcal{L}\mathrm{ip}^n$ and $(c^0)^{(k)} = c^k$.*

Proof. For $n = 0$ this was shown in 1.5. For $n \geq 1$, by 1.7 applied to $\ell \circ c^0$ we have that

$$\ell\left(\frac{1}{t}\left(\frac{c^0(t) - c^0(0)}{t} - c^1(0)\right)\right)$$

is locally bounded, and hence by [5, 5.2.7] the set

$$\left\{\frac{1}{t}\left(\frac{c^0(t) - c^0(0)}{t} - c^1(0)\right) : t \in I\right\}$$

is bounded. Thus $\frac{c^0(t)-c^0(0)}{t}$ converges even Mackey to $c^1(0)$. Now the general statement follows by induction. □

2. Completeness

Do we really need the knowledge of a candidate for the derivative, as in 1.9? In finite dimensional analysis one often uses the Cauchy condition to prove convergence. Here we will replace the Cauchy condition again by a stronger condition, which provides information about the quality of being Cauchy:

A net $(x_\gamma)_{\gamma\in\Gamma}$ in E is called *Mackey-Cauchy* provided that there exist a bounded (absolutely convex) set B and a net $(\mu_{\gamma,\gamma'})_{(\gamma,\gamma')\in\Gamma\times\Gamma}$ in $\mathbb{R}$ converging to 0, such that $x_\gamma - x_{\gamma'} \in \mu_{\gamma,\gamma'} B$. As in 1.6 one shows that for a net x_γ in E_B this is equivalent to the condition that x_γ is Cauchy in the normed space E_B.

2.1. Lemma. The difference quotient is Mackey-Cauchy. *Let $c : \mathbb{R} \to E$ be scalarly a $\mathcal{L}\mathrm{ip}^1$-curve. Then $t \mapsto \frac{c(t)-c(0)}{t}$ is a Mackey-Cauchy net for $t \to 0$.*

Proof. For $\mathcal{L}\mathrm{ip}^1$-curves this is a immediate consequence of 1.7 but we only assume it to be scalarly $\mathcal{L}\mathrm{ip}^1$. It is enough to show that $\frac{1}{t-s}\left(\frac{c(t)-c(0)}{t} - \frac{c(s)-c(0)}{s}\right)$ is bounded on bounded subsets in $\mathbb{R} \setminus \{0\}$. We may test this with continuous linear functionals, and hence may assume that $E = \mathbb{R}$. Then by the fundamental theorem of calculus we have

$$\begin{aligned}\frac{1}{t-s}\left(\frac{c(t)-c(0)}{t} - \frac{c(s)-c(0)}{s}\right) &= \int_0^1 \frac{c'(tr) - c'(sr)}{t-s}\,dr \\ &= \int_0^1 \frac{c'(tr) - c'(sr)}{tr - sr}\,r\,dr.\end{aligned}$$

Since $\frac{c'(tr)-c'(sr)}{tr-sr}$ is locally bounded by assumption, the same is true for the integral, and we are done. □

2.2. Lemma. Mackey Completeness. *For a space E the following conditions are equivalent:*

1. *Every Mackey-Cauchy net converges in E;*
2. *Every Mackey-Cauchy sequence converges in E;*
3. *For every absolutely convex closed bounded set B the space E_B is complete;*
4. *For every bounded set B there exists an absolutely convex bounded set $B' \supseteq B$ such that $E_{B'}$ is complete.*

A space satisfying the equivalent conditions is called Mackey complete. Note that a sequentially complete space is Mackey complete.

Proof. (1) ⇒ (2), and (3) ⇒ (4) are trivial.

(2) ⇒ (3) Since E_B is normed, it is enough to show sequential completeness. So let (x_n) be a Cauchy sequence in E_B. Then (x_n) is Mackey-Cauchy in E and hence converges in E to some point x. Since $p_B(x_n - x_m) \to 0$ there exists for every $\varepsilon > 0$ an $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $p_B(x_n - x_m) < \varepsilon$, and hence $x_n - x_m \in \varepsilon B$. Taking the limit for $m \to \infty$, and using closedness of B we conclude that $x_n - x \in \varepsilon B$ for all $n > N$. In particular $x \in E_B$ and $x_n \to x$ in E_B.

(4) ⇒ (1) Let $(x_\gamma)_{\gamma\in\Gamma}$ be a Mackey-Cauchy net in E. So there is some net $\mu_{\gamma,\gamma'} \to 0$, such that $x_\gamma - x_{\gamma'} \in \mu_{\gamma,\gamma'} B$ for some bounded set B. Let γ_0 be arbitrary. By (4) we may assume that B is absolutely convex and contains x_{γ_0}, and that E_B is complete. For $\gamma \in \Gamma$ we have that $x_\gamma = x_{\gamma_0} + x_\gamma - x_{\gamma_0} \in x_{\gamma_0} + \mu_{\gamma,\gamma_0} B \in E_B$, and $p_B(x_\gamma - x_{\gamma'}) \leq \mu_{\gamma,\gamma'} \to 0$. So (x_γ) is a Cauchy net in E_B, hence converges in E_B, and thus also in E. □

2.3. Corollary. Scalar testing of differentiable curves. *Let E be Mackey complete and $c : \mathbb{R} \to E$ be a curve for which $\ell \circ c$ is $\mathcal{L}\mathrm{ip}^n$ for all $\ell \in E^*$. Then c is $\mathcal{L}\mathrm{ip}^n$.*

Proof. For $n = 0$ this was shown in 1.5 without using any completeness, so let $n \geq 1$. Since we have shown in 2.1 that the difference quotient is a Mackey-Cauchy net we conclude that the derivative c' exists, and hence $(\ell \circ c)' = \ell \circ c'$. So we may apply the induction hypothesis to conclude that c' is $\mathcal{L}\mathrm{ip}^{n-1}$, and consequently c is $\mathcal{L}\mathrm{ip}^n$. □

Next we turn to integration. For continuous curves $c : [0, 1] \to E$ one can show completely analogously to 1-dimensional analysis that the *Riemann sums* $R(c, \mathcal{Z}, \xi)$, defined by $\sum_k (t_k - t_{k-1})c(\xi_k)$, where $0 = t_0 < t_1 < \cdots < t_n = 1$ is a partition $\mathcal{Z}$ of $[0, 1]$ and $\xi_k \in [t_{k-1}, t_k]$, form a Cauchy net with respect to the partial strict ordering given by the size of the *mesh* $\max\{|t_k - t_{k-1}| : 0 < k < n\}$. So under the assumption of sequential completeness we have a *Riemann integral* of curves. A second way to see this is the following reduction to the 1-dimensional case.

2.4. Lemma. *Let $L(E^*_{equi}, \mathbb{R})$ be the space of all linear functionals on E^* which are bounded on equicontinuous sets, equipped with the complete locally convex topology of uniform convergence on these sets. There is a natural topological embedding $\delta : E \to L(E^*_{equi}, \mathbb{R})$ given by $\delta(x)(\ell) := \ell(x)$.*

Proof. The space $L(E^*_{\mathrm{equi}}, \mathbb{R})$ is complete, since this is true for the space of all bounded mappings (see 2.15) in which it is obviously closed.

Let $\mathcal{U}$ be a basis of absolutely convex closed 0-neighborhoods in E. Then the family of *polars* $U^o := \{\ell \in E^* : |\ell(x)| \leq 1 \text{ for all } x \in U\}$, with $U \in \mathcal{U}$ form a basis for the equicontinuous sets, and hence the *bipolars* $U^{oo} := \{\ell^* \in L(E^*_{\mathrm{equi}}, \mathbb{R}) : |\ell^*(\ell)| \leq 1 \text{ for all } \ell \in U^o\}$ form a basis of 0-neighborhoods in $L(E^*_{\mathrm{equi}}, \mathbb{R})$. By the bipolar theorem [5, 7.4.7] we have $U = \delta^{-1}(U^{oo})$ for all $U \in \mathcal{U}$. This shows that δ is a homeomorphism onto its image. □

2.5. Lemma. Integral of continuous curves. *Let $c : \mathbb{R} \to E$ be a continuous curve in a locally convex vector space. Then there is a unique differentiable curve $\int c : \mathbb{R} \to \widehat{E}$ in the completion $\widehat{E}$ of E such that $(\int c)(0) = 0$ and $(\int c)' = c$.*

Proof. We show uniqueness first. Let $c_1 : \mathbb{R} \to \widehat{E}$ be a curve with derivative c and $c_1(0) = 0$. For every $\ell \in E^*$ the composite $\ell \circ c_1$ is an anti-derivative of $\ell \circ c$ with initial value 0, so it is uniquely determined, and since E^* separates points c_1 is also uniquely determined.

Now we show the existence. By the previous lemma we have that $\widehat{E}$ is (isomorphic to) the closure of E in the obviously complete space $L(E^*_{\text{equi}}, \mathbb{R})$. We define $(\int c)(t) : E^* \to \mathbb{R}$ by $\ell \mapsto \int_0^t (\ell \circ c)(s)ds$. It is a bounded linear functional on E^*_{equi} since for an equicontinuous and hence bounded subset $\mathcal{E} \subseteq E^*$ the set $\{(\ell \circ c)(s) : \ell \in \mathcal{E}, s \in [0, t]\}$ is bounded. So $\int c : \mathbb{R} \to L(E^*_{\text{equi}}, \mathbb{R})$. NewLect:Mo/Di

Now we show that $\int c$ is differentiable with derivative $\delta \circ c$.

$$\left(\frac{(\int c)(t + r) - (\int c)(r)}{t} - (\delta \circ c)(r)\right)(\ell) =$$
$$= \frac{1}{t}\left(\int_0^{t+r} (\ell \circ c)(s)ds - \int_0^r (\ell \circ c)(s)ds - t(\ell \circ c)(r)\right) =$$
$$= \frac{1}{t}\int_r^{r+t} \Big((\ell \circ c)(s) - (\ell \circ c)(r)\Big)ds = \int_0^1 \ell\Big(c(r + ts) - c(r)\Big)ds.$$

Let $\mathcal{E} \subseteq E^*$ be equicontinuous, and let $\varepsilon > 0$. Then there exists a neighborhood U of 0 such that $|\ell(U)| < \varepsilon$ for all $\ell \in \mathcal{E}$. For sufficiently small t, all $s \in [0, 1]$ and fixed r we have $c(r + ts) - c(r) \in U$. So $|\int_0^1 \ell(c(r + ts) - c(r))ds| < \varepsilon$. This shows that the difference quotient of $\int c$ at r converges to $\delta(c(r))$ uniformly on equicontinuous subsets.

It remains to show that $(\int c)(t) \in \widehat{E}$. By the mean value theorem 1.4 the difference quotient $\frac{1}{t}((\int c)(t) - (\int c)(0))$ is contained in the closed convex hull in $L(E^*_{\text{equi}}, \mathbb{R})$ of the subset $\{c(s) = (\int c)'(s) : 0 < s < t\}$ of E. So it lies in $\widehat{E}$. □

Definition of the integral. For continuous curves $c : \mathbb{R} \to E$ the *definite integral* $\int_a^b c \in \widehat{E}$ is given by $\int_a^b c = (\int c)(b) - (\int c)(a)$.

2.6. Corollary. Basics on the integral. *For a continuous curve $c : \mathbb{R} \to E$ we have:*

1. (1) $\ell(\int_a^b c) = \int_a^b (\ell \circ c)$ *for all* $\ell \in E^*$.
2. (2) $\int_a^b c + \int_b^d c = \int_a^d c$.
3. (3) $\int_a^b (c \circ \varphi)\varphi' = \int_{\varphi(a)}^{\varphi(b)} c$ *for* $\varphi \in C^1(\mathbb{R}, \mathbb{R})$.
4. (4) $\int_a^b c$ *lies in the closed convex hull in* $\widehat{E}$ *of the set* $\{(b - a)c(t) : a < t < b\}$ *in* E.
5. (5) $\int_a^b : C(\mathbb{R}, E) \to \widehat{E}$ *is linear.*
6. (6) *(Fundamental theorem of calculus.) For each C^1-curve $c : \mathbb{R} \to E$ we have* $c(s) - c(t) = \int_t^s c'$. □

We are mainly interested in smooth curves and we can test for this by applying linear functionals if the space is Mackey complete, see 2.3. So let us try to show that the integral for such curves lies in E if E is Mackey-complete. So let $c : [0, 1] \to E$ be a smooth or just a $\mathcal{L}$ip-curve, and take a partition $\mathcal{Z}$ with mesh $\mu(\mathcal{Z})$ at most δ. If we have a second partition, then we can take the common refinement. Let $[a, b]$ be one interval of the original partition with intermediate point t, and let

$a = t_0 < t_1 < \cdots < t_n = b$ be the refinement. Note that $|b-a| \leq \delta$ and hence $|t - t_k| \leq \delta$. Then we can estimate as follows.

$$(b-a)\,c(t) - \sum_k (t_k - t_{k-1}) c(t_k) = \sum_k (t_k - t_{k-1})\,(c(t) - c(t_k)) = \sum_k \mu_k b_k,$$

where $b_k := \frac{c(t)-c(t_k)}{\delta}$ is contained in the *absolutely convex Lipschitz bound*

$$B := \left\langle \left\{ \frac{c(t)-c(s)}{t-s} : t, s \in [0,1] \right\} \right\rangle_{abs.conv}$$

of c and $\mu_k := (t_k - t_{k-1})\delta \geq 0$ and satisfies $\sum_k \mu_k = (b-a)\delta$. Hence we have for the Riemann sums with respect to the original partition $\mathcal{Z}_1$ and the refinement $\mathcal{Z}'$ that $R(c, \mathcal{Z}_1) - R(c, \mathcal{Z}')$ lies in $\delta \cdot B$. So $R(c, \mathcal{Z}_1) - R(c, \mathcal{Z}_2) \in 2\delta B$ for any two partitions $\mathcal{Z}_1$ and $\mathcal{Z}_2$ of mesh at most δ, i.e. the Riemann sums form a Mackey-Cauchy net with coefficients $\mu_{Z_1, \mathcal{Z}_2} := 2 \max\{\mu(\mathcal{Z}_1), \mu(\mathcal{Z}_2)\}$ and we have proved:

2.7. Proposition. Integral of Lipschitz curves. *Let $c : [0,1] \to E$ be a Lipschitz curve into a Mackey complete space. Then the Riemann integral exists in E as (Mackey)-limit of the Riemann sums.* □

2.8. Now we have to discuss the relationship between differentiable curves and Mackey convergent sequences. Recall that a sequence (x_n) converges if and only if there exists a continuous curve c (e.g. a reparameterization of the infinite polygon) and $t_n \searrow 0$ with $c(t_n) = x_n$. The corresponding result for smooth curves uses the following notion.

Definition. We say that a sequence x_n in a locally convex space E *converges fast* to x in E, or *falls fast* towards x, if for each $k \in \mathbb{N}$ the sequence $n^k(x_n - x)$ is bounded.

Special curve lemma. *Let x_n be a sequence which converges fast to x in E.*

Then the infinite polygon through the x_n can be parameterized as a smooth curve $c : \mathbb{R} \to E$ such that $c(\frac{1}{n}) = x_n$ and $c(0) = x$.

Proof. Let $\varphi : \mathbb{R} \to [0,1]$ be a smooth function, which is 0 on $\{t : t \leq 0\}$ and 1 on $\{t : t \geq 1\}$. The parameterization c is defined as follows:

$$c(t) := \begin{cases} x & \text{for } t \leq 0, \\ x_{n+1} + \varphi\left(\frac{t - \frac{1}{n+1}}{\frac{1}{n} - \frac{1}{n+1}}\right)(x_n - x_{n+1}) & \text{for } \frac{1}{n+1} \leq t \leq \frac{1}{n}, \\ x_1 & \text{for } t \geq 1 \end{cases}.$$

Obviously, c is smooth on $\mathbb{R} \setminus \{0\}$, and the p-th derivative of c for $\frac{1}{n+1} \leq t \leq \frac{1}{n}$ is given by

$$c^{(p)}(t) = \varphi^{(p)}\left(\frac{t - \frac{1}{n+1}}{\frac{1}{n} - \frac{1}{n+1}}\right)(n(n+1))^p (x_n - x_{n+1}).$$

Since x_n converges fast to x, we have that $c^{(p)}(t) \to 0$ for $t \to 0$, because the first factor is bounded and the second goes to zero. Hence c is smooth on $\mathbb{R}$, by the following lemma. □

2.9. Lemma. Differentiable extension to an isolated point. *Let $c : \mathbb{R} \to E$ be continuous and differentiable on $\mathbb{R} \setminus \{0\}$, and assume that the derivative $c' : \mathbb{R} \setminus \{0\} \to E$ has a continuous extension to $\mathbb{R}$. Then c is differentiable at 0 and $c'(0) = \lim_{t \to 0} c'(t)$.*

Proof. Let $a := \lim_{t\to 0} c'(t)$. By the mean value theorem 1.4 we have $\frac{c(t)-c(0)}{t} \in \langle c'(s) : 0 \neq |s| \leq |t|\rangle_{\text{closed, convex}}$. Since c' is assumed to be continuously extendable to 0 we have that for any closed convex 0-neighborhood U there exists a $\delta > 0$ such that $c'(t) \in a + U$ for all $0 < |t| \leq \delta$. Hence $\frac{c(t)-c(0)}{t} - a \in U$, i.e. $c'(0) = a$. □

The next result shows that we can pass through certain sequences $x_n \to x$ even with given velocities $v_n \to 0$.

2.10. Corollary. *If $x_n \to x$ fast and $v_n \to 0$ fast in E, then there exist a smoothly parameterized polygon $c : \mathbb{R} \to E$ and $t_n \to 0$ in $\mathbb{R}$ such that $c(t_n + t) = x_n + tv_n$ for t in a neighborhood of 0 depending on n.*

Proof. Consider the sequence y_n defined by

$$y_{2n} := x_n + \tfrac{1}{4n(2n+1)} v_n \quad \text{and} \quad y_{2n+1} := x_n - \tfrac{1}{4n(2n+1)} v_n.$$

It is easy to show that y_n converges fast to x, and the parameterization c of the polygon through the y_n (using a function φ which satisfies $\varphi(t) = t$ for t near $1/2$) has the claimed properties, where

$$t_n := \tfrac{4n+1}{4n(2n+1)} = \frac{1}{2}\left(\frac{1}{2n} + \frac{1}{2n+1}\right). \quad \square$$

As first application 2.10 we can give the following sharpening of 1.3.

2.11. Corollary. Bounded linear maps. *A linear mapping $\ell : E \to F$ between locally convex vector spaces is bounded (or bornological), i.e. it maps bounded sets to bounded ones, if and only if it maps smooth curves in E to smooth curves in F.*

Proof. As in the proof of 1.3 one shows using 1.7 that a bounded linear map preserves $\mathcal{L}\text{ip}^k$-curves. Conversely, assume that a linear map $f : E \to F$ carries smooth curves to locally bounded curves. Take a bounded set B, and assume that $f(B)$ is unbounded. Then there is a sequence (b_n) in B and some $\lambda \in F'$ such that $|(\lambda \circ f)(b_n)| \geq n^{n+1}$. The sequence $(n^{-n} b_n)$ converges fast to 0, hence lies on some compact part of a smooth curve by 2.8. Consequently, $(\lambda \circ f)(n^{-n} b_n) = n^{-n}(\lambda \circ f)(b_n)$ is bounded, a contradiction. □

2.12. Definition. The c^∞*-topology* on a locally convex space E is the final topology with respect to all smooth curves $\mathbb{R} \to E$. Its open sets will be called c^∞*-open*. We will treat this topology in more detail in section 4: In general it describes neither a topological vector space 4.20 and 4.26, nor a uniform structure 4.27. However, by 4.4 and 4.6 the finest locally convex topology coarser than the c^∞-topology is the bornologification of the locally convex topology. Motivate

Let (μ_n) be a sequence of real numbers converging to ∞. Then a sequence (x_n) in E is called μ*-converging* to x if the sequence $(\mu_n(x_n - x))$ is bounded in E.

2.13. Theorem. c$^\infty$-open subsets. *Let $\mu_n \to \infty$ be a real valued sequence and $k \in \mathbb{N}_\infty$. Then a subset $U \subseteq E$ is open for the c^∞-topology if it satisfies any of the following equivalent conditions:*

1. *All inverse images under $\mathcal{L}\text{ip}^k$-curves are open in $\mathbb{R}$;*
2. *All inverse images under μ-converging sequences are open in $\mathbb{N}_\infty$;*
3. *The traces to E_B are open in E_B for all absolutely convex bounded subsets $B \subseteq E$.*

Note that for closed subsets an equivalent statement reads as follows: A set A is c^∞-closed if and only if for every sequence $x_n \in A$, which is μ-converging (respectively M-converging, resp. fast falling) towards x, the point x belongs to A.

The topology described in (2) is also called *Mackey-closure topology*. It is not the Mackey topology discussed in duality theory.

Proof. (1) $\Rightarrow$ (2) Suppose (x_n) is μ-converging to $x \in U$, but $x_n \notin U$ for infinitely many n. Then we may choose a subsequence again denoted by (x_n), which is fast falling to x, hence lies on some compact part of a smooth curve c as described in 2.8. Then $c(\frac{1}{n}) = x_n \notin U$ but $c(0) = x \in U$. This is a contradiction.

(2) $\Rightarrow$ (3) A sequence (x_n), which converges in E_B to x with respect to p_B, is Mackey convergent, hence has a μ-converging subsequence. Note that E_B is normed, and hence it is enough to consider sequences.

(3) $\Rightarrow$ (1) Let $c : \mathbb{R} \to E$ be $\mathcal{L}$ip. By 1.5 c factors locally as continuous curve over some E_B, hence $c^{-1}(U)$ is open. □

Let us show next that the c^∞-topology and c^∞-completeness are intimately related.

2.14. Theorem. Convenient vector spaces. *Let E be a locally convex vector space. E is said to be c^∞-*complete *or* convenient *if one of the following equivalent (completeness) conditions is satisfied:*

1. *Any Lipschitz curve in E is locally Riemann integrable.*
2. *For any $c_1 \in C^\infty(\mathbb{R}, E)$ there is $c_2 \in C^\infty(\mathbb{R}, E)$ with $c_2' = c_1$ (existence of an* anti-derivative*).*
3. *E is c^∞-closed in any locally convex space.*
4. *If $c : \mathbb{R} \to E$ is a curve such that $\ell \circ c : \mathbb{R} \to \mathbb{R}$ is smooth for all $\ell \in E^*$, then c is smooth.*
5. *Any Mackey-Cauchy sequence converges; i.e. E is Mackey complete, see 2.2.*
6. *If B is bounded closed absolutely convex, then E_B is a Banach space. This property is called* locally complete *in* [Jarchow, 1981, p196].
7. *Any continuous linear mapping from a normed space into E has a continuous extension to the completion of the normed space.*

Condition (4) says that in a convenient vector space one can recognize smooth curves by investigating compositions with continuous linear functionals. Condition (5) and (6) say via 2.2.4 that c^∞-completeness is a bornological concept. In [Frölicher, Kriegl, 1988] a convenient vector space is always considered with its bornological topology — an equivalent but not isomorphic category.

Proof. In 2.3 we showed (5) $\Rightarrow$ (4), in 2.7 we got (5) $\Rightarrow$ (1), and in 2.2 we had (5) $\Rightarrow$ (6).

(1) $\Rightarrow$ (2) A smooth curve is Lipschitz, thus locally Riemann integrable. By 2.6.1 the indefinite Riemann integral equals the "weakly defined" integral of lemma 2.5, hence is an anti-derivative.

(2) $\Rightarrow$ (3) Let E be a topological linear subspace of F. To show that E is c^∞-closed we use 2.13. Let $x_n \to x_\infty$ be fast falling, $x_n \in E$ but $x_\infty \in F$. By 2.8 the polygon c through (x_n) can be smoothly symmetrically parameterized in F. Hence c' is smooth and has values in the vector space generated by $\{x_n : n \neq \infty\}$,

which is contained in E. Its anti-derivative c_2 is up to a constant equal to c, and by (2) $x_1 - x_\infty = c(1) - c(0) = c_2(1) - c_2(0)$ lies in E. So $x_\infty \in E$.

(4) $\Rightarrow$ (3) Let E be a topological linear subspace of F as before. We use again 2.13 in order to show that E is c^∞-closed in F. So let $x_n \to x_\infty$ be fast falling, $x_n \in E$ for $n \neq 0$, but $x_\infty \in F$. By 2.8 the polygon c through (x_n) can be smoothly symmetrically parameterized in F, and $c(t) \in E$ for $t \neq 0$. We consider $\tilde{c}(t) := tc(t)$. This is a curve in E which is smooth in F, so it is scalarwise smooth in E, thus smooth in E by (4). Then $x_\infty = \tilde{c}'(0) \in E$.

(3) $\Rightarrow$ (5) Let F be the completion $\widehat{E}$ of E. Any Mackey Cauchy sequence in E has a limit in F, and since E is by assumption c^∞-closed in F the limit lies in E. Hence, the sequence converges in E.

(6) $\Rightarrow$ (7) Let $f : F \to E$ be a continuous mapping on a normed space F. Since the image of the unit ball is bounded, it is a bounded mapping into E_B for some closed absolutely convex B. But into E_B it can be extended to the completion, since E_B is complete.

(7) $\Rightarrow$ (1) Let $c : \mathbb{R} \to E$ be a Lipschitz curve. Then c is locally a continuous curve into E_B for some absolutely convex bounded set B by 1.5. The inclusion of E_B into E has a continuous extension to the completion of E_B, and c is Riemann integrable in this Banach space, so also in E. □ NewLect: Do/Mo

2.15. Theorem. Inheritance of c^∞-completeness. *The following constructions preserve c^∞-completeness: limits, direct sums, strict inductive limits of sequences of closed embeddings, as well as formation of $\ell^\infty(X,\ \ )$, where X is a set together with a family $\mathcal{B}$ of subsets of X containing the finite ones, which are called bounded and $\ell^\infty(X, F)$ denotes the space of all functions $f : X \to F$, which are bounded on all $B \in \mathcal{B}$, supplied with the topology of uniform convergence on the sets in $\mathcal{B}$.*

Note that the definition of the topology of uniform convergence as initial topology shows, that adding all subsets of finite unions of elements in $\mathcal{B}$ to $\mathcal{B}$ does not change this topology. Hence, we may always assume that $\mathcal{B}$ has this stability property; this is the concept of a *bornology on a set.*

Proof. The projective limit [**5**, 4.8.1] of a functor $\mathcal{F}$ is the c^∞-closed linear subspace Appendix: lim, colim

$$\Big\{(x_\alpha) \in \prod \mathcal{F}(\alpha) : \mathcal{F}(f)x_\alpha = x_\beta \text{ for all } f : \alpha \to \beta\Big\},$$

hence is c^∞-complete, since the product of c^∞-complete factors is obviously c^∞-complete.

Since the coproduct [**5**, 4.6.1] of spaces X_α is the topological direct sum, and has as bounded sets those which are contained and bounded in some finite subproduct, it is c^∞-complete if all factors are.

For colimits this is in general not true. For strict inductive limits of sequences of closed embeddings it is true, since bounded sets are contained and bounded in some step, see [**5**, 4.8.1].

For the result on $\ell^\infty(X, F)$ we consider first the case, where X itself is bounded. Then c^∞-completeness can be proved as in [**5**, 3.2.3] or reduced to this result. In fact let $\mathcal{B}$ be bounded in $\ell^\infty(X, F)$. Then $\mathcal{B}(X)$ is bounded in F and hence contained in some absolutely convex bounded set B, for which F_B is a Banach space. So we may assume that $\mathcal{B} := \{f \in \ell^\infty(X, F) : f(X) \subseteq B\}$. The space

$\ell^\infty(X,F)_{\mathcal{B}}$ is just the space $\ell^\infty(X,F_B)$ with the supremum norm, which is a Banach space by [5, 3.2.3]. In fact, we have the implications

$$\|f\|_\infty := \sup\{p_B(f(x)) : x \in X\} < \lambda \Rightarrow \frac{f(x)}{\lambda} \in B \forall x \in X$$
$$\Rightarrow p_B\left(\frac{f(x)}{\lambda}\right) \le 1 \forall x \in X \Rightarrow \|f\|_\infty \le \lambda,$$

i.e.

$$\{\lambda : \|f\|_\infty < \lambda\} \subseteq \{\lambda : f \in \lambda\mathcal{B}\} \subseteq \{\lambda : \|f\|_\infty \le \lambda\}$$

and hence

$$\underbrace{\inf\{\lambda : \|f\|_\infty < \lambda\}}_{=\|f\|_\infty} \ge \underbrace{\inf\{\lambda : f \in \lambda\mathcal{B}\}}_{=p_{\mathcal{B}}(f)} \ge \underbrace{\inf\{\lambda : \|f\|_\infty \le \lambda\}}_{=\|f\|_\infty}.$$

Let now X and $\mathcal{B}$ be arbitrary. Then the restriction maps $\ell^\infty(X,F) \to \ell^\infty(B,F)$ give an embedding ι of $\ell^\infty(X,F)$ into the product $\prod_{B\in\mathcal{B}} \ell^\infty(B,F)$. Since this product is complete, by what we have shown above, it is enough to show that this embedding has a closed image. So let $f_\alpha|_B$ converge to some f_B in $\ell^\infty(B,F)$. Define $f(x) := f_{\{x\}}(x)$. For any $B \in \mathcal{B}$ containing x we have that $f_B(x) = (\lim_\alpha f_\alpha|_B)(x) = \lim_\alpha(f_\alpha(x)) = \lim_\alpha f_\alpha|_{\{x\}} = f_{\{x\}}(x) = f(x)$, and $f(B)$ is bounded for all $B \in \mathcal{B}$, since $f|_B = f_B \in \ell^\infty(B,F)$. □

Example. In general, a quotient and an inductive limit of c^∞-complete spaces need not be c^∞-complete. In fact, let $E_D := \{x \in \mathbb{R}^{\mathbb{N}} : \operatorname{supp} x \subseteq D\}$ for any
CITAT subset $D \subseteq \mathbb{N}$ of *density* $\operatorname{dens} D := \limsup\{\frac{|D\cap[1,n]|}{n}\} = 0$. It can be shown that $E := \bigcup_{\operatorname{dens} D=0} E_D \subset \mathbb{R}^{\mathbb{N}}$ is the inductive limit of the Fréchet subspaces $E_D \cong \mathbb{R}^D$. It cannot be c^∞-complete, since finite sequences are contained in E and are dense in $\mathbb{R}^{\mathbb{N}} \supset E$.

3. Smooth Mappings and the Exponential Law

A particular case of the exponential law for continuous mappings is the following

3.1. Lemma. *A map $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous if and only if the associated mapping $f^\vee : \mathbb{R} \to C(\mathbb{R},\mathbb{R})$ is continuous, where $C(\mathbb{R},\mathbb{R})$ carries the usual Fréchet-topology of uniform convergence on compact subsets of $\mathbb{R}$.*

Proof. ($\Rightarrow$) Obviously $f^\vee$ has values $f^\vee(t) : s \mapsto f(t,s)$ in $C(\mathbb{R},\mathbb{R})$. It is continuous, since for $t_0 \in \mathbb{R}$, compact $J \subseteq \mathbb{R}$ and $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(t,s) - f(t_0,s)| < \varepsilon$ for all $|t-t_0| < \delta$ and $s \in I$, i.e. $\|(f^\vee(t) - f^\vee(t_0))|_J\|_\infty \le \varepsilon$ for $|t - t_0| < \delta$.

($\Leftarrow$) Let $(t_0,s_0) \in \mathbb{R}^2$ and $\varepsilon > 0$ and choose a compact neighborhood J of s_0 such that $|f^\vee(t_0)(s) - f^\vee(t_0)(s_0)| < \varepsilon$ for all $s \in J$. Since $f^\vee$ is assumed to be continuous there exists a $\delta > 0$ auch that $\|(f^\vee(t) - f^\vee(t_0))|_J\|_\infty \le \varepsilon$ for $|t - t_0| < \delta$, and hence

$$|f(t,s) - f(t_0,s_0)| \le |f^\vee(t)(s) - f^\vee(t_0)(s)| + |f^\vee(t_0)(s) - f^\vee(t_0)(s_0)| \le 2\varepsilon$$

for all $|t - t_0| < \delta$ and all $s \in J$. □

Now let us start proving the exponential law $C^\infty(U \times V, F) \cong C^\infty(U, C^\infty(V,F))$ first for $U = V = F = \mathbb{R}$.

3.2. Theorem. Simplest case of exponential law. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an arbitrary mapping. Then all iterated partial derivatives exist and are continuous if and only if the associated mapping $f^\vee : \mathbb{R} \to C^\infty(\mathbb{R},\mathbb{R})$ exists as a smooth curve,* *where $C^\infty(\mathbb{R},\mathbb{R})$ is considered as the Fréchet space with the topology of uniform convergence of each derivative on compact sets. Furthermore, we have $(\partial_1 f)^\vee = d(f^\vee)$ and $(\partial_2 f)^\vee = d\circ f^\vee = d_*(f^\vee)$.*

Proof. We have several possibilities to prove this result. Either we show Mackey convergence of the difference quotients, via the boundedness of $\frac{1}{t}\left(\frac{c(t)-c(0)}{t} - c'(0)\right)$, and then use the trivial exponential law $\ell^\infty(X\times Y,\mathbb{R}) \cong \ell^\infty(X,\ell^\infty(Y,\mathbb{R}))$; or we use exponential law $C(\mathbb{R}^2,\mathbb{R})\cong C(\mathbb{R},C(\mathbb{R},\mathbb{R}))$ of 3.1. We choose the latter method.

Proof of ($\Leftarrow$) Let $g := f^\vee : \mathbb{R}\to C^\infty(\mathbb{R},\mathbb{R})$ be smooth. Then both curves dg and $d\circ g = d_* g$ are smooth (use 1.3 and that d is continuous and linear). An easy calculation shows that the partial derivatives of $f = g^\wedge$ exist and are given by $\partial_1 g^\wedge = (dg)^\wedge$ and $\partial_2 g^\wedge = (d\circ g)^\wedge$. So one obtains inductively that all iterated derivatives of f exist and are continuous by 3.1, since they are associated to smooth curves $\mathbb{R}\to C^\infty(\mathbb{R},\mathbb{R})\to C(\mathbb{R},\mathbb{R})$.

Proof of ($\Rightarrow$) First observe that $f^\vee : \mathbb{R}\to C^\infty(\mathbb{R},\mathbb{R})$ makes sense and that for all $t\in\mathbb{R}$ we have

$$d^p(f^\vee(t)) = (\partial_2^p f)^\vee(t).$$

Next we claim that $f^\vee : \mathbb{R}\to C^\infty(\mathbb{R},\mathbb{R})$ is differentiable, with derivative $d(f^\vee) = (\partial_1 f)^\vee$, or equivalently that for all a the curve

$$c : t\mapsto \begin{cases} \frac{f^\vee(t+a)-f^\vee(a)}{t} & \text{for } t\neq 0\\ (\partial_1 f)^\vee(a) & \text{otherwise}\end{cases}$$

is continuous (at 0) as curve $\mathbb{R}\to C^\infty(\mathbb{R},\mathbb{R})$. Without loss of generality we may assume that $a=0$. Since $C^\infty(\mathbb{R},\mathbb{R})$ carries the initial structure with respect to the linear mappings $d^p : C^\infty(\mathbb{R},\mathbb{R})\to C(\mathbb{R},\mathbb{R})$ we have to show that $d^p\circ c : \mathbb{R}\to C(\mathbb{R},\mathbb{R})$ is continuous, or equivalently by the exponential law for continuous maps, that $(d^p\circ c)^\wedge : \mathbb{R}^2\to\mathbb{R}$ is continuous. For $t\neq 0$ and $s\in\mathbb{R}$ we have

$$\begin{aligned}(d^p\circ c)^\wedge(t,s) &= d^p(c(t))(s) = d^p\left(\frac{f^\vee(t)-f^\vee(0)}{t}\right)(s)\\ &= \frac{\partial_2^p f(t,s) - \partial_2^p f(0,s)}{t} && \text{by (1)}\\ &= \int_0^1 \partial_1\partial_2^p f(t\,\tau,s)\,d\tau && \text{by the fundamental theorem.}\end{aligned}$$

For $t=0$ we have

$$\begin{aligned}(d^p\circ c)^\wedge(0,s) &= d^p(c(0))(s) = d^p((\partial_1 f)^\vee(0))(s)\\ &= (\partial_2^p(\partial_1 f))^\vee(0)(s) && \text{by (1)}\\ &= \partial_2^p\partial_1 f(0,s)\\ &= \partial_1\partial_2^p f(0,s) && \text{by the theorem of Schwarz.}\end{aligned}$$

So we see that $(d^p\circ c)^\wedge(t,s) = \int_0^1 \partial_1\partial_2^p f(t\,\tau,s)\,d\tau$ for all (t,s). This function is continuous in (t,s), since $\partial_1\partial_2^p f : \mathbb{R}^2\to\mathbb{R}$ is assumed to be continuous, hence $(t,s,\tau)\mapsto \partial_1\partial_2^p f(t\,\tau,s)$ is continuous, and therefore also $(t,s)\mapsto(\tau\mapsto\partial_1\partial_2^p f(t\,\tau,s))$ from $\mathbb{R}^2\to C([0,1],\mathbb{R})$ by 3.1. Composition with the continuous linear mapping $\int_0^1 : C([0,1],\mathbb{R})\to\mathbb{R}$ gives the continuity of $(d^p\circ c)^\wedge$.

Now we proceed by induction. By the induction hypothesis applied to $\partial_1 f$, we obtain that $d(f^\vee) = (\partial_1 f)^\vee$ and $(\partial_1 f)^\vee : \mathbb{R}\to C^\infty(\mathbb{R},\mathbb{R})$ is n times differentiable, so $f^\vee$ is $(n+1)$-times differentiable. □

In order to proceed to more general cases of the exponential law we need a definition of C^∞-maps defined on infinite dimensional spaces. This definition should at least guarantee the chain rule, and so one could take the weakest notion that satisfies the chain rule. However, consider the following

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3.3. Example. We consider the following 3-fold "singular covering" $f : \mathbb{R}^2 \to \mathbb{R}^2$ given in polar coordinates by $(r, \varphi) \mapsto (r, 3\varphi)$. In cartesian coordinates we obtain the following formula for the values of f:

$$\begin{aligned}(r\,\cos(3\varphi), r\sin(3\varphi)) &= r\Big((\cos\varphi)^3 - 3\cos\varphi(\sin\varphi)^2, 3\sin\varphi(\cos\varphi)^2 - (\sin\varphi)^3\Big)\\ &= \left(\frac{x^3-3xy^2}{x^2+y^2}, \frac{3x^2y-y^3}{x^2+y^2}\right).\end{aligned}$$

Note that the composite from the left with any orthonormal projection is just the composite of the first component of f with a rotation from the right (Use that f intertwines the rotation with angle δ and the rotation with angle 3δ).
Obviously, the map f is smooth on $\mathbb{R}^2 \setminus \{0\}$. It is homogeneous of degree 1, and hence the directional derivative is $f'(0)(v) = \frac{\partial}{\partial t}|_{t=0} f(tv) = f(v)$. However, both components are nonlinear with respect to v and thus are not differentiable at $(0, 0)$.

Obviously, $f : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous.

We claim that f is differentiable along differentiable curves, i.e. $(f \circ c)'(0)$ exists, provided $c'(0)$ exists.
Only the case $c(0) = 0$ is not trivial. Since c is differentiable at 0 the curve c_1 defined by $c_1(t) := \frac{c(t)}{t}$ for $t \neq 0$ and $c'(0)$ for $t = 0$ is continuous at 0. Hence $\frac{f(c(t))-f(c(0))}{t} = \frac{f(t\,c_1(t))-0}{t} = f(c_1(t))$. This converges to $f(c_1(0))$, since f is continuous.

Furthermore, if $f'(x)(v)$ denotes the directional derivative, which exists everywhere, then $(f \circ c)'(t) = f'(c(t))(c'(t))$. Indeed for $c(t) \neq 0$ this is clear and for $c(t) = 0$ it follows from $f'(0)(v) = f(v)$.

The directional derivative of the 1-homogeneous mapping f is 0-homogeneous: In fact, for $s \neq 0$ we have

$$f'(sx)(v) = \frac{\partial}{\partial t}\Big|_{t=0} f(s\,x + tv) = s\,\frac{\partial}{\partial t}\Big|_{t=0} f(x + \frac{t}{s}v) = s\,f'(x)(\frac{1}{s}v) = f'(x)(v).$$

For any $s \in \mathbb{R}$ we have $f'(s\,v)(v) = \frac{\partial}{\partial t}|_{t=0} f(s\,v + tv) = \frac{\partial}{\partial t}|_{t=s} t\,f(v) = f(v)$.

Using this homogeneity we show next, that it is also continuously differentiable along continuously differentiable curves. So we have to show that $(f \circ c)'(t) \to (f \circ c)'(0)$ for $t \to 0$. Again only the case $c(0) = 0$ is interesting. As before we factor c as $c(t) = t\,c_1(t)$. In the case, where $c'(0) = c_1(0) \neq 0$ we have for $t \neq 0$ that

$$\begin{aligned}(f \circ c)'(t) - (f \circ c)'(0) &= f'(t\,c_1(t))(c'(t)) - f'(0)(c_1(0))\\ &= f'(c_1(t))(c'(t)) - f'(c_1(0))(c_1(0))\\ &= f'(c_1(t))(c'(t)) - f'(c_1(0))(c'(0)),\end{aligned}$$

which converges to 0 for $t \to 0$, since $(f')^\wedge$ is continuous (and even smooth) on $(\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2$.
In the other case, where $c'(0) = c_1(0) = 0$ we consider first the values of t, for which $c(t) = 0$. Then

$$\begin{aligned}(f \circ c)'(t) - (f \circ c)'(0) &= f'(0)(c'(t)) - f'(0)(c'(0))\\ &= f(c'(t)) - f(c'(0)) \to 0,\end{aligned}$$

since f is continuous. For the remaining values of t, where $c(t) \neq 0$, we factor $c(t) = \|c(t)\|\, e(t)$, with $e(t) \in \{x : \|x\| = 1\}$. Then

$$(f \circ c)'(t) - (f \circ c)'(0) = f'(e(t))(c'(t)) - 0 \to 0,$$

since $f'(x)(c'(t)) \to 0$ for $t \to 0$ uniformly for $\|x\| = 1$, since $c'(t) \to 0$.

Furthermore, $f \circ c$ is smooth for all c which are smooth and nowhere infinitely flat. In fact, a smooth curve c with $c^{(k)}(0) = 0$ for $k < n$ can be factored as $c(t) = t^n c_n(t)$ with smooth c_n, by Taylor's formula with integral remainder. Since $c^{(n)}(0) = n!\, c_n(0)$, we may assume that n is chosen maximal and hence $c_n(0) \neq 0$. But then $(f \circ c)(t) = t^n \cdot (f \circ c_n)(t)$, and $f \circ c_n$ is smooth.

A completely analogous argument shows also that $f \circ c$ is real analytic for all real analytic curves $c : \mathbb{R} \to \mathbb{R}^2$.

However, let us show that $f \circ c$ is not Lipschitz differentiable even for smooth curves c. For $x \neq 0$ we have

$$(\partial_2)^2 f(x, 0) = \left(\tfrac{\partial}{\partial s}\right)^2|_{s=0} f(x, s) = x \left(\tfrac{\partial}{\partial s}\right)^2|_{s=0} f(1, \tfrac{1}{x}s) =$$
$$= \tfrac{1}{x}\left(\tfrac{\partial}{\partial s}\right)^2|_{s=0} f(1, s) =: \tfrac{a}{x} \neq 0.$$

Now we choose a smooth curve c which passes for each n in finite time t_n through $(\frac{1}{n^{2n+1}}, 0)$ with locally constant velocity vector $(0, \frac{1}{n^n})$, by 2.10. Then for small t we get

$$(f \circ c)'(t_n + t) = \partial_1 f(c(t_n + t))\, \underbrace{\mathrm{pr}_1(c'(t_n + t))}_{=0} + \partial_2 f(c(t_n + t))\, \mathrm{pr}_2(c'(t_n + t))$$
$$(f \circ c)''(t_n) = 0 + (\partial_2)^2 f(c(t_n))\, (\mathrm{pr}_2(c'(t_n)))^2 = a\, \frac{n^{2n+1}}{n^{2n}} = n\, a,$$

which is unbounded.

So although preservation of (continuous) differentiability of curves is not enough to ensure differentiability of a function $\mathbb{R}^2 \to \mathbb{R}$, we now prove that smoothness can be tested with smooth curves.

3.4. Boman's theorem. [Boman, 1967] *For a mapping $f : \mathbb{R}^2 \to \mathbb{R}$ the following assertions are equivalent:*

(1) *All iterated partial derivatives exist and are continuous.*
(2) *For $v \in \mathbb{R}^2$ the iterated directional derivatives*
$$d_v^n f(x) := (\tfrac{\partial}{\partial t})^n|_{t=0}(f(x + tv))$$
exist and are continuous with respect to x.
(3) *For $v \in \mathbb{R}^2$ the iterated directional derivatives*
$$d_v^n f(x) := (\tfrac{\partial}{\partial t})^n|_{t=0}(f(x + tv))$$
exist and are locally bounded with respect to x.
(4) *For all smooth curves $c : \mathbb{R} \to \mathbb{R}^2$ the composite $f \circ c$ is smooth.*

Proof.

(1) $\Rightarrow$ (4) is a direct consequence of the classical chain rule, namely that $(f \circ c)'(t) = \partial_1 f(c(t)) \cdot x'(t) + \partial_2 f(c(t)) \cdot y'(t)$, where $c = (x, y)$.

(4) $\Rightarrow$ (3) Obviously, $d_v^p f(x) := (\frac{d}{dt})^p|_{t=0} f(x + tv)$ exists, since $t \mapsto x + tv$ is a smooth curve. Suppose $d_v^p f$ is not locally bounded. So we may find a sequence x_n which converges fast to x, and such that $|d_v^p f(x_n)| \geq 2^{n^2}$. Let c be a smooth

curve with $c(t+t_n) = x_n + \frac{t}{2^n}v$ locally for some sequence $t_n \to 0$, by 2.8. Then $(f\circ c)^{(p)}(t_n) = d_v^p f(x_n)\frac{1}{2^{np}}$ is unbounded, which is a contradiction.

(3) $\Rightarrow$ (2) We prove this by induction on p:

$$d_v^p f(\quad + tv) - d_v^p f(\quad) = t\int_0^1 d_v^{p+1} f(\quad + t\tau v)d\tau \to 0$$

for $t \to 0$ uniformly on bounded sets. Suppose now that $|d_v^p f(x_n) - d_v^p f(x)| \geq \varepsilon$ for some sequence $x_n \to x$. Without loss of generality we may assume that $d_v^p f(x_n) - d_v^p f(x) \geq \varepsilon$. Then by the uniform convergence there exists a $\delta > 0$ such that $d_v^p f(x_n + tv) - d_v^p f(x+tv) \geq \frac{\varepsilon}{2}$ for $|t| \leq \delta$. Integration $\int_0^\delta dt$ yields

$$\Big(d_v^{p-1} f(x_n + \delta v) - d_v^{p-1} f(x_n)\Big) - \Big(d_v^{p-1} f(x + \delta v) - d_v^{p-1} f(x)\Big) \geq \tfrac{\varepsilon\delta}{2},$$

but by induction hypothesis the left hand side converges towards

$$\Big(d_v^{p-1} f(x + \delta v) - d_v^{p-1} f(x)\Big) - \Big(d_v^{p-1} f(x + \delta v) - d_v^{p-1} f(x)\Big) = 0.$$

(2) $\Rightarrow$ (1) We remark now that for a smooth map $g : \mathbb{R}^2 \to \mathbb{R}$ we have by the chain rule

$$d_v g(x+tv) = \frac{d}{dt}g(x+tv) = \partial_1 g(x+tv)\cdot v_1 + \partial_2 g(x+tv)\cdot v_2$$

and by induction that

$$d_v^p g(x) = \sum_{i=0}^p \binom{p}{i} v_1^i v_2^{p-i} \partial_1^i \partial_2^{p-i} g(x).$$

Hence, we can calculate the iterated derivatives $\partial_1^i\partial_2^{p-i}g(x)$ for $0 \leq i \leq p$ from $p+1$ many derivatives $d_{v^j}^p g(x)$ provided the v^j are chosen in such a way, that the *Vandermonde's determinant* $\det((v_1^j)^i(v_2^j)^{p-i})_{ij} \neq 0$. For this choose $v_2 = 1$ and all the v_1 pairwise distinct, then $\det((v_1^j)^i(v_2^j)^{p-i})_{ij} = \prod_{j>k}(v_1^j - v_1^k) \neq 0$.

To complete the proof we use convolution by an *approximation of unity*. So let $\varphi \in C^\infty(\mathbb{R}^2,\mathbb{R})$ have compact support, $\int_{\mathbb{R}^2}\varphi = 1$, and $\varphi(y) \geq 0$ for all y. Define $\varphi_\varepsilon(x) := \frac{1}{\varepsilon^2}\varphi(\frac{1}{\varepsilon}x)$, and let

$$f_\varepsilon(x) := (f\star\varphi_\varepsilon)(x) = \int_{\mathbb{R}^2} f(x-y)\,\varphi_\varepsilon(y)\,dy = \int_{\mathbb{R}^2} f(x-\varepsilon y)\varphi(y)dy.$$

Since the *convolution* $f_\varepsilon := f\star\varphi_\varepsilon$ of a continuous function f with a smooth function φ_ε with compact support is differentiable with directional derivative $d_v(f\star\varphi_\varepsilon)(x) = (f\star d_v\varphi_\varepsilon)(x)$, we obtain that f_ε is smooth. And since $f\star\varphi_\varepsilon \to f$ in $C(\mathbb{R}^2,\mathbb{R})$ for $\varepsilon \to 0$ and any continuous function f, we conclude $d_v^p f_\varepsilon = d_v^p f \star \varphi_\varepsilon \to d_v^p f$ uniformly on compact sets.

By what we said above for smooth g, the iterated derivatives of f_ε are linear combinations of the derivatives $d_v^p f_\varepsilon$ for $p+1$ many vectors v, where the coefficients depend only on the v's. So we conclude that the iterated partial derivatives of f_ε form a Cauchy sequence in $C(\mathbb{R}^2,\mathbb{R})$, and hence converge to continuous functions f^α. Thus, all iterated derivatives $\partial^\alpha f$ of f exist and are equal to these continuous functions f^α, by the following lemma 3.5 recursively applied to $c_\varepsilon(s) := \partial^\alpha f_\varepsilon(x + s\,v)$. $\square$

3.5. Lemma. *Let $c_\varepsilon : \mathbb{R} \to E$ be C^1 into a locally convex space E such that $c_\varepsilon \to c$ and $c_\varepsilon' \to c^1$ uniformly on bounded subsets of $\mathbb{R}$ for $\varepsilon \to 0$. Then $c : \mathbb{R} \to E$ is C^1 and $c' = c^1$. With other words, the injection $c \mapsto (c, c')$, $C^1(\mathbb{R}, E) \to \ell^\infty(\mathbb{R}, E)^2$ has closed image.*

Proof. Since $C(\mathbb{R},E)$ is closed in $\ell^\infty(\mathbb{R},E)$ the curves c and c^1 are continuous, Remains to show that for fixed $s\in\mathbb{R}$ the curve

$$\gamma : t \mapsto \begin{cases} \frac{c(s+t)-c(s)}{t} & \text{for } t\neq 0 \\ c^1(s) & \text{otherwise} \end{cases}$$

is continuous (at 0). The corresponding curve γ_ε for c_ε can be rewritten as $\gamma_\varepsilon(t) = \int_0^1 c_\varepsilon'(s+\tau\, t)\, d\tau$, which converges by assumption for $\varepsilon\to 0$ uniformly on compact sets to the continuous curve $t\mapsto \int_0^1 c^1(s+\tau\, t)\, d\tau$. Pointwise it converges to $\gamma(t)$, hence γ is continuous. □

For the vector valued case of the exponential law we need a locally convex structure on $C^\infty(\mathbb{R},E)$.

3.6. Definition. Space of curves. Let $C^\infty(\mathbb{R},E)$ be the locally convex vector space of all smooth curves in E, with the pointwise vector operations, and with the topology of uniform convergence on compact sets of each derivative separately. This is the initial topology with respect to the linear mappings $C^\infty(\mathbb{R},E) \xrightarrow{d^k} C^\infty(\mathbb{R},E) \to \ell^\infty(K,E)$, where k runs through $\mathbb{N}$, where K runs through all compact subsets of $\mathbb{R}$, and where $\ell^\infty(K,E)$ carries the topology of uniform convergence, see also 2.15.

Note that the derivatives $d^k : C^\infty(\mathbb{R},E)\to C^\infty(\mathbb{R},E)$, the point evaluations $\mathrm{ev}_t : C^\infty(\mathbb{R},E)\to E$ and the pull backs $g^* : C^\infty(\mathbb{R},E)\to C^\infty(\mathbb{R},E)$ for all $g\in C^\infty(\mathbb{R},\mathbb{R})$ are continuous and linear. For the later one uses that obviously $g^* : \ell^\infty(Y,E)\to \ell^\infty(X,E)$ is continuous for bounded mappings $g : X\to Y$.

3.7. Lemma. *A space E is c^∞-complete if and only if $C^\infty(\mathbb{R},E)$ is so.*

Proof. ($\Rightarrow$) The mapping $c\mapsto (c^{(n)})_{n\in\mathbb{N}}$ is by definition an embedding of $C^\infty(\mathbb{R},E)$ into the c^∞-complete product $\prod_{n\in\mathbb{N}}\ell^\infty(\mathbb{R},E)$. Its image is a closed subspace by lemma 3.5.

($\Leftarrow$) Consider the continuous linear mapping const : $E\to C^\infty(\mathbb{R},E)$ given by $x\mapsto (t\mapsto x)$. It has as continuous left inverse the evaluation at any point (e.g. $\mathrm{ev}_0 : C^\infty(\mathbb{R},E)\to E$, $c\mapsto c(0)$). Hence, E can be identified with the closed subspace of $C^\infty(\mathbb{R},E)$ given by the constant curves, and is thereby itself c^∞-complete. □

3.8. Lemma. Curves into limits. *A curve into a c^∞-closed subspace of a space is smooth if and only if it is smooth into the total space. In particular, a curve is smooth into a projective limit if and only if all its components are smooth.*

Proof. Since the derivative of a smooth curve is the Mackey limit of the difference quotient, the c^∞-closedness implies that this limit belongs to the subspace. Thus, we deduce inductively that all derivatives belong to the subspace, and hence the curve is smooth into the subspace.

The result on projective limits now follows, since obviously a curve is smooth into a product, if all its components are smooth. □

We show now that the bornology, but obviously not the topology, on function spaces can be tested with the linear functionals on the range space.

3.9. Lemma. Bornology of $C^\infty(\mathbb{R},E)$. *The family*

$$\{\ell_* : C^\infty(\mathbb{R},E)\to C^\infty(\mathbb{R},\mathbb{R}) : \ell\in E^*\}$$

generates the bornology of $C^\infty(\mathbb{R},E)$. This also holds for E^* replaced by E'.

A set in $C^\infty(\mathbb{R}, E)$ is bounded if and only if each derivative is uniformly bounded on compact subsets.

Proof. A set $\mathcal{B} \subseteq C^\infty(\mathbb{R}, E)$ is bounded if and only if the sets $\{d^n c(t) : t \in I, c \in \mathcal{B}\}$ are bounded in E for all $n \in \mathbb{N}$ and compact subsets $I \subset \mathbb{R}$.

This is furthermore equivalent to the condition that the set $\{\ell(d^n c(t)) = d^n(\ell \circ c)(t) : t \in I, c \in \mathcal{B}\}$ is bounded in $\mathbb{R}$ for all $\ell \in E^*$, $n \in \mathbb{N}$, and compact subsets $I \subset \mathbb{R}$ and in turn equivalent to: $\{\ell \circ c : c \in \mathcal{B}\}$ is bounded in $C^\infty(\mathbb{R}, \mathbb{R})$.

For E^* replaced by $E' \supseteq E^*$ the statement holds, since ℓ_* is bounded for all $\ell \in E'$ by the explicit description of the bounded sets. □

3.10. Proposition. Vector valued simplest exponential law. *For a mapping $f : \mathbb{R}^2 \to E$ into a locally convex space (which need not be c^∞-complete) the following assertions are equivalent:*

(1) *f is smooth along smooth curves.*
(2) *All iterated directional derivatives $d_v^p f$ exist and are locally bounded.*
(3) *All iterated partial derivatives $\partial^\alpha f$ exist and are locally bounded.*
(4) *$f^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, E)$ exists as a smooth curve.*

Are the coordinate axes for v in (2) enough?

Proof. We prove this result first for c^∞-complete spaces E.

We could do this either by carrying over the proofs of 3.2 and 3.4 to the vector valued situation, or we reduce the vector valued one by linear functionals to the scalar valued situation. We choose here the second way.

Each of the statements (1-4) is valid if and only if the corresponding statement with f replaced by $\ell \circ f$ is valid for all $\ell \in E^*$. Only (4) needs some arguments: In fact, $f^\vee(t) \in C^\infty(\mathbb{R}, E)$ if and only if $\ell_*(f^\vee(t)) = (\ell \circ f)^\vee(t) \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $\ell \in E^*$ by 2.14. Since $C^\infty(\mathbb{R}, E)$ is c^∞-complete, its bornologically isomorphic image in $\prod_{\ell \in E^*} C^\infty(\mathbb{R}, \mathbb{R})$ is c^∞-closed. So $f^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, E)$ is smooth if and only if $\ell_* \circ f^\vee = (\ell \circ f)^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$ is smooth for all $\ell \in E^*$. Note, that local boundedness of all iterated partial derivatives is equivalent to their continuity, since if for a function g the partial derivatives $\partial_1 g$ and $\partial_2 g$ exist and are locally bounded then g is continuous:

$$\begin{aligned} g(x,y) - g(0,0) &= g(x,y) - g(x,0) + g(x,0) - g(0,0) \\ &= y\partial_2 g(x, r_2 y) + x\partial_1 g(r_1 x, 0) \end{aligned}$$

for suitable r_1, $r_2 \in [0,1]$, which goes to 0 with (x,y). So the proof is reduced to the scalar valid case, which was proved in 3.2 and 3.4.

Now the general case. For the existence of certain derivatives we know by 1.9 that it is enough that we have some candidate in the space, which is the corresponding derivative of the map considered as map into the c^∞-completion (or even some larger space). Since the derivatives required in (1-4) depend linearly on each other, this is true. In more detail this means:

(1) ⇒ (2) is obvious.

(2) ⇒ (3) is the fact that ∂^α is a universal linear combination of $d_v^{|\alpha|} f$.

(3) ⇒ (1) follows from the chain rule which says that $(f \circ c)^{(p)}(t)$ is a universal linear combination of $\partial_{i_1} \dots \partial_{i_q} f(c(t)) c_{i_1}^{(p_1)}(t) \dots c_{i_q}^{(p_q)}(t)$ for $i_j \in \{1, 2\}$ and $\sum p_j = p$, see also 10.4.

(3) ⇔ (4) holds by 1.9 since $(\partial_1 f)^\vee = d(f^\vee)$ and $(\partial_2 f)^\vee = d \circ f^\vee = d_*(f^\vee)$. □

3.11. For the general case of the exponential law we need a notion of smooth mappings and a locally convex topology on the corresponding function spaces. Of course, it would be also handy to have a notion of smoothness for locally defined mappings. Since the idea is to test smoothness with smooth curves, such curves should have locally values in the domains of definition, and hence these domains should be c^∞-open.

Definition. Smooth mappings and spaces of them. A mapping $f : E \supseteq U \to F$ defined on a c^∞-open subset U is called *smooth* (or C^∞) if it maps smooth curves in U to smooth curves in F.

Let $C^\infty(U, F)$ denote the locally convex *space of all smooth mappings* $U \to F$ with pointwise linear structure and the initial topology with respect to all mappings $c^* : C^\infty(U, F) \to C^\infty(\mathbb{R}, F)$ for $c \in C^\infty(\mathbb{R}, U)$.

For $U = E = \mathbb{R}$ this coincides with our old definition. Obviously, any composition of smooth mappings is also smooth.

Lemma. *The space $C^\infty(U, F)$ is the (inverse) limit of spaces $C^\infty(\mathbb{R}, F)$, one for each $c \in C^\infty(\mathbb{R}, U)$, where the connecting mappings are pull backs g^* along reparameterizations $g \in C^\infty(\mathbb{R}, \mathbb{R})$.*

Note that this limit is the closed linear subspace in the product

$$\prod_{c \in C^\infty(\mathbb{R}, U)} C^\infty(\mathbb{R}, F)$$

consisting of all (f_c) with $f_{c\circ g} = f_c \circ g$ for all c and all $g \in C^\infty(\mathbb{R}, \mathbb{R})$.

Proof. The mappings $c^* : C^\infty(U, F) \to C^\infty(\mathbb{R}, F)$ define a continuous linear embedding $C^\infty(U, F) \to \lim_c C^\infty(\mathbb{R}, F)$, since for the connecting mappings g^* we have $c^*(f) \circ g = f \circ c \circ g = (c \circ g)^*(f)$. It is surjective since for any $(f_c) \in \lim_c C^\infty(\mathbb{R}, F)$ one has $f_c = f \circ c$ where f is defined as $x \mapsto f_{\mathrm{const}_x}(0)$. $\square$

3.12. Theorem. Cartesian closedness. *Let $U_i \subseteq E_i$ be c^∞-open subsets in locally convex spaces, which need not be c^∞-complete. Then a mapping $f : U_1 \times U_2 \to F$ is smooth if and only if the canonically associated mapping $f^\vee : U_1 \to C^\infty(U_2, F)$ exists and is smooth.*

Proof. We have the following implications:

$f^\vee : U_1 \to C^\infty(U_2, F)$ is smooth.
$\Leftrightarrow$ $f^\vee \circ c_1 : \mathbb{R} \to C^\infty(U_2, F)$ is smooth for all smooth curves c_1 in U_1, by 3.11.
$\Leftrightarrow$ $c_2^* \circ f^\vee \circ c_1 : \mathbb{R} \to C^\infty(\mathbb{R}, F)$ is smooth for all smooth curves c_i in U_i, by 3.11 and 3.8.
$\Leftrightarrow$ $f \circ (c_1 \times c_2) = (c_2^* \circ f^\vee \circ c_1)^\wedge : \mathbb{R}^2 \to F$ is smooth for all smooth curves c_i in U_i, by 3.10.
$\Leftrightarrow$ $f : U_1 \times U_2 \to F$ is smooth.

Here the last equivalence is seen as follows: Each curve into $U_1 \times U_2$ is of the form $(c_1, c_2) = (c_1 \times c_2) \circ \Delta$, where Δ is the diagonal mapping. Conversely, $f \circ (c_1 \times c_2) : \mathbb{R}^2 \to F$ is smooth for all smooth curves c_i in U_i, since the product and the composite of smooth mappings is smooth by 3.11 (and by 3.4). $\square$

3.13. Corollary. Consequences of cartesian closedness. *Let E, F, G, etc. be locally convex spaces, and let U, V be c^∞-open subsets of such. Then the following canonical mappings are smooth.*

(1) $\mathrm{ev} : C^\infty(U,F)\times U \to F$, $(f,x)\mapsto f(x)$;
(2) $\mathrm{ins} : E \to C^\infty(F, E\times F)$, $x\mapsto (y\mapsto (x,y))$;
(3) $(\quad)^\wedge : C^\infty(U, C^\infty(V,G)) \to C^\infty(U\times V, G)$;
(4) $(\quad)^\vee : C^\infty(U\times V, G) \to C^\infty(U, C^\infty(V,G))$;
(5) $\mathrm{comp} : C^\infty(F,G)\times C^\infty(U,F) \to C^\infty(U,G)$, $(f,g)\mapsto f\circ g$;
(6) $C^\infty(\quad,\quad) : C^\infty(E_2,E_1)\times C^\infty(F_1,F_2) \to$
$\to C^\infty(C^\infty(E_1,F_1), C^\infty(E_2,F_2))$, $(f,g)\mapsto (h\mapsto g\circ h\circ f)$;
(7) $\prod : \prod C^\infty(E_i,F_i) \to C^\infty(\prod E_i, \prod F_i)$, *for any index set.*

Proof. (1) The mapping associated to *ev* via cartesian closedness is the identity on $C^\infty(U,F)$, which is C^∞, thus *ev* is also C^∞.

(2) The mapping associated to *ins* via cartesian closedness is the identity on $E\times F$, hence *ins* is C^∞.

(3) The mapping associated to $(\quad)^\wedge$ via cartesian closedness is the smooth composition of evaluations $\mathrm{ev}\circ(\mathrm{ev}\times \mathrm{Id}) : (f;x,y)\mapsto f(x)(y)$.

(4) We apply cartesian closedness twice to get the associated mapping $(f;x;y)\mapsto f(x,y)$, which is just a smooth evaluation mapping.

(5) The mapping associated to *comp* via cartesian closedness is $(f,g;x)\mapsto f(g(x))$, which is the smooth mapping $\mathrm{ev}\circ(\mathrm{Id}\times\mathrm{ev})$.

(6) The mapping associated to the one in question by applying cartesian closed is $(f,g,h)\mapsto g\circ h\circ f$, which is appart permutation of the variables the C^∞-mapping $\mathrm{comp}\circ(\mathrm{Id}\times\mathrm{comp})$.

(7) Up to a flip of factors the mapping associated via cartesian closedness is the product of the evaluation mappings $C^\infty(E_i,F_i)\times E_i \to F_i$. □

Next we generalize 3.4 to finite dimensions.

3.14. Corollary. [Boman, 1967]. *The smooth mappings on open subsets of $\mathbb{R}^n$ in the sense of definition 3.11 are exactly the usual smooth mappings.*

Proof. ($\Leftarrow$) is obvious by the usual chain rule.

($\Rightarrow$) Both conditions are of local nature, so we may assume that the open subset of $\mathbb{R}^n$ is an open box and (by reparametrizing with a diffeomorphism in usual sense) even $\mathbb{R}^n$ itself.

If $f:\mathbb{R}^n\to F$ is smooth along smooth curves then by cartesian closedness 3.12, for each coordinate the respective associated mapping $f^{\vee_i} : \mathbb{R}^{n-1}\to C^\infty(\mathbb{R},F)$ is smooth along smooth curves. Moreover the first partial derivative $\partial_i f$ exists and we have $\partial_i f = (d\circ f^{\vee_i})^\wedge$ (c.f. 3.2), so all first partial derivatives exist and are smooth along smooth curves. Inductively, all iterated partial derivatives exist and are smooth along smooth curves, thus continuous, so f is smooth in the usual sense. □

3.15. Differentiation of an integral. We return to the question of differentiating an integral. So let $f : E\times\mathbb{R}\to F$ be smooth, and let $\widehat{F}$ be the completion of the locally convex space F. Then we may form the function $f_0 : E\to\widehat{F}$ defined by $x\mapsto \int_0^1 f(x,t)\,dt$. We claim that it is smooth, and that the directional derivative is given by $d_v f_0(x) = \int_0^1 d_v(f(\quad,t))(x)\,dt$. By cartesian closedness 3.12 the associated mapping $f^\vee : E\to C^\infty(\mathbb{R},F)$ is smooth, so the mapping $f_0 := \int_0^1 \circ f^\vee : E\to\widehat{F}$

is smooth since integration is a bounded linear operator, and

$$\begin{aligned}
d_v f_0(x) &= \tfrac{\partial}{\partial s}\big|_{s=0} f_0(x+sv) = \tfrac{\partial}{\partial s}\big|_{s=0} \Big(\int_0^1 \circ f^\vee\Big)(x+sv)\\
&= \int_0^1 \Big(\tfrac{\partial}{\partial s}\big|_{s=0} f^\vee(x+sv)\Big)(t)\,dt = \int_0^1 \mathrm{ev}_t\Big(\tfrac{\partial}{\partial s}\big|_{s=0} f^\vee(x+sv)\Big)\,dt\\
&= \int_0^1 \tfrac{\partial}{\partial s}\big|_{s=0}\Big(\mathrm{ev}_t\Big(f^\vee(x+sv)\Big)\Big)\,dt = \int_0^1 \tfrac{\partial}{\partial s}\big|_{s=0} f(x+sv,t)dt\\
&= \int_0^1 d_v(f(\quad,t))(x)\,dt.
\end{aligned}$$

We want to generalize this to functions f defined only on some c^∞-open subset $U \subseteq E\times\mathbb{R}$, so we have to show that the natural domain $U_0 := \{x \in E : \{x\}\times[0,1] \subseteq U\}$ of f_0 is c^∞-open in E. We will do this in lemma 4.15. From then on the proof runs exactly the same way as for globally defined functions, since for $x_0 \in U_0$ there exists a bounded open interval $J \supseteq [0,1]$ such that $\{x_0\}\times\overline{J} \subseteq U$ and hence $f^\vee$ is defined on a c^∞-neighborhood of x_0 and smooth into $C^\infty(J,F) \to C([0,1],F)$. So we obtain the

Proposition. *Let $f : E\times\mathbb{R} \supseteq U \to F$ be smooth with c^∞-open $U \subseteq E\times\mathbb{R}$. Then $x \mapsto \int_0^1 f(x,t)\,dt$ is smooth on the c^∞-open set $U_0 := \{x \in E : \{x\}\times[0,1] \subseteq U\}$ with values in the completion $\widehat{F}$ and $d_v f_0(x) = \int_0^1 d_v(f(\quad,t))(x)\,dt$ for all $x \in U_0$ and $v \in E$.* □

Now we want to define the derivative of a general smooth map and prove the chain rule for them.

3.16. Corollary. Smoothness of the difference quotient. *For a smooth curve $c : \mathbb{R} \to E$ the difference quotient*

$$(t,s) \mapsto \begin{cases} \dfrac{c(t)-c(s)}{t-s} & \text{for } t \neq s\\ c'(t) & \text{for } t = s\end{cases}$$

is a smooth mapping $\mathbb{R}^2 \to E$. Cf. 1.7 and 2.1.

Proof. By 2.5 we have $f : (t,s) \mapsto \frac{c(t)-c(s)}{t-s} = \int_0^1 c'(s+r(t-s))\,dr$, and by 3.15 it is smooth $\mathbb{R}^2 \to \widehat{E}$. The left hand side has values in E, and for $t \neq s$ this is also true for all iterated directional derivatives. It remains to consider the derivatives for $t = s$. The iterated directional derivatives are given by 3.15 as

$$\begin{aligned}
d^p_{(v,w)} f(t,s) &= d^p_{(v,w)} \int_0^1 c'(\underbrace{s+r(t-s)}_{rt+(1-r)s})\,dr\\
&= \int_0^1 \left(\tfrac{d}{du}\right)^p\big|_{u=0} c'(\underbrace{r\,(t+u\,v)+(1-r)\,(s+u\,w)}_{u\,(r\,v+(1-r)\,w)+(r\,t+(1-r)\,s})\,dr\\
&= \int_0^1 (r\,v+(1-r)\,w)^p\, c^{(p+1)}(r\,t+(1-r)\,s)\,dr
\end{aligned}$$

The later integrand is for $t = s$ just $\int_0^1 (r\,v+(1-r)\,w)^p\,dr\, c^{(p+1)}(t) \in E$. Hence $d^p_{(v,w)} f(t,s) \in E$. By 3.10 the mapping f is smooth into E. □

3.17. Definition. Spaces of linear mappings. Let $L(E,F)$ denote the *space of all bounded* (equivalently smooth by 2.11) *linear mappings* from E to F. It is a closed linear subspace of $C^\infty(E,F)$ since f is linear if and only if for all x, $y\in E$ and $\lambda\in\mathbb{R}$ we have $(\operatorname{ev}_x+\lambda\operatorname{ev}_y-\operatorname{ev}_{x+\lambda y})f=0$. We equip it with this topology and linear structure.

So a mapping $f:U\to L(E,F)$ is smooth if and only if the composite mapping $U\xrightarrow{f}L(E,F)\to C^\infty(E,F)$ is smooth.

3.18. Theorem. Chain rule. *Let E and F be locally convex spaces, and let $U\subseteq E$ be c^∞-open. Then the differentiation operator*

$$d:C^\infty(U,F)\to C^\infty(U,L(E,F)),$$
$$df(x)v:=\lim_{t\to 0}\frac{f(x+tv)-f(x)}{t},$$

exists, is linear and bounded (smooth). Also the chain rule holds:

$$d(f\circ g)(x).v=df(g(x)).dg(x).v.$$

Proof. Since $t\mapsto x+tv$ is a smooth curve we know that $d^{\wedge\wedge}:C^\infty(U,F)\times U\times E\to F$ exists. We want to show that it is smooth, so let $(f,x,v):\mathbb{R}\to C^\infty(U,F)\times U\times E$ be a smooth curve. Then

$$d^{\wedge\wedge}(f(t),x(t),v(t))=\lim_{s\to 0}\frac{f(t)(x(t)+sv(t))-f(t)(x(t))}{s}=\partial_2 h(t,0),$$

which is smooth in t, where the smooth mapping $h:\mathbb{R}^2\supseteq\{(t,s):x(t)+sv(t)\in U\}\to F$ is given by $(t,s)\mapsto f^\wedge(t,x(t)+sv(t))$. By cartesian closedness 3.12 the mapping $d^\wedge:C^\infty(U,F)\times U\to C^\infty(E,F)$ is smooth.

Now we show that this mapping has values in the subspace $L(E,F)$: $d^\wedge(f,x)$ is obviously homogeneous. It is additive, because we may consider the smooth mapping $(t,s)\mapsto f(x+tv+sw)$ and compute as follows, using 3.14.

$$\begin{aligned}df(x)(v+w)&=\tfrac{\partial}{\partial t}\big|_0 f(x+t(v+w))\\&=\tfrac{\partial}{\partial t}\big|_0 f(x+tv+0w)+\tfrac{\partial}{\partial t}\big|_0 f(x+0v+tw)=df(x)v+df(x)w.\end{aligned}$$

So we see that $d^\wedge:C^\infty(U,F)\times U\to L(E,F)$ is smooth, and the mapping $d:C^\infty(U,F)\to C^\infty(U,L(E,F))$ is smooth by 3.12 and obviously linear.

We first prove the chain rule for a smooth curve c instead of g. We have to show that the curve

$$t\mapsto\begin{cases}\frac{f(c(t))-f(c(0))}{t} & \text{for } t\neq 0\\ df(c(0)).c'(0) & \text{for } t=0\end{cases}$$

is continuous at 0. It can be rewritten as $t\mapsto\int_0^1 df(c(0)+s(c(t)-c(0))).c_1(t)\,ds$, where c_1 is the (by 3.16) smooth curve given by

$$t\mapsto\begin{cases}\frac{c(t)-c(0)}{t} & \text{for } t\neq 0\\ c'(0) & \text{for } t=0\end{cases}.$$

Since $h:\mathbb{R}^2\to E\times E$ given by

$$(t,s)\mapsto(c(0)+s(c(t)-c(0)),c_1(t))$$

is smooth, there exist open neighborhoods I of $[0,1]$ and J of 0 in $\mathbb{R}$ such that map $t\mapsto\Big(s\mapsto df(c(0)+s(c(t)-c(0))).c_1(t)\Big)$ is smooth $J\to C^\infty(I,F)$, and hence $t\mapsto\int_0^1 df(c(0)+s(c(t)-c(0))).c_1(t)\,ds$ is smooth as in 3.15, and hence continuous.

For general g we have

$$\begin{aligned} d(f\circ g)(x)(v) &= \tfrac{\partial}{\partial t}\big|_0 (f\circ g)(x+tv) = (df)(g(x+0v))(\tfrac{\partial}{\partial t}\big|_0 (g(x+tv))) \\ &= (df)(g(x))(dg(x)(v)). \quad \square \end{aligned}$$

3.19. Lemma. *Two locally convex spaces are locally diffeomorphic if and only if they are linearly diffeomorphic.*
Any smooth and 1-homogeneous mapping is linear.

Proof. By the chain rule the derivatives at corresponding points give the linear diffeomorphisms.

For a 1-homogeneous mapping f one has $df(0)v = \frac{\partial}{\partial t}\big|_0 f(tv) = f(v)$, and this is linear in v. $\square$

4. The c^∞-Topology

4.1. Definition. *A locally convex vector space E is called bornological if and only if the following equivalent conditions are satisfied:*

(1) *Any bounded linear mapping $T : E \to F$ in any locally convex space F is continuous; It is sufficient to know this for all Banach spaces F.*
(2) *Every bounded seminorm on E is continuous.*
(3) *Every absolutely convex bornivorous subset is a 0-neighborhood.*

A *radial subset* U (i.e. $[0,1]U \subseteq U$) of a locally convex space E is called *bornivorous* if it *absorbs* each bounded set, i.e. for every bounded B there exists $r > 0$ such that $rU \supseteq B$.

We will make use of the following simple fact: Let $A, B \subseteq E$ be subsets of a vector space E with A absolutely convex. Then A absorbs B if and only if the Minkowski-funktional p_A is bounded on B.

Proof.

($1 \Rightarrow 2$) Let p be a bounded seminorm. Then the canonical projection $T : E \to E/\ker p \subseteq \widehat{E/\ker p}$ is bounded and hence continuous by (1). Hence, $p = \tilde{p}\circ T$ is continuous, where $\tilde{p}$ denotes the canonical norm on the completion $\widehat{E/\ker p}$ induced from p.

($2 \Rightarrow 3$), since the Minkowski-functional p generated by an absolutely convex bornivorous subset is a bounded seminorm.

($3 \Rightarrow 1$) Let $T : E \to F$ be bounded linear and $V \subseteq F$ be a absolutely convex 0-neighborhood. Then $T^{-1}(V)$ is absolutely convex and bornivorous, thus by (3) a 0-neighborhood, i.e. T is continuous. $\square$

4.2. Lemma. Bornologification. *The bornologification E_{born} of a locally convex space can be described in the following equivalent ways:*

(1) *It is the finest locally convex structure having the same bounded sets;*
(2) *It is the final locally convex structure with respect to the inclusions $E_B \to E$, where B runs through all bounded (closed) absolutely convex subsets.*

Moreover, E_{born} is bornological. For any locally convex vector space F the continuous linear mappings $E_{born} \to F$ are exactly the bounded linear mappings $E \to F$. The continuous seminorms on E_{born} are exactly the bounded seminorms of E. An absolutely convex set is a 0-neighborhood in E_{born} if and only if it is bornivorous, i.e. absorbs bounded sets.

Proof. Let E_{born} be the vector space E supplied with the topology described in (1) and E_{fin} be E supplied with the final toplogy described in (2).

($E_{\text{fin}} \to E_{\text{born}}$ is continuous), since all bounded absolutely convex sets B in E are bounded in E_{born}, thus the inclusions $E_B \to E_{\text{born}}$ are bounded and hence continuous since E_B is normed. Thus, the final structure on E induced by the inclusions $E_B \to E$ is finer than the structure of E_{born}.

($E_{\text{born}} \to E_{\text{fin}}$ is continuous). It is obviously bounded, since the construction the bounded subsets B of E_{born} are bounded in E, hence contained in bounded absolutely convex $B \subseteq E$ and hence bounded in $E_B \to E_{\text{fin}}$.
Hence, E_{fin} has exactly the same bounded sets as E, and so E_{born} is by definition finer than E_{fin}.

$E_{\text{born}} = E_{\text{fin}}$ is bornological by (1) in 4.1: Let $T : E \to F$ be bounded linear, then $T|_{E_B} : E_B \to E \to F$ is bounded and hence $T : E_{\text{fin}} \to F$ is continuous.

The remaining statements now follow, since E_{born} and E have the same bounded seminorms, the same bounded linear mappings with values in locally convex spaces and the same bornivorous absolutely convex subsets. And on the bornological space E_{born} these are by 4.1 exactly the continuous seminorms, the continuous linear mappings and the absolutely convex 0-neighborhoods. □

4.3. Corollary. Bounded seminorms. *For a seminorm p and a sequence $\mu_n \to \infty$ the following statements are equivalent:*

(1) *p is bounded;*
(2) *p is bounded on compact sets;*
(3) *p is bounded on M-converging sequences;*
(4) *p is bounded on μ-converging sequences;*
(5) *p is bounded on images of bounded intervals under $\mathcal{L}\text{ip}^k$-curves (for fixed $0 \le k \le \infty$).*

The corresponding statement for subsets of E is the following. For a RADIAL *subset $U \subseteq E$ (i.e., $[0,1] \cdot U \subseteq U$) the following properties are equivalent:*

(1) *U is bornivorous.*
(1') *For all absolutely convex bounded sets B, the trace $U \cap E_B$ is a 0-neighborhood in E_B.*
(2) *U absorbs all compact subsets in E.*
(3) *U absorbs all Mackey convergent sequences.*
(3') *U absorbs all sequences converging Mackey to 0.*
(4) *U absorbs all μ-convergent sequences (for a fixed μ).*
(4') *U absorbs all sequences which are μ-converging to 0.*
(5) *U absorbs the images of bounded sets under $\mathcal{L}\text{ip}^k$-curves (for a fixed $0 \le k \le \infty$).*

Proof. We prove the statement on radial subsets, for seminorms p it then follows since p is bounded on a subset $A \subseteq E$ if and only if the radial set $U := \{x \in E : p(x) \le 1\}$ absorbs A (using the equality $K \cdot U = \{x \in E : p(x) \le K\}$).

($\boxed{1}$') $\Leftrightarrow$ ($\boxed{1}$) $\Rightarrow$ ($\boxed{2}$) $\Rightarrow$ ($\boxed{3}$) $\Rightarrow$ ($\boxed{4}$) $\Rightarrow$ ($\boxed{4}$'), ($\boxed{3}$) $\Rightarrow$ ($\boxed{3}$') $\Rightarrow$ ($\boxed{4}$'), ($\boxed{2}$) $\Rightarrow$ ($\boxed{5}$), are trivial.

($\boxed{5}$) $\Rightarrow$ ($\boxed{4}$') Suppose that (x_n) is μ-converging to 0 but is not absorbed by U. Then for each $m \in \mathbb{N}$ there is an $n_m \in \mathbb{N}$ with $x_{n_m} \notin mU$ and by passing to a subsequence $(n_{m_k})_k$ of $(n_m)_m$ we may assume that $k \mapsto 1/\mu_{n_{m_k}}$ is fast falling. The sequence $(x_{n_{m_k}} = \frac{1}{\mu_{n_{m_k}}} \mu_{n_{m_k}} x_{n_{m_k}})_k$ is then fast falling and lies on some compact part of a smooth curve by the special curve lemma $\boxed{2.8}$. The set U absorbs this by ($\boxed{5}$), a contradiction to $x_{m_{m_k}} \notin m_k U$ with $m_k \geq k \to \infty$.

($\boxed{4}$') $\Rightarrow$ ($\boxed{1}$) Suppose U does not absorb some bounded B. Hence, there are $b_n \in B$ with $b_n \notin \mu_n^2 U$. However, $\frac{b_n}{\mu_n}$ is μ-convergent to 0, so it is contained in KU for some $K > 0$. Equivalently, $b_n \in \mu_n KU \subseteq \mu_n^2 U$ for all $\mu_n \geq K$, which gives a contradiction. □

4.4. Corollary. Bornologification as locally-convex-ification.
The bornologification of E is the finest locally convex topology with one (hence all) of the following properties:

(1) *It has the same bounded sets as E.*
(2) *It has the same Mackey converging sequences as E.*
(3) *It has the same μ-converging sequences as E (for some fixed μ).*
(4) *It has the same $\mathcal{L}\mathrm{ip}^k$-curves as E (for some fixed $0 \leq k \leq \infty$).*
(5) *It has the same bounded linear mappings from E into arbitrary locally convex spaces.*
(6) *It has the same continuous linear mappings from normed spaces into E.*

Proof. Since the bornologification has the same bounded sets as the original topology, the other objects are also the same: they depend only on the bornology – this would not be true for compact sets, e.g. the bornologification of the topology of pointwise convergence on the dual of any infinite dimensional Banach space is that of uniform convergence on the unit ball, but the dual unit ball is only compact for the former.

Conversely, we consider a topology τ which has for one of the above mentioned types the same objects as the original one. Then τ has by $\boxed{4.3}$ the same bornivorous absolutely convex subsets as the original one. Hence, any 0-neighborhood of τ has to be bornivorous for the original topology, and hence is by $\boxed{4.2}$ a 0-neighborhood of the bornologification of the original topology. □

4.5. Lemma. *Let E be a bornological locally convex vector space, $U \subseteq E$ a convex subset. Then U is open for the locally convex topology of E if and only if U is open for the c^∞-topology.*
Furthermore, an absolutely convex subset U of E is a 0-neighborhood for the locally convex topology if and only if it is so for the c^∞-topology.

Proof. ($\Rightarrow$) The c^∞-topology is finer than the locally convex topology, cf. $\boxed{4.2}$.

($\Leftarrow$) Let first U be an absolutely convex 0-neighborhood for the c^∞-topology. Hence, U absorbs Mackey-0-sequences by $\boxed{2.13}$. By $\boxed{4.1.3}$ we have to show that U is bornivorous, in order to obtain that U is a 0-neighborhood for the locally convex topology. But this follows immediately from $\boxed{4.3}$.

Let now U be convex and c^∞-open, let $x \in U$ be arbitrary. We consider the c^∞-open absolutely convex set $W := (U - x) \cap (x - U)$ which is a 0-neighborhood of the locally convex topology by the argument above. Then $x \in W + x \subseteq U$. So U is open in the locally convex topology. □

4.6. Corollary. *The bornologification of a locally convex space E is the finest locally convex topology coarser than the c^∞-topology on E.* □

4.7. In 2.12 we defined the c^∞-topology on an arbitrary locally convex space E as the final topology with respect to the smooth curves $c : \mathbb{R} \to E$. Now we will compare the c^∞-topology with other refinements of a given locally convex topology. We first specify those refinements.

Definition. Let E be a locally convex vector space.

(i) We denote by kE the *Kelley-fication* of the locally convex topology of E, i.e. the vector space E together with the final topology induced by the inclusions of the subsets being compact for the locally convex topology.

(ii) We denote by sE the vector space E with the final topology induced by the curves being continuous for the locally convex topology, or equivalently the sequences $\mathbb{N}_\infty \to E$ converging in the locally convex topology. The equivalence holds since the infinite polygon through a converging sequence can be continuously parameterized by a compact interval.

(iii) We recall that by $c^\infty E$ we denote the vector space E with its c^∞-topology, i.e. the final topology induced by the smooth curves.

Using that smooth curves are continuous and that converging sequences $\mathbb{N}_\infty \to E$ have compact images, the following identities are continuous: $c^\infty E \to sE \to kE \to E$.

If the locally convex topology of E coincides with the topology of $c^\infty E$, resp. sE, resp. kE then we call E smoothly generated, resp. sequentially generated, resp. compactly generated.

4.8. Example. *On $E = \mathbb{R}^J$ all the refinements of the locally convex topology described in 4.7 above are different, i.e. $c^\infty E \neq sE \neq kE \neq E$, provided the cardinality of the index set J is at least that of the continuum.*

Proof. It is enough to show this for J equipotent to the continuum, since $\mathbb{R}^{J_1}$ is a direct summand in $\mathbb{R}^{J_2}$ for $J_1 \subseteq J_2$.

($c^\infty E \neq sE$) We may take as index set J the set c_0 of all real sequences converging to 0. Define a sequence (x^n) in E by $(x^n)_j := j_n$. Since every $j \in J$ is a 0-sequence we conclude that the x^n converge to 0 in the locally convex topology of the product, hence also in sE. Assume now that the x^n converge towards 0 in $c^\infty E$. Then by 4.9 some subsequence converges Mackey to 0. Thus, there exists an unbounded sequence of reals λ_n with $\{\lambda_n x^n : n \in \mathbb{N}\}$ bounded. Let j be a 0-sequence with $\{j_n \lambda_n : n \in \mathbb{N}\}$ unbounded (e.g. $(j_n)^{-2} := 1 + \max\{|\lambda_k| : k \leq n\}$). Then the j-th coordinate $j_n\lambda_n$ of $\lambda_n x^n$ is not bounded with respect to n, a contradiction.

($sE \neq kE$) Consider in E the subset

$$A := \Big\{x \in \{0,1\}^J : x_j = 1 \text{ for at most countably many } j \in J\Big\}.$$

It is clearly closed with respect to the converging sequences, hence closed in sE. But it is not closed in kE since it is dense in the compact set $\{0,1\}^J$.

($kE \neq E$) Consider in E the subsets

$$A_n := \big\{x \in E : |x_j| < n \text{ for at most } n \text{ many } j \in J\big\}.$$

Each A_n is closed in E since its complement is the union of the open sets $\{x \in E : |x_j| < n \text{ for all } j \in J_o\}$ where J_o runs through all subsets of J with $n+1$ elements. We show that the union $A := \bigcup_{n\in\mathbb{N}} A_n$ is closed in kE. So let K be a compact subset of E; then $K \subseteq \prod \mathrm{pr}_j(K)$, and each $\mathrm{pr}_j(K)$ is compact, hence bounded in $\mathbb{R}$. Since the family $(\{j \in J : \mathrm{pr}_j(K) \subseteq [-n,n]\})_{n\in\mathbb{N}}$ covers J, there has to exist an $N \in \mathbb{N}$ and infinitely many $j \in J$ with $\mathrm{pr}_j(K) \subseteq [-N,N]$. Thus $K \cap A_n = \emptyset$ for all $n > N$, and hence, $A \cap K = \bigcup_{n\le N} A_n \cap K$ is closed. Nevertheless, A is not closed in E, since 0 is in $\bar{A}$ but not in A. □

4.9. c^∞-convergent sequences. By 2.13 every M-convergent sequence gives a continuous mapping $\mathbb{N}_\infty \to c^\infty E$ and hence converges in $c^\infty E$. Conversely, a sequence converging in $c^\infty E$ is not necessarily Mackey convergent, see [Frölicher, Kriegl, 1985]. However, one has the following result.

Lemma. *A sequence (x_n) is convergent to x in the c^∞-topology if and only if every subsequence has a subsequence which is Mackey convergent to x.*

Proof. ($\Leftarrow$) is true for any topological convergence. In fact if x_n would not converge to x, then there would be a neighborhood U of x and a subsequence of x_n which lies outside of U and hence cannot have a subsequence converging to x.

($\Rightarrow$) It is enough to show that (x_n) has a subsequence which converges Mackey to x, since every subsequence of a c^∞-convergent sequence is clearly c^∞-convergent to the same limit. Without loss of generality we may assume that $x \notin A := \{x_n : n \in \mathbb{N}\}$. Hence, A cannot be c^∞-closed, and thus there is a sequence $n_k \in \mathbb{N}$ such that (x_{n_k}) converges Mackey to some point $x' \notin A$. The set $\{n_k : k \in \mathbb{N}\}$ cannot be bounded, and hence we may assume that the n_k are strictly increasing by passing to a subsequence. But then (x_{n_k}) is a subsequence of (x_n) which converges in $c^\infty E$ to x and Mackey to x' hence also in $c^\infty E$. Thus $x' = x$. □

Remark. A consequence of this lemma is, that there is no topology in general having as convergent sequences exactly the M-convergent ones, since this topology obviously would have to be coarser than the c^∞-topology.

One can use this lemma also to show that the c^∞-topology on a locally convex vector space gives a so called *arc-generated vector space*. See [Frölicher, Kriegl, 1988, 2.3.9 and 2.3.13] for a discussion of this.

Let us now describe several important situations where at least some of these topologies coincide. For the proof we will need the following

4.10. Lemma. [Averbukh, Smolyanov, 1968] *For any locally convex space E the following statements are equivalent:*

(1) *The sequential closure of any subset is formed by all limits of sequences in the subset.*
(2) *For any given double sequence $(x_{n,k})$ in E with $x_{n,k}$ convergent to some x_k for $n \to \infty$ and k fixed and x_k convergent to some x, there are strictly increasing sequences $i \mapsto n(i)$ and $i \mapsto k(i)$ with $x_{n(i),k(i)} \to x$ for $i \to \infty$.*

Proof. (1⇒2) Take an $a_0 \in E$ different from $k \cdot (x_{n+k,k} - x)$ and from $k \cdot (x_k - x)$ for all k and n. Define $A := \{a_{n,k} := x_{n+k,k} - \frac{1}{k} \cdot a_0 : n, k \in \mathbb{N}\}$. Then x is in the sequential closure of A, since $x_{n+k,k} - \frac{1}{k} \cdot a_0$ converges to $x_k - \frac{1}{k} \cdot a_0$ as $n \to \infty$, and $x_k - \frac{1}{k} \cdot a_0$ converges to $x - 0 = x$ as $k \to \infty$. Hence, by (1) there has to exist a sequence $i \mapsto (n_i, k_i)$ with a_{n_i,k_i} convergent to x. By passing to a subsequence we may suppose that $i \mapsto k_i$ and $i \mapsto n_i$ are increasing. Assume that $i \mapsto k_i$ is bounded, hence finally constant. Then a subsequence $x_{n_i+k_i,k_i} - \frac{1}{k_i}\cdot a_0$ is converging to $x_k - \frac{1}{k}\cdot a_0 \neq x$ if $i \mapsto n_i$ is unbounded, and to $x_{n+k,k} - \frac{1}{k}\cdot a_0 \neq x$ if $i \mapsto n_i$ is bounded, which both yield a contradiction. Thus, $i \mapsto k_i$ can be chosen strictly increasing. But then

$$x_{n_i+k_i,k_i} = a_{n_i,k_i} + \tfrac{1}{k_i}a_0 \to x.$$

(1) $\Leftarrow$ (2) is obvious. □

4.11. Theorem. *For any bornological vector space E the following implications hold:*

1. (1) *$c^\infty E = E$ provided the closure of subsets in E is formed by all limits of sequences in the subset; hence in particular if E is metrizable.*
2. (2) *$c^\infty E = E$ provided E is the strong dual of a Fréchet Schwartz space;*
3. (3) *$c^\infty E = kE$ provided E is the strict inductive limit of a sequence of Fréchet spaces.*
4. (4) *$c^\infty E = sE$ provided E satisfies the M-convergence condition, i.e. every sequence converging in the locally convex topology is M-convergent.*
5. (5) *$sE = E$ provided E is the strong dual of a Fréchet Montel space;*

Proof. (1) Using the lemma 4.10 above one obtains that the closure and the sequential closure coincide, hence $sE = E$. It remains to show that $sE \to c^\infty E$ is (sequentially) continuous. So suppose a sequence converging to x is given, and let (x_n) be an arbitrary subsequence. Then $x_{n,k} := k(x_n - x) \to k\cdot 0 = 0$ for $n \to \infty$, and hence by lemma 4.10 there are subsequences k_i, n_i with $k_i \cdot (x_{n_i} - x) \to 0$, i.e. $i \mapsto x_{n_i}$ is M-convergent to x. Thus, the original sequence converges in $c^\infty E$ by 4.9.

(3) Let E be the strict inductive limit of the Fréchet spaces E_n. By [5, 4.8.1] every E_n carries the trace topology of E, hence is closed in E, and every bounded subset of E is contained in some E_n. Thus, every compact subset of E is contained as compact subset in some E_n. Since E_n is a Fréchet space such a subset is even compact in $c^\infty E_n$ and hence compact in $c^\infty E$. Thus, the identity $kE \to c^\infty E$ is continuous.

(4) is valid, since the M-closure topology is the final one induced by the M-converging sequences.

(5) Let E be the dual of any Fréchet Montel space F. By 52.29 E is bornological. First we show that $kE = sE$. Let $K \subseteq E = F'$ be compact for the locally convex topology. Then K is bounded, hence equicontinuous since F is barrelled by [5, 5.2.2]. Since F is separable by 52.27 the set K is metrizable in the weak topology $\sigma(E, F)$ by 52.21. By [5, 7.4.12] this weak topology coincides with the topology of uniform convergence on precompact subsets of F. Since F is a Montel space, this latter topology is the strong one, and even the bornological one, as remarked at the beginning. Thus, the (metrizable) topology on K is the initial one induced by the converging sequences. Hence, the identity $kE \to sE$ is continuous, and therefore $sE = kE$.

It remains to show $kE = E$. Since F is Montel the locally convex topology of the strong dual coincides with the topology of uniform convergence on precompact subsets of F. Since F is metrizable this topology coincides with the so-called equicontinuous weak*-topology, cf. 52.22, which is the final topology induced by the inclusions of the equicontinuous subsets. These subsets are by the Alaoğlu-Bourbaki theorem [**5**, 7.4.12] relatively compact in the topology of uniform convergence on precompact subsets. Thus, the locally convex topology of E is compactly generated.

(2) By (5), and since Fréchet Schwartz spaces are Montel by 52.24, we have $sE = E$ and it remains to show that $c^\infty E = sE$. So let (x_n) be a sequence converging to 0 in E. Then the set $\{x_n : n \in \mathbb{N}\}$ is relatively compact, and by [**11**, 4.4.39] it is relatively compact in some Banach space E_B. Hence, at least a subsequence has to be convergent in E_B. Clearly its Mackey limit has to be 0. This shows that (x_n) converges to 0 in $c^\infty E$, and hence $c^\infty E = sE$. One can even show that E satisfies the Mackey convergence condition, see 52.28. $\square$

4.12. Example. We give now a non-metrizable example to which 4.11.1 applies. Let E denote the subspace of $\mathbb{R}^J$ of all sequences with countable support. Then the closure of subsets of E is given by all limits of sequences in the subset, but for non-countable J the space E is not metrizable. This was proved in [Balanzat, 1960].

4.13. Remark. The conditions 4.11.1 and 4.11.2 are rather disjoint since every locally convex space, that has a countable basis of its bornology and for which the *sequential adherence* of subsets (the set of all limits of sequences in it) is sequentially closed, is normable as the following proposition shows:

Proposition. *Let E be a non-normable bornological locally convex space that has a countable basis of its bornology. Then there exists a subset of E whose sequential adherence is not sequentially closed.*

Proof. Let $\{B_k : k \in \mathbb{N}_0\}$ be an increasing basis of the von Neumann bornology with $B_0 = \{0\}$. Since E is non-normable we may assume that B_k does not absorb B_{k+1} for all k. Now choose $b_{n,k} \in \frac{1}{n} B_{k+1}$ with $b_{n,k} \notin B_k$. We consider the double sequence $\{b_{k,0} - b_{n,k} : n, k \geq 1\}$. For fixed k the sequence $b_{n,k}$ converges by construction (in $E_{B_{k+1}}$) to 0 for $n \to \infty$. Thus, $b_{k,0} - 0$ is the limit of the sequence $b_{k,0} - b_{n,k}$ for $n \to \infty$, and $b_{k,0}$ converges to 0 for $k \to \infty$. Suppose $b_{k(i),0} - b_{n(i),k(i)}$ converges to 0. So it has to be bounded, thus there must be an $N \in \mathbb{N}$ with $B_1 - \{b_{k(i),0} - b_{n(i),k(i)} : i \in \mathbb{N}\} \subseteq B_N$. Hence, $b_{n(i),k(i)} = b_{k(i),0} - (b_{k(i),0} - b_{n(i),k(i)}) \in B_N$, i.e. $k(i) < N$. This contradicts 4.10.2. $\square$

4.14. Lemma. *Let U be a c^∞-open subset of a locally convex space, let $\mu_n \to \infty$ be a real sequence, and let $f : U \to F$ be a mapping which is bounded on each μ-converging sequence in U. Then f is bounded on every* bornologically compact subset *(i.e. compact in some E_B) of U.*

Proof. Let $K \subseteq E_B \cap U$ be compact in E_B for some bounded absolutely convex set B. Assume that $f(K)$ is not bounded. By composing with linear functionals we may assume that $F = \mathbb{R}$. So there is a sequence (x_n) in K with $|f(x_n)| \to \infty$. Since K is compact in the normed space E_B we may assume that (x_n) converges to $x \in K$. By passing to a subsequence we may even assume that (x_n) is μ-converging. Contradiction. $\square$

4.15. Lemma. *Let U be c^∞-open in $E \times \mathbb{R}$ and $K \subseteq \mathbb{R}$ be compact. Then $U_0 := \{x \in E : \{x\} \times K \subseteq U\}$ is c^∞-open in E.*

Proof. Let $x:\mathbb{R}\to E$ be a smooth curve in E with $x(0)\in U_0$, i.e. $(x(0),t)\in U$ for all $t\in K$. We have to show that $x(s)\in U_0$ for all s near 0. So consider the smooth map $x\times\mathbb{R}:\mathbb{R}\times\mathbb{R}\to E\times\mathbb{R}$. By assumption $(x\times\mathbb{R})^{-1}(U)$ is open in $c^\infty(\mathbb{R}^2)=\mathbb{R}^2$. It contains the compact set $\{0\}\times K$ and hence also a $W\times K$ for some neighborhood W of 0 in $\mathbb{R}$. But this amounts in saying that $x(W)\subseteq U_0$. □

4.16. The c^∞-topology of a product. Consider the product $E\times F$ of two locally convex vector spaces. Since the projections onto the factors are linear and continuous, and hence smooth, we always have that the identity mapping $c^\infty(E\times F)\to c^\infty(E)\times c^\infty(F)$ is continuous. It is not always a homeomorphism: Just take a bounded separately continuous bilinear functional, which is not continuous (like the evaluation map) on a product of spaces where the c^∞-topology is the bornological topology, cf. 4.20.

However, if one of the factors is finite dimensional the product is well behaved:

Corollary. *For any locally convex space E the c^∞-topology of $E\times\mathbb{R}^n$ is the product topology of the c^∞-topologies of the two factors, so that we have $c^\infty(E\times\mathbb{R}^n)=c^\infty(E)\times\mathbb{R}^n$.*

Proof. This follows recursively from the special case $E\times\mathbb{R}$, for which we can proceed as follows. Take a c^∞-open neighborhood U of some point $(x,t)\in E\times\mathbb{R}$. Since the inclusion map $s\mapsto(x,s)$ from $\mathbb{R}$ into $E\times\mathbb{R}$ is continuous and affine, the inverse image of U in $\mathbb{R}$ is an open neighborhood of t. Let's take a smaller compact neighborhood K of t. Then by the previous lemma $U_0:=\{y\in E:\{y\}\times K\subseteq U\}$ is a c^∞-open neighborhood of x, and hence $U_0\times K$ is a neighborhood of (x,t) in $c^\infty(E)\times\mathbb{R}$, what was to be shown. □

4.17. Lemma. *Let U be c^∞-open in a locally convex space and $x\in U$. Then the star $\operatorname{st}_x(U):=\{x+v:x+\lambda v\in U$ for all $|\lambda|\le 1\}$ with center x in U is again c^∞-open.*

Proof. Let $c:\mathbb{R}\to E$ be a smooth curve with $c(0)\in\operatorname{st}_x(U)$. The smooth mapping $f:(t,s)\mapsto(1-s)x+sc(t)$ maps $\{0\}\times\{s:|s|\le 1\}$ into U. So there exists $\delta>0$ with $f\Big(\{(t,s):|t|<\delta,|s|\le 1\}\Big)\subseteq U$. Thus, $c(t)\in\operatorname{st}_x(U)$ for $|t|<\delta$. □

4.18. Lemma. *The (absolutely) convex hull of a c^∞-open set is again c^∞-open.*

Proof. Let U be c^∞-open in a locally convex vector space E.
For each $x\in U$ the set

$$U_x:=\{x+t(y-x):t\in[0,1],y\in U\}=U\cup\bigcup_{0<t\le 1}(x+t(U-x))$$

is c^∞-open. The convex hull can be constructed by applying n times the operation $U\mapsto\bigcup_{x\in U}U_x$ and taking the union over all $n\in\mathbb{N}$, which respects c^∞-openness.

The absolutely convex hull can be obtained by forming first $\{\lambda:|\lambda|=1\}.U=\bigcup_{|\lambda|=1}\lambda U$ which is c^∞-open, and then forming the convex hull. □

4.19. Corollary. *Let E be a bornological convenient vector space containing a nonempty c^∞-open subset which is either locally compact or metrizable in the c^∞-topology. Then the c^∞-topology on E is locally convex. In the first case E is finite dimensional, in the second case E is a Fréchet space.*

Proof. Let $U \subseteq E$ be a c^∞-open metrizable subset. We may assume that $0 \in U$. Then there exists a countable neighborhood basis of 0 in U consisting of c^∞-open sets. This is also a neighborhood basis of 0 for the c^∞-topology of E. We take the absolutely convex hulls of these open sets, which are again c^∞-open by 4.18, and obtain by 4.5 a countable neighborhood basis for the bornologification of the locally convex topology, so the latter is metrizable and Fréchet, and by 4.11 it equals the c^∞-topology.

If U is locally compact in the c^∞-topology we may find a c^∞-open neighborhood V of 0 with compact closure $\overline{V}$ in the c^∞-topology. By lemma 4.18 the absolutely convex hull of V is also c^∞-open, and by 4.5 it is also open in the bornologification E_{born} of E. The set $\overline{V}$ is then also compact in E_{born}, hence precompact. So the absolutely convex hull of $\overline{V}$ is also precompact by [**5**, 6.4.3]. Therefore, the absolutely convex hull of V is a precompact neighborhood of 0 in E_{born}, thus E is finite dimensional by [**5**, 4.4.5]. So $E_{\text{born}} = c^\infty(E)$. □

Now we describe classes of spaces where $c^\infty E \neq E$ or where $c^\infty E$ is not even a topological vector space. Finally, we give an example where the c^∞-topology is not completely regular.

4.20. Proposition. *Let E and F be bornological locally convex vector spaces. If there exists a bilinear smooth mapping $m : E \times F \to \mathbb{R}$ that is not continuous with respect to the locally convex topologies, then $c^\infty(E \times F)$ is not a topological vector space.*

We shall show in lemma 5.5 below that multilinear mappings are smooth if and only if they are bounded.

Proof. Suppose that addition $c^\infty(E\times F)\times c^\infty(E\times F) \to c^\infty(E\times F)$ is continuous with respect to the product topology. Using the continuous inclusions $c^\infty E \to c^\infty(E \times F)$ and $c^\infty F \to c^\infty(E \times F)$ we can factor the identity as $c^\infty E \times c^\infty F \to c^\infty(E \times F) \times c^\infty(E \times F) \xrightarrow{+} c^\infty(E \times F)$ and hence $c^\infty E \times c^\infty F = c^\infty(E \times F)$.

In particular, $m : c^\infty E \times c^\infty F = c^\infty(E \times F) \to \mathbb{R}$ ist continuous. Thus, for every $\varepsilon > 0$ there are 0-neighborhoods U and V with respect to the c^∞-topology such that $m(U\times V) \subseteq (-\varepsilon, \varepsilon)$. Then also $m(\langle U\rangle \times \langle V\rangle) \subseteq (-\varepsilon, \varepsilon)$ where $\langle \quad \rangle$ denotes the absolutely convex hull. By 4.5 one concludes that m is continuous with respect to the locally convex topology, a contradiction. □

4.21. Corollary. *Let E be a non-normable bornological locally convex space. Then $c^\infty(E \times E')$ is not a topological vector space.*

Proof. By 4.20 it is enough to show that $\text{ev} : E \times E' \to \mathbb{R}$ is not continuous for the bornological topologies on E and E'; if it were so there was be a neighborhood U of 0 in E and a neighborhood U' of 0 in E' such that $\text{ev}(U \times U') \subseteq [-1, 1]$. Since U' is absorbing, U is scalarwise bounded, hence a bounded neighborhood. Thus, E is normable. □

4.22. Remark. In particular, for a Fréchet Schwartz space E (e.g. $\mathbb{R}^{\mathbb{N}}$) and its dual E' we have $c^\infty(E \times E') \neq c^\infty E \times c^\infty E'$, since by 4.11 we have $c^\infty E = E$ and $c^\infty E' = E'$, so equality would contradict corollary 4.21.

In order to get a large variety of spaces where the c^∞-topology is not a topological vector space topology the next three technical lemmas will be useful.

4.23. Lemma. *Let E be a locally convex vector space. Suppose a double sequence $b_{n,k}$ in E exists which satisfies the following two conditions:*

(b') *For every sequence $k \mapsto n_k$ the sequence $k \mapsto b_{n_k,k}$ has no accumulation point in $c^\infty E$.*
(b") *For all k the sequence $n \mapsto b_{n,k}$ converges to 0 in $c^\infty E$.*

Suppose furthermore that a double sequence $c_{n,k}$ in E exists that satisfies the following two conditions:

(c') *For every 0-neighborhood U in $c^\infty E$ there exists some k_0 such that $c_{n,k} \in U$ for all $k \geq k_0$ and all n.*
(c") *For all k the sequence $n \mapsto c_{n,k}$ has no accumulation point in $c^\infty E$.*

Then $c^\infty E$ is not a topological vector space.

Proof. Assume that the addition $c^\infty E \times c^\infty E \to c^\infty E$ is continuous. In this proof convergence is meant always with respect to $c^\infty E$. We may without loss of generality assume that $c_{n,k} \neq 0$ for all n, k, since by (c") we may delete for each n all those $c_{n,k}$ which are equal to 0. Then we consider $A := \{b_{n,k} + \varepsilon_{n,k} c_{n,k} : n, k \in \mathbb{N}\}$ where the $\varepsilon_{n,k} \in \{-1, 1\}$ are chosen in such a way that $0 \notin A$.

We first show that A is closed in the sequentially generated topology $c^\infty E$: Let $b_{n_i,k_i} + \varepsilon_{n_i,k_i} c_{n_i,k_i} \to x$, and assume first that (k_i) is unbounded. By passing if necessary to a subsequence we may even assume that $i \mapsto k_i$ is strictly increasing. Then $c_{n_i,k_i} \to 0$ by (c'), hence $b_{n_i,k_i} \to x$ by the assumption that addition is continuous, which is a contradiction to (b'). Thus, (k_i) is bounded, and we may assume it to be constant. Now suppose that (n_i) is unbounded. Then $b_{n_i,k} \to 0$ by (b"), and hence $\varepsilon_{n_i,k} c_{n_i,k} \to x$, and for a subsequence where ε is constant one has $c_{n_i,k} \to \pm x$, which is a contradiction to (c"). Thus, n_i is bounded as well, and we may assume it to be constant. Hence, $x = b_{n,k} + \varepsilon_{n,k} c_{n,k} \in A$.

By the assumed continuity of the addition there exists an open and symmetric 0-neighborhood U in $c^\infty E$ with $U + U \subseteq E \setminus A$. For K sufficiently large and n arbitrary one has $c_{n,K} \in U$ by (c'). For such a fixed K and N sufficiently large $b_{N,K} \in U$ by (b'). Thus, $b_{N,K} + \varepsilon_{N,K} c_{N,K} \notin A$, which is a contradiction. □

Let us now show that many spaces have a double sequence $c_{n,k}$ as in the above lemma.

4.24. Lemma. *Let E be an infinite dimensional metrizable locally convex space. Then a double sequence $c_{n,k}$ subject to the conditions (c') and (c") of 4.23 exists.*

Proof. If E is normable we choose a sequence (c_n) in the unit ball without accumulation point and define $c_{n,k} := \frac{1}{k} c_n$. If E is not normable we take a countable increasing family of non-equivalent seminorms p_k generating the locally convex topology, and we choose $c_{n,k}$ with $p_k(c_{n,k}) = \frac{1}{k}$ and $p_{k+1}(c_{n,k}) > n$. □

Next we show that many spaces have a double sequence $b_{n,k}$ as in lemma 4.23.

4.25. Lemma. *Let E be a non-normable bornological locally convex space having a countable basis of its bornology. Then a double sequence $b_{n,k}$ subject to the conditions (b') and (b") of 2.11 exists.*

Proof. Let B_n ($n \in \mathbb{N}$) be absolutely convex sets forming an increasing basis of the bornology. Since E is not normable the sets B_n can be chosen such that B_n does not absorb B_{n+1}. Now choose $b_{n,k} \in \frac{1}{n} B_{k+1}$ with $b_{n,k} \notin B_k$. □

Using these lemmas one obtains the

4.26. Proposition. *For the following bornological locally convex spaces the c^∞-topology is not a vector space topology:*

(i) *Every bornological locally convex space that contains as c^∞-closed subspaces an infinite dimensional Fréchet space and a space which is non-normable in the bornological topology and having a countable basis of its bornology.*
(ii) *Every strict inductive limit of a strictly increasing sequence of infinite dimensional Fréchet spaces.*
(iii) *Every product for which at least $2^{\aleph_0}$ many factors are non-zero.*
(iv) *Every coproduct for which at least $2^{\aleph_0}$ many summands are non-zero.*

Proof. (i) follows directly from the last 3 lemmas.

(ii) Let E be the strict inductive limit of the spaces E_n $(n \in \mathbb{N})$. Then E contains the infinite dimensional Fréchet space E_1 as subspace. The subspace generated by points $x_n \in E_{n+1} \setminus E_n$ $(n \in \mathbb{N})$ is bornologically isomorphic to $\mathbb{R}^{(\mathbb{N})}$, hence its bornology has a countable basis. Thus, by (i) we are done.

(iii) Such a product E contains the Fréchet space $\mathbb{R}^\mathbb{N}$ as complemented subspace. We want to show that $\mathbb{R}^{(\mathbb{N})}$ is also a subspace of E. For this we may assume that the index set J is $\mathbb{R}^\mathbb{N}$ and all factors are equal to $\mathbb{R}$. Now consider the linear subspace E_1 of the product generated by the elements $x^n \in E = \mathbb{R}^J$, where $(x^n)_j := j(n)$ for every $j \in J = \mathbb{R}^\mathbb{N}$. The linear map $\mathbb{R}^{(\mathbb{N})} \to E_1 \subseteq E$ that maps the n-th unit vector to x^n is injective, since for a given finite linear combination $\sum t_n x^n = 0$ the j-th coordinate for $j(n) := \operatorname{sign}(t_n)$ equals $\sum |t_n|$. It is continuous since $\mathbb{R}^{(\mathbb{N})}$ carries the finest locally convex structure. So it remains to show that it is a bornological embedding. We have to show that any bounded $B \subseteq E_1$ is contained in a subspace generated by finitely many x^n. Otherwise, there would exist a strictly increasing sequence (n_k) and $b^k = \sum_{n \le n_k} t^k_n x^n \in B$ with $t^k_{n_k} \ne 0$. Define an index j recursively by $j(n) := n|t^k_n|^{-1} \cdot \operatorname{sign}\big(\sum_{m<n} t^k_m j(m)\big)$ if $n = n_k$ and $j(n) := 0$ if $n \ne n_k$ for all k. Then the absolute value of the j-th coordinate of b^k evaluates as follows:

$$
\begin{aligned}
|(b^k)_j| &= \Big|\sum_{n \le n_k} t^k_n j(n)\Big| = \Big|\sum_{n<n_k} t^k_n j(n) + t^k_{n_k} j(n_k)\Big| \\
&= \Big|\sum_{n<n_k} t^k_n j(n)\Big| + |t^k_{n_k} j(n_k)| \ge |t^k_{n_k} j(n_k)| = n_k.
\end{aligned}
$$

Hence, the j-th coordinates of $\{b^k : k \in \mathbb{N}\}$ are unbounded with respect to $k \in \mathbb{N}$, thus B is unbounded.

(iv) We can not apply lemma 4.23 since every double sequence has countable support and hence is contained in the dual $\mathbb{R}^{(A)}$ of a Fréchet Schwartz space $\mathbb{R}^A$ for some countable subset $A \subset J$. It is enough to show (iv) for $\mathbb{R}^{(J)}$ where $J = \mathbb{N} \cup c_0$. Let $A := \{j_n(e_n + e_j) : n \in \mathbb{N}, j \in c_0, j_n \ne 0 \text{ for all } n\}$, where e_n and e_j denote the unit vectors in the corresponding summand. The set A is c^∞-closed, since its intersection with finite subsums is finite. Suppose there exists a symmetric c^∞-open 0-neighborhood U with $U + U \subseteq E \setminus A$. Then for each n there exists a $j_n \ne 0$ with $j_n e_n \in U$. We may assume that $n \mapsto j_n$ converges to 0 and hence defines an element $j \in c_0$. Furthermore, there has to be an $N \in \mathbb{N}$ with $j_N e_j \in U$, thus $j_N(e_N + e_j) \in (U + U) \cap A$, in contradiction to $U + U \subseteq E \setminus A$. □

Remark. A nice and simple example where one either uses (i) or (ii) is $\mathbb{R}^{\mathbb{N}} \oplus \mathbb{R}^{(\mathbb{N})}$. The locally convex topology on both factors coincides with their c^∞-topology (the first being a Fréchet (Schwartz) space, cf. (i) of 4.11, the second as dual of the first, cf. (ii) of 4.11); but the c^∞-topology on their product is not even a vector space topology.

From (ii) it follows also that each space $C^\infty_c(M, \mathbb{R})$ of smooth functions with compact support on a non-compact separable finite dimensional manifold M has the property, that the c^∞-topology is not a vector space topology.

4.27. Although the c^∞-topology on a convenient vector space is always functionally separated, hence Hausdorff, it is not always completely regular as the following example shows.

Example. The c^∞-topology is not completely regular. *The c^∞-topology of $\mathbb{R}^J$ is not completely regular if the cardinality of J is at least $2^{\aleph_0}$.*

Proof. It is enough to show this for an index set J of cardinality $2^{\aleph_0}$, since the corresponding product is a complemented subspace in every product with larger index set. We prove the theorem by showing that every function $f : \mathbb{R}^J \to \mathbb{R}$ which is continuous for the c^∞-topology is also continuous with respect to the locally convex topology. Hence, the completely regular topology associated to the c^∞-topology is the locally convex topology of E. That these two topologies are different was shown in 4.8. We use the following theorem of [Mazur, 1952]: Let $E_0 := \{x \in \mathbb{R}^J : \operatorname{supp}(x) \text{ is countable}\}$, and let $f : E_0 \to \mathbb{R}$ be sequentially continuous. Then there is some countable subset $A \subset J$ such that $f(x) = f(x_A)$, where in this proof x_A is defined as $x_A(j) := x(j)$ for $j \in A$ and $x_A(j) = 0$ for $j \notin A$. Every sequence which is converging in the locally convex topology of E_0 is contained in a metrizable complemented subspace $\mathbb{R}^A$ for some countable A and therefore is even M-convergent. Thus, this theorem of Mazur remains true if f is assumed to be continuous for the M-closure topology. This generalization follows also from the fact that $c^\infty E_0 = E_0$, cf. 4.12. Now let $f : \mathbb{R}^J \to \mathbb{R}$ be continuous for the c^∞-topology. Then $f|E_0 : E_0 \to \mathbb{R}$ is continuous for the c^∞-topology, and hence there exists a countable set $A_0 \subset J$ such that $f(x) = f(x_{A_0})$ for any $x \in E_0$. We want to show that the same is true for arbitrary $x \in \mathbb{R}^J$. In order to show this we consider for $x \in \mathbb{R}^J$ the map $\varphi_x : 2^J \to \mathbb{R}$ defined by $\varphi_x(A) := f(x_A) - f(x_{A \cap A_0})$ for any $A \subseteq J$, i.e. $A \in 2^J$. For countable A one has $x_A \in E_0$, hence $\varphi_x(A) = 0$. Furthermore, φ_x is sequentially continuous where one considers on 2^J the product topology of the discrete factors $2 = \{0, 1\}$. In order to see this consider a converging sequence of subsets $A_n \to A$, i.e. for every $j \in J$ one has for the characteristic functions $\chi_{A_n}(j) = \chi_A(j)$ for n sufficiently large. Then $\{n(x_{A_n} - x_A) : n \in \mathbb{N}\}$ is bounded in $\mathbb{R}^J$ since for fixed $j \in J$ the j-th coordinate equals 0 for n sufficiently large. Thus, x_{A_n} converges Mackey to x_A, and since f is continuous for the c^∞-topology $\varphi_x(A_n) \to \varphi_x(A)$. Now we can apply another theorem of [Mazur, 1952]: Any function $f : 2^J \to \mathbb{R}$ that is sequentially continuous and is zero on all countable subsets of J is identically 0, provided the cardinality of J is smaller than the first inaccessible cardinal. Thus, we conclude that $0 = \varphi_x(J) = f(x) - f(x_{A_0})$ for all $x \in \mathbb{R}^J$. Hence, f factors over the metrizable space $\mathbb{R}^{A_0}$ and is therefore continuous for the locally convex topology. □

In general, the trace of the c^∞-topology on a linear subspace is not its c^∞-topology. However, for c^∞-closed subspaces this is true:

4.28. Lemma. Closed embedding lemma. *Let E be a linear c^∞-closed subspace of F. Then the trace of the c^∞-topology of F on E is the c^∞-topology on E*

Proof. Since the inclusion is continuous and hence bounded it is c^∞-continuous. Therefore, it is enough to show that it is closed for the c^∞-topologies. So let $A \subseteq E$ be $c^\infty E$-closed. And let $x_n \in A$ converge Mackey towards x in F. Then $x \in E$, since E is assumed to be c^∞-closed, and hence x_n converges Mackey to x in E. Since A is c^∞-closed in E, we have that $x \in A$. □

We will give an example in 4.33 below which shows that c^∞-closedness of the subspace is essential for this result. Another example will be given in 4.36.

4.29. Theorem. The c^∞-completion. *For any locally convex space E there exists a unique (up to a bornological isomorphism) convenient vector space $\tilde{E}$ and a bounded linear injection $i : E \to \tilde{E}$ with the following universal property:*

> *Each bounded linear mapping $\ell : E \to F$ into a convenient vector space F has a unique bounded extension $\tilde{\ell} : \tilde{E} \to F$ such that $\tilde{\ell} \circ i = \ell$.*

Furthermore, $i(E)$ is dense for the c^∞-topology in $\tilde{E}$.

Proof. Let $\tilde{E}$ be the c^∞-closure of E in the locally convex completion $\widehat{E_{\mathrm{born}}}$ of the bornologification E_{born} of E. The inclusion $i : E \to \tilde{E}$ is bounded (but not continuous in general). By 4.28 the c^∞-topology on $\tilde{E}$ is the trace of the c^∞-topology on $\widehat{E_{\mathrm{born}}}$. Hence, $i(E)$ is dense also for the c^∞-topology in $\tilde{E}$.

Using the universal property of the locally convex completion the mapping ℓ has a unique continuous extension $\hat{\ell} : \widehat{E_{\mathrm{born}}} \to \widehat{F}$ into the locally convex completion of F, whose restriction to $\tilde{E}$ has values in F, since F is c^∞-closed in $\widehat{F}$, so it is the desired $\tilde{\ell}$. Uniqueness follows, since $i(E)$ is dense for the c^∞-topology in $\tilde{E}$. □

4.30. Proposition. c^∞-completion via c^∞-dense embeddings. *Let E be c^∞-dense and bornologically embedded into a c^∞-complete locally convex space F. If $E \to F$ has the extension property for bounded linear functionals, then F is bornologically isomorphic to the c^∞-completion of E.*

Proof. We have to show that $E \to F$ has the universal property for extending bounded linear maps T into c^∞-complete locally convex spaces G. Since we are only interested in bounded mappings, we may take the bornologification of G and hence may assume that G is bornological. Consider the following diagram

$$\begin{tikzcd}
E \arrow[rr, hook] \arrow[dd, "T"'] & & F \arrow[d, dashed] \arrow[ddll, dotted] \arrow[ddrr, dotted, "\widetilde{\lambda\circ T}"] & & \\
& & \prod_{G'}\mathbb{R} \arrow[drr, "\mathrm{pr}_\lambda"] & & \\
G \arrow[urr, "\delta"] \arrow[rrrr, "\lambda"] & & & & \mathbb{R}
\end{tikzcd}$$

The arrow δ, given by $\delta(x)_\lambda := \lambda(x)$, is a *bornological embedding*, i.e. the image of a set is bounded if and only if the set is bounded, since $B \subseteq G$ is bounded if and only if $\lambda(B) \subseteq \mathbb{R}$ is bounded for all $\lambda \in G'$, i.e. $\delta(B) \subseteq \prod_{G'} \mathbb{R}$ is bounded.

By assumption, the dashed arrow on the right hand side exists, hence by the universal property of the product the dashed vertical arrow (denoted $\tilde{T}$) exists. It remains to show that it has values in the image of δ. Since $\tilde{T}$ is bounded we have

$$\tilde{T}(F) = \tilde{T}(\overline{E}^{c^\infty}) \subseteq \overline{\tilde{T}(E)}^{c^\infty} \subseteq \overline{\delta(G)}^{c^\infty} = \delta(G),$$

since G is c^∞-complete and hence also $\delta(G)$, which is thus c^∞-closed.

The uniqueness follows, since as a bounded linear map $\tilde{T}$ has to be continuous for the c^∞-topology (since it preserves the smooth curves by 2.11 which in turn generate the c^∞-topology), and E lies dense in F with respect to this topology. □

4.31. Proposition. Inductive representation of bornological locally convex spaces. *For a locally convex space E the bornologification E_{born} is by 4.2 the colimit of all the normed spaces E_B for the absolutely convex bounded sets B. The colimit of the respective completions $\tilde{E}_B$ is the linear subspace of the c^∞-completion $\tilde{E}$ consisting of all limits in $\tilde{E}$ of Mackey Cauchy sequences in E.*

Proof. Let $E^{(1)}$ be the *Mackey adherence* of E in the c^∞-completion $\tilde{E}$, by which we mean the limits in $\tilde{E}$ of all sequences in E which converge Mackey in $\tilde{E}$. Then $E^{(1)}$ is a subspace of the locally convex completion $\widehat{E_{\text{born}}}$. For every absolutely convex bounded set $B \subseteq E$ we have the continuous inclusion $E_B \to E_{\text{born}}$, and by passing to the c^∞-completion we get mappings $\widehat{E_B} = \widetilde{E_B} \to \tilde{E}$. These mappings commute with the inclusions $\widehat{E_B} \to \widehat{E_{B'}}$ for $B \subseteq B'$ and have values in the Mackey adherence of E, since every point in $\widehat{E_B}$ is the limit of a sequence in E_B, and hence its image is the limit of this Mackey Cauchy sequence in E.

We claim that the Mackey adherence $E^{(1)}$ together with these mappings has the universal property of the colimit $\varinjlim_B \widehat{E_B}$. In fact, let $T : E^{(1)} \to F$ be a linear mapping, such that $\widehat{E_B} \to E^{(1)} \to F$ is continuous for all B. In particular $T|_E : E \to F$ has to be bounded, and hence $T|_{E_{\text{born}}} : E_{\text{born}} \to F$ is continuous. Thus, it has a unique continuous extension $\widehat{T} : E^{(1)} \subseteq \widehat{E_{\text{born}}} \to \widehat{F}$, and it remains to show that this extension is T. So take a point $x \in E^{(1)}$. Then there exists a sequence (x_n) in E, which converges Mackey to x. Thus, the x_n form a Cauchy-sequence in some E_B and hence converge to some y in $\widehat{E_B}$. Then $\iota_B(y) = x$, since the mapping $\iota_B : \widehat{E_B} \to E^{(1)}$ is continuous. Since the trace of T to $\widehat{E_B}$ is continuous $T(x_n)$ converges to $T(\iota_B(y)) = T(x)$ and $T(x_n) = \widehat{T}(x_n)$ converges to $\widehat{T}(x)$, i.e. $T(x) = \widehat{T}(x)$. □

In spite of (1) in 4.36 we can use the Mackey adherence to describe the c^∞-closure in the following inductive way:

4.32. Proposition. Mackey adherences. *For ordinal numbers α the Mackey adherence $A^{(\alpha)}$ of order α is defined recursively by:*

$$A^{(\alpha)} := \begin{cases} \text{M-Adh}(A^{(\beta)}) & \text{if } \alpha = \beta + 1 \\ \bigcup_{\beta<\alpha} A^{(\beta)} & \text{if } \alpha \text{ is a limit ordinal number.} \end{cases}$$

Then the closure $\overline{A}$ of A in the c^∞-topology coincides with $A^{(\omega_1)}$, where ω_1 denotes the first uncountable ordinal number, i.e. the set of all countable ordinal numbers.

Proof. Let us first show that $A^{(\omega_1)}$ is c^∞-closed. So take a sequence $x_n \in A^{(\omega_1)} = \bigcup_{\alpha<\omega_1} A^{(\alpha)}$, which converges Mackey to some x. Then there are $\alpha_n < \omega_1$ with $x_n \in A^{(\alpha_n)}$. Let $\alpha := \sup_n \alpha_n$. Then α is a again countable and hence less than

ω_1. Thus, $x_n \in A^{(\alpha_n)} \subseteq A^{(\alpha)}$, and therefore $x \in \text{M-Adh}(A^{(\alpha)}) = A^{(\alpha+1)} \subseteq A^{(\omega_1)}$ since $\alpha + 1 \le \omega_1$.

It remains to show that $A^{(\alpha)}$ is contained in $\overline{A}$ for all α. We prove this by transfinite induction. So assume that for all $\beta < \alpha$ we have $A^{(\beta)} \subseteq \overline{A}$. If α is a limit ordinal number then $A^{(\alpha)} = \bigcup_{\beta<\alpha} A^{(\beta)} \subseteq \overline{A}$. If $\alpha = \beta + 1$ then every point in $A^{(\alpha)} = \text{M-Adh}(A^{(\beta)})$ is the Mackey-limit of some sequence in $A^{(\beta)} \subseteq \overline{A}$, and since $\overline{A}$ is c^∞-closed, this limit has to belong to it. So $A^{(\alpha)} \subseteq \overline{A}$ in all cases. □

Example where ω_1-many steps are indeed needed!

4.33. Example. *The trace of the c^∞-topology is not the c^∞-topology and the Mackey-adherence is not the c^∞-closure, in general.*

Proof. Consider $E = \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{(\mathbb{N})}$, $A := \{a_{n,k} := (\frac{1}{n}\chi_{\{1,..,k\}}, \frac{1}{k}\chi_{\{n\}}) : n, k \in \mathbb{N}\} \subseteq E$. Let F be the linear subspace of E generated by A. We show that the closure of A with respect to the c^∞-topology of F is strictly smaller than that with respect to the trace topology of the c^∞-topology of E.

A is closed in the c^∞-topology of F: Assume that a sequence (a_{n_j,k_j}) is M-converging to (x, y). Then the second component of a_{n_j,k_j} has to be bounded. Thus, $j \mapsto n_j$ has to be bounded and may be assumed to have constant value n_∞. If $j \mapsto k_j$ were unbounded, then $(x, y) = (\frac{1}{n_\infty}\chi_{\mathbb{N}}, 0)$, which is not an element of F. Thus, $j \mapsto k_j$ has to be bounded too and may be assumed to have constant value k_∞. Thus, $(x, y) = a_{n_\infty,k_\infty} \in A$.

A is not closed in the trace topology since (0,0) is contained in the closure of A with respect to the c^∞-topology of E: For $k \to \infty$ and fixed n the sequence $a_{n,k}$ is M-converging to $(\frac{1}{n}\chi_{\mathbb{N}}, 0)$, and $\frac{1}{n}\chi_{\mathbb{N}}$ is M-converging to 0 for $n \to \infty$. □

4.34. Example. We consider the space $\ell^\infty(X) := \ell^\infty(X, \mathbb{R})$ as defined in 2.15 for a set X together with a family $\mathcal{B}$ of subsets called bounded. We have the subspace $C_c(X) := \{f \in \ell^\infty(X) : \operatorname{supp} f \text{ is finite}\}$. And we want to calculate its c^∞-closure in $\ell^\infty(X)$.

Claim: The c^∞-closure of $C_c(X)$ equals

$$c_0(X) := \{f \in \ell^\infty(X) : f|_B \in c_0(B) \text{ for all } B \in B\},$$

provided that X is countable.

Proof. The right hand side is just the intersection $c_0(X) := \bigcap_{B\in\mathcal{B}} \iota_B^{-1}(c_0(B))$, where $\iota_B : \ell^\infty(X) \to \ell^\infty(B)$ denotes the restriction map. We use the notation $c_0(X)$, since in the case where X is bounded this is exactly the space $\{f \in \ell^\infty(X) : \{x : |f(x)| \ge \varepsilon\} \text{ is finite for all } \varepsilon > 0\}$. In particular, this applies to the bounded space $\mathbb{N}$, where $c_0(\mathbb{N}) = c_0$. Since $\ell^\infty(X)$ carries the initial structure with respect to these maps $c_0(X)$ is closed. It remains to show that $C_c(X)$ is c^∞-dense in $c_0(X)$. So take $f \in c_0(X)$. Let $\{x_1, x_2, \dots\} := \{x : f(x) \ne 0\}$.

We consider first the case, where there exists some $\delta > 0$ such that $|f(x_n)| \ge \delta$ for all n. Then we consider the functions $f_n := f \cdot \chi_{x_1,\dots,x_n} \in C_c(X)$. We claim that $n(f - f_n)$ is bounded in $\ell^\infty(X, \mathbb{R})$. In fact, let $B \in \mathcal{B}$. Then $\{n : x_n \in B\} = \{n : x_n \in B \text{ and } |f(x_n)| \ge \delta\}$ is finite. Hence, $\{n(f - f_n)(x) : x \in B\}$ is finite and thus bounded, i.e. f_n converges Mackey to f.

Now the general case. We set $X_n := \{x \in X : |f(x)| \ge \frac{1}{n}\}$ and define $f_n := f \cdot \chi_{X_n}$. Then each f_n satisfies the assumption of the particular case with $\delta = \frac{1}{n}$ and hence is a Mackey limit of a sequence in $C_c(X)$. Furthermore, $n(f - f_n)$ is uniformly bounded by 1, since for $x \in X_n$ it is 0 and otherwise $|n(f-f_n)(x)| = n|f(x)| < 1$. So after forming the Mackey adherence (i.e. adding the limits of all Mackey-convergent sequences contained in the set, see 4.32 for a formal definition) twice, we obtain $c_0(X)$. □

Now we want to show that $c_0(X)$ is in fact the c^∞-completion of $C_c(X)$.

4.35. Example. $\mathbf{c_0(X)}$. We claim that $c_0(X)$ is the c^∞-completion of the subspace $C_c(X)$ in $\ell^\infty(X)$ formed by the finite sequences.
We may assume that the bounded sets of X are formed by those subsets B, for which $f(B)$ is bounded for all $f \in \ell^\infty(X)$. Obviously, any bounded set has this property, and the space $\ell^\infty(X)$ is not changed by adding these sets. Furthermore, the restriction map $\iota_B : \ell^\infty(X) \to \ell^\infty(B)$ is also bounded for such a B, since using the closed graph theorem [**5**, 5.3.3] we only have to show that $\mathrm{ev}_b \circ \iota_B = \iota_{\{b\}}$ is bounded for every $b \in B$, which is obviously the case.

By proposition 4.30 it is enough to show the universal property for bounded linear functionals. We only have to show that in analogy to Banach-theory the dual $C_c(X)'$ is just

$$\ell^1(X) := \{g : X \to \mathbb{R} : \operatorname{supp} g \text{ is bounded and } g \text{ is absolutely summable}\}.$$

In fact, any such g acts even as bounded linear functional on $\ell^\infty(X, \mathbb{R})$ by $\langle g, f\rangle := \sum_x g(x)\, f(x)$, since a subset is bounded in $\ell^\infty(X)$ if and only if it is uniformly bounded on all bounded sets $B \subseteq X$. Conversely, let $\ell : C_c(X) \to \mathbb{R}$ be bounded and linear and define $g : X \to \mathbb{R}$, by $g(x) := \ell(e_x)$, where e_x denotes the function given by $e_x(y) := 1$ for $x = y$ and 0 otherwise. Obviously $\ell(f) = \langle g, f\rangle$ for all $f \in C_c(X)$. Suppose indirectly that $\operatorname{supp} g = \{x : \ell(e_x) \neq 0\}$ is not bounded. Then there exists a sequence $x_n \in \operatorname{supp} g$ and a function $f \in \ell^\infty(X)$ such that $|f(x_n)| \geq n$. In particular, the only bounded subsets of $\{x_n : n \in \mathbb{N}\}$ are the finite ones. Hence $\{\frac{n}{|g(x_n)|} e_{x_n} : n \in \mathbb{N}\}$ is bounded in $C_c(X)$, but the image under ℓ is not. Furthermore, g has to be absolutely summable since the set of finite subsums of $\sum_x \operatorname{sign} g(x)\, e_x$ is uniformly bounded and hence bounded in $C_c(X)$ and its image under ℓ are the subsums of $\sum_x |g(x)|$.

$\ell^1(X)' = \ell^\infty(X)$

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4.36. Corollary. Counter-examples on $\mathbf{c^\infty}$-topology. *The following statements are* **false***:*

1. *The c^∞-closure of a subset (or of a linear subspace) is given by the Mackey adherence, i.e. the set formed by all limits of sequences in this subset which are Mackey convergent in the total space.*
2. *A subset U of E that contains a point x and has the property, that every sequence which M-converges to x belongs to it finally, is a c^∞-neighborhood of x.*
3. *A c^∞-dense subspace of a c^∞-complete space has this space as c^∞-completion.*
4. *If a subspace E is c^∞-dense in the total space, then it is also c^∞-dense in each linear subspace lying in between.*
5. *The c^∞-topology of a linear subspace is the trace of the c^∞-topology of the whole space.*
6. *Every bounded linear functional on a linear subspace can be extended to such a functional on the whole space.*
7. *A linear subspace of a bornological locally convex space is bornological.*
8. *The c^∞-completion preserves embeddings.*

Proof. (1) For this we give an example, where the Mackey adherence of $C_c(X)$ is not all of $c_0(X)$.

Let $X = \mathbb{N} \times \mathbb{N}$, and take as bounded sets all sets of the form $B_\mu := \{(n,k) : n \le \mu(k)\}$, where μ runs through all functions $\mathbb{N} \to \mathbb{N}$. Let $f : X \to \mathbb{R}$ be defined by $f(n,k) := \frac{1}{k}$. Obviously, $f \in c_0(X)$, since for given $j \in \mathbb{N}$ and function μ the set of points $(n,k) \in B_\mu$ for which $f(n,k) = \frac{1}{k} \ge \frac{1}{j}$ is the finite set $\{(n,k) : k \le j, n \le \mu(k)\}$.
Assume there is a sequence $f_n \in C_c(X)$ Mackey convergent to f. By passing to a subsequence we may assume that $n^2(f - f_n)$ is bounded. Now choose $\mu(k)$ to be larger than all of the finitely many n, with $f_k(n,k) \ne 0$. If $k^2(f - f_k)$ is bounded on B_μ, then in particular $\{k^2(f - f_k)(\mu(k), k) : k \in \mathbb{N}\}$ has to be bounded, but $k^2(f - f_k)(\mu(k), k) = k^2 \frac{1}{k} - 0 = k$.

(2) Let A be a set for which (1) fails, and choose x in the c^∞-closure of A but not in the M-adherence of A. Then $U := E \setminus A$ satisfies the assumptions of (2). In fact, let x_n be a sequence which converges Mackey to x, and assume that it is not finally in U. So we may assume without loss of generality that $x_n \notin U$ for all n, but then $A \ni x_n \to x$ would imply that x is in the Mackey adherence of A. However, U cannot be a c^∞-neighborhood of x. In fact, such a neighborhood must meet A since x is assumed to be in the c^∞-closure of A.

(3) Let F be a locally convex vector space whose Mackey adherence in its c^∞-completion E is not all of E, e.g. $C_c(X) \subseteq c_0(X)$ as in (1). Choose a $y \in E$ that is not contained in the Mackey adherence of F, and let F_1 be the subspace of E generated by $F \cup \{y\}$. We claim that $F_1 \subseteq E$ cannot be the c^∞-completion although F_1 is obviously c^∞-dense in the convenient vector space E. In order to see this we consider the linear map $\ell : F_1 \to \mathbb{R}$ characterized by $\ell(F) = 0$ and $\ell(y) = 1$. Clearly ℓ is well defined.

$\ell : F_1 \to \mathbb{R}$ is bornological: For any bounded $B \subseteq F_1$ there exists an N such that $B \subseteq F + [-N, N]y$. Otherwise, $b_n = x_n + t_n y \in B$ would exist with $t_n \to \infty$ and $x_n \in F$. This would imply that $b_n = t_n(\frac{x_n}{t_n} + y)$, and thus $-\frac{x_n}{t_n}$ would converge Mackey to y; a contradiction.

Now assume that a bornological extension $\bar{\ell}$ to E exists. Then $F \subseteq \ker(\bar{\ell})$ and $\ker(\bar{\ell})$ is c^∞-closed, which is a contradiction to the c^∞-denseness of F in E. So $F_1 \subseteq E$ does not have the universal property of a c^∞-completion.

This shows also that (6) fails.

(4) Furthermore, it follows that F is $c^\infty F_1$-closed in F_1, although F and hence F_1 are c^∞-dense in E.

(5) The trace of the c^∞-topology of E to F_1 cannot be the c^∞-topology of F_1, since for the first one F is obviously dense.

(7) Obviously, the trace topology of the bornological topology on E cannot be bornological on F_1, since otherwise the bounded linear functionals on F_1 would be continuous and hence extendable to E.

(8) Furthermore, the extension of the inclusion $\iota : F \oplus \mathbb{R} \cong F_1 \to E$ to the completion is given by $(x,t) \in E \oplus \mathbb{R} \cong \tilde{F} \oplus \mathbb{R} = \tilde{F}_1 \mapsto x + ty \in E$ and has as kernel the linear subspace generated by $(y, -1)$. Hence, the extension of an embedding to the c^∞-completions need not be an embedding anymore, in particular the c^∞-completion functor does not preserve injectivity of morphisms. □

5. Uniform Boundedness Principles and Multilinearity

5.1. The category of locally convex spaces and smooth mappings. The category of all smooth mappings between bornological vector spaces is a subcategory of the category of all smooth mappings between locally convex spaces which is equivalent to it, since a locally convex space and its bornologification 4.4 have the same bounded sets and smoothness depends only on the bornology by 1.8. So it is also cartesian closed, but the topology on $C^\infty(E, F)$ from 3.11 has to be bornologized. For an example showing the necessity see [Kriegl, 1983, p. 297] or [**11**, 5.4.19]: The topology on $C^\infty(\mathbb{R}, \mathbb{R}^{(\mathbb{N})})$ is not bornological, in fact $\{c = (c_n)_n \in C^\infty(\mathbb{R}, \mathbb{R}^{(\mathbb{N})}) : |c_n^{(n)}(0)| < 1\}$ is absolutely convex, bornivorous but not a 0-neighborhood.

We will in general, however, work in the category of locally convex spaces and smooth mappings, so function spaces carry the topology of 3.11.

The category of bounded (equivalently continuous) linear mappings between bornological vector spaces is in the same way equivalent to the category of all bounded linear mappings between all locally convex spaces, since a linear mapping is smooth if and only if it is bounded, by 2.11. It is closed under formation of colimits and under quotients (this is an easy consequence of 4.1.1). The Mackey-Ulam theorem [Jarchow, 1981, 13.5.4] tells us that a product of non trivial bornological vector spaces is bornological if and only if the index set does not admit a Ulam measure, i.e. a non trivial $\{0, 1\}$-valued measure on the whole power set. A cardinal admitting a Ulam measure has to be strongly inaccessible, so we can restrict set theory to exclude measurable cardinals.

Let $L(E_1, \dots, E_n; F)$ denote the *space of all bounded n-linear mappings* from $E_1 \times \dots \times E_n \to F$ with the topology of uniform convergence on bounded sets in $E_1 \times \dots \times E_n$.

5.2. Proposition. Exponential law for L. *There are natural bornological isomorphisms*

$$L(E_1, \dots, E_{n+k}; F) \cong L(E_1, \dots, E_n; L(E_{n+1}, \dots, E_{n+k}; F)).$$

Proof. We proof this for bilinear maps, the general case is completely analogous. We already know that bilinearity translates into linearity into the space of linear functions. Remains to prove boundedness. So let $\mathcal{B} \subseteq L(E_1, E_2; F)$ be given. Then $\mathcal{B}$ is bounded if and only if $\mathcal{B}(B_1 \times B_2) \subseteq F$ is bounded for all bounded $B_i \subseteq E_i$. This however is equivalent to $\mathcal{B}^\vee(B_1)$ is contained and bounded in $L(E_2, F)$ for all bounded $B_1 \subseteq E_1$, i.e. $\mathcal{B}^\vee$ is contained and bounded in $L(E_1, L(E_2, F))$. □

Recall that we have already put a structure on $L(E, F)$ in 3.17, namely the initial one with respect to the inclusion in $C^\infty(E, F)$. Let us now show that bornologically these definitions agree:

5.3. Lemma. Structure on L. *A subset is bounded in $L(E, F) \subseteq C^\infty(E, F)$ if and only if it is uniformly bounded on bounded subsets of E, i.e. $L(E, F) \to C^\infty(E, F)$ is initial.*

Proof. Let $\mathcal{B} \subseteq L(E, F)$ be bounded in $C^\infty(E, F)$, and assume that it is not uniformly bounded on some bounded set $B \subseteq E$. So there are $f_n \in \mathcal{B}$, $b_n \in B$, and $\ell \in F^*$ with $|\ell(f_n(b_n))| \geq n^n$. Then the sequence $n^{1-n} b_n$ converges fast to 0, and hence lies on some compact part of a smooth curve c by the special curve lemma 2.8. So $\mathcal{B}$ cannot be bounded, since otherwise $C^\infty(\ell, c) = \ell_* \circ c^* : C^\infty(E, F) \to$

$C^\infty(\mathbb{R},\mathbb{R}) \to \ell^\infty(\mathbb{R},\mathbb{R})$ would have bounded image, i.e. $\{\ell \circ f_n \circ c : n \in \mathbb{N}\}$ would be uniformly bounded on any compact interval.

Conversely, let $\mathcal{B} \subseteq L(E,F)$ be uniformly bounded on bounded sets and hence in particular on compact parts of smooth curves. We have to show that $d^n \circ c^* : L(E,F) \to C^\infty(\mathbb{R},F) \to \ell^\infty(\mathbb{R},F)$ has bounded image. But for linear smooth maps we have by the chain rule 3.18, recursively applied, that $d^n(f\circ c)(t) = f(c^{(n)}(t))$, and since $c^{(n)}$ is still a smooth curve we are done. □

Let us now generalize this result to multilinear mappings. For this we first characterize bounded multilinear mappings in the following two ways:

5.4. Lemma. *A multilinear mapping is bounded if and only if it is bounded on each sequence which converges Mackey to 0.*

Proof. Suppose that $f : E_1 \times \ldots \times E_k \to F$ is not bounded on some bounded set $B \subseteq E_1 \times \ldots \times E_k$. By composing with a linear functional we may assume that $F = \mathbb{R}$. So there are $b_n \in B$ with $\lambda_n^{k+1} := |f(b_n)| \to \infty$. Then $|f(\frac{1}{\lambda_n} b_n)| = \lambda_n \to \infty$, but $(\frac{1}{\lambda_n} b_n)$ is Mackey convergent to 0. □

5.5. Lemma. Bounded multilinear mappings are smooth. *Let $f : E_1 \times \ldots \times E_n \to F$ be a multilinear mapping. Then f is bounded if and only if it is smooth. For the derivative we have the product rule:*

$$df(x_1,\ldots,x_n)(v_1,\ldots,v_n) = \sum_{i=1}^n f(x_1,\ldots,x_{i-1},v_i,x_{i+1},\ldots,x_n).$$

In particular, we get for $f : E \supseteq U \to \mathbb{R}$, $g : E \supseteq U \to F$ and $x \in U$, $v \in E$ the Leibniz formula

$$(f\cdot g)'(x)(v) = f'(x)(v)\cdot g(x) + f(x)\cdot g'(x)(v).$$

Proof. We use induction on n. The case $n = 1$ is corollary 2.11. The induction goes as follows:

f is bounded
$\iff$ $f(B_1 \times \ldots \times B_n) = f^\vee(B_1 \times \ldots \times B_{n-1})(B_n)$ is bounded for all bounded sets B_i in E_i;
$\iff$ $f^\vee(B_1 \times \ldots \times B_{n-1}) \subseteq L(E_n,F) \subseteq C^\infty(E_n,F)$ is bounded, by 5.3;
$\iff$ $f^\vee : E_1 \times \ldots \times E_{n-1} \to C^\infty(E_n,F)$ is bounded;
$\iff$ $f^\vee : E_1 \times \ldots \times E_{n-1} \to C^\infty(E_n,F)$ is smooth by the inductive assumption;
$\iff$ $f : E_1 \times \ldots \times E_n \to F$ is smooth by cartesian closedness 3.13.

The formula for the derivative follows by direct evaluation of the directional difference quotient.

The particular case follows by application to the scalar multiplication $\mathbb{R} \times F \to F$. □ Di/Mi

Now let us show that also the structures coincide:

5.6. Proposition. Structure on space of multilinear maps. *The injection of $L(E_1,\ldots,E_n;F) \to C^\infty(E_1 \times \ldots \times E_n, F)$ is a bornological embedding.*

Proof. We can show this by induction. In fact, let $\mathcal{B} \subseteq L(E_1,\ldots,E_n;F)$. Then

$\mathcal{B}$ is bounded

$\iff$ $\mathcal{B}(B_1 \times \ldots \times B_n) = \mathcal{B}^\vee(B_1 \times \ldots \times B_{n-1})(B_n)$ is bounded for all bounded B_i in E_i;
$\iff$ $\mathcal{B}^\vee(B_1 \times \ldots \times B_{n-1}) \subseteq L(E_n, F) \subseteq C^\infty(E_n, F)$ is bounded, by 5.3;
$\iff$ $\mathcal{B}^\vee \subseteq C^\infty(E_1 \times \ldots \times E_{n-1}, C^\infty(E_n, F))$ is bounded by the inductive assumption;
$\iff$ $\mathcal{B} \subseteq C^\infty(E_1 \times \ldots \times E_n, F)$ is bounded by cartesian closedness 3.13. $\square$

Algebraic Tensor Product

Remark. The importance of the tensor product is twofold. First it allows linearizing of multi-linear mappings and secondly it allows to calculate function spaces.

We will consider the spaces of linear and multi-linear mappings between vector spaces. If we supply all vector spaces E, $E_1, \ldots, E_n$, F with the finest locally convex topology (i.e. the final locally convex topology with respect to the inclusions of all finite dimensional subspaces - on which the topology is unique) then all linear mappings are continuous and all multi-linear mappings are bounded (but not necessarily continuous as the evaluation map $\mathrm{ev} : E^* \times E \to \mathbb{K}$ on an infinite dimensional vector space E shows) and hence it is consistent to denote the corresponding function spaces by $L(E, F) = \mathcal{L}(E, F)$ and $L(E_1, \ldots E_n; F)$.

In more detail the first feature is:

3.1 Proposition. Linearization. *Given two linear spaces E and F, then there exists a solution $\otimes : E \times F \to E \otimes F$ – called the* ALGEBRAIC TENSOR PRODUCT *of E and F – to the following universal problem:*

$$\begin{tikzcd}
E \times F \arrow[rr, "\otimes"] \arrow[dr, "T"'] & & E \otimes F \arrow[dl, dotted, "!"', "\tilde{T}"] \\
& G &
\end{tikzcd}$$

Here $\otimes : E \times F \to E \otimes F$ and $T : E \times F \to G$ are bilinear and $\tilde{T}$ is linear.

Proof. In order to find $E \otimes F$ one considers first the case, where $G = \mathbb{R}$. Then we have that $\otimes^* : (E \otimes F)^* \to L(E, F; \mathbb{R})$ should be an isomorphism. Hence $E \otimes F$ could be realized as subspace of $(E \otimes F)^{**} \cong L(E, F; \mathbb{R})^*$. Obviously to each bilinear functional $T : E \times F \to \mathbb{R}$ corresponds the linear map $\mathrm{ev}_T : L(E, F; \mathbb{R})^* \to \mathbb{R}$. The map $\otimes : E \times F \to E \otimes F \subseteq L(E, F; \mathbb{R})^*$ has to be such that $\mathrm{ev}_T \circ \otimes = T$ for all bilinear functionals $T : E \times F \to \mathbb{R}$, i.e. $\otimes(x, y)(T) = (\mathrm{ev}_T \circ \otimes)(x, y) = T(x, y)$. Thus we have proved the existence of $\tilde{T} := \mathrm{ev}_T$ for $G = \mathbb{R}$. But uniqueness can be true only on the linear subspace generated by the image of $\otimes$, and hence we denote this subspace $E \otimes F$.

For bilinear mappings $T : E \times F \to G$ into an arbitrary vector space G, we consider the following diagram, which has quite some similarities with that used in the construction of the c^∞-completion in 2.31:

$$\begin{tikzcd}
E\times F \arrow[r, "\otimes"] \arrow[dd, "T"'] & E\otimes F \arrow[r, hook] \arrow[d, dotted, "(2)"] \arrow[ddl, dotted, "\widetilde{T}"', "(3)"] & L(E,F;\mathbb{R})^* \arrow[dl, dotted, "(1)"'] \arrow[dd, "\mathrm{ev}_{\lambda\circ T}"] \\
 & \prod_{G'}\mathbb{R} \arrow[dr, "\mathrm{pr}_\lambda"] & \\
G \arrow[ur, dotted, "\delta"'] \arrow[rr, "\lambda"] & & \mathbb{R}
\end{tikzcd}$$

The right dashed arrow (1) and δ exist uniquely by the universal property of the product in the center. The arrow (2) exists uniquely as restriction of (1) to the subspace $E\otimes F$. Finally (3) exists, since the generating subset $\otimes(E\times F)$ in $E\otimes F$ is mapped to $T(E\times F)\subseteq G$ and since δ is injective. $\square$

Note that $\otimes$ extends to a functor, by defining $T\otimes S$ via the following diagram:

$$\begin{tikzcd}
E_1\times F_1 \arrow[r, "\otimes"] \arrow[d, "T\times S"'] & E_1\otimes F_1 \arrow[d, dotted, "!"', "T\otimes S"] \\
E_2\times F_2 \arrow[r, "\otimes"'] & E_2\otimes F_2
\end{tikzcd}$$

Furthermore one easily proves the existence of the following natural isomorphisms:

$$E\otimes\mathbb{R}\cong E$$
$$E\otimes F\cong F\otimes E$$
$$(E\otimes F)\otimes G\cong E\otimes(F\otimes G)$$

In analogy to the exponential law for smooth mappings or continuous mappings, we show now the existence of a natural isomorphism

$$L(E,F;G)\cong L(E,L(F,G))$$

again denoted by $(\_)^\vee$ with inverse isomorphism $(\_)^\wedge$ given by the same formula as above.

In fact for a bilinear mapping $T:E\times F\to G$, the mapping $T^\vee$ has values in $L(F,G)$, since $T(x,\_)$ is linear, and it is linear, since $L(F,G)$ carries the initial vector space structure with respect to the evaluations ev_y and $\mathrm{ev}_y\circ T^\vee=T(\_,y)$ is also linear. The same way one shows that the converse implication is also true.

Note that if both spaces E and F are finite dimensional, then so is $L(E,F;\mathbb{R})$ and hence also the dual $L(E,F;\mathbb{R})^*$. But then $E\otimes F$ is finite dimensional too (in fact $\dim(E\otimes F)=\dim E\cdot\dim F$), as we will see in 3.30, and hence $E\otimes F=(E\otimes F)^{**}=L(E,F;\mathbb{R})^*$.

If one factor is infinite dimensional and the other one is not 0, then this is not true. In fact take $F=\mathbb{R}$, then $E\otimes\mathbb{R}\cong E$ whereas $L(E,\mathbb{R};\mathbb{R})^*\cong L(E,L(\mathbb{R},\mathbb{R}))^*\cong L(E,\mathbb{R})^*=E^{**}$.

Projective Tensor Product

We turn first to the property of making bilinear continuous mappings into linear ones. We call the corresponding solution the PROJECTIVE TENSOR PRODUCT of E and F and denote it by $E\otimes_\pi F$. Obviously it can be obtained by taking the algebraic tensor product and supplying it with the finest locally convex topology such that $E\times F\to E\otimes F$ is continuous. This topology exists since the union of locally convex topologies is locally convex and $E \times F \to E \otimes F$ is continuous for the weak topology on $E \otimes F$ generated by those linear functionals which correspond to continuous bilinear functionals on $E \times F$. It has the universal property, since the inverse image of a locally convex topology under a linear mapping $\tilde{T}$ is again a locally convex topology, such that $\otimes$ is continuous, provided the associated bilinear mapping T is continuous. However, it is not obvious that this topology is separated, and we prove that now. We will denote the SPACE OF CONTINUOUS LINEAR MAPPINGS from E to F by $\mathcal{L}(E, F)$, and the SPACE OF CONTINUOUS MULTI-LINEAR MAPPINGS by $\mathcal{L}(E_1, \ldots, E_n; F)$. If all $E_1, \ldots, E_n$ are the same space E, we will also write $\mathcal{L}^n(E; F)$.

3.3 Lemma. *$E \otimes_\pi F$ is Hausdorff provided E and F are.*

Proof. It is enough to show that the set $E^* \times F^*$ separates points in $E \otimes F$ or even in $L(E, F; \mathbb{R})^*$. So let $0 \neq z = \sum_k x_k \otimes y_k$ be given. By replacing linear dependent x_k by the corresponding linear combinations and using bilinearity of $\otimes$, we may assume that the x_k are linearly independent. Now choose $x^* \in E^*$ and $y^* \in F^*$ be such that $x^*(x_k) = \delta_{1,k}$ and $y^*(y_1) = 1$. Then $(x^* \otimes y^*)(z) = 1 \neq 0$. □

Since a bilinear mapping is continuous iff it is so at 0, a 0-neighborhood basis in $E \otimes_\pi F$ is given by all those absolutely convex sets, for which the inverse image under $\otimes$ is a 0-neighborhood in $E \times F$. A basis is thus given by the absolutely convex hulls denoted $U \otimes V$ of the images of $U \times V$ under $\otimes$, where U resp. V runs through a 0-neighborhood basis of E resp. F. We only have to show that these sets $U \otimes V$ are absorbing. So let $z = \sum_k x_k \otimes y_k \in E \otimes F$ be arbitrary. Then there are $a_k > 0$ and $b_k > 0$ such that $x_k \in a_k U$ and $y_k \in b_k V$ and hence $z = \sum_{k \le K} a_k\, b_k \frac{x_k}{a_k} \otimes \frac{y_k}{b_k} \in (\sum_k a_k\, b_k) \cdot \langle U \otimes V \rangle_{\text{abs.conv.}}$. The Minkowski-functionals $p_{U \otimes V}$ form a base of the seminorms of $E \otimes_\pi F$ and we will denote them by $\pi_{U,V}$. In terms of the Minkowski-functionals p_U and p_V of U and V we obtain that $z \in (\sum_k p_U(x_k)\, p_V(y_k))\, U \otimes V$ for any $z = \sum_k x_k \otimes y_k$ since $x_k \in p_U(x_k) \cdot U$ for closed U, and thus $p_{U \otimes V}(z) \le \inf\{\sum_k p_U(x_k)\, p_V(y_k) : z = \sum_k x_k \otimes y_k\}$. We now show the converse:

3.4 Proposition. Seminorms of the projective tensor product.

$$p_{U \otimes V}(z) = \inf\left\{ \sum_k p_U(x_k) \cdot p_V(y_k) : z = \sum_k x_k \otimes y_k \right\}.$$

Proof. Let $z \in \lambda \cdot U \otimes V$ with $\lambda > 0$. Then $z = \lambda \sum \lambda_k u_k \otimes v_k$ with $u_k \in U$, $v_k \in V$ and $\sum_k |\lambda_k| = 1$. Hence $z = \sum x_k \otimes v_k$, where $x_k = \lambda \lambda_k u_k$, and $\sum_k p_U(x_k) \cdot p_V(v_k) \le \sum \lambda |\lambda_k| = \lambda$. Taking the infimum of all λ gives now that $p_{U \otimes V}(z)$ is greater or equal to the infimum on the right side. □

3.5 Corollary. *$E \otimes_\pi F$ is normable (metrizable) provided E and F are.*

3.6 Lemma. The semi-norms of decomposable tensors.

$$p_{U,V}(x \otimes y) = p_U(x) \cdot p_V(y).$$

Proof. According to [5, 7.1.8] there are $x^* \in E^*$ and $y^* \in F^*$ such that $|x^*| \le p_U$ and $|y^*| \le p_V$ and $x^*(x) = p_U(x)$ and $y^*(y) = p_V(y)$. If $x \otimes y = \sum_k x_k \otimes y_k$, then

$$p_{U \otimes V}(x \otimes y) \le p_U(x) \cdot p_V(y) = x^*(x) \cdot y^*(y) = (x^* \otimes y^*)(x \otimes y) =$$
$$= \sum_k x^*(x_k) \cdot y^*(y_k) \le \sum_k p_U(x_k) \cdot p_V(y_k),$$

and taking the infimum gives the desired result. □

3.7 Remark. Functorality. Given two continuous linear maps $T_1 : E_1 \to F_1$ and $T_2 : E_2 \to F_2$ we can consider bilinear continuous map given by composing $T_1 \times T_2 : E_1 \times E_2 \to F_1 \times F_2$ with $\otimes : F_1 \times F_2 \to F_1 \otimes F_2$. By the universal property of $E_1 \times E_2 \to E_1 \otimes E_2$ we obtain a continuous linear map denoted by $T_1 \otimes T_2 : E_1 \otimes E_2 \to F_1 \otimes F_2$.

$$\begin{tikzcd} E_1 \times E_2 \arrow[r, "\otimes"] \arrow[d, "T_1 \times T_2"'] & E_1 \otimes E_2 \arrow[d, dotted, "T_1 \otimes T_2"] \\ F_1 \times F_2 \arrow[r, "\otimes"'] & F_1 \otimes F_2 \end{tikzcd}$$

By the uniqueness of the linearization one obtains immediately that $\otimes$ is a functor. Because of the uniqueness of universal solutions one sees easily that one has natural isomorphisms $\mathbb{R} \otimes E \cong E$, $E \otimes F \cong F \otimes E$ and $(E \otimes F) \otimes G \cong E \otimes (F \otimes G)$.

3.14 Example. $\otimes_\pi$ does not preserve embeddings.
In fact consider the isometric embedding $\ell^2 \to C(K)$, where K is the closed unit-ball of $(\ell^2)^*$ supplied with its compact topology of pointwise convergence, see the corollary to the Alaoğlu-Bourbaki-theorem in [**5**, 7.4.12]. This subspace has however no topological complement, since $C(K)$ has the Dunford-Pettis property (see [**12**, 20.7.8], i.e. $x_n^*(x_n) \to 0$ for every two sequences $x_n \to 0$ in $\sigma(E, E^*)$ and $x_n^* \to 0$ in $\sigma(E^*, E^{**})$), but no infinite dimensional reflexive Banach space like ℓ^2 has it (e.g. $x_n := e_n$, $x_n^* := e_n$) and hence cannot be a complemented subspace of $C(K)$, see [**12**, 20.7].

Suppose now that $\ell^2 \otimes_\pi (\ell^2)^* \to C(K) \otimes_\pi (\ell^2)^*$ were an embedding. The duality mapping $\mathrm{ev} : \ell^2 \times (\ell^2)^* \to \mathbb{R}$ yields a continuous linear mapping $s : \ell^2 \otimes_\pi (\ell^2)^* \to \mathbb{R}$ and would hence have a continuous linear extension $\tilde{s} : C(K) \otimes (\ell^2)^* \to \mathbb{R}$. The corresponding bilinear map would give a continuous mapping $\tilde{s}^\vee : C(K) \to (\ell^2)^{**} \cong \ell^2$, which is a left inverse to the embedding $\ell^2 \to C(K)$, a contradiction.

In connection with the second usage of tensor products we would expect that for the product $E^{\mathbb{N}} = (\mathbb{R} \otimes_\pi E)^{\mathbb{N}} = \mathbb{R}^{\mathbb{N}} \otimes_\pi E$, i.e. we are looking for preservation of certain products. But even purely algebraically this fails to be true. In fact take the coproduct $E = \mathbb{R}^{(\mathbb{N})}$. Using that $\mathbb{R}^{\mathbb{N}} \otimes (\_)$ is left-adjoint and hence preserves colimits we get $\mathbb{R}^{\mathbb{N}} \otimes \mathbb{R}^{(\mathbb{N})} \cong (\mathbb{R}^{\mathbb{N}} \otimes \mathbb{R})^{(\mathbb{N})} \cong (\mathbb{R}^{\mathbb{N}})^{(\mathbb{N})}$, which is strictly smaller than $(\mathbb{R}^{(\mathbb{N})})^{\mathbb{N}}$. However in both spaces the union $\bigcup_n E^n$ is dense, so after taking completions there should be some chance. In order to work with completions we have to show preservation of dense embeddings. To obtain such a result we need a dual characterization of such mappings. And this we treat next.

5.7. Bornological tensor product. It is natural to consider the universal problem of linearizing bounded bilinear mappings. The solution is given by the *bornological tensor product* $E \otimes_\beta F$, i.e. the algebraic tensor product with the finest locally convex topology such that $E \times F \to E \otimes F$ is bounded. A 0-neighborhood basis of this topology is given by those absolutely convex sets, which absorb $B_1 \otimes B_2$ for all bounded $B_1 \subseteq E_1$ and $B_2 \subseteq E_2$. Note that this topology is bornological since it is the finest locally convex topology with given bounded linear mappings on it.

Theorem. *The bornological tensor product is the left adjoint functor to the Hom-functor $L(E, \quad)$ on the category of bounded linear mappings between locally convex* *spaces, and one has the following bornological isomorphisms:*

$$L(E \otimes_\beta F, G) \cong L(E, F; G) \cong L(E, L(F, G))$$
$$E \otimes_\beta \mathbb{R} \cong E$$
$$E \otimes_\beta F \cong F \otimes_\beta E$$
$$(E \otimes_\beta F) \otimes_\beta G \cong E \otimes_\beta (F \otimes_\beta G)$$

Furthermore, the bornological tensor product preserves colimits. It neither preserves embeddings nor countable products.

Proof. We show first that this topology has the universal property for bounded bilinear mappings $f : E_1 \times E_2 \to F$. Let U be an absolutely convex zero neighborhood in F, and let B_1, B_2 be bounded sets. Then $f(B_1 \times B_2)$ is bounded, hence it is absorbed by U. Then $\tilde{f}^{-1}(U)$ absorbs $\otimes(B_1 \times B_2)$, where $\tilde{f} : E_1 \otimes E_2 \to F$ is the canonically associated linear mapping. So $\tilde{f}^{-1}(U)$ is in the zero neighborhood basis of $E_1 \otimes_\beta E_2$ described above. Therefore, $\tilde{f}$ is continuous.

A similar argument for sets of mappings shows that the first isomorphism $L(E \otimes_\beta F, G) \cong L(E, F; G)$ is bornological.

The topology on $E_1 \otimes_\beta E_2$ is finer than the projective tensor product topology, and so it is Hausdorff. The rest of the positive results is clear.

The counter-example for embeddings given for the projective tensor product works also, since all spaces involved are Banach.

Since the bornological tensor-product preserves coproducts it cannot preserve products. In fact $(\mathbb{R} \otimes_\beta \mathbb{R}^{(\mathbb{N})})^{\mathbb{N}} \cong (\mathbb{R}^{(\mathbb{N})})^{\mathbb{N}}$ whereas $\mathbb{R}^{\mathbb{N}} \otimes_\beta \mathbb{R}^{(\mathbb{N})} \cong (\mathbb{R}^{\mathbb{N}} \otimes_\beta \mathbb{R})^{(\mathbb{N})} \cong (\mathbb{R}^{\mathbb{N}})^{(\mathbb{N})}$. □

5.8. Proposition. Projective versus bornological tensor product. *If every bounded bilinear mapping on $E \times F$ is continuous then $E \otimes_\pi F = E \otimes_\beta F$. In particular, we have $E \otimes_\pi F = E \otimes_\beta F$ for any two metrizable spaces, and for a normable space F we have $E_{born} \otimes_\pi F = E \otimes_\beta F$.*

Proof. Recall that $E \otimes_\pi F$ carries the finest locally convex topology such that $\otimes : E \times F \to E \otimes F$ is continuous, whereas $E \otimes_\beta F$ carries the finest locally convex topology such that $\otimes : E \times F \to E \otimes F$ is bounded. So we have that $\otimes : E \times F \to E \otimes_\beta F$ is bounded and hence by assumption continuous, and thus the topology of $E \otimes_\pi F$ is finer than that of $E \otimes_\beta F$. Since the converse is true in general, we have equality.

In [5, 3.1.6] it is shown that in metrizable locally convex spaces the convergent sequences coincide with the Mackey-convergent ones. Now let $T : E \times F \to G$ be bounded and bilinear. We have to show that T is continuous. So let (x_n, y_n) be a convergent sequence in $E \times F$. Without loss of generality we may assume that its limit is $(0, 0)$. So there are $\mu_n \to \infty$ such that $\{\mu_n(x_n, y_n) : n \in \mathbb{N}\}$ is bounded and hence also $T\Big(\{\mu_n(x_n, y_n) : n \in \mathbb{N}\}\Big) = \Big\{\mu_n^2 T(x_n, y_n) : n \in \mathbb{N}\Big\}$, i.e. $T(x_n, y_n)$ converges even Mackey to 0.

If F is normable and $T : E_{born} \times F \to G$ is bounded bilinear then $T^\vee : E_{born} \to L(F, G)$ is bounded, and since E_{born} is bornological it is even continuous. Clearly, for normed spaces F the evaluation map $\mathrm{ev} : L(F, G) \times F \to G$ is continuous, and hence $T = \mathrm{ev} \circ (T^\vee \times F) : E_{born} \times F \to G$ is continuous. Thus, $E_{born} \otimes_\pi F = E \otimes_\beta F$. □

Note that the bornological tensor product is invariant under bornologification, i.e. $E_{born} \otimes_\beta F_{born} \cong E \otimes_\beta F$. So it is no loss of generality to assume that both spaces are bornological. Keep however in mind that the corresponding identity for the projective tensor product does not hold. Another possibility to obtain the identity $E \otimes_\pi F = E \otimes_\beta F$ is to assume that E and F are bornological and every separately continuous bilinear mapping on $E \times F$ is continuous. In fact, every bounded bilinear mapping is obviously separately bounded, and since E and F are assumed to be bornological, it has to be separately continuous. We want to find another class beside the Fréchet spaces (see [**5**, 5.2.8]) which satisfies these assumptions.

3.47 Theorem. Continuity versus separately continuity. *Let E and F be two barreled spaces with a countable base of bornology. Then every separately continuous bilinear map $E \times F \to G$ is continuous.*

Proof. Let A_n and B_n be a basis of the bornologies of E and F. Let $T : E \times F \to G$ be separately continuous. Then $T^\vee : E \to \mathcal{L}(F, G)$ is continuous for the topology of pointwise convergence on $\mathcal{L}(F, G)$. Thus $T^\vee(A_k)$ is bounded for this topology, and since F is barreled it is equi-continuous. Thus for every 0-neighborhood W in G there exists a 0-neighborhood V_k in F with $T(A_k \times V_k) \subseteq W$. By symmetry there exists a 0-neighborhood U_k in E with $T(U_k \times B_k) \subseteq W$. We have to show that this implies for gDF-spaces E and F the continuity of T, see [**12**, 15.6.1]. Since E is quasi-normable, we can find for every 0-neighborhood U_n a 0-neighborhood U'_n such that for every $\rho > 0$ there is some $k(n, \rho) \in \mathbb{N}$ with $U'_n \subseteq \rho U_n + A_{k(n,\rho)}$. Since A_k is a basis of bounded sets there exist $\rho_n > 0$ such that $U := \bigcap_n \rho_n U'_n$ is a 0-neighborhood in the topology generated by $\{A_n\}$, see [**12**, 12.3.2]. And this topology coincides with the given topology since E is gDF, by [**12**, 12.3.6]. Let $W_n := V_{k(n,1/\rho_n)}$. Then $V := \langle \bigcup_n \frac{1}{\rho_n} W_n \cap B_n$ is a 0-neighborhood again by [**12**, 12.3.6] and by the description of a 0-neighborhood basis of the topology induced by $\{B_n\}_n$ given in [**12**, 12.3.1]. We claim that $T(U \times V) \subseteq W$. In fact take $x \in U$ and $y \in V$. Then y is an absolutely convex combination of $y_n \in \frac{1}{\rho_n} W_n \cap B_n$. Since $x \in \rho_n U'_n \subseteq U_n + \rho_n A_{k(n,1/\rho_n)}$ there are $u_n \in U_n$ and $a_n \in A_{k(n,1/\rho_n)}$ with $x = u_n + \rho_n a_n$. So

$$T(x, y_n) = T(u_n, y_n) + T(\rho_n a_n, y_n) \in T(U_n \times B_n) + \rho_n T(A_{k(n,1/\rho_n)} \times \frac{1}{\rho_n} W_n) \subseteq 2\,W$$

Hence the same is true for the absolutely convex combination $T(x, y)$, i.e. $T(U \times V) \subseteq 2\,W$. $\square$

3.48 Corollary. Projective versus bornological tensor product for LB-spaces. *Let E and F be countable inductive limits of Banach spaces (e.g. the duals of metrizable spaces with their bornological topology, i.e. the bornologification of the strong topology). Then $E \otimes_\pi F \cong E \otimes_\beta F$.*

Proof. Let $T : E \times F \to G$ be bounded. Since both spaces are bornological T is separately continuous and since both spaces are barreled and DF it is continuous. This is enough to guarantee the equality of the two tensor products by $\boxed{3.39}$. $\square$

5.9. Corollary. *The following mappings are bounded multilinear.*

(1) $\lim : \mathrm{Nat}(\mathcal{F}, \mathcal{G}) \to L(\lim \mathcal{F}, \lim \mathcal{G})$, *where $\mathcal{F}$ and $\mathcal{G}$ are two functors on the same index category, and where* $\mathrm{Nat}(\mathcal{F}, \mathcal{G})$ *denotes the space of all natural transformations with the structure induced by the embedding into* $\prod_i L(\mathcal{F}(i), \mathcal{G}(i))$.
(2) $\mathrm{colim} : \mathrm{Nat}(\mathcal{F}, \mathcal{G}) \to L(\mathrm{colim}\, \mathcal{F}, \mathrm{colim}\, \mathcal{G})$.

(3)
$$\begin{aligned} L : L(E_1, F_1) \times \ldots \times L(E_n, F_n) \times L(F, E) \to \\ \to L(L(F_1, \ldots, F_n; F), L(E_1, \ldots, E_n; E)) \\ (T_1, \ldots, T_n, T) \mapsto (S \mapsto T \circ S \circ (T_1 \times \ldots \times T_n)); \end{aligned}$$

(4) $\bigotimes\limits^n_\beta : L(E_1, F_1) \times \ldots \times L(E_n, F_n) \to L(E_1 \otimes_\beta \cdots \otimes_\beta E_n, F_1 \otimes_\beta \cdots \otimes_\beta F_n)$.

(5) $\bigwedge^n : L(E, F) \to L(\bigwedge^n E, \bigwedge^n F)$, *where* $\bigwedge^n E$ *is the linear subspace of all alternating tensors in* $\bigotimes^n_\beta E$*. It is the universal solution of*

$$L\Big(\bigwedge^n E, F\Big) \cong L^n_{alt}(E; F),$$

where $L^n_{alt}(E; F)$ *is the space of all bounded* n*-linear alternating mappings* $E \times \ldots \times E \to F$*. This space is a direct summand of* $L^n(E; F) := L(E, \ldots, E; F)$.

(6) $\bigvee^n : L(E, F) \to L(\bigvee^n E, \bigvee^n F)$, *where* $\bigvee^n E$ *is the linear subspace of all symmetric tensors in* $\bigotimes^n_\beta E$*. It is the universal solution of*

$$L\Big(\bigvee^n E, F\Big) \cong L^n_{sym}(E; F),$$

where $L^n_{sym}(E; F)$ *is the space of all bounded* n*-linear symmetric mappings* $E \times \ldots \times E \to F$*. This space is also a direct summand of* $L^n(E; F)$.

(7) $\bigotimes_\beta : L(E, F) \to L(\bigotimes_\beta E, \bigotimes_\beta F)$, *where* $\bigotimes_\beta E := \coprod_{n=0}^\infty \bigotimes\limits^n{}_\beta E$ *is the tensor algebra of* E*. Note that it has the universal property of prolonging bounded linear mappings with values in locally convex spaces, which are algebras with bounded operations, to continuous algebra homomorphisms:*

$$L(E, F) \cong \mathrm{Alg}\Big(\bigotimes_\beta E, F\Big).$$

(8) $\bigwedge : L(E, F) \to L(\bigwedge E, \bigwedge F)$, *where* $\bigwedge E := \coprod_{n=0}^\infty \bigwedge^n E$ *is the exterior algebra. It has the universal property of prolonging bounded linear mappings to continuous algebra homomorphisms into graded-commutative algebras, i.e. algebras in the sense above, which are as vector spaces a coproduct* $\coprod_{n\in\mathbb{N}} E_n$ *and the multiplication maps* $E_k \times E_l \to E_{k+l}$ *and for* $x \in E_k$ *and* $y \in E_l$ *one has* $x \cdot y = (-1)^{kl} y \cdot x$.

(9) $\bigvee : L(E, F) \to L(\bigvee E, \bigvee F)$, *where* $\bigvee E := \coprod_{n=0}^\infty \bigvee^n E$ *is the symmetric algebra. It has the universal property of prolonging bounded linear mappings to continuous algebra homomorphisms into commutative algebras.*

Recall that the symmetric product is given as the image of the *symmetrizer* $\mathrm{sym} : E \otimes_\beta \cdots \otimes_\beta E \to E \otimes_\beta \cdots \otimes_\beta E$ given by

$$x_1 \otimes \cdots \otimes x_n \to \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

Similarly the wedge product is given as the image of the *alternator*

$$\mathrm{alt} : E \otimes_\beta \cdots \otimes_\beta E \to E \otimes_\beta \cdots \otimes_\beta E$$

$$\text{given by } x_1 \otimes \cdots \otimes x_n \to \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \mathrm{sign}(\sigma)\, x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

Symmetrizer and alternator are bounded projections, so both subspaces are complemented in the tensor product.

Proof. All results follow easily by flipping coordinates until only a composition of products of evaluation maps remains.

In particular, consider the following diagrams:

(1)

$$\begin{tikzcd}
\operatorname{Nat}(\mathcal{F},\mathcal{G}) \times \lim \mathcal{F} \arrow[r] \arrow[d, "\mathrm{pr}_i \times \mathrm{pr}_i"] & \mathcal{G} \arrow[d, "\mathrm{pr}_i"] \\
L(\mathcal{F}(i),\mathcal{G}(i)) \times \mathcal{F}(i) \arrow[r, "\mathrm{ev}"] & \mathcal{G}(i)
\end{tikzcd}$$

(2)

$$\begin{tikzcd}
\mathcal{F}(i) \arrow[r, "\mathrm{inj}_i"] & \operatorname{colim} \mathcal{F} \arrow[r, dotted] & L(\operatorname{Nat}(\mathcal{F},\mathcal{G}), \operatorname{colim} \mathcal{G})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}(i) \times \operatorname{Nat}(\mathcal{F},\mathcal{G}) \arrow[r, dotted] \arrow[d, "\mathrm{Id} \times \mathrm{pr}_i"'] & \operatorname{colim} \mathcal{G} \\
\mathcal{F}(i) \times L(\mathcal{F}(i),\mathcal{G}(i)) \arrow[r, "\mathrm{ev}"] & \mathcal{G}(i) \arrow[u, "\mathrm{inj}_i"]
\end{tikzcd}$$

(3)

$$\begin{tikzcd}
\Big(\prod_i L(E_i,F_i)\Big) \times L(F,E) \times L(F_1,\dots,F_n;F) \times \prod_i E_i \arrow[r, dotted] \arrow[d, "\cong"] & E \\
L(F,E) \times L(F_1,\dots,F_n;F) \times \prod_i (L(E_i,F_i) \times E_i) \arrow[d, "\mathrm{Id} \times \mathrm{ev} \times \ldots \times \mathrm{ev}"] & \\
L(F,E) \times L(F_1,\dots,F_n;F) \times \prod_i F_i \arrow[r, "\mathrm{ev}"'] & L(F,E) \times F \arrow[uu, "\mathrm{ev}"]
\end{tikzcd}$$

(4)

$$\begin{tikzcd}
E_1 \times \ldots \times E_n \arrow[r, dotted] & L(L(E_1,F_1),\dots,L(E_n,F_n); F_1 \otimes_\beta \cdots \otimes_\beta F_n)
\end{tikzcd}$$

$$\begin{tikzcd}
E_1 \times \ldots \times E_n \times L(E_1,F_1) \times \ldots \times L(E_n,F_n) \arrow[r, dotted] \arrow[d, "\cong"] & F_1 \otimes_\beta \cdots \otimes_\beta F_n \\
L(E_1,F_1) \times E_1 \times \ldots \times L(E_n,F_n) \times E_n \arrow[r, "\mathrm{ev} \times \ldots \times \mathrm{ev}"] & F_1 \times \ldots \times F_n \arrow[u]
\end{tikzcd}$$

(5)

$$\begin{tikzcd}
L(E,F) \arrow[r, dotted] \arrow[d, "\Delta"] & L(\bigwedge^n E, \bigwedge^n F) \\
L(E,F) \times \ldots \times L(E,F) \arrow[r, "\bigotimes^n"] & L(\bigotimes_\beta^n E, \bigotimes_\beta^n F) \arrow[u, "L(\mathrm{incl},\mathrm{alt})"']
\end{tikzcd}$$

The projection $L^n(E;F) \to L^n_{\mathrm{alt}}(E;F)$ is given by the alternator

$$T \mapsto \Big((v_1,\dots,v_n) \mapsto \frac{1}{n!} \sum_\sigma \operatorname{sign}(\sigma)\, T(v_{\sigma(1)},\dots,v_{\sigma(n)})\Big).$$

The universal proporty follows from the diagram:

$$\begin{tikzcd} E\times\ldots\times E \arrow[r, "\otimes"] \arrow[dr, "f"'] & E\otimes_\beta\cdots\otimes_\beta E \arrow[r, "alt"] \arrow[d, "\tilde f"] & \bigwedge^n E \arrow[dl, "\tilde f|_{\bigwedge^n E}"] \\ & F & \end{tikzcd}$$

([6])

$$\begin{tikzcd} L(E,F) \arrow[r, dotted] \arrow[d, "\Delta"] & L(\bigvee^n E, \bigvee^n F) \\ L(E,F)\times\ldots\times L(E,F) \arrow[r, "\bigotimes_\beta^n"] & L(\bigotimes_\beta^n E, \bigotimes_\beta^n F) \arrow[u, "L(\mathrm{incl},\mathrm{alt})"'] \end{tikzcd}$$

The projection $L^n(E;F) \to L^n_{\mathrm{sym}}(E;F)$ symmetrizer

$$T \mapsto \Big((v_1,\ldots,v_n) \mapsto \frac{1}{n!}\sum_\sigma T(v_{\sigma(1)},\ldots,v_{\sigma(n)})\Big).$$

The universal property follows from the diagram:

$$\begin{tikzcd} E\times\ldots\times E \arrow[r, "\otimes"] \arrow[dr, "f"'] & E\otimes_\beta\cdots\otimes_\beta E \arrow[r, "sym"] \arrow[d, "\tilde f"] & \bigvee^n E \arrow[dl, "\tilde f|_{\bigvee^n E}"] \\ & F & \end{tikzcd}$$

([7])

$$\begin{tikzcd} L(E,F) \arrow[r, dotted, "\bigotimes"] \arrow[d, "(\bigotimes^n)_n"] & L(\bigotimes_\beta E, \bigotimes_\beta F) \\ \prod_n L(\bigotimes_\beta^n E, \bigotimes_\beta^n F) \arrow[r, "\prod_n \mathrm{incl}_*"] & \prod_n L(\bigotimes_\beta^n E, \coprod_n \bigotimes_\beta^n F) \arrow[u, "\cong"'] \end{tikzcd}$$

The universal property holds, since to $T\in L(E,F)$ we can associate $\sum_n \mu_n\circ\bigotimes^n T$, where $\mu_n : \bigotimes F \to F$ denotes the n-fold multiplication of the algebra F.

([8])

$$\begin{tikzcd} L(E,F) \arrow[r, dotted, "\bigwedge"] \arrow[d, "(\bigwedge^n)_n"] & L(\bigwedge E, \bigwedge F) \\ \prod_n L(\bigwedge^n E, \bigwedge^n F) \arrow[r, "\prod_n \mathrm{incl}_*"] & \prod_n L(\bigwedge^n E, \coprod_n \bigwedge^n F) \arrow[u, "\cong"'] \end{tikzcd}$$

([9])

$$\begin{tikzcd} L(E,F) \arrow[r, dotted, "\bigvee"] \arrow[d, "(\bigvee^n)_n"] & L(\bigvee E, \bigvee F) \\ \prod_n L(\bigvee^n E, \bigvee^n F) \arrow[r, "\prod_n \mathrm{incl}_*"] & \prod_n L(\bigvee^n E, \coprod_n \bigvee^n F) \arrow[u, "\cong"'] \end{tikzcd}$$

□

5.12. Theorem. Taylor formula. *Let $f : U \to F$ be smooth, where U is c^∞-open in E. Then for each segment $[x, x+y] = \{x+ty : 0 \le t \le 1\} \subseteq U$ we have*

$$f(x+y) = \sum_{k=0}^{n} \frac{1}{k!} d^k f(x) y^k + \int_0^1 \frac{(1-t)^n}{n!} d^{n+1} f(x+ty) y^{n+1} dt,$$

where $y^k = (y, \dots, y) \in E^k$.

Proof. Recall that we can form iterated derivatives as follows:

$$\begin{aligned}
f &: E \supseteq U \to F \\
df &: E \supseteq U \to L(E,F) \\
d(df) &: E \supseteq U \to L(E, L(E,F)) \cong L(E,E;F) \\
&\vdots \\
d(\dots(d(df))\dots) &: E \supseteq U \to L(E, \dots, L(E,F)\dots) \cong L(E,\dots,E;F)
\end{aligned}$$

Thus, the iterated derivative $d^n f(x)(v_1, \dots, v_n)$ is given by

$$\tfrac{\partial}{\partial t_1}|_{t_1=0} \cdots \tfrac{\partial}{\partial t_n}|_{t_n=0} f(x + t_1 v_1 + \dots + t_n v_n) = \partial_1 \dots \partial_n \tilde{f}(0, \dots, 0),$$

where $\tilde{f}(t_1, \dots, t_n) := f(x + t_1 v_1 + \dots + t_n v_n)$.

This Taylor formula is an assertion on the smooth curve $t \mapsto f(x+ty)$. Using functionals λ we can reduce it to the scalar valued case since $(\frac{d}{dt})^k|_{t=0} \lambda(f(x+ty)) = \lambda(f^{(k)}(x) y^k)$, or we proceed directly by induction on n: The first step is (6) in 2.6, and the induction step is partial integration of the remainder integral. □

5.11. Proposition. Symmetry of higher derivatives. *Let $f : E \supseteq U \to F$ be smooth. The n-th derivative $f^{(n)}(x) = d^n f(x)$, considered as an element of $L^n(E;F)$, is symmetric, so lies in the space $L^n_{sym}(E;F) \cong L(\bigvee^n E; F)$*

Proof. The result now follows from the finite dimensional property, since the iterated derivative $d^n f(x)(v_1, \dots, v_n)$ is given by

$$\tfrac{\partial}{\partial t_1}|_{t_1=0} \cdots \tfrac{\partial}{\partial t_n}|_{t_n=0} f(x + t_1 v_1 + \dots + t_n v_n) = \partial_1 \dots \partial_n \tilde{f}(0, \dots, 0),$$

where $\tilde{f}(t_1, \dots, t_n) := f(x + t_1 v_1 + \dots + t_n v_n)$. □

5.13. Corollary. *The following subspaces are direct summands:*

$$L(E_1, \dots, E_n; F) \subseteq C^\infty(E_1 \times \dots \times E_n, F),$$
$$L^n_{sym}(E;F) \xrightarrow{\Delta^*} C^\infty(E,F).$$

Note that direct summand is meant in the bornological category, i.e. the embedding admits a left-inverse in the category of bounded linear mappings, or, equivalently, with respect to the bornological topology it is a topological direct summand.

Proof. The projection for $L(E,F) \subseteq C^\infty(E,F)$ is $f \mapsto df(0)$. The statement on L^n follows by induction using the exponential laws 3.13 and 5.2.

The second embedding is given by $\triangle^*$, which is bounded and linear $C^\infty(E \times \dots \times E, F) \to C^\infty(E,F)$. Here $\Delta : E \to E \times \dots \times E$ denotes the diagonal mapping

$x \mapsto (x, \dots, x)$.

$$\begin{tikzcd}
L^k_{\text{sym}}(E;F) \arrow[r, hook] \arrow[d, dotted] & L^k(E;F) \arrow[d, hook] \\
C^\infty(E,F) & C^\infty(E\times\dots\times E, F) \arrow[l, "\Delta^*"]
\end{tikzcd}$$

A bounded linear left inverse $C^\infty(E,F) \to L^k_{\text{sym}}(E;F)$ is given by $f \mapsto \frac{1}{k!}d^k f(0)$, since each $f = \Delta^*(\tilde f)$ in the image of Δ^* is k-homogeneous and so $d^k f(0)v^k = \left(\frac{d}{dt}\right)^k f(tv)|_{t=0} = ((\frac{d}{dt})^k t^k)|_{t=0} f(v) = k!\, f(v) = k!\,\tilde f\, v^k$ and by the polarization formula $\boxed{7.13}$ $\tilde f = \frac{1}{k!}d^k f(0)$.

5.15. Definition. A smooth mapping $f : E \to F$ is called a *polynomial* if some derivative $d^p f$ vanishes on E. The largest p such that $d^p f \neq 0$ is called the degree of the polynomial. The mapping f is called a *monomial of degree p* if it is of the form $f(x) = \tilde f(x, \dots, x)$ for some $\tilde f \in L^p_{\text{sym}}(E;F)$.

5.16. Lemma. Polynomials versus monomials.

(1) *The smooth p-homogeneous maps are exactly the monomials of degree p.*
(2) *The symmetric multilinear mapping representing a monomial is unique.*
(3) *A smooth mapping is a polynomial of degree $\le p$ if and only if its restriction to each one dimensional subspace is a polynomial of degree $\le p$.*
(4) *The polynomials are exactly the finite sums of monomials.*

Proof. ($\boxed{1}$) Every monomial of degree p is clearly smooth and p-homogeneous. If f is smooth and p-homogeneous, then

$$(d^p f)(0)(x, \dots, x) = (\tfrac{\partial}{\partial t})^p\big|_{t=0} f(tx) = (\tfrac{\partial}{\partial t})^p\big|_{t=0} t^p f(x) = p! f(x).$$

($\boxed{2}$) The symmetric multilinear mapping $g \in L^p_{\text{sym}}(E;F)$ representing a monomial f is uniquely determined by the polarization formula $\boxed{7.13}$.

($\boxed{3}$) & ($\boxed{4}$) Let the restriction of f to each one dimensional subspace be a polynomial of degree $\le p$, i.e., we have $\ell(f(tx)) = \sum_{k=0}^p \frac{t^k}{k!}\,(\frac{\partial}{\partial t})^k\big|_{t=0} \ell(f(tx))$ for $x \in E$ and $\ell \in F'$. So $f(x) = \sum_{k=0}^p \frac{1}{k!} d^k f(0\cdot x)(x, \dots, x)$ and hence is a finite sum of monomials.
For the derivatives of a monomial q of degree k we have $q^{(j)}(tx)(v_1, \dots, v_j) = k(k-1)\dots(k-j+1)t^{k-j}\tilde q(x, \dots, x, v_1, \dots, v_j)$. Hence, any such finite sum is a polynomial in the sense of $\boxed{5.15}$.
Finally, any such polynomial has obviously a polynomial as trace on each one dimensional subspace. □

5.17. Lemma. Spaces of polynomials. *The space* $\text{Poly}^p(E,F)$ *of polynomials of degree $\le p$ is isomorphic to $\bigoplus_{k\le p} L(\bigvee^k E;F)$ and is a direct summand in $C^\infty(E,F)$ with a complement given by the smooth functions which are p-flat at 0.*

Proof. By $\boxed{5.16}$ the mapping $\bigoplus_{k\le p} L(\bigvee^k E;F) \to C^\infty(E,F)$ given on the summands by $L(\bigvee^k E;F) \cong L^k_{\text{sym}}(E,F) \xrightarrow{\Delta^*} C^\infty(E,F)$ has $\text{Poly}^p(E,F)$ as image. A retraction to it is given by $\bigoplus_{k\le p} \frac{1}{k!}d^k|_0$, since $\frac{1}{k!}d^k|_0$ is by $\boxed{5.9.6}$ together with $\boxed{5.13}$ a retraction to the inclusion of the summand $L(\bigvee^k E;F)$ which is 0 when composed with the inclusion of the summands $L(\bigvee^j E;F)$ for $j \neq k$ by the formula for $q^{(k)}(x)$ given in the proof of $\boxed{5.16}$. □

Remark. The corresponding statement is false for infinitely flat functions. E.g. the short exact sequence $E \to C^\infty(\mathbb{R},\mathbb{R}) \to \mathbb{R}^\mathbb{N}$ does not split, where E denotes the space of smooth functions which are *infinitely flat at 0* and where the projection is given by the Taylor-coefficients. Otherwise, $\mathbb{R}^\mathbb{N}$ would be a subspace of $C^\infty([0,1],\mathbb{R})$ (compose the section with the restriction map from $C^\infty(\mathbb{R},\mathbb{R}) \to C^\infty([0,1],\mathbb{R})$) and hence would have the restriction of the supremum norm as continuous norm.

$$\begin{tikzcd} C^\infty(\mathbb{R},\mathbb{R}) \arrow[r, two heads] & C^\infty([0,1],\mathbb{R}) \arrow[d, two heads] \\ \mathbb{R}^\mathbb{N} \arrow[u, dotted] \arrow[r, "\mathrm{Id}"] & \mathbb{R}^\mathbb{N} \end{tikzcd}$$

This is however easily seen to be not the case.

5.14. Remark. Recall that for finite dimensional spaces $E = \mathbb{R}^n$ a polynomial into a (locally convex) vector space F is just a finite sum

$$\sum_{k\in\mathbb{N}^n} a_k x^k,$$

where $a_k \in F$ and $x^k := \prod_{i=1}^n x_i^{k_i}$. Thus, it is just an element in the algebra generated by the coordinate projections pr_i tensorized with F. Since every (continuous) linear functional on $E = \mathbb{R}^n$ is a finite linear combination of coordinate projections, this algebra is also the algebra generated by E'. For a general locally convex space E we define the algebra $P_f(E)$ of *finite type polynomials* to be the subalgebra of $C^\infty(E,\mathbb{R}) \subseteq \mathbb{R}^E$ generated by E'.

This is not in general the algebra of polynomials as defined in 5.15. Take for example the square of the norm $\|\ \ \|^2 : E \to \mathbb{R}$ on some infinite dimensional Hilbert space E. Its derivative is given by $x \mapsto (v \mapsto 2\langle x, v\rangle)$, and hence is linear. The second derivative is $x \mapsto ((v,w) \mapsto 2\langle v,w\rangle)$ and hence constant. Thus, the third derivative vanishes.
This function is not a finite type polynomial. Otherwise, it would be continuous for the weak topology $\sigma(E,E')$. Hence, the unit ball would be a 0-neighborhood for the weak topology, which is not true, since it is compact for it.

Note that for $E = \ell^2$ the space $\bigvee^2 E'$ is not even dense in $(\bigvee^2 E)' = L^2_{\mathrm{sym}}(E,\mathbb{R})$ and hence $P_f(\ell^2)$ is not dense in $\mathrm{Poly}(\ell^2,\mathbb{R})$: Otherwise $f := \|\ \ \|^2 \in L^2_{\mathrm{sym}}(E,\mathbb{R}) \subseteq L^2(E,\mathbb{R}) \cong L(E,E')$ could be approximated by elements in $\bigvee^2 E' \subseteq \bigotimes^2 E'$. However $\check{f} : \ell^2 \to (\ell^2)' \cong \ell^2$ is the identity and elements in $\bigotimes^2 E'$ correspond to finite dimensional operators, so they approximate only compact operators.

Note that the series $\sum_k x_k^2$ converges pointwise and even uniformly for $x = (x_k)_k$ in compact subsets of ℓ^2. In fact, every compact set K is contained in the absolutely convex hull of a 0-sequence x^n. In particular $\mu_k := \sup\{|x_k^n| : n \in \mathbb{N}\} \to 0$ for $k \to \infty$ (otherwise, we can find an $\varepsilon > 0$ and $k_j \to \infty$ and $n_j \in \mathbb{N}$ with $\|x^{n_j}\|_2 \geq |x_{k_j}^{n_j}| \geq \varepsilon$. Since $x^n \in \ell^2 \subseteq c_0$, we conclude that $n_j \to \infty$, which yields a contradiction to $\|x^n\|_2 \to 0$). Thus

$$K \subseteq \langle x^n : n \in \mathbb{N}\rangle_{\text{absolutely convex}} \subseteq \langle \mu_n e^n\rangle_{\text{absolutely convex}},$$

and hence $\sum_{k\geq n} |x_k| \leq \max\{\mu_k : k \geq n\}$ for all $x \in K$.
The series does not converge uniformly on bounded sets. To see this choose $x = e_k$.

5.10. Lemma. *Let E be a convenient vector space. Then $E' \hookrightarrow P_f(E) := \langle E'\rangle_{alg} \subseteq C^\infty(E,\mathbb{R})$ is the free commutative algebra over the vector space E', i.e.*

to every linear mapping $f : E' \to A$ in a commmutative algebra, there exists a unique algebra homomorphism $\tilde f : P_f(E) \to A$.

Proof. The solution of this universal problem is given by the symmetric algebra $\bigvee E' := \coprod_{k=0}^\infty \bigvee^k E'$ described in 5.9.9. In particular we have an algebra homomorphism $\tilde\imath : \bigvee E' \to P_f(E)$, which is onto, since by definition $P_f(E)$ is generated by E'. It remains to show that it is injective. So let $\sum_{k=1}^N \alpha_k \in \bigvee E'$, i.e. $\alpha_k \in \bigvee^k E'$, with $\tilde\imath(\sum_{k=1}^N \alpha_k) = 0$. Thus all derivatives $\tilde\imath(\alpha_k)$ at 0 of this mapping in $P_f(E) \subseteq C^\infty(E,\mathbb{R})$ vanish. So it remains to show that $\bigotimes_\beta^k E' \to L(E,\dots,E;\mathbb{R})$ is injective, since then by 5.13 also $\bigvee^k E' \to P_f(E) \subseteq C^\infty(E,\mathbb{R})$ is injective.

$$\begin{tikzcd}
\bigvee^k E' \arrow[r, tail] \arrow[d, hook] & L^k_{\text{sym}}(E,\mathbb{R}) \arrow[r, hook, "\Delta^*"] \arrow[d, hook] & C^\infty(E,\mathbb{R}) \\
\bigotimes^k E' \arrow[r, tail] & L^k(E,\mathbb{R})
\end{tikzcd}$$

We prove by induction that the mapping $E_1' \otimes_\beta \cdots \otimes_\beta E_n' \to L(E_1,\dots,E_n;\mathbb{R})$, $\alpha \mapsto \tilde\alpha$ is injective. For $n = 0$ and $n = 1$ this is obvious. So let $n \ge 2$ and let $\alpha = \sum_k \alpha_k \otimes x^k$, where $\alpha_k \in E_1' \otimes_\beta \cdots \otimes_\beta E_{n-1}'$ and $x^k \in E_n'$. We may assume that $(x^k)_k$ is linearly independent and hence may choose $x_j \in E_n$ with $x^k(x_j) = \delta_j^k$ and get $0 = \tilde\alpha(y^1,\dots,y^{n-1},x_j) = \tilde\alpha_j(y^1,\dots,y^{n-1})$ for all $y^1,\dots,y^{n-1}$. Hence $\tilde\alpha_j = 0$ and by induction hypothesis $\alpha_j = 0$ for all j and so $\alpha = 0$. $\square$

Note, however, that the injective mapping $\bigvee E' \to C^\infty(E,\mathbb{R})$ is not a bornological embedding in general:
Otherwise also $\bigvee^2 E' \to L^2_{\text{sym}}(E,\mathbb{R})$ would be such an embedding. Take $E = \ell^2$ and consider $\mathcal{B} = \{z_n : n \in \mathbb{N}\} \subseteq \bigvee^2 \ell^2$ where $z_n := \sum_{k=1}^n e_k \otimes e_k$. The bilinear form $\tilde z_n \in L^2_{\text{sym}}(\ell^2,\mathbb{R})$ associated to z_n ist given by $\tilde z_n(x,y) = \sum_{k\le n} e_k(x)\cdot e_k(y) = \sum_{k\le n} x^k\, y^k$. Thus the operator norm of $\tilde z_n$ is

$$\|\tilde z_n\| = \sup\Big\{\sum_{k\le n} x^k\, y^k : \|x\|_2 \le 1, \|y\|_2 \le 1\Big\} = 1.$$

The projective tensor norm of z_n is

$$\|z_n\|_\pi = \inf\Big\{\sum_k \|a_k\|_2\,\|b_k\|_2 : z = \sum_k a_k \otimes b_k\Big\} \ge n,$$

since by Hölders inequality

$$\begin{aligned}
\sum_k \|a_k\|_2\,\|b_k\|_2 &\ge \sum_k \|a_k\cdot b_k\|_1 = \sum_{k,j} |a_k^j \cdot b_k^j| \\
&\ge \sum_j \Big|\sum_k a_k^j\cdot b_k^j\Big| = \sum_j \Big|\Big(\sum_k a_k \otimes b_k\Big)^\sim (e_j,e_j)\Big| \\
&= \sum_j |\tilde z_n(e_j,e_j)| = \sum_{j\le n} 1 = n.
\end{aligned}$$

5.18. Theorem. Uniform boundedness principle. *If all E_i are convenient vector spaces, and if F is a locally convex space, then the bornology on the space $L(E_1,\dots,E_n;F)$ consists of all pointwise bounded sets.*

So a mapping into $L(E_1,\dots,E_n;F)$ is smooth if and only if all composites with evaluations at points in $E_1\times\ldots\times E_n$ are smooth.

Proof. Let us first consider the case $n = 1$. So let $\mathcal{B} \subseteq L(E, F)$ be a pointwise bounded subset. By lemma 5.3 we have to show that it is uniformly bounded on each bounded subset B of E. We may assume that B is closed absolutely convex, and thus E_B is a Banach space, since E is convenient. By the classical uniform boundedness principle, see [5, 5.2.2], the set $\mathcal{B}|_{E_B}$ is bounded in $L(E_B, F)$, and thus $\mathcal{B}$ is bounded on B.

The smoothness detection principle: Clearly it suffices to recognize smooth curves. If $c : \mathbb{R} \to L(E, F)$ is such that $\mathrm{ev}_x \circ c : \mathbb{R} \to F$ is smooth for all $x \in E$, then clearly $\mathbb{R} \xrightarrow{c} L(E, F) \xrightarrow{j} \prod_E F$ is smooth. We will show that $(j \circ c)'$ has values in $L(E, F) \subseteq \prod_E F$. Clearly, $(j \circ c)'(s)$ is linear $E \to F$. The family of mappings $\frac{c(s+t)-c(s)}{t} : E \to F$ is pointwise bounded for s fixed and t in a compact interval, so by the first part it is uniformly bounded on bounded subsets of E. It converges pointwise to $(j \circ c)'(s)$, so this is also a bounded linear mapping $E \to F$. By the first part $j : L(E, F) \to \prod_E F$ is a bornological embedding, so c is differentiable into $L(E, F)$. Smoothness follows now by induction on the order of the derivative.

The multilinear case follows from the exponential law 5.2 by induction on n: Let $\mathcal{B} \subseteq L(E_1, \dots, E_n; F)$ be pointwise bounded. Then $\mathcal{B}(x_1, \dots, x_{n-1}, \quad)$ is pointwise bounded in $L(E_n, F)$ for all $x_1, \dots, x_{n-1}$. So by the case $n = 1$ it is bounded in the locally convex space $L(E_n, F)$ and by induction hypothesis $\check{\mathcal{B}}$ is bounded in $L(E_1, \dots, E_{n-1}; L(E_n, F))$. By 5.2 $\mathcal{B}$ is bounded. □

5.19. Theorem. Multilinear mappings on convenient vector spaces. *A multilinear mapping from convenient vector spaces to a locally convex space is bounded if and only if it is separately bounded.*

Proof. Let $f : E_1 \times \dots \times E_n \to F$ be n-linear and separately bounded, i.e. $x_i \mapsto f(x_1, \dots, x_n)$ is bounded for each i and all fixed x_j for $j \neq i$. Then $f^\vee : E_1 \times \dots \times E_{n-1} \to L(E_n, F)$ is $(n-1)$-linear. By 5.18 the bornology on $L(E_n, F)$ consists of the pointwise bounded sets, so $f^\vee$ is separately bounded. By induction on n it is bounded. The bornology on $L(E_n, F)$ consists also of the subsets which are uniformly bounded on bounded sets by lemma 5.3, so f is bounded. □

We will now derive an infinite dimensional version of 3.4, which gives us minimal requirements for a mapping to be smooth.

5.20. Theorem. *Let E be a convenient vector space. An arbitrary mapping $f : E \supseteq U \to F$ is smooth if and only if all unidirectional iterated derivatives $d_v^p f(x) = (\frac{\partial}{\partial t})^p|_0 f(x + tv)$ exist, $x \mapsto d_v^p f(x)$ is bounded on sequences which are Mackey converging in U, and $v \mapsto d_v^p f(x)$ is bounded on fast falling sequences.*

Proof. A smooth mapping obviously satisfies this requirement. Conversely, from 3.4 we see that f is smooth restricted to each finite dimensional subspace, and the iterated directional derivatives $d_{v_1} \dots d_{v_n} f(x)$ exist and are bounded multilinear mappings in $v_1, \dots, v_n$ by 5.4, since they are universal linear combinations of the unidirectional iterated derivatives $d_v^p f(x)$, compare with the proof of 3.4. So $d^n f : U \to L^n(E; F)$ is bounded on Mackey converging sequences with respect to the pointwise bornology on $L^n(E; F)$. By the uniform boundedness principle 5.18 together with lemma 4.14 the mapping $d^n f : U \times E^n \to F$ is bounded on sets which are contained in a product of a BORNOLOGICALLY COMPACT SET in U - i.e. a set in U which is contained and compact in some E_B - and a bounded set in E^n.

Now let $c:\mathbb{R}\to U$ be a smooth curve. We have to show that $\frac{f(c(t))-f(c(0))}{t}$ converges to $f'(c(0))(c'(0))$. It suffices to check that

$$\frac{1}{t}\left(\frac{f(c(t))-f(c(0))}{t}-f'(c(0))(c'(0))\right)$$

is locally bounded with respect to t. Integrating along the segment from $c(0)$ to $c(t)$ we see that this expression equals

$$\begin{aligned}\frac{1}{t}\int_0^1\left(f'\Big(c(0)+s(c(t)-c(0))\Big)\left(\frac{c(t)-c(0)}{t}\right)-f'(c(0))(c'(0))\right)ds=\\=\int_0^1 f'\Big(c(0)+s(c(t)-c(0))\Big)\left(\frac{\frac{c(t)-c(0)}{t}-c'(0)}{t}\right)ds\\+\int_0^1\int_0^1 f''\Big(c(0)+rs(c(t)-c(0))\Big)\left(s\frac{c(t)-c(0)}{t},c'(0)\right)dr\,ds.\end{aligned}$$

The first integral is bounded since $df:U\times E\to F$ is bounded on the product of the bornologically compact set $\{c(0)+s(c(t)-c(0)):0\le s\le 1, t \text{ near } 0\}$ in U and the bounded set $\left\{\frac{1}{t}\left(\frac{c(t)-c(0)}{t}-c'(0)\right):t\text{ near }0\right\}$ in E (use 1.6).

The second integral is bounded since $d^2f:U\times E^2\to F$ is bounded on the product of the bornologically compact set $\{c(0)+rs(c(t)-c(0)):0\le r,s\le 1, t\text{ near }0\}$ in U and the bounded set $\left\{\left(s\frac{c(t)-c(0)}{t},c'(0)\right):0\le s\le 1,t\text{ near }0\right\}$ in E^2.

Thus $f\circ c$ is differentiable in F with derivative $df\circ(c,c')$. Since $df((x,v)+t(y,w))=df(x+ty,v)+t\,df(x+ty,w)$ the mapping $df:U\times E\to F$ satisfies again the assumptions of the theorem, so we may iterate. □

5.21. The following result shows that bounded multilinear mappings are the right ones for uses like homological algebra, where multilinear algebra is essential and where one wants a kind of 'continuity'. With continuity itself it does not work. The same results hold for convenient algebras and modules, one just may take c^∞-completions of the tensor products.

So by a *bounded algebra* A we mean a (real or complex) algebra which is also a locally convex vector space, such that the multiplication is a bounded bilinear mapping. Likewise, we consider *bounded modules* over bounded algebras, where the action is bounded bilinear.

Lemma. [Cap et. al., 1993]. *Let A be a bounded algebra, M a bounded right A-module and N a bounded left A-module.*

(1) *There are a locally convex vector space $M\otimes_A N$ and a bounded bilinear map $b:M\times N\to M\otimes_A N$, $(m,n)\mapsto m\otimes_A n$ such that $b(ma,n)=b(m,an)$ for all $a\in A$, $m\in M$ and $n\in N$ which has the following universal property: If E is a locally convex vector space and $f:M\times N\to E$ is a bounded bilinear map such that $f(ma,n)=f(m,an)$ then there is a unique bounded linear map $\tilde f:M\otimes_A N\to E$ with $\tilde f\circ b=f$. The space of all such f is denoted by $L^A(M,N;E)$, a closed linear subspace of $L(M,N;E)$.*

(2) *We have a bornological isomorphism*
$$L^A(M,N;E)\cong L(M\otimes_A N,E).$$

(3) *Let B be another bounded algebra such that N is a bounded right B-module and such that the actions of A and B on N commute. Then $M\otimes_A N$ is in a canonical way a bounded right B-module.*

(4) *If in addition P is a bounded left B-module then there is a natural bornological isomorphism $M \otimes_A (N \otimes_B P) \cong (M \otimes_A N) \otimes_B P$.*

Proof. We construct $M \otimes_A N$ as follows: Let $M \otimes_\beta N$ be the algebraic tensor product of M and N equipped with the (bornological) topology mentioned in 5.7 and let V be the locally convex closure of the subspace generated by all elements of the form $ma \otimes n - m \otimes an$, and define $M \otimes_A N$ to be $M \otimes_A N := (M \otimes_\beta N)/V$. As $M \otimes_\beta N$ has the universal property that bounded bilinear maps from $M \times N$ into arbitrary locally convex spaces induce bounded and hence continuous linear maps on $M \otimes N$, (1) is clear.

(2) By (1) the bounded linear map $b^* : L(M \otimes_A N, E) \to L^A(M, N; E)$ is a bijection. Thus, it suffices to show that its inverse is bounded, too. From 5.7 we get a bounded linear map $\varphi : L(M, N; E) \to L(M \otimes_\beta N, E)$ which is inverse to the map induced by the canonical bilinear map. Now let $L^{\mathrm{ann}(V)}(M \otimes_\beta N, E)$ be the closed linear subspace of $L(M \otimes_\beta N, E)$ consisting of all maps which annihilate V. Restricting φ to $L^A(M, N; E)$ we get a bounded linear map $\varphi : L^A(M, N; E) \to L^{\mathrm{ann}(V)}(M \otimes_\beta N, E)$.

Let $\psi : M \otimes_\beta N \to M \otimes_A N$ be the the canonical projection. Then ψ induces a well defined linear map $\hat{\psi} : L^{\mathrm{ann}(V)}(M \otimes_\beta N, E) \to L(M \otimes_A N, E)$, and $\hat{\psi} \circ \varphi$ is inverse to b^*. So it suffices to show that $\hat{\psi}$ is bounded.

This is the case if and only if the associated map $L^{\mathrm{ann}(V)}(M \otimes_\beta N, E) \times (M \otimes_A N) \to E$ is bounded. This in turn is equivalent to boundedness of the associated map $M \otimes_A N \to L(L^{\mathrm{ann}(V)}(M \otimes_\beta N, E), E)$ which sends x to the evaluation at x and is clearly bounded.

(3) Let $\rho : B^{op} \to L(N, N)$ be the right action of B on N and let $\Phi : L^A(M, N; M \otimes_A N) \cong L(M \otimes_A N, M \otimes_A N)$ be the isomorphism constructed in (2). We define the right module structure on $M \otimes_A N$ as:

$$\begin{aligned} B^{op} \xrightarrow{\rho} L(N, N) \xrightarrow{\mathrm{Id}\times} L(M \times N, M \times N) \xrightarrow{b_*} \\ \longrightarrow L^A(M, N; M \otimes_A N) \xrightarrow{\Phi} L(M \otimes_A N, M \otimes_A N). \end{aligned}$$

This map is obviously bounded and easily seen to be an algebra homomorphism.

(4) Straightforward computations show that both spaces have the following universal property: For a locally convex vector space E and a trilinear map $f : M \times N \times P \to E$ which satisfies $f(ma, n, p) = f(m, an, p)$ and $f(m, nb, p) = f(m, n, bp)$ there is a unique linear map prolonging f. □

5.22. Lemma. Uniform $\mathcal{S}$-boundedness principle. *Let E be a locally convex space, and let $\mathcal{S}$ be a point separating set of bounded linear mappings with common domain E. Then the following conditions are equivalent.*

(1) *If F is a Banach space (or even a c^∞-complete locally convex space) and $f : F \to E$ is a linear mapping with $\lambda \circ f$ bounded for all $\lambda \in \mathcal{S}$, then f is bounded.*
(2) *If $B \subseteq E$ is absolutely convex such that $\lambda(B)$ is bounded for all $\lambda \in \mathcal{S}$ and the normed space E_B generated by B is complete, then B is bounded in E.*
(3) *Let (b_n) be an unbounded sequence in E with $\lambda(b_n)$ bounded for all $\lambda \in \mathcal{S}$, then there is some $(t_n) \in \ell^1$ such that $\sum t_n b_n$ does not converge in E for the initial locally convex topology induced by $\mathcal{S}$.*

Definition. We say that E satisfies the *uniform $\mathcal{S}$-boundedness principle* if these equivalent conditions are satisfied.

Proof. (1) $\Rightarrow$ (3) : Suppose that (3) is not satisfied. So let (b_n) be an unbounded sequence in E such that $\lambda(b_n)$ is bounded for all $\lambda \in \mathcal{S}$, and such that for all $(t_n) \in \ell^1$ the series $\sum t_n\, b_n$ converges in E for the initial locally convex topology induced by $\mathcal{S}$. We define a linear mapping $f : \ell^1 \to E$ by $f((t_n)_n) = \sum t_n\, b_n$, i.e. $f(e_n) = b_n$. It is easily checked that $\lambda \circ f$ is bounded, hence by (1) the image of the closed unit ball, which contains all b_n, is bounded. Contradiction.

(3) $\Rightarrow$ (2): Let $B \subseteq E$ be absolutely convex such that $\lambda(B)$ is bounded for all $\lambda \in \mathcal{S}$ and that the normed space E_B generated by B is complete. Suppose that B is unbounded. Then B contains an unbounded sequence (b_n), so by (3) there is some $(t_n) \in \ell^1$ such that $\sum t_n\, b_n$ does not converge in E for the initial locally convex topology induced by $\mathcal{S}$. But $\sum t_n\, b_n$ is a Cauchy sequence in E_B, since $\sum_{k=n}^m t_n b_n \in (\sum_{k=n}^m |t_n|) \cdot B$, and thus converges even bornologically, a contradiction.

(2) $\Rightarrow$ (1): Let F be convenient, and let $f : F \to E$ be linear such that $\lambda \circ f$ is bounded for all $\lambda \in \mathcal{S}$. It suffices to show that $f(B)$, the image of an absolutely convex bounded set B in F with F_B complete, is bounded. By assumption, $\lambda(f(B))$ is bounded for all $\lambda \in \mathcal{S}$ and the normed space $E_{f(B)}$ is a quotient of the Banach space F_B, hence complete.

$$\begin{aligned}\widetilde{q_B}(y) &= \inf\{q_B(x) : y = f(x)\} = \inf\{\lambda : y = f(x),\, x \in \lambda\, B\}\\ &= \inf\{\lambda : y \in \lambda\, f(B)\} = q_{f(B)}(y).\end{aligned}$$

By (2) the set $f(B)$ is bounded. □

5.23. Lemma. *A convenient vector space E satisfies the uniform $\mathcal{S}$-boundedness principle for each point separating set $\mathcal{S}$ of bounded linear mappings on E if and only if there exists no strictly weaker ultrabornological topology than the bornological topology of E.*

Proof. ($\Rightarrow$) Let τ be an ultrabornological topology on E which is weaker than the natural bornological topology. Consider $\mathcal{S} := \{\mathrm{Id} : E \to (E, \tau)\}$ and the identity $(E, \tau) \to E$. Since every ultra-bornological space is an inductive limit of Banach spaces, cf. 52.31, it is enough to show that for each of these Banach spaces F the mapping $F \to (E, \tau) \to E$ is continous. By 5.22.1 this is the case.

($\Leftarrow$) If $\mathcal{S}$ is a point separating set of bounded linear mappings, the ultrabornological topology given by the inductive limit of the spaces E_B with B satisfying the assumptions of 5.22.2 equals the natural bornological topology of E. Hence, 5.22.2 is satisfied. □

5.24. Theorem. Webbed spaces have the uniform boundedness property. *A locally convex space which is webbed satisfies the uniform $\mathcal{S}$-boundedness principle for any point separating set $\mathcal{S}$ of bounded linear mappings.*

Proof. Since the bornologification of a webbed space is webbed, cf. [5, 5.3.3], we may assume that E is bornological, and hence that every bounded linear mapping on it is continuous, see 4.1.1. Now the closed graph principle [5, 5.3.3] applies to any mapping satisfying the assumptions of 5.22.1. □

5.25. Lemma. Stability of the uniform boundedness principle. *Let $\mathcal{F}$ be a set of bounded linear mappings $f : E \to E_f$ between locally convex spaces, let $\mathcal{S}_f$ be* *a point separating set of bounded linear mappings on E_f for every $f \in \mathcal{F}$, and let $\mathcal{S} := \bigcup_{f\in\mathcal{F}} f^*(\mathcal{S}_f) = \{g \circ f : f \in \mathcal{F}, g \in \mathcal{S}_f\}$. If $\mathcal{F}$ generates the bornology and E_f satisfies the uniform $\mathcal{S}_f$-boundedness principle for all $f \in \mathcal{F}$, then E satisfies the uniform $\mathcal{S}$-boundedness principle.*

Proof. We check the condition 5.22.1. So assume $h : F \to E$ is a linear mapping for which $g \circ f \circ h$ is bounded for all $f \in \mathcal{F}$ and $g \in \mathcal{S}_f$. Then $f \circ h$ is bounded by the uniform $\mathcal{S}_f$-boundedness principle for E_f. Consequently, h is bounded since $\mathcal{F}$ generates the bornology of E. □

5.26. Theorem. Smooth uniform boundedness principle. *Let E and F be convenient vector spaces, and let U be c^∞-open in E. Then $C^\infty(U, F)$ satisfies the uniform $\mathcal{S}$-boundedness principle where $\mathcal{S} := \{\mathrm{ev}_x : x \in U\}$.*

Proof. For $E = F = \mathbb{R}$ this follows from 5.24, since $C^\infty(U, \mathbb{R})$ is a Fréchet space. The general case then follows from 5.25. □

41. Jets and Whitney Topologies

Jet spaces or jet bundles consist of the invariant expressions of Taylor developments up to a certain order of smooth mappings between manifolds. Their invention goes back to Ehresmann [Ehresmann, 1951.]

41.1. Jets between convenient vector spaces. Let E and F be convenient vector spaces, and let $U \subseteq E$ and $V \subseteq F$ be c^∞-open subsets. For $0 \le k \le \infty$ the space of *k-jets* from U to V is defined by

$$J^k(U, V) := U \times V \times \mathrm{Poly}^k(E, F), \text{ where } \mathrm{Poly}^k(E, F) = \prod_{j=1}^{k} L^j_{\mathrm{sym}}(E; F).$$

We shall use the source and image projections $\alpha : J^k(U, V) \to U$ and $\beta : J^k(U, V) \to V$, and we shall consider $J^k(U, V) \to U \times V$ as a trivial bundle, with fibers $J^k_x(U, V)_y$ for $(x, y) \in U \times V$. Moreover, we have obvious projections $\pi^k_l : J^k(U, V) \to J^l(U, V)$ for $k > l$, given by truncation at order l. For a smooth mapping $f : U \to V$ the *k-jet extension* is defined by

$$j^k f(x) = j^k_x f := (x, f(x), df(x), \frac{1}{2!}d^2 f(x), \dots, \frac{1}{j!}d^j f(x), \dots),$$

the Taylor expansion of f at x of order k. If $k < \infty$ then $j^k : C^\infty(U, F) \to J^k(U, F)$ is smooth with a smooth right inverse (the polynomial), see 5.17. If $k = \infty$ then j^k need not be surjective for infinite dimensional E, see 15.4. For later use, we consider now the *truncated composition*

$$\bullet : \mathrm{Poly}^k(F, G) \times \mathrm{Poly}^k(E, F) \to \mathrm{Poly}^k(E, G),$$

where $p \bullet q$ is the composition $p \circ q$ of the polynomials p, q (formal power series in case $k = \infty$) without constant terms, and without all terms of order $> k$. Obviously, $\bullet$ is polynomial for finite k and is real analytic for $k = \infty$ since then each component is polynomial. Now let $U \subset E$, $V \subset F$, and $W \subset G$ be open subsets, and consider the fibered product

$$\begin{aligned} J^k(U, V) \times_U J^k(W, U) &= \{ (\sigma, \tau) \in J^k(U, V) \times J^k(W, U) : \alpha(\sigma) = \beta(\tau) \} \\ &= U \times V \times W \times \mathrm{Poly}^k(E, F) \times \mathrm{Poly}^k(G, E). \end{aligned}$$

Then the mapping

$$\bullet : J^k(U,V) \times_U J^k(W,U) \to J^k(W,V),$$
$$\sigma \bullet \tau = (\alpha(\sigma),\beta(\sigma),\bar\sigma) \bullet (\alpha(\tau),\beta(\tau),\bar\tau) := (\alpha(\tau),\beta(\sigma),\bar\sigma \bullet \bar\tau),$$

is a real analytic mapping, called the *fibered composition of jets.*

Let $U, U' \subset E$ and $V \subset F$ be open subsets, and let $g : U' \to U$ be a smooth diffeomorphism. We define a mapping $J^k(g,V) : J^k(U,V) \to J^k(U',V)$ by $J^k(g,V)(\sigma) = \sigma \bullet j^k g(g^{-1}(x))$, which also satisfies $J^k(g,V)(j^k f(x)) = j^k(f \circ g)(g^{-1}(\alpha(\sigma)))$. If $g' : U'' \to U'$ is another diffeomorphism, then clearly $J^k(g',V) \circ J^k(g,V) = J^k(g \circ g', V)$, and $J^k(\quad, V)$ is a contravariant functor acting on diffeomorphisms between open subsets of E. Since the truncated composition $\bar\sigma \mapsto \bar\sigma \bullet j^k_{g^{-1}(x)} g$ is linear, the mapping $J^k_x(g,F) := J^k(g,F)|J^k_x(U,F) : J^k_x(U,F) \to J^k_{g^{-1}(x)}(U',F)$ is also linear.

Now let $U \subset E$, $V \subset F$, and $W \subset G$ be c^∞-open subsets, and let $h : V \to W$ be a smooth mapping. Then we define $J^k(U,h) : J^k(U,V) \to J^k(U,W)$ by $J^k(U,h)\sigma = j^k h(\beta(\sigma)) \bullet \sigma$, which satisfies $J^k(U,h)(j^k f(x)) = j^k(h \circ f)(x)$. Clearly, $J^k(U, \quad)$ is a covariant functor acting on smooth mappings between c^∞-open subsets of convenient vector spaces. The mapping $J^k_x(U,h)_y : J^k_x(U,V)_y \to J^k_x(U,W)_{h(y)}$ is linear if and only if h is affine or $k = 1$ or $U = \emptyset$.

41.3. Jets between manifolds. Now let M and N be smooth manifolds with smooth atlas (U_α, u_α) and (V_β, v_β), modeled on convenient vector spaces E and F, respectively. Then we may glue the open subsets $J^k(u_\alpha(U_\alpha), v_\beta(V_\beta))$ of convenient vector spaces via the chart change mappings

$$J^k(u_{\alpha'} \circ u_\alpha^{-1}, v_\beta \circ v_{\beta'}^{-1}) : J^k(u_{\alpha'}(U_\alpha \cap U_{\alpha'}), v_{\beta'}(V_\beta \cap V_{\beta'})) \to$$
$$\to J^k(u_\alpha(U_\alpha \cap U_{\alpha'}), v_\beta(V_\beta \cap V_{\beta'})),$$

and we obtain a smooth fiber bundle $J^k(M,N) \to M \times N$ with standard fiber $\mathrm{Poly}^k(E,F)$. With the identification topology $J^k(M,N)$ is Hausdorff, since it is a fiber bundle and the usual argument for gluing fiber bundles applies which was given, e.g., in 28.12.

Theorem. *If M and N are smooth manifolds, modeled on convenient vector spaces E and F, respectively. Let $0 \le k \le \infty$. Then the following results hold.*

(1) *$(J^k(M,N), (\alpha,\beta), M \times N, \mathrm{Poly}^k(E,F))$ is a fiber bundle with standard fiber $\mathrm{Poly}^k(E,F)$, with the smooth group $GL^k(E) \times GL^k(F)$ as structure group, where $(\gamma,\chi) \in GL^k(E) \times GL^k(F)$ acts on $\sigma \in \mathrm{Poly}^k(E,F)$ by $(\gamma,\chi).\sigma = \chi \bullet \sigma \bullet \gamma^{-1}$.*
(2) *If $f : M \to N$ is a smooth mapping then $j^k f : M \to J^k(M,N)$ is also smooth, called the k-jet extension of f. We have $\alpha \circ j^k f = \mathrm{Id}_M$ and $\beta \circ j^k f = f$.*
(3) *If $g : M' \to M$ is a diffeomorphism then also the induced mapping $J^k(g,N) : J^k(M,N) \to J^k(M',N)$ is a diffeomorphism.*
(4) *If $h : N \to N'$ is a smooth mapping then $J^k(M,h) : J^k(M,N) \to J^k(M,N')$ is also smooth. Thus, $J^k(M, \quad)$ is a covariant functor from the category of smooth manifolds and smooth mappings into itself which respects each of the following classes of mappings: initial mappings, embeddings, closed embeddings, splitting embeddings, fiber bundle projections. Furthermore, $J^k(\quad,\quad)$ is a contra-covariant bifunctor, where we have to restrict in the first variable to the category of diffeomorphisms.*

(5) *For $k \geq l$, the projections $\pi^k_l : J^k(M,N) \to J^l(M,N)$ are smooth and natural, i.e., they commute with the mappings from (3) and (4).*

(6) *$(J^k(M,N), \pi^k_l, J^l(M,N), \prod_{i=l+1}^k L^i_{sym}(E;F))$ are fiber bundles for all $l \leq k$. For finite k the bundle $(J^k(M,N), \pi^k_{k-1}, J^{k-1}(M,N), L^k_{sym}(E,F))$ is an affine bundle. The first jet space $J^1(M,N) \to M \times N$ is a vector bundle. It is isomorphic to the bundle $(L(TM,TN), (\pi_M, \pi_N), M \times N)$, see 29.4 and 29.5. Moreover, we have $J^1_0(\mathbb{R}, N) = TN$ and $J^1(M,\mathbb{R})_0 = T^*M$.*

(7) *Truncated composition is a smooth mapping*

$$\bullet : J^k(N,P) \times_N J^k(M,N) \to J^k(M,P).$$

Proof. (1) is already proved. (2), (3), (5), and (7) are obvious from 41.1, mainly by the functorial properties of $J^k(\quad,\quad)$.

(4) It is clear from 41.1 that $J^k(M,h)$ is a smooth mapping. The rest follows by looking at special chart representations of h and the induced chart representations for $J^k(M,h)$.

It remains to show (6), and here we concentrate on the affine bundle. Let $a_1 + a \in GL(E) \times \prod_{i=2}^k L^i_{\text{sym}}(F;F)$, $\sigma + \sigma_k \in \text{Poly}^{k-1}(E,F) \times L^k_{\text{sym}}(E;F)$, and $b_1 + b \in GL(E) \times \prod_{i=2}^k L^i_{\text{sym}}(E;E)$, then the only term of degree k containing σ_k in $(a_1 + a) \bullet (\sigma + \sigma_k) \bullet (b_1 + b)$ is $a_1 \circ \sigma_k \circ b_1^k$, which depends linearly on σ_k. To this the degree k-components of compositions of the lower order terms of σ with the higher order terms of a and b are added, and these may be quite arbitrary. So an affine bundle results.

We have $J^1(M,N) = L(TM,TN)$ since both bundles have the same transition functions. Finally,

$$J^1_0(\mathbb{R}, N) = L(T_0\mathbb{R}, TN) = TN \quad \text{and} \quad J^1(M,\mathbb{R})_0 = L(TM, T_0\mathbb{R}) = T^*M. \qquad \square$$

41.4. Jets of sections of fiber bundles. If $(p : E \to M, S)$ is a fiber bundle, let (U_α, u_α) be a smooth atlas of M such that $(U_\alpha, \psi_\alpha : E|U_\alpha \to U_\alpha \times S)$ is a fiber bundle atlas. If we glue the smooth manifolds $J^k(U_\alpha, S)$ via $(\sigma \mapsto j^k(\psi_{\alpha\beta}(\alpha(\sigma),\quad))) \bullet \sigma : J^k(U_\alpha \cap U_\beta, S) \to J^k(U_\alpha \cap U_\beta, S)$, we obtain the smooth manifold $J^k(E)$, which for finite k is the space of all k-jets of local sections of E.

Theorem. *In this situation we have:*

(1) *$J^k(E)$ is a splitting closed submanifold of $J^k(M,E)$, namely the set of all $\sigma \in J^k_x(M,E)$ with $J^k(M,p)(\sigma) = j^k(\text{Id}_M)(x)$.*

(2) *$J^1(E)$ of sections is an affine subbundle of the vector bundle $J^1(M,E) = L(TM,TE)$. In fact, we have*

$$J^1(E) = \{\, \sigma \in L(TM,TE) : Tp \circ \sigma = \text{Id}_{TM} \,\}.$$

(3) *For k finite $(J^k(E), \pi^k_{k-1}, J^{k-1}(E))$ is an affine bundle.*

(4) *If $p : E \to M$ is a vector bundle, then $(J^k(E), \alpha, M)$ is also a vector bundle. If $\phi : E \to E'$ is a homomorphism of vector bundles covering the identity, then $J^k(\varphi)$ is of the same kind.*

Proof. Locally $J^k(E)$ in $J^k(M,E)$ looks like $u_\alpha(U_\alpha) \times \text{Poly}^k(F_M, F_S)$ in $u_\alpha(U_\alpha) \times (u_\alpha(U_\alpha) \times v_\beta(V_\beta)) \times \text{Poly}^k(F_M, F_M \times F_S)$, where F_M and F_S are modeling spaces of M and S, respectively, and where (V_β, v_β) is a smooth atlas for S. The rest is clear. $\square$

6. Some Spaces of Smooth Functions

6.1. Proposition. *Let M be a smooth finite dimensional paracompact manifold. Then the space $C^\infty(M,\mathbb{R})$ of all smooth functions on M is a convenient vector space in any of the following (bornologically) isomorphic descriptions, and it satisfies the uniform boundedness principle for the point evaluations.*

(1) *The initial structure with respect to the cone*
$$C^\infty(M,\mathbb{R}) \xrightarrow{c^*} C^\infty(\mathbb{R},\mathbb{R})$$
for all $c \in C^\infty(\mathbb{R},M)$.

(2) *The initial structure with respect to the cone*
$$C^\infty(M,\mathbb{R}) \xrightarrow{(u_\alpha^{-1})^*} C^\infty(\mathbb{R}^n,\mathbb{R}),$$
where (U_α,u_α) is a smooth atlas with $u_\alpha(U_\alpha)=\mathbb{R}^n$.

(3) *The initial structure with respect to the cone*
$$C^\infty(M,\mathbb{R}) \xrightarrow{j^k} C(M \leftarrow J^k(M,\mathbb{R}))$$
for all $k \in \mathbb{N}$, where $J^k(M,\mathbb{R})$ is the bundle of k-jets of smooth functions on M, where j^k is the jet prolongation, and where all the spaces of continuous sections are equipped with the compact open topology.

It is easy to see that the cones in (2) and (3) induce even the same locally convex topology which is sometimes called the *compact C^∞ topology*, if $C^\infty(\mathbb{R}^n,\mathbb{R})$ is equipped with its usual Fréchet topology. From (2) we see also that with the bornological topology $C^\infty(M,\mathbb{R})$ is nuclear by 52.35, and is a Fréchet space if and only if M is separable.

Proof. For all three descriptions the initial locally convex topology is convenient, since the spaces are closed linear subspaces in the relevant products of the right hand sides:

(1) For this structure $C^\infty(M,\mathbb{R}) = \varprojlim_{c C^\infty(\mathbb{R},M)} C^\infty(\mathbb{R},\mathbb{R})$, where the connecting mappings are given by g^* for $g \in C^\infty(\mathbb{R},\mathbb{R})$. Obviously, $(c^*)_{c\in C^\infty(\mathbb{R},M)}$ has values in this inductive limit and induces the structure of (1) on $C^\infty(M,\mathbb{R})$. This mapping is bijective, since to $(f_c)_{c\in C^\infty(\mathbb{R},\mathbb{R})} \in \varprojlim_c C^\infty(\mathbb{R},\mathbb{R})$ we can associate $f : M \to \mathbb{R}$ given by $f(x) = f_{\mathrm{const}_x}(0)$. Then $c^*(f) = f_c$, since $\mathrm{const}_{c(t)} = c \circ \mathrm{const}_t$. Moreover $\mathrm{const}_x^*(f) = \mathrm{const}_{f(x)}$, so we found the inverse.

(2) For this structure $C^\infty(M,\mathbb{R}) = \varprojlim_u C^\infty(\mathbb{R}^n,\mathbb{R})$, where u run through all smooth open embeddings $\mathbb{R}^n \to M$ and where the connecting mappings are given by g^* for smooth embeddings $g \in C^\infty(\mathbb{R}^n,\mathbb{R}^n)$. Obviously, $(u^*)_u$ has values in this inductive limit and induces the structure of (2) on $C^\infty(M,\mathbb{R})$, since locally such u coincide with some $(u_\alpha)^{-1}$ and $C^\infty(\mathbb{R}^n,\mathbb{R})$ carries the initial structure with resperct to $\mathrm{incl}_V^* : C^\infty(\mathbb{R}^n,\mathbb{R}) \to C^\infty(V,\mathbb{R})$, where the V form some open covering of $\mathbb{R}^n$ This mapping $(u^*)_u$ is bijective, since to $(f_u)_u \in \varprojlim_u C^\infty(\mathbb{R}^n,\mathbb{R})$ we can associate $f : M \to \mathbb{R}$ given by $f(x) = f_u(t)$, where $u : \mathbb{R}^m \to M$ is some smooth open embedding with $u(t) = x$. This definition does not depend on the choice of (u,t) since two such embeddings can be locally reparametrized into each another. As before this gives the required invese.

(3) First note for vector bundles $p : E \to M$ the compact open topology turns $C(M \leftarrow E)$ into a locally convex space. In fact for a neighborhood subbasis of this topology it is enough to consider the convex sets $N_{K,U} := \{\sigma \in C(M \leftarrow E) : \sigma(K) \subseteq U\}$ for compact subsets K contained in trivializing open subsets V of the

basis and open sets $U \subseteq E$ of the form $\psi^{-1}(V \times W)$, where $\psi : p^{-1}(V) \to V \times \mathbb{R}^k$ is the trivialization and $W \subseteq \mathbb{R}^k$ is open and convex in the typical fiber. This shows also, that the topology is the initial one induced by the restriction maps $\mathrm{incl}_K^* : C(M \leftarrow E) \to C(K \leftarrow E|_K) \cong C(K, \mathbb{R}^k) \subseteq \ell^\infty(K, \mathbb{R}^k)$. So it is enough to show closednes of the image of $C^\infty(M,\mathbb{R}) \to \prod_{k,K} C(K, \prod_{j=0}^k L^j_{\mathrm{sym}}(\mathbb{R}^m, \mathbb{R}^k))$ where the K are assumed to be compact in some chart domain in M. This is clearly the case.

Thus, the uniform boundedness principle for the point evaluations holds for all structures since it holds for all right hand sides (for $C(M \leftarrow J^k(M,\mathbb{R}))$ we may reduce to a connected component of M, and we then have a Fréchet space). So the identity is bibounded between all structures. $\square$

6.2. Spaces of smooth functions with compact supports. For a smooth finite dimensional Lindelöf (equivalently, separable metrizable) Hausdorff manifold M we denote by $C_c^\infty(M,\mathbb{R})$ the vector space of all smooth functions with compact supports in M.

Corollary. *The following convenient structures on the space $C_c^\infty(M,\mathbb{R})$ are all isomorphic:*

(1) *Let $C_K^\infty(M,\mathbb{R})$ be the space of all smooth functions on M with supports contained in the fixed compact subset $K \subseteq M$, a closed linear subspace of $C^\infty(M,\mathbb{R})$. Let us consider the final convenient vector space structure on the space $C_c^\infty(M,\mathbb{R})$ induced by the cone*

$$C_K^\infty(M,\mathbb{R}) \hookrightarrow C_c^\infty(M,\mathbb{R})$$

where K runs through a basis for the compact subsets of M. Then the space $C_c^\infty(M,\mathbb{R})$ is even the strict inductive limit of a sequence of Fréchet spaces $C_K^\infty(M,\mathbb{R})$.

(2) *We equip $C_c^\infty(M,\mathbb{R})$ with the initial structure with respect to the inclusion $C_c^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$ and the cone*

$$C_c^\infty(M,\mathbb{R}) \xrightarrow{x^*} C_c(\mathbb{N},\mathbb{R}) = \coprod_{n\in\mathbb{N}} \mathbb{R}^n = \mathbb{R}^{(\mathbb{N})},$$

where $x = (x_n)_n$ runs through all sequences in M without accumulation point.

(3) *The initial structure with respect to the cone*

$$C_c^\infty(M,\mathbb{R}) \xrightarrow{j^k} C_c(M \leftarrow J^k(M,\mathbb{R}))$$

for all $k \in \mathbb{N}$, where $J^k(M,\mathbb{R})$ is the bundle of k-jets of smooth functions on M, where j^k is the jet prolongation, and where the spaces of continuous sections with compact support are equipped with the inductive limit topology with steps $C_K(M \leftarrow J^k(M,\mathbb{R})) \subseteq C(M \leftarrow J^k(M,\mathbb{R}))$.

For M with only finitely many connected components which are all non-compact, this is also true for

(4) *the convenient vector space structure induced by $c^* : C_c^\infty(M,\mathbb{R}) \to C_c^\infty(\mathbb{R},\mathbb{R})$, where $c : \mathbb{R} \to M$ run through the proper smooth curves.*

The space $C_c^\infty(M,\mathbb{R})$ satisfies the uniform boundedness principle for the point evaluations.

First Proof. We show that in all four descriptions the space $C^\infty_c(M,\mathbb{R})$ is convenient and satisfies the uniform boundedness principle for point evaluations, hence the identity is bibounded for all structures:

In (1) we may assume that the basis of compact subsets of M is countable, since M is Lindelöf, hence has only countable many connected components and these are metrizable, so the inductive limit is a strict inductive limit of a sequence of Fréchet spaces, hence $C^\infty_c(M,\mathbb{R})$ is convenient and webbed by [**5**, 5.3.3] and [**5**, 5.3.3] and satisfies the uniform boundedness principle by 5.24.

In (2)–(4) the space is a closed subspace of the product of $C^\infty(M,\mathbb{R})$ and spaces on the right hand side which are strict inductive limits of Fréchet spaces, hence convenient and satisfy the uniform boundedness principle:
In (2) closedness follows, since for smoothness of $f : M \to \mathbb{R}$ follows from the inclusion into $C^\infty(M,\mathbb{R})$, and compactness of the support follows because this can be tested along sequences without accumulation point.
In (3) closedness follows, since $C^\infty(M,\mathbb{R})$ is closed in $\prod_k C(M \leftarrow J^k(M,\mathbb{R}))$ by the proof of 6.1 and the support is that of $f = f^0 \in C_c(M \leftarrow J^0(M,\mathbb{R})) = C_c(M,\mathbb{R})$.
In (4) this follows from (2), since every smooth curve in M coincides locally with a proper smooth curve and if $A \subseteq M$ is closed and not compact then there exists some end $e \in \varprojlim_U \pi(U)$ (where $\pi(U)$ denotes the finite set of (non-compact) connected components of $M \setminus \overline{U}$ for open relative compact $U \subseteq M$) which is in the closure of A in the compact topology of the Freudenthal-compactification $M \cup \varprojlim_U \pi(U)$ with the sets $e_K \cup \{e' \in \varprojlim_U \pi(U) : e'_K = e_K\}$ for the open relative compact sets $U \subseteq M$ as neighborhoodbasis of e. See [H.Freudenthal: Über die Enden topologischer Räume und Gruppen, Math. Zeitschrift 33 (1931) 692-713] und [Frank Reymond: the end point compactification of manifolds, Pacific J. Math. 10 (1960) 947-963]. Thus for every compact $K_n \subseteq M$ there exists a point $a_n \in e_{K_n} \cap A$. Since $e_{K_{n+1}} \subseteq e_{K_n}$ there is a curve in the connected component $e_{K_n} \subseteq M \setminus K_n$ connecting a_n with a_{n+1} we may piece these curves smoothly together to obtain a proper smooth curve $c : \mathbb{R} \to M$ with $c(\pm n) = a_n$. □

Second Proof.

(1→2) For this we consider for sequences $x = (x_n)_n$ without accumulation point the diagram

$$\begin{tikzcd}
C^\infty(M,\mathbb{R}) \arrow[d, "x^*"'] & C^\infty_K(M,\mathbb{R}) \arrow[l, hook'] \arrow[r, hook, "(1)"] \arrow[d, "x^*"] \arrow[dr, dotted] & C^\infty_c(M,\mathbb{R}) \arrow[d, "x^*"', "(2)"] \\
\prod_{n\in\mathbb{N}} \mathbb{R} & \mathbb{R}^{x^{-1}(K)} \arrow[l, hook'] \arrow[r, hook] & \coprod_{n\in\mathbb{N}} \mathbb{R},
\end{tikzcd}$$

where $x^{-1}(K) := \{n : x_n \in K\}$ is by assumption finite. Then obviously the identity on $C^\infty_c(M,\mathbb{R})$ is bounded from the structure (1) to the structure (2).

(1→3) We consider the diagram:

$$\begin{tikzcd}
C^\infty(M,\mathbb{R}) \arrow[d, "j^k"'] & C^\infty_K(M,\mathbb{R}) \arrow[l, hook'] \arrow[r, hook, "(1)"] \arrow[d, "j^k"'] \arrow[dr, dotted] & C^\infty_c(M,\mathbb{R}) \arrow[d, "j^k"', "(3)"] \\
C(M \leftarrow J^k(M,\mathbb{R})) & C_K(M \leftarrow J^k(M,\mathbb{R})) \arrow[l, hook'] \arrow[r, hook] & C_c(M \leftarrow J^k(M,\mathbb{R}))
\end{tikzcd}$$

Obviously, the identity on $C_c^\infty(M,\mathbb{R})$ is bounded from the structure (1) into the structure (3).

(1→4) follows from the diagram

$$\begin{tikzcd}
C^\infty(M,\mathbb{R}) \arrow[d, "e^*"'] & C_K^\infty(M,\mathbb{R}) \arrow[l, hook'] \arrow[r, hook, "(1)"] \arrow[d, "e^*"] \arrow[dr, dotted] & C_c^\infty(M,\mathbb{R}) \arrow[d, "e^*"', "(2)"] \\
C^\infty(\mathbb{R},\mathbb{R}) & C_{e^{-1}(K)}^\infty(\mathbb{R},\mathbb{R}) \arrow[l, hook'] \arrow[r, hook] & C_c^\infty(\mathbb{R},\mathbb{R})
\end{tikzcd}$$

with proper $e:\mathbb{R}\to M$.

(2→1) Now let $\mathcal{B}\subseteq C_c^\infty(M,\mathbb{R})$ be bounded in the structure of (2). We claim that $\mathcal{B}$ is contained in some $C_{K_n}^\infty(M,\mathbb{R})$, where K_n form an exhaustion of M by compact subsets such that K_n is contained in the interior of K_{n+1}. Otherwise there would be $x_n\notin K_n$ and $f_n\in\mathcal{B}$ with $f_n(x_n)\neq 0$. Then $x^*(\mathcal{B})$ ist not bounded in $\coprod_{\mathbb{N}}\mathbb{R}=\varinjlim_n\mathbb{R}^n$, since this limit is regular, but $x^*(f_n)(n)=f_n(x_n)\neq 0$. Since $C_c^\infty(M,\mathbb{R})\to C^\infty(M,\mathbb{R})$ is bounded, $\mathcal{B}$ is also bounded in $C_{K_n}^\infty(M,\mathbb{R})$ and hence in the structure (1).

(3→1) Now let $\mathcal{B}\subseteq C_c^\infty(M,\mathbb{R})$ be bounded in the structure of (3). Then $\mathcal{B}=j^0(\mathcal{B})$ is bounded in $C_c(M\leftarrow J^0(M,\mathbb{R}))=C_c(M,\mathbb{R})=\varinjlim_K C_K(M,\mathbb{R})$ and since this limit is regular there exists a compact $K\subseteq M$ such that $\mathcal{B}\subseteq C_K(M,\mathbb{R})$. But then also $\mathcal{B}\subseteq C_K^\infty(M,\mathbb{R})$. Since $j^k(\mathcal{B})\subseteq C_K(M\leftarrow J^k(M,\mathbb{R}))\subseteq C_c(M\leftarrow J^k(M,\mathbb{R}))$ is bounded we get that $\mathcal{B}\subseteq C_c^\infty(M,\mathbb{R})$ is bounded in the structure (3).

(4→2) Let now M have only finitely many connected components which are all non-compact and let $\mathcal{B}\subseteq C_c^\infty(M,\mathbb{R})$ be bounded for the structure (4). Since every smooth curve in M coincides locally with a proper smooth curve the set $\mathcal{B}$ is bounded in $C^\infty(M,\mathbb{R})$. Suppose there were a sequence $x=(x_n)_n$ without accumulation point for which $x^*(\mathcal{B})$ is not bounded in $\coprod_{n\in\mathbb{N}}\mathbb{R}^n$. Since $\mathrm{ev}_{x_n}(\mathcal{B})$ is bounded there are infinitely many $n\in\mathbb{N}$ for which $f_n\in\mathcal{B}$ exists with $f_n(x_n)\neq 0$. Since we only have finitely many connected components we may assume that all x_n are in the same non-compact connected component. Now we may choose a proper smooth curve c passing through a subsequence of the x_n and hence $c^*(\mathcal{B})$ would not be bounded in $C_c^\infty(\mathbb{R},\mathbb{R})$.

For the uniform boundedness principle we refer to the first proof. □

Remark. Note that the locally convex topologies described in (1) and (3) are distinct: The continuous dual of $(C_c^\infty(\mathbb{R},\mathbb{R}),(1))$ is the space of all distributions (generalized functions), whereas the continuous dual of $(C_c^\infty(\mathbb{R},\mathbb{R}),(3))$ are all distributions of finite order, i.e., globally finite derivatives of continuous functions.

If M is only assumed to be a smooth paracompact Hausdorff manifold, then we can still consider the structure on $C_c^\infty(M,\mathbb{R})$ given in 1. It will no longer be an inductive limit of a sequence of Fréchet spaces but will still satisfy the uniform boundedness principle for the point-evaluations, by [11, 3.4.4]. since

$$\begin{aligned}
C_c^\infty(M,\mathbb{R}) &= \varinjlim_K C_K^\infty(M,\mathbb{R}) = \varinjlim_K \bigoplus_i C_{K\cap M_i}^\infty(M_i,\mathbb{R}) \cong \\
&\cong \coprod_i \varinjlim_K C_{K\cap M_i}^\infty(M_i,\mathbb{R}) = \coprod_i C_c^\infty(M_i,\mathbb{R}),
\end{aligned}$$

where the M_i are the connected components and these are Lindelöf.

54. Differentiabilities discussed by Keller [13]

54.1 Remark. (e,g. [2, 6.1.4]) Recall that for Banach spaces E and F a mapping $f : E \supseteq U \to F$ defined on an open subset U of E is called (Fréchet-)differentiable at $x \in U$ iff there exists a continuous linear operator $\ell : E \to F$, such that

$$\frac{f(x+v)-f(x)-\ell(v)}{\|v\|} \to 0 \text{ for } v \to 0.$$

Existence of ℓ implies its unicity, and hence it is denoted $f'(x)$ and called the (Fréchet-)derivative of f at x.

In order to calculate $f'(x)$ we may consider the directional derivatives

$$d_v f(x) := \lim_{t\searrow 0} \frac{f(x+tv)-f(x)}{t}.$$

Note that this is $\mathbb{R}^+$-homogeneous with respect to v. If f is Fréchet differentiable at x with derivative $f'(x)$, then $d_v f(x)$ exists and equals $f'(x)(v)$, since

$$\begin{aligned}\frac{f(x+tv)-f(x)}{t} - f'(x)(v) &= \frac{f(x+tv)-f(x)-f'(x)(tv)}{t}\\ &= \frac{f(x+tv)-f(x)-\ell(tv)}{\|tv\|}\,\|v\| \to 0.\end{aligned}$$

The converse direction is not the case, but one has:

54.2 Lemma. *(e.g. [2, 6.1.6]) Let E and F be Banach spaces, $U \subseteq E$ be open, $x \in U$. Then $f : E \supseteq U \to F$ is Fréchet differentiable at x iff the following conditions are satisfied:*

1. *$\forall v \in E\ \exists d_v f(x)$;*
2. *$v \mapsto d_v f(x)$ is linear and continuous;*
3. *$\frac{f(x+tv)-f(x)}{t} \to d_v f(x)$ for $t \searrow 0$ uniformly for v in the unit-sphere.*

Proof. ($\Rightarrow$) was shown just before this lemma.

($\Leftarrow$) We claim that $v \mapsto d_v f(x)$ is the Fréchet derivative of f. So consider an arbitrary $v \neq 0$ and put $t := \|v\|$, $w := \frac{1}{t}v$. Then

$$\begin{aligned}\frac{f(x+v)-f(x)-d_v f(x)}{\|v\|} &= \frac{f(x+t\,w)-f(x)-d_{tw} f(x)}{t}\\ &= \frac{f(x+t\,w)-f(x)}{t} - d_w f(x) \to 0\\ &\text{for } t=\|v\| \to 0 \text{ uniformly for } \|w\|=1. \quad \square\end{aligned}$$

Definition. The straight forward generalization of this notion to mappings between locally convex spaces is the following:

A mapping $f : E \supseteq U \to F$ defined on an open subset U of a locally convex space E is called ($\mathcal{B}$-)differentiable at $x \in U$, iff for all $v \in E$ the directional derivative $d_v f(x) := \lim_{t\searrow 0} \frac{f(x+tv)-f(x)}{t}$ exists, this convergence is uniformly for $x \in B$, for any $B \in \mathcal{B}$, where $\mathcal{B}$ is some given set of bounded subsets of E, and $v \mapsto d_v f(x)$ is linear and continuous. In [13] the following particular cases for $\mathcal{B}$ are treated:

's' the finite subsets (leading to so called simple (or pointwise) convergence).

'k' the compact subsets. These are in general not stable under formation of closed convex hulls.
'pk' the precompact subsets. These are in contrast stable under formation of closed convex hulls.
'b' the bounded sets.

It is called CONTINUOUSLY ($\mathcal{B}$-)DIFFERENTIABLE ($C^1_{\mathcal{B}}$ for short), iff it is differentiable at each point $x \in U$ and $x \mapsto (v \mapsto f'(x)(v))$ is continuous from U to $\mathcal{L}_{\mathcal{B}}(E,F) := \{\ell : E \to F | \ell \text{ is linear and continuous}\}$, where we put the topology of uniform convergence on sets $B \in \mathcal{B}$ on $\mathcal{L}(E,F)$.

A mapping $f : E \supseteq U \to F$ is called GÂTEAUX DIFFERENTIABLE at $x \in U$, iff for all $v \in E$ the directional derivative $d_v f(x)$ exists and is linear in v (and most often it is also required to be continuous).

Moreover, it is sufficient to assume the continuity of the directional derivative to get differentiability:

54.3 Lemma. *Let $f : E \supseteq U \to F$ be as in 54.2 and assume that for all $x \in U$ and $v \in E$ the directional derivative $d_v f(x)$ exists and $x \mapsto d_v f(x)$ defines a continuous mapping $f' : E \supseteq U \to \mathcal{L}_{\mathcal{B}}(E,F)$.*

Then f is $\mathcal{B}$-differentiable on U and f' is its derivative.

Proof. By 54.2 we only have to show that $\frac{f(x+tv)-f(x)}{t} \to d_v f(x)$ for $t \searrow 0$ uniformly for v in $B \in \mathcal{B}$. So we consider the difference and get by the principal theorem of calculus:

$$\begin{aligned}\frac{f(x+t\,v)-f(x)}{t} - d_v f(x) &= \int_0^1 \frac{1}{t}\frac{d}{ds} f(x+s\,t\,v)\,ds - d_v f(x)\,ds \\ &= \int_0^1 \Big(d_v f(x+s\,t\,v) - d_v f(x)\Big)(v)\,ds,\end{aligned}$$

which converges as required, since $d_v f : U \to \mathcal{L}_{\mathcal{B}}(E,F)$ is assumed to be continuous. □

This observation has been used by [13] to compare various differentiability notions given in the literature.

However, the problem with this type of definition, is to show the chain-rule for $C^1_{\mathcal{B}}$: Let $f : E \to F$ and $g : F \to G$ be $C^1_{\mathcal{B}}$. We would like to have that $g \circ f : E \to G$ is $C^1_{\mathcal{B}}$ and its derivative should be $(g \circ f)'(x) = g'(f(x)) \circ f'(x)$. Obviously $f' : E \to \mathcal{L}(E,F)$ is continuous and also $g' \circ f : E \to F \to \mathcal{L}(F,G)$. Thus we would need that the composition $\circ : \mathcal{L}(F,G) \times \mathcal{L}(E,F) \to \mathcal{L}(E,G)$ is continuous. We have seen that even for $E = G = \mathbb{R}$ this is only the case, iff F is normed.

For this reason limit structures where used instead of topology by several authors. The coarsest reasonable structure is that of continuous convergence (denoted c), i.e. one calls a filter $\mathcal{F}$ on $\mathcal{L}(E,F)$ to be convergent to $\ell \in \mathcal{L}(E,F)$, iff for each filter $\mathcal{E}$ in E converging to some $x \in E$ the image filter $\mathcal{F}(\mathcal{E})$ converges to $\ell(x)$ in F. This definition turns $\mathcal{L}(E,F)$ into a convergence vector space denoted $\mathcal{L}_c(E,F)$. This is (by definition) the weakest convergence structure on $\mathcal{L}(E,F)$ which makes $\mathrm{ev} : \mathcal{L}(E,F) \times E \to F$ continuous. Moreover, a mapping $f : X \to \mathcal{L}_c(E,F)$ on a topological space X is continuous, iff the associated mapping $\hat{f} : X \times E \to F$ is continuous.

Using some convergence structure Λ on $\mathcal{L}(E,F)$ (like continuous convergence) one can define $f : E \supseteq U \to F$ to be C^1_Λ, iff it is Gâteaux-differentiable and the derivative $f' : E \supseteq U \to \mathcal{L}_\Lambda(E, F)$ is continuous. For C^1_c mappings one can easily show the chain-rule. However, in Banach spaces one does not recover classical Fréchet differentiability (for which the inverse and implicit function theorem can be shown) but something weaker, see the following example of Smolyanov (12.13).

According to [13] one has the following implications, where qb denotes the limit structure of quasi-bounded convergence, which I will not explain here.

$$\begin{tikzcd}
C^1_s & C^1_k \arrow[l] & C^1_{pk} \arrow[l] & C^1_b \arrow[l] \\
 & & C^1_c \arrow[u] & C^1_{qb} \arrow[l] \arrow[u]
\end{tikzcd}$$

The two smaller frames indicate groups of definitions which are equivalent for mappings between Fréchet spaces. And the large frame indicates that all definitions are equivalent for Fréchet-Schwarz spaces.

54.4 Higher Order Differentiability. In order to define differentiability of higher order we need appropriate spaces of multi-linear mappings in which the higher derivatives should take values.

For the concepts of $C^n_{\mathcal{B}}$ [13] introduces the spaces $\mathcal{H}^n_{\mathcal{B}}(E, F)$ (for hyper-continuity) defined recursively by

$$\mathcal{H}^0_{\mathcal{B}}(E, F) := F$$
$$\mathcal{H}^{n+1}_{\mathcal{B}}(E, F) := \mathcal{L}_{\mathcal{B}}(E, \mathcal{H}^n_{\mathcal{B}}(E, F))$$

For C^n_c he considers $\mathcal{L}^n_c(E, F)$ as space of all continuous n-linear mappings $E \times \ldots \times E \to F$ with the convergence structure c of continuous convergence.

54.5 Definition. Let $\mathcal{B}$ be some family of bounded sets on E. A mapping $f : E \supseteq U \to F$ is called $C^n_{\mathcal{B}}$ iff it is n-times Gâteaux differentiable, i.e. all the n-fold iterated directional derivatives $d_{v_n} \ldots d_{v_1} f(x)$ exist and $(v_1, \ldots, v_n) \mapsto d_{v_n} \ldots d_{v_1} f(x)$ is n-linear and defines a continuous mapping $f^{(n)}(x) : E \supseteq U \to \mathcal{H}^n_{\mathcal{B}}(E, F)$.

It is called $C^\infty_{\mathcal{B}}$, if it is $C^n_{\mathcal{B}}$ for all $n \in \mathbb{N}$.

Similarly, let Λ be a convergence structure on $\mathcal{L}^k(E, F)$ for all $k \le n$. Then f is called C^n_Λ, iff it is n-times Gâteaux-differentiable and the n-th derivative $f^{(n)} : E \supseteq U \to \mathcal{L}^n_\Lambda(E, F)$ is continuous. It is called C^∞_Λ, if it is C^n_Λ for all $n \in \mathbb{N}$.

Again one has the same implications for C^n instead of C^1.

$$\begin{tikzcd}
C^n_s & C^n_k \arrow[l] & C^n_{pk} \arrow[l] & C^n_b \arrow[l] \\
 & & C^n_c \arrow[u] & C^n_{qb} \arrow[l] \arrow[u]
\end{tikzcd}$$

One gets the following dependencies by using that from the continuity of a higher derivative with respect to some convergence structure one can deduce continuity of lower derivatives with respect to certain stronger convergence structures:

$$\begin{tikzcd}
C^\infty_s & C^\infty_k \arrow[l] & C^\infty_{pk} \arrow[l] & C^\infty_b \arrow[l] \\
 & & C^\infty_c \arrow[u] & C^\infty_{qb} \arrow[l] \arrow[u]
\end{tikzcd}$$

Where this time the definitions in the smaller frame are equivalent for all lcs's, and for Fréchet spaces all given definitions are equivalent. This has become popular as "In Fréchet spaces all concepts of smoothness coincide" although strictly speaking this is not true: Gâteaux-smoothness is strictly weaker and tame-smoothness and the concepts of C^∞_Δ and C^∞_Θ (see [13]) are strictly stronger.

54.6 Remark. In order to compare the concepts of smoothness to be found in [13] with our smoothness we first have to compare the spaces of (multi-)linear mappings. For the following results [15] is the appropriate reference.

54.7 Lemma. *Let $\mathcal{B}$ be some set of bounded subsets of a locally convex space E, containing the finite subsets and being stable under the formation of finite unions and subsets.*

We denote with $L_{\mathcal{B}}(E,F)$ the space of all bounded linear mappings with the topology of uniform convergence on each bounded subset $B\in\mathcal{B}$. A 0-neighborhood-basis of this locally convex topology is given by the sets $N_{B,V} := \{f : f(B)\subseteq V\}$, where $B\in\mathcal{B}$ and V runs through the 0-neighborhoods in F. Note that $\mathcal{L}_{\mathcal{B}}(E,F)$ is the topological subspace of this space formed by the continuous linear mappings.

A subset $\mathcal{F}\subseteq L_{\mathcal{B}}(E,F)$ is bounded, iff it is uniformly bounded on bounded subsets $B\in\mathcal{B}$. In fact, $N_{B,V}$ absorbs $\mathcal{F}$ $\Leftrightarrow$ $\exists k$: $B_{B,k\,V} = k\,N_{B,V}\supseteq\mathcal{F}$, i.e. $\mathcal{F}(B)\subseteq k\,V$.

54.8 Corollary. *The bornology of $L_{\mathcal{B}}(E,F)$ is that of $L(E,F)$ provided $\mathcal{B}$ is any of the families mentioned in 4.3. And if E is c^∞-complete then this is true for all $\mathcal{B}$ between s and b.*

Proof. By what we said just before, $\mathcal{F}\subseteq L_{\mathcal{B}}(E,F)$ is bounded, iff $\mathcal{F}(B)$ is bounded for all $B\in\mathcal{B}$, or equivalently, iff $\mathcal{F}(B)$ is absorbed by any 0-neighborhood V in F, i.e. the absolutely convex set $U := \bigcap_{f\in\mathcal{F}} f^{-1}(V)$ absorbs all B. Now we may apply 4.3 and, in the c^∞-complete case, 5.18. □

54.9 Corollary. *Let $\mathcal{B}$ be any of the bornologies in 54.8. Then the inclusion $\mathcal{H}^n_{\mathcal{B}}(E,F)\to L(E,\dots,E;F)$ is well-defined, bounded and linear.*

Proof. For $n=0$ nothing is to be shown.

For $n=1$ we have that $\mathcal{H}^1_{\mathcal{B}}(E,F)=\mathcal{L}_{\mathcal{B}}(E,F)\subseteq L_{\mathcal{B}}(E,F)\overset{b}{\cong} L(E,F)$ by 54.8. With induction we get for $n+1$ the following sequence of bounded mappings:

$$\begin{aligned}
\mathcal{H}^{n+1}_{\mathcal{B}}(E,F) &\cong \mathcal{L}_{\mathcal{B}}(E,\mathcal{H}^n_{\mathcal{B}}(E,F))\to\\
&\to L(E,\mathcal{H}^n_{\mathcal{B}}(E,F))\to\\
&\to L(E,L(E,\dots,E;F))\cong L(E,\dots,E;F)
\end{aligned}$$

□

54.10 Theorem. *For a mapping $f: E\supseteq U\to F$ from a c^∞-open subset E of a lcs E with values in an lcs F the following statements are equivalent:*

(1) *f is C^∞;*
(2) *All the iterated directional derivatives $d^n f(x)(v_1, \dots, v_n)$ exist and are bounded on M-converging sequences in $U \times E^n$;*
(3) *The iterated directional derivatives $d^n f(x)(v_1, \dots, v_n)$ exist and define a mapping $d^n f : E \supseteq U \to L(E, \dots, E; F)$ which is bounded on M-converging sequences (or bornologically compact subsets of U);*

If E is c^∞-complete then this is further equivalent to

(4) *The iterated unidirectional derivatives $d_v^n f(x)$ exist and are separately bounded in x and in v on M-converging sequences.*

Proof. ($4 \Rightarrow 3 \Rightarrow 1$) In the proof of 5.20 we have shown that for c^∞-complete lcs E a mappings satisfying (4) satisfies (3) as well.
Then we showed without using any completeness condition that from (3) the chain rule for curves $c : \mathbb{R} \to U$ follows and hence (1).

($1 \Rightarrow 3$) follows from the chain rule given in 3.18, since then $d^n f : E \supseteq U \to L(E, \dots, E; F)$ is C^∞ and hence continuous on bornologically compact sets $K \subseteq E_B \subseteq E$.

($3 \Rightarrow 4$) and ($3 \Rightarrow 2$) are trivial, since bounded subsets of $L(E, \dots, E; F)$ are bounded on M-converging sequences, see below.

($2 \Rightarrow 3$) It was shown in 5.20 that from (2) we conclude that $d^n f : E \supseteq U \to L(E, \dots, E; F)$ exists and is bounded on M-converging sequences with respect to the pointwise topology on $L(E, \dots, E; F)$. But by assumption this is even true for the topology of uniform convergence on M-converging sequences, and this is induces the same bornology as that of uniform convergence on bounded sets by 54.8. □

54.11 Proposition. *Let η be some real sequence converging to ∞ and $f : E \supseteq U \to F$ be a mapping from an open subset U of a lcs E with value in an lcs F.
If $f \in C^\infty_{\mathcal{B}}$, where $\mathcal{B}$ contains all η-sequences, then $f \in C^\infty$.
If E is c^∞-complete, then $f \in C^\infty_s$ implies $f \in C^\infty$.*

Proof. By assumption we have that f is infinite often Gâteaux differentiable and $f^{(n)} : E \supseteq U \to \mathcal{H}^n_{\mathcal{B}}(E, F)$ is continuous. Since $\mathcal{H}^n_{\mathcal{B}}(E, F) \to L(E, \dots, E; F)$ is well-defined and bounded by 54.9 the result follows from 54.10. □

54.12 Theorem. *Let $f : E \supseteq U \to F$ with U open in an lcs E and F an lcs. If $c^\infty(E) = E$ then $f \in C^\infty \Leftrightarrow f \in C^\infty_{\mathcal{B}}$, with any $\mathcal{B}$ as in 54.11.*

Proof. Because of 54.11 we only have to show ($\Rightarrow$). By 3.18 we have the directional derivative $df : E \supseteq U \to L(E, F)$ which is C^∞ as well. So f is infinitely often Gâteaux differentiable and it remains to show that $d^n f : E \supseteq U \to L(E, \dots, E; F)$ is well-defined and continuous into $\mathcal{H}^n_b(E, F)$. Since $d^n f$ is smooth, we have that $d^n f : c^\infty(U) = c^\infty(E)|_U \to c^\infty(L(E, \dots, E; F)) \to L(E, \dots, E; F)_{\text{born}}$ is continuous, and since $c^\infty E = E$, we get that $d^n f : E \supseteq U \to L(E, \dots, E; F)_{\text{born}}$ is continuous. Since $c^\infty E = E$ we have that E is bornological, so $\mathcal{L}_{\mathcal{B}}(E, F)_{\text{born}} =$

$L(E,F)_{\text{born}} = L(E,F_{\text{born}})_{\text{born}}$ and by induction we get

$$\begin{aligned} L(E,\dots,E,E;F)_{\text{born}} &\cong L(E,L(E,\dots,E;F)_{\text{born}})_{\text{born}} \\ &= \mathcal{L}_{\mathcal{B}}(E,L(E,\dots,E;F)_{\text{born}})_{\text{born}} \\ &= \mathcal{L}_{\mathcal{B}}(E,\mathcal{L}_{\mathcal{B}}^{n-1}(E,F)_{\text{born}})_{\text{born}} \\ &= \mathcal{L}_{\mathcal{B}}(E,\mathcal{L}_{\mathcal{B}}^{n-1}(E,F))_{\text{born}} \\ &= \mathcal{L}_{\mathcal{B}}^{n+1}(E,F)_{\text{born}}. \end{aligned}$$

So the derivatives are continuous into $\mathcal{L}^n_{\mathcal{B}}(E,F)$. □

55. Silva-Differentiability

See [9]. The idea here is to use the normed spaces E_B with B bounded in E and $F_q := F/q^{-1}(0)$ for continuous seminorms q on F associated with each locally convex space, and in fact for E we only need a CONVEX BORNOLOGICAL SPACE (CBS, for short) E (i.e. a vector space together with a bornology which is invariant under addition, homotheties and formation of convex hulls).

55.1 Definition. Let E and F be cbs's. A mapping $f : E \to F$ is called SILVA DIFFERENTIABLE AT $x \in E \Leftrightarrow \forall A \subseteq E$ absolutely convex bounded $\exists B \subseteq F$ absolutely convex bounded such that $f(x + \_) - f(x) : E_A \to F_B$ is locally around 0 defined and Fréchet differentiable at 0.

Equivalently, $\forall A \subseteq E$ absolutely convex bounded with $x \in A$ $\exists B \subseteq F$ absolutely convex bounded such that $f : E_A \to F_B$ is locally around x defined and Fréchet differentiable at x.

In fact, $f(x + \_) - f(x) : E_A \to F_B$ has some local property at 0 provided $f : E_{A_x} \to F_{B_{-f(x)}}$ has the same property at x, where $A_x := \langle\{x\} \cup A\rangle_{\text{abs.conv.}}$, since $f(x + \_) - f(x) = (\_ - f(x)) \circ f \circ (x + \_)$ and $x + \_ : E_A \to E_{A_x}$ is affine and bounded because $A \subseteq A_x$. Conversely, $f : E_A \to F_B$ has some local property at $x \in A$ provided $f(x + \_) : E_{A_{-x}} \to F_{b_{f(x)}}$ has the same property at 0, since $f(\_) = (\_ + f(x)) \circ (f(x + \_) - f(x)) \circ (\_ - x)$ and $\_ - x : E_A \to E_{A_{-x}}$ is affine and bounded as before.

Note that in this situation the derivatives at x of the restrictions of $f : E \to F$ to locally defined mappings $E_A \to F_B$ fit together to define a bounded linear mapping $f'(x) : E \to F$. Thus the definition of Silva-differentiability of f at x can be rephrased as in [9, 1.1.1]: $\exists \ell : E \to F$ bounded and linear, such that for $r(h) := f(x + h) - f(x) - \ell \cdot h$ one has:
$\forall A \subseteq E$ absolutely convex bounded $\exists B \subseteq F$ absolutely convex bounded such that $\exists \varepsilon > 0 : r(\varepsilon A) \subseteq B$ and $p_B(r(h))/p_A(h) \to 0$ for $p_A(h) \to 0$.

55.2 Definition. Let E and F be cbs's and $f : E \to F$. Then f is called SILVA DIFFERENTIABLE iff it is Silva-differentiable at each point $x \in E$. Note that the B in the definition 55.1 of differentiability at x may depend not only on the given A but also on x. Thus a Silva differentiable f need not have locally differentiable restrition $E_A \to F_B$ for some B.

55.3 Definition. Let E and F be cbs's and $f : E \to F$. Then f is called M-CONTINUOUS at $x \in E$ iff $\forall A \subseteq E$ absolutely convex bounded $\exists B \subseteq F$ absolutely convex bounded such that $f(x + \_) - f(x) : E_A \to F_B$ is defined locally around 0 and continuous at 0, i.e. $\exists \varepsilon > 0$ with $f(x + \varepsilon A) \subseteq B$ and $p_B(f(x + h) - f(x)) \to 0$ for $p_A(h) \to 0$.

Equivalently, $\forall A \subseteq E$ absolutely convex bounded with $x \in A$ $\exists B \subseteq F$ absolutely convex bounded such that $f : E_A \to F_B$ is locally around x defined and continuous at x.

The mapping f is called M-continuous, iff it is so at every point $x \in E$.

55.4 Definition. Let E and F be cbs's and $f : E \to F$. Then f is called continuously Silva differentiable (S^1 for short) iff it is Silva differentiable and $f' : E \to L(E, F)$ is M-continuous, where $L(E, F)$ denotes the cbs of bounded linear mappings from E to F with the bornology formed by the subsets being uniformly bounded in F on each bounded subset of E.

55.5 Definition. Let E and F be cbs's and $f : E \to F$. Then f is called $n+1$-times continuously Silva differentiable (S^{n+1} for short) if it is S^n and the n-th derivative $f^{(n)} : E \to L(E, \dots, E; F)$ is S^1, or equivalently, avoiding the higher derivative, if f is S^1 and its derivative $f' : E \to L(E, F)$ is S^n.

The mapping f is called S^∞ iff f is n-times Silva-differentiable for all $n \in \mathbb{N}$.

55.6 Definition. Let E be a cbs, F an lcs and $f : E \to F$. The f is called Silva differentiable in the enlarged sense, iff $\forall x \in E$ there exists a bounded linear $\ell : E \to F$ such that $r_x(h) := f(x+h) - f(x) - \ell \cdot h$ is a remainder in the following sense: For every $A \subseteq E$ absolutely convex bounded and every continuous seminorm q on F we have

$$q(r_a(h))/p_A(h) \to 0 \text{ for all } h \to 0 \text{ in } E_A.$$

For complete F this condition is equivalent to $E_A \to E \to F \to F/\operatorname{Ker}(q)$ being differentiable between normed spaces since then F is embedded as closed subspace of $\prod_q \widehat{F/\operatorname{Ker}(q)}$

55.7 Definition. Analogously, for $n \in \mathbb{N} \cup \{\infty\}$, one may define n-times (continuously) Silva differentiable in the enlarged sense (S^n_e for short) and this is for complete (and in case $n = \infty$ even for c^∞-complete) F equivalent to $E_B \to E \to F \to F/\operatorname{Ker}(q)$ being n-times (continuously) differentiable between normed spaces. Thus for a locally convex space E and convenient vector spaces F a mapping $f : E \to F$ is S^∞_e for the von Neumann bornology on E if and only if it is C^∞.

55.8 Remark. This definition makes problems with the chain-rule $E \to F \to G$ even if the space F in the middle is a locally convex space, since for $F \to G$ we only have properties on F_B but the restriction of $E \to F$ to E_A need not have values in F_B for some B.

55.9 Example. Note that S^n implies S^n_e (see [9, 1.4.8]), but not conversely even for $f : E \to \mathbb{R}$, see [9, 2.5.2].

55.10 Definition. Let $p \in \mathbb{N} \cup \{\infty\}$ and E and F be cbs's. A mapping $f : E \to F$ is called locally p-times differentiable between normed spaces at a point $x \in E$ iff $\forall A \subseteq E$ absolutely convex bounded $\exists \varepsilon > 0$ $\exists B \subseteq F$ absolutely convex bounded such that $f(x + \varepsilon A) \subseteq B$ and $f : x + \{z \in E_A : \|z\|_A < \varepsilon\} \to F_B$ is p-times differentiable. Note that here in contrast to definitions 55.1–55.4 the bounded set B is locally independent on x.

55.11 Proposition. *Let E and F be cbs's and F be polar, i.e. the lcs-closure of bounded sets is bounded. Then f $p+1$-times continuously Silva differentiable implies f locally p-times continuously differentiable between normed spaces.*

55.12 Example. There exists scalar valued mappings which are locally C^∞ between normed spaces but are not S^∞, see [9, 2.5].

55.13 Corollary. *Let $f : E \to F$ be smooth and $K \subseteq E$ be bornologically-compact. Then the image $f(K)$ in F is bornologically compact. Moreover, if $K \subseteq E_B$ is compact we find a bounded absolutely convex set $A \subseteq F$ such that $f : E_B \supseteq K \to F_A$ is a contraction.*

Proof. Since $f : E \to F$ is smooth, we have that $g := \ell \circ f : E_B \to \mathbb{R}$ is C^∞ in various senses. In particular it is continuous, and from continuity of $g' : E_B \to L(E_B, \mathbb{R})$ we deduce locally Lipschitzness of g, since

$$\begin{aligned}|g(y) - g(x)| &= \left|\int_0^1 g'(x + t(y-x))(y-x)\,dt\right|\\ &\le \int_0^1 |g'(x + t(y-x))(y-x)|\,dt\\ &\le \sup\Big\{\|g'(x+t(y-x))\| : t \in [0,1]\Big\} \cdot \|y-x\|\end{aligned}$$

Since $K \subseteq E_B$ is compact we get a Lipschitz bound of $\ell \circ f$ on K for each $\ell \in E'$ (see below) and hence $\{\frac{f(x)-f(y)}{\|x-y\|_B} : x, y \in K\}$ is bounded in F. Let A be the absolutely convex hull of this set, then $f : E_B \supseteq K \to F_A$ is a contraction, and hence continuous and thus $f(K)$ is compact in F_A.

A locally Lipschitzian mapping on a normed space is Lipschitzian on each compact subset: Otherwise we would find x_n and y_n with $|f(x_n) - f(y_n)|/\|x_n - y_n\|$ unbounded. Without loss of generality we may assume that $x_n \to x_\infty$ and $y_n \to y_\infty$. If $x_\infty \neq y_\infty$ then by continuity of f we get boundedness of the difference quotient. And if $x_\infty = y_\infty$ this contradicts the local Lipschitzness of f at x_∞. □

55.14 Proposition. [15] *Let E and F be convenient vector spaces and $f : E \to F$. Then f is C^∞ $\Leftrightarrow$ $\forall K \subseteq E$, absolutely convex, bornologically compact, $\forall x \in K$ $\forall n \in \mathbb{N}$ ($n \neq \infty$) $\exists J \subseteq F$, absolutely convex, bornologically compact such that $f : E_K \to F_J$ is C^n locally around x, i.e. f is locally n-times continuously differentiable between normed spaces for the bornologies of bornologically compact sets.*

Proof. ($\Leftarrow$) Let $c : \mathbb{R} \to E$ be C^∞, let $I \subseteq \mathbb{R}$ be a bounded open interval, $t_0 \in I$ and $n \in \mathbb{N}$. Since $\delta c : \mathbb{R}^2 \to E$ given by $\delta c(t,s) := \int_0^1 c'(t + r(s-t))\,dr$ is smooth the image is bornologically-compact. By 55.13 $\delta c(I \times I) \cup \dots \cup \delta c^{(n)}(I \times I) \cup \{c(t_0), \dots, c^{(n)}(t_0)\}$ is a bornologically-compact set K and hence compact in some E_B.

Then there exists a sequence $x_n \to 0$ in E_B such that K is in the closed absolutely convex hull of $\{x_n : n \in \mathbb{N}\}$. The closed convex hull B' of this sequence is compact in E_B, so K is in the unit-sphere of $E_{B'}$ with bornologically compact B'.

Now we can deduce recursively that $c : I \to E_{B'}$ is C^n and hence the composite $f \circ c : I \to F$ is C^n.

($\Rightarrow$) Let K be bornologically compact and $n \in \mathbb{N}$. It suffices to show the existence of a bornologically compact $K_n \subseteq F$ such that $f : E_K \supseteq o(E_K) \to F_{K_n}$ is C^n_s, i.e. $x \mapsto f^{(k)}(x)(v_1, \dots v_k)$, $oE_K \to F_{K_n}$ is continuous for all $k \leq n$.

Since these derivatives are smooth $E_B \to F$ there exists some bornologically compact $K_n \subseteq F$, such that they are Lipschitz $E_B \supseteq K \to E_{K_n}$ by what we proved in 55.13. Hence they are continuous $K \subseteq E_K \to E_B \to E_{K_n}$. □

55.15 Remark. Ulrich Seip defined f to be smooth iff it is smooth along all smooth mappings $c : \mathbb{R}^n \to U$ (by Boman [**?**] $n = 1$ suffices) and all derivatives are continuous on compact subsets $U \times E^n$. This is weaker than C^∞_c, since continuity $f^{(n)} : U \to \mathcal{L}^n_c(E, F)$ is required only on compact subsets of U.

However, it is not clear, whether all compact subsets are bounding.

Chapter IV
Smoothly Realcompact Spaces

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As motivation for the developments in this chapter let us tell a mathematical short story which was posed as an exercise in [Milnor, Stasheff, 1974, p.11]. For a finite dimensional Hausdorff second countable manifold M, one can prove that the space of algebra homomorphisms $\operatorname{Hom}(C^\infty(M,\mathbb{R}),\mathbb{R})$ equals M as follows. The kernel of a homomorphism $\varphi : C^\infty(M,\mathbb{R}) \to \mathbb{R}$ is an ideal of codimension 1 in $C^\infty(M,\mathbb{R})$. The zero sets $Z_f := f^{-1}(0)$ for $f \in \ker\varphi$ form a filter of closed sets, since $Z_f \cap Z_g = Z_{f^2+g^2}$, which contains a compact set Z_f for a function f which is proper (i.e., compact sets have compact inverse images). Thus $\bigcap_{f\in\ker\varphi} Z_f$ is not empty, it contains at least one point $x_0 \in M$. But then for any $f \in C^\infty(M,\mathbb{R})$ the function $f - \varphi(f)1$ belongs to the kernel of φ, so vanishes on x_0 and we have $f(x_0) = \varphi(f)$.

This question has many rather complicated (partial) answers in any infinite dimensional setting which are given in this chapter. One is able to prove that the answer is positive surprisingly often, but the proofs are involved and tied intimately to the class of spaces under consideration. The existing counter-examples are based on rather trivial reasons. We start with setting up notation and listing some interesting algebras of functions on certain spaces.

First we recall the topological theory of realcompact spaces from the literature and discuss the connections to the concept of smooth realcompactness. For an algebra homomorphism $\varphi : \mathcal{A} \to \mathbb{R}$ on some algebra of functions on a space X we investigate when $\varphi(f) = f(x)$ for some $x \in X$ for one function f, later for countably many, and finally for all $f \in \mathcal{A}$. We study stability of smooth realcompactness under pullback along injective mappings, and also under (left) exact sequences. Finally we discuss the relation between smooth realcompactness and bounding sets, i.e. sets on which every function of the algebra is bounded. In this chapter, the ordering principle for sections and results is based on the amount of evaluating properties obtained and we do not aim for linearly ordered proofs. So we will often use results presented later in the text. We believe that this is here a more transparent presentation than the usual one. Most of the material in this chapter can also be found in the theses' [Biström, 1993] and [Adam, 1993].

17. Basic Concepts and Topological Realcompactness

17.1. The setting. In [Hewitt, 1948, p.85] those completely regular topological spaces were considered under the name Q-spaces, for which each real valued algebra homomorphism on the algebra of all continuous functions is the evaluation at some point of the space. Later on these spaces where called realcompact spaces.

Accordingly, we call a 'space' *smoothly realcompact* if this is true for 'the' algebra of smooth functions. There are other algebras for which this question is interesting, like polynomials, real analytic functions, C^k-functions. So we will treat the question in the following setting. Let

X be a set;

$\mathcal{A} \subseteq \mathbb{R}^X$ a point-separating subalgebra with unit; If X is a topological space we also require that $\mathcal{A} \subseteq C(X, \mathbb{R})$; If $X = E$ is a locally convex vector space we also assume that $\mathcal{A}$ is invariant under all translations and contains the dual E^* of all continuous linear functionals;

$X_{\mathcal{A}}$ the set X equipped with the initial topology with respect to $\mathcal{A}$;

$\varphi : \mathcal{A} \to \mathbb{R}$ an algebra homomorphism preserving the unit;

$Z_f := \{x \in X : f(x) = \varphi(f)\}$ for $f \in \mathcal{A}$;

$\operatorname{Hom} \mathcal{A}$ be the set of all real valued algebra homomorphisms $\mathcal{A} \to \mathbb{R}$ preserving the unit.

Moreover,

φ is called $\mathcal{F}$*-evaluating* for some subset $\mathcal{F} \subseteq \mathcal{A}$ if there exists an $x \in X$ with $\varphi(f) = f(x)$ for all $f \in \mathcal{F}$; equivalently $\bigcap_{f \in \mathcal{F}} Z_f \neq \emptyset$;

φ is called $\mathfrak{m}$*-evaluating* for a cardinal number $\mathfrak{m}$ if φ is $\mathcal{F}$-evaluating for all $\mathcal{F} \subseteq \mathcal{A}$ with cardinality of $\mathcal{F}$ at most $\mathfrak{m}$; This is most important for $\mathfrak{m} = 1$ and for $\mathfrak{m} = \omega$, the first infinite cardinal number;

φ is said to be $\bar{1}$-evaluating if $\varphi(f) \in \overline{f(X)}$ for all $f \in \mathcal{A}$.

φ is said to be evaluating if φ is $\mathcal{A}$-evaluating, i.e., $\varphi = \operatorname{ev}_x$ for some $x \in X$;

$\operatorname{Hom}_\omega \mathcal{A}$ is the set of all ω-evaluating homomorphisms in $\operatorname{Hom} \mathcal{A}$;

$\mathcal{A}$ is called $\mathfrak{m}$-evaluating if φ is $\mathfrak{m}$-evaluating for each algebra homomorphism $\varphi \in \operatorname{Hom} \mathcal{A}$;

$\mathcal{A}$ is called evaluating if φ is evaluating for algebra homomorphism $\varphi \in \operatorname{Hom} \mathcal{A}$;

X is called $\mathcal{A}$-realcompact if $\mathcal{A}$ is evaluating; i.e., each algebra homomorphism $\varphi \in \operatorname{Hom} \mathcal{A}$ is the evaluation at some point in X.

The algebra $\mathcal{A}$ is called

inversion closed if $1/f \in \mathcal{A}$ for all $f \in \mathcal{A}$ with $f(x) > 0$ for every $x \in X$; equivalently, if $1/f \in \mathcal{A}$ for all $f \in \mathcal{A}$ with f nowhere 0 (use $f^2 > 0$).

bounded inversion closed if $1/f \in \mathcal{A}$ for $f \in \mathcal{A}$ with $f(x) > \varepsilon$ for some $\varepsilon > 0$ and all $x \in X$;

$C^{(\infty)}$-algebra if $h \circ f \in \mathcal{A}$ for all $f \in \mathcal{A}$ and $h \in C^\infty(\mathbb{R}, \mathbb{R})$;

C^∞-algebra if $h \circ (f_1, \dots, f_n) \in \mathcal{A}$ for all $f_j \in \mathcal{A}$ and $h \in C^\infty(\mathbb{R}^n, \mathbb{R})$;

C^∞_{lfs}-algebra if it is a C^∞-algebra which is closed under locally finite sums, with respect to a specified topology on X. This holds if $\mathcal{A}$ is local, i.e., it contains any function f such that for each $x \in X$ there is some $f_x \in \mathcal{A}$ with $f = f_x$ near x.

C^∞_{lfcs}-algebra if it is a C^∞-algebra which is closed under locally finite countable sums.

Interesting algebras are the following, where in this chapter in the notation we shall generally omit the range space $\mathbb{R}$.

$$\begin{tikzcd}
 & & & C_b \arrow[rr, hook] & & C \\
 & C^\infty_{\mathrm{lfcs}} \arrow[rr, hook] & & \mathcal{C}^\infty \arrow[r, hook] & C^\infty \cap C \arrow[ur, hook] \arrow[dr, hook] & \\
P_f \arrow[r, hook] \arrow[ur, hook] & P \arrow[r, hook] \arrow[dr, hook] & \mathcal{C}^\omega \arrow[r, hook] \arrow[ur, hook] & C^\omega \cap C \arrow[ur, hook] \arrow[dr, hook] & & C^\infty \\
 & & C^\omega_{\mathrm{conv}} \arrow[rr, hook] & & C^\omega \arrow[ur, hook] &
\end{tikzcd}$$

$C(X) = C(X, \mathbb{R})$, the algebra of continuous functions on a topological space X. It has all the properties from above.

$C_b(X) = C_b(X, \mathbb{R})$, the algebra of bounded continuous functions on a topological space X. It is only bounded inversion closed and a C^∞-algebra, in general.

$C^\infty(X) = C^\infty(X, \mathbb{R})$, the algebra of smooth functions on a Frölicher space X, see 23.1, or on a smooth manifold X, see section 27. It has all properties from above, where we may use the c^∞-topology.

$C^\infty(E) \cap C(E)$, the algebra of smooth and continuous functions on a locally convex space E. It has all properties from above, where we use the locally convex topology on E.

$\mathcal{C}^\infty(E) = \mathcal{C}^\infty(E, \mathbb{R})$, the algebra of smooth functions, all of whose derivatives are continuous on a locally convex space E. It has all properties from above, again for the locally convex topology on E.

$C^\omega(X) = C^\omega(X, \mathbb{R})$, the algebra of real analytic functions on a real analytic manifold X. It is only inversion closed.

$C^\omega(E) \cap C(E)$, the algebra of real analytic and continuous functions on a locally convex space E. It is only inversion closed.

$\mathcal{C}^\omega(E) = \mathcal{C}^\omega(E, \mathbb{R})$, the algebra of real analytic functions, all of whose derivatives are continuous on a locally convex space E. It is only inversion closed.

$C^\omega_{\mathrm{conv}}(E) = C^\omega_{\mathrm{conv}}(E, \mathbb{R})$, the algebra of globally convergent power series on a locally convex space E.

$P_f(E) = \mathrm{Poly}_f(E, \mathbb{R})$, the algebra of finite type polynomials on a locally convex space E, i.e. the algebra $\langle E' \rangle_{\mathrm{Alg}}$ generated by E'. This is the free commutative algebra generated by the vector space E', see 18.12. It has none of the properties from above.

$P(E) = \mathrm{Poly}(E, \mathbb{R})$, the algebra of polynomials on a locally convex space E, see 5.15, 5.17, i.e. the homogeneous parts are given by bounded symmetric multilinear mappings. No property from above holds.

$C^\infty_{\mathrm{lfcs}}(E) = C^\infty_{\mathrm{lfcs}}(E, \mathbb{R})$, the C^∞_{lfcs}-algebra (see below) generated by E', and hence also called $(E')^\infty_{\mathrm{lfcs}}$. Only the C^∞_{lfs}-property does not hold.

17.2. Results. *For completely regular topological spaces X and $\mathcal{A} = C(X)$ the following holds:*

(9) *See* [Engelking, 1989, 3.11.16]. *The realcompactification νX of a completely regular space X is defined as the realcompact space* $\mathrm{Hom}(C(X)) \subseteq \mathbb{R}^{C(X)}$ *of all $\mathbb{R}$-valued algebra-homomorphisms mit the topology of pointwise convergence. It is the closure of $\delta(X)$ in $\mathbb{R}^{C(X)}$. It has the universal*

property of extending continuous functions $f : X \to Y$ into realcompact spaces Y uniquely to continuous functions $\tilde{f} : \nu X \to Y$.

(10) *See* [Engelking, 1989, 3.11.10]. *The space νX is homeomorphic to the subspace $\bigcap_{f\in C(X)} \bar{f}^{-1}(\mathbb{R}) = \bigcap\{h^{-1}((0,1]) : h \in C(\beta X, [0,1]), h|_X > 0\} \subseteq \beta X$ where $\bar{f}$ denotes the extension of $f : Y \to \mathbb{R} \to \mathbb{R}_\infty$ to the Stone-Čech-compactification βX.*

(11) *For every $y \in \beta X \setminus \nu X$ there exists countably many closed neighborhoods U_n with $\nu X \cap \bigcap_n U_n = \emptyset$.*

(12) *For every $y \in \nu X \setminus X$ and closed neighborhoods U_n of y we have $X \cap \bigcap_n U_n \neq \emptyset$.*

(1) *Due to* [Hewitt, 1948, p.85 + p.60] *&* [Shirota, 1952, p.24]*, see also* [Engelking, 1989, 3.12.22.g & 3.11.3]. *The space X is called realcompact if all algebra homomorphisms in* Hom $C(X)$ *are evaluations at points of X, equivalently, if X is a closed subspace of a product of $\mathbb{R}$'s.*

(2) *Due to* [Hewitt, 1948, p.61] *&* [Katětov, 1951, p.82]*, see also* [Engelking, 1989, 3.11.4 & 3.11.5]. *Hence every closed subspace of a product of realcompact spaces is realcompact.*

(3) *Due to* [Hewitt, 1948, p.85]*, see also* [Engelking, 1989, 3.11.12]. *Each Lindelöf space is realcompact.*

(18) *See* [Engelking, 1989, 3.11.H] *The realcompact spaces are exactly the (projective) limits of Lindelöf spaces.*

(17) *See* [Engelking, 1989, 3.11.8] *If $f : X \to Y$ is continuous X and $Z \subseteq Y$ realcompact and Y T_2. Then $f^{-1}(Z)$ is realcompact. In particular functionally open subsets are realcompact.*

(15) *See* [Engelking, 1989, 3.11.1] *A topological space is compact if and only if it is pseudocompact and realcompact.*

(16) *See* [Engelking, 1989, 3.11.2] *An example for a non-realcompact space is the space $\Omega = [0, \Omega)$ of countable ordinals*

(5) *Due to* [Hewitt, 1950, p.170, p.175] *&* [Mackey, 1944]*, see also* [Engelking, 1989, 3.11.D.a]. *Discrete spaces are realcompact if and only if their cardinality is non-measurable.*

(8) *Due to* [Dieudonné, 1939,] *see also* [Engelking, 1989, 8.5.13.a]. *For a topological space X the following statements are equivalent:*
   (a) *X admits a complete uniformity, i.e. X is Dieudonné complete;*
   (b) *X is closed embedable into a product of complete metrizable spaces;*
   (c) *X is closed embedable into a product of metrizable spaces;*
   (d) *X is a projective limit of complete metrizable spaces;*
   (e) *X is a projective limit of metrizable spaces;*

(19) *Stone's Theorem: Let X be metrizable and $\mathcal{U}$ an open covering of X. Then there exists an open locally finite and σ-discrete refinement of $\mathcal{U}$. Note, that σ-discrete means that it is a countable union of discrete sets of subsets, i.e. every point in X has a neighborhood that intersections at most one of the subsets.*

(14) *See* [Engelking, 1989, 5.1.J.e] *and* [Engelking, 1989, 8.5.13.h]. *A topological space is realcompact if and only if it is Dieudonné complete and each closed discrete subspace is realcompact.*

(7) [Shirota, 1952]*, see also* [Engelking, 1989, 5.5.10 & 8.5.13.h]. *A topological space of non-measurable cardinality is realcompact if and only if it is Dieudonné complete.*

(13) *See* [Engelking, 1989, 8.5.13b] *or* [Engelking, 1989, 5.1.J.f]. *Every paracompact space is Dieudonné complete.*

(4) *Due to* [Katětov, 1951, p.82]*, see also* [Engelking, 1989, 5.5.10]*. Paracompact spaces are realcompact if and only if all closed discrete subspaces are realcompact.*

(6) *Hence Banach spaces (or even Fréchet spaces) are realcompact if and only if their density (i.e., the cardinality of a maximal discrete or of a minimal dense subset) or their cardinality is non-measurable.*

Realcompact spaces where introduced by [Hewitt, 1948, p.85] under the name Q-compact spaces. The equivalence in (1) is due to [Shirota, 1952, p.24]. The results (1) and (2) are proved in [Engelking, 1989] for a different notion of realcompactness, which was shown to be equivalent to the original one by [Katětov, 1951], see also [Engelking, 1989, 3.12.22.g].

Proof. (9) See 17.3. Let $\nu X := \operatorname{Hom}(C(X)) \subseteq \mathbb{R}^{C(X)}$ and $\delta : X \to \nu X$ be given by $x \mapsto \operatorname{ev}_x := (f(x))_f$. Then every $f \in C(X)$ extends along δ to $\tilde{f} := \operatorname{pr}_f : \mathbb{R}^{C(X)} \to \mathbb{R}$. Obviously $\operatorname{Hom}(C(X))$ is closed in $\mathbb{R}^{C(X)}$ and X is dense in $\operatorname{Hom}(C(X))$, since for $\varphi \in \nu(X)$, $f_1, \dots, f_n \in C(X)$ and $\varepsilon > 0$ we find an $x \in X$ with $\varphi(f_i) = f_i(x)$ for all i. Thus the extension $\tilde{f}$ is unique.

Now let Y be a realcompact space, then $\delta_Y : Y \to \nu(Y)$ is a homeomorphism and any continuous $f : X \to Y$ induces a continuous map $f^{**} : \mathbb{R}^{C(X)} \to \mathbb{R}^{C(Y)}$ which thus maps $\overline{\delta(X)} = \nu X$ into $\overline{\delta(f(X))} \subseteq \nu Y$. This extension $\tilde{f} : \nu X \to \nu Y$ is unique, since X is dense in νX.

Furthermore νX is realcompact: Let $\varphi : C(\nu X) \to \mathbb{R}$ be an algebra-homomorphism. then $\psi := \varphi \circ (\delta^*)^{-1} : C(X) \to \mathbb{R}$ is an algebra-homomorphism and hence an element of $\nu(X)$, i.e. $\varphi(g) = (\psi \circ \delta^*)(g) = \psi(g \circ \delta) = g(\psi) = \operatorname{ev}_\psi(g)$, since $g = \widetilde{g \circ \delta} = \operatorname{ev}_{\delta^*(g)}$.

(10) Consider the subspace $\gamma X := \bigcap_{f \in C(X)} \bar{f}^{-1}(\mathbb{R}) \subseteq \beta X$. Obviously X is dense in γX and any $f \in C(X)$ extends (uniquely) to $\bar{f} : \gamma X \to \mathbb{R}$. We show the universal property of the realcompactification for $\delta : X \to \gamma X$: So let $f : X \to Y$ be continuous into a realcompact space which is closed in $\mathbb{R}^{C(Y)}$. Then f extends to a continuous map $\gamma X \to \mathbb{R}^{C(Y)}$ and as before it has values in $\overline{\delta(Y)} = Y$.

Furthermore,

$$\begin{aligned} \beta X \setminus \nu(X) &= \bigcup \Big\{ \bar{f}^{-1}(\infty) : f \in C(X, \mathbb{R}) \Big\} \\ &= \bigcup \Big\{ \bar{f}^{-1}(\infty) : f \in C(X, [1, +\infty)) \Big\} \\ &= \bigcup \Big\{ \bar{f}^{-1}(0) : f \in C(X, (0, 1]) \Big\}, \end{aligned}$$

where the second equality follows since for $f \in C(X, \mathbb{R})$ with $\bar{f}(y) = \infty$ we can consider $g := 1 + |f| \in C(X, [1, +\infty))$ with $\bar{g} = 1 + |\bar{f}|$, hence $\bar{g}(y) = \infty$, and the third follows by considering $1/f$ instead of f.

(11) Let $y \in \beta X \setminus \nu X$. By (10) there exists an $f \in C(X)$ with $\bar{f}(y) = \infty$. Thus $U_n := \{z \in \beta X : |\bar{f}(z)| \geq n\}$ are closed neighborhoods of y in βX and $\bigcap_n U_n = \bar{f}^{-1}(\infty) \subseteq \beta X \setminus \nu X$.

(12) Let U_n be closed neigborhoods of $y \in \nu X \setminus X$. Since νX is completely regular there are $f_n \in C(\nu X, \mathbb{R})$ with $f_n(y) = 1$ and $f_n(\nu X \setminus U_n) = 0$. Suppose $X \cap \bigcap_n U_n = \emptyset$. Then $\sum_n f_n$ is locally finite on X hence $f := \sum_{n=1}^\infty f_n \in C(X, \mathbb{R})$ but for its extension $\bar{f}(y) = \infty$, so $y \notin \nu X$ by (10).

(1) If X is realcompact, then $X \cong \nu(X) \subseteq \mathbb{R}^{C(X)}$ is a closed subspace of a product of $\mathbb{R}$. Conversely, let $\iota : X \to \mathbb{R}^J$ be a closed embedding. This may be extended to

a continuous mapping $\tilde{\iota} : \nu X \to \mathbb{R}^J$ along $\delta : X \to \nu X$. Since δ has dense image $\tilde{\iota}$ has values in $\overline{\iota(X)} = \iota(X)$, hence $\iota^{-1} \circ \tilde{\iota} \circ \delta = \mathrm{Id}$ and thus δ is a closed dense embedding, i.e. an isomorphism.

(2) follows trivially from (1).

(3) follows from 18.11 and 18.24.

(18) By (3) and (2) each limit of Lindelöf spaces is realcompact. Conversely, realcompact spaces $X = \nu X = \mathrm{Hom}(C(X))$ embed into $\prod_A \mathrm{Hom}(A)$, where A runs through the finitely generated (hence countable) subrings A of $C(X)$ and $\mathrm{Hom}(A) \subset \mathbb{R}^A$ denotes the ring-homomorphisms preserving the unit. Note that homogeneity follows by considering rings containing f and λ.

(17) We have $f^{-1}(Z) \cong \mathrm{graph}(f|_{f^{-1}(Z)}) = \mathrm{graph}(f) \cap X \times Z$, a closed subspace of the product $X \times Z$ of realcompact spaces, and hence realcompact by (2). Since open subsets $O \subseteq \mathbb{R}$ are Lindelöf, they are realcompact by (3) and so is their inverse image under a continuous mapping in any realcompact space.

(15) Every compact space is Lindelöf hence realcompact by (3) and obviously pseudocompact, i.e. every continuous $f : X \to \mathbb{R}$ is bounded. Conversely, let f be realcompact and pseudocompact and $y \in \beta X \setminus \nu X = \beta X \setminus X$. Then by (10) there exists an $f \in C(X, \mathbb{R})$ with $\bar{f}(y) = \infty$, but since f is by assumption bounded its extension to βX is bounded, a contradiction.

(16) Obvioulsy Ω is not compact, since $\{[0, \alpha) : \alpha \in \Omega\}$ form an open cover. It is pseudocompact, since otherwise there would be a continuous unbounded function $f : \Omega \to \mathbb{R}$ and hence countable ordinals α_n with $|f(\alpha_n)| \geq n$. But then $\alpha_\infty := \lim_n \alpha_n$ is also a countable ordinal with $f(\alpha_\infty) = \infty$.

(5) Let X be discrete. Hence $C(X) = \mathbb{R}^X$ is complete and by 18.9 any algebra-homomorphism on $C(X)$ is bounded. We claim that the algebra-homomorphisms φ correspond uniquely to $\{0,1\}$-valued probability measures μ on $\mathcal{P}(X)$ via $\mu(A) := \varphi(\chi_A)$. Since $\chi_{\bigsqcup_i A_i} = \sum_i \chi_{A_i}$ we get the countable additivity. From $\mu(A) = \varphi(\chi_A) = \varphi(\chi_A^2) = \varphi(\chi_A)^2 = \mu(A)^2$ we get that μ is $\{0,1\}$-valued and in particular positiv. Moreover $\mu(X) = \varphi(1) = 1$. Conversely, any μ can be extended to all (=measurable) functions as usual (see [5, 4.12.2]) by defining $\varphi(f) := \sup\{\int g\, d\mu : g$ ist simple and $g \leq f\}$ for $f \geq 0$ and $\mu(f) := \mu(f_+) - \mu(f_-)$ for general $f = f_+ - f_-$. Since μ is $\{0,1\}$-valued this extension φ is multiplicative: In fact, using $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$ gives $\mu(A \cap B) = 1 \Leftrightarrow \mu(A) = 1 = \mu(B)$. The point-evaluations ev_x correspond to the point measures $\mu(A) = 1$ iff $x \in A$. Thus there exists an algebra-homomorphism being not a point evaluation if and only if there exists such a measure μ with $\mu(\{x\}) = 0$ for all $x \in X$, i.e. iff the cardinality of X is measurable.

(8) (a⇒d) Let the complete uniformity of X be given by a directed set of quasimetrics $d : X \times X \to \mathbb{R}$. Then X embeds as uniform space into the product $\prod_d X/\sim_d$ of metrizable spaces, where the equivalence relation $\sim_d$ is given by $x_1 \sim_d x_2 \Leftrightarrow d(x_1, x_2) = 0$. As connecting mappings $X/\sim_{d_1} \to X/\sim_{d_2}$ for $d_1 \geq d_2$ we have the canonical quotient mappings and X is a dense subspace of the corresponding projective limit of their completions $\widehat{X/\sim_d}$: In fact, let z be in the completion of $X/\sim_d$ then for $\varepsilon > 0$ there is some $x \in X$ with $d([x], z) < \varepsilon$. Since the uniformity is complete X coincides with the limit $\varprojlim_d \widehat{X/\sim_d}$.

(d⇒e), (e⇒c) and (b⇒a) are obvious.

(c⇒b) It is enough to embed any metrizable space X closed into a product of complete metrizable spaces. For this consider the completion $\tilde{X}$ and for each $x \in \tilde{X} \setminus$

X the closed embedding embeddings $\tilde{X}\setminus\{x\}\to\tilde{X}\times\mathbb{R}$ given by $y\mapsto(y,1/d(y,x)))$ and $X\to\prod_{x\in\tilde{X}\setminus X}\tilde{X}\setminus\{x\}$.

(19) Let $\preceq$ be a well-ordering on $\mathcal{U}$ and d a metric for X. For $n\in\mathbb{N}$ and $U\in\mathcal{U}$ let

$$\mathcal{V}_n:=\{V_{U,n}:U\in\mathcal{U}\},\ \text{where}$$
$$V_{U,n}:=\bigcup\Big\{U_{\frac{1}{2^n}}(y):\ \min_{y\in W\in\mathcal{U}}W=U\supseteq U_{\frac{3}{2^n}}(y);\ y\in X\setminus\bigcup_{k<n}\bigcup\mathcal{V}_k\Big\}$$

is recursively defined. Then $\mathcal{V}:=\bigcup_{n=1}^\infty\mathcal{V}_n$ has the required properties:

- $V_{U,n}\subseteq U$ is open.
- For each $x\in X$ let U be minimal with $x\in U$ and $n\in\mathbb{N}$ be such that $B_{3/2^n}(x)\subseteq U$. Then either $x\in\bigcup_{k<n}\bigcup\mathcal{V}_k$, or $x\in V_{U,n}$, hence $\mathcal{V}$ is a covering of X.
- $x_1\in V_{U_1,n}$, $x_2\in V_{U_2,n}$ with $U_1\neq U_2$ implies $d(x_1,x_2)>1/2^n$ (hence $\mathcal{V}_n$ is discrete, since every $1/2^{n+1}$-ball meets at most one member of $\mathcal{V}_n$): Let $U_1\preceq U_2$. Thus there are points y_i as above with $x_i\in U_{1/2^n}(y_i)\subseteq V_{U_i,n}$. In particular, $U_{3/2^n}(y_1)\subseteq U_1$ and $y_2\notin U_1$ since $U_1\prec U_2$. So
$$d(x_1,x_2)\geq d(y_1,y_2)-d(y_1,x_1)-d(y_2,x_2)\geq\frac{3}{2^n}-\frac{1}{2^n}-\frac{1}{2^n}=\frac{1}{2^n}.$$
- The set $\mathcal{V}$ is locally finite: For $x\in X$ there are $m,n\geq1$ and $U\in\mathcal{U}$ with $U_{1/2^n}(x)\subseteq V_{U,m}$. We claim that
$$\forall j\geq n+m\,\forall W\in\mathcal{U}:U_{1/2^{n+m}}(x)\cap V_{W,j}=\emptyset.$$
The y in the definition of $V_{W,j}$ do not belong to $V_{U,m}\supseteq U_{1/2^n}(x)$ for all $j\geq n+m>m$ and hence $d(x,y)\geq1/2^n$ and thus $U_{1/2^{n+m}}(x)\cap U_{1/2^j}(y)=\emptyset$.

(14) If X is realcompact, then X is a closed subspace of $\mathbb{R}^{C(X)}$ and hence Dieudonné-complete by (8) and each closed subspace of X is realcompact by (2). Conversely, let X be Dieudonné complete and any discrete closed subspace realcompact. By the proof of (8, $a\Rightarrow d$) the space X is the projective limit of the metrizable spaces $X/\sim_d$. Let $A\subseteq X/\sim_d$ be closed and discrete. Then $\{a\in A:\pi^{-1}(a)\}$ is a discrete family with closed union in X where π denotes the natural quotient mapping. By choosing a section $\sigma:A\to\pi^{-1}(A)$ to $\pi|_{\pi^{-1}(A)}$ we get a closed discrete and hence relacompact subspace $\sigma(A)$ of X which is homeomorphic to A. So by (2) we may assume without loss of generality that X is metrizable. For $y\in\beta X\setminus X$ let $\mathcal{U}:=\{X\setminus\overline{U}^{\beta X}:U\text{ is a neighborhood of }y\text{ in }\beta X\}$, an open covering of X. By 19 we finde a σ-discrete (locally finite open) covering $\mathcal{F}=\bigcup_n\mathcal{F}_n$ of X which is a refinement of $\mathcal{U}$, in particular $\mathcal{F}_n$ is discrete and $y\notin\overline{F}^{\beta X}$ for any $F\in\mathcal{F}_n$. By passing to the closures $\overline{F}$ in X of the F we may assume w.l.o.g. that the F are closed in X. Let $F_n:=\bigcup\mathcal{F}_n$.
We claim that there exists an $f\in C(\beta X,I)$ with $f(y)=0$ and $f|_X>0$ (then by (10) we have $y\notin\nu X$, so $X=\nu X$ is realcompact):
If $y\notin\overline{F_n}^{\beta X}$ for all n, then there exist $f_n\in C(\beta X,[0,1])$ with $f_n(y)=0$ and $f_n|_{F_n}=1$ and thus $f:=\sum_n\frac{1}{2^n}f_n\in C(\beta X,[0,1])$ with $f(y)=0$ and $f|_X>0$.
Otherwise $y\in\overline{F_n}^{\beta X}$ for some n and we consider the quotient mapping $\pi:F_n\cup\{y\}\to\mathcal{F}_n\cup\{y\}$. Since $\mathcal{F}_n$ is discrete, the elements $F\in\mathcal{F}_n$ are open subsets of F_n and since $F_n\subseteq F_n\cup\{y\}$ is open also in $F_n\cup\{y\}$ and hence $\{F\}$ is open in $\mathcal{F}_n\cup\{y\}$. So $\mathcal{F}_n\subseteq\mathcal{F}_n\cup\{y\}$ is discrete. The restriction mapping

$\mathrm{incl}^* : C_b(\mathcal{F}_n \cup \{y\}) \to C_b(\mathcal{F}_n)$ is bijective, since for every $f \in C_b(\mathcal{F}_n)$ the composite $f \circ \pi \in C_b(F_n)$ and hence extends to a bounded continuous function on βX. Its restriction to $F_n \cup \{y\}$ factors over π to a continuous bounded functions on $\mathcal{F}_n \cup \{y\}$ and since F_n is dense in $F_n \cup \{y\}$ this extension is unique. Note that the functions in $C_b(\mathcal{F}_n)$ separate points, since for the discrete subspace $\mathcal{F}_n$ this is obvious and for $F \in \mathcal{F}_n$ we have $y \notin \overline{F}^{\beta X}$ and hence an $f \in C_b(\beta X, I)$ exists with $f|_F = 0$ and $f(y) = 1$. Now we replace f by $x \mapsto \sup(f(F))$ for $x \in F \in \mathcal{F}_n$. Then f is continuous (at y) and constant on the $F \in \mathcal{F}_n$ hence factors over π. So we have an injective continuous mapping $\mathcal{F}_n \cup \{y\} \to \mathrm{Alg}(C_b(\mathcal{F}_n \cup \{y\})) \cong \mathrm{Alg}(C_b(\mathcal{F}_n)) = \beta(\mathcal{F}_n)$. Since $\mathcal{F}_n$ is realcompact (choose points in each $F \in \mathcal{F}_n$ to obtain a discrete closed subspace of X and use 5) there exists an $f \in C(\beta(\mathcal{F}_n), [0,1])$ with $f(y) = 0$ and $f|_{\mathcal{F}_n} > 0$ again by (10). Now $f \circ \pi|_{F_n}$ can be extended to X by the theorem of Tietze and Urysohn (even as positive function, since we may replace f with $\max(f, h)$, where $h \in C(X, I)$ with $h|_{F_n} = 0$ and $h|_{f^{-1}(0)} = 1$) and furtheron to βX and this extension vanishes on y, since it coincides on the dense subset F_n with $f \circ \pi$ and the later one vanishes on y.

(7) follows from 14 and 5.

(13) We show that the uniformity given by all continuous pseudo-metrics $d : X \times X \to \mathbb{R}$ is complete. So let $x_i \in X$ be Cauchy for all d. In particular, for $d_f(y_1, y_2) := |f(y_1) - f(y_2)|$ for any $f \in C_b(X, \mathbb{R})$. So $\delta(x_i)(f) = f(x_i)$ is Cauchy in $\mathbb{R}^{C_b(X)}$ and hence converges to some $y \in \beta X$. Suppose $y \notin X$. Since X is paracompact there is a partition $\mathcal{F}$ of unity with $y \notin \overline{f^{-1}((0,1])}^{\beta X}$ for all $f \in \mathcal{F}$. Let $d_0(x_1, x_2) := \sum_{f \in \mathcal{F}} |f(x_1) - f(x_2)|$. Note that this sum is locally finite, since $\mathcal{F}$ is it. So d_0 is a continuous pseudo-metric on X. For every $x \in X$ and $f \in \mathcal{F}$ with $f(x) \neq 0$ there exists a neighborhood of y in βX on which f vanishes. Let U be the finite intersection of these neighborhoods. Then $x_i \in U$ finally. We claim that d_0 has no continuous extension $\tilde{d}_0$ to y. Otherwise for $x \in X$ we have $\tilde{d}_0(x, y) = \lim_i d_0(x, x_i)$ and $d_0(x, x_i) = \sum_f |f(x) - f(x_i)| \geq \sum_{f(x) \neq 0} f(x) = 1$. In particular $\tilde{d}_0(x_i, y) \geq 1$ in contradiction to $\tilde{d}_0(x_i, y) \to \tilde{d}_0(y, y) = 0$, so $y \in X$.
Note that $d(y, x_i) \to 0$ for each continuous pseudo-metric $d : X \times X \to \mathbb{R}$, since $x_i \to y$ in βX and hence in X and d is continuous, so $d(y, x_i) = \lim_j d(x_j, x_i) \leq \varepsilon$, since $(x_i)_i$ is assumed to be Cauchy.

(4) follows from 13 and 14.

(6) Banach-spaces (or even Fréchet spaces) are metrizable hence paracompact. So by 4 they are realcompact iff all their closed discrete subsets are non-measurable, i.e. their density is non-measurable. □

17.3. Lemma. [Kriegl, Michor, Schachermayer, 1989, 2.2, 2.3]. *Let $\mathcal{A}$ be $\bar{1}$-evaluating. Then we have a topological embedding*

$$\delta : X_{\mathcal{A}} \hookrightarrow \prod_{\mathcal{A}} \mathbb{R}, \quad \mathrm{pr}_f \circ \delta := f,$$

with dense image in the closed subset $\mathrm{Hom}\,\mathcal{A} \subseteq \prod_{\mathcal{A}} \mathbb{R}$. *Hence X is $\mathcal{A}$-realcompact if and only if δ has closed image.*

Proof. The topology of $X_{\mathcal{A}}$ is by definition initial with respect to all $f = \mathrm{pr}_f \circ \delta$, hence δ is an embedding. Obviously $\mathrm{Hom}\,\mathcal{A} \subseteq \prod_{\mathcal{A}} \mathbb{R}$ is closed. Let $\varphi : \mathcal{A} \to \mathbb{R}$ be an algebra-homomorphism. For $f \in \mathcal{A}$ consider Z_f. If $\mathcal{A}$ is 1-evaluating then by 18.8 for any finite subset $\mathcal{F} \subseteq \mathcal{A}$ there exists an $x_{\mathcal{F}} \in \bigcap_{f \in \mathcal{F}} Z_f$. Thus $\delta(x_{\mathcal{F}})_f = \varphi(f)$ for all $f \in \mathcal{F}$. If $\mathcal{A}$ is only $\bar{1}$-evaluating, then we get as in the proof of 18.3 for every $\varepsilon > 0$ a point $x_{\mathcal{F}} \in X$ such that $|f(x_{\mathcal{F}}) - \varphi(f)| < \varepsilon$ for all $f \in \mathcal{F}$. Thus $\delta(x_{\mathcal{F}})$ lies in the corresponding neighborhood of $(\varphi(f))_f$. Thus $\delta(X)$ is dense in $\operatorname{Hom}\mathcal{A}$.

Now X is $\mathcal{A}$-realcompact if and only if δ has $\operatorname{Hom}\mathcal{A}$ as image, and hence if and only if the image of δ is closed. □

17.4. Theorem. [Kriegl, Michor, Schachermayer, 1989, 2.4] & [Adam, Biström, Kriegl, 1995, 3.1]. *The topology of pointwise convergence on* $\operatorname{Hom}_\omega \mathcal{A}$ *is realcompact. If* $X_{\mathcal{A}}$ *is not realcompact then there exists an* ω*-evaluating homomorphism* φ *which is not evaluating.*

Proof. We first show the weaker statement, that: If $X_{\mathcal{A}}$ is not realcompact then there exists a non-evaluating φ, i.e., X is not $\mathcal{A}$-realcompact.
Assume that X is $\mathcal{A}$-realcompact, then $\mathcal{A}$ is 1-evaluating and hence by lemma 17.3 $\delta : X_{\mathcal{A}} \to \prod_{\mathcal{A}} \mathbb{R}$ is a closed embedding. Thus by 17.2.1 the space $X_{\mathcal{A}}$ is realcompact.

Now we give a proof of the stronger statement that $\operatorname{Hom}_\omega \mathcal{A}$ is realcompact:
We supply all sets of homomorphisms with the topology of pointwise convergence. Let $\mathcal{M} \subseteq 2^{\mathcal{A}}$ be the family of all countable subsets of $\mathcal{A}$ containing the unit. For $M \in \mathcal{M}$, consider the topological space $\operatorname{Hom}_\omega\langle M\rangle$, where $\langle M\rangle$ denotes the subalgebra generated by M. Obviously the family $(\delta_f)_{f\in M}$, where $\delta_f(\varphi) = \varphi(f)$, is a countable subset of $C(\operatorname{Hom}_\omega\langle M\rangle)$ that separates the points in $\operatorname{Hom}_\omega\langle M\rangle$. Hence $\operatorname{Hom}_\omega\langle M\rangle = \operatorname{Hom}(C(\operatorname{Hom}_\omega\langle M\rangle))$ by 18.25, since $C(\operatorname{Hom}_\omega\langle M\rangle)$ is ω-evaluating by 18.11, i.e. $\operatorname{Hom}_\omega\langle M\rangle$ is realcompact. Now $\operatorname{Hom}_\omega \mathcal{A}$ is an inverse limit of the spaces $\operatorname{Hom}_\omega\langle M\rangle$ for $M \in \mathcal{M}$. Since $\operatorname{Hom}_\omega\langle M\rangle$ is Hausdorff, we obtain that $\operatorname{Hom}_\omega \mathcal{A}$ as a closed subset of a product of realcompact spaces is realcompact by 17.2.2.

Since X is not realcompact in the topology $X_{\mathcal{A}}$, which is that induced from the embedding into $\operatorname{Hom}_\omega \mathcal{A}$, we have that $X \neq \operatorname{Hom}_\omega \mathcal{A}$ and the statement is proved. □

17.5. Counter-example. [Kriegl, Michor, 1993, 3.6.2]. *The locally convex space* $\mathbb{R}^\Gamma_{count}$ *of all points in the product with countable carrier is not* C^∞*-realcompact, if* Γ *is uncountable and not measurable.*

Proof. By [Engelking, 1989, 3.10.17 & 3.11.2] the space $X := \mathbb{R}^\Gamma_{count}$ is not realcompact, in fact every c^∞-continuous function on it extends to a continuous function on $\mathbb{R}^\Gamma$, see the proof of 4.27. Since the projections are smooth, X_{C^∞} is the product topology. So the result follows from 17.4. □

17.6. Theorem. [Kriegl, Michor, Schachermayer, 1989, 3.2] & [Garrido, Gómez, Jaramillo, 1994, 1.8]. *Let* X *be a realcompact and completely regular topological space, let* $\mathcal{A}$ *be uniformly dense in* $C(X)$ *(e.g.* X *is* $\mathcal{A}$*-paracompact) and* $\bar{1}$*-evaluating. Then* X *is* $\mathcal{A}$*-realcompact.*

In [Kriegl, Michor, Schachermayer, 1989] it is shown that C^∞_{lfcs}-algebra $\mathcal{A}$ is uniformly dense in $C(X)$ if and only if $\mathcal{A} \cap C_b(X)$ is uniformly dense in $C_b(X)$. One may find also other equivalent conditions there.

Proof. Since $\mathcal{A} \subseteq C(X)$ we have that the identity $X \to X_{\mathcal{A}}$ is continuous, and hence $\mathcal{A} \subseteq C(X_{\mathcal{A}}) \subseteq C(X)$. For each of these point-separating algebras we consider the natural inclusion δ of X into the product of factors $\mathbb{R}$ over the algebra, given by $\operatorname{pr}_f \circ \delta = f$. It is a uniform embedding for the uniformity induced on X by this algebra and the complete product uniformity on $\prod \mathbb{R}$ with basis formed by the sets $U_{f,\varepsilon} := \{(u,v) : |\operatorname{pr}_f(u) - \operatorname{pr}_f(v)| < \varepsilon\}$ with $\varepsilon > 0$.

The condition that $\mathcal{A} \subseteq C$ is dense implies that the uniformities generated by $C(X)$, by $C(X_{\mathcal{A}})$ and by $\mathcal{A}$ coincide and hence we will consider X as a uniform space endowed with this uniform structure in the sequel. In fact for an arbitrarily given continuous map f and $\varepsilon > 0$ choose a $g \in \mathcal{A}$ such that $|g(x) - f(x)| < \varepsilon$ for all $x \in X$. Then

$$\begin{aligned}\{(x,y) : |f(x) - f(y)| < \varepsilon\} &\subseteq \{(x,y) : |g(x) - g(y)| < 3\varepsilon\}\\ &\subseteq \{(x,y) : |f(x) - f(y)| < 5\varepsilon\}.\end{aligned}$$

Since X is realcompact, $\delta_C(X) = \operatorname{Hom}(C(X))$ and hence X is closed in $\prod_{C(X)} \mathbb{R}$ and so the uniform structure on X is complete. Thus, also the image $\delta_{\mathcal{A}}(X)$ is a complete uniform subspace of $\prod_{\mathcal{A}} \mathbb{R}$ and so it is closed with respect to the product topology, i.e. X is $\mathcal{A}$-realcompact by 17.3. $\square$

17.7. In the case of a locally convex vector space the last result 17.6 can be slightly generalized to:

Result. [Biström, Lindström, 1993b, Thm.6]. *For E a realcompact locally convex vector space, let $E' \subseteq \mathcal{A} \subseteq C(E)$ be a ω-evaluating $C^{(\infty)}$-algebra which is invariant under translations and homotheties. Moreover, we assume that there exists a 0-neighborhood U in E such that for each $f \in C(E)$ there exists $g \in \mathcal{A}$ with $\sup_{x\in U} |f(x) - g(x)| < \infty$.*

Then E is $\mathcal{A}$-realcompact.

18. Evaluation Properties of Homomorphisms

In this section we consider first properties near the evaluation property at single functions, then evaluation properties for homomorphisms on countable many functions, and finally direct situations where all homomorphisms are point evaluations.

18.1. Remark. *If φ in $\operatorname{Hom}\mathcal{A}$ is 1-evaluating (i.e., $\varphi(f) \in f(X)$ for all f in $\mathcal{A}$), then φ is $\bar{1}$-evaluating.* $\square$

18.2. Lemma. [Biström, Bjon, Lindström, 1991, p.181]. *For a topological space X the following assertions are equivalent:*

1. *φ is $\bar{1}$-evaluating;*
2. *There exists $\tilde{x}$ in the Stone-Čech compactification βX with $\varphi(f) = \tilde{f}(\tilde{x})$ for all $f \in \mathcal{A}$.*

Here $\tilde{f}$ denotes the extension of $f : X \to \mathbb{R} \hookrightarrow \mathbb{R}_\infty$ to the Stone-Čech-compactification βX with values in the 1-point compactification $\mathbb{R}_\infty$ of $\mathbb{R}$.

In [Garrido, Gómez, Jaramillo, 1994, 1.3] it is shown for a subalgebra of $C_b(\mathbb{R})$ that $\tilde{x}$ need not be unique.

Proof. For $f \in \mathcal{A}$ and $\varepsilon > 0$ let $U(f,\varepsilon) := \{x \in X : |\varphi(f) - f(x)| < \varepsilon\}$. Then $\mathcal{U} := \{U(f,\varepsilon) : f \in \mathcal{A}, \varepsilon > 0\}$ is a filter basis on X. Consider X as embedded into βX and take an ultrafilter $\tilde{\mathcal{U}}$ on βX that is finer than $\mathcal{U}$. For $f := (f_1 - \varphi(f_1))^2 + (f_2 - \varphi(f_2))^2$ we have in fact

$$U(f_1,\varepsilon_1) \cap U(f_2,\varepsilon_2) \supseteq U(f, \min\{\varepsilon_1,\varepsilon_2\}^2).$$

Let $\tilde{x} \in \beta X$ be the point to which $\tilde{\mathcal{U}}$ converges. For an arbitrary function f in $\mathcal{A}$ the filter $f(\mathcal{U})$ converges to $\varphi(f)$ by construction. But $\tilde{f}(\tilde{\mathcal{U}}) \geq \tilde{f}(\mathcal{U}) = f(\mathcal{U})$, so $\varphi(f) = \tilde{f}(\tilde{x})$. The converse is obvious since $\varphi(f) = \tilde{f}(\tilde{x}) \in \tilde{f}(\beta X) \subseteq \overline{f(X)} \subseteq \mathbb{R}_\infty$, and $\varphi(f) \in \mathbb{R}$. □

18.3. Lemma. [Adam, Biström, Kriegl, 1995, 4.1]. *An algebra homomorphism φ is $\bar{1}$-evaluating if and only if φ extends (uniquely) to an algebra homomorphism on $\mathcal{A}^\infty$, the C^∞-algebra generated by $\mathcal{A}$.*

Proof. For C^∞-algebras $\mathcal{A}$, we have that

$$\varphi(h \circ (f_1, \dots, f_n)) = h(\varphi(f_1), \dots, \varphi(f_n))$$

for all $h \in C^\infty(\mathbb{R}^n, \mathbb{R})$ and $f_1, \dots, f_n$ in $\mathcal{A}$.

In fact set $a := (\varphi(f_1), \dots, \varphi(f_n)) \in \mathbb{R}^n$. Then

$$h(x) - h(a) = \int_0^1 \sum_{j \leq n} \partial_j h(a + t(x-a))\, dt \cdot (x_j - a_j) = \sum_{j \leq n} h_j^a(x) \cdot (x_j - a_j),$$

where $h_j^a(x) := \int_0^1 \partial_j h(a + t(x-a)) dt$. Applying φ to this equation composed with the f_i one obtains

$$\varphi(h \circ (f_1, \dots, f_n)) - h(\varphi(f_1), \dots, \varphi(f_n)) = \\ = \sum_{j \leq n} \varphi(h_j^a \circ (f_1, \dots, f_n)) \cdot (\varphi(f_j) - \varphi(f_j)) = 0.$$

($\Rightarrow$) We define $\tilde{\varphi}(h \circ (f_1, \dots, f_n)) := h(\varphi(f_1), \dots, \varphi(f_n))$. By what we have shown above (1-preserving) algebra homomorphisms are C^∞-algebra homomorphisms and hence this is the only candidate for an extension. This map is well defined. Indeed, let $h \circ (f_1, \dots, f_n) = k \circ (g_1, \dots, g_m)$. For each $\varepsilon > 0$ there is a point $x \in E$ such that $|\varphi(f_i) - f_i(x)| < \varepsilon$ for $i = 1, ..., n$, and $|\varphi(g_j) - g_j(x)| < \varepsilon$ for $j = 1, ..., m$. In fact by 18.2 there is a point $\tilde{x} \in \beta X$ with $\varphi(f) = \tilde{f}(\tilde{x})$ for

$$f := \sum_{i=1}^n (f_i - \varphi(f_i))^2 + \sum_{j=1}^m (g_j - \varphi(g_j))^2,$$

and hence $\varphi(f_i) = \tilde{f}_i(\tilde{x})$ and $\varphi(g_j) = \tilde{g}_j(\tilde{x})$. Now approximate $\tilde{x}$ by $x \in X$.
By continuity of h and k we obtain that

$$h(\varphi(f_1), \dots, \varphi(f_n)) = k(\varphi(f_1), \dots, \varphi(f_m)),$$

and we therefore have a well defined extension of φ. This extension is a homomorphism, since for every polynomial θ on $\mathbb{R}^m$ (or even for $\theta \in C^\infty(\mathbb{R}^m)$) and $g_i := h_i \circ (f_1^i, \dots, f_{n_i}^i) \in \mathcal{A}^\infty$ we have

$$\begin{aligned}
\tilde{\varphi}(\theta \circ (g_1, \dots, g_m)) &= \tilde{\varphi}(\theta \circ (h_1 \times \dots \times h_m) \circ (f_1^1, \dots, f_{n_m}^m)) \\
&= (\theta \circ (h_1 \times \dots \times h_m))(\varphi(f_1^1), \dots, \varphi(f_{n_m}^m)) \\
&= \theta(h_1(\varphi(f_1^1), \dots, \varphi(f_{n_1}^1)), \dots, h_m(\varphi(f_1^m), \dots, \varphi(f_{n_m}^m)) \\
&= \theta(\tilde{\varphi}(g_1), \dots, \tilde{\varphi}(g_m)).
\end{aligned}$$

($\Leftarrow$) Suppose there is some $f \in \mathcal{A}$ with $\varphi(f) \notin \overline{f(X)}$. Then we may find an $h \in C^\infty(\mathbb{R})$ with $h(\varphi(f)) = 1$ and $\operatorname{carr} h \cap f(X) = \emptyset$. Since $\mathcal{A}^\infty$ is a C^∞-algebra, we conclude from what we said above that $\tilde{\varphi}(h \circ f) = h(\varphi(f)) = 1$. But since $h \circ f = 0$ we arrive at a contradiction. □

18.4. Proposition. [Garrido, Gómez, Jaramillo, 1994, 1.2]. *If $\mathcal{A}$ is bounded inversion closed and $\varphi \in \operatorname{Hom}\mathcal{A}$ then φ is $\bar{1}$-evaluating.*

Proof. We assume indirectly that there is a function $f \in \mathcal{A}$ with $\varphi(f) \notin \overline{f(X)}$. Let $\varepsilon := \inf_{x\in X} |\varphi(f) - f(x)|$ and $g(x) := \frac{1}{\varepsilon}(\varphi(f) - f(x))$. Then $g \in \mathcal{A}$, $\varphi(g) = 0$ and $|g(x)| = \frac{1}{\varepsilon}|\varphi(f) - f(x)| \geq 1$ for each $x \in X$. Thus $1/g \in \mathcal{A}$. But then $1 = \varphi(g \cdot 1/g) = \varphi(g)\varphi(1/g) = 0$ gives a contradiction. □

18.5. Lemma. *Any $C^{(\infty)}$-algebra is bounded inversion closed.*

Moreover, it is stable under composition with smooth locally defined functions, which contain the closure of the image in its domain of definition.

Proof. Let $\mathcal{A}$ be a C^∞-algebra (resp. $C^{(\infty)}$-algebra), n a natural number (resp. $n = 1$), $U \subseteq \mathbb{R}^n$ open, $h \in C^\infty(U,\mathbb{R})$, $f := (f_1,\dots,f_n)$, with $f_i \in \mathcal{A}$ such that $\overline{f(X)} \subseteq U$, then $h \circ f \in \mathcal{A}$. Indeed, choose $\rho \in C^\infty(\mathbb{R})$ with $\rho|_{\overline{f(X)}} = 1$ and $\operatorname{supp}\rho \subseteq U$. Then $k := \rho \cdot h$ is a globally smooth function and $h \circ f = k \circ f \in \mathcal{A}$. □

18.6. Lemma. *Any inverse closed algebra $\mathcal{A}$ is 1-evaluating.*

By 18.10 the converse is wrong.

Proof. Let $f \in \mathcal{A}$ and assume indirectly that $Z_f = \emptyset$. Let $g := f - \varphi(f)$. Then $g \in \mathcal{A}$ and $g(x) \neq 0$ for all $x \in X$, by which $1/g \in \mathcal{A}$ since $\mathcal{A}$ is inverse-closed. But then $1 = \varphi(g \cdot 1/g) = \varphi(g)\varphi(1/g) = 0$, which is a contradiction. □

18.7. Proposition. [Biström, Jaramillo, Lindström, 1995, Lem.14] *&* [Adam, Biström, Kriegl, 1995, 4.2]. *For φ in $\operatorname{Hom}\mathcal{A}$ the following statements are equivalent:*

(1) *φ is 1-evaluating.*
(2) *φ extends to a unique (1-evaluating) homomorphism on the algebra $\mathcal{RA} := \{f/g : f,g \in \mathcal{A}, 0 \notin g(X)\}$.*
(3) *φ extends to a unique (1-evaluating) homomorphism on the following C^∞-algebra $\mathcal{A}^{\langle\infty\rangle}$ constructed from $\mathcal{A}$:*

$$\mathcal{A}^{\langle\infty\rangle} := \{h \circ (f_1,\dots,f_n) : f_i \in \mathcal{A}, (f_1,\dots,f_n)(X) \subseteq U, \quad U \text{ open in some } \mathbb{R}^n, h \in C^\infty(U)\}.$$

Proof. (1) ⇒ (3) We define $\varphi(h \circ (f_1,\dots,f_n)) := h(\varphi(f_1),\dots,\varphi(f_n))$. Since there exists by 18.8 an x with $\varphi(f_i) = f_i(x)$, we have $(\varphi(f_1),\dots,\varphi(f_n)) \in U$, hence the right side makes sense. The rest follows in the same way as in the proof of 18.3.

(3) ⇒ (2) Existence is obvious, since $\mathcal{RA} \subseteq \mathcal{A}^{\langle\infty\rangle}$, and uniqueness follows from the definition of $\mathcal{RA}$.

(2) ⇒ (1) Since $\mathcal{RA}$ is inverse-closed, the extension of φ to this algebra is 1-evaluating by 18.6, hence the same is true for φ on $\mathcal{A}$. □

18.8. Lemma. *Every 1-evaluating homomorphism is finitely evaluating.*

Proof. Let $\mathcal{F}$ be a finite subset of $\mathcal{A}$. Define a function $f : X \to \mathbb{R}$ by

$$f := \sum_{g\in\mathcal{F}} (g - \varphi(g))^2.$$

Then $f \in \mathcal{A}$ and $\varphi(f) = 0$. By assumption there is a point $x \in X$ with $\varphi(f) = f(x)$. Hence $g(x) = \varphi(g)$ for all $g \in \mathcal{F}$. □

18.9. Theorem. Automatic boundedness. [Kriegl, Michor, 1993] *&* [Arias-de-Reyna, 1988] *Every 1-evaluating homomorphism* $\varphi \in \operatorname{Hom} \mathcal{A}$ *is positive, i.e.,* $0 \le \varphi(f)$ *for all* $0 \le f \in \mathcal{A}$. *Moreover we even have* $\varphi(f) > 0$ *for* $f \in \mathcal{A}$ *with* $f(x) > 0$ *for all* $x \in X$.

Every positive homomorphism $\varphi \in \operatorname{Hom} \mathcal{A}$ *is bounded for any convenient algebra structure on* $\mathcal{A}$.

A convenient algebra structure on A is a locally convex topology, which turns $\mathcal{A}$ into a convenient vector space and such that the multiplication $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is bounded, compare 5.21.

Proof. Positivity: Let $f_1 \le f_2$. By 17 and 18.8 there exists an $x \in X$ such that $\varphi(f_i) = f_i(x)$ for $i = 1, 2$. Thus $\varphi(f_1) = f_1(x) \le f_2(x) = \varphi(f_2)$. Note that if $f(x) > 0$ for all x, then $\varphi(f) > 0$.

Boundedness: Suppose f_n is a bounded sequence, but $|\varphi(f_n)|$ is unbounded. By replacing f_n by f_n^2 we may assume that $f_n \ge 0$ and hence also $\varphi(f_n) \ge 0$. Choosing a subsequence we may even assume that $\varphi(f_n) \ge 2^n$. Now consider $\sum_n \frac{1}{2^n} f_n$. This series converges Mackey, and since the bornology on $\mathcal{A}$ is by assumption complete the limit is an element $f \in \mathcal{A}$. Applying φ yields

$$\varphi(f) = \varphi\left(\sum_{n=0}^{N} \frac{1}{2^n} f_n + \sum_{n>N} \frac{1}{2^n} f_n\right) = \sum_{n=0}^{N} \frac{1}{2^n} \varphi(f_n) + \varphi\left(\sum_{n>N} \frac{1}{2^n} f_n\right) \ge$$
$$\ge \sum_{n=0}^{N} \frac{1}{2^n} \varphi(f_n) + 0 = \sum_{n=0}^{N} \frac{1}{2^n} \varphi(f_n),$$

where we used the monotonicity of φ applied to $\sum_{n>N} \frac{1}{2^n} f_n \ge 0$. Thus the series $N \mapsto \sum_{n=0}^{N} \frac{1}{2^n} \varphi(f_n)$ is bounded and increasing, hence converges, but its summands are bounded by 1 from below. This is a contradiction. □

18.10. Lemma. *For a locally convex vector space* E *the algebra* $P_f(E)$ *is 1-evaluating.*

More on the algebra $P_f(E)$ can be found in 18.27, 18.28, and 18.12.

Proof. Every finite type polynomial p is a polynomial in a finite number of linearly independent functionals $\ell_1, \dots, \ell_n$ in E'. So there is for each $i = 1, \dots, n$ some point $a_i \in E$ such that $\ell_i(a_i) = \varphi(\ell_i)$ and $\ell_j(a_i) = 0$ for all $j \ne i$. Let $a = a_1 + \dots + a_n \in E$. Then $\ell_i(a) = \ell_i(a_i) = \varphi(\ell_i)$ for $i = 1, \dots, n$ hence $\varphi(p) = p(a)$. □

Countably Evaluating Homomorphisms

18.11. Theorem. *Idea of* [Arias-de-Reyna, 1988, proof of thm.8], [Adam, Biström, Kriegl, 1995, 2.5]. *For a topological space* X *any* C^∞_{lfcs}*-algebra* $\mathcal{A} \subseteq C(X)$ *is closed under composition with local smooth functions and is* ω*-evaluating.*

Note that this does not apply to C^ω.

Proof. We first show closedness under local smooth functions (and hence in particular under inversion), i.e. if $h \in C^\infty(U)$, where $U \subseteq \mathbb{R}^n$ is open and $f := (f_1, \dots, f_n)$ with $f_i \in \mathcal{A}$ has values in U, then $h \circ f \in \mathcal{A}$.
Consider a smooth partition of unity $\{h_j : j \in \mathbb{N}\}$ of U, such that $\operatorname{supp} h_j \subseteq U$.

Then $h_j \cdot h$ is a smooth function on $\mathbb{R}^n$ vanishing outside $\operatorname{supp} h_j$. Hence $(h_j \cdot h) \circ f \in \mathcal{A}$. Since we have

$$\operatorname{carr}\Big((h_j \cdot h) \circ f\Big) \subseteq f^{-1}(\operatorname{carr} h_j),$$

the family $\{\operatorname{carr}((h_j \cdot h) \circ f) : j \in \mathbb{N}\}$ is locally finite, f is continuous, and since $1 = \sum_{j\in\mathbb{N}} h_j$ on U we obtain that $h \circ f = \sum_{j\in\mathbb{N}} (h_j \cdot h) \circ f \in \mathcal{A}$.

By 18.6 we have that φ is 1-evaluating, hence finitely evaluating by 18.8. We now show that φ is countably evaluating:
For this take a sequence $(f_n)_n$ in $\mathcal{A}$. Then $h_n : x \mapsto (f_n(x) - \varphi(f_n))^2$ belongs to $\mathcal{A}$ and $\varphi(h_n) = 0$. We have to show that there exists an $x \in X$ with $h_n(x) = 0$ for all n. Assume that this were not true, i.e. for all $x \in X$ there exists an n with $h_n(x) > 0$. Take $h \in C^\infty(\mathbb{R}, [0, 1])$ with $\operatorname{carr} h = \{t : t > 0\}$ and let $g_n : x \mapsto h(h_n(x)) \cdot h(\frac{1}{n} - h_1(x)) \cdot \dots \cdot h(\frac{1}{n} - h_{n-1}(x))$. Then $g_n \in \mathcal{A}$ and the sum $\sum_n \frac{1}{2^n} g_n$ is locally finite, hence defines a function $g \in \mathcal{A}$. Since φ is 1-evaluating there exists for any n an $x_n \in X$ with $h_n(x_n) = \varphi(h_n) = 0$ and $\varphi(g_n) = g_n(x_n)$. Hence

$$\varphi(g_n) = g_n(x_n) = h(h_n(x_n)) \cdot h(\tfrac{1}{n} - h_1(x_n)) \cdot \dots \cdot h(\tfrac{1}{n} - h_{n-1}(x_n)) = 0.$$

By assumption on the h_n and h we have that $g > 0$. Hence by 18.9 $\varphi(g) > 0$, since φ is 1-evaluating. Let N be so large that $1/2^N < \varphi(g)$. Again since $\mathcal{A}$ is 1-evaluating, there is some $a \in X$ such that $\varphi(g) = g(a)$ and $\varphi(g_j) = g_j(a)$ for $j \le N$. Then

$$\frac{1}{2^N} < \varphi(g) = g(a) = \sum_n \frac{1}{2^n} g_n(a) = \sum_{n\le N} \frac{1}{2^n}\varphi(g_n) + \sum_{n>N} \frac{1}{2^n} g_n(a) \le 0 + \frac{1}{2^N}$$

gives a contradiction. □

18.12. Counter-example. [Biström, Jaramillo, Lindström, 1995, Prop.17]. *For any non-reflexive weakly realcompact locally convex space (and any non-reflexive Banach space) E the algebra $P_f(E)$ of finite type polynomials is not ω-evaluating.*

Moreover, $E_\mathcal{A}$ is realcompact, but E is not $\mathcal{A}$-realcompact, for $\mathcal{A} = P_f(E)$, so that the converse of the assertion in 17.4 holds only under the additional assumptions of 17.6.

As example we may take $E = \ell^1$, which is non-reflexive, but by 18.27 weakly realcompact.

By 18.10 the algebra $P_f(E)$ is 1-evaluating and hence by 18.7 it has the same homomorphisms as $RP_f(E)$, $P_f(E)^\infty$ or even $P_f(E)^{\langle\infty\rangle}$. So these algebras are not ω-evaluating for spaces E as above.

Proof. By the universal property 5.10 of $P_f(E)$ we get $\operatorname{Hom} P_f(E) \cong (E')^\times$, the space of (not necessarily bounded) linear functionals on E'. For weakly realcompact E by 18.27 we have $\operatorname{Hom}_\omega P_f(E) = E$. So if $P_f(E)$ were ω-evaluating then even $E = \operatorname{Hom} P_f(E)$. Any bounded subset of E is obviously P_f-bounding and hence by 20.2 relatively compact in the weak topology, since $E_{P_f(E)} = (E, \sigma(E, E'))$. Since E is not semi-reflexive, this is a contradiction, see [Jarchow, 1981, 11.4.1].

If we have a (not necessarily weakly compact) Banach space, we can replace in the argument above 20.2 by the following version given in [Biström, 1993, 5.10]: If $\operatorname{Hom}_\omega P_f(E) = \operatorname{Hom} P_f(E)$ then every $\mathcal{A}$-bounding set with complete closed convex hull is relatively compact in the weak topology. □

18.13. Lemma. *The C^∞_{lfs}-algebra $\mathcal{A}^\infty_{lfs}$ generated by an algebra $\mathcal{A}$ can be obtained in two steps as $(\mathcal{A}^\infty)_{lfs}$. Also the C^∞_{lfcs}-algebra $\mathcal{A}^\infty_{\mathrm{lfcs}}$ can be obtained in two steps as $(\mathcal{A}^\infty)_{\mathrm{lfcs}}$.*

Proof. We prove the result only for countable sums, the general case is easier. We have to show that $(\mathcal{A}^\infty)_{\mathrm{lfcs}}$ is closed under composition with smooth mappings. So take $h \in C^\infty(\mathbb{R}^n)$ and $\sum_{j\ge1} f_{i,j} \in (\mathcal{A}^\infty)_{\mathrm{lfcs}}$ for $i = 1,\dots,n$. We put $h_0 := 0$ and $h_k := h \circ (\sum_{j=1}^k f_{1,j},\dots,\sum_{j=1}^k f_{n,j}) \in \mathcal{A}^\infty$ and obtain

$$h \circ \Big(\sum_{j\ge1} f_{1,j},\dots,\sum_{j\ge1} f_{n,j}\Big) = \sum_{k\ge1}(h_k - h_{k-1}),$$

where the right member is locally finite and hence an element of $(\mathcal{A}^\infty)_{\mathrm{lfcs}}$. □

18.14. Theorem. [Adam, Biström, Kriegl, 1995, 4.3]. *A homomorphism φ in* $\operatorname{Hom}\mathcal{A}$ *is ω-evaluating if and only if φ extends (uniquely) to an algebra homomorphism on the C^∞_{lfcs}-algebra $\mathcal{A}^\infty_{\mathrm{lfcs}}$ generated by $\mathcal{A}$, which can be obtained in two steps as $(\mathcal{A}^\infty)_{\mathrm{lfcs}}$ (and this extension is ω-evaluating by 18.11).*

Proof. ($\Rightarrow$) The algebra $\mathcal{A}^\infty_{\mathrm{lfcs}}$ is the union of the algebras obtained by a finite iteration of passing to $\mathcal{A}_{lfcs}$ and $\mathcal{A}^\infty$, where $\mathcal{A}_{\mathrm{lfcs}} := \{f : f = \sum_n f_n, f_n \in \mathcal{A}$, the sum is locally finite$\}$. To $\mathcal{A}^\infty$ it extends by 18.3. It is countably evaluating there, since in any $f \in \mathcal{A}^\infty$ only finitely many elements of $\mathcal{A}$ are involved. Remains to show that φ can be extended to $\mathcal{A}_{lfcs}$ and that this extension is also countably evaluating.

For a locally finite sum $f = \sum_k f_k$ we define $\varphi(f) := \sum_k \varphi(f_k)$. This makes sense, since there exists an $x \in X$ with $\varphi(f_n) = f_n(x)$, and since $\sum_n f_n$ is point finite, we have that the sum $\sum_n \varphi(f_n) = \sum_n f_n(x)$ is in fact finite. It is well defined, since for $\sum_n f_n = \sum_n g_n$ we can choose an $x \in X$ with $\varphi(f_n) = f_n(x)$ and $\varphi(g_n) = g_n(x)$ for all n, and hence $\sum_n \varphi(f_n) = \sum_n f_n(x) = \sum_n g_n(x) = \sum_n \varphi(g_n)$. The extension is a homomorphism, since for the product for example we have

$$\begin{aligned}\varphi\Big(\big(\sum_n f_n\big)\big(\sum_k g_k\big)\Big) &= \varphi\Big(\sum_{n,k} f_n\, g_k\Big) = \sum_{n,k}\varphi(f_n\, g_k) = \\ &= \sum_{n,k}\varphi(f_n)\,\varphi(g_k) = \Big(\sum_n \varphi(f_n)\Big)\Big(\sum_k \varphi(g_k)\Big).\end{aligned}$$

Remains to show that the extension is countably evaluating. So let $f^k = \sum_n f_n^k$ be given. By assumption there exists an x such that $\varphi(f_n^k) = f_n^k(x)$ for all n and all k. Thus $\varphi(f^k) = \sum_n \varphi(f_n^k) = \sum_n f_n^k(x) = f^k(x)$ for all k.

($\Leftarrow$) Since $\mathcal{A}^\infty_{lfcs}$ is a C^∞_{lfcs}-algebra we conclude from 18.11 that the extension of φ is countably evaluating. □

18.15. Proposition. [Garrido, Gómez, Jaramillo, 1994, 1.10]. *Let φ in* $\operatorname{Hom}\mathcal{A}$ *be 1-evaluating, and let $f_n \in \mathcal{A}$ be such that $\sum_n \lambda_n f_n^j \in \mathcal{A}$ for all $\lambda \in \ell^1$ and $j \in \{1,2\}$.*

Then φ is $\{f_n : n \in \mathbb{N}\}$-evaluating.

For a convenient algebra structure on $\mathcal{A}$ and $\{f_n : n \in \mathbb{N}\}$ bounded in $\mathcal{A}$ the second condition holds, as used in 18.26.

It would be interesting to know if the assumption for $j = 2$ can be removed, since then the application in 18.26 to finite type polynomials would work.

Proof. Choose a positive absolutely summable sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that the sequences $(\lambda_n\,\varphi(f_n))_{n\in\mathbb{N}}$ and $(\lambda_n\,\varphi(f_n)^2)_{n\in\mathbb{N}}$ are summable. Then the sum

$$g := \sum_{j=1}^{\infty} \lambda_j (f_j - \varphi(f_j))^2 \in \mathcal{A}.$$

If there exists $x \in X$ with $g(x) = 0$, we are done. If not, then consider the (positive) function

$$h := \sum_{j=1}^{\infty} \frac{1}{2^j} \lambda_j (f_j - \varphi(f_j))^2 \in \mathcal{A}.$$

For every $n \in \mathbb{N}$ there exists $x_n \in X$ such that $\varphi(f_j) = f_j(x_n)$ for all $j \le n$, $\varphi(g) = g(x_n)$ and $\varphi(h) = h(x_n)$. But then for all $n \in \mathbb{N}$ we have by 18.9 that

$$0 < 2^n \varphi(h) = \sum_{j>n} 2^{n-j} \lambda_j\, \varphi(f_j - \varphi(f_j))^2 \le \sum_{j>n} \lambda_j\, \varphi(f_j - \varphi(f_j))^2 = \varphi(g),$$

a contradiction. □

18.16. Corollary. [Biström, Jaramillo, Lindström, 1995, Prop.9]. *Let E be a Banach space and $\mathcal{A}$ a 1-evaluating algebra containing $P(E)$. Then for each $\varphi \in \operatorname{Hom}\mathcal{A}$, each $f \in \mathcal{A}$, and each sequence $(p_n)_{n\in\mathbb{N}}$ in $P(E)$ with uniformly bounded degree, there exists $a \in E$ with $\varphi(f) = f(a)$ and $\varphi(p_n) = p_n(a)$ for all $n \in \mathbb{N}$.*

Proof. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of positive reals such that $\{\lambda_n p_n : n \in \mathbb{N}\}$ is bounded. Then by 18.15 the set $\{f, p_n\}$ is evaluated. □

18.17. Theorem. [Adam, Biström, Kriegl, 1995, 3.3]. *Let $(f_\gamma)_{\gamma\in\Gamma}$ be a family in $\mathcal{A}$ such that $\sum_{\gamma\in\Gamma} z_\gamma f_\gamma^j$ is a pointwise convergent sum in $\mathcal{A}$ for all $z = (z_\gamma) \in \ell^\infty(\Gamma)$ and $j = 1, 2$. Let $|\Gamma|$ be non-measurable, and let φ be ω-evaluating.*

Then φ is $\{f_\gamma : \gamma \in \Gamma\}$-evaluating.

We will apply this in particular if $\{f_\gamma : \gamma \in \Gamma\}$ is locally finite, and $\mathcal{A}$ stable under locally finite sums. Note that we can always add finitely many $f \in \mathcal{A}$ to $\{f_\gamma : \gamma \in \Gamma\}$.

Again it would be nice to get rid of the assumption for $j = 2$.

Proof. Let $x \in X$ and set $z_\gamma := \operatorname{sign}(f_\gamma(x))$ for all $\gamma \in \Gamma$. Then $z = (z_\gamma) \in \ell^\infty(\Gamma)$ and $\sum_{\gamma\in\Gamma} |f_\gamma(x)| = \sum_{\gamma\in\Gamma} z_\gamma f_\gamma(x) < \infty$, i.e. $(f_\gamma(x))_{\gamma\in\Gamma} \in \ell^1(\Gamma)$. Next observe that $(\varphi(f_\gamma))_{\gamma\in\Gamma} \in c_0(\Gamma)$, since otherwise there exists some $\varepsilon > 0$ and a countable set $\Lambda \subseteq \Gamma$ with $|\varphi(f_\gamma)| \ge \varepsilon$ for each $\gamma \in \Lambda$. By the countably evaluating property of φ there is a point $x \in X$ with $|f_\gamma(x)| = |\varphi(f_\lambda| \ge \varepsilon$ for each $\gamma \in \Lambda$, violating the condition $(f_\gamma(x))_{\gamma\in\Gamma} \in \ell^1(\Gamma)$. Since as a vector in $c_0(\Gamma)$ it has countable support and since φ is countably evaluating we get even $(\varphi(f_\gamma))_{\gamma\in\Gamma} \in \ell^1(\Gamma)$. Therefore we may consider g, defined by

$$X \ni x \mapsto g(x) := \Big((f_\gamma(x) - \varphi(f_\gamma))^2\Big)_{\gamma\in\Gamma} \in \ell^1(\Gamma).$$

This gives a map $g^* : \ell^\infty(\Gamma) = \ell^1(\Gamma)' \to \mathcal{A}$, by

$$g^*(z) : x \mapsto \langle z, g(x)\rangle = \sum_{\gamma\in\Gamma} z_\gamma \cdot (f_\gamma(x) - \varphi(f_\gamma))^2,$$

since $(\varphi(f_\gamma)_{\gamma\in\Gamma} \in \ell^1(\Gamma)$. Let $\Phi : \ell^\infty(\Gamma) \to \mathbb{R}$ be the linear map $\Phi := \varphi \circ g^* : \ell^\infty(\Gamma) \to \mathcal{A} \to \mathbb{R}$. By the countably evaluating property of φ, for any sequence (z_n) in $\ell^\infty(\Gamma)$ there exists an $x \in X$ such that $\Phi(z_n) = \varphi(g^*(z_n)) = g^*(z_n)(x) =$

$\langle z_n, g(x)\rangle$ for all n. For non-measurable $|\Gamma|$ the weak topology on $\ell^1(\Gamma)$ is realcompact by [Edgar, 1979, p.575]. By 18.19 there exists a point $c \in \ell^1(\Gamma)$ such that $\Phi(z) = \langle z, c\rangle$ for all $z \in \ell^\infty(\Gamma)$. For each standard unit vector $e_\gamma \in \ell^\infty(\Gamma)$ we have $0 = \Phi(e_\gamma) = \langle e_\gamma, c\rangle = c_\gamma$. Hence $c = 0$ and therefore $\Phi = 0$. For the constant vector 1 in $\ell^\infty(\Gamma)$, we get $0 = \Phi(1) = \varphi(g^*(1))$. Since φ is 1-evaluating there exists an $a \in X$ with $\varphi(g^*(1)) = g^*(1)(a) = \langle 1, g(a)\rangle = \sum_{\gamma\in\Gamma}(f_\gamma(a) - \varphi(f_\gamma))^2$, hence $\varphi(f_\gamma) = f_\gamma(a)$ for each $\gamma \in \Gamma$. □

18.18. Valdivia gives in [Valdivia, 1982] a characterization of the locally convex spaces which are realcompact in their weak topologies. Let us mention some classes of spaces that are *weakly realcompact*:

Result.

(1) *All locally convex spaces E with $\sigma(E', E)$-separable E'.*
(2) *All weakly Lindelöf locally convex spaces, and hence in particular all weakly countably determined Banach spaces, see* [Vašák, 1981]. *In particular this applies to $c_0(X)$ for locally compact metrizable X by* [Corson, 1961, p.5].
(3) *The Banach spaces E with angelic weak* dual unit ball* [Edgar, 1979, p.564].
Note that $(E^*, weak^*)$ is angelic :⇔ for $B \subseteq E^*$ bounded the weak*-closure is obtained by weak*-convergent sequences in B, i.e. sequentially for the weak*-topology.
(4) *$\ell^1(\Gamma)$ for $|\Gamma|$ non-measurable. Furthermore the spaces $C[0,1]$, ℓ^∞, $L^\infty[0,1]$, the space JL of* [Johnson, Lindenstrauss, 1974] *(a short exact sequence $c_0 \to JL \to \ell^2(\Gamma)$ exists), the space $D[0,1]$ or right-continuous functions having left sided limits, by* [Edgar, 1979, p.575] *and* [Edgar, 1977]. *All these spaces are not weakly Lindelöf.*
(5) *All closed subspaces of products of the spaces listed above.*
(6) *Not weakly realcompact are $C[0,\omega_1]$ and $\ell^\infty_{count}[0,1]$, the space of bounded functions on $[0,1]$ with countable support, by* [Edgar, 1979].

18.19. Lemma. [Corson, 1961]. *If E is a weakly realcompact locally convex space, then every linear countably evaluating $\Phi : E' \to \mathbb{R}$ is given by a point-evaluation ev_x on E' with $x \in E$.*

Proof. Since $\Phi : E' \to \mathbb{R}$ is countably evaluating it is linear and $\mathcal{F} := \{Z_K : K \subseteq E' \text{ countable}\}$ does not contain the empty set and generates a filter. We claim that this filter is Cauchy with respect to the uniformity defined by the weakly continuous real functions on E:
To see this, let $f : E \to \mathbb{R}$ be weakly continuous. For each $r \in \mathbb{R}$, let $L_r := \{x \in E : f(x) < r\}$ and similarly $U_r := \{x \in E : f(x) > r\}$. By [Jarchow, 1981, 8.1.4] we have that E is $\sigma(E'^*, E')$-dense in E'^*. Thus there are open disjoint subsets $\tilde{L}_r$ and $\tilde{U}_r$ on E'^* having trace L_r and U_r on E (take the complements of the closures of the complements). Let $\mathcal{B} \subseteq E'$ be an algebraic basis of E'. Then the map $\chi : E'^* \to \mathbb{R}^{\mathcal{B}}$, $l \mapsto (l(x'))_{x'\in\mathcal{B}}$ is a topological isomorphism for $\sigma(E'^*, E')$. By [Bockstein, 1948] there exists a countable subset $K_r \subseteq \mathcal{B} \subseteq E'$, such that the images under $\mathrm{pr}_{K_r} : \mathbb{R}^{\mathcal{B}} \to \mathbb{R}^{K_r}$ of the open sets $\tilde{L}_r$ and $\tilde{U}_r$ are disjoint. Let $K = \bigcup_{r\in\mathbb{Q}} K_r$. For $\varepsilon > 0$ we have that $Z_K \times Z_K \subseteq \{(x_1, x_2) : f(x_1) = f(x_2)\} \subseteq \{(x_1, x_2) : |f(x_1) - f(x_2)| < \varepsilon\}$, i.e. the filter generated by $\mathcal{F}$ is Cauchy. In fact, let $x_1, x_2 \in Z_K$. Then $x'(x_1) = \varphi(x') = x'(x_2)$ for all $x' \in K$. Suppose $f(x_1) \neq f(x_2)$. Without loss of generality we find a $r \in \mathbb{Q}$ with $f(x_1) < r < f(x_2)$, i.e. $x_1 \in L_r$ and $x_2 \in U_r$. But then $x'(x_1) \neq x'(x_2)$ for all $x' \in K_r \subseteq K$ gives a contradiction.

By realcompactness of $(E, \sigma(E, E'))$ the uniform structure generated by the weakly continuous functions $E \to \mathbb{R}$ is complete (see [Gillman, Jerison, 1960, p.226]) and hence the filter $\mathcal{F}$ converges to a point $a \in E$. Thus $a \in Z_K$ for all countable $K \subseteq E'$, and in particular $\Phi(x') = x'(a)$ for all $x' \in E'$. $\square$

18.20. Proposition. [Biström, Jaramillo, Lindström, 1995, Thm.10]. *Let E be a Banach space, let $\mathcal{A} \supseteq C^\omega_{conv}(E)$ be 1-evaluating, let $f \in \mathcal{A}$, and let $\mathcal{F}$ be a countable subset of $C^\omega_{conv}(E)$.*

Then $\{f\} \cup \mathcal{F}$ is evaluating. In particular, $\mathcal{R}C^\omega_{conv}(E)$ (see 18.7.2) is ω-evaluating for every Banach space E.

Proof. Let $(p_n)_{n\in\mathbb{N}}$ be a sequence in $P(E)$ and $(k_n)_{n\in\mathbb{N}}$ a sequence of odd natural numbers with $k_1 = 1$ and $k_{n+1} > 2k_n(1 + \deg p_n)$ for $n \in \mathbb{N}$. Then $|p_n^{k_n}(x)| \leq \|p_n\|^{k_n} \cdot \|x\|^{k_n \deg p_n}$ for every $x \in E$. Set

$$g := \sum_{n=1}^\infty \frac{1}{\lambda_n} \cdot \frac{1}{2^n} \cdot \frac{1}{n^{2k_n \deg p_n}} (p_n^{k_n} - \varphi(p_n^{k_n}))^2,$$

where $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence of reals with

$$\lambda_n > \|p_n\|^{2k_n} + 2|\varphi(p_n^{k_n})| \cdot \|p_n\|^{k_n} + (\varphi(p_n^{2k_n}))^2 \text{ for all } n \in \mathbb{N}.$$

Then

$$\begin{aligned} g(x) &\leq \sum_{n=1}^\infty \frac{1}{2^n} \cdot \frac{1}{n^{2k_n \deg p_n}} \left(\|x\|^{k_n 2 \deg p_n} + \|x\|^{k_n \deg p_n} + 1\right) \\ &\leq \sum_{n=1}^\infty \frac{1}{2^n} \left(\|\frac{x}{n}\|^{2k_n \deg p_n} + \|\frac{x}{n}\|^{k_n \deg p_n} + 1\right) < \infty \text{ for all } x \in E \end{aligned}$$

Since g is pointwise convergent, it is a function in $C^\omega_{\mathrm{conv}}(E)$. By the technique used in 18.15 we obtain that there exists $x \in E$ with $\varphi(f) = f(x)$ and $\varphi(p_n^{k_n}) = p_n^{k_n}(x)$ for all $n \in \mathbb{N}$. As for each $n \in \mathbb{N}$ the number k_n is odd, it follows that $\varphi(p_n) = p_n(x)$ for all $n \in \mathbb{N}$. Since each $g \in \mathcal{F}$ is a sum $\sum_{n\in\mathbb{N}} p_{n,g}$ of homogeneous polynomials $p_{n,g} \in P(E)$ of degree n for $n \in \mathbb{N}$, there exists $x \in E$ with $\varphi(g) = g(x)$ for all $g \in \mathcal{F}$, and $\varphi(p_{n,g}) = p_{n,g}(x)$ for all $n \in \mathbb{N}$, whence $\varphi(g) = \sum_{n\in\mathbb{N}} \varphi(p_{n,g})$ for all $g \in \mathcal{F}$. Let $a \in E$ with $\varphi(f) = f(a)$ and $\varphi(p_{n,g}) = p_{n,g}(a)$ for all $n \in \mathbb{N}$ and all $g \in \mathcal{F}$. Then

$$\varphi(g) = \sum_{n\in\mathbb{N}} \varphi(p_{n,g}) = \sum_{n\in\mathbb{N}} p_{n,g}(a) = g(a) \text{ for all } g \in \mathcal{F}. \quad \square$$

18.21. Result. [Adam, Biström, Kriegl, 1995, 2.1]. *Given two infinite cardinals $\mathfrak{m} < \mathfrak{n}$, let $E := \{x \in \mathbb{R}^{\mathfrak{n}} : |\operatorname{supp} x| \leq \mathfrak{m}\}$ Then for any algebra $\mathcal{A} \subseteq C(E)$, containing the natural projections $(\mathrm{pr}_\gamma)_{\gamma\in\mathfrak{n}}$, there is a homomorphism φ on $\mathcal{A}$ that is $\mathfrak{m}$-evaluating but not $\mathfrak{n}$-evaluating.*

Evaluating Homomorphisms

18.22. Proposition. [Garrido, Gómez, Jaramillo, 1994, 1.7]. *Let X be a closed subspace of a product $\mathbb{R}^\Gamma$. Let $\mathcal{A} \subseteq C(X)$ be a subalgebra containing the projections $\mathrm{pr}_\gamma|_X : X \subseteq \mathbb{R}^\Gamma \to \mathbb{R}$, and let $\varphi \in \operatorname{Hom} \mathcal{A}$ be $\bar{1}$-evaluating.*

Then φ is $\mathcal{A}$-evaluating.

Proof. Set $a_\gamma = \varphi(\mathrm{pr}_\gamma|_X)$. Then the point $a = (a_\gamma)_{\gamma\in\Gamma}$ is an element in X. Otherwise, since X is closed there exists a finite set $J \subseteq \Gamma$ and $\varepsilon > 0$ such that no point y with $|y_\gamma - a_\gamma| < \varepsilon$ for all $\gamma \in J$ is contained in X. Set $p(x) := \sum_{\gamma\in J}(\mathrm{pr}_\gamma(x) - a_\gamma)^2$ for $x \in X$. Then $p \in \mathcal{A}$ and $\varphi(p) = 0$. By assumption there is an $x \in X$, such that $|\varphi(p) - p(x)| < \varepsilon^2$, but then $|\mathrm{pr}_\gamma(x) - a_\gamma| < \varepsilon$ for all $\gamma \in J$, a contradiction. Thus $a \in X$ and $\varphi(g) = g(a)$ for all g in the algebra $\mathcal{A}_0$ generated by all functions $\mathrm{pr}_\gamma|_X$.

By the assumption and by 18.2 there exists a point $\tilde{x}$ in the Stone-Čech compactification βX such that $\varphi(f) = \tilde{f}(\tilde{x})$ for all $f \in \mathcal{A}$, where $\tilde{f}$ is the unique continuous extension $\beta X \to \mathbb{R}_\infty$ of f. We claim that $\tilde{x} = a$. This holds if $\tilde{x} \in X$ since the pr_γ separate points on X. So let $\tilde{x} \in \beta X \setminus X$. Then $\tilde{x}$ is the limit of an ultrafilter $\mathcal{U}$ in X. Since $\mathcal{U}$ does not converge to a, there is a neighborhood of a in X, without loss of generality of the form $U = \{x \in X : f(x) > 0\}$ for some $f \in \mathcal{A}_0$. But then the complement of U is in the ultrafilter $\mathcal{U}$, thus $\tilde{f}(\tilde{x}) \le 0$. But this contradicts $\tilde{f}(\tilde{x}) = \varphi(f) = f(a)$ for all $f \in \mathcal{A}_0$. □

18.23. Corollary. [Kriegl, Michor, 1993, 1]. *If $\mathcal{A}$ is finitely generated then each 1-evaluating $\varphi \in \operatorname{Hom}\mathcal{A}$ is evaluating.*

Finitely generated can even be meant in the sense of $C^{\langle\infty\rangle}$-algebra, see the proof. This applies to the algebras $\mathcal{R}P$, C^ω, C^ω_{conv} and C^∞ on $\mathbb{R}^n$ (or a closed submanifold of $\mathbb{R}^n$).

Proof. Let $\mathcal{F} \subseteq \mathcal{A}$ be a finite subset which generates $\mathcal{A}$ in the sense that $\mathcal{A} \subseteq \mathcal{F}^{\langle\infty\rangle} := (\langle\mathcal{F}\rangle_{\mathrm{Alg}})^{\langle\infty\rangle}$, compare 18.7.3. By 18.7 again we have that φ restricted to $\langle\mathcal{F}\rangle_{\mathrm{Alg}}$ extends to $\tilde{\varphi} \in \operatorname{Hom}\mathcal{F}^{\langle\infty\rangle}$ by $\varphi(h \circ (f_1,\dots,f_n)) = h(\varphi(f_1),\dots,\varphi(f_n))$ for $f_i \in \mathcal{F}$, $h \in C^\infty(U,\mathbb{R})$ where $(f_1,\dots,f_n)(X) \subseteq U$ and U is open in $\mathbb{R}^n$. For $f \in \mathcal{A}$ there exists $x \in X$ such that $\varphi = \mathrm{ev}_x$ on f and on $\mathcal{F}$, which implies that $\tilde{\varphi}(f) = f(x) = \varphi(f)$. Finally note that if $\varphi = \mathrm{ev}_x$ on $\mathcal{F}$ then $\tilde{\varphi} = \mathrm{ev}_x$ on $\mathcal{F}^{\langle\infty\rangle}$, thus $\varphi = \mathrm{ev}_x$ on $\mathcal{A}$. □

18.24. Proposition. [Biström, Bjon, Lindström, 1992, Prop.4]. *Let $\varphi \in \operatorname{Hom}\mathcal{A}$ be ω-evaluating and X be Lindelöf (for some topology finer than $X_{\mathcal{A}}$).*

Then φ is evaluating.

This applies to any ω-evaluating algebra on a separable Fréchet space, [Arias-de-Reyna, 1988, 8].
It applies also to $\mathcal{A} = C^\infty_{\mathrm{lfcs}}(E)$ for any weakly Lindelöf space by 18.27. In particular, for $1 < p \le \infty$ the space $\ell^p(\Gamma)$ is weakly Lindelöf by 18.18.1 as weak*-dual of the normed space ℓ^q with $q := 1/(1 - \frac{1}{p})$ and the same holds for the spaces $(\ell^1(\Gamma), \sigma(\ell^1(\Gamma), c_0(\Gamma)))$. Furthermore it is true for $(\ell^1(\Gamma), \sigma(\ell^1(\Gamma), \ell^\infty(\Gamma)))$ by [Edgar, 1979], and for $(c_0(\Gamma), \sigma(c_0(\Gamma), \ell^1(\Gamma)))$ by [Corson, 1961, p.5].

Proof. By the sequentially evaluating property of $\mathcal{A}$ the family $(Z_f)_{f\in\mathcal{A}}$ of closed sets $Z_f = \{x \in X : f(x) = \varphi(f)\}$ has the countable intersection property. Since X is Lindelöf, the intersection of all sets in this collection is non-empty. Thus φ is a point evaluation with a point in this intersection. □

18.25. Proposition. *Let $\mathcal{A}$ be an algebra which contains a countable point-separating subset.*

Then every ω-evaluating φ in $\operatorname{Hom}\mathcal{A}$ is $\mathcal{A}$ is evaluating.

If a Banach space E has weak*-separable dual and $D \subseteq E'$ is countable and weak*-dense, then D is point-separating, since for $x \neq 0$ there is some $\ell \in E'$ with $\ell(x) = 1$

and since $\{x' \in E' : x'(x) > 0\}$ is open in the weak*-topology also an $\ell \in D$ with $\ell(x) > 0$. The converse is true as well, see [Biström, 1993, p.28].

Thus 18.25 applies to all Banach-spaces with weak*-separable dual and the algebras RP, C^ω, RC^ω_{conv}, C^∞.

Proof. Let $\{f_n\}_n$ be a countable subset of $\mathcal{A}$ separating the points of X. Let $f \in \mathcal{A}$. Since $\mathcal{A}$ is ω-evaluating there exists a point $x_f \in X$ with $f(x_f) = \varphi(f)$ and $f_n(x_f) = \varphi(f_n)$. Since the f_n are point-separating this point x_f is uniquely determined and hence independent on $f \in \mathcal{A}$. $\square$

18.26. Proposition. [Arias-de-Reyna,1988, Thm.8] *for C^m on separable Banach spaces;* [Gómez, Llavona, 1988, Thm.1] *for ω-evaluating algebras on locally convex spaces with w^*-separable dual;* [Adam, 1993, 6.40]. *Let E be a convenient vector space, let $\mathcal{A} \supseteq P$ be an algebra containing a point separating bounded sequence of homogeneous polynomials of fixed degree.*

Then each 1-evaluating homomorphism is evaluating.

In particular this applies to c_0 and ℓ^p for $1 \le p \le \infty$. It also applies to a dual of a separable Fréchet space, since then any dense countable subset of E can be made equicontinuous on E' by [Biström, 1993, 4.13].

Proof. Let $\{p_n : n \in \mathbb{N}\}$ be a point-separating bounded sequence. By the polarization formulas given in 7.13 this is equivalent to boundedness of the associated multilinear symmetric mappings, hence $\{p_n : n \in \mathbb{N}\}$ satisfies the assumptions of 18.15 and thus $\{p_n : n \in \mathbb{N}\}$ is evaluated. Now the result follows as in 18.25. $\square$

18.27. Theorem. [Adam, Biström, Kriegl, 1995, 5.1]. *A locally convex space E is weakly realcompact if and only if $E = \operatorname{Hom}_\omega P_f(E) (= \operatorname{Hom} C^\infty_{\text{lfcs}}(E))$.*

Proof. By 18.14 we have $\operatorname{Hom}_\omega P_f(E) = \operatorname{Hom} C^\infty_{\text{lfcs}}(E)$.
($\Rightarrow$) Let E be weakly realcompact. Since E is $\sigma(E'^*, E')$-dense in E'^* (see [Jarchow, 1981, 8.1.4]), it follows from 18.19 that any $\varphi \in \operatorname{Hom}_\omega P_f(E) = \operatorname{Hom} C^\infty_{\text{lfcs}}(E)$ is E'-evaluating and hence also evaluating on the algebra $P_f(E)$ generated by E'.
($\Leftarrow$) By 17.4 the space $\operatorname{Hom}_\omega(P_f(E))$ is realcompact in the topology of pointwise convergence. Since $E = \operatorname{Hom}_\omega P_f(E)$ and $\sigma(E, E')$ equals the topology of pointwise convergence on $\operatorname{Hom}_\omega(P_f(E))$, we have that $(E, \sigma(E, E'))$ is realcompact. $\square$

18.28. Proposition. [Biström, Jaramillo, Lindström, 1995, Thm.13]. *Let E be a Banach space with the Dunford-Pettis property that does not contain a copy of ℓ^1. Then $P_f(E)$ is dense in $P(E)$ for the topology of uniform convergence on bounded sets.*

A Banach space E is said to have the *Dunford-Pettis property* [Diestel, 1984, p.113] if $x_n^* \to 0$ in $\sigma(E', E''))$ and $x_n \to 0$ in $\sigma(E, E')$ implies $x_n^*(x_n) \to 0$. Well known Banach spaces with the Dunford-Pettis property are $L^1(\mu)$, $C(K)$ for any compact K, and $\ell^\infty(\Gamma)$ for any Γ. Furthermore $c_0(\Gamma)$ and $\ell^1(\Gamma)$ belong to this class since if E' has the Dunford-Pettis property then also E has. According to [Aron, 1976, p.215], the space ℓ^1 is not contained in $C(K)$ if and only if K is dispersed, i.e. $K^{(\alpha)} = \emptyset$ for some α, or equivalently whenever its closed subsets admit isolated points.

Proof. According to [Carne, Cole, Gamelin, 1989, theorem 7.1], the restriction of any $p \in P(E)$ to a weakly compact set is weakly continuous if E has the Dunford-Pettis property and, consequently, sequentially weakly continuous. By [Llavona, 1986, theorems 4.4.7 and 4.5.9], such a polynomial p is weakly uniformly continuous on bounded sets if E, in addition, does not contain a copy of ℓ_1. The assertion therefore follows from [Llavona, 1986, theorem 4.3.7]. $\square$

18.29. Theorem. [Garrido, Gómez, Jaramillo, 1994, 2.4] & [Adam, Biström, Kriegl, 1995, 3.4]. *Let E be $\ell^{2n}(\Gamma)$ for some n and some Γ of non-measurable cardinality. Let $P(E) \subseteq \mathcal{A} \subseteq C(E)$.*

Then every 1-evaluating homomorphism φ is evaluating.

Proof. For $f \in \mathcal{A}$ let $\mathcal{A}_f$ be the algebra generated by f and all i-homogeneous polynomials in $P(E)$ with degree $i \le 4n+2$. Take a sequence (p_n) of continuous polynomials with degree $i \le 2n+1$. Then there is a sequence (t_n) in $\mathbb{R}_+$ such that $\{t_n p_n : n \in \mathbb{N}\}$ is bounded, hence φ is by 18.15 evaluating on it, i.e. φ is ω-evaluating on $\mathcal{A}_f$.

Given $z = (z_\gamma) \in \ell^\infty(\Gamma)$ and $x \in E$, set

$$f_{z,j}(x) := f(x)^j + \sum_{\gamma\in\Gamma} z_\gamma \operatorname{pr}_\gamma(x)^{(2n+1)j},$$

where $j = 1, 2$. Then $f_{z,j} \in \mathcal{A}_f$ and we can apply 18.17. Thus there is a point $x_f \in E$ with $\varphi(f) = f(x_f)$ and $\varphi(\operatorname{pr}_\gamma)^{2n+1} = \operatorname{pr}_\gamma(x_f)^{2n+1}$ for all $\gamma \in \Gamma$. Hence $\varphi(\operatorname{pr}_\gamma) = \operatorname{pr}_\gamma(x_f)$, and since $(\operatorname{pr}_\gamma)_{\gamma\in\Gamma}$ is point separating, x_f is uniquely determined and thus not depending on f and we are finished. $\square$

18.30. Proposition. *Let $E = c_0(\Gamma)$ with Γ non-measurable. If one of the following conditions is satisfied, then φ is evaluating:*

(1) [Biström, 1993, 2.22] & [Adam, Biström, Kriegl, 1995, 5.4]. *$C^\infty_{lfs}(E) \subseteq \mathcal{A}$ and φ is ω-evaluating.*
(2) [Garrido, Gómez, Jaramillo, 1994, 2.7]. *$P(E) \subseteq \mathcal{A}$, every $f \in \mathcal{A}$ depends only on countably many coordinates and φ is 1-evaluating.*

Proof. (1) Since φ is ω-evaluating, it follows that $(\varphi(\operatorname{pr}_\gamma))_{\gamma\in\Gamma} \in c_0(\Gamma)$, where $\operatorname{pr}_\gamma : c_0(\Gamma) \to \mathbb{R}$ are the natural coordinate projections (see the proof of 18.17). Fix n and consider the function $f_n : c_0(\Gamma) \to \mathbb{R}$ defined by the locally finite product

$$f_n(x) := \prod_{\gamma\in\Gamma} h\Big(n \cdot (\operatorname{pr}_\gamma(x) - \varphi(\operatorname{pr}_\gamma))\Big),$$

where $h \in C^\infty(\mathbb{R}, [0,1])$ is chosen such that $h(t) = 1$ for $|t| \le 1/2$ and $h(t) = 0$ for $|t| \ge 1$. Note that a locally finite product $f := \prod_{i\in I} f_i$ (i.e. locally only finitely many factors f_i are unequal to 1) can be written as locally finite sum $f = \sum_J g_J$, where $g_i := f_i - 1$ and for finite subsets $J \subseteq I$ let $g_J := \prod_{j\in J} g_j \in \mathcal{A}$ and the index J runs through all finite subsets of I. Since I is at least countable, the set of these indices has the same cardinality as I has.

By means of 18.17 $\varphi(f_n) = \prod_{\gamma\in\Gamma} h(0) = 1$ for all n. Now let $f \in \mathcal{A}$. Then there exists a $x_f \in E$ with $\varphi(f) = f(x_f)$ and $1 = \varphi(f_n) = f_n(x_f)$. Hence $|n \cdot (\operatorname{pr}_\gamma(x_f) - \varphi(\operatorname{pr}_\gamma))| \le 1$ for all n, i.e. $\operatorname{pr}_\gamma(x_f) = \varphi(\operatorname{pr}_\gamma)$ for all $\gamma \in \Gamma$. Since $(\operatorname{pr}_\gamma)_{\gamma\in\Gamma}$ is point separating, the point $x_f \in E$ is unique and thus does not depend on f.

(2) By 18.15 or 18.16 the restriction of φ to the algebra generated by $\{\operatorname{pr}_\gamma : \gamma \in \Gamma\}$ is ω-evaluating. Since $c_0(K)$ is weakly-realcompact by [Corson, 1961] for locally compact metrizable K and hence in particular for discrete K, we have by

18.19 that φ is evaluating on this algebra, i.e. there exists $a = (a_\gamma)_{\gamma\in\Gamma} \in E$ with $a_\gamma = \mathrm{pr}_\gamma(a) = \varphi(\mathrm{pr}_\gamma)$ for all $\gamma \in \Gamma$.

Every $f \in \mathcal{A}(E)$ depends only on countably many coordinates, i.e. there exists a countable $\Gamma_f \subseteq \Gamma$ and a function $\tilde f : c_0(\Gamma_f) \to \mathbb{R}$ with $\tilde f \circ \mathrm{pr}_{\Gamma_f} = f$. Let

$$\mathcal{A}_f := \{g \in \mathbb{R}^{c_0(\Gamma_f)} : g \circ \mathrm{pr}_{\Gamma_f} \in \mathcal{A}\}$$

and let $\tilde\varphi : \mathcal{A}_f \to \mathbb{R}$ be given by $\tilde\varphi = \varphi \circ \mathrm{pr}_{\Gamma_f}$. Since Γ_f is countable there is by 18.15 an $x^f \in c_0(\Gamma_f)$ with $\tilde\varphi(\tilde f) = \tilde f(x^f)$ and $a_\gamma = \varphi(\mathrm{pr}_\gamma) = \tilde\varphi(\mathrm{pr}_\gamma) = \mathrm{pr}_\gamma(x^f) = x^f_\gamma$ for all $\gamma \in \Gamma_f$. Thus $\mathrm{pr}_{\Gamma_f}(a) = x$ and

$$\varphi(f) = \varphi(\tilde f \circ \mathrm{pr}_{\Gamma_f}) = \tilde\varphi(\tilde f) = \tilde f(\mathrm{pr}_{\Gamma_f}(a)) = f(a). \quad \square$$

18.31. Proposition. [Garrido, Gómez, Jaramillo, 1994, 2.7]. *Each $f \in C^\omega(c_0(\Gamma))$ depends only on countably many coordinates.*

Proof. Let $f : c_0(\Gamma) \to \mathbb{R}$ be real analytic. So there is a ball $B_\varepsilon(0) \subseteq c_0(\Gamma)$ such that $f(x) = \sum_{n=1}^\infty p_n(x)$ for all $x \in B_r(0)$, where $p_n \in L^n_{\mathrm{sym}}(c_0(\Gamma);\mathbb{R})$ for all $n \in \mathbb{N}$.

By 18.28 the space $P_f(c_0(\Gamma))$ is dense in $P(c_0(\Gamma))$ for the topology of uniform convergence on bounded sets, since $c_0(\Gamma)$ has the Dunford-Pettis property and does not contain ℓ^1 as topological linear subspace. Thus we have that for any $n, k \in \mathbb{N}$ there is some $q_{nk} \in P_f(c_0(\Gamma))$ with

$$\sup\{|p_n(x) - q_{nk}(x)| : x \in B_\varepsilon(0)\} < \frac{1}{k}.$$

Since each $q \in P_f(c_0(\Gamma))$ is a polynomial form in elements of $\ell^1(\Gamma)$, there is a countable set $\Lambda_{nk} \subseteq \Gamma$ such that q_{nk} only depends on the coordinates with index in Λ_{nk}, whence p_n on $B_\varepsilon(0)$ only depends on coordinates with index in $\Lambda_n := \bigcup_{k\in\mathbb{N}} \Lambda_{nk}$. Set $\Lambda := \bigcup_{n\in\mathbb{N}} \Lambda_n$ and let $\iota_\Lambda : c_0(\Lambda) \to c_0(\Gamma)$ denote the natural injection given by $(\iota_\Lambda(x))_\gamma = x_\gamma$ if $\gamma \in \Lambda$ and $(\iota_\Lambda(x))_\gamma = 0$ otherwise. By construction $f = f \circ \iota_\Lambda \circ \mathrm{pr}_\Lambda$ on $B_\varepsilon(0)$. Since both functions are real analytic and agree on $B_\varepsilon(0)$, they also agree on $c_0(\Gamma)$. $\square$

18.32. Example. [Garrido, Gómez, Jaramillo, 1994, 2.6]. *For uncountable Γ the space $c_0(\Gamma) \setminus \{0\}$ is not C^ω-realcompact.*

But for non-measurable Γ the whole space $c_0(\Gamma)$ is C^ω-evaluating by 18.30 and 18.31.

Proof. Let $\Omega := c_0(\Gamma) \setminus \{0\}$, let $f : \Omega \to \mathbb{R}$ be real analytic and consider any sequence $(u^m)_{m\in\mathbb{N}}$ in Ω with $u^m \to 0$. For each $m \in \mathbb{N}$ there exists $\varepsilon_m > 0$ and homogeneous P^n_m in $P(c_0(\Gamma))$ of degree n for all n, such that, for $\|h\| < \varepsilon_m$

$$f(u^m + h) = f(u^m) + \sum_{n=1}^\infty P^n_m(h).$$

As carried out in 18.31, each P^n_m only depends on coordinates with index in some countable set $\Lambda^n_m \subseteq \Gamma$. The set $\Lambda := (\bigcup_{n,m\in\mathbb{N}} \Lambda^n_m) \cup (\bigcup_{m\in\mathbb{N}} \operatorname{supp} u^m)$ is countable. Let $\gamma \in \Gamma \setminus \Lambda$. Then, since $P^n_m(e_\gamma) = 0$ and $u^m + te_\gamma \neq 0$ for all $m, n \in \mathbb{N}$ and $t \in \mathbb{R}$, we get $f(u^m + te_\gamma) = f(u^m)$ for all $|t| < \varepsilon_m$. Thus $f(u^m + te_\gamma) = f(u^m)$ for every $t \in \mathbb{R}$, since the function $t \mapsto f(u^m + te_\gamma)$ is real analytic on $\mathbb{R}$. In particular, $f(u^m + e_\gamma) = f(u^m)$ and, since $u^m + e_\gamma \to e_\gamma$, there exists

$$\varphi(f) := \lim_{m\in\mathbb{N}} f(u^m) = \lim_{m\in\mathbb{N}} f(u^m + e_\gamma) = f(e_\gamma).$$

Then φ is an algebra homomorphism, since a common γ can be found for finitely many f. And since $\ell_1(\Gamma) \subseteq C^\omega(\Omega)$ is point separating the homomorphism φ cannot be an evaluation at some point of Ω. □

18.33. Example. [Biström, Jaramillo, Lindström, 1995, Prop.16]. *The algebra $C^\omega_{conv}(\ell^\infty)$ is not 1-evaluating.*

Proof. Suppose that $C^\omega_{\text{conv}}(\ell^\infty)$ is 1-evaluating. By 20.3 the unit ball B_{c_0} of c_0 is C^ω_{conv}-bounding in ℓ^∞. By 18.20 the algebra $C^\omega_{\text{conv}}(\ell^\infty)$ is ω-evaluating and, since $(\ell^\infty)'$ admits a point separating sequence, we have $\ell^\infty = \operatorname{Hom}(C^\omega_{\text{conv}}(\ell^\infty))$ by 18.25. Hence by 20.2, every C^ω_{conv}-bounding set in ℓ^∞ is relatively compact in the initial topology induced by $C^\omega_{\text{conv}}(\ell^\infty)$ and in particular relatively $\sigma(\ell^\infty, (\ell^\infty)')$-compact. Therefore, since the topologies $\sigma(c_0, \ell^1)$ and $\sigma(\ell^\infty, (\ell^\infty)')$ coincide on c_0, we have that B_{c_0} is $\sigma(c_0, \ell^1)$-compact, which contradicts the non-reflexivity of c_0 by by [Jarchow, 1981, 11.4.4]. □

19. Stability of Smoothly Realcompact Spaces

In this section stability of evaluation properties along certain mappings are studied which furnish some large classes of smoothly realcompact spaces.

19.1. Proposition. *Let $\mathcal{A}_X$ and $\mathcal{A}_Y$ be algebras of functions on sets X and Y as in 17.1, let $T : X \to Y$ be injective with $T^*(\mathcal{A}_Y) \subseteq \mathcal{A}_X$, and suppose that Y is $\mathcal{A}_Y$-realcompact. Then we have:*

(1) [Jaramillo, 1992, 5]. *If $\mathcal{A}_X$ is 1-evaluating and $\mathcal{A}_Y$ is 1-isolating on Y, then X is $\mathcal{A}_X$-realcompact and $\mathcal{A}_X$ is 1-isolating on X.*
(2) [Biström, Lindström, 1993a, Thm.2]. *If $\mathcal{A}_X$ is ω-evaluating and $\mathcal{A}_Y$ is ω-isolating on Y, then X is $\mathcal{A}_X$-realcompact and $\mathcal{A}_X$ is ω-isolating on X.*

We say that $\mathcal{A}_X$ is *1-isolating* on X if for every $x \in X$ there is an $f \in \mathcal{A}_X$ with $\{x\} = f^{-1}(f(x))$.

Similarly $\mathcal{A}_X$ is called *ω-isolating* on X if for every $x \in X$ there exists a sequence $(f_n)_n$ in $\mathcal{A}_X$ such that $\{x\} = \bigcap_n f_n^{-1}(f_n(x))$. This was called $\mathcal{A}$-countably separated in [Biström, Lindström, 1993a].

Proof. There is a point $y \in Y$ with $\psi = \mathrm{ev}_y$. Let $\mathcal{G} \subseteq \mathcal{A}_Y$ be such that $\{y\} = \bigcap_{g\in\mathcal{G}} g^{-1}(g(y))$, where $\mathcal{G}$ is either a single function or countably many functions. Let $f \in \mathcal{A}_X$ be arbitrary. By assumption there exists $x_f \in X$ with $\varphi(f) = f(x_f)$ and $\varphi(T^*(g)) = T^*(g)(x_f)$ for all $g \in \mathcal{G}$. Since $g(y) = \psi(g) = \varphi(T^*(g) = T^*(g)(x_f) = g(T(x_f))$ for all $g \in \mathcal{G}$, we obtain that $y = T(x_f)$. Since T is injective, we get that x_f does not depend on f, and hence φ is evaluating. □

19.2. Lemma. *If E is a convenient vector space which admits a bounded point-separating sequence in the dual E' then the algebra $P(E)$ of polynomials is 1-isolating on E.*

Proof. Let $\{x'_n : n \in \mathbb{N}\} \subseteq E'$ be such a sequence and let $a \in E$ be arbitrary. Then the series $x \mapsto \sum_{n=1}^\infty 2^{-n} x'_n(x-a)^2$ converges in $P(E)$, since $x'_n(\quad -a)^2$ is bounded and $\sum_{n=1}^\infty 2^{-n} < \infty$. It gives a polynomial which vanishes exactly at a. □

19.3. Examples. [Garrido, Gómez, Jaramillo, 1994, 2.4 and 2.5.2]. *Any super-reflexive Banach space X of non-measurable cardinality is $\mathcal{A}_X$-realcompact, for each*

1-isolating and 1-evaluating algebra $\mathcal{A}_X$ as in 17.1 which contains the algebra of rational functions $\mathcal{R}P(X)$, see 18.7.2.

A Banach-space E is called *super-reflexive*, if all Banach-spaces F which are finitely representable in E (i.e. for any finite dimensional subspace F_1 and $\varepsilon > 0$ there exists a isomorphism $T : F_1 \cong E_1 \subseteq E$ onto a subspace E_1 of E with $\|T\| \cdot \|T^{-1}\| \le 1+\varepsilon$) are reflexive (see [Enflo, Lindenstrauss, Pisier, 1975]). This is by [Enflo, 1972] equivalent to the existence of an equivalent *uniformly convex norm*, i.e. $\inf\{2 - \|x+y\| : \|x\| = \|y\| = 1, \|x-y\| \ge \varepsilon\} > 0$ for all $0 < \varepsilon < 2$. In [Enflo, Lindenstrauss, Pisier, 1975] it is shown that superreflexivity has the 3-space property.

Proof. A super-reflexive Banach space injects continuously and linearly into $\ell^p(\Gamma)$ for some $p > 1$ and some Γ by [John, Torunczyk, Zizler, 1981, p.133] and hence into some $\ell^{2n}(\Gamma)$. We apply 19.1.1 to the situation $X := E \to \ell^{2n}(\Gamma) =: Y$, which is possible because the algebra $P(Y)$ is 1-isolating on Y, since the $2n$-th power of the norm is a polynomial and can be used as isolating function. By 18.6 the algebra $\mathcal{R}P(Y)$ is 1-evaluating, and by 18.29 it is thus evaluating on Y. □

19.4. Lemma.

1. *Every 1-isolating algebra is ω-isolating.*
2. *If X is $\mathcal{A}$-regular and $X_{\mathcal{A}}$ has first countable topology then $\mathcal{A}$ is ω-isolating.*
3. *If for a convenient vector space the dual $(E', \sigma(E', E))$ is separable then the algebra $P_f(E)$ of finite type polynomials is ω-isolating on E.*

Proof. (1) is trivial.

(2) Let $x \in X$ be given and consider a countable neighborhood base $(U_n)_n$ of x. Since X is assumed to be $\mathcal{A}$-regular, there exist $f_n \in \mathcal{A}$ with $f_n(y) = 0$ for $y \notin U_n$ and $f_n(x) = 1$. Thus $\bigcap_n f_n^{-1}(f_n(x)) = \{x\}$.

(3) Let $\{x'_n : n \in \mathbb{N}\}$ be dense in $(E', \sigma(E', E))$ and $0 \ne x \in E$. Then there is some $x' \in E'$ with $x'(x) = 1$. By the denseness there is some n such that $|x'_n(x) - x'(x)| < 1$ and hence $x'_n(x) > 0$. So $\{0\} = \bigcap_n (x'_n)^{-1}(0)$. □

19.5. Example. *For Γ of non-measurable cardinality, the Banach space $E := c_0(\Gamma)$ is $C^\infty_{lfs}(E)$-paracompact by 16.15, and hence any ω-evaluating algebra $\mathcal{A} \supseteq C^\infty_{lfs}(E)$ is ω-isolating and evaluating.*

Proof. The Banach space E is $C^\infty_{lfs}(E)$-paracompact by 16.16. By 17.6 the space E is $\mathcal{A}$-realcompact for any $\mathcal{A} \supseteq C^\infty_{lfs}(E)$ and is ω-isolating by 19.4.2. □

19.6. Example. *Let K be a compact space of non-measurable cardinality with $K^{(\omega_0)} = \emptyset$.*

Then the Banach space $C(K)$ is C^∞-paracompact by 16.20.1, hence $C^\infty(C(K))$ is ω-isolating and $C(K)$ is C^∞-realcompact.

Proof. We use the exact sequence

$$c_0(K \setminus K') \cong \{f \in C(K) : F|_{K'} = 0\} \to C(K) \to C(K')$$

to obtain that $C(K)$ is C^∞-paracompact, see 16.19. By 17.6 the space E is C^∞-realcompact, is ω-isolating by 19.4.2. □

19.7. Example. [Biström, Lindström, 1993a, Corr.3bac]. *The following locally convex space are $\mathcal{A}$-realcompact for each ω-evaluating algebra $\mathcal{A} \supseteq C^\infty_{lfs}$, if their cardinality is non-measurable.*

(1) *Weakly compactly generated (WCG) Banach spaces, in particular separable Banach spaces and reflexive ones. More generally weakly compactly determined (WCD) Banach spaces.*
(2) *$C(K)$ for Valdivia-compact spaces K, i.e. compact subsets $K \subseteq \mathbb{R}^\Gamma$ with $K \cap \{x \in \mathbb{R}^\Gamma : \operatorname{supp} x$ countable$\}$ being dense in K.*
(3) *The dual of any realcompact Asplund Banach space.*

Proof. All three classes of spaces inject continuous and linearly into some $c_0(\Gamma)$ with non-measurable Γ by 53.21. Now we apply 19.5 for the algebra C^∞_{lfs} on $c_0(\Gamma)$ to see that the conditions of 19.1.2 for the range space $Y = c_0(\Gamma)$ are satisfied. So 19.1.2 implies the result. □

19.8. Proposition. *Let $T : X \to Y$ be a closed embedding between topological spaces equipped with algebras of continuous functions such that $T^*(\mathcal{A}_Y) \subseteq \mathcal{A}_X$. Let $\varphi \in \operatorname{Hom} \mathcal{A}_X$ such that $\psi := \varphi \circ T^*$ is $\mathcal{A}_Y$-evaluating.*

(1) [Kriegl, Michor, 1993, 8]. *If φ is 1-evaluating on $\mathcal{A}_X$ and $\mathcal{A}_Y$ has 1-small zerosets on Y then φ is $\mathcal{A}_X$-evaluating, and $\mathcal{A}_X$ has 1-small zerosets on X.*
(2) [Biström, Lindström, 1993b, p.178]. *If φ is ω-evaluating on $\mathcal{A}_X$ and $\mathcal{A}_Y$ has ω-small zerosets on Y then φ is $\mathcal{A}_X$-evaluating, and $\mathcal{A}_X$ has ω-small zerosets on X.*

Let $\mathfrak{m}$ be a cardinal number (often 1 or ω). We say that there are $\mathfrak{m}$-*small* $\mathcal{A}_Y$-zerosets on Y or $\mathcal{A}_Y$ has $\mathfrak{m}$-small zerosets on Y if for every $y \in Y$ and neighborhood U of y there exists a subset $\mathcal{G} \subseteq \mathcal{A}_Y$ with $\bigcap_{g\in\mathcal{G}} g^{-1}(g(y)) \subseteq U$ and $|\mathcal{G}| \leq \mathfrak{m}$.
For $\mathfrak{m} = 1$ this was called large $\mathcal{A}$-carriers in [Kriegl, Michor, 1993], and for $\mathfrak{m} = \omega$ it was called weakly $\mathcal{A}$-countably separated in [Biström, Lindström, 1993b, p.178].

Proof. Let $y \in Y$ be a point with $\psi = \mathrm{ev}_y$. Since Y admits $\mathfrak{m}$-small $\mathcal{A}_Y$-zerosets there exists for every neighborhood U of y a set $\mathcal{G} \subseteq \mathcal{A}_Y$ of functions with $\bigcap_{g\in\mathcal{G}} g^{-1}(g(y)) \subseteq U$ with $|\mathcal{G}| \leq \mathfrak{m}$. Let now $f \in \mathcal{A}_X$ be arbitrary. Since $\mathcal{A}_X$ is assumed to be $\mathfrak{m}$-evaluating, there exists a point $x_{f,U}$ such that $f(x_{f,U}) = \varphi(f)$ and $g(T(x_{f,U})) = T^*(g)(x_{f,U}) = \varphi(T^*g) = \psi(g) = g(y)$ for all $g \in \mathcal{G}$, hence $T(x_{f,U}) \in U$. Thus the net $T(x_{f,U})$ converges to y for $U \to y$ and since T is a closed embedding there exists a unique x with $T(x) = y$ and $x = \lim_U x_{f,U}$. Consequently $f(x) = f(\lim_U x_{f,U}) = \lim_U f(x_{f,U}) = \lim_U \varphi(f) = \varphi(f)$ since f is continuous.

The additional assertions are obvious. □

19.9. Corollary. [Adam, Biström, Kriegl, 1995, 5.6]. *Let E be a locally convex space, $\mathcal{A} \supseteq E'$, and let $\varphi \in \operatorname{Hom} \mathcal{A}$ be ω-evaluating. Assume φ is E'-evaluating (this holds if $(E, \sigma(E, E'))$ is realcompact by 18.27, e.g.). Let E admit ω-small $((E')^\infty \cap \mathcal{A})_{lfs} \cap \mathcal{A}$-zerosets. Then φ is evaluating on $\mathcal{A}$.*

In particular, if E is realcompact in the weak topology and admits ω-small C^∞_{lfs}-zerosets then $E = \operatorname{Hom}_\omega C^\infty_{lfs}(E)$.

Proof. We may apply 19.8.2 to $X = Y := E$, $\mathcal{A}_X = \mathcal{A}$ and $\mathcal{A}_Y := ((E')^\infty \cap \mathcal{A})_{lfs} \cap \mathcal{A}$. Note that φ is evaluating on $\mathcal{A}_Y$ by 18.17 and that $C^\infty_{lfs}(E) = ((E')^\infty)_{lfs}$ by 18.13. □

19.10. Lemma. [Adam, Biström, Kriegl, 1995, 5.5].

(1) *If a space is $\mathcal{A}$-regular then it admits 1-small $\mathcal{A}$-zerosets (and in turn also ω-small $\mathcal{A}$-zerosets).*
(2) *For any cardinality $\mathfrak{m}$, any $\mathfrak{m}$-isolating algebra $\mathcal{A}$ has $\mathfrak{m}$-small $\mathcal{A}$-zerosets.*
(3) *If a topological space X is first countable and admits ω-small $\mathcal{A}$-zerosets then $\mathcal{A}$ is ω-isolating.*
(4) *Any Lindelöf locally convex space admits ω-small P_f-zerosets.*

The converse to (1) is false for $P_f(E)$, where E is an infinite dimensional separable Banach space E, see [Adam, Biström, Kriegl, 1995, 5.5].
The converse to (2) is false for $P_f(\mathbb{R}^\Gamma)$ with uncountable Γ, see [Adam, Biström, Kriegl, 1995, 5.5].

Proof. (1) and (2) are obvious.

(3) Let $x \in X$ and $\mathcal{U}$ a countable neighborhood basis of x. For every $U \in \mathcal{U}$ there is a countable set $\mathcal{G}_U \subseteq \mathcal{A}$ with $\bigcap_{g\in\mathcal{G}_U} g^{-1}(g(y)) \subseteq U$. Then $\mathcal{G} := \bigcup_{U\in\mathcal{U}} \mathcal{G}_U$ is countable and

$$\bigcap_{g\in\mathcal{G}} g^{-1}(g(y)) \subseteq \bigcap_{U\in\mathcal{U}} \bigcap_{g\in\mathcal{G}_U} g^{-1}(g(y)) \subseteq \bigcap_{U\in\mathcal{U}} U = \{y\}$$

(4) Take a point x and an open set U with $x \in U \subseteq E$. For each $y \in E \setminus U$ let $p_y \in E' \subseteq P_f(E)$ with $p_y(x) = 0$ and $p_y(y) = 1$. Set $V_y := \{z \in E : p_y(z) > 0\}$. By the Lindelöf property, there is a sequence (y_n) in $E \setminus U$ such that $\{U\} \cup \{V_{y_n}\}_{n\in\mathbb{N}}$ is a cover of E. Hence for each $y \in E \setminus U$ there is some $n \in \mathbb{N}$ such that $y \in V_{y_n}$, i.e. $p_{y_n}(y) > 0 = p_{y_n}(x)$. □

19.11. Theorem. [Kriegl, Michor, 1993] *&* [Biström, Lindström, 1993b, Prop.4]. *Let $\mathfrak{m}$ be 1 or an infinite cardinal and let X be a closed subspace of $\prod_{i\in I} X_i$, let $\mathcal{A}$ be an algebra of functions on X and let $\mathcal{A}_i$ be algebras on X_i, respectively, such that $\mathrm{pr}_i^*(\mathcal{A}_i) \subseteq \mathcal{A}$ for all i.*

If each X_i admits $\mathfrak{m}$-small $\mathcal{A}_i$-zerosets then X admits $\mathfrak{m}$-small $\mathcal{A}$-zerosets.

If in addition $\varphi \in \operatorname{Hom} \mathcal{A}$ is $\mathfrak{m}$-evaluating on $\mathcal{A}$ and $\varphi_i := \varphi \circ \mathrm{pr}_i^* \in \operatorname{Hom} \mathcal{A}_i$ is evaluating on $\mathcal{A}_i$ for all i, then φ is evaluating $\mathcal{A}$ on X.

Proof. We consider $Y := \prod_i X_i$ and the algebra $\mathcal{A}_Y$ generated by $\bigcup_i\{f_i \circ \mathrm{pr}_i : f_i \in \mathcal{A}_{X_i}\}$, where $\mathrm{pr}_j : \prod_i X_i \to X_j$ denotes the canonical projection.

Now we prove the first statement for $\mathcal{A}_Y$. Let $x \in Y$ and U a neighborhood of $x = (x_i)_i$ in Y. Thus there exists a neighborhood in $\prod_i X_i$ contained in U, which we may assume to be of the form $\prod_i U_i$ with $U_i = X_i$ for all but finitely many i. Let $\mathcal{F}$ be the finite set of those exceptional i. For each $i \in \mathcal{F}$ we choose a set $\mathcal{G}_i \subseteq \mathcal{A}$ with $\bigcap_{g\in\mathcal{G}_i} g^{-1}(g(x_i)) \subseteq U_i$. Without loss of generality we may assume $g(x_i) = 0$ and $g \geq 0$ (replace g by $x \mapsto (g(x) - g(x_i))^2$). For any $g \in \prod_{i\in\mathcal{F}} \mathcal{G}_i$ we define $\tilde{g} \in \mathcal{A}_Y$ by $\tilde{g} := \sum_{i\in\mathcal{F}} g_i \circ \mathrm{pr}_i \in \mathcal{A}_Y$. Then $\tilde{g}(x) = \sum_{i\in\mathcal{F}} g_i(x) = 0$

$$\bigcap_{g\in\prod_{i\in\mathcal{F}}\mathcal{G}_i} \tilde{g}^{-1}(0) \subseteq U,$$

since for $z \notin U$ we have $z_i \notin U_i$ for at least one $i \in \mathcal{F}$. Note that $|\prod_{i\in\mathcal{F}} \mathcal{G}_i| \le \mathfrak{m}$, since $\mathfrak{m}$ is either 1 or infinite.

That $\mathcal{A}_Y$ is evaluating follows trivially since $\varphi_i := \varphi \circ \mathrm{pr}_i{}^* : \mathcal{A}_{X_i} \to \mathcal{A}_X \to \mathbb{R}$ is an algebra homomorphism and $\mathcal{A}_{X_i}$ is evaluating, so there exists a point $a_i \in X_i$ such that $\varphi(f_i \circ \mathrm{pr}_i) = (\varphi \circ \mathrm{pr}_i{}^*)(f_i) = f_i(a_i)$ for all $f_i \in \mathcal{A}_{X_i}$. Let $a := (a_i)_i$. Then obviously every $f \in \mathcal{A}_Y$ is evaluated at a.

If now X is a closed subspace of the product $Y := \prod_i X_i$ then we can apply 19.8.1 and 19.8.2. □

19.12. Theorem 19.11 is usually applied as follows. Let $\mathcal{U}$ be a zero-neighborhood basis of a locally convex space E. Then E embeds into $\prod_{U\in\mathcal{U}} \widehat{E_{(U)}}$, where $\widehat{E_{(U)}}$ denotes the completion of the Banach space $E_{(U)} := E/\ker p_U$, where p_U denotes the Minkowski functional of U. If E is complete, then this is a closed embedding, and in order to apply 19.11 we have to find an appropriate basis $\mathcal{U}$ and for each $U \in \mathcal{U}$ an algebra $\mathcal{A}_U$ on $\widehat{E_{(U)}}$, which pulls back to $\mathcal{A}$ along the canonical projections $\pi_U : E \to E_{(U)} \subseteq \widehat{E_{(U)}}$, such that the Banach space $\widehat{E_{(U)}}$ is $\mathcal{A}_U$-realcompact and has $\mathfrak{m}$-small $\mathcal{A}_U$-zerosets.

Examples.

(1) [Kriegl, Michor, 1993]. *A complete, trans-separable (i.e. contained in product of separable normed spaces) locally convex space is $\mathcal{A}$-realcompact for every 1-evaluating algebra $\mathcal{A} \supseteq \bigcup_U \pi_U^*(P_f)$.*
Note that for products of separable Banach spaces one has $C^\infty = \mathcal{C}^\infty$, see [Adam, 1993, 9.18] *&* [Kriegl, Michor, 1993].

(2) [Biström, 1993, 4.5]. *A complete, Hilbertizable (i.e. there exists a basis of Hilbert seminorms, in particular nuclear spaces) locally convex space is $\mathcal{A}$-realcompact for every 1-evaluating $\mathcal{A} \supseteq \bigcup_U \pi_U^*(P)$.*

(3) [Biström, Lindström, 1993b, Cor.3]. *Every complete non-measurable WCG locally convex space is C^∞-realcompact.*

(4) [Biström, Lindström, 1993b, Cor.5]. *Any reflexive non-measurable Fréchet space is $C^\infty = \mathcal{C}^\infty$-realcompact.*

(5) [Biström, Lindström, 1993b, Cor.4]. *Any complete non-measurable infra-Schwarz space is $\mathcal{C}^\infty$-realcompact.*

(6) [Biström, 1993, 4.16-4.18]. *Every countable coproduct of locally convex spaces, and every countable ℓ^p-sum or c_0-sum of Banach-spaces injects continuously into the corresponding product. Thus from $\mathcal{A}$ being ω-isolating and evaluating on each factor, we deduce the same for the total space by 19.1.2 if $\mathcal{A}$ is ω-evaluating on it.*

A locally convex space is usually called WCG if there exists a sequence of absolutely convex, weakly-compact subsets, whose union is dense.

Proof. (1) We have for $\widehat{E_{(U)}}$ that it is $\mathcal{A}$-realcompact for every 1-evaluating $\mathcal{A} \supseteq P$ by 18.26 and P_f is 1-isolating by 19.2 and hence has 1-small zero sets by 19.10.2.

For a product E of metrizable spaces the two algebras $C^\infty(E)$ and $\mathcal{C}^\infty(E)$ coincide: For every countable subset $\mathcal{A}$ of the index set, the corresponding product is separable and metrizable, hence C^∞-realcompact. Thus there exists a point $x_{\mathcal{A}}$ in this countable product such that $\varphi(f) = f(x_{\mathcal{A}})$ for all f which factor over the projection to that countable subproduct. Since for $\mathcal{A}_1 \subseteq \mathcal{A}_2$ the projection of $x_{\mathcal{A}_2}$

to the product over $\mathcal{A}_1$ is just $x_{\mathcal{A}_1}$ (use the coordinate projections for f), there is a point x in the product, whose projection to the subproduct with index set $\mathcal{A}$ is just $x_{\mathcal{A}}$. Every Mackey continuous function, and in particular every C^∞-function, depends only on countable many coordinates, thus factors over the projection to some subproduct with countable index set $\mathcal{A}$, hence $\varphi(f) = f(x_{\mathcal{A}}) = f(x)$. This can be shown by the same proof as for a product of factors $\mathbb{R}$ in 4.27, since the result of [Mazur, 1952] is valid for a product of separable metrizable spaces.

(2) By 19.3 we have that $\ell^2(\Gamma)$ is $\mathcal{A}$-realcompact for every 1-evaluating $\mathcal{A} \supseteq P$ and P is 1-isolating.

(3) For every U the Banach space $\widehat{E_{(U)}}$ is then WCG, hence as in 19.7.1 there is a SPRI, and by 53.20 a continuous linear injection into some $c_0(\Gamma)$. By 19.5 any ω-evaluating algebra $\mathcal{A}$ on $c_0(\Gamma)$ which contains C^∞_{lfs} is evaluating and ω-isolating. By 19.1.2 this is true for such stable algebras on $\widehat{E_{(U)}}$, and hence by 19.11 for E.

(4) Here $E_{(U)}$ embeds into $C(K)$, where $K := (U^o, \sigma(E', E''))$ is Talagrand compact [Cascales, Orihuela, 1987, theorem 12] and hence Corson compact [Negrepontis, 1984, 6.23]. Thus by 19.7.2 we have PRI. Now we proceed as in (3).

(5) Any complete infra-Schwarz space is a closed subspace of a product of reflexive and hence WCG Banach spaces, since weakly compact mappings factor over such spaces by [Jarchow, 1981, p.374]. Hence we may proceed as in (3). □

Short Exact Sequences

In the following we will consider exact sequences of locally convex spaces

$$0 \to H \xrightarrow{\iota} E \xrightarrow{\pi} F,$$

i.e. $\iota : H \to E$ is a embedding of a closed subspace and π has $\iota(H)$ as kernel. Let algebras $\mathcal{A}_H$, $\mathcal{A}_E$ and $\mathcal{A}_F$ on H, E and F be given, which satisfy $\pi^*(\mathcal{A}_F) \subseteq \mathcal{A}_E$ and $\iota^*(\mathcal{A}_E) \supseteq \mathcal{A}_H$, the latter one telling us that $\mathcal{A}_H$ functions on H can be extended to $\mathcal{A}_E$ functions on E. This is a very strong requirement, since by 21.11 not even polynomials of degree 2 on a closed subspace of a Banach space can be extended to a smooth function. The only algebra, where we have the extension property in general is that of finite type polynomials. So we will apply the following theorem in 19.14 and 19.15 to situations, where $\mathcal{A}_H$ is of quite different type then $\mathcal{A}_E$ and $\mathcal{A}_F$.

19.13. Theorem. [Adam, Biström, Kriegl, 1995, 6.1]. *Let $0 \to H \xrightarrow{\iota} E \xrightarrow{\pi} F$ be an exact sequence of locally convex spaces equipped with algebras satisfying*

(i) $\pi^*(\mathcal{A}_F) \subseteq \mathcal{A}_E$ *and* $\iota^*(\mathcal{A}_E) \supseteq \mathcal{A}_H$.
(ii) $\mathcal{A}_F$ *is ω-isolating on* F.
(iii) $\mathcal{A}_E$ *is translation invariant.*

Then we have:

(1) *If $\mathcal{A}_H$ is ω-isolating on H then $\mathcal{A}_E$ is ω-isolating on E.*
(2) *If H has ω-small $\mathcal{A}_H$-zerosets then E has ω-small $\mathcal{A}_E$-zerosets.*

If in addition

(iv) $\operatorname{Hom}_\omega \mathcal{A}_F = F$ *and* $\operatorname{Hom}_\omega \mathcal{A}_H = H$,

then we have:

(3) *If $\varphi \in \operatorname{Hom} \mathcal{A}_E$ is ω-evaluating on $\mathcal{A}_E$ then φ is evaluating on $\mathcal{A}_0 := \{f \in \mathcal{A}_E : \iota^*(f) \in \mathcal{A}_H\}$.*
(4) *If $\varphi \in \operatorname{Hom} \mathcal{A}_E$ is ω-evaluating on $\mathcal{A}_E$ and if $\mathcal{A}_H$ is ω-isolating on H then φ is evaluating on $\mathcal{A}_E$; i.e., $E = \operatorname{Hom}_\omega \mathcal{A}_E$.*

Proof. Let $x \in E$. Since $\mathcal{A}_E$ is translation invariant, we may assume $x = 0$. By (ii) there is a sequence (g_n) in $\mathcal{A}_F$ which isolates $\pi(x)$ in F, i.e. $g_n(\pi(x)) = 0$ and $\bigcap g_n^{-1}(0) = \{\pi(x)\}$.

(1) By the special assumption in 19.13.1 there exist countable many $h_n \in \mathcal{A}_H$ which isolate 0 in H. According to (i) $\pi^*(g_n) \in \mathcal{A}_E$ and there exist $\tilde{h}_n \in \mathcal{A}_E$ with $\tilde{h}_n \circ \iota = h_n$. By (iii) we have that $f_n := \tilde{h}_n(\quad - x) \in \mathcal{A}_E$. Now the functions $\pi^*(g_n)$ together with the sequence (f_n) isolate x. Indeed, if $x' \in E$ is such that $(g_n \circ \pi)(x') = (g_n \circ \pi)(x)$ for all n, then $\pi(x') = \pi(x)$, i.e. $x' - x \in H$. From $f_n(x') = f_n(x)$ we conclude that $h_n(x' - x) = \tilde{h}_n(x' - x) = f_n(x') = f_n(x) = h_n(0)$, and hence $x' = x$.

(2) Let U be a 0-neighborhood in E. By the special assumption there are countably many $h_n \in \mathcal{A}_H$ with $0 \in \bigcap_n Z(h_n) \subseteq U \cap H$. As before consider the sequence of functions $f_n := \tilde{h}_n(\quad - x)$. The common kernel of the functions in the sequences (f_n) and $(\pi^*(g_n))$ contains x and is contained in $\pi^{-1}(\pi(x)) = x + H$ and hence in $(x + U) \cap (x + H) \subseteq x + U$.

Now the remaining two statements:
Let $\varphi \in \operatorname{Hom}_\omega \mathcal{A}_E$. Then $\varphi \circ \pi^* : \mathcal{A}_F \to \mathbb{R}$ is a ω-evaluating homomorphism, and hence by (iv) given by the evaluation at a point $y \in F$. By (ii) there is a sequence (g_n) in $\mathcal{A}_F$ which isolates y. Since φ is ω-evaluating there exists a point $x \in E$, such that $g_n(y) = \varphi(\pi^*(g_n)) = \pi^*(g_n)(x) = g_n(\pi(x))$ for all n. Hence $y = \pi(x)$. Since φ obviously evaluates each countable set in $\mathcal{A}_E$ at a point in $\pi^{-1}(y) \cong K$, φ induces a ω-evaluating homomorphism $\varphi_H : \mathcal{A}_H \to \mathbb{R}$ by $\varphi_H(\iota^*(f)) := \varphi(f(\quad - x))$ for $f \in \mathcal{A}_0$. In fact let $f, \bar{f} \in \mathcal{A}_0$ with $\iota^*(f) = \iota^*(\bar{f})$. Then φ evaluates $f(\quad - x)$, $\bar{f}(\quad - x)$ and all $\pi^*(g_n)$ at some common point $\bar{x}$. So $\pi(\bar{x}) = y = \pi(x)$, hence $\bar{x} - x \in H$ and $f(\bar{x} - x) = \bar{f}(\bar{x} - x)$.
By (iv), φ_H is given by the evaluation at a point $z \in H$.

(3) Here we have that $\mathcal{A}_H = \iota^*(\mathcal{A}_0)$, and hence

$$\varphi(f) = \varphi_H(\iota^*(f(\quad + x)) = \iota^*(f(\quad + x)(z)) = f(\iota(z) + x)$$

for all $f \in \mathcal{A}_0$. So φ is evaluating on $\mathcal{A}_0$.

(4) We show that $\varphi = \delta_{\iota(z)+x}$ on $\mathcal{A}_E$. Indeed, by the special assumption there is a sequence (h_n) in $\mathcal{A}_H$ which isolates z. By (i) and (iii), we may find $f_n \in \mathcal{A}_E$ such that $h_n = \iota^*(f_n(\quad + x))$. The sequences $(\pi^*(g_n))$ and (f_n) isolate $z + x$. So let $f \in \mathcal{A}_E$ be arbitrary. Then there exists a point $z' \in E$, such that $\varphi = \delta_{z'}$ for all these functions, hence $z' = \iota(z) + x$. □

19.14. Corollary. [Adam, Biström, Kriegl, 1995, 6.3]. *Let $0 \to H \xrightarrow{\iota} E \xrightarrow{\pi} F$ be a left exact sequence of locally convex spaces and let $\mathcal{A}_F$ and $\mathcal{A}_E \supseteq E'$ be algebras on F and E, respectively, that satisfy all the assumptions (i-iv) of 19.13 not involving $\mathcal{A}_H$. Let furthermore $\varphi : \mathcal{A}_E \to \mathbb{R}$ be ω-evaluating and $\varphi \circ \pi^*$ be evaluating on $\mathcal{A}_F$. Then we have*

(1) *The homomorphism φ is $\mathcal{A}_E$-evaluating if $(H, \sigma(H, H'))$ is realcompact and admits ω-small P_f-zerosets.*
(2) *The homomorphism φ is $\mathcal{A}_0$-evaluating if $(H, \sigma(H, \iota^*(\mathcal{A}_0)))$ is Lindelöf and $\mathcal{A}_0 \subseteq \mathcal{A}_E$ is some subalgebra.*

(3) *The homomorphism φ is E'-evaluating if $(H, \sigma(H, H'))$ is realcompact.*

Proof. We will apply 19.13.3. For this we choose appropriate subalgebras $\mathcal{A}_0 \subseteq \mathcal{A}_E$ and put $\mathcal{A}_H := \iota^*(\mathcal{A}_0)$. Then (i-iii) of 19.13 is satisfied. Remains to show for (iv) that $\mathrm{Hom}_\omega(\mathcal{A}_H) = H$ in the three cases:

(1) Let $\mathcal{A}_0 := \mathcal{A}_E$. Then we have $\mathrm{Hom}_\omega(\mathcal{A}_H) = H$ by 19.9 using 18.27.

(2) If $H_{\mathcal{A}_H} = (H, \sigma(H, \mathcal{A}_H))$ is Lindelöf, then $H = \mathrm{Hom}_\omega(\mathcal{A}_H)$, by 18.24.

(3) Let $\mathcal{A}_0 := P_f(E)$. Then $\mathcal{A}_H := \iota^*(\mathcal{A}_0) = P_f(H)$ by Hahn-Banach. If H is $\sigma(H, H')$-realcompact, then $H = \mathrm{Hom}_\omega(\mathcal{A}_H)$, by 18.27. □

19.15. Theorem. [Adam, Biström, Kriegl, 1995, 6.4 and 6.5]. *Let $c_0(\Gamma) \xrightarrow{\iota} E \xrightarrow{\pi} F$ be a short exact sequence of locally convex spaces where $\mathcal{A}_E$ is translation invariant and contains $(\pi^*(\mathcal{A}_F) \cup E')^\infty_{lfs}$, and where F is $\mathcal{A}_F$-regular.*

Then $\iota^*(\mathcal{A}_E)$ contains the algebra $\mathcal{A}_{c_0(\Gamma)}$ which is generated by all functions $x \mapsto \prod_{\gamma\in\Gamma} \eta(x_\gamma)$, where $\eta \in C^\infty(\mathbb{R}, \mathbb{R})$ is 1 near 0.

If $\mathcal{A}_F$ is ω-isolating on F then $\mathcal{A}_E$ is ω-isolating on E. If in addition $F = \mathrm{Hom}_\omega \mathcal{A}_F$ and Γ is non-measurable then $E = \mathrm{Hom}_\omega \mathcal{A}_E$.

Proof. Let us show that the function $x \mapsto \prod_{\gamma\in\Gamma} \eta(x_\gamma)$ can be extended to a function in $\mathcal{A}_E$.
Remark that this product is locally finite, since $x \in c_0(\Gamma)$ and $\eta = 1$ locally around 0. Let p be an extension of the supremum norm $\| \quad \|_\infty$ on $c_0(\Gamma)$ to a continuous seminorm on E, and let $\tilde{p}$ be the corresponding quotient seminorm on F defined by $\tilde{p}(y) := \inf\{p(x) : \pi(x) = y\}$. Let furthermore ℓ_γ be a continuous linear extensions of $\mathrm{pr}_\gamma : c_0(\Gamma) \to \mathbb{R}$ which satisfy $|\ell_\gamma(x)| \le p(x)$ for all $x \in E$. Finally let $\varepsilon > 0$ be such that $\eta(t) = 1$ for $|t| \le \varepsilon$.

We show first, that for the open subset $\{x \in E : \tilde{p}(\pi(x)) < \varepsilon\}$ the product $\prod_{\gamma\in\Gamma} \eta(\ell_\gamma(x))$ is locally finite as well. So let $\tilde{p}(\pi(x)) < \varepsilon$ and $3\,\delta := \varepsilon - \tilde{p}(\pi(x))$. We claim that

$$\Gamma_x := \{\gamma : |\ell_\gamma(x)| \ge \tilde{p}(\pi(x)) + 2\delta\}$$

is finite. In fact by definition of the quotient seminorm $\tilde{p}(\pi(x)) := \inf\{p(x+y) : y \in c_0(\Gamma)\}$ there is a $y \in c_0(\Gamma)$ such that $p(x+y) \le \tilde{p}(\pi(x)) + \delta$. Since $y \in c_0(\Gamma)$ the set $\Gamma_0 := \{\gamma : |y_\gamma| \ge \delta\}$ is finite. For all $\gamma \notin \Gamma_0$ we have

$$|\ell_\gamma(x)| \le |\ell_\gamma(x+y)| + |\ell_\gamma(y)| \le p(x+y) + |y_\gamma| < \tilde{p}(\pi(x)) + 2\,\delta,$$

hence $\Gamma_x \subseteq \Gamma_0$ is finite.
Now take $z \in E$ with $p(z - x) \le \delta$. Then for $\gamma \notin \Gamma_x$ we have

$$|\ell_\gamma(z)| \le |\ell_\gamma(x)| + |\ell_\gamma(z-x)| < \tilde{p}(\pi(x)) + 2\,\delta + p(z-x) \le \tilde{p}(\pi(x)) + 3\,\delta = \varepsilon,$$

hence $\eta(\ell_\gamma(z)) = 1$ and the product is a locally finite.

In order to obtain the required extension to all of E, we choose $0 < \varepsilon' < \varepsilon$ and a function $g \in \mathcal{A}_F$ with carrier contained inside $\{z : \tilde{p}(z) \le \varepsilon'\}$ and with $g(0) = 1$. Then $f : E \to \mathbb{R}$ defined by

$$f(x) := g(\pi(x)) \prod_{\gamma\in\Gamma} \eta(\ell_\gamma(x))$$

is an extension belonging to $\langle \pi^*(\mathcal{A}_F) \cup (E')^\infty_{lfs} \rangle_{\mathrm{Alg}} \subseteq (\pi^*(\mathcal{A}_F) \cup E')^\infty_{lfs} \subseteq \mathcal{A}_E$.

Let us now show that we can find such an extension with arbitrary small carrier, and hence that E is $\mathcal{A}_E$-regular.
So let an arbitrary seminorm p on E be given. Then there exists a $\delta > 0$ such

that $\delta\, p|_{c_0(\Gamma)} \le \|\ \ \|_\infty$. Let q be an extension of $\|\ \ \|_\infty$ to a continuous seminorm on E. By replacing p with $\max\{q, \delta\, p\}$ we may assume that $p|_{c_0(\Gamma)} = \|\ \ \|_\infty$ and the unit ball of the original p contains the δ-ball of the new p. Let again $\tilde p$ be the corresponding quotient norm on F.

Then the construction above with some $0 < \varepsilon' < \varepsilon < \varepsilon'' \le \delta/3$, for $\eta \in C^\infty(\mathbb{R}, \mathbb{R})$ with $\eta(t) = 1$ for $|t| \le \varepsilon$ and $\eta(t) = 0$ for $|t| > \varepsilon'' > \varepsilon$ and $g \in C^\infty(F, \mathbb{R})$ with $\operatorname{carr}(g) \subseteq \{y \in F : \tilde p(y) \le \varepsilon' < \varepsilon\}$ gives us a function $f \in \mathcal{A}_E$ and it remains to show that the carrier of f is contained in the δ-ball of p. So let $x \in E$ be such that $f(x) \ne 0$. Then on one hand $g(\pi(x)) \ne 0$ and hence $\tilde p(\pi(x)) \le \varepsilon'$ and on the other hand $\eta(\ell_\gamma(x)) \ne 0$ for all $\gamma \in \Gamma$ and hence $|\ell_\gamma(x)| \le \varepsilon''$. We have a unique continuous linear mapping $T : \ell^1(\Gamma) \to E'$, which maps pr_γ to ℓ_γ, and satisfies $|T(y^*)(z)| \le \|y^*\|\, p(z)$ for all $z \in E$ since the unit ball of $\ell^1(\Gamma)$ is the closed absolutely convex hull of $\{\mathrm{pr}_\gamma : \gamma \in \Gamma\}$. By Hahn-Banach there is some $\ell \in E'$ be such that $|\ell(z)| \le p(z)$ for all z and $\ell(x) = p(x)$. Hence $\iota^*(\ell) = \ell|_{c_0(\Gamma)}$ is in the unit ball of $\ell^1(\Gamma)$, and hence $|T(\iota^*(\ell))(x)| \le \varepsilon''$, since $|\ell_\gamma(x)| \le \varepsilon''$. Moreover $|T(\iota^*(\ell))(z)| \le p(z)$. Then $\ell_0 := (T \circ \iota^* - 1)(\ell) = T(\ell|_{c_0(\Gamma)}) - \ell \in E'$ vanishes on $c_0(\Gamma)$ and $|\ell_0(z)| \le 2\, p(z)$ for all z. Hence $|\ell_0(x)| \le 2\, \tilde p(\pi(x)) \le 2\,\varepsilon'$. So $p(x) = |\ell(x)| \le |T(\iota^*(\ell))(x)| + |\ell_0(x)| \le \varepsilon'' + 2\,\varepsilon' < \delta$.

Because of the extension property $\mathcal{A}_{c_0(\Gamma)} \subseteq \iota^*(\mathcal{A}_E)$ and since $c_0(\Gamma)$ is $\mathcal{A}_{c_0(\Gamma)}$-regular and hence by 19.10.1 ω-isolated, we can apply 19.13.1 to obtain the statement on ω-isolatedness. The evaluating property now follows from 19.13.4 using that $\operatorname{Hom}_\omega \mathcal{A}_{c_0(\Gamma)} = c_0(\Gamma)$ by 18.30.1. □

19.16. The class c_0-ext. We shall show in 19.18 that in the short exact sequence of 19.15 we can in fact replace $c_0(\Gamma)$ by spaces from a huge class which we now define.

Definition. Let c_0*-ext* be the class of spaces H, for which there are short exact sequences $c_0(\Gamma_j) \to H_j \to H_{j+1}$ for $j = 1, ..., n$, with $|\Gamma_j|$ non-measurable, $H_{n+1} = \{0\}$ and $T : H \to H_1$ an operator whose kernel is weakly realcompact and has ω-small P_f-zerosets (By 18.18.1 and 19.2 these two conditions are satisfied, if it has for example a weak*-separable dual).

Of course all spaces which admit a continuous linear injection into some $c_0(\Gamma)$ with non-measurable Γ belong to c_0-ext. Besides these there are other natural spaces in c_0-ext. For example let K be a compact space with $|K|$ non-measurable and $K^{(\omega_0)} = \emptyset$, where ω_0 is the first infinite ordinal and $K^{(\omega_0)}$ the corresponding ω_0-th derived set. Then the Banach space $C(K)$ belongs to c_0-ext, but is in general not even injectable into some $c_0(\Gamma)$, see [Godefroy, Pelant, et. al., 1988]. In fact, from $K^{(\omega)} = \emptyset$ and the compactness of K, we conclude that $K^{(n)} = \emptyset$ for some integer n. We have the short exact sequence

$$c_0(K \setminus K^{(1)}) \cong E \xrightarrow{\iota} C(K) \xrightarrow{\pi} C(K)/E \cong C(K^{(1)}),$$

where $E := \{f \in C(K) : f|_{K^{(1)}} = 0\}$. By using 19.15 inductively the space $C(K)$ is C^∞_{lfs}-regular. Also it is again an example of a Banach space E with $E = \operatorname{Hom} C^\infty(E)$ that we are able to obtain without using the quite complicated result 16.20.1 that it admits C^∞-partition of unity.

19.17. Lemma. Pushout. [Adam, Biström, Kriegl, 1995, 6.6]. *Let a closed subspace $\iota : H \hookrightarrow E$ and a continuous linear mapping $T : H \to H_1$ of locally convex spaces be given.*

Then the pushout of ι and T is the locally convex space $E_1 := H_1 \times E/\{(Tz, -z) : z \in H\}$. The natural mapping $\iota_1 : H \to E_1$, given by $u \mapsto [(u, 0)]$ is a closed embedding and the natural mapping $T_1 : E \to E_1$ given by $T_1(x) := [(0, x)]$ is continuous and linear. Moreover, if T is a quotient mapping then so is T_1.

Given a short exact sequence $H \xrightarrow{\iota} E \xrightarrow{\pi} F$ of locally convex vector spaces and a continuous linear map $T : H \to H_1$ then we obtain by this construction a short exact sequence $H_1 \xrightarrow{\iota_1} E_1 \xrightarrow{\pi_1} F$ and a (unique) extension $T_1 : E \to E_1$ of T, with $\ker T = \ker T_1$, such that the following diagram commutes

$$\begin{tikzcd}
\ker T \arrow[r, equal] \arrow[d, hook] & \ker T_1 \arrow[r, two heads] \arrow[d, hook] & 0 \arrow[d, hook] \\
H \arrow[r, hook, "\iota"] \arrow[d, "T"] & E \arrow[r, two heads, "\pi"] \arrow[d, dotted, "T_1"] & F \arrow[d, equal] \\
H_1 \arrow[r, hook, dotted, "\iota_1"] & E_1 \arrow[r, two heads, dotted, "\pi_1"] & F
\end{tikzcd}$$

Proof. Since H is closed in E the space E_1 is a Hausdorff locally convex space. The mappings ι_1 and T_1 are clearly continuous and linear. And ι_1 is injective, since $(u, 0) \in \{(T(z), -z) : z \in H\}$ implies $0 = z$ and $u = T(z) = T(0) = 0$. In order to see that ι_1 is a topological embedding let U be an absolutely convex 0-neighborhood in H_1. Since ι is a topological embedding there is a 0-neighborhood W in E with $W \cap H = T^{-1}(U)$. Now consider the image of $U \times W \subseteq H_1 \times E$ under the quotient map $H_1 \times E \to E_1$. This is a 0-neighborhood in E_1 and its inverse image under ι_1 is contained in $2U$. Indeed, if $[(u, 0)] = [(x, z)]$ with $u \in H_1$, $x \in U$ and $z \in W$, then $x - u = T(z)$ and $z \in H \cap W$, by which $u = x - T(z) \in U - U = 2U$. Hence ι_1 embeds H_1 topologically into E_1.

We have the universal property of a pushout, since for any two continuous linear mappings $\alpha : E \to G$ and $\beta : H_1 \to G$ with $\beta \circ T = \alpha \circ \iota$, there exists a unique linear mapping $\gamma : E_1 \to G$, given by $[(u, x)] \mapsto \alpha(x) - \beta(u)$ with $\gamma \circ T_1 = \alpha$ and $\gamma \circ \iota_1 = \beta$. Since $H_1 \oplus E \to E_1$ is a quotient mapping γ is continuous as well.

Let now $\pi : E \to F$ be a continuous linear mapping with kernel H, e.g. π the natural quotient mapping $E \to F := E/H$. Then by the universal property we get a unique continuous linear $\pi_1 : E_1 \to F$ with $\pi_1 \circ T_1 = \pi$ and $\pi_1 \circ \iota_1 = 0$. We have $\iota_1(H_1) = \ker(\pi_1)$, since $0 = \pi_1[(u, z)] = \pi(z)$ if and only if $z \in H$, i.e. if and only if $[(u, z)] = [(u + Tz, 0)]$ lies in the image of ι_1. If π is a quotient map then clearly so is π_1. In particular the image of ι_1 is closed.

Since $T(x) = 0$ if and only if $[(0, x)] = [(0, 0)]$, we have that $\ker T = \ker T_1$. Assume now, in addition, that T is a quotient map. Given any $[(y, x)] \in E_1$, there is then some $z \in H$ with $T(z) = y$. Thus $T_1(x + z) = [(0, x + z)] = [(T(z), x)] = [(y, x)]$ and T_1 is onto. Remains to prove that T_1 is final, which follows by categorical reasoning. In fact let $g : E_1 \to G$ be a mapping with $g \circ T_1$ continuous and linear. Then $g \circ \iota_1 : H_1 \to G$ is a mapping with $(g \circ \iota_1) \circ T = g \circ T_1 \circ \iota$ continuous and linear and since T is final also $g \circ \iota_1$ is continuous. Thus g composed with the quotient mapping $H_1 \oplus E \to E_1$ is continuous and linear and thus also g itself. □

19.18. Theorem. [Adam, Biström, Kriegl, 1995, 6.7]. *Let $H \xrightarrow{\iota} E \xrightarrow{\pi} F$ be a short exact sequence of locally convex spaces, let F be C^∞_{lfs}-regular and let H be of class c_0-ext, see* 19.16.

If $C^\infty_{lfs}(F)$ is ω-isolating on F then $C^\infty_{lfs}(E)$ is ω-isolating on E. If, in addition, $F = \operatorname{Hom}_\omega C^\infty_{lfs}(F)$ then $E = \operatorname{Hom}_\omega C^\infty_{lfs}(E)$.

Proof. Since H is of class c_0-ext there are short exact sequences $c_0(\Gamma_j) \to H_j \to H_{j+1}$ for $j = 1, ..., n$ such that $|\Gamma_j|$ is non-measurable, $H_{n+1} = \{0\}$, and $T : H \to H_1$ is an operator whose kernel is weakly realcompact and has ω-small P_f-zerosets. We proceed by induction on the length of the resolution

$$H_0 := H \to H_1 \twoheadrightarrow \cdots \twoheadrightarrow H_{n+1} = \{0\}.$$

According to 19.17 we have for every continuous linear $T : H_j \to H_{j+1}$ the following diagram

$$\begin{tikzcd}
\ker T \arrow[r, equal] \arrow[d, hook] & \ker T_1 \arrow[r, two heads] \arrow[d, hook] & 0 \arrow[d, hook] \\
H_j \arrow[r, hook, "\iota_j"] \arrow[d, "T"] & E_j \arrow[r, two heads, "p_j"] \arrow[d, dotted, "T_1"] & F \arrow[d, equal] \\
H_{j+1} \arrow[r, hook, dotted, "\iota_{j+1}"] & E_{j+1} \arrow[r, two heads, dotted, "\pi_{j+1}"] & F
\end{tikzcd}$$

For $j > 0$ we have that $\ker T = c_0(\Gamma)$ for some none-measurable Γ, and T and T_1 are quotient mappings. So let as assume that we have already shown for the bottom row, that E_{j+1} has the required properties and is in addition C^∞_{lfs}-regular. Then by the exactness of the middle column we get the same properties for E_j using 19.15. If $j = 0$, then the kernel is by assumption weakly paracompact and admits ω-small P_f-zerosets. Thus applying 19.14.1 and 19.13.1 to the left exact middle column we get the required properties for $E = E_0$. $\square$

A Class of C^∞_{lfs}-Realcompact Locally Convex Spaces

19.19. Definition. Following [Adam, Biström, Kriegl, 1995] let RZ denote the class of all locally convex spaces E which admit ω-small C^∞_{lfs}-zerosets and have the property that $E = \operatorname{Hom}_\omega \mathcal{A}$ for each translation invariant algebra $\mathcal{A}$ with $C^\infty_{lfs}(E) \subseteq \mathcal{A} \subseteq C(E)$. In particular this applies to the algebras C, $\mathcal{C}^\infty$ and $C^\infty \cap C$.

Note that for every continuous linear $T : E \to F$ we have $T^* : C^\infty_{lfs}(F) \to C^\infty_{lfs}(E)$. In fact we have $T^*(F') \subseteq E'$, hence $T^* : (F')^\infty \to (E')^\infty$ and $T^*(\sum_i f_i)$ is again locally finite, if T is continuous and $\sum_i f_i$ is it.

A locally convex space E with ω-small C^∞_{lfs}-zerosets belongs to RZ if and only if $E = \operatorname{Hom}_\omega C^\infty_{lfs}(E) = \operatorname{Hom} C^\infty_{lfs}(E)$. In fact by 18.11 we have $\operatorname{Hom}_\omega C^\infty_{lfs}(E) = \operatorname{Hom} C^\infty_{lfs}(E)$. Now let $\mathcal{A} \supseteq C^\infty_{lfs}(E)$ and let $\varphi \in \operatorname{Hom}_\omega \mathcal{A}$ be countably evaluating. Then by 19.8.2 applied to $X = Y = E$, $\mathcal{A}_X := \mathcal{A}$ and $\mathcal{A}_Y := C^\infty_{lfs}(E)$ the homomorphism φ is evaluating on $\mathcal{A}$.

Note that by 19.10.3 for metrizable E the condition of having ω-small C^∞_{lfs}-zerosets can be replaced by C^∞_{lfs} being ω-isolating. Moreover, by 19.10.1 it is enough to assume that E is C^∞_{lfs}-regular in order that E belongs to RZ.

19.20. Proposition. *The class RZ is closed under formation of arbitrary products and closed subspaces.*

Proof. This is a direct corollary of 19.11. $\square$

19.21. Proposition. [Adam, Biström, Kriegl, 1995]. *Every locally convex space that admits a linear continuous injection into a metrizable space of class RZ is itself of class RZ.*

Proof. Use 19.1.2 and 19.10.3. □

19.22. Corollary. [Adam, Biström, Kriegl, 1995]. *The countable locally convex direct sum of a sequence of metrizable spaces in RZ belongs to RZ.*

The class of Banach spaces in RZ is closed under forming countable c_0-sums and ℓ_p-sums with $1 \le p \le \infty$.

Proof. By 19.20 the class RZ is stable under (countable) products. And 19.21 applies since a countable product of metrizable is again metrizable. □

19.23. Corollary. [Adam, Biström, Kriegl, 1995]. *Among the complete locally convex spaces the following belong to the class RZ:*

(1) *All trans-separable (i.e. subspaces of products of separable Banach spaces) locally convex spaces;*
(2) *All Hilbertizable locally convex spaces;*
(3) *All non-measurable WCG locally convex spaces;*
(4) *All non-measurable reflexive Fréchet spaces;*
(5) *All non-measurable infra-Schwarz locally convex spaces.*

Proof. By 19.20, 19.5, and 19.21 we see that every complete locally convex space E belongs to RZ, if it admits a zero-neighborhood basis $\mathcal{U}$ such that each Banach space $\widehat{E_{(U)}}$ for $U \in \mathcal{U}$ injects into some $c_0(\Gamma_U)$ with non-measurable Γ_U. Apply this to the examples 19.12.1-19.12.5. □

19.24. Proposition. [Adam, Biström, Kriegl, 1995]. *Let $0 \to H \hookrightarrow E \to F$ be an exact sequence. Let F be in RZ and let C^∞_{lfs} be ω-isolating on F.*

Then E is in RZ under any of the following assumptions.

(1) *The sequence $0 \to H \to E \to F \to 0$ is exact, H is in c_0-ext and F is C^∞_{lfs}-regular; Here it follows also that C^∞_{lfs} is ω-isolating on E.*
(2) *The sequence $0 \to H \to E \to F \to 0$ is exact, $H = c_0(\Gamma)$ for some none-measurable Γ and F is C^∞_{lfs}-regular; Here it follows also that E is C^∞_{lfs}-regular.*
(3) *The weak topology on H is realcompact and H admits ω-small P_f-zerosets.*
4 *The class c_0-ext is a subclass of RZ.*

Proof. (1) This is 19.18.

(2) follows directly from 19.15 applied to the algebra $\mathcal{A} = C^\infty_{lfs}$.

(3) By 19.13.2 the space E has ω-small C^∞_{lfs}-zerosets. By 19.14.1 we have assumption (iv) in 19.13, and then by 19.13.4 we have $E = \operatorname{Hom}_\omega(C^\infty_{lfs}(E))$. Thus E belongs to RZ.

(4) Since every space E in c_0-ext is obtained by applying finitely many constructions as in (2) and a last one as in (3) we get it for E. □

19.25. Remark. [Adam, Biström, Kriegl, 1995]. The class RZ is 'quite big'. By 19.24.4 we have that c_0-ext is a subclass of RZ. Also the following spaces are in RZ:

The space $C(K)$ where K is the one-point compactification of the topological disjoint union of a sequence of compact spaces K_n with $K_n^{(\omega)} = \emptyset$. In fact we have a continuous injection given by the countable product of the restriction maps $C(K) \to C(K_n)$. Hence the result follows from 19.24.4 using also the remark in 19.16 for the $C(K_n)$, followed by 19.20 for the product and by 19.21 for $C(K)$. Remark that in such a situation we might have $K^{(\omega)} = \{\infty\} \neq \emptyset$.

The space $D[0,1]$ of all functions $f : [0,1] \to \mathbb{R}$ which are right continuous and have left limits and endowed with the sup norm is in RZ. Indeed it contains $C[0,1]$ as a subspace and $D[0,1]/C[0,1] \cong c_0[0,1]$ according to [Corson, 1961]. By 18.27 we have that $C[0,1]$ is weakly Lindelöf and P_f is ω-isolating, since $\{\mathrm{ev}_t : t \in \mathbb{Q} \cap [0,1]\}$ are point-separating. Now we use 19.24.3.

Open Problem. Is $\ell^\infty(\Gamma)$ in RZ for $|\Gamma|$ non-measurable, i.e. is $C^\infty_{lfs}(\ell^\infty(\Gamma))$ ω-isolating on $\ell^\infty(\Gamma)$ and is $\mathrm{Hom}_\omega\, C^\infty_{lfs}(\ell^\infty(\Gamma)) = \ell^\infty(\Gamma)$?

If this is true, then every complete locally convex space E of non-measurable cardinality would be in RZ, since every Banach space E is a closed subspace of $\ell^\infty(\Gamma)$, where Γ is the closed unit-ball of E'.

20. Sets on which all Functions are Bounded

In this last section the relationship of evaluation properties and bounding sets, i.e. sets on which every function of the algebra is bounded, are studied.

20.1. Proposition. [Kriegl, Nel, 1990, 2.2]. *Let $\mathcal{A}$ be a convenient algebra, and $B \subseteq X$ be $\mathcal{A}$-bounding. Then $p_B : f \mapsto \sup\{|f(x)| : x \in B\}$ is a bounded seminorm on $\mathcal{A}$.*

A subset $B \subseteq X$ is called *$\mathcal{A}$-bounding* if $f(B) \subseteq \mathbb{R}$ is bounded for all $f \in \mathcal{A}$.

Proof. Since B is bounding, we have that $p_B(f) < \infty$. Now assume there is some bounded set $\mathcal{F} \subseteq \mathcal{A}$, for which $p_B(\mathcal{F})$ is not bounded. Then we may choose $f_n \in \mathcal{F}$, such that $p_B(f_n) \geq \sqrt{n2^n}$. Note that $\{f^2 : f \in \mathcal{F}\}$ is bounded, since multiplication is assumed to be bounded. Furthermore $p_B(f^2) = \sup\{|f(x)|^2 : x \in B\} = \sup\{|f(x)| : x \in B\}^2 = p_B(f)^2$, since $t \mapsto t^2$ is a monotone bijection $\mathbb{R}_+ \to \mathbb{R}_+$, hence $p_B(f_n^2) \geq n2^n$. Now consider the series $\sum_{n=0}^\infty \frac{1}{2^n} f_n^2$. This series is Mackey-Cauchy, since $(2^{-n})_n \in \ell^1$ and $\{f_n^2 : n \in \mathbb{N}\}$ is bounded. Since $\mathcal{A}$ is assumed to be convenient, we have that this series is Mackey convergent. Let $f \in \mathcal{A}$ be its limit. Since all summands are non-negative we have

$$p_B(f) = p_B\Big(\sum_{n=0}^\infty \frac{1}{2^n} f_n^2\Big) \geq p_B(\frac{1}{2^n} f_n^2) = \frac{1}{2^n} p_B(f_n)^2 \geq n,$$

for all $n \in \mathbb{N}$, a contradiction. □

20.2. Proposition. [Kriegl, Nel, 1990, 2.3] *for $\mathcal{A}$-paracompact,* [Biström, Bjon, Lindström, 1993, Prop.2]. *If X is $\mathcal{A}$-realcompact then every $\mathcal{A}$-bounding subset of X is relatively compact in $X_\mathcal{A}$.*

Proof. Consider the diagram

$$X_\mathcal{A} \xhookrightarrow{\cong} \operatorname{Hom}(\mathcal{A}) \hookrightarrow \prod_\mathcal{A} \mathbb{R}$$

and let $B \subseteq X$ be $\mathcal{A}$-bounding. Then its image in $\prod_\mathcal{A} \mathbb{R}$ is relatively compact by Tychonoff's theorem. Since $\operatorname{Hom}(\mathcal{A}) \subseteq \prod_\mathcal{A} \mathbb{R}$ is closed, we have that B is relatively compact in $X_\mathcal{A}$. □

20.3. Proposition. [Biström, Jaramillo, Lindström, 1995, Prop.7]. *Every function $f = \sum_{n=0}^\infty p_n \in C^\omega_{conv}(\ell^\infty)$ converges uniformly on the bounded sets in c_0. In particular, each bounded set in c_0 is C^ω_{conv}-bounding in l^∞.*

Proof. Take $f = \sum_{n=0}^\infty p_n \in C^\omega_{\text{conv}}(\ell^\infty)$. According to 7.14, the function f may be extended to a holomorphic function $\tilde f \in H(\ell^\infty \otimes \mathbb{C})$ on the complexification. [Josefson, 1978] showed that each holomorphic function on $\ell^\infty \otimes \mathbb{C}$ is bounded on every bounded set in $c_0 \otimes \mathbb{C}$. Hence, the restriction $\tilde f|_{c_0 \otimes \mathbb{C}}$ is a holomorphic function on $c_0 \otimes \mathbb{C}$ which is bounded on bounded subsets. By 7.15 its Taylor series at zero $\sum_{n=0}^\infty \widetilde{p_n}$ converges uniformly on each bounded subset of $c_0 \otimes \mathbb{C}$. The statement then follows by restricting to the bounded subsets of the real space c_0. □

20.4. Result. [Biström, Jaramillo, Lindström, 1995, Corr.8]. *Every weakly compact set in c_0, in particular the set $\{e_n : n \in \mathbb{N}\} \cup \{0\}$ with e_n the unit vectors, is RC^ω_{conv}-bounding in l^∞.*

20.5. Result. [Biström, Jaramillo, Lindström, 1995, Thm.5]. *Let $\mathcal{A}$ be a functorial algebra on the category of continuous linear maps between Banach spaces with $RP \subseteq$*

$\mathcal{A}$. Then, for every Banach space E, the $\mathcal{A}$-bounding sets are relatively compact in E if there is a function in $\mathcal{A}(\ell^\infty)$ that is unbounded on the set of unit vectors in ℓ^∞.

20.6. Result.

(1) [Biström, Jaramillo, 1994, Thm.2] *&* [Biström, 1993, p.73, Thm.5.23]. *In all Banach spaces the C^∞_{lfcs}-bounding sets are relatively compact.*
(2) [Biström, Jaramillo, 1994, p.5] *&* [Biström, 1993, p.74,Cor.5.24]. *Any C^∞_{lfcs}-bounding set in a locally convex space E is precompact and therefore relatively compact if E, in addition, is quasi-complete.*
(3) [Biström, Jaramillo, 1994, Cor.4] *&* [Biström, 1993, p.74, 5.25]. *Let E be a quasi-complete locally convex space. Then E and $E_{C^\infty_{\mathrm{lfcs}}}$ have the same compact sets. Furthermore $x_n \to x$ in E if and only if $f(x_n) \to f(x)$ for all $f \in C^\infty_{\mathrm{lfcs}}(E)$.*

Chapter V
Extensions and Liftings of Mappings

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In this chapter we will consider various extension and lifting problems. In the first section we state the problems and give several counter-examples: We consider the subspace F of all functions which vanish of infinite order at 0 in the nuclear Fréchet space $E := C^\infty(\mathbb{R},\mathbb{R})$, and we construct a smooth function on F that has no smooth extension to E, and a smooth curve $\mathbb{R} \to F'$ that has not even locally a smooth lifting along $E' \to F'$. These results are based on E. Borel's theorem which tells us that $\mathbb{R}^\mathbb{N}$ is isomorphic to the quotient E/F and the fact that this quotient map $E \to \mathbb{R}^\mathbb{N}$ has no continuous right inverse. Also the result 16.8 of [Seeley, 1964] is used which says that, in contrast to F, the subspace $\{f \in C^\infty(\mathbb{R},\mathbb{R}) : f(t) = 0 \text{ for } t \le 0\}$ of E is complemented.

In section 22 we characterize in terms of a simple boundedness condition on the difference quotients those functions $f : A \to \mathbb{R}$ on an arbitrary subset $A \subseteq \mathbb{R}$ which admit a smooth extension $\tilde{f} : \mathbb{R} \to \mathbb{R}$ as well as those which admit an m-times differentiable extension $\tilde{f}$ having locally Lipschitzian derivatives. This results are due to [Frölicher, Kriegl, 1993] and are much stronger than Whitney's extension theorem, which holds for closed subsets only and needs the whole jet and conditions on it. There is, however, up to now no analog in higher dimensions, since difference quotients are defined only on lattices.

Section 23 gives an introduction to smooth spaces in the sense of Frölicher. These are sets together with curves and functions which compose into $C^\infty(\mathbb{R},\mathbb{R})$ and determine each other by this. They are very useful for chasing smoothness of mappings which sometimes leave the realm of manifolds.

In section 23 it is shown that there exist free convenient vector spaces over Frölicher spaces, this means that to every such space X one can associate a convenient vector space λX together with a smooth map $\iota_X : X \to \lambda X$ such that for any convenient vector space E the map $(\iota_X)^* : L(\lambda X, E) \to C^\infty(X, E)$ is a bornological isomorphism. The space λX can be obtained as the c^∞-closure of the linear subspace spanned by the image of the canonical map $X \to C^\infty(X,\mathbb{R})'$. In the case where X is a finite dimensional smooth manifold we prove that the linear subspace generated by $\{\ell \circ \mathrm{ev}_x : x \in X, \ell \in E'\}$ is c^∞-dense in $C^\infty(X,E)'$. From this we conclude that the free convenient vector space over a manifold X is the space of distributions with compact support on X.

In the last 3 sections we discuss germs of smooth, holomorphic, and real analytic functions on convex sets with non-empty interior, following [Kriegl, 1997]. Let us recall the finite dimensional situation for smooth maps, so let first $E = F = \mathbb{R}$ and X be a non-trivial closed interval. Then a map $f : X \to \mathbb{R}$ is usually called smooth, if it is infinite often differentiable on the interior of X and the one-sided derivatives of all orders exist. The later condition is equivalent to the condition, that all derivatives extend continuously from the interior of X to X. Furthermore, by Whitney's extension theorem these maps turn out to be the restrictions to X of smooth functions on (some open neighborhood of X in) $\mathbb{R}$. In case where $X \subseteq \mathbb{R}$ is more general, these conditions fall apart. Now what happens if one changes to $X \subseteq \mathbb{R}^n$. For closed convex sets with non-empty interior the corresponding conditions to the one dimensional situation still agree. In case of holomorphic and real analytic maps the germ on such a subset is already defined by the values on the subset. Hence, we are actually speaking about germs in this situation. In infinite dimensions we will consider maps on just those convex subsets. So we do not claim greatest achievable generality, but rather restrict to a situation which is quite manageable. We will show that even in infinite dimensions the conditions above often coincide, and that real analytic and holomorphic maps on such sets are often germs of that class. Furthermore, we have exponential laws for all three classes, more precisely, the maps on a product correspond uniquely to maps from the first factor into the corresponding function space on the second.

21. Extension and Lifting Properties

21.1. Remark. The extension property. The general extension problem is to find an arrow $\tilde{f}$ making a diagram of the following form commutative:

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & Y \arrow[dl, dotted, "\tilde{f}"] \\ & Z & \end{tikzcd}$$

We will consider problems of this type for smooth, for real-analytic and for holomorphic mappings between appropriate spaces, e.g., Frölicher spaces as treated in section 23.

Let us first sketch a step by step approach to the general problem for the smooth mappings at hand.

If for a given mapping $i : X \to Y$ an extension $\tilde{f} : Y \to Z$ exists for all $f \in C^\infty(X, Z)$, then this says that the restriction operator $i^* : C^\infty(Y, Z) \to C^\infty(X, Z)$ is surjective.

Note that a mapping $i : X \to Y$ has the extension property for all $f : X \to Z$ with values in an arbitrary space Z if and only if i is a section, i.e. there exists a mapping $\widetilde{\mathrm{Id}_X} : Y \to X$ with $\widetilde{\mathrm{Id}_X} \circ i = \mathrm{Id}_X$. (Then $\tilde{f} := f \circ \widetilde{\mathrm{Id}_X}$ is the extension of a general mapping f).

A particularly interesting case is $Z = \mathbb{R}$. A mapping $i : X \to Y$ with the extension property for all $f : X \to \mathbb{R}$ is said to have the *scalar valued extension property*. Such a mapping is necessarily initial: In fact let $g : Z \to X$ be a mapping with $i \circ g : X \to Y$ being smooth. Then $f \circ g = \tilde{f} \circ i \circ g$ is smooth for all $f \in C^\infty(X, \mathbb{R})$ and hence g is smooth, since the functions $f \in C^\infty(X, \mathbb{R})$ generate the smooth structure on the Frölicher space X.

More generally, we consider the same question for any convenient vector space $Z = E$. Let us call this the *vector valued extension property.* Assume that we have already shown the scalar valued extension property for $i : X \to Y$, and thus we have an operator $S : C^\infty(X, \mathbb{R}) \to C^\infty(Y, \mathbb{R})$ between convenient vector spaces, which is a right inverse to $i^* : C^\infty(Y, \mathbb{R}) \to C^\infty(X, \mathbb{R})$. It is reasonable to hope that S will be linear (which can be easily checked). So the next thing would be to check, whether it is bounded. By the uniform boundedness theorem it is enough to show that $\mathrm{ev}_y \circ S : C^\infty(X, \mathbb{R}) \to C^\infty(Y, \mathbb{R}) \to Y$ given by $f \mapsto \tilde f(y)$ is smooth, and usually this is again easily checked. By dualization we get a bounded linear operator $S^* : C^\infty(Y, \mathbb{R})' \to C^\infty(X, \mathbb{R})'$ which is a left inverse to $i^{**} : C^\infty(X, \mathbb{R})' \to C^\infty(Y, \mathbb{R})'$. Now in order to solve the vector valued extension problem we use the free convenient vector space λX over a smooth space X given in 23.6. Thus any $f \in C^\infty(X, E)$ corresponds to a bounded linear $\tilde f : \lambda X \to E$. It is enough to extend $\tilde f$ to a bounded linear operator $\lambda Y \to E$ given by $\tilde f \circ S^*$. So we only need that $S^*|_{\lambda Y}$ has values in λX, or equivalently, that $S^* \circ \delta_Y : Y \to C^\infty(Y, \mathbb{R})' \to C^\infty(X, \mathbb{R})'$, given by $y \mapsto (f \mapsto \tilde f(y))$, has values in λX. In the important cases (e.g. finite dimensional manifolds X), where $\lambda X = C^\infty(X, \mathbb{R})'$, this is automatically satisfied. Otherwise it is by the uniform boundedness principle enough to find for given $y \in Y$ a bounding sequence $(x_k)_k$ in X (i.e. every $f \in C^\infty(X, \mathbb{R})$ is bounded on $\{x_k : k \in \mathbb{N}\}$) and an absolutely summable sequence $(a_k)_k \in \ell^1$ such that $\tilde f(y) = \sum_k a_k f(x_k)$ for all $f \in C^\infty(X, \mathbb{R})$. Again we can hope that this can be achieved in many cases.

21.2. Proposition. *Let $i : X \to Y$ be a smooth mapping, which satisfies the vector valued extension property. Then there exists a bounded linear extension operator $C^\infty(X, E) \to C^\infty(Y, E)$.*

Proof. Since i is smooth, the mapping $i^* : C^\infty(Y, E) \to C^\infty(X, E)$ is a bounded linear operator between convenient vector spaces. Its kernel is $\ker(i^*) = \{f \in C^\infty(Y, E) : f \circ i = 0\}$. And we have to show that the sequence

$$0 \longrightarrow \ker(i^*) \hookrightarrow C^\infty(Y, E) \xrightarrow{i^*} C^\infty(X, E) \longrightarrow 0$$

splits via a bounded linear operator $\sigma : C^\infty(X, E) \ni f \mapsto \tilde f \in C^\infty(Y, E)$, i.e. a bounded linear extension operator.

By the exponential law 3.13 a mapping $\sigma \in L(C^\infty(X, E), C^\infty(Y, E))$ would correspond to $\tilde\sigma \in C^\infty(Y, L(C^\infty(X, E), E))$ and $\sigma \circ i^* = \mathrm{Id}$ translates to $\tilde\sigma \circ i = \widetilde{\mathrm{Id}} = \delta : X \to L(C^\infty(X, E), E)$, given by $x \mapsto (f \mapsto f(x))$, i.e. $\tilde\sigma$ must be a solution of the following vector valued extension problem:

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "\delta"'] & & Y \arrow[dl, dotted, "\tilde\sigma"] \\ & L(C^\infty(X, E), E) & \end{tikzcd}$$

By the vector valued extension property such a $\tilde\sigma$ exists. □

21.3. The lifting property. Dual to the extension problem, we have the lifting problem, i.e. we want to find an arrow $\tilde f$ making a diagram of the following form commutative:

$$\begin{tikzcd} X & & Y \arrow[ll, "p"'] \\ & Z \arrow[ul, "f"] \arrow[ur, dotted, "\tilde f"'] & \end{tikzcd}$$

Note that in this situation it is too restrictive to search for a bounded linear or even just a smooth lifting operator $T : C^\infty(Z, X) \to C^\infty(Z, Y)$. If such an operator exists for some $Z \neq \emptyset$, then $p : Y \to X$ has a smooth right inverse namely the dashed arrow in the following diagram:

$$\begin{tikzcd}
X \arrow[rr, "\mathrm{Id}"] \arrow[dr, dashed] \arrow[ddd, "\mathrm{const}^*"'] & & X \\
& Y \arrow[ur, "p"'] & \\
& C^\infty(Z,Y) \arrow[u, "\mathrm{ev}_z"'] \arrow[dr, "p_*"] & \\
C^\infty(Z,X) \arrow[ur, "T"'] \arrow[rr, "\mathrm{Id}"] & & C^\infty(Z,X) \arrow[uuu, "\mathrm{ev}_z"']
\end{tikzcd}$$

Again the first important case is, when $Z = \mathbb{R}$. If X and Y are even convenient vector spaces, then we know that the image of a convergent sequence $t_n \to t$ under a smooth curve $c : \mathbb{R} \to Y$ is Mackey convergent. And since one can find by the general curve lemma a smooth curve passing through sufficiently fast falling subsequences of a Mackey convergent sequence, the first step could be to check whether such sequences can be lifted. If bounded sets (or at least sequences) can be lifted, then the same is true for Mackey convergent sequences. However, this is not always true as we will show in 21.9.

21.4. Remarks. The scalar valued extension property for bounded linear mappings on a c^∞-dense linear subspace is true if and only if the embedding represents the c^∞-completion by 4.30. In this case it even has the vector valued extension property by 4.29.

That in general bounded linear functionals on a (dense or c^∞-closed subspace) may not be extended to bounded (equivalently, smooth) linear functionals on the whole space was shown in 4.36.6.

The scalar valued extension problem is true for the c^∞-closed subspace of an uncountable product formed by all points with countable support, see 4.27 (and 4.12). As a consequence this subspace is not smoothly real compact, see 17.5.

Let E be not smoothly regular and U be a corresponding 0-neighborhood. Then the closed subset $X := \{0\} \cup (E \setminus U) \subseteq Y := E$ does not have the extension property for the smooth mapping $f = \chi_{\{0\}} : X \to \mathbb{R}$.

Let E be not smoothly normal and A_0, A_1 be the corresponding closed subsets. Then the closed subset $X := A_1 \cup A_2 \subseteq Y := E$ does not have the extension property for the smooth mapping $f = \chi_{A_1} : X \to \mathbb{R}$.

If $q : E \to F$ is a quotient map of convenient vector spaces one might expect that for every smooth curve $c : \mathbb{R} \to F$ there exists (at least locally) a smooth lifting, i.e. a smooth curve $c : \mathbb{R} \to E$ with $q \circ c = c$. And if $\iota : F \to E$ is an embedding of a convenient subspace one might expect that for every smooth function $f : F \to \mathbb{R}$ there exists a smooth extension to E. In this section we give examples showing that both properties fail. As convenient vector spaces we choose spaces of smooth real functions and their duals. We start with some lemmas.

21.5. Lemma. *Let $E := C^\infty(\mathbb{R},\mathbb{R})$, let $q : E \to \mathbb{R}^\mathbb{N}$ be the infinite jet mapping at 0, given by $q(f) := (f^{(n)}(0))_{n\in\mathbb{N}}$, and let $F \xrightarrow{\iota}$ be the kernel of q, consisting of all smooth functions which are flat of infinite order at 0.*

Then the following sequence is exact:

$$0 \to F \xrightarrow{\iota} E \xrightarrow{q} \mathbb{R}^\mathbb{N} \to 0.$$

Moreover, $\iota^* : E' \to F'$ is a quotient mapping between the strong duals. Every bounded linear mapping $s : \mathbb{R}^\mathbb{N} \to E$ the composite $q\circ s$ factors over $pr_N : \mathbb{R}^\mathbb{N} \to \mathbb{R}^N$ for some $N \in \mathbb{N}$, and so the sequence does not split.

Proof. The mapping $q : E \to \mathbb{R}^\mathbb{N}$ is a quotient mapping by the open mapping theorem [**5**, 5.3.5] & [**5**, 5.3.3], since both spaces are Fréchet and q is surjective by Borel's theorem 15.4. The inclusion ι is an embedding of Fréchet spaces, so the adjoint ι^* is a quotient mapping for the strong duals 52.28. Note that these duals are bornological by 52.29.

Let $s : \mathbb{R}^\mathbb{N} \to E$ be an arbitrary bounded linear mapping. Since $\mathbb{R}^\mathbb{N}$ is bornological s has to be continuous. The set $U := \{g \in E : |g(t)| \le 1 \text{ for } |t| \le 1\}$ is a 0-neighborhood in the locally convex topology of E. So there has to exist an $N \in \mathbb{N}$ such that $s(V) \subseteq U$ with $V := \{x \in \mathbb{R}^\mathbb{N} : |x_n| < \frac{1}{N} \text{ for all } n \le N\}$. We show that $q\circ s$ factors over $\mathbb{R}^N$. So let $x \in \mathbb{R}^\mathbb{N}$ with $x_n = 0$ for all $n \le N$. Then $k\cdot x \in V$ for all $k \in \mathbb{N}$, hence $k\cdot s(x) \in U$, i.e. $|s(x)(t)| \le \frac{1}{k}$ for all $|t| \le 1$ and $k \in \mathbb{N}$. Hence $s(x)(t) = 0$ for $|t| \le 1$ and therefore $q(s(x)) = 0$.

Suppose now that there exists a bounded linear mapping $\rho : E \to F$ with $\rho\circ\iota = \mathrm{Id}_F$. Define $s(q(x)) := x - \iota\rho x$. This definition makes sense, since q is surjective and $q(x) = q(x')$ implies $x - x' \in F$ and thus $x - x' = \rho(x - x')$. Moreover s is a bounded linear mapping, since q is a quotient map, as surjective continuous map between Fréchet spaces; and $(q\circ s)(q(x)) = q(x) - q(\iota(\rho(x))) = q(x) - 0$. □

21.6. Proposition. [**11**, 7.1.5] *Let $\iota^* : E' \to F'$ the quotient map of 21.5. The curve $c : \mathbb{R} \to F'$ defined by $c(t) := \mathrm{ev}_t$ for $t \ge 0$ and $c(t) = 0$ for $t < 0$ is smooth but has no smooth lifting locally around 0. In contrast, bounded sets and Mackey convergent sequences are liftable.*

Proof. By the uniform boundedness principle 5.18 c is smooth provided $\mathrm{ev}_f \circ c : \mathbb{R} \to \mathbb{R}$ is smooth for all $f \in F$. Since $(\mathrm{ev}_f \circ c)(t) = f(t)$ for $t \ge 0$ an $(\mathrm{ev}_f \circ c)(t) = 0$ for $t \le 0$ this obviously holds.

Assume first that there exists a global smooth lifting of c, i.e. a smooth curve $e : \mathbb{R} \to E'$ with $\iota^* \circ e = c$. By exchanging the variables, c corresponds to a bounded linear mapping $\tilde c : F \to E$ and e corresponds to a bounded linear mapping $\tilde e : E \to E$ with $\tilde e \circ \iota = \tilde c$. The curve c was chosen in such a way that $\tilde c(f)(t) = f(t)$ for $t \ge 0$ and $\tilde c(f)(t) = 0$ for $t \le 0$.

We show now that such an extension $\tilde e$ of $\tilde c$ cannot exist. In 16.8 we have shown the existence of a retraction s to the embedding of the subspace $F_+ := \{f \in F : f(t) = 0 \text{ for } t \le 0\}$ of E. For $f \in F$ one has $s(\tilde e(f)) = s(\tilde c(f)) = \tilde c(f)$ since $\tilde c(E) \subseteq F_+$. Now let $\Psi : E \to E$, $\Psi(f)(t) := f(-t)$ be the reflection at 0. Then $\Psi(F) \subseteq F$ and $f = \tilde c(f) + \Psi(\tilde c(\Psi(f)))$ for $f \in F$. We claim that $\rho := s\circ\tilde e + \Psi\circ s\circ\tilde e\circ\Psi : E \to F$ is a retraction to the inclusion, and this is a contradiction with 21.5. In fact

$$\rho(f) = (s\circ\tilde e)(f) + (\Psi\circ s\circ\tilde e\circ\Psi)(f) = \tilde c(f) + \Psi(\tilde c(\Psi(f))) = f$$

for all $f \in F$. So we have proved that c has no global smooth lifting.

Assume now that $c|_I$ has a smooth lifting $e_0 : U \to E'$ for some open neighborhood I of 0. Trivially $c|_{\mathbb{R}\setminus\{0\}}$ has a smooth lifting e_1 defined by the same formula as c. Take now a smooth partition $\{f_0, f_1\}$ of the unity subordinated to the open covering $\big\{(-\varepsilon,\varepsilon), \mathbb{R}\setminus\{0\}\big\}$ of $\mathbb{R}$, i.e. $f_0 + f_1 = 1$ with $\operatorname{supp}(f_0) \subseteq (-\varepsilon,\varepsilon)$ and $0 \notin \operatorname{supp}(f_1)$. Then $f_0 e_0 + f_1 e_1$ gives a global smooth lifting of c, in contradiction with the case treated above.

Let now $B \subseteq F'$ be bounded. Without loss of generality we may assume that $B = U^o$ for some 0-neighborhood U in F. Since F is a subspace of the Fréchet space E, the set U can be written as $U = F \cap V$ for some 0-neighborhood V in E. Then the bounded set $V^o \subseteq E'$ is mapped onto $B = U^o$ by the Hahn-Banach theorem. □

21.7. Proposition. [11, 7.1.7] *Let $\iota : F \to E$ be as in 21.5. The function $\varphi : F \to \mathbb{R}$ defined by $\varphi(f) := f(f(1))$ for $f(1) \geq 0$ and $\varphi(f) := 0$ for $f(1) < 0$ is smooth but has no smooth extension to E and not even to a neighborhood of F in E.*

Proof. We first show that φ is smooth. Using the bounded linear $\tilde{c} : F \to E$ associated to the smooth curve $c : \mathbb{R} \to F'$ of 21.6 we can write φ as the composite $\operatorname{ev} \circ (\tilde{c}, \operatorname{ev}_1)$ of smooth maps.

Assume now that a smooth global extension $\psi : E \to \mathbb{R}$ of φ exists. Using a fixed smooth function $h : \mathbb{R} \to [0,1]$ with $h(t) = 0$ for $t \leq 0$ and $h(t) = 1$ for $t \geq 1$, we then define a map $\sigma : E \to E$ as follows:

$$(\sigma g)(t) := \psi\big(g + \big(t - g(1)\big)h\big) - \big(t - g(1)\big)h(t).$$

Obviously $\sigma g \in E$ for any $g \in E$, and using cartesian closedness 3.12 one easily verifies that σ is a smooth map. For $f \in F$ one has, using that $\big(f + \big(t - f(1)\big)h\big)(1) = t$, the equations

$$(\sigma f)(t) = \big(f + \big(t - f(1)\big)h\big)(t) - \big(t - f(1)\big)h(t) = f(t)$$

for $t \geq 0$ and $(\sigma f)(t) = 0 - (t - f(1))h(t) = 0$ for $t \leq 0$. This means $\sigma f = \tilde{c} f$ for $f \in F$. So one has $\tilde{c} = \sigma \circ \iota$ with σ smooth. Differentiation gives $\tilde{c} = \tilde{c}'(0) = \sigma'(0) \circ \iota$, and $\sigma'(0)$ is a bounded linear mapping $E \to E$. But in the proof of 21.6 it was shown that such an extension of $\tilde{c}$ does not exist.

Let us now assume that a local extension to some neighborhood of F in E exists. This extension could then be multiplied with a smooth function $E \to \mathbb{R}$ being 1 on F and having support inside the neighborhood (E as nuclear Fréchet space has smooth partitions of unity see 16.10) to obtain a global extension. □

21.8. Remark. As a corollary it is shown in [Frölicher, Kriegl, 1988, 7.1.6] that the category of smooth spaces is not locally cartesian closed, since pullbacks do not commute with coequalizers.

Furthermore, this examples shows that the structure curves of a quotient of a Frölicher space need not be liftable as structure curves and the structure functions on a subspace of a Frölicher space need not be extendable as structure functions.

In fact, since Mackey-convergent sequences are liftable in the example, one can show that every $f : F' \to \mathbb{R}$ is smooth, provided $f \circ \iota^*$ is smooth, see [Frölicher,Kriegl, 1988, 7.1.8].

21.9. Example. In [Jarchow, 1981, 11.6.4] a Fréchet Montel space is given, which has ℓ^1 as quotient. The standard basis in ℓ^1 cannot have a bounded lift, since in a Montel space every bounded set is by definition relatively compact, hence the standard basis would be relatively compact.

21.10. Result. [Jarchow, 1981, remark after 9.4.5]. *Let $q : E \to F$ be a quotient map between Fréchet spaces. Then (Mackey) convergent sequences lift along q.*

This is not true for general spaces. In [11, 7.2.10] it is shown that the quotient map $\coprod_{\mathrm{dens}\, A=0} \mathbb{R}^A \to E := \{x \in \mathbb{R}^{\mathbb{N}} : \mathrm{dens}(\mathrm{carr}(x)) = 0\}$ does not lift Mackey-converging sequences. Note, however, that this space is not convenient. We do not know whether smooth curves can be lifted over quotient mappings, even in the case of Banach spaces.

21.11. Example. *There exists a short exact sequence $\ell^2 \xrightarrow{\iota} E \to \ell^2$, which does not split, see 13.18.6. The square of the norm on the subspace ℓ^2 does not extend to a smooth function on E.*

Proof. Assume indirectly that a smooth extension of the square of the norm exists. Let $2b$ be the second derivative of this extension at 0, then $b(x, y) = \langle x, y\rangle$ for all $x, y \in \ell^2$, and hence the following diagram commutes

$$\begin{tikzcd} \ell^2 \arrow[r, hook, "\iota"] \arrow[d, "\cong"', "\sharp"] & E \arrow[d, "b^\vee"] \\ (\ell^2)^* & E^* \arrow[l, two heads, "\iota^*"'] \end{tikzcd}$$

giving a retraction to ι. □

22. Whitney's Extension Theorem Revisited

Whitney's extension theorem [Whitney, 1934] concerns extensions of jets and not of functions. In particular it says, that a real-valued function f from a closed subset $A \subseteq \mathbb{R}$ has a smooth extension if and only if there exists a (not uniquely determined) sequence $f_n : A \to \mathbb{R}$, such that the formal Taylor series satisfies the appropriate remainder conditions, see 22.1. Following [Frölicher, Kriegl, 1993], we will characterize in terms of a simple boundedness condition on the difference quotients those functions $f : A \to \mathbb{R}$ on an arbitrary subset $A \subseteq \mathbb{R}$ which admit a smooth extension $\tilde f : \mathbb{R} \to \mathbb{R}$ as well as those which admit an m-times differentiable extension $\tilde f$ having locally Lipschitzian derivatives.

We shall use Whitney's extension theorem in the formulation given in [Stein, 1970]. In order to formulate it we recall some definitions.

22.1. Notation on jets. An *m-jet* on A is a family $F = (F^k)_{k\le m}$ of continuous functions on A. With $J^m(A, \mathbb{R})$ one denotes the vector space of all m-jets on A.

The canonical map $j^m : C^\infty(\mathbb{R}, \mathbb{R}) \to J^m(A, \mathbb{R})$ is given by $f \mapsto (f^{(k)}|_A)_{k\le m}$.

For $k \le m$ one has the *'differentiation operator'* $D^k : J^m(A, \mathbb{R}) \to J^{m-k}(A, \mathbb{R})$ given by $D^k : (F^i)_{i\le m} \mapsto (F^{i+k})_{i\le m-k}$.

For $a \in A$ the *Taylor-expansion operator* $T^m_a : J^m(A, \mathbb{R}) \to C^\infty(\mathbb{R}, \mathbb{R})$ of order m at a is defined by $T^m_a((F^i)_{i\le m}) : x \mapsto \sum_{k\le m} \frac{(x-a)^k}{k!} F^k(a)$.

Finally the *remainder operator* $R^m_a : J^m(A, \mathbb{R}) \to J^m(A, \mathbb{R})$ at a of order m is given by $F \mapsto F - j^m(T^m_a F)$.

In [Stein, 1970, p.176] the space $\mathcal{L}\mathrm{ip}(m+1,A)$ denotes all m-jets on A for which there exists a constant $M>0$ such that

$$|F^j(a)| \le M \text{ and } \left|(R_a^m F)^j(b)\right| \le M\,|a-b|^{m+1-j}$$

for all $a,b\in A$ and all $j\le m$.

The smallest constant M defines a norm on $\mathcal{L}\mathrm{ip}(m+1,A)$.

22.2. Whitney's Extension. The construction of Whitney for finite order m goes as follows, see [Malgrange, 1966], [Tougeron, 1972] or [Stein, 1970]:

First one picks a special partition of unity Φ for $\mathbb{R}^n\setminus A$ satisfying in particular $\operatorname{diam}(\operatorname{supp}\varphi)\le 2\,d(\operatorname{supp}\varphi,A)$ for $\varphi\in\Phi$. For every $\varphi\in\Phi$ one chooses a nearest point $a_\varphi\in A$, i.e. a point a_φ with $d(\operatorname{supp}\varphi,A)=d(\operatorname{supp}\varphi,a_\varphi)$. The extension $\tilde F$ of the jet F is then defined by

$$\tilde F(x) := \begin{cases} F^0(x) & \text{for } x\in A\\ \sum_{\varphi\in\Phi'}\varphi(x)T^m_{a_\varphi}F(x) & \text{otherwise,}\end{cases}$$

where the set Φ' consists of all $\varphi\in\Phi$ such that $d(\operatorname{supp}\varphi,A)\le 1$.

The version of [Stein, 1970, theorem 4, p. 177] of Whitney's extension theorem is:

Whitney's Extension Theorem. *Let m be an integer and A a compact subset of $\mathbb{R}$. Then the assignment $F\mapsto\tilde F$ defines a bounded linear mapping $\mathcal{E}^m:\mathcal{L}\mathrm{ip}(m+1,A)\to\mathcal{L}\mathrm{ip}(m+1,\mathbb{R})$ such that $\mathcal{E}^m(F)|_A=F^0$.*

In order that $\mathcal{E}^m$ makes sense, one has to identify $\mathcal{L}\mathrm{ip}(m+1,\mathbb{R})$ with a space of functions (and not jets), namely those functions on $\mathbb{R}$ which are m-times differentiable on $\mathbb{R}$ and the m-th derivative is Lipschitzian. In this way $\mathcal{L}\mathrm{ip}(m+1,\mathbb{R})$ is identified with the space $\mathcal{L}\mathrm{ip}^m(\mathbb{R},\mathbb{R})$ in 1.2 (see also 12.10).

Remark. The original condition of [Whitney, 1934] which guarantees a C^m-extension is:

$$(R_a^m F)^k(b) = o(|a-b|^{m-k}) \text{ for } a,b\in A \text{ with } |a-b|\to 0 \text{ and } k\le m.$$

In the following A will be an arbitrary subset of $\mathbb{R}$.

22.3. Difference Quotients. The definition of difference quotients $\delta^k f$ given in 12.4 works also for functions $f:A\to\mathbb{R}$ defined on arbitrary subsets $A\subseteq\mathbb{R}$. The natural domain of definition of $\delta^k f$ is the subset $A^{<k>}$ of A^{k+1} of pairwise distinct points, i.e.

$$A^{<k>} := \left\{(t_0,\dots,t_k)\in A^{k+1} : t_i\ne t_j \text{ for all } i\ne j\right\}.$$

The following product rule can be found for example in [Verde-Star, 1988] or [Frölicher, Kriegl, 1993, 3.3].

22.4. The Leibniz product rule for difference quotients.

$$\delta^k(f\cdot g)\,(t_0,\dots,t_k) = \sum_{i=0}^k \binom{k}{i}\,\delta^i f(t_0,\dots,t_i)\cdot\delta^{k-i}g(t_i,\dots,t_k)$$

Proof. This is easily proved by induction on k. $\square$

We will make strong use of interpolation polynomials as they have been already used in the proof of lemma 12.4. The following descriptions are valid for them:

22.5. Lemma. Interpolation polynomial. *Let $f : A \to E$ be a function with values in a vector space E and let $(t_0, \dots, t_m) \in A^{<m>}$. Then there exists a unique polynomial $P^m_{(t_0,\dots,t_m)} f$ of degree at most m which takes the values $f(t_j)$ on t_j for all $j = 0, \dots, m$. It can be written in the following ways:*

$$P^m_{(t_0,\dots,t_m)} f : t \mapsto \sum_{k=0}^{m} \frac{1}{k!} \delta^k f(t_0,\dots,t_k) \prod_{j=0}^{k-1} (t - t_j) \qquad (Newton)$$

$$t \mapsto \sum_{k=0}^{m} f(t_k) \prod_{j \neq k} \frac{t - t_j}{t_k - t_j} \qquad (Lagrange).$$

See, for example, [11, 1.3.7] for a proof of the first description. The second one is obvious.

22.6. Lemma. *For pairwise distinct points $a, b, t_1, \dots, t_m$ and $k \le m$ one has:*

$$\begin{aligned}
&\Big(P^m_{(a,t_1,\dots,t_m)} f - P^m_{(b,t_1,\dots,t_m)} f\Big)^{(k)}(t) = \\
&\quad = (a-b) \tfrac{1}{(m+1)!}\, \delta^{m+1} f(a,b,t_1,\dots,t_m) \cdot \\
&\qquad \cdot k! \sum_{i_1 < \dots < i_k} (t - t_1) \cdot \dots \cdot \widehat{(t - t_{i_1})} \cdot \dots \cdot \widehat{(t - t_{i_k})} \cdot \dots \cdot (t - t_m).
\end{aligned}$$

Proof. For the interpolation polynomial we have

$$\begin{aligned}
&P^m_{(a,t_1,\dots,t_m)} f(t) = P^m_{(t_1,\dots,t_m,a)} f(t) = \\
&\quad = f(t_1) + \dots + (t - t_1) \cdot \dots \cdot (t - t_{m-1}) \tfrac{1}{(m-1)!}\, \delta^{m-1} f\,(t_1,\dots,t_m) \\
&\qquad\qquad + (t - t_1) \cdot \dots \cdot (t - t_m) \tfrac{1}{m!}\, \delta^m f\,(t_1,\dots,t_m,a).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&P^m_{(a,t_1,\dots,t_m)} f(t) - P^m_{(b,t_1,\dots,t_m)} f(t) = \\
&\quad = 0 + \dots + 0 + (t - t_1) \cdot \dots \cdot (t - t_m) \tfrac{1}{m!}\, \delta^m f\,(t_1,\dots,t_m,a) \\
&\qquad\qquad - (t - t_1) \cdot \dots \cdot (t - t_m) \tfrac{1}{m!}\, \delta^m f\,(t_1,\dots,t_m,b) \\
&\quad = (t - t_1) \cdot \dots \cdot (t - t_m) \tfrac{1}{m!} \tfrac{a-b}{m+1}\, \delta^{m+1} f\,(t_1,\dots,t_m,a,b) \\
&\quad = (a-b) \cdot (t - t_1) \cdot \dots \cdot (t - t_m) \tfrac{1}{(m+1)!}\, \delta^{m+1} f\,(a,b,t_1,\dots,t_m).
\end{aligned}$$

Differentiation using the product rule 22.4 gives the result. □

22.7. Proposition. *Let $f : A \to \mathbb{R}$ be a function, whose difference quotient of order $m+1$ is bounded on $A^{<m+1>}$. Then the approximation polynomial $P^m_{\mathbf{a}} f$ converges to some polynomial denoted by $P^m_{\mathbf{x}} f$ of degree at most m if the point $\mathbf{a} \in A^{<m>}$ converges to $\mathbf{x} \in A^{m+1}$.*

Proof. We claim that $P^m_{\mathbf{a}} f$ is a Cauchy net for $A^{<m>} \ni \mathbf{a} \to \mathbf{x}$. Since $P^m_{\mathbf{a}} f$ is symmetric in the entries of $\mathbf{a}$ we may assume without loss of generality that the entries x_j of $\mathbf{x}$ satisfy $x_0 \le x_1 \le \dots \le x_m$. For a point $\mathbf{a} \in A^{<m>}$ which is close to $\mathbf{x}$ and any two coordinates i and j with $x_i < x_j$ we have $a_i < a_j$. Let $\mathbf{a}$ and $\mathbf{b}$ be two points close to $\mathbf{x}$. Let J be a set of indices on which $\mathbf{x}$ is constant. If the set $\{a_j : j \in J\}$ differs from the set $\{b_j : j \in J\}$, then we may order them as in the proof of lemma 12.4 in such a way that $a_i \neq b_j$ for $i \le j$ in J. If the two sets are equal we order both strictly increasing and thus have $a_i < a_j = b_j$ for $i < j$ in J. Since $\mathbf{x}$ is constant on J the distance $|a_i - b_j| \le |a_i - x_i| + |x_j - b_j|$ goes to zero as $\mathbf{a}$ and $\mathbf{b}$ approach $\mathbf{x}$. Altogether we obtained that $a_i \neq b_j$ for all $i < j$ and applying now 22.6 for $k = 0$ inductively one obtains:

$$\begin{aligned}
&P^m_{(a_0,\dots,a_m)} f(t) - P^m_{(b_0,\dots,b_m)} f(t) = \\
&\qquad = \sum_{j=0}^{m} \Big(P^m_{(a_0,\dots,a_{j-1},b_j,\dots,b_m)} f(t) - P^m_{(a_0,\dots,a_j,b_{j+1},\dots,b_m)} f(t) \Big) \\
&\qquad = \sum_{j=0}^{m} (a_j - b_j)(t-a_0)\dots(t-a_{j-1})(t-b_{j+1})\dots(t-b_m)\cdot \\
&\qquad\qquad\qquad \cdot \tfrac{1}{(m+1)!}\delta^{m+1} f(a_0,\dots,a_j,b_j,\dots,b_m).
\end{aligned}$$

Where those summands with $a_j = b_j$ have to be defined as 0. Since $a_j - b_j \to 0$ the claim is proved and thus also the convergence of $P^m_{\mathbf{a}} f$. □

22.8. Definition of $\mathcal{L}\mathrm{ip}^k$ function spaces. Let E be a convenient vector space, let A be a subset of $\mathbb{R}$ and k a natural number or 0. Then we denote with $\mathcal{L}\mathrm{ip}^k_{\mathrm{ext}}(A,E)$ the vector space of all maps $f : A \to E$ for which the difference quotient of order $k+1$ is bounded on bounded subsets of $A^{<k>}$. As in 12.10 – but now for arbitrary subsets $A \subseteq \mathbb{R}$ – we put on this space the initial locally convex topology induced by $f \mapsto \delta^j f \in \ell^\infty(A^{\langle j\rangle}, E)$ for $0 \le j \le k+1$, where the spaces ℓ^∞ carry the topology of uniform convergence on bounded subsets of $A^{\langle j\rangle} \subseteq \mathbb{R}^{j+1}$.

In case where $A = \mathbb{R}$ the elements of $\mathcal{L}\mathrm{ip}^k_{\mathrm{ext}}(A,\mathbb{R})$ are exactly the k-times differentiable functions on $\mathbb{R}$ having a locally Lipschitzian derivative of order $k+1$ and the locally convex space $\mathcal{L}\mathrm{ip}^k_{\mathrm{ext}}(A,\mathbb{R})$ coincides with the convenient vector space $\mathcal{L}\mathrm{ip}^k(\mathbb{R},\mathbb{R})$ studied in section 12.

If k is infinite, then $\mathcal{L}\mathrm{ip}^\infty_{\mathrm{ext}}(A,E)$ or alternatively $C^\infty_{\mathrm{ext}}(A,E)$ denotes the intersection of $\mathcal{L}\mathrm{ip}^j_{\mathrm{ext}}(A,E)$ for all finite j.

If $A = \mathbb{R}$ then the elements of $C^\infty_{\mathrm{ext}}(\mathbb{R},\mathbb{R})$ are exactly the smooth functions on $\mathbb{R}$ and the space $C^\infty_{\mathrm{ext}}(\mathbb{R},\mathbb{R})$ coincides with the usual Fréchet space $C^\infty(\mathbb{R},\mathbb{R})$ of all smooth functions.

22.9. Proposition. Uniform boundedness principle for $\mathcal{L}\mathrm{ip}^k_{\mathrm{ext}}$. *For any finite or infinite k and any convenient vector space E the space $\mathcal{L}\mathrm{ip}^k_{\mathrm{ext}}(A,E)$ is also convenient. It carries the initial structure with respect to*

$$\ell_* : \mathcal{L}\mathrm{ip}^k_{\mathrm{ext}}(A,E) \to \mathcal{L}\mathrm{ip}^k_{\mathrm{ext}}(A,\mathbb{R}) \text{ for } \ell \in E'.$$

Moreover, it satisfies the $\{\mathrm{ev}_x : x \in A\}$-uniform boundedness principle. If E is Fréchet then so is $\mathcal{L}\mathrm{ip}^k_{\mathrm{ext}}(A,E)$.

Proof. We consider the commutative diagram

$$\begin{tikzcd}
\mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A,E) \arrow[r, "\ell_*"] \arrow[d, "\delta^j"'] & \mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A,\mathbb{R}) \arrow[d, "\delta^j"] \\
\ell^\infty(A^{<j>},E) \arrow[r, "\ell_*"] & \ell^\infty(A^{<j>},\mathbb{R})
\end{tikzcd}$$

Obviously the bornology is initial with respect to the bottom arrows for $\ell \in E'$ and by definition also with respect to the vertical arrows for $j \le k+1$. Thus also the top arrows form an initial source. By 2.15 the spaces in the bottom row are c^∞-complete and are metrizable if E is metrizable. Since the boundedness of the difference quotient of order $k+1$ implies that of order $j \le k+1$, also $\mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A,E)$ is convenient, and it is Fréchet provided E is. The uniform boundedness principle follows also from this diagram, using the stability property 5.25 and that the Fréchet and hence webbed space $\ell^\infty(A^{<j>}, \mathbb{R})$ has it by 5.24. $\square$

22.10. Proposition. *For a convenient vector space E the following operators are well-defined bounded linear mappings:*

(1) *The restriction operator $\mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A_1, E) \to \mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A_2, E)$ defined by $f \mapsto f|_{A_2}$ for $A_2 \subseteq A_1$.*
(2) *For $g \in \mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A, \mathbb{R})$ the multiplication operator*

$$\begin{aligned} \mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A, E) &\to \mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A, E) \\ f &\mapsto g \cdot f. \end{aligned}$$

(3) *The gluing operator*

$$\mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A_1, E) \times_{A_1 \cap A_2} \mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A_2, E) \to \mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A, E)$$

defined by $(f_1, f_2) \mapsto f_1 \cup f_2$ for any covering of A by relatively open subsets $A_1 \subseteq A$ and $A_2 \subseteq A$.

The fibered product (pull back) $\mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A_1, E) \times_{A_1 \cap A_2} \mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A_2, E) \to \mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A, E)$ is the subspace of $\mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A_1, E) \times \mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A_2, E)$ formed by all (f_1, f_2) with $f_1 = f_2$ on $A_0 := A_1 \cap A_2$.

Proof. It is enough to consider the particular case where $E = \mathbb{R}$. The general case follows easily by composing with ℓ_* for each $\ell \in E'$.
(1) is obvious.
(2) follows from the Leibniz formula 22.4.
(3) First we show that the gluing operator has values in $\mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A, \mathbb{R})$. Suppose the difference quotient $\delta^j f$ is not bounded for some $j \le m+1$, which we assume to be minimal. So there exists a bounded sequence $\mathbf{x}^n \in A^{<j>}$ such that $(\delta^j f)(\mathbf{x}^n)$ converges towards infinity. Since A is compact we may assume that $\mathbf{x}^n$ converges to some point $\mathbf{x}^\infty \in A^{(j+1)}$. If $\mathbf{x}^\infty$ does not lie on the diagonal, there are two indices $i_1 \neq i_2$ and some $\delta > 0$, such that $|\mathbf{x}^n{}_{i_1} - \mathbf{x}^n{}_{i_2}| \ge \delta$. But then

$$\delta^j f(\mathbf{x}^n)(\mathbf{x}^n{}_{i_1} - \mathbf{x}^n{}_{i_2}) = \tfrac{1}{j}\left(\delta^{j-1} f(\dots, \widehat{\mathbf{x}^n{}_{i_2}}, \dots) - \delta^{j-1} f(\dots, \widehat{\mathbf{x}^n{}_{i_1}}, \dots)\right).$$

Which is a contradiction to the boundedness of $\delta^{j-1} f$ and hence the minimality of j. So $\mathbf{x}^\infty = (x^\infty, \dots, x^\infty)$ and since the covering $\{A_1, A_2\}$ of A is open x^∞ lies in A_i for $i = 1$ or $i = 2$. Thus we have that $\mathbf{x}^n \in A_i{}^{<j>}$ for almost all n, and hence $\delta^j f(\mathbf{x}^n) = \delta^j f_i(\mathbf{x}^n)$, which is bounded by assumption on f_i.

Because of the uniform boundedness principle 22.9 it only remains to show that $(f_1, f_2) \mapsto f(a)$ is bounded, which is obvious since $f(a) = f_i(a)$ for some i depending on the location of a. $\square$

22.11. Remark. If A is finite, we define an extension $\tilde{f} : \mathbb{R} \to E$ of the given function $f : A \to E$ as the interpolation polynomial of f at all points in A. For infinite compact sets $A \subset \mathbb{R}$ we will use Whitney's extension theorem 22.2, where we will replace the Taylor polynomial in the definition 22.2 of the extension by the interpolation polynomial at appropriately chosen points near a_φ. For this we associate to each point $a \in A$ a sequence $\mathbf{a} = (a_0, a_1, \dots)$ of points in A starting from the given point $a_0 = a$.

22.12. Definition of $a \mapsto \mathbf{a}$. Let A be a closed infinite subset of $\mathbb{R}$, and let $a \in A$. Our aim is to define a sequence $\mathbf{a} = (a_0, a_1, a_2, \dots)$ in a certain sense close to a. The construction is by induction and goes as follows: $a_0 := a$. For the induction step we choose for every non-empty finite subset $F \subset A$ a point a_F in the closure of $A \setminus F$ having minimal distance to F. In case F does not contain an accumulation point the set $A \setminus F$ is closed and hence $a_F \notin F$, otherwise the distance of $A \setminus F$ to F is 0 and a_F is an accumulation point in F. In both cases we have for the distances $d(a_F, F) = d(A \setminus F, F)$. Now suppose $(a_0, \dots, a_{j-1})$ is already constructed. Then let $F := \{a_0, \dots, a_{j-1}\}$ and define $a_j := a_F$.

Lemma. *Let $\mathbf{a} = (a_0, \dots)$ and $\mathbf{b} = (b_0, \dots)$ be constructed as above.*

If $\{a_0, \dots, a_k\} \neq \{b_0, \dots, b_k\}$ then we have for all $i, j \le k$ the estimates

$$\begin{aligned} |a_i - b_j| &\le (i + j + 1)\, |a_0 - b_0| \\ |a_i - a_j| &\le \max\{i, j\}\, |a_0 - b_0| \\ |b_i - b_j| &\le \max\{i, j\}\, |a_0 - b_0|. \end{aligned}$$

Proof. First remark that if $\{a_0, \dots, a_i\} = \{b_0, \dots, b_i\}$ for some i, then the same is true for all larger i, since the construction of a_{i+1} depends only on the set $\{a_0, \dots, a_i\}$. Furthermore the set $\{a_0, \dots, a_i\}$ contains at most one accumulation point, since for an accumulation point a_j with minimal index j we have by construction that $a_j = a_{j+1} = \dots = a_i$.

We now show by induction on $i \in \{1, \dots, k\}$ that

$$\begin{aligned} d(a_{i+1}, \{a_0, \dots, a_i\}) &\le |a_0 - b_0|, \\ d(b_{i+1}, \{b_0, \dots, b_i\}) &\le |a_0 - b_0|. \end{aligned}$$

We proof this statement for a_{i+1}, it then follows for b_{i+1} by symmetry.
In case where $\{a_0, \dots, a_i\} \supseteq \{b_0, \dots, b_i\}$ we have that $\{a_0, \dots, a_i\} \supset \{b_0, \dots, b_i\}$ by assumption. Thus some of the elements of $\{b_0, \dots, b_i\}$ have to be equal and hence are accumulation points. So $\{a_0, \dots, a_i\}$ contains an accumulation point, and hence $a_{i+1} \in \{a_0, \dots, a_i\}$ and the claimed inequality is trivially satisfied.
In the other case there exist some $j \le i$ such that $b_j \notin \{a_0, \dots, a_i\}$. We choose the minimal j with this property and obtain

$$d(a_{i+1}, \{a_0, \dots, a_i\}) := d(A \setminus \{a_0, \dots, a_i\}, \{a_0, \dots, a_i\}) \le d(b_j, \{a_0, \dots, a_i\}).$$

If $j = 0$, then this can be further estimated as follows

$$d(b_j, \{a_0, \dots, a_i\}) \le |a_0 - b_0|.$$

Otherwise $\{b_0, \dots, b_{j-1}\} \subseteq \{a_0, \dots, a_j\}$ and hence we have

$$d(b_j, \{a_0, \dots, a_i\}) \le d(b_j, \{b_0, \dots, b_{j-1}\}) \le |a_0 - b_0|$$

by induction hypothesis. Thus the induction is completed.

From the proven inequalities we deduce by induction on $k := \max\{i, j\}$ that

$$|a_i - a_j| \le \max\{i, j\}\, |a_0 - b_0|$$

and similarly for $|b_j - b_i|$:
For $k = 0$ this is trivial. Now for $k > 0$. We may assume that $i > j$. Let $i' < i$ be such that $|a_i - a_{i'}| = d(a_i, \{a_0, \dots, a_{i-1}\}) \le |a_0 - b_0|$. Thus by induction hypothesis $|a_{i'} - a_j| \le (k - 1)\, |a_0 - b_0|$ and hence

$$|a_i - a_j| \le |a_i - a_{i'}| + |a_{i'} - a_j| \le k\, |a_0 - b_0|.$$

By the triangle inequality we finally obtain

$$|a_i - b_j| \le |a_i - a_0| + |a_0 - b_0| + |b_0 - b_j| \le (i + 1 + j)\,|a_0 - b_0|. \quad \square$$

22.13. Finite Order Extension Theorem. *Let E be a convenient vector space, A a subset of $\mathbb{R}$ and m be a natural number or 0. A function $f : A \to E$ admits an extension to $\mathbb{R}$ which is m-times differentiable with locally Lipschitzian m-th derivative if and only if its difference quotient of order $m+1$ is bounded on bounded sets.*

Proof. Without loss of generality we may assume that A is infinite. We consider first the case that A is compact and $E = \mathbb{R}$.

So let $f : A \to \mathbb{R}$ be in $\mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}$. We want to apply Whitney's extension theorem 22.2. So we have to find an m-jet F on A. For this we define

$$F^k(a) := (P^m_{\mathbf{a}} f)^{(k)}(a),$$

where $\mathbf{a}$ denotes the sequence obtained by this construction starting with the point $a$and where $P^m_{\mathbf{a}} f$ denotes the interpolation polynomial of f at the first $m+1$ points of $\mathbf{a}$ if these are all different; if not, at least one of these $m+1$ points is an accumulation point of A and then $P^m_{\mathbf{a}} f$ is taken as limit of interpolation polynomials according to 22.7.

Let Φ be the partition of unity mentioned in 22.2 and Φ' the subset specified there. Then we define $\tilde f$ analogously to 22.2 where $\mathbf{a}_\varphi$ denotes the sequence obtained by construction 22.12 starting with the point a_φ chosen in 22.2:

$$\tilde f(x) := \begin{cases} f(x) & \text{for } x \in A \\ \sum_{\varphi \in \Phi'} \varphi(x) P^m_{\mathbf{a}_\varphi} f(x) & \text{otherwise.} \end{cases}$$

In order to verify that F belongs to $\mathcal{L}\mathrm{ip}(m+1, A)$ we need the Taylor polynomial

$$T^m_a F(x) := \sum_{k=0}^m \frac{(x-a)^k}{k!} F^k(a) = \sum_{k=0}^m \frac{(x-a)^k}{k!} (P^m_{\mathbf{a}} f)^{(k)}(a) = P^m_{\mathbf{a}} f(x),$$

where the last equation holds since $P^m_{\mathbf{a}} f$ is a polynomial of degree at most m. This shows that our extension $\tilde f$ coincides with the classical extension $\tilde F$ given in 22.2 of the m-jet F constructed from f.

The remainder term $R^m_a F := F - j^m(T^m_a F)$ is given by:

$$(R^m_a F)^k(b) = F^k(b) - (T^m_a F)^{(k)}(b) = (P^m_{\mathbf{b}} f)^{(k)}(b) - (P^m_{\mathbf{a}} f)^{(k)}(b)$$

We have to show that for some constant M one has $\left|(R^m_a F)^k(b)\right| \le M|a-b|^{m+1-k}$ for all $a, b \in A$ and all $k \le m$.

In order to estimate this difference we write it as a telescoping sum of terms which can written by 22.6 as

$$\Big(P^m_{(a_0,\dots,a_{i-1},b_i,b_{i+1},\dots,b_m)} f - P^m_{(a_0,\dots,a_{i-1},a_i,b_{i+1},\dots,b_m)} f\Big)^{(k)}(t) =$$

$$= \frac{k!}{(m+1)!}\, \delta^{m+1} f(a_0,\dots,a_i,b_i,\dots,b_m)\cdot$$

$$\cdot (b_i - a_i) \sum_{i_1<\dots<i_k} (t-a_0)\dots\widehat{(t-a_{i_1})}\dots\widehat{(t-b_{i_k})}\dots(t-b_m).$$

Note that this formula remains valid also in case where the points are not pairwise different. This follows immediately by passing to the limit with the help of 22.7.

We have estimates for the distance of points in $\{a_0,\dots,a_m;b_0,\dots,b_m\}$ by 22.12 and so we obtain the required constant M as follows

$$
\begin{aligned}
|(R_a^m F)^k(b)| \le \frac{k!}{(m+1)!}\sum_{i=0}^{m}(2i+1)\,|b-a|^{m+1-k}\\
\sum_{i_1<\dots<i_k} 1\cdot 2\cdot\dots\cdot\widehat{(1+i_1)}\dots\widehat{i_k}\cdot\dots\cdot m\cdot\\
\cdot\max\{|\delta^{m+1}f(\{a_0,\dots,a_m,b_0,\dots,b_m\}^{<m+1>})|\}.
\end{aligned}
$$

In case, where E is an arbitrary convenient vector space we define an extension $\tilde f$ for $f\in\mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A,E)$ by the same formula as before. Since Φ' is locally finite, this defines a function $\tilde f:\mathbb{R}\to E$. In order to show that $\tilde f\in\mathcal{L}\mathrm{ip}^m(\mathbb{R},E)$ we compose with an arbitrary $\ell\in E'$. Then $\ell\circ\tilde f$ is just the extension of $\ell\circ f$ given above, thus belongs to $\mathcal{L}\mathrm{ip}^m(\mathbb{R},\mathbb{R})$.

Let now A be a closed subset of $\mathbb{R}$. Then let the compact subsets $A_n\subset\mathbb{R}$ be defined by $A_1:=A\cap[-2,2]$ and $A_n:=[-n+1,n-1]\cup(A\cap[-n-1,n+1])$ for $n>1$. We define recursively functions $f_n\in\mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A_n,E)$ as follows: Let f_1 be a $\mathcal{L}\mathrm{ip}^m$-extension of $f|_{A_1}$. Let $f_n:A_n\to\mathbb{R}$ be a $\mathcal{L}\mathrm{ip}^m$-extension of the function which equals f_{n-1} on $[-n+1,n-1]$ and which equals f on $A\cap[-n-1,n+1]$. This definition makes sense, since the two sets

$$
\begin{aligned}
A_n\setminus[-n+1,n-1]&=A\cap\big([-n-1,n+1]\setminus[-n+1,n-1]\big),\\
A_n\setminus\big([-n-1,-n]\cup[n,n+1]\big)&=[-n+1,n-1]\cup\big(A\cap[-n,n]\big)
\end{aligned}
$$

form an open cover of A_n, and their intersection is contained in the set $A\cap[-n,n]$ on which f and f_{n-1} coincide. Now we apply 22.10. The sequence f_n converges uniformly on bounded subsets of $\mathbb{R}$ to a function $\tilde f:\mathbb{R}\to E$, since $f_j=f_n$ on $[-n,n]$ for all $j>n$. Since each f_n is $\mathcal{L}\mathrm{ip}^m$, so is $\tilde f$. Furthermore, $\tilde f$ is an extension of f, since $\tilde f=f_n$ on $[-n,n]$ and hence on $A\cap[-n+1,n-1]$ equal to f.

Finally the case, where $A\subseteq\mathbb{R}$ is completely arbitrary. Let $\bar A$ denote the closure of A in $\mathbb{R}$. Since the first difference quotient is bounded on bounded subsets of A one concludes that f is Lipschitzian and hence uniformly continuous on bounded subsets of A, moreover, the values $f(a)$ form a Mackey Cauchy net for $A\ni a\to\bar a\in\mathbb{R}$. Thus f has a unique continuous extension $\tilde f$ to $\bar A$, since the limit $\tilde f(\bar a):=\lim_{a\to\bar a}f(a)$ exists in E, because E is convenient. Boundedness of the difference quotients of order j of $\tilde f$ can be tested by composition with linear continuous functionals, so we may assume $E=\mathbb{R}$. Its value at $(\tilde t_0,\dots,\tilde t_j)\in\bar A^{<j>}$ is the limit of $\delta^j f(t_0,\dots,t_j)$, where $A^{<j>}\ni(t_0,\dots,t_j)$ converges to $(\tilde t_0,\dots,\tilde t_j)$, since in the explicit formula for δ^j the factors $f(t_i)$ converge to $\tilde f(\tilde t_i)$. Now we may apply the result for closed A to obtain the required extension. □

22.14. Extension Operator Theorem. *Let E be a convenient vector space and let m be finite. Then the space $\mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A,E)$ of functions having an extension in the sense of 22.13 is a convenient vector space and there exists a bounded linear extension operator from $\mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A,E)$ to $\mathcal{L}\mathrm{ip}^m(\mathbb{R},E)$.*

Proof. This follows from 21.2.

Explicitly the proof runs as follows: For any convenient vector space E we have to construct a bounded linear operator

$$T:\mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A,E)\to\mathcal{L}\mathrm{ip}^m(\mathbb{R},E)$$

satisfying $T(f)|_A = f$ for all $f \in \mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A,E)$. Since $\mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A,E)$ is a convenient vector space, this is by 12.12 via a flip of variables equivalent to the existence of a $\mathcal{L}\mathrm{ip}^m$-curve

$$\tilde{T} : \mathbb{R} \to L(\mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A,E),E)$$

satisfying $\tilde{T}(a)(f) = T(f)(a) = f(a)$. Thus $\tilde{T}$ should be a $\mathcal{L}\mathrm{ip}^m$-extension of the map $e : A \to L(\mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A,E),E)$ defined by $e(a)(f) := f(a) = \mathrm{ev}_a(f)$.

By the vector valued finite order extension theorem 22.13 it suffices to show that this map e belongs to $\mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A, L(\mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A,E),E))$. So consider the difference quotient $\delta^{m+1}e$ of e. Since, by the linear uniform boundedness principle 5.18, boundedness in $L(F,E)$ can be tested pointwise, we consider

$$\begin{aligned}\delta^{m+1}e(a_0,\dots,a_{m+1})(f) &= \delta^{m+1}(\mathrm{ev}_f \circ e)(a_0,\dots,a_{m+1})\\ &= \delta^{m+1}f(a_0,\dots,a_{m+1}).\end{aligned}$$

This expression is bounded for $(a_0,\dots,a_{m+1})$ varying in bounded sets, since $f \in \mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A,E)$. □

In order to obtain a extension theorem for smooth mappings, we use a modification of the original construction of [Whitney, 1934]. In particular we need the following result.

22.15. Result. [Malgrange, 1966, lemma 4.2], also [Tougeron, 1972, lemme 3.3]. *There exist constants c_k, such that for any compact set $K \subset \mathbb{R}$ and any $\delta > 0$ there exists a smooth function h_δ on $\mathbb{R}$ which satisfies*

(1) *$h_\delta = 1$ locally around K and $h_\delta(x) = 0$ for $d(x,K) \geq \delta$;*
(2) *for all $x \in \mathbb{R}$ and $k \geq 0$ one has: $\left|h_\delta^{(k)}(x)\right| \leq \frac{c_k}{\delta^k}$.*

22.16. Lemma. *Let A be compact and A_{acc} be the compact set of accumulation points of A. We denote by $C^\infty_A(\mathbb{R},\mathbb{R})$ the set of smooth functions on $\mathbb{R}$ which vanish on A. For finite m we denote by $C^m_A(\mathbb{R},\mathbb{R})$ the set of C^m-functions on $\mathbb{R}$, which vanish on A, are m-flat on A_{acc} and are smooth on the complement of A_{acc}. Then $C^\infty_A(\mathbb{R},\mathbb{R})$ is dense in $C^{m+1}_A(\mathbb{R},\mathbb{R})$ with respect to the structure of $C^m(\mathbb{R},\mathbb{R})$.*

Proof. Let $\varepsilon > 0$ and let $g \in C^{m+1}_A(\mathbb{R},\mathbb{R})$ be the function which we want to approximate. By Taylor's theorem we have for $f \in C^{m+1}(\mathbb{R},\mathbb{R})$ the equation

$$f(x) - \sum_{i=0}^{k} \frac{f^{(i)}(a)}{i!}(x-a)^i = (x-a)^{k+1}\frac{f^{(k+1)}(\xi)}{(k+1)!}$$

for some ξ between a and x. If we apply this equation for $j \leq m$ and $k = m-j$ to $g^{(j)}$ for some point $a \in A_{acc}$ we obtain

$$\left|g^{(j)}(x) - 0\right| \leq |x-a|^{m+1-j}\left|\frac{g^{(m+1)}}{(m+1-j)!}(\xi)\right|$$

Taking the infimum over all $a \in A_{acc}$ we obtain a constant

$$K := \sup\left\{\left|\frac{g^{(m+1)}}{(m+1-j)!}(\xi)\right| : d(\xi, A_{acc}) \leq 1\right\}$$
$$\text{satisfying} \qquad \left|g^{(j)}(x)\right| \leq K \cdot d(x, A_{acc})^{m+1-j}$$

for all x with $d(x,A) \leq 1$.

We choose $0 < \delta < 1$ depending on ε such that $\delta \cdot \max\{c_i : i \leq m\} \cdot K \cdot 2^m \leq \varepsilon$, and let h_δ be the function given in 22.15 for $K := A_{acc}$. The function $(1-h_\delta)\cdot g$ is smooth, since on $\mathbb{R}\setminus A_{acc}$ both factors are smooth and on a neighborhood of A_{acc} one has $h_\delta = 1$. The function $(1-h_\delta)\cdot g$ equals g on $\{x : d(x, A_{acc}) \ge \delta\}$, since h_δ vanishes on this set. So it remains to show that the derivatives of $h_\delta \cdot g$ up to order m are bounded by ε on $\{x : d(x, A_{acc}) \le \delta\}$. By the Leibniz rule we have:

$$(h_\delta \cdot g)^{(j)} = \sum_{i=0}^{j} \binom{j}{i} h_\delta^{(i)} g^{(j-i)}.$$

The i-th summand can be estimated as follows:

$$\left|h_\delta^{(i)}(x) g^{(j-i)}(x)\right| \le \frac{c_i}{\delta^i} K\, d(x, A_{acc})^{m+1+i-j} \le c_i\, K\, \delta^{m+1-j}$$

An estimate for the derivative now is

$$\begin{aligned}\left|(h_\delta \cdot g)^{(j)}(x)\right| &\le \sum_{i=0}^{j} \binom{j}{i} c_i\, K\, \delta^{m+1-j}\\ &\le 2^j\, K\, \delta^{m+1-j} \max\{c_i : 0 \le i \le j\} \le \varepsilon. \qquad \square\end{aligned}$$

22.17. Smooth Extension Theorem. *Let E be a Fréchet space (or, slightly more general, a convenient vector space satisfying Mackey's countability condition) A function $f : A \to E$ admits a smooth extension to $\mathbb{R}$ if and only if each of its difference quotients is bounded on bounded sets.*

A convenient vector space is said to satisfy *Mackey's countability condition* if for every sequence of bounded sets $B_n \subseteq E$ there exists a sequence $\lambda_n > 0$ such that $\bigcup_{n\in\mathbb{N}} \lambda_n B_n$ is bounded in E.

Proof. We consider first the case, where $E = \mathbb{R}$. For $k \ge 0$ let $\tilde f^k$ be a $\mathcal{L}\text{ip}^k$-extension of f according to 22.13. The difference $\tilde f^{k+1} - \tilde f^k$ is an element of $C^k_A(\mathbb{R}, \mathbb{R})$: It is by construction C^k and on $\mathbb{R}\setminus A$ smooth. At an accumulation point a of A the Taylor expansion of $\tilde f^k$ of order $j \le k$ is just the approximation polynomial $P^j_{(a,\dots,a)} f$ by 22.13. Thus the derivatives up to order k of $\tilde f^{k+1}$ and $\tilde f^k$ are equal in a, and hence the difference is k-flat at a. Locally around any isolated point of A, i.e. a point $a \in A\setminus A_{acc}$, the extension $\tilde f^k$ is just the approximation polynomial $P^k_{\mathbf{a}}$ and hence smooth. In order to see this, use that for x with $|x-a| < \frac14 d(a, A\setminus\{a\})$ the point $\mathbf{a}_\varphi$ has as first entry a for every φ with $x \in \operatorname{supp}\varphi$: Let $b \in A\setminus\{a\}$ and $y \in \operatorname{supp}\varphi$ be arbitrary, then

$$\begin{aligned}|b-x| &\ge |b-a| - |a-x| \ge d(a, A\setminus\{a\}) - |a-x| > (4-1)\,|a-x|\\ |b-y| &\ge |b-x| - |x-y| > 3\,|a-x| - \operatorname{diam}(\operatorname{supp}\varphi)\\ &\ge 3\, d(a, \operatorname{supp}\varphi) - 2\, d(a, \operatorname{supp}\varphi) = d(a, \operatorname{supp}\varphi)\\ &\Rightarrow\ d(b, \operatorname{supp}\varphi) > d(a, \operatorname{supp}\varphi)\ \Rightarrow\ a_\varphi = a.\end{aligned}$$

By lemma 22.16 there exists an $h_k \in C^\infty_A(\mathbb{R}, \mathbb{R})$ such that

$$\left|(\tilde f^{k+1} - \tilde f^k - h_k)^{(j)}(x)\right| \le \frac{1}{2^k} \text{ for all } j \le k-1.$$

Now we consider the function $\tilde f := \tilde f^0 + \sum_{k\ge0}(\tilde f^{k+1} - \tilde f^k - h_k)$. It is the required smooth extension of f, since the summands $\tilde f^{k+1} - \tilde f^k - h_k$ vanish on A, and since for any n it can be rewritten as $\tilde f = \tilde f^n + \sum_{k<n} h_k + \sum_{k\ge n}(\tilde f^{k+1} - \tilde f^k - h_k)$, where the first summand is C^n, the first sum is C^∞, and the derivatives up to order $n-1$ of the terms of the second sum are uniformly summable.

Now we prove the vector valued case, where E satisfies Mackey's countability condition. It is enough to show the result for compact subsets $A \subset \mathbb{R}$, since the generalization arguments given in the proof of 22.13 can be applied equally in the smooth case. First one has to give a vector valued version of 22.16: Let a function $g \in \mathcal{L}\mathrm{ip}^m(\mathbb{R}, E)$ with compact support be given, which vanishes on A, is m-flat on A_{acc} and smooth on the complement of A_{acc}. Then for every $\varepsilon > 0$ there exists a $h \in C^\infty(\mathbb{R}, \mathbb{R})$, which equals 1 on a neighborhood of A_{acc} and such that $\delta^m(h \cdot g)(\mathbb{R}^{m+1})$ is contained in ε times the absolutely convex hull of the image of $\delta^{m+1}g$.

The proof of this assertion is along the lines of that of 22.16. One only has to define K as the absolutely convex hull of the image of $\delta^{m+1}g$ and choose $0 < \delta < 1$ such that $\delta \cdot \max\{c_i : i \le m\} \cdot 2^m \le \varepsilon$.

Now one proceeds as in scalar valued part: Let $\tilde f^k$ be the $\mathcal{L}\mathrm{ip}^k$-extension of f according to 22.13. Then $g_k := \tilde f^{k+1} - \tilde f^k$ satisfies the assumption of the vector valued version of 22.16. Let K_k be the absolutely convex hull of the bounded image of $\delta^{k+1}g_k$. By assumption on E there exist $\lambda_n > 0$ such that $K := \bigcup_{k\in\mathbb{N}} \lambda_k \cdot K_k$ is bounded. Hence we may choose an $h_k \in C^\infty_A(\mathbb{R}, \mathbb{R})$ such that $\delta^k(h_k \cdot g_k)(\mathbb{R}^{\langle k+1\rangle}) \subseteq \frac{\lambda_k}{2^k} K_k$. Now the extension $\tilde f$ is given by

$$\tilde f = \tilde f^0 + \sum_{k\ge 0} h_k \cdot g_k = \tilde f^n + \sum_{k<n} (1 - h_k) \cdot g_k + \sum_{k\ge n} h_k \cdot g_k$$

and the result follows as above using convergence in the Banach space E_K. □

22.18. Remark. The restriction operator $\mathcal{L}\mathrm{ip}^m(\mathbb{R}, E) \to \mathcal{L}\mathrm{ip}^m_{\mathrm{ext}}(A, E)$ is a quotient mapping. We constructed a section for it, which is bounded and linear in the finite order case. It is unclear, whether it is possible to obtain a bounded linear section also in the smooth case, even if $E = \mathbb{R}$.

If the smooth extension theorem were true for any arbitrary convenient vector space E, then it would also give the extension operator theorem for the smooth case. Thus in order to obtain a counter-example to the latter one, the first step might be to find a counter-example to the vector valued extension theorem. In the particular cases, where the values lie in a Fréchet space E the vector valued smooth extension theorem is however true.

22.19. Proposition. *Let A be the image of a strictly monotone bounded sequence $\{a_n : n \in \mathbb{N}\}$. Then a map $f : A \to \mathbb{R}$ has a $\mathcal{L}\mathrm{ip}^m$-extension to $\mathbb{R}$ if and only if the sequence $\delta^k f(a_n, a_{n+1}, \dots, a_{n+k})$ is bounded for $k = m + 1$ if m is finite, respectively for all k if $m = \infty$.*

Proof. By [**11**, 1.3.10], the difference quotient $\delta^k f(a_{i_0}, \dots, a_{i_k})$ is an element of the convex hull of the difference quotients $\delta^k f(a_n, \dots, a_{n+k})$ for all $\min\{i_0, \dots, i_k\} \le n \le n + k \le \max\{i_0, \dots, i_k\}$. So the result follows from the extension theorems 22.13 and 22.17. □

For explicit descriptions of the boundedness condition for $\mathcal{L}\mathrm{ip}^k$-mappings defined on certain sequences and low k see [Frölicher, Kriegl, 1993, Sect. 6].

23. Frölicher Spaces and Free Convenient Vector Spaces

The central theme of this book is 'infinite dimensional manifolds'. But many natural examples suggest that this is a quite restricted notion, and it will be very helpful to have at hand a much more general and also easily useable concept, namely smooth spaces as they were introduced by [Frölicher, 1980, 1981]. We follow his line of development, replacing technical arguments by simple use of cartesian closedness of smooth calculus on convenient vector spaces, and we call them Frölicher spaces.

23.1. The category of Frölicher spaces. A *Frölicher space* or a space with *smooth structure* is a triple $(X, \mathcal{C}_X, \mathcal{F}_X)$ consisting of a set X, a subset $\mathcal{C}_X$ of the set of all mappings $\mathbb{R} \to X$, and a subset $\mathcal{F}_X$ of the set of all functions $X \to \mathbb{R}$, with the following two properties:

(1) A function $f : X \to \mathbb{R}$ belongs to $\mathcal{F}_X$ if and only if $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $c \in \mathcal{C}_X$.
(2) A curve $c : \mathbb{R} \to X$ belongs to $\mathcal{C}_X$ if and only if $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $f \in \mathcal{F}_X$.

Note that a set X together with any subset $\mathcal{F}$ of the set of functions $X \to \mathbb{R}$ generates a unique Frölicher space $(X, \mathcal{C}_X, \mathcal{F}_X)$, where we put in turn:

$$\mathcal{C}_X := \{c : \mathbb{R} \to X : f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } f \in \mathcal{F}\},$$
$$\mathcal{F}_X := \{f : X \to \mathbb{R} : f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } c \in \mathcal{C}_X\},$$

so that $\mathcal{F} \subseteq \mathcal{F}_X$. The set $\mathcal{F}$ will be called a *generating set of functions for the Frölicher space*. A locally convex space is convenient if and only if it is a Frölicher space with the smooth curves and smooth functions from section 1 by 2.14. Furthermore, c^∞-open subsets U of convenient vector spaces E are Frölicher spaces, where $\mathcal{C}_U = C^\infty(\mathbb{R}, U)$ and $\mathcal{F}_U = C^\infty(U, \mathbb{R})$. Here we can use as generating set $\mathcal{F}$ of functions the restrictions of any set of bounded linear functionals which generates the bornology of E, see 2.14.4.

A mapping $\varphi : X \to Y$ between two Frölicher spaces is called *smooth* if the following three equivalent conditions hold

(3) For each $c \in \mathcal{C}_X$ the composite $\varphi \circ c$ is in $\mathcal{C}_Y$.
(4) For each $f \in \mathcal{F}_Y$ the composite $f \circ \varphi$ is in $\mathcal{F}_X$.
(5) For each $c \in \mathcal{C}_X$ and for each $f \in \mathcal{F}_Y$ the composite $f \circ \varphi \circ c$ is in $C^\infty(\mathbb{R}, \mathbb{R})$.

Note that $\mathcal{F}_Y$ can be replaced by any generating set of functions. The set of all smooth mappings from X to Y will be denoted by $C^\infty(X, Y)$. Then we have $C^\infty(\mathbb{R}, X) = \mathcal{C}_X$ and $C^\infty(X, \mathbb{R}) = \mathcal{F}_X$. Frölicher spaces and smooth mappings form a category.

23.2. Theorem. *The category of Frölicher spaces and smooth mappings has the following properties:*

(1) *Complete, i.e., arbitrary limits exist. The underlying set is formed as in the category of sets as a certain subset of the cartesian product, and the smooth structure is generated by the smooth functions on the factors.*
(2) *Cocomplete, i.e., arbitrary colimits exist. The underlying set is formed as in the category of set as a certain quotient of the disjoint union, and the smooth functions are exactly those which induce smooth functions on the cofactors.*
(3) *Cartesian closed, which means: The set $C^\infty(X, Y)$ carries a canonical smooth structure described by*

$$C^\infty(X, Y) \xrightarrow{C^\infty(c, f)} C^\infty(\mathbb{R}, \mathbb{R}) \xrightarrow{\lambda} \mathbb{R}$$

where $c \in C^\infty(\mathbb{R}, X)$, where f is in $C^\infty(Y, \mathbb{R})$ or in a generating set of functions, and where $\lambda \in C^\infty(\mathbb{R}, \mathbb{R})'$. With this structure the exponential law holds:

$$C^\infty(X \times Y, Z) \cong C^\infty(X, C^\infty(Y, Z)).$$

Proof. Obviously, the limits and colimits described above have all required universal properties.

We have the following implications:

(1) "" $\varphi^\vee : X \to C^\infty(Y, Z)$ is smooth.
(2) "$\Leftrightarrow$" $\varphi^\vee \circ c_X : \mathbb{R} \to C^\infty(Y, Z)$ is smooth for all smooth curves $c_X \in C^\infty(\mathbb{R}, X)$, by definition.
(3) "$\Leftrightarrow$" $C^\infty(c_Y, f_Z) \circ \varphi^\vee \circ c_X : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$ is smooth for all smooth curves $c_X \in C^\infty(\mathbb{R}, X)$, $c_Y \in C^\infty(\mathbb{R}, Y)$, and smooth functions $f_Z \in C^\infty(Z, \mathbb{R})$, by definition.
(4) "$\Leftrightarrow$" $f_Z \circ \varphi \circ (c_X \times c_Y) = f_Z \circ (c_Y^* \circ \varphi^\vee \circ c_X)^\wedge : \mathbb{R}^2 \to \mathbb{R}$ is smooth for all smooth curves c_X, c_Y, and smooth functions f_Z, by the simplest case of cartesian closedness of smooth calculus 3.10.
(5) "$\Rightarrow$" $\varphi : X \times Y \to Z$ is smooth, since each curve into $X \times Y$ is of the form $(c_X, c_Y) = (c_X \times c_Y) \circ \Delta$, where Δ is the diagonal mapping.
(6) "$\Rightarrow$" $\varphi \circ (c_X \times c_Y) : \mathbb{R}^2 \to Z$ is smooth for all smooth curves c_X and c_Y, since the product and the composite of smooth mappings is smooth.

As in the proof of 3.13 it follows in a formal way that the exponential law is a diffeomorphism for the smooth structures on the mapping spaces. □

23.3. Remark. By [11, 2.4.4] the convenient vector spaces are exactly the linear Frölicher spaces for which the smooth linear functionals generate the smooth structure, and which are separated and 'complete'. On a locally convex space which is not convenient, one has to saturate to the scalarwise smooth curves and the associated functions in order to get a Frölicher space.

23.4 Proposition. *Let X be a Frölicher space and E a convenient vector space. Then $C^\infty(X, E)$ is a convenient vector space with the smooth structure described in 23.2.3.*

Proof. We consider the locally convex topology on $C^\infty(X, E)$ induced by $c^* : C^\infty(X, E) \to C^\infty(\mathbb{R}, E)$ for all $c \in C^\infty(\mathbb{R}, X)$. As in 3.11 one shows that this describes $C^\infty(X, E)$ as inverse limit of spaces $C^\infty(\mathbb{R}, E)$, which are convenient by 3.7. Thus also $C^\infty(X, E)$ is convenient by 2.15. By 2.14.4, 3.8, 3.9 and 3.7 its smooth curves are exactly those $\gamma : \mathbb{R} \to C^\infty(X, E)$, for which

$$\mathbb{R} \xrightarrow{\gamma} C^\infty(X, E) \xrightarrow{c^*} C^\infty(\mathbb{R}, E) \xrightarrow{f_*} C^\infty(\mathbb{R}, \mathbb{R}) \xrightarrow{\lambda} \mathbb{R}$$

is smooth for all $c \in C^\infty(\mathbb{R}, X)$, for all f in the generating set E' of functions, and all $\lambda \in C^\infty(\mathbb{R}, \mathbb{R})$. This is the smooth structure described in 23.2.3. □

23.5. Related concepts: Holomorphic Frölicher spaces. They can be defined in a way similar as smooth Frölicher spaces in 23.1, with the following changes: As curves one has to take mappings from the complex unit disk. Then the results analogous to 23.2 hold, where for the proof one has to use the holomorphic exponential law 7.22 instead of the smooth one 3.10, see [Siegl, 1995] and [Siegl, 1997].

The concept of holomorphic Frölicher spaces is not without problems: Namely finite dimensional complex manifolds are holomorphic Frölicher spaces if they are Stein, and compact complex manifolds are never holomorphic Frölicher spaces. But arbitrary subsets A of complex convenient vector spaces E are holomorphic Frölicher spaces with the initial structure, again generated by the restrictions of bounded complex linear functionals. Note that *analytic subsets* of complex convenient spaces, i.e., locally zero sets of holomorphic mappings, are holomorphic spaces. But usually, as analytic sets, holomorphic functions on them are restrictions of holomorphic functions defined on neighborhoods, whereas as holomorphic spaces they admit more holomorphic functions, as the following example shows:

Example. Neil's parabola $P := \{z_1^2 - z_2^3 = 0\} \subset \mathbb{C}^2$ has the holomorphic curves $a : \mathbb{D} \to P \subset \mathbb{C}^2$ of the form $a = (b^3, b^2)$ for holomorphic $b : \mathbb{D} \to \mathbb{C}$: If $a(z) = (z^k a_1(z), z^l a_2(z))$ with $a(0) = 0$ and $a_i(0) \neq 0$, then $k = 3n$ and $l = 2n$ for some $n > 0$ and (a_1, a_2) is still a holomorphic curve in $P \setminus 0$, so $(a_1, a_2) = (c^3, c^2)$ by the implicit function theorem, then $b(z) = z^n c(z)$ is the solution. Thus, $z \mapsto (z^3, z^2)$ is biholomorphic $\mathbb{C} \to P$. So z is a holomorphic function on P which cannot be extended to a holomorphic function on a neighborhood of 0 in $\mathbb{C}^2$, since this would have infinite differential at 0.

23.6. Theorem. Free Convenient Vector Space. [11, 5.1.1] *For every Frölicher space X there exists a free convenient vector space λX, i.e. a convenient vector space λX together with a smooth mapping $\delta_X : X \to \lambda X$, such that for every smooth mapping $f : X \to G$ with values in a a convenient vector space G there exists a unique linear bounded mapping $\tilde{f} : \lambda X \to G$ with $\tilde{f} \circ \delta_X = f$. Moreover $\delta^* : L(\lambda X, G) \cong C^\infty(X, G)$ is an isomorphisms of convenient vector spaces and δ is an initial morphism.*

Proof. In order to obtain a candidate for λX, we put $G := \mathbb{R}$ and thus should have $(\lambda X)' = L(\lambda X, \mathbb{R}) \cong C^\infty(X, \mathbb{R})$ and hence λX should be describable as subspace of $(\lambda X)'' \cong C^\infty(X, \mathbb{R})'$. In fact every $f \in C^\infty(E, \mathbb{R})$ acts as bounded linear functional $\mathrm{ev}_f : C^\infty(X, \mathbb{R})' \to \mathbb{R}$ and if we define $\delta_X : X \to C^\infty(X, \mathbb{R})'$ to be $\delta_X : x \mapsto \mathrm{ev}_x$ then $\mathrm{ev}_f \circ \delta_X = f$ and δ_X is smooth, since by the uniform boundedness principle 5.18 it is sufficient to check that $\mathrm{ev}_f \circ \delta_X = f : X \to C^\infty(X, \mathbb{R})' \to \mathbb{R}$ is smooth for all $f \in C^\infty(X, \mathbb{R})$. In order to obtain uniqueness of the extension $\tilde{f} := \mathrm{ev}_f$, we have to restrict it to the c^∞-closure of the linear span of $\delta_X(X)$. So let λX be this closure and let $f : X \to G$ be an arbitrary smooth mapping with values in some convenient vector space. Since δ belongs to C^∞ we have that $\delta^* : L(\lambda X, G) \to C^\infty(X, G)$ is well defined and it is injective since the linear subspace generated by the image of δ is c^∞-dense in λX by construction. To show surjectivity consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "\delta"] \arrow[dd, "f"'] & \lambda X \arrow[r, hook] \arrow[d, dotted, "(2)"] \arrow[ddl, dotted, "\tilde{f}"', "(3)"] & C^\infty(X)' \arrow[dl, dotted, "(1)"'] \arrow[dd, "\mathrm{ev}_{\lambda\circ f}"] \\
& \prod_{G'} \mathbb{R} \arrow[dr, "\mathrm{pr}_\lambda"] & \\
G \arrow[ur, "\delta"'] \arrow[rr, "\lambda"] & & \mathbb{R}
\end{tikzcd}$$

Note that (2) has values in $\delta(G)$, since this is true on the ev_x, which generate by definition a c^∞-dense subspace of λX.

Remains to show that this bijection is a bornological isomorphism. In order to show that the linear mapping $C^\infty(X,G) \to L(\lambda X, G)$ is bounded we can reformulate this equivalently using 3.12, the universal property of λX and the uniform boundedness principle 5.18 in turn:

$$\begin{aligned} & C^\infty(X,G) \to L(\lambda X, G) \text{ is } L \\ \iff & \lambda X \to L(C^\infty(X,G),G) \text{ is } L \\ \iff & X \to L(C^\infty(X,G),G) \text{ is } C^\infty \\ \iff & X \to L(C^\infty(X,G),G) \xrightarrow{\mathrm{ev}_f} G \text{ is } C^\infty \end{aligned}$$

and since the composition is just f we are done.

Conversely we have to show that $L(\lambda X, G) \to C^\infty(X,G)$ belongs to L. Composed with $\mathrm{ev}_x : C^\infty(X,G) \to G$ this yields the bounded linear map $\mathrm{ev}_{\delta(x)} : L(\lambda X, G) \to G$. Thus this follows from the uniform boundedness principle 5.26.

That δ_X is initial follows immediately from the fact that the structure of X is initial with respect to family $\{f = \mathrm{ev}_o \circ \delta_X : f \in C^\infty(X,\mathbb{R})\}$. □

Remark. The corresponding result with the analogous proof is true for holomorphic Frölicher spaces, $\mathcal{L}\mathrm{ip}^k$-spaces, and ℓ^∞-spaces. For the first see [Siegl, 1997] for the last two see [Frölicher, Kriegl, 1988].

23.7. Corollary. *Let X be a Frölicher space such that the functions in $C^\infty(X,\mathbb{R})$ separate points on X. Then X is diffeomorphic as Frölicher space to a subspace of the convenient vector space $\lambda(X) \subseteq C^\infty(X,\mathbb{R})'$ with the initial smooth structure (generated by the restrictions of linear bounded functionals, among other possibilities).* □

We have constructed the free convenient vector space λX as the c^∞-closure of the linear subspace generated by the point evaluations in $C^\infty(X,\mathbb{R})'$. This is not very constructive, in particular since adding Mackey-limits of sequences (or even nets) of a subspace does not always give its Mackey-closure. In important cases (like when X is a finite dimensional smooth manifold) one can show however that not only $\lambda X = C^\infty(X,\mathbb{R})'$, but even that every element of λX is the Mackey-limit of a sequence of linear combinations of point evaluations, and that $C^\infty(X,\mathbb{R})'$ is the space of distributions of compact support.

23.8. Proposition. *Let E be a convenient vector space and X a finite dimensional smooth separable manifold. Then for every $\ell \in C^\infty(X,E)'$ there exists a compact set $K \subseteq X$ such that $\ell(f) = 0$ for all $f \in C^\infty(X,E)$ with $f|_K = 0$.*

Proof. Since X is separable its compact bornology has a countable basis $\{K_n : n \in \mathbb{N}\}$ of compact sets. Assume now that no compact set has the claimed property. Then for every $n \in \mathbb{N}$ there has to exist a function $f_n \in C^\infty(X,E)$ with $f_n|_{K_n} = 0$ but $\ell(f_n) \neq 0$. By multiplying f_n with $\frac{n}{\ell(f_n)}$ we may assume that $\ell(f_n) = n$. Since every compact subset of X is contained in some K_n one has that $\{f_n : n \in \mathbb{N}\}$ is bounded in $C^\infty(X,E)$, but $\ell(\{f_n : n \in \mathbb{N}\}$ is not; this contradicts the assumption that ℓ is bounded. □

23.9. Remark. The proposition above remains true if X is a finite dimensional smooth paracompact manifold with non-measurably many components. In order to show this generalization one uses that for the partition $\{X_j : j \in J\}$ by the non-measurably many components one has $C^\infty(X,E) \cong \prod_{j\in J} C^\infty(X_j,E)$, and the fact that an ℓ belongs to the dual of such a product if it is a finite sum of elements of the duals of the factors. Now the result follows from 23.8 since the components of a paracompact manifold are paracompact and hence separable.

For such manifolds X the dual $C^\infty(X,\mathbb{R})'$ is the space of distributions with compact support. In fact, in case X is connected, $C^\infty(X,\mathbb{R})'$ is the space of all linear functionals which are continuous for the classically considered topology on $C^\infty(X,\mathbb{R})$ by 6.1; and in case of an arbitrary X this result follows using the isomorphism $C^\infty(X,\mathbb{R}) \cong \prod_j C^\infty(X_j,\mathbb{R})$ where the X_j denote the connected components of X.

23.10. Theorem. [11, 5.1.7] *Let E be a convenient vector space and X a finite dimensional separable smooth manifold. Then the Mackey-adherence of the linear subspace generated by $\{\ell \circ \mathrm{ev}_x : x \in X,\ \ell \in E'\}$ is $C^\infty(X,E)'$.*

Proof. The proof is in several steps.

(Step 1) There exist $g_n \in C^\infty(\mathbb{R},\mathbb{R})$ with $\operatorname{supp}(g_n) \subseteq [-\frac{2}{n},\frac{2}{n}]$ such that for every $f \in C^\infty(\mathbb{R},E)$ the set $\{n\cdot\big(f - \sum_{k\in\mathbb{Z}} f(r_{n,k}) g_{n,k}\big) : n \in \mathbb{N}\}$ is bounded in $C^\infty(\mathbb{R},E)$, where $r_{n,k} := \frac{k}{2^n}$ and $g_{n,k}(t) := g_n(t - r_{n,k})$.

We choose a smooth $h : \mathbb{R} \to [0,1]$ with $\operatorname{supp}(h) \subseteq [-1,1]$ and $\sum_{k\in\mathbb{Z}} h(t-k) = 1$ for all $t \in \mathbb{R}$ and we define $Q^n : C^\infty(\mathbb{R},E) \to C^\infty(\mathbb{R},E)$ by setting

$$Q^n(f)(t) := \sum_k f(\tfrac{k}{n}) h(tn - k).$$

Let $K \subseteq \mathbb{R}$ be compact. Then

$$n(Q^n(f) - f)(t) = \sum_k (f(\tfrac{k}{n}) - f(t)) \cdot n \cdot h(tn-k) \in B_1(f, K + \tfrac{1}{n}\operatorname{supp}(h))$$

for $t \in K$, where $B_n(f,K_1)$ denotes the absolutely convex hull of the bounded set $\delta^n f(K_1^{\langle n\rangle})$.

To get similar estimates for the derivatives we use convolution. Let $h_1 : \mathbb{R} \to \mathbb{R}$ be a smooth function with support in $[-1,1]$ and $\int_{\mathbb{R}} h_1(s)ds = 1$. Then for $t \in K$ one has

$$(f * h_1)(t) := \int_{\mathbb{R}} f(t-s)h_1(s)ds \ \in\ B_0(f, K + \operatorname{supp}(h_1)) \cdot \|h_1\|_1,$$

where $\|h_1\|_1 := \int_{\mathbb{R}} |h_1(s)|ds$. For smooth functions $f,h : \mathbb{R} \to \mathbb{R}$ one has $(f*h)^{(k)} = f*h^{(k)}$; one immediately deduces that the same holds for smooth functions $f : \mathbb{R} \to E$ and one obtains $(f*h_1)(t) - f(t) = \int_{\mathbb{R}}(f(t-s) - f(t))h_1(s)ds \in \operatorname{diam}(\operatorname{supp}(h_1)) \cdot \|h_1\|_1 \cdot B_1(f, K + \operatorname{supp}(h_1))$ for $t \in K$, where $\operatorname{diam}(S) := \sup\{|s| : s \in S\}$. Using now $h_n(t) := n \cdot h_1(nt)$ we obtain for $t \in K$:

$$\begin{aligned}
(Q^m(f) * h_n - f)^{(k)}(t) &= (Q^m(f) * h_n^{(k)} - f * h_n^{(k)})(t) + (f^{(k)} * h_n - f^{(k)})(t)\\
&= (Q^m(f) - f) * h_n^{(k)}(t) + (f^{(k)} * h_n - f^{(k)})(t)\\
&\in B_0(Q^m(f) - f, K + \operatorname{supp}(h_n)) \cdot \|h_n^{(k)}\|_1 +\\
&\quad + B_1(f^{(k)}, K + \operatorname{supp}(h_n)) \cdot \operatorname{diam}(\operatorname{supp}(h_n)) \cdot \|h_n\|_1\\
&\subseteq \tfrac{1}{m} n^k \cdot B_1(f, K + \operatorname{supp}(h_n) + \tfrac{1}{m}\operatorname{supp}(h)) \cdot \|h_1^{(k)}\|_1\\
&\quad + n \cdot B_1(f^{(k)}, K + \operatorname{supp}(h_n)) \cdot \|h_n\|_1.
\end{aligned}$$

Let now $m := 2^n$ and $P^n(f) := Q^m(f) * h_n$. Then

$$n \cdot \big(P^n(f) - f\big)^{(k)}(t) \in n^{k+1} 2^{-n} \cdot B_1(f, K + (\tfrac{1}{n} + \frac{1}{2^n})[-1,1]) \|h_1^{(k)}\|_1 \\ + B_1(f^{(k)}, K + \tfrac{1}{n}[-1,1]) \|h_1\|_1$$

for $t \in K$ and the right hand side is uniformly bounded for $n \in \mathbb{N}$.

With $g_n(t) := \int_{\mathbb{R}} h(s2^n - k) h_n(t + k2^{-n} - s)ds = \int_{\mathbb{R}} h(s2^n) h_n(t-s)ds$ we obtain

$$\begin{aligned} P^n(f)(t) = (Q^{2^n}(f) * h_n)(t) &= \sum_k f(k2^{-n}) h(t2^n - k) * h_n \\ &= \sum_k f(k2^{-n}) \int_{\mathbb{R}} h(s2^n - k) h_n(t-s)ds \\ &= \sum_k f(k2^{-n}) g_n(t - k2^{-n}). \end{aligned}$$

Thus $r_{n,k} := k2^{-n}$ and the g_n have all the claimed properties.

(Step 2) For every $m \in \mathbb{N}$ and every $f \in C^\infty(\mathbb{R}^m, E)$ the set

$$\Big\{ n \cdot \Big(f - \sum_{k_1 \in \mathbb{Z}, \dots, k_m \in \mathbb{Z}} f(r_{n;k_1,\dots,k_m}) g_{n;k_1,\dots,k_m} \Big) : n \in \mathbb{N} \Big\}$$

is bounded in $C^\infty(\mathbb{R}^m, E)$, where $r_{n;k_1,\dots,k_m} := (r_{n,k_1}, \dots, r_{n,k_m})$ and

$$g_{n;k_1,\dots,k_m}(x_1, \dots, x_m) := g_{n,k_1}(x_1) \cdot \dots \cdot g_{n,k_m}(x_m).$$

We prove this statement by induction on m. For $m = 1$ it was shown in step 1. Now assume that it holds for m and $C^\infty(\mathbb{R}, E)$ instead of E. Then by induction hypothesis applied to $f^\vee : C^\infty(\mathbb{R}^m, C^\infty(\mathbb{R}, E))$ we conclude that

$$\Big\{ n \cdot \Big(f - \sum_{k_1 \in \mathbb{Z}, \dots, k_m \in \mathbb{Z}} f(r_{n;k_1,\dots,k_m}, \quad) g_{n;k_1,\dots,k_m} \Big) : n \in \mathbb{N} \Big\}$$

is bounded in $C^\infty(\mathbb{R}^{m+1}, E)$. Thus it remains to show that

$$\Big\{ n \sum_{k_1,\dots,k_m} g_{n;k_1,\dots,k_m} \Big(f(r_{n;k_1,\dots,k_m}, \quad) - \sum_{k_{m+1}} f(r_{n;k_1,\dots,k_m}, r_{k_{m+1}}) g_{n,k_{m+1}} \Big) : n \in \mathbb{N} \Big\}$$

is bounded in $C^\infty(\mathbb{R}^{m+1}, E)$. Since the support of the $g_{n;k_1,\dots,k_m}$ is locally finite only finitely many summands of the outer sum are non-zero on a given compact set. Thus it is enough to consider each summand separately. By step (1) we know that the linear operators $h \mapsto n\big(h - \sum_k h(r_{n,k}) g_{n,k}\big)$, $n \in \mathbb{N}$, are pointwise bounded. So they are bounded on bounded sets, by the linear uniform boundedness principle 5.18. Hence

$$\Big\{ n \cdot \Big(f(r_{n;k_1,\dots,k_m}, \quad) - \sum_{k_{m+1}} f(r_{n;k_1,\dots,k_m}, r_{k_{m+1}}) g_{n,k_{m+1}} \Big) : n \in \mathbb{N} \Big\}$$

is bounded in $C^\infty(\mathbb{R}^{m+1}, E)$. Using that the multiplication $\mathbb{R} \times E \to E$ is bounded one concludes immediately that also the multiplication with a map $g \in C^\infty(X, \mathbb{R})$ is bounded from $C^\infty(X, E) \to C^\infty(X, E)$ for any Frölicher space X. Thus the proof of step (2) is complete.

(Step 3) For every $\ell \in C^\infty(X, E)'$ there exist $x_{n,k} \in X$ and $\ell_{n,k} \in E'$ such that $\{n(\ell - \sum_k \ell_{n,k} \circ \mathrm{ev}_{x_{n,k}}) : n \in \mathbb{N}\}$ is bounded in $C^\infty(X, E)'$, where in the sum only finitely many terms are non-zero. In particular the subspace generated by $\ell_E \circ \mathrm{ev}_x$ for $\ell_E \in E'$ and $x \in X$ is c^∞-dense.

By 23.8 there exists a compact set K with $f|_K = 0$ implying $\ell(f) = 0$. One can cover K by finitely many relatively compact $U_j \cong \mathbb{R}^m$ ($j = 1 \dots N$). Let $\{h_j : j = 0 \dots N\}$ be a partition of unity subordinated to $\{X \setminus K, U_1, \dots, U_N\}$. Then $\ell(f) = \sum_{j=1}^N \ell(h_j \cdot f)$ for every f. By step (2) the set

$$\Big\{n(h_j f - \sum h_j f(r_{n,k_1,\dots,k_m}) g_{n,k_1,\dots,k_m} : n \in \mathbb{N}\Big\}$$

is bounded in $C^\infty(U_j, E)$. Since $\operatorname{supp}(h_j)$ is compact in U_j this is even bounded in $C^\infty(X, E)$ and for fixed n only finitely many $r_{n,k_1,\dots,k_m}$ belong to $\operatorname{supp}(h_j)$. Thus the above sum is actually finite and the supports of all functions in the bounded subset of $C^\infty(U_j, E)$ are included in a common compact subset. Applying ℓ to this subset yields that $\big\{n\big((\ell(h_j f) - \sum \ell_{n,k_1,\dots,k_m} \circ \operatorname{ev}(r_{n,k_1,\dots,k_m})\big) : n \in \mathbb{N}\big\}$ is bounded in $\mathbb{R}$, where $\ell_{n,k_1,\dots,k_m}(x) := \ell\big(h_j(r_{n,k_1,\dots,k_m}) g_{n;k_1,\dots,k_m} \cdot x\big)$.

To complete the proof one only has to take as $x_{n,k}$ all the $r_{n,k_1,\dots,k_m}$ for the finitely many charts $U_j \cong \mathbb{R}^m$ and as $\ell_{n,k}$ the corresponding functionals $\ell_{n,k_1,\dots,k_m} \in E'$. □

23.11. Corollary. [11, 5.1.8] *Let X be a finite dimensional separable smooth manifold. Then the free convenient vector space λX over X is equal to $C^\infty(X, \mathbb{R})'$.* □

23.12. Remark. In [Kriegl, Nel, 1990] it was shown that the free convenient vector space over the long line L is not $C^\infty(L, \mathbb{R})'$ and the same for the space E of points with countable support in an uncountable product of $\mathbb{R}$.

In [Adam, 1995, 2.2.6] it is shown that the isomorphism $\delta^* : L(C^\infty(X, \mathbb{R})', G) \cong C^\infty(X, G)$ is even a topological isomorphism for (the) natural topologies on all spaces under consideration provided X is a finite dimensional separable smooth manifold. Furthermore, the corresponding statement holds for holomorphic mappings, provided X is a separable complex manifold modeled on polycylinders. For Riemannian surfaces X it is shown in [Siegl, 1997, 2.11] that the free convenient vector space for holomorphic mappings is the Mackey adherence of the linear subspace of $\mathcal{H}(X, \mathbb{C})'$ generated by the point evaluations ev_x for $x \in X$. In [Siegl, 1997, 2.52] the same is shown for pseudo-convex subsets of $X \subseteq \mathbb{C}^n$. Reflexivity of the space of scalar valued functions implies that the linear space generated by the point evaluations is dense in the dual of the function space with respect to its bornological topology by [Siegl, 1997, 3.3]. And conversely if $\Lambda(X)$ is this dual, then the function space is reflexive. Thus $\Lambda(E) \neq C^\infty(E, \mathbb{R})'$ for non-reflexive convenient vector spaces E. Partial positive results for infinite dimensional spaces have been obtained in [Siegl, 1997, section 3].

23.13. Remark. On can define *convenient co-algebras* dually to convenient algebras, as a convenient vector space E together with a compatible co-algebra structure, i.e. two bounded linear mappings

> $\mu : E \to E \tilde{\otimes}_\beta E$, called co-multiplication, into the c^∞-completion 4.29 of the bornological tensor product 5.9;
> and $\varepsilon : E \to \mathbb{R}$, called co-unit,

such that one has the following commutative diagrams:

$$\begin{tikzcd}
E\tilde{\otimes}_\beta E \arrow[r, "\mu\tilde{\otimes}_\beta \mathrm{Id}"] & (E\tilde{\otimes}_\beta E)\tilde{\otimes}_\beta E \arrow[r, "\cong"] & E\tilde{\otimes}_\beta(E\tilde{\otimes}_\beta E) \arrow[d, "\mathrm{Id}\tilde{\otimes}_\beta\mu"] \\
E \arrow[u, leftarrow, "\mu"'] \arrow[rr, "\mu"] & & E\tilde{\otimes}_\beta E
\end{tikzcd}$$

$$\begin{tikzcd} & E\tilde{\otimes}_\beta E \arrow[dr, "\varepsilon\tilde{\otimes}\mathrm{Id}"] & \\ E \arrow[ur, "\mu"] \arrow[rr, "\cong"] & & \mathbb{R}\tilde{\otimes}_\beta E \end{tikzcd}$$

In words, the co-multiplication has to be co-associative and ε has to be a co-unit with respect to μ.

If, in addition, the following diagram commutes

$$\begin{tikzcd} E\tilde{\otimes}_\beta E \arrow[rr, "\cong"] & & E\tilde{\otimes}_\beta E \\ & E \arrow[ul, "\mu"] \arrow[ur, "\mu"'] & \end{tikzcd}$$

then the co-algebra is called *co-commutative*.

Morphisms $g : E \to F$ between convenient co-algebras E and F are bounded linear mappings for which the following diagrams commute:

$$\begin{tikzcd} E\tilde{\otimes}_\beta E \arrow[r, "g\tilde{\otimes}g"] & F\tilde{\otimes}_\beta F \\ E \arrow[u, "\mu_E"] \arrow[r, "g"] & F \arrow[u, "\mu_F"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{R} \arrow[r, "\mathrm{Id}"] & \mathbb{R} \\ E \arrow[u, "\varepsilon_E"] \arrow[r, "g"] & F \arrow[u, "\varepsilon_F"'] \end{tikzcd}$$

A *co-idempotent* in a convenient co-algebra E, is an element $x \in E$ satisfying $\varepsilon(x) = 1$ and $\mu(x) = x \otimes x$. They correspond bijectively to convenient co-algebra morphisms $\mathbb{R} \to E$, see [**11**, 5.2.7].

In [**11**, 5.2.4] it was shown that $\lambda(X \times Y) \cong \lambda(X)\tilde{\otimes}\lambda(Y)$ using only the universal property of the free convenient vector space. Thus $\lambda(\Delta) : \lambda(X) \to \lambda(X \times X) \cong \lambda(X)\tilde{\otimes}\lambda(X)$ of the diagonal mapping $\Delta : X \to X \times X$ defines a co-multiplication on $\lambda(X)$ with co-unit $\lambda(\mathrm{const}) : \lambda(X) \to \lambda(\{*\}) \cong \mathbb{R}$. In this way λ becomes a functor from the category of Frölicher spaces into that of convenient co-algebras, see [**11**, 5.2.5]. In fact this functor is left-adjoint to the functor I, which associates to each convenient co-algebra the Frölicher space of co-idempotents with the initial structure inherited from the co-algebra, see [**11**, 5.2.9].

Furthermore, it was shown in [**11**, 5.2.18] that any co-idempotent element e of $\lambda(X)$ defines an algebra-homomorphism $C^\infty(X, \mathbb{R}) \cong \lambda(X)' \xrightarrow{\mathrm{ev}_e} R$. Thus the equality $I(\lambda(X)) = X$, i.e. every co-idempotent $e \in \lambda(X)$ is given by ev_x for some $x \in X$, is thus satisfied for smoothly realcompact spaces X, as they are treated in chapter IV.

24. Smooth Mappings on Non-Open Domains

In this section we will discuss smooth maps $f : E \supseteq X \to F$, where E and F are convenient vector spaces and X are certain not necessarily open subsets of E. We consider arbitrary subsets $X \subseteq E$ as Frölicher spaces with the initial smooth structure induced by the inclusion into E, i.e., a map $f : E \supseteq X \to F$ is smooth if and only if for all smooth curves $c : \mathbb{R} \to X \subseteq E$ the composite $f \circ c : \mathbb{R} \to F$ is a smooth curve.

24.1. Lemma. Convex sets with non-void interior.
Let $K \subseteq E$ be a convex set with non-void c^∞-interior K^o. Then the segment $(x, y] := \{x + t(y - x) : 0 < t \le 1\}$ is contained in K^o for every $x \in K$ and $y \in K^0$.

The interior K^o is convex and open even in the locally convex topology. And K is closed if and only if it is c^∞-closed.

Proof. Let $y_0 := x + t_0(y - x)$ be an arbitrary point on the segment $(x, y]$, i.e., $0 < t_0 \le 1$. Then $x + t_0(K^o - x)$ is an c^∞-open neighborhood of y_0, since homotheties are c^∞-continuous. It is contained in K, since K is convex.

In particular, the c^∞-interior K^o is convex, hence it is not only c^∞-open but open in the locally convex topology 4.5.

Without loss of generality we now assume that $0 \in K^o$. We claim that the closure of K is the set $\{x : tx \in K^o \text{ for } 0 < t < 1\}$. This implies the statement on closedness. Let $U := K^o$ and consider the Minkowski-functional $p_U(x) := \inf\{t > 0 : x \in tU\}$. Since U is convex, the function p_U is convex, see [5, 2.3.6]. Using that U is c^∞-open it can easily be shown that $U = \{x : p_U(x) < 1\}$. From 13.2 we conclude that p_U is c^∞-continuous, and furthermore that it is even continuous for the locally convex topology. Hence, the set $\{x : tx \in K^o \text{ for } 0 < t < 1\} = \{x : p_U(x) \le 1\} = \{x : p_K(x) \le 1\}$ is the closure of K in the locally convex topology by [5, 2.3.6]. □

24.2. Theorem. Derivative of smooth maps.
Let $K \subseteq E$ be a convex subset with non-void interior K^o, and let $f : K \to \mathbb{R}$ be a smooth map. Then $f|_{K^o} : K^o \to F$ is smooth, and its derivative $(f|_{K^o})'$ extends (uniquely) to a smooth map $K \to L(E, F)$.

Proof. Only the extension property is to be shown. Let us first try to find a candidate for $f'(x)(v)$ for $x \in K$ and $v \in E$ with $x + v \in K^o$. By convexity the smooth curve $c_{x,v} : t \mapsto x + t^2 v$ has for $0 < |t| < 1$ values in K^o and $c_{x,v}(0) = x \in K$, hence $f \circ c_{x,v}$ is smooth. In the special case where $x \in K^o$ we have by the chain rule that $(f \circ c_{x,v})'(t) = f'(x)(c_{x,v}(t))(c'_{x,v}(t))$, hence $(f \circ c_{x,v})''(t) = f''(c_{x,v}(t))(c'_{x,v}(t), c'_{x,v}(t)) + f'(c_{x,v}(t))(c''_{x,v}(t))$, and for $t = 0$ in particular $(f \circ c_{x,v})''(0) = 2\, f'(x)(v)$. Thus we define

$$2\, f'(x)(v) := (f \circ c_{x,v})''(0) \text{ for } x \in K \text{ and } v \in K^o - x.$$

Note that for $0 < \varepsilon < 1$ we have $f'(x)(\varepsilon\, v) = \varepsilon\, f'(x)(v)$, since $c_{x,\varepsilon\, v}(t) = c_{x,v}(\sqrt{\varepsilon}\, t)$.

Let us show next that $f'(\quad)(v) : \{x \in K : x + v \in K^o\} \to \mathbb{R}$ is smooth. So let $s \mapsto x(s)$ be a smooth curve in K, and let $v \in K^0 - x(0)$. Then $x(s) + v \in K^o$ for all sufficiently small s. And thus the map $(s, t) \mapsto c_{x(s),v}(t)$ is smooth from some neighborhood of $(0, 0)$ into K. Hence $(s, t) \mapsto f(c_{x(s),v}(t))$ is smooth and also its second derivative $s \mapsto (f \circ c_{x(s),v})''(0) = 2\, f'(x(s))(v)$.

In particular, let $x_0 \in K$ and $v_0 \in K^o - x_0$ and $x(s) := x_0 + s^2 v_0$. Then

$$2f'(x_0)(v) := (f \circ c_{x_0,v})''(0) = \lim_{s\to 0}(f \circ c_{x(s),v})''(0) = \lim_{s\to 0} 2\, f'(x(s))(v),$$

with $x(s) \in K^o$ for $0 < |s| < 1$. Obviously this shows that the given definition of $f'(x_0)(v)$ is the only possible smooth extension of $f'(\quad)(v)$ to $\{x_0\} \cup K^o$.

Now let $v \in E$ be arbitrary. Choose a $v_0 \in K^o - x_0$. Since the set $K^o - x_0 - v_0$ is a c^∞-open neighborhood of 0, hence absorbing, there exists some $\varepsilon > 0$ such that $v_0 + \varepsilon v \in K^o - x_0$. Thus

$$f'(x)(v) = \tfrac{1}{\varepsilon} f'(x)(\varepsilon v) = \tfrac{1}{\varepsilon}\big(f'(x)(v_0 + \varepsilon v) - f'(x)(v_0)\big)$$

for all $x \in K^0$. By what we have shown above the right side extends smoothly to $\{x_0\} \cup K^o$, hence the same is true for the left side. I.e. we define $f'(x_0)(v) := \lim_{s\to 0} f'(x(s))(v)$ for some smooth curve $x : (-1, 1) \to K$ with $x(s) \in K^o$ for $0 < |s| < 1$. Then $f'(x)$ is linear as pointwise limit of $f'(x(s)) \in L(E, \mathbb{R})$ and is bounded by the Banach-Steinhaus theorem (applied to E_B). This shows at the same time, that the definition does not depend on the smooth curve x, since for $v \in x_0 + K^o$ it is the unique extension.

In order to show that $f' : K \to L(E, F)$ is smooth it is by 5.18 enough to show that

$$s \mapsto f'(x(s))(v), \quad \mathbb{R} \xrightarrow{x} K \xrightarrow{f'} L(E, F) \xrightarrow{\mathrm{ev}_x} F$$

is smooth for all $v \in E$ and all smooth curves $x : \mathbb{R} \to K$. For $v \in x_0 + K^o$ this was shown above. For general $v \in E$, this follows since $f'(x(s))(v)$ is a linear combination of $f'(x(s))(v_0)$ for two $v_0 \in x_0 + K^o$ not depending on s locally. □

By 24.2 the following lemma applies in particular to smooth maps.

24.3. Lemma. Chain rule. *Let $K \subseteq E$ be a convex subset with non-void interior K^o, let $f : K \to \mathbb{R}$ be smooth on K^o and let $f' : K \to L(E, F)$ be an extension of $(f|_{K^o})'$, which is continuous for the c^∞-topology of K, and let $c : \mathbb{R} \to K \subseteq E$ be a smooth curve. Then $(f \circ c)'(t) = f'(c(t))(c'(t))$.*

Proof.

Claim Let $g : K \to L(E, F)$ be continuous along smooth curves in K, then $\hat{g} : K \times E \to F$ is also continuous along smooth curves in $K \times E$.
In order to show this let $t \mapsto (x(t), v(t))$ be a smooth curve in $K \times E$. Then $g \circ x : \mathbb{R} \to L(E, F)$ is by assumption continuous (for the bornological topology on $L(E, F)$) and $v^* : L(E, F) \to C^\infty(\mathbb{R}, F)$ is bounded and linear 3.13 and 3.17. Hence, the composite $v^* \circ g \circ x : \mathbb{R} \to C^\infty(\mathbb{R}, F) \to C(\mathbb{R}, F)$ is continuous. Thus, $(v^* \circ g \circ x)^\wedge : \mathbb{R}^2 \to F$ is continuous, and in particular when restricted to the diagonal in $\mathbb{R}^2$. But this restriction is just $g \circ (x, v)$.

Now choose a $y \in K^o$. And let $c_s(t) := c(t) + s^2(y - c(t))$. Then $c_s(t) \in K^o$ for $0 < |s| \le 1$ and $c_0 = c$. Furthermore, $(s, t) \mapsto c_s(t)$ is smooth and $c_s'(t) = (1 - s^2)c'(t)$. And for $s \neq 0$

$$\frac{f(c_s(t)) - f(c_s(0))}{t} = \int_0^1 (f \circ c_s)'(t\tau) d\tau = (1 - s^2) \int_0^1 f'(c_s(t\tau))(c'(t\tau)) d\tau .$$

Now consider the specific case where $c(t) := x + tv$ with x, $x + v \in K$. Since f is continuous along $(t, s) \mapsto c_s(t)$, the left side of the above equation converges to $\frac{f(c(t)) - f(c(0))}{t}$ for $s \to 0$. And since $f'(\cdot)(v)$ is continuous along $(t, \tau, s) \mapsto c_s(t\tau)$ we have that $f'(c_s(t\tau))(v)$ converges to $f'(c(t\tau))(v)$ uniformly with respect to $0 \le \tau \le 1$ for $s \to 0$. Thus, the right side of the above equation converges to $\int_0^1 f'(c(t\tau))(v) d\tau$. Hence, we have

$$\frac{f(c(t)) - f(c(0))}{t} = \int_0^1 f'(c(t\tau))(v) d\tau \to \int_0^1 f'(c(0))(v) d\tau = f'(c(0))(c'(0))$$

for $t \to 0$.

Now let $c : \mathbb{R} \to K$ be an arbitrary smooth curve. Then $(s, t) \mapsto c(0) + s(c(t) - c(0))$ is smooth and has values in K for $0 \le s \le 1$. By the above consideration we have for $x = c(0)$ and $v = (c(t) - c(0))/t$ that

$$\frac{f(c(t)) - f(c(0))}{t} = \int_0^1 f'\Big(c(0) + \tau(c(t) - c(0))\Big)\Big(\frac{c(t) - c(0)}{t}\Big)$$

which converges to $f'(c(0))(c'(0))$ for $t \to 0$, since f' is continuous along smooth curves in K and thus $f'(c(0) + \tau(c(t) - c(0))) \to f'(c(0))$ uniformly on the bounded set $\{\frac{c(t)-c(0)}{t} : t \text{ near } 0\}$. Thus, $f \circ c$ is differentiable with derivative $(f \circ c)'(t) = f'(c(t))(c'(t))$. □

Since f' can be considered as a map $df : E \times E \supseteq K \times E \to F$ it is important to study sets $A \times B \subseteq E \times F$. Clearly, $A \times B$ is convex provided $A \subseteq E$ and $B \subseteq F$ are. Remains to consider the openness condition. In the locally convex topology $(A \times B)^o = A^o \times B^o$, which would be enough to know in our situation. However, we are also interested in the corresponding statement for the c^∞-topology. This topology on $E \times F$ is in general not the product topology $c^\infty E \times c^\infty F$. Thus, we cannot conclude that $A \times B$ has non-void interior with respect to the c^∞-topology on $E \times F$, even if $A \subseteq E$ and $B \subseteq F$ have it. However, in case where $B = F$ everything is fine.

24.4. Lemma. Interior of a product.
Let $X \subseteq E$. Then the interior $(X \times F)^o$ of $X \times F$ with respect to the c^∞-topology on $E \times F$ is just $X^o \times F$, where X^o denotes the interior of X with respect to the c^∞-topology on E.

Proof. Let W be the saturated hull of $(X \times F)^o$ with respect to the projection $\mathrm{pr}_1 : E \times F \to E$, i.e. the c^∞-open set $(X \times F)^o + \{0\} \times F \subseteq X \times F$. Its projection to E is c^∞-open, since it agrees with the intersection with $E \times \{0\}$. Hence, it is contained in X^o, and $(X \times F)^o \subseteq X^o \times F$. The converse inclusion is obvious since pr_1 is continuous. □

24.5. Theorem. Smooth maps on convex sets.
Let $K \subseteq E$ be a convex subset with non-void interior K^o, and let $f : K \to F$ be a map. Then f is smooth if and only if f is smooth on K^o and all derivatives $(f|_{K^o})^{(n)}$ extend continuously to K with respect to the c^∞-topology of K.

Proof. ($\Rightarrow$) It follows by induction using 24.2 that $f^{(n)}$ has a smooth extension $K \to L^n(E; F)$.

($\Leftarrow$) By 24.3 we conclude that for every $c : \mathbb{R} \to K$ the composite $f \circ c : \mathbb{R} \to F$ is differentiable with derivative $(f \circ c)'(t) = f'(c(t))(c'(t)) =: df(c(t), c'(t))$.

The map df is smooth on the interior $K^o \times E$, linear in the second variable, and its derivatives $(df)^{(p)}(x, w)(y_1, w_1; \ldots, y_p, w_p)$ are universal linear combinations of

$$f^{(p+1)}(x)(y_1, \ldots, y_p; w) \text{ and of } f^{(k+1)}(x)(y_{i_1}, \ldots, y_{i_k}; w_{i_0}) \text{ for } k \le p.$$

These summands have unique extensions to $K \times E$. The first one is continuous along smooth curves in $K \times E$, because for such a curve $(t \mapsto (x(t), w(t))$ the extension $f^{(k+1)} : K \to L(E^k, L(E, F))$ is continuous along the smooth curve x, and $w^* : L(E, F) \to C^\infty(\mathbb{R}, F)$ is continuous and linear, so the mapping $t \mapsto (s \mapsto f^{(k+1)}(x(t))(y_{i_1}, \ldots, y_{i_k}; w(s)))$ is continuous from $\mathbb{R} \to C^\infty(\mathbb{R}, F)$ and thus as map from $\mathbb{R}^2 \to F$ it is continuous, and in particular if restricted to the diagonal. And the other summands only depend on x, hence have a continuous extension by assumption.

So we can apply 24.3 inductively using 24.4, to conclude that $f \circ c : \mathbb{R} \to F$ is smooth. □

In view of the preceding theorem 24.5 it is important to know the c^∞-topology $c^\infty X$ of X, i.e. the final topology generated by all the smooth curves $c : \mathbb{R} \to X \subseteq E$. So the first question is whether this is the trace topology $c^\infty E|_X$ of the c^∞-topology of E.

24.6. Lemma. The c^∞-topology is the trace topology.
In the following cases of subsets $X \subseteq E$ the trace topology $c^\infty E|X$ equals the topology $c^\infty X$:

(1) *X is $c^\infty E$-open.*
(2) *X is convex and locally c^∞-closed.*
(3) *The topology $c^\infty E$ is sequential and $X \subseteq E$ is convex and has non-void interior.*

(3) applies in particular to the case where E is metrizable, see 4.11. A topology is called sequential if and only if the closure of any subset equals its adherence, i.e. the set of all accumulation points of sequences in it. By 2.13 and 2.8 the adherence of a set X with respect to the c^∞-topology, is formed by the limits of all Mackey-converging sequences in X.

Proof. Note that the inclusion $X \to E$ is by definition smooth, hence the identity $c^\infty X \to c^\infty E|_X$ is always continuous.

(1) Let $U \subseteq X$ be $c^\infty X$-open and let $c : \mathbb{R} \to E$ be a smooth curve with $c(0) \in U$. Since X is $c^\infty E$-open, $c(t) \in X$ for all small t. By composing with a smooth map $h : \mathbb{R} \to \mathbb{R}$ which satisfies $h(t) = t$ for all small t, we obtain a smooth curve $c \circ h : \mathbb{R} \to X$, which coincides with c locally around 0. Since U is $c^\infty X$-open we conclude that $c(t) = (c \circ h)(t) \in U$ for small t. Thus, U is $c^\infty E$-open.

(2) Let $A \subseteq X$ be $c^\infty X$-closed. And let $\bar{A}$ be the $c^\infty E$-closure of A. We have to show that $\bar{A} \cap X \subseteq A$. So let $x \in \bar{A} \cap X$. Since X is locally $c^\infty E$-closed, there exists a $c^\infty E$-neighborhood U of $x \in X$ with $U \cap X$ c^∞-closed in U. For every $c^\infty E$-neighborhood U of x we have that x is in the closure of $A \cap U$ in U with respect to the $c^\infty E$-topology (otherwise some open neighborhood of x in U does not meet $A \cap U$, hence also not A). Let $a_n \in A \cap U$ be Mackey converging to $a \in U$. Then $a_n \in X \cap U$ which is closed in U thus $a \in X$. Since X is convex the infinite polygon through the a_n lies in X and can be smoothly parameterized by the special curve lemma 2.8. Using that A is $c^\infty X$-closed, we conclude that $a \in A$. Thus, $A \cap U$ is $c^\infty U$-closed and $x \in A$.

(3) Let $A \subseteq X$ be $c^\infty X$-closed. And let $\bar{A}$ denote the closure of A in $c^\infty E$. We have to show that $\bar{A} \cap X \subseteq A$. So let $x \in \bar{A} \cap X$. Since $c^\infty E$ is sequential there is a Mackey converging sequence $A \ni a_n \to x$. By the special curve lemma 2.8 the infinite polygon through the a_n can be smoothly parameterized. Since X is convex this curve gives a smooth curve $c : \mathbb{R} \to X$ and thus $c(0) = x \in A$, since A is $c^\infty X$-closed. □

24.7. Example. The c^∞-topology is not trace topology.
Let $A \subseteq E$ be such that the c^∞-adherence $\operatorname{Adh}(A)$ of A is not the whole c^∞-closure $\bar{A}$ of A. So let $a \in \bar{A} \setminus \operatorname{Adh}(A)$. Then consider the convex subset $K \subseteq E \times \mathbb{R}$ defined by $K := \{(x,t) \in E \times \mathbb{R} : t \geq 0 \text{ and } (t = 0 \Rightarrow x \in A \cup \{a\})\}$ which has non-empty interior $E \times \mathbb{R}^+$. However, the topology $c^\infty K$ is not the trace topology of $c^\infty(E \times \mathbb{R})$ which equals $c^\infty(E) \times \mathbb{R}$ by 4.15.

Note that this situation occurs quite often, see 4.13 and 4.36 where A is even a linear subspace.

Proof. Consider $A = A \times \{0\} \subseteq K$. This set is closed in $c^\infty K$, since $E \cap K$ is closed in $c^\infty K$ and the only point in $(K \cap E) \setminus A$ is a, which cannot be reached by a Mackey converging sequence in A, since $a \notin \operatorname{Adh}(A)$.

It is however not the trace of a closed subset in $c^\infty(E) \times \mathbb{R}$. Since such a set has to contain A and hence $\bar{A} \ni a$. □

24.8. Theorem. Smooth maps on subsets with collar.
Let $M \subseteq E$ have a smooth collar, i.e., the boundary ∂M of M is a smooth submanifold of E and there exists a neighborhood U of ∂M and a diffeomorphism $\psi : \partial M \times \mathbb{R} \to U$ which is the identity on ∂M and such that $\psi(M \times \{t \in \mathbb{R} : t \geq 0\}) = M \cap U$. Then every smooth map $f : M \to F$ extends to a smooth map $\tilde{f} : M \cup U \to F$. Moreover, one can choose a bounded linear extension operator $C^\infty(M, F) \to C^\infty(M \cup U, F)$, $f \mapsto \tilde{f}$.

Proof. By 16.8 there is a continuous linear right inverse S to the restriction map $C^\infty(\mathbb{R}, \mathbb{R}) \to C^\infty(I, \mathbb{R})$, where $I := \{t \in \mathbb{R} : t \geq 0\}$. Now let $x \in U$ and $(p_x, t_x) := \psi^{-1}(x)$. Then $f(\psi(p_x, \cdot)) : I \to F$ is smooth, since $\psi(p_x, t) \in M$ for $t \geq 0$. Thus, we have a smooth map $S(f(\psi(p_x, \cdot))) : \mathbb{R} \to F$ and we define $\tilde{f}(x) := S(f(\psi(p_x, \cdot)))(t_x)$. Then $\tilde{f}(x) = f(x)$ for all $x \in M \cap U$, since for such an x we have $t_x \geq 0$. Now we extend the definition by $\tilde{f}(x) = f(x)$ for $x \in M^o$. Remains to show that $\tilde{f}$ is smooth (on U). So let $s \mapsto x(s)$ be a smooth curve in U. Then $s \mapsto (p_s, t_s) := \psi^{-1}(x(s))$ is smooth. Hence, $s \mapsto (t \mapsto f(\psi(p_s, t))$ is a smooth curve $\mathbb{R} \to C^\infty(I, F)$. Since S is continuous and linear the composite $s \mapsto (t \mapsto S(f\psi(p_s, \cdot))(t))$ is a smooth curve $\mathbb{R} \to C^\infty(\mathbb{R}, F)$ and thus the associated map $\mathbb{R}^2 \to F$ is smooth, and also the composite $\tilde{f}(x_s)$ of it with $s \mapsto (s, t_s)$.

The existence of a bounded linear extension operator follows now from 21.2. $\square$

In particular, the previous theorem applies to the following convex sets:

24.9. Proposition. Convex sets with smooth boundary have a collar.
Let $K \subseteq E$ be a closed convex subset with non-empty interior and smooth boundary ∂K. Then K has a smooth collar as defined in 24.8.

Proof. Without loss of generality let $0 \in K^o$.

In order to show that the set $U := \{x \in E : tx \notin K \text{ for some } t > 0\}$ is c^∞-open let $s \mapsto x(s)$ be a smooth curve $\mathbb{R} \to E$ and assume that $t_0 x(0) \notin K$ for some $t_0 > 0$. Since K is closed we have that $t_0 x(s) \notin K$ for all small $|s|$.

For $x \in U$ let $r(x) := \sup\{t \geq 0 : tx \in K^o\} > 0$, i.e. $r = \frac{1}{p_{K^o}}$ as defined in the proof of 24.1 and $r(x)x$ is the unique intersection point of $\partial K \cap (0, +\infty)x$. We claim that $r : U \to \mathbb{R}^+$ is smooth. So let $s \mapsto x(s)$ be a smooth curve in U and $x_0 := r(x(0))x(0) \in \partial K$. Choose a local diffeomorphism $\psi : (E, x_0) \to (E, 0)$ which maps ∂K locally to some closed hyperplane $F \subseteq E$. Any such hyperplane is the kernel of a continuous linear functional $\ell : E \to \mathbb{R}$, hence $E \cong F \times \mathbb{R}$.

We claim that $v := \psi'(x_0)(x_0) \notin F$. If this were not the case, then we consider the smooth curve $c : \mathbb{R} \to \partial K$ defined by $c(t) = \psi^{-1}(-tv)$. Since $\psi'(x_0)$ is injective its derivative is $c'(0) = -x_0$ and $c(0) = x_0$. Since $0 \in K^o$, we have that $x_0 + \frac{c(t)-c(0)}{t} \in K^o$ for all small $|t|$. By convexity $c(t) = x_0 + t\frac{c(t)-c(0)}{t} \in K^o$ for small $t > 0$, a contradiction.

So we may assume that $\ell(\psi'(x)(x)) \neq 0$ for all x in a neighborhood of x_0.

For s small $r(x(s))$ is given by the implicit equation $\ell(\psi(r(x(s))x(s))) = 0$. So let $g : \mathbb{R}^2 \to \mathbb{R}$ be the locally defined smooth map $g(t, s) := \ell(\psi(tx(s)))$. For $t \neq 0$ its first partial derivative is $\partial_1 g(t, s) = \ell(\psi'(tx(s))(x(s))) \neq 0$. So by the classical implicit function theorem the solution $s \mapsto r(x(s))$ is smooth.

Now let $\Psi : U \times \mathbb{R} \to U$ be the smooth map defined by $(x, t) \mapsto e^{-t} r(x)x$. Restricted to $\partial K \times \mathbb{R} \to U$ is injective, since $tx = t'x'$ with $x, x' \in \partial K$ and $t, t' > 0$ implies $x = x'$ and hence $t = t'$. Furthermore, it is surjective, since the inverse mapping is given by $x \mapsto (r(x)x, \ln(r(x)))$. Use that $r(\lambda x) = \frac{1}{\lambda} r(x)$. Since this inverse is also smooth, we have the required diffeomorphism Ψ. In fact, $\Psi(x,t) \in K$ if and only if $e^{-t} r(x) \le r(x)$, i.e. $t \le 0$. $\square$

That 24.8 is far from being best possible shows the

24.10. Proposition. *Let $K \subseteq \mathbb{R}^n$ be the quadrant $K := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \ge 0, \dots, x_n \ge 0\}$. Then there exists a bounded linear extension operator $C^\infty(K, F) \to C^\infty(\mathbb{R}^n, F)$ for each convenient vector space F.*

This can be used to obtain the same result for submanifolds with convex corners sitting in smooth finite dimensional manifolds.

Proof. Since $K = (\mathbb{R}_+)^n \subseteq \mathbb{R}^n$ and the inclusion is the product of inclusions $\iota : \mathbb{R}_+ \hookrightarrow \mathbb{R}$ we can use the exponential law 23.2.3 to obtain $C^\infty(K, F) \cong C^\infty((\mathbb{R}_+)^{n-1}, C^\infty(\mathbb{R}_+, F))$. By Seeley's theorem 16.8 we have a bounded linear extension operator $S : C^\infty(\mathbb{R}_+, F) \to C^\infty(\mathbb{R}, F)$. We now proceed by induction on n. So we have an extension operator $S_{n-1} : C^\infty((\mathbb{R}_+)^{n-1}, G) \to C^\infty(\mathbb{R}^{n-1}, G)$ for the convenient vector space $G := C^\infty(\mathbb{R}, F)$ by induction hypothesis. The composite gives up to natural isomorphisms the required extension operator

$$C^\infty(K, F) \cong C^\infty((\mathbb{R}_+)^{n-1}, C^\infty(\mathbb{R}_+, F)) \xrightarrow{S_*} C^\infty((\mathbb{R}_+)^{n-1}, C^\infty(\mathbb{R}, F)) \to$$
$$\xrightarrow{S_{n-1}} C^\infty(\mathbb{R}^{n-1}, C^\infty(\mathbb{R}, F)) \cong C^\infty(\mathbb{R}^n, F). \quad \square$$

25. Real Analytic Mappings on Non-Open Domains

In this section we will consider real analytic mappings defined on the same type of convex subsets as in the previous section.

25.1. Theorem. Power series in Fréchet spaces. *Let E be a Fréchet space and (F, F') be a dual pair. Assume that a Baire vector space topology on E' exists for which the point evaluations are continuous. Let f_k be k-linear symmetric bounded functionals from E to F, for each $k \in \mathbb{N}$. Assume that for every $\ell \in F'$ and every x in some open subset $W \subseteq E$ the power series $\sum_{k=0}^\infty \ell(f_k(x^k))t^k$ has positive radius of convergence. Then there exists a 0-neighborhood U in E, such that $\{f_k(x_1, \dots, x_k) : k \in \mathbb{N}, x_j \in U\}$ is bounded and thus the power series $x \mapsto \sum_{k=0}^\infty f_k(x^k)$ converges Mackey on some 0-neighborhood in E.*

Proof. Choose a fixed but arbitrary $\ell \in F'$. Then $\ell \circ f_k$ satisfy the assumptions of 7.14 for an absorbing subset in a closed cone C with non-empty interior. Since this cone is also complete metrizable we can proceed with the proof as in 7.14 to obtain a set $A_{K,r} \subseteq C$ whose interior in C is non-void. But this interior has to contain a non-void open set of E and as in the proof of 7.14 there exists some $\rho_\ell > 0$ such that for the ball U_{ρ_ℓ} in E with radius ρ_ℓ and center 0 the set $\{\ell(f_k(x_1, \dots, x_k)) : k \in \mathbb{N}, x_j \in U_{\rho_\ell}\}$ is bounded.

Now let similarly to 9.6

$$A_{K,r,\rho} := \bigcap_{k \in \mathbb{N}} \bigcap_{x_1, \dots x_n \in U_\rho} \{\ell \in F' : |\ell(f_k(x_1, \dots, x_k))| \le K r^k\}$$

for $K, r, \rho > 0$. These sets $A_{K,r,\rho}$ are closed in the Baire topology, since evaluation at $f_k(x_1, \dots, x_k)$ is assumed to be continuous.

By the first part of the proof the union of these sets is F'. So by the Baire property, there exist $K, r, \rho > 0$ such that the interior U of $A_{K,r,\rho}$ is non-empty. As in the proof of 9.6 we choose an $\ell_0 \in U$. Then for every $\ell \in F'$ there exists some $\varepsilon > 0$ such that $\ell_\varepsilon := \varepsilon\ell \in U - \ell_0$. So $|\ell(y)| \le \frac{1}{\varepsilon}(|\ell_\varepsilon(y) + \ell_0(y)| + |\ell_0(y)|) \le \frac{2}{\varepsilon} K r^n$ for every $y = f_k(x_1, \dots, x_k)$ with $x_i \in U_\rho$. Thus, $\{f_k(x_1, \dots, x_k) : k \in \mathbb{N}, x_i \in U_{\frac{\rho}{r}}\}$ is bounded.

On every smaller ball we have therefore that the power series with terms f_k converges Mackey. □

Note that if the vector spaces are real and the assumption above hold, then the conclusion is even true for the complexified terms by 7.14.

25.2. Theorem. Real analytic maps $I \to \mathbb{R}$ are germs.
Let $f : I := \{t \in \mathbb{R} : t \ge 0\} \to \mathbb{R}$ be a map. Suppose $t \mapsto f(t^2)$ is real analytic $\mathbb{R} \to \mathbb{R}$. Then f extends to a real analytic map $\tilde{f} : \tilde{I} \to \mathbb{R}$, where $\tilde{I}$ is an open neighborhood of I in $\mathbb{R}$.

Proof. We show first that f is smooth. Consider $g(t) := f(t^2)$. Since $g : \mathbb{R} \to \mathbb{R}$ is assumed to be real analytic it is smooth and clearly even. We claim that there exists a smooth map $h : \mathbb{R} \to \mathbb{R}$ with $g(t) = h(t^2)$ (this is due to [Whitney, 1943]). In fact, by $h(t^2) := g(t)$ a continuosu map $h : \{t :\in \mathbb{R} : t \ge 0\} \to \mathbb{R}$ is uniquely determined. Obviously, $h|_{\{t\in\mathbb{R}:t>0\}}$ is smooth. Differentiating for $t \ne 0$ the defining equation gives $h'(t^2) = \frac{g'(t)}{2t} =: g_1(t)$. Since g is smooth and even, g' is smooth and odd, so $g'(0) = 0$. Thus

$$t \mapsto g_1(t) = \frac{g'(t) - g'(0)}{2t} = \frac{1}{2}\int_0^1 g''(ts)\, ds$$

is smooth. Hence, we may define h' on $\{t \in \mathbb{R} : t \ge 0\}$ by the equation $h'(t^2) = g_1(t)$ with even smooth g_1. By induction we obtain continuous extensions of $h^{(n)} : \{t \in \mathbb{R} : t > 0\} \to \mathbb{R}$ to $\{t \in \mathbb{R} : t \ge 0\}$, and hence h is smooth on $\{t \in \mathbb{R} : t \ge 0\}$ and so can be extended to a smooth map $h : \mathbb{R} \to \mathbb{R}$.

From this we get $f(t^2) = g(t) = h(t^2)$ for all t. Thus, $h : \mathbb{R} \to \mathbb{R}$ is a smooth extension of f.

Composing with the exponential map $\exp : \mathbb{R} \to \mathbb{R}^+$ shows that f is real analytic on $\{t : t > 0\}$, and has derivatives $f^{(n)}$ which extend by 24.5 continuously to maps $I \to \mathbb{R}$. It is enough to show that $a_n := \frac{1}{n!} f^{(n)}(0)$ are the coefficients of a power series p with positive radius of convergence and for $t \in I$ this map p coincides with f.

Claim. We show that a smooth map $f : I \to \mathbb{R}$, which has a real analytic composite with $t \mapsto t^2$, is the germ of a real analytic mapping.
Consider the real analytic curve $c : \mathbb{R} \to I$ defined by $c(t) = t^2$. Thus, $f \circ c$ is real analytic. By the chain rule the derivative $(f \circ c)^{(p)}(t)$ is for $t \ne 0$ a universal linear combination of terms $f^{(k)}(c(t)) c^{(p_1)}(t) \cdots c^{(p_k)}(t)$, where $1 \le k \le p$ and $p_1 + \ldots + p_k = p$. Taking the limit for $t \to 0$ and using that $c^{(n)}(0) = 0$ for all $n \ne 2$ and $c''(0) = 2$ shows that there is a universal constant c_p satisfying $(f \circ c)^{(2p)}(0) = c_p \cdot f^{(p)}(0)$. Take as $f(x) = x^p$ to conclude that $(2p)! = c_p \cdot p!$. Now we use 9.2 to show that the power series $\sum_{k=0}^\infty \frac{1}{k!} f^{(k)}(0) t^k$ converges locally. So choose a sequence (r_k) with $r_k t^k \to 0$ for all $t > 0$. Define a sequence $(\bar{r}_k)$ by $\bar{r}_{2n} = \bar{r}_{2n+1} := r_n$ and let $\bar{t} > 0$. Then $\bar{r}_k \bar{t}^k = r_n t^n$ for $2n = k$ and $\bar{r}_k \bar{t}^k = r_n t^n \bar{t}$ for $2n + 1 = k$, where $t := \bar{t}^2 > 0$, hence $(\bar{r}_k)$ satisfies the same assumptions as (r_k) and thus by 9.3 $(1 \Rightarrow 3)$ the sequence $\frac{1}{k!}(f \circ c)^{(k)}(0) \bar{r}_k$ is bounded. In particular,

this is true for the subsequence

$$\tfrac{1}{(2p)!}(f\circ c)^{(2p)}(0)\bar r_{2p} = \tfrac{c_p}{(2p)!}f^{(p)}(0)r_p = \tfrac{1}{p!}f^{(p)}(0)r_p.$$

Thus, by 9.3 ($1 \Leftarrow 3$) the power series with coefficients $\frac{1}{p!}f^{(p)}(0)$ converges locally to a real analytic function $\tilde f$.

Remains to show that $\tilde f = f$ on J. But since $\tilde f\circ c$ and $f\circ c$ are both real analytic near 0, and have the same Taylor series at 0, they have to coincide locally, i.e. $\tilde f(t^2) = f(t^2)$ for small t. □

Note however that the more straight forward attempt of a proof of the first step, namely to show that $f\circ c$ is smooth for all $c:\mathbb{R}\to\{t\in\mathbb{R}: t\geq 0\}$ by showing that for such c there is a smooth map $h:\mathbb{R}\to\mathbb{R}$, satisfying $c(t) = h(t)^2$, is doomed to fail as the following example shows.

25.3. Example. A smooth function without smooth square root. *Let $c:\mathbb{R}\to\{t\in\mathbb{R}: t\geq 0\}$ be defined by the general curve lemma 12.2 using pieces of parabolas $c_n : t\mapsto \frac{2n}{2^n}t^2 + \frac{1}{4^n}$. Then there is no smooth square root of c.*

Proof. The curve c constructed in 12.2 has the property that there exists a converging sequence t_n such that $c(t+t_n) = c_n(t)$ for small t. Assume there were a smooth map $h:\mathbb{R}\to\mathbb{R}$ satisfying $c(t) = h(t)^2$ for all t. At points where $c(t)\neq 0$ we have in turn:

$$\begin{aligned} c'(t) &= 2h(t)h'(t)\\ c''(t) &= 2h(t)h''(t) + 2h'(t)^2\\ 2c(t)c''(t) &= 4h(t)^3h''(t) + c'(t)^2. \end{aligned}$$

Choosing t_n for t in the last equation gives $h''(t_n) = 2n$, which is unbounded in n. Thus h cannot be C^2. □

25.4. Definition. (Real analytic maps $I\to F$)
Let $I\subseteq\mathbb{R}$ be a non-trivial interval. Then a map $f: I\to F$ is called real analytic if and only if the composites $\ell\circ f\circ c:\mathbb{R}\to\mathbb{R}$ are real analytic for all real analytic $c:\mathbb{R}\to I\subseteq\mathbb{R}$ and all $\ell\in F'$. If I is an open interval then this definition coincides with 10.3.

25.5. Lemma. Bornological description of real analyticity. *Let $I\subseteq\mathbb{R}$ be a compact interval. A curve $c: I\to E$ is real analytic if and only if c is smooth and the set $\{\frac{1}{k!}c^{(k)}(a)\,r_k : a\in I, k\in\mathbb{N}\}$ is bounded for all sequences (r_k) with $r_k\,t^k\to 0$ for all $t>0$.*

Proof. We use 9.3. Since both sides can be tested with $\ell\in E'$ we may assume that $E=\mathbb{R}$.

($\Rightarrow$) By 25.2 we may assume that $c:\tilde I\to\mathbb{R}$ is real analytic for some open neighborhood $\tilde I$ of I. Thus, the required boundedness condition follows from 9.3.

($\Leftarrow$) By 25.2 we only have to show that $f: t\mapsto c(t^2)$ is real analytic. For this we use again 9.3. So let $K\subseteq\mathbb{R}$ be compact. Then the Taylor series of f is obtained by that of c composed with t^2. Thus, the composite f satisfies the required boundedness condition, and hence is real analytic. □

This characterization of real analyticity can not be weakened by assuming the boundedness conditions only for single pointed K as the map $c(t) := e^{-1/t^2}$ for

$t \neq 0$ and $c(0) = 0$ shows. It is real analytic on $\mathbb{R} \setminus \{0\}$ thus the condition is satisfied at all points there, and at 0 the power series has all coefficients equal to 0, hence the condition is satisfied there as well.

25.6. Corollary. Real analytic maps into inductive limits.
Let $T_\alpha : E \to E_\alpha$ be a family of bounded linear maps that generates the bornology on E. Then a map $c : I \to F$ is real analytic if and only if all the composites $T_\alpha \circ c : I \to F_\alpha$ are real analytic.

Proof. This follows either directly from 25.5 or from 25.2 by using the corresponding statement for maps $\mathbb{R} \to E$, see 9.9. □

25.7. Definition. (Real analytic maps $K \to F$)
For an arbitrary subset $K \subseteq E$ let us call a map $f : E \supseteq K \to F$ real analytic if and only if $\lambda \circ f \circ c : I \to \mathbb{R}$ is a real analytic (resp. smooth) for all $\lambda \in F'$ and all real analytic (resp. smooth) maps $c : I \to K$, where $I \subset \mathbb{R}$ is some compact non-trivial interval. Note however that it is enough to use all real analytic (resp. smooth) curves $c : \mathbb{R} \to K$ by 25.2.

With $C^\omega(K, F)$ we denote the vector space of all real analytic maps $K \to F$. And we topologize this space with the initial structure induced by the cone $c^* : C^\omega(K, F) \to C^\omega(\mathbb{R}, F)$ (for all real analytic $c : \mathbb{R} \to K$) and the cone $c^* : C^\omega(K, F) \to C^\infty(\mathbb{R}, F)$ (for all smooth $c : \mathbb{R} \to K$). The space $C^\omega(\mathbb{R}, F)$ should carry the structure of 11.2 and the space $C^\infty(\mathbb{R}, F)$ that of 3.6.

For an open $K \subseteq E$ the definition for $C^\omega(K, F)$ given here coincides with that of 10.3.

25.8. Proposition. $C^\omega(K, F)$ is convenient. *Let $K \subseteq E$ and F be arbitrary. Then the space $C^\omega(K, F)$ is a convenient vector space and satisfies the $\mathcal{S}$-uniform boundedness principle 5.22, where $\mathcal{S} := \{\mathrm{ev}_x : x \in K\}$.*

Proof. Since both spaces $C^\omega(\mathbb{R}, \mathbb{R})$ and $C^\infty(\mathbb{R}, \mathbb{R})$ are c^∞-complete and satisfy the uniform boundedness principle for the set of point evaluations the same is true for $C^\omega(K, F)$, by 5.25. □

25.9. Theorem. Real analytic maps $K \to F$ are often germs.
Let $K \subseteq E$ be a convex subset with non-empty interior of a Fréchet space and let (F, F') be a complete dual pair for which a Baire topology on F' exists, as required in 25.1. Let $f : K \to F$ be a real analytic map. Then there exists an open neighborhood $U \subseteq E_\mathbb{C}$ of K and a holomorphic map $\tilde{f} : U \to F_\mathbb{C}$ such that $\tilde{f}|_K = f$.

Proof. By 24.5 the map $f : K \to F$ is smooth, i.e. the derivatives $f^{(k)}$ exist on the interior K^0 and extend continuously (with respect to the c^∞-topology of K) to the whole of K. So let $x \in K$ be arbitrary and consider the power series with coefficients $f_k = \frac{1}{k!} f^{(k)}(x)$. This power series has the required properties of 25.1, since for every $\ell \in F'$ and $v \in K^o - x$ the series $\sum_k \ell(f_k(v^k))t^k$ has positive radius of convergence. In fact, $\ell(f(x + tv))$ is by assumption a real analytic germ $I \to \mathbb{R}$, by 24.8 hence locally around any point in I it is represented by its converging Taylor series at that point. Since $(x, v - x] \subseteq K^o$ and f is smooth on this set, $(\frac{d}{dt})^k(\ell(f(x + tv)) = \ell(f^{(k)}(x + tv)(v^k)$ for $t > 0$. Now take the limit for $t \to 0$ to conclude that the Taylor coefficients of $t \mapsto \ell(f(x + tv))$ at $t = 0$ are exactly $k!\ell(f_k)$. Thus, by 25.1 the power series converges locally and hence represents a holomorphic map in a neighborhood of x. Let $y \in K^o$ be an arbitrary point in this neighborhood. Then $t \mapsto \ell(f(x + t(y - x)))$ is real analytic $I \to \mathbb{R}$ and hence the series converges at $y - x$ towards $f(y)$. So the restriction of the power series to the interior of K coincides with f.

We have to show that the extensions f_x of $f : K \cap \tilde{U}_x \to F_{\mathbb{C}}$ to star shaped neighborhoods $\tilde{U}_x$ of x in $E_{\mathbb{C}}$ fit together to give an extension $\tilde{f} : \tilde{U} \to F_{\mathbb{C}}$. So let $\tilde{U}_x$ be such a domain for the extension and let $U_x := \tilde{U}_x \cap E$.

For this we claim that we may assume that U_x has the following additional property: $y \in U_x \Rightarrow [0,1]y \subseteq K^o \cup U_x$. In fact, let $U_0 := \{y \in U_x : [0,1]y \subseteq K^o \cup U_x\}$. Then U_0 is open, since $f : (t,s) \mapsto ty(s)$ being smooth, and $f(t,0) \in K^o \cup U_x$ for $t \in [0,1]$, implies that a $\delta > 0$ exists such that $f(t,s) \in K^o \cup U_x$ for all $|s| < \delta$ and $-\delta < t < 1 + \delta$. The set U_0 is star shaped, since $y \in U_0$ and $s \in [0,1]$ implies that $t(x + s(y - x)) \in [x, t'y]$ for some $t' \in [0,1]$, hence lies in $K^o \cup U_x$. The set U_0 contains x, since $[0,1]x = \{x\} \cup [0,1)x \subseteq \{x\} \cup K^o$. Finally, U_0 has the required property, since $z \in [0,1]y$ for $y \in U_0$ implies that $[0,1]z \subseteq [0,1]y \subseteq K^o \cup U_x$, i.e. $z \in U_0$.

Furthermore, we may assume that for $x + iy \in \tilde{U}_x$ and $t \in [0,1]$ also $x + ity \in \tilde{U}_x$ (replace $\tilde{U}_x$ by $\{x + iy : x + ity \in \tilde{U}_x \text{ for all } t \in [0,1]\}$).

Now let $\tilde{U}_1$ and $\tilde{U}_2$ be two such domains around x_1 and x_2, with corresponding extensions f_1 and f_2. Let $x + iy \in \tilde{U}_1 \cap \tilde{U}_2$. Then $x \in U_1 \cap U_2$ and $[0,1]x \subseteq K^o \cup U_i$ for $i = 1, 2$. If $x \in K^o$ we are done, so let $x \notin K^o$. Let $t_0 := \inf\{t > 0 : tx \notin K^o\}$. Then $t_0 x \in U_i$ for $i = 1, 2$ and by taking t_0 a little smaller we may assume that $x_0 := t_0 x \in K^o \cap U_1 \cap U_2$. Thus, $f_i = f$ on $[x_0, x_i]$ and the f_i are real analytic on $[x_0, x]$ for $i = 1, 2$. Hence, $f_1 = f_2$ on $[x_0, x]$ and thus $f_1 = f_2$ on $[x, x + iy]$ by the 1-dimensional uniqueness theorem. □

That the result corresponding to 24.8 is not true for manifolds with real analytic boundary shows the following

25.10. Example. No real analytic extension exists.
Let $I := \{t \in \mathbb{R} : t \geq 0\}$, $E := C^\omega(I, \mathbb{R})$, and let $ev : E \times \mathbb{R} \supseteq E \times I \to \mathbb{R}$ be the real analytic map $(f,t) \mapsto f(t)$. Then there is no real analytic extension of ev to a neighborhood of $E \times I$.

Proof. Suppose there is some open set $U \subseteq E \times \mathbb{R}$ containing $\{(0,t) : t \geq 0\}$ and a C^ω-extension $\varphi : U \to \mathbb{R}$. Then there exists a c^∞-open neighborhood V of 0 and some $\delta > 0$ such that U contains $V \times (-\delta, \delta)$. Since V is absorbing in E, we have for every $f \in E$ that there exists some $\varepsilon > 0$ such that $\varepsilon f \in V$ and hence $\frac{1}{\varepsilon}\varphi(\varepsilon f, \cdot) : (-\delta, \delta) \to \mathbb{R}$ is a real analytic extension of f. This cannot be true, since there are $f \in E$ having a singularity inside $(-\delta, \delta)$. □

The following theorem generalizes 11.17.

25.11. Theorem. Mixing of C^∞ and C^ω.
Let (E, E') be a complete dual pair, let $X \subseteq E$, let $f : \mathbb{R} \times X \to \mathbb{R}$ be a mapping that extends for every B locally around every point in $\mathbb{R} \times (X \cap E_B)$ to a holomorphic map $\mathbb{C} \times (E_B)_{\mathbb{C}} \to \mathbb{C}$, and let $c \in C^\infty(\mathbb{R}, X)$. Then $c^* \circ f^\vee : \mathbb{R} \to C^\omega(X, \mathbb{R}) \to C^\infty(\mathbb{R}, \mathbb{R})$ is real analytic.

Proof. Let $I \subseteq \mathbb{R}$ be open and relatively compact, let $t \in \mathbb{R}$ and $k \in \mathbb{N}$. Now choose an open and relatively compact $J \subseteq \mathbb{R}$ containing the closure $\bar{I}$ of I. By 1.8 there is a bounded subset $B \subseteq E$ such that $c|_J : J \to E_B$ is a $\mathcal{L}\mathrm{ip}^k$-curve in the Banach space E_B generated by B. Let X_B denote the subset $X \cap E_B$ of the Banach space E_B. By assumption on f there is a holomorphic extension $f : V \times W \to \mathbb{C}$

of f to an open set $V \times W \subseteq \mathbb{C} \times (E_B)_{\mathbb{C}}$ containing the compact set $\{t\} \times c(\bar{I})$. By cartesian closedness of the category of holomorphic mappings $f^\vee : V \to \mathcal{H}(W, \mathbb{C})$ is holomorphic. Now recall that the bornological structure of $\mathcal{H}(W, \mathbb{C})$ is induced by that of $C^\infty(W, \mathbb{C}) := C^\infty(W, \mathbb{R}^2)$. Furthermore, $c^* : C^\infty(W, \mathbb{C}) \to \mathcal{L}\mathrm{ip}^k(I, \mathbb{C})$ is a bounded $\mathbb{C}$-linear map (see tyhe proof of 11.17). Thus, $c^* \circ f^\vee : V \to \mathcal{L}\mathrm{ip}^k(I, \mathbb{C})$ is holomorphic, and hence its restriction to $\mathbb{R} \cap V$, which has values in $\mathcal{L}\mathrm{ip}^k(I, \mathbb{R})$, is (even topologically) real analytic by 9.5. Since $t \in \mathbb{R}$ was arbitrary we conclude that $c^* \circ f^\vee : \mathbb{R} \to \mathcal{L}\mathrm{ip}^k(I, \mathbb{R})$ is real analytic. But the bornology of $C^\infty(\mathbb{R}, \mathbb{R})$ is generated by the inclusions into $\mathcal{L}\mathrm{ip}^k(I, \mathbb{R})$, by the uniform boundedness principles 5.26 for $C^\infty(\mathbb{R}, \mathbb{R})$ and 12.9 for $\mathcal{L}\mathrm{ip}^k(\mathbb{R}, \mathbb{R})$, and hence $c^* \circ f^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$ is real analytic. □

This can now be used to show cartesian closedness with the same proof as in 11.18 for certain non-open subsets of convenient vector spaces. In particular, the previous theorem applies to real analytic mappings $f : \mathbb{R} \times X \to \mathbb{R}$, where $X \subseteq E$ is convex with non-void interior. Since for such a set the intersection X_B with E_B has the same property and since E_B is a Banach space, the real analytic mapping is the germ of a holomorphic mapping.

25.12. Theorem. Exponential law for real analytic germs.
Let K and L be two convex subsets with non-empty interior in convenient vector spaces. A map $f : K \to C^\omega(L, F)$ is real analytic if and only if the associated mapping $\hat{f} : K \times L \to F$ is real analytic.

Proof. ($\Rightarrow$) Let $c = (c_1, c_2) : \mathbb{R} \to K \times L$ be C^α (for $\alpha \in \{\infty, \omega\}$) and let $\ell \in F'$. We have to show that $\ell \circ \hat{f} \circ c : \mathbb{R} \to \mathbb{R}$ is C^α. By cartesian closedness of C^α it is enough to show that the map $\ell \circ \hat{f} \circ (c_1 \times c_2) : \mathbb{R}^2 \to \mathbb{R}$ is C^α. This map however is associated to $\ell_* \circ (c_2)_* \circ f \circ c_1 : \mathbb{R} \to K \to C^\omega(L, F) \to C^\alpha(\mathbb{R}, \mathbb{R})$, hence is C^α by assumption on f and the structure of $C^\omega(L, F)$.

($\Leftarrow$) Let conversely $f : K \times L \to F$ be real analytic. Then obviously $f(x, \cdot) : L \to F$ is real analytic, hence $f^\vee : K \to C^\omega(L, F)$ makes sense. Now take an arbitrary C^α-map $c_1 : \mathbb{R} \to K$. We have to show that $f^\vee \circ c_1 : \mathbb{R} \to C^\omega(L, F)$ is C^α. Since the structure of $C^\omega(L, F)$ is generated by $C^\beta(c_1, \ell)$ for C^β-curves $c_2 : \mathbb{R} \to L$ (for $\beta \in \{\infty, \omega\}$) and $\ell \in F'$, it is by 9.3 enough to show that $C^\beta(c_2, \ell) \circ f^\vee \circ c_1 : \mathbb{R} \to C^\beta(\mathbb{R}, \mathbb{R})$ is C^α. For $\alpha = \beta$ it is by cartesian closedness of C^α maps enough to show that the associate map $\mathbb{R}^2 \to \mathbb{R}$ is C^α. Since this map is just $\ell \circ f \circ (c_1 \times c_2)$, this is clear. In fact, take for $\gamma \le \alpha, \gamma \in \{\infty, \omega\}$ an arbitrary C^γ-curve $d = (d_1, d_2) : \mathbb{R} \to \mathbb{R}^2$. Then $(c_1 \times c_2) \circ (d_1, d_2) = (c_1 \circ d_1, c_2 \circ d_2)$ is C^γ, and so the composite with $\ell \circ f$ has the same property.

It remains to show the mixing case, where c_1 is real analytic and c_2 is smooth or conversely. First the case c_1 real analytic, c_2 smooth. Then $\ell \circ f \circ (c_1 \times \mathrm{Id}) : \mathbb{R} \times L \to \mathbb{R}$ is real analytic, hence extends to some holomorphic map by 25.9, and by 25.11 the map

$$C^\infty(c_2, \ell) \circ f^\vee \circ c_1 = c_2^* \circ (\ell \circ f \circ (c_1 \times \mathrm{Id}))^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$$

is real analytic. Now the case c_1 smooth and c_2 real analytic. Then $\ell \circ f \circ (\mathrm{Id} \times c_2) : K \times \mathbb{R} \to \mathbb{R}$ is real analytic, so by the same reasoning as just before applied to $\tilde{f}$ defined by $\tilde{f}(x, y) := f(y, x)$, the map

$$C^\infty(c_1, \ell) \circ (\tilde{f})^\vee \circ c_2 = c_1^* \circ (\ell \circ \tilde{f} \circ (\mathrm{Id} \times c_2))^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$$

is real analytic. By 11.16 the associated mapping

$$(c_1^* \circ (\ell \circ \tilde{f} \circ (\operatorname{Id} \times c_2))^\vee)^\sim = C^\omega(c_2, \ell) \circ \tilde{f} \circ c_1 : \mathbb{R} \to C^\omega(\mathbb{R}, \mathbb{R})$$

is smooth. □

The following example shows that theorem 25.12 does not extend to arbitrary domains.

25.13. Example. The exponential law for general domains is false.
Let $X \subseteq \mathbb{R}^2$ be the graph of the map $h : \mathbb{R} \to \mathbb{R}$ defined by $h(t) := e^{-t^{-2}}$ for $t \neq 0$ and $h(0) = 0$. Let, furthermore, $f : \mathbb{R} \times X \to \mathbb{R}$ be the mapping defined by $f(t, s, r) := \frac{r}{t^2+s^2}$ for $(t, s) \neq (0, 0)$ and $f(0, 0, r) := 0$. Then $f : \mathbb{R} \times X \to \mathbb{R}$ is real analytic, however the associated mapping $f^\vee : \mathbb{R} \to C^\omega(X, \mathbb{R})$ is not.

Proof. Obviously, f is real analytic on $\mathbb{R}^3 \setminus \{(0, 0)\} \times \mathbb{R}$. If $u \mapsto (t(u), s(u), r(u))$ is real analytic $\mathbb{R} \to \mathbb{R} \times X$, then $r(u) = h(s(u))$. Suppose s is not constant and $t(0) = s(0) = 0$, then we have that $r(u) = h(u^n s_0(u))$ cannot be real analytic, since it is not constant but the Taylor series at 0 is identical 0, a contradiction. Thus, $s = 0$ and $r = h \circ s = 0$, therefore $u \mapsto f(t(u), s(u), r(u)) = 0$ is real analytic.

Remains to show that $u \mapsto f(t(u), s(u), r(u))$ is smooth for all smooth curves $(t, s, r) : \mathbb{R} \to \mathbb{R} \times X$. Since $f(t(u), s(u), r(u)) = \frac{h(s(u))}{t(u)^2+s(u)^2}$ it is enough to show that $\varphi : \mathbb{R}^2 \to \mathbb{R}$ defined by $\varphi(t, s) = \frac{h(s)}{t^2+s^2}$ is smooth. This is obviously the case, since each of its partial derivatives is of the form $h(s)$ multiplied by some rational function of t and s, hence extends continuously to $\{(0, 0)\}$.

Now we show that $f^\vee : \mathbb{R} \to C^\omega(X, \mathbb{R})$ is not real analytic. Take the smooth curve $c : u \mapsto (u, h(u))$ into X and consider $c^* \circ f^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$, which is given by $t \mapsto (s \mapsto f(t, c(s)) = \frac{h(s)}{t^2+s^2})$. Suppose it is real analytic into $C([-1, +1], \mathbb{R})$. Then it has to be locally representable by a converging power series $\sum a_n t^n \in C([-1, +1], \mathbb{R})$. So there has to exist a $\delta > 0$ such that $\sum a_n(s) z^n = \frac{h(s)}{s^2} \sum_{k=0}^\infty (-1)^k (\frac{z}{s})^{2k}$ converges for all $|z| < \delta$ and $|s| < 1$. This is impossible, since at $z = si$ there is a pole. □

26. Holomorphic Mappings on Non-Open Domains

In this section we will consider holomorphic maps defined on two types of convex subsets. First the case where the set is contained in some real part of the vector space and has non-empty interior there. Recall that for a subset $X \subseteq \mathbb{R} \subseteq \mathbb{C}$ the space of germs of holomorphic maps $X \to \mathbb{C}$ is the complexification of that of germs of real analytic maps $X \to \mathbb{R}$, 11.2. Thus, we give the following

26.1. Definition. (Holomorphic maps $K \to F$)
Let $K \subseteq E$ be a convex set with non-empty interior in a real convenient vector space. And let F be a complex convenient vector space. We call a map $f : E_\mathbb{C} \supseteq K \to F$ holomorphic if and only if $f : E \supseteq K \to F$ is real analytic.

26.2. Lemma. Holomorphic maps can be tested by functionals.
Let $K \subseteq E$ be a convex set with non-empty interior in a real convenient vector space. And let F be a complex convenient vector space. Then a map $f : K \to F$ is holomorphic if and only if the composites $\ell \circ f : K \to \mathbb{C}$ are holomorphic for all $\ell \in L_\mathbb{C}(E, \mathbb{C})$, where $L_\mathbb{C}(E, \mathbb{C})$ denotes the space of $\mathbb{C}$-linear maps.

Proof. ($\Rightarrow$) Let $\ell \in L_\mathbb{C}(F, \mathbb{C})$. Then the real and imaginary part $\operatorname{Re} \ell, \operatorname{Im} \ell \in L_\mathbb{R}(F, \mathbb{R})$ and since by assumption $f : K \to F$ is real analytic so are the composites

$\operatorname{Re} \ell \circ f$ and $\operatorname{Im} \ell \circ f$, hence $\ell \circ f : K \to \mathbb{R}^2$ is real analytic, i.e. $\ell \circ f : K \to \mathbb{C}$ is holomorphic.

($\Leftarrow$) We have to show that $\ell \circ f : K \to \mathbb{R}$ is real analytic for every $\ell \in L_{\mathbb{R}}(F, \mathbb{R})$. So let $\tilde{\ell} : F \to \mathbb{C}$ be defined by $\tilde{\ell}(x) = i\ell(x) + \ell(ix)$. Then $\tilde{\ell} \in L_{\mathbb{C}}(F, \mathbb{C})$, since $i\tilde{\ell}(x) = -\ell(x) + i\ell(ix) = \tilde{\ell}(ix)$. Note that $\ell = \operatorname{Im} \circ \tilde{\ell}$. By assumption, $\tilde{\ell} \circ f : K \to \mathbb{C}$ is holomorphic, hence its imaginary part $\ell \circ f : K \to \mathbb{R}$ is real analytic. □

26.3. Theorem. Holomorphic maps $K \to F$ are often germs.
Let $K \subseteq E$ be a convex subset with non-empty interior in a real Fréchet space E and let F be a complex convenient vector space such that F' carries a Baire topology as required in 25.1. Then a map $f : E_{\mathbb{C}} \supseteq K \to F$ is holomorphic if and only if it extends to a holomorphic map $\tilde{f} : \tilde{K} \to F$ for some neighborhood $\tilde{K}$ of K in $E_{\mathbb{C}}$.

Proof. Using 25.9 we conclude that f extends to a holomorphic map $\tilde{f} : \tilde{K} \to F_{\mathbb{C}}$ for some neighborhood $\tilde{K}$ of K in $E_{\mathbb{C}}$. The map $\operatorname{pr} : F_{\mathbb{C}} \to F$, given by $\operatorname{pr}(x, y) = x + iy \in F$ for $(x, y) \in F^2 = F \otimes_{\mathbb{R}} \mathbb{C}$, is $\mathbb{C}$-linear and restricted to $F \times \{0\} = F$ it is the identity. Thus, $\operatorname{pr} \circ \tilde{f} : \tilde{K} \to F_{\mathbb{C}} \to F$ is a holomorphic extension of f.

Conversely, let $\tilde{f} : \tilde{K} \to F$ be a holomorphic extension to a neighborhood $\tilde{K}$ of K. So it is enough to show that the holomorphic map $\tilde{f}$ is real analytic. By 7.19 it is smooth. So it remains to show that it is real analytic. For this it is enough to consider a topological real analytic curve in $\tilde{K}$ by 10.4. Such a curve is extendable to a holomorphic curve $\tilde{c}$ by 9.5, hence the composite $\tilde{f} \circ \tilde{c}$ is holomorphic and its restriction $\tilde{f} \circ c$ to $\mathbb{R}$ is real analytic. □

26.4. Definition. (Holomorphic maps on complex vector spaces)
Let $K \subseteq E$ be a convex subset with non-empty interior in a complex convenient vector space. And map $f : E \supseteq K \to F$ is called holomorphic iff it is real analytic and the derivative $f'(x)$ is $\mathbb{C}$-linear for all $x \in K^o$.

26.5. Theorem. Holomorphic maps are germs.
Let $K \subseteq E$ be a convex subset with non-empty interior in a complex convenient vector space. Then a map $f : E \supseteq K \to F$ into a complex convenient vector space F is holomorphic if and only if it extends to a holomorphic map defined on some neighborhood of K in E.

Proof. Since $f : K \to F$ is real analytic, it extends by 25.9 to a real analytic map $\tilde{f} : E \supseteq U \to F$, where we may assume that U is connected with K by straight line segments. We claim that $\tilde{f}$ is in fact holomorphic. For this it is enough to show that $f'(x)$ is $\mathbb{C}$-linear for all $x \in U$. So consider the real analytic mapping $g : U \to F$ given by $g(x) := if'(x)(v) - f'(x)(iv)$. Since it is zero on K^o it has to be zero everywhere by the uniqueness theorem. □

26.6. Remark. (There is no definition for holomorphy analogous to 25.7)
In order for a map $K \to F$ to be holomorphic it is not enough to assume that all composites $f \circ c$ for holomorphic $c : \mathbb{D} \to K$ are holomorphic, where $\mathbb{D}$ is the open unit disk. Take as K the closed unit disk, then $c(\mathbb{D}) \cap \partial K = \phi$. In fact let $z_0 \in \mathbb{D}$ then $c(z) = (z - z_0)^n (c_n + (z - z_0) \sum_{k>n} c_k (z - z_0)^{k-n-1})$ for z close to z_0, which covers a neighborhood of $c(z_0)$. So the boundary values of such a map would be completely arbitrary.

26.7. Lemma. Holomorphy is a bornological concept.
Let $T_\alpha : E \to E_\alpha$ be a family of bounded linear maps that generates the bornology

on E. Then a map $c : K \to F$ is holomorphic if and only if all the composites $T_\alpha \circ c : I \to F_\alpha$ are holomorphic.

Proof. It follows from 25.6 that f is real analytic. And the $\mathbb{C}$-linearity of $f'(x)$ can certainly be tested by point separating linear functionals. □

26.8. Theorem. Exponential law for holomorphic maps.
Let K and L be convex subsets with non-empty interior in complex convenient vector spaces. Then a map $f : K \times L \to F$ is holomorphic if and only if the associated map $f^\vee : K \to H(L, F)$ is holomorphic.

Proof. This follows immediately from the real analytic result 25.12, since the $\mathbb{C}$-linearity of the involved derivatives translates to each other, since we obviously have $f'(x_1, x_2)(v_1, v_2) = ev_{x_2}((f^\vee)'(x_1)(v_1)) + (f^\vee(x_1))'(x_2)(v_2)$ for $x_1 \in K$ and $x_2 \in L$. □

52. Appendix: Functional Analysis

The aim of this appendix is the following. This book needs prerequisites from functional analysis, in particular about locally convex spaces, which are beyond usual knowledge of non-specialists. We have used as unique reference the book [Jarchow, 1981]. In this appendix we try to sketch these results and to connect them to more widespread knowledge in functional analysis: for this we decided to use [Schaefer, 1971].

52.1. Basic concepts. A *locally convex space* E is a vector space together with a Hausdorff topology such that addition $E \times E \to E$ and scalar multiplication $\mathbb{R} \times E \to E$ (or $\mathbb{C} \times E \to E$) are continuous and 0 has a basis of neighborhoods consisting of (absolutely) convex sets. Equivalently, the topology on E can be described by a system $\mathcal{P}$ of (continuous) seminorms. A *seminorm* $p : E \to \mathbb{R}$ is specified by the following properties: $p(x) \geq 0$, $p(x+y) \leq p(x) + p(y)$, and $p(\lambda x) = |\lambda| p(x)$.

A set B in a locally convex space E is called *bounded* if it is absorbed by each 0-neighborhood, equivalently, if each continuous seminorm is bounded on B. The family of all bounded subsets is called the *bornology* of E. The *bornologification of a locally convex space* is the finest locally convex topology with the same bounded sets, which is treated in detail in 4.2 and 4.4. A locally convex space is called *bornological* if it is stable under the bornologification, see also 4.1. The *ultra-bornologification of a locally convex space* is the finest locally convex topology with the same bounded absolutely convex sets for which E_B is a Banach space.

52.2. Result. [Jarchow, 1981, 6.3.2] & [Schaefer, 1971, I.1.3] *The Minkowski functional $q_A : x \mapsto \inf\{t > 0 : x \in t.A\}$ of a convex absorbing set A containing 0 is a convex function.*

A subset A in a vector space is called *absorbing* if $\bigcup\{rA : r > 0\}$ is the whole space.

52.3. Result. [Jarchow, 1981, 6.4.2.(3)] *For an absorbing radial set U in a locally convex space E the closure is given by $\{x \in E : q_U(x) \leq 1\}$, where q_U is the Minkowski functional.*

52.4. Result. [Jarchow, 1981, 3.3.1] *Let X be a set and let F be a Banach space. Then the space $\ell^\infty(X, F)$ of all bounded mappings $X \to F$ is itself a Banach space, supplied with the supremum norm.*

52.5. Result. [Jarchow, 1981, 3.5.6, p66] & [Schaefer, 1971, I.3.6] *A Hausdorff topological vector space E is finite dimensional if and only if it admits a precompact neighborhood of 0.*

A subset K of E is called *precompact* if finitely many translates of any neighborhood of 0 cover K.

52.6. Result. [Jarchow, 1981, 6.7.1, p112] & [Schaefer, 1971, II.4.3] *The absolutely convex hull of a precompact set is precompact.*

A set B in a vector space E is called *absolutely convex* if $\lambda x + \mu y \in B$ for $x, y \in B$ and $|\lambda| + |\mu| \leq 1$. By E_B we denote the linear span of B in E, equipped with the Minkowski functional q_B. This is a normed space.

52.7. Result. [Jarchow, 1981, 4.1.4] & [Horvath, 1966] *A basis of neighborhoods of 0 of the direct sum $\mathbb{C}^{(\mathbb{N})}$ is given by the sets of the form $\{(z_k)_k \in \mathbb{C}^{(\mathbb{N})} : |z_k| \leq \varepsilon_k$ for all $k\}$ where $\varepsilon_k > 0$.*

The *direct sum* $\bigoplus_i E_i$, also called the *coproduct* $\coprod_i E_i$ of locally convex spaces E_i is the subspace of the cartesian product formed by all points with only finitely many non-vanishing coordinates supplied with the finest locally convex topology for which the inclusions $E_j \to \coprod_i E_i$ are continuous. It solves the universal problem for a coproduct: For continuous linear mappings $f_i : E_i \to F$ into a locally convex space there is a unique continuous linear mapping $f : \coprod_i E_i \to F$ with $f \circ \mathrm{incl}_j = f_j$ for all j. The bounded sets in $\bigoplus_i E_i$ are exactly those which are contained and bounded in a finite subsum. If all spaces E_i are equal to E and the index set is Γ, we write $E^{(\Gamma)}$ for the direct sum.

52.8. Result. [Jarchow, 1981, 4.6.1, 4.6.2, 6.6.9] & [Schaefer, 1971, II.6.4 and II.6.5] *Let E be the strict inductive limit of a sequence of locally convex vector spaces E_n. Then every E_n carries the trace topology of E, and every bounded subset of E is contained in some E_n, i.e., the inductive limit is regular.*

Let E be a functor from a small (index) category into the category of all locally convex spaces with continuous linear mappings as morphisms. The *colimit* $\operatorname{colim} E$ of the functor E is the unique (up to isomorphism) locally convex space together with continuous linear mappings $l_i : E(i) \to \operatorname{colim} E$ which solves the following universal problem: Given continuous linear $g_i : E(i) \to F$ into a locally convex space F with $g_j \circ E(f) = g_i$ for each morphism $f : i \to j$ in the index category. Then there exists a unique continuous linear mapping $g : \operatorname{colim} E \to F$ with $g \circ l_i = g_i$ for all i.

$$\begin{tikzcd}
i \arrow[dd, "f"'] & E(i) \arrow[dd, "E(f)"'] \arrow[dr, "l_i"] \arrow[drr, "g_i"] & & \\
& & \operatorname{colim} E \arrow[r, dotted, "g"] & F \\
j & E(j) \arrow[ur, "l_j"] \arrow[urr, "g_j"'] & &
\end{tikzcd}$$

The colimit is given as the locally convex quotient of the direct sum $\coprod_i E(i)$ by the closed linear subspace generated by all elements of the form $\mathrm{incl}_i(x) - (\mathrm{incl}_j \circ E(f))(x)$ for all $x \in E(i)$ and $f : i \to j$ in the index category. Compare [Jarchow, 1981, p.82 & p.110], but we force here inductive limits to be Hausdorff. A *directed set* Γ is a partially ordered set such that for any two elements there is another one that is larger that the two. The *inductive limit* is the colimit of a functor from a directed set (considered as a small category); one writes $\varinjlim_j E_j$ for this. A *strict inductive limit* is the inductive limit of a functor E on the directed set $\mathbb{N}$ such that $E(n < n+1) : E(n) \to E(n+1)$ is the topological embedding of a closed linear subspace.

The dual notions (with the arrows between locally convex spaces reversed) are called the *limit* $\lim E$ of the functor E, and the *projective limit* $\varprojlim_j E_j$ in the case of a directed set. It can be described as the linear subset of the cartesian product $\prod_i E(i)$ consisting of all $(x_i)_i$ with $E(f)(x_i) = x_j$ for all $f : i \to j$ in the index category.

52.9. Result. [Jarchow, 1981, 5.1.4+11.1.6] *&* [Schaefer, 1971, III.5.1, Cor. 1] *Every separately continuous bilinear mapping on Fréchet spaces is continuous.*

A *Fréchet* space is a complete locally convex space with a metrizable topology, equivalently, with a countable base of seminorms. See [Jarchow, 1981, 2.8.1] or [Schaefer, 1971, p.48].

Closed graph and open mapping theorems. These are well known if Banach spaces or even Fréchet spaces are involved. We need a wider class of situations where these theorems hold; those involving webbed spaces. Webbed spaces were introduced for exactly this reason by de Wilde in his thesis, see [de Wilde, 1978]. We do not give their (quite lengthy) definition here, only the results and the permanence properties.

52.10. Result. Closed Graph Theorem. [Jarchow, 1981, 5.4.1] *Any closed linear mapping from an inductive limit of Baire locally convex spaces into a webbed locally convex space is continuous.*

52.11. Result. Open Mapping Theorem. [Jarchow, 1981, 5.5.2] *Any continuous surjective linear mapping from a webbed locally convex space into an inductive limit of Baire locally convex spaces vector spaces is open.*

52.12. Result. *The Fréchet spaces are exactly the webbed spaces with the Baire property.*

This corresponds to [Jarchow, 1981, 5.4.4] by noting that Fréchet spaces are Baire.

52.13. Result. [Jarchow, 1981, 5.3.3] *Projective limits and inductive limits of sequences of webbed spaces are webbed.*

52.14. Result. *The bornologification of a webbed space is webbed.*

This follows from [Jarchow, 1981, 13.3.3 and 5.3.1.(d)] since the bornologification is coarser that the ultrabornologification, [Jarchow, 1981, 13.3.1].

52.15. Definition. [Jarchow, 1981, 6.8] *For a zero neighborhood U in a locally convex vector space E we denote by $\widetilde{E_{(U)}}$ the completed quotient of E with the Minkowski functional of U as norm.*

52.16. Result. Hahn-Banach Theorem. [Jarchow, 1981, 7.3.3] *Let E be a locally convex vector space and let $A \subset E$ be a convex set, and let $x \in E$ be not in the closure of A. Then there exists a continuous linear functional ℓ with $\ell(x)$ not in the closure of $\ell(A)$.*

This is a consequence of the usual Hahn-Banach theorem, [Schaefer, 1971,II.9.2]

52.17. Result. [Jarchow, 1981, 7.2.4] *Let $x \in E$ be a point in a normed space. Then there exists a continuous linear functional $x' \in E^*$ of norm 1 with $x'(x) = \|x\|$.*

This is another consequence of the usual Hahn-Banach theorem, cf. [Schaefer, 1971, II.3.2].

52.18. Result. Bipolar Theorem. [Jarchow, 1981, 8.2.2] *Let E be a locally convex vector space and let $A \subset E$. Then the bipolar A^{oo} in E with respect to the dual pair (E, E^*) is the closed absolutely convex hull of A in E.*

For a duality $\langle \quad , \quad \rangle$ between vector spaces E and F and a set $A \subseteq E$ the *polar* of A is $A^o := \{y \in F : |\langle x, y\rangle| \leq 1 \text{ for all } x \in A\}$. The *weak topology* $\sigma(E, F)$ is the locally convex topology on E generated by the seminorms $x \mapsto |\langle x, y\rangle|$ for all $y \in F$.

52.19. Result. [Schaefer, 1971, IV.3.2] *A subset of a locally convex vector space is bounded if and only if every continuous linear functional is bounded on it.*

This follows from [Jarchow, 1981, 8.3.4], since the weak topology $\sigma(E, E')$ and the given topology are compatible with the duality, and a subset is bounded for the weak topology, if and only if every continuous linear functional is bounded on it.

52.20. Result. Alaoğlu-Bourbaki Theorem. [Jarchow, 1981, 8.5.2 & 8.5.1.b] *&* [Schaefer, 1971, III.4.3 and II.4.5] *An equicontinuous subset K of E' has compact closure in the topology of uniform convergence on precompact subsets; On K the latter topology coincides with the weak topology $\sigma(E', E)$.*

52.21. Result. [Jarchow, 1981, 8.5.3, p157] *&* [Schaefer, 1971, III.4.7] *Let E be a separable locally convex vector space. Then each equicontinuous subset of E' is metrizable in the weak* topology $\sigma(E', E)$.*

A topological space is called *separable* if it contains a dense countable subset.

52.22. Result. Banach Dieudonné theorem. [Jarchow, 1981, 9.4.3, p182] *&* [Schaefer, 1971, IV.6.3] *On the dual of a metrizable locally convex vector space E the topology of uniform convergence on precompact subsets of E coincides with the so-called equicontinuous weak*-topology which is the final topology induced by the inclusions of the equicontinuous subsets.*

52.23. Result. [Jarchow, 1981, 10.1.4] *In metrizable locally convex spaces the convergent sequences coincide with the Mackey-convergent ones.*

For Mackey convergence see 1.6.

52.24. Result. [Jarchow, 1981, 10.4.3, p202] *&* [Horvath, 1966, p277] *In Schwartz spaces bounded sets are precompact.*

A locally convex space E is called *Schwartz* if each absolutely convex neighborhood U of 0 in E contains another one V such that the induced mapping $E_{(U)} \to E_{(V)}$ maps U into a precompact set.

52.25. Result. Uniform boundedness principle. [Jarchow, 1981, 11.1.1] *(*[Schaefer, 1971, IV.5.2] *for $F = \mathbb{R}$) Let E be a barrelled locally convex vector space and F be a locally convex vector space. Then every pointwise bounded set of continuous linear mappings from E to F is equicontinuous.*

Note that each Fréchet space is barrelled, see [Jarchow, 1981, 11.1.5].

A locally convex space is called *barrelled* if each closed absorbing absolutely convex set is a 0-neighborhood.

52.26. Result. [Jarchow, 1981, 11.5.1, 13.4.5] *&* [Schaefer, 1971, IV.5.5] *Montel spaces are reflexive.*

By a *Montel space* we mean (following [Jarchow, 1981, 11.5]) a locally convex vector space which is barrelled and in which every bounded set is relatively compact. A locally convex space E is called *reflexive* if the canonical embedding of E into the strong dual of the strong dual of E is a topological isomorphism.

52.27. Result. [Jarchow, 1981, 11.6.2, p231] *Fréchet Montel spaces are separable.*

52.28. Result. [Jarchow, 1981, 12.5.8, p266] *In the strong dual of a Fréchet Schwartz space every converging sequence is Mackey converging.*

The *strong dual of a locally convex space* E is the dual E^* of all continuous linear functionals equipped with the topology of uniform convergence on bounded subsets of E.

52.29. Result. *Fréchet Montel spaces have a bornological strong dual.*

Proof. By [**5**, 7.4.21] a Fréchet Montel space E is reflexive, thus it's strong dual E'_β is also reflexive by [Jarchow, 1981, 11.4.5.(f)]. So it is barrelled by [Jarchow, 1981, 11.4.2]. By [Jarchow, 1981, 13.4.4] or [Schaefer, 1971, IV.6.6] the strong dual E'_β of a metrizable locally convex vector space E is bornological if and only if it is barrelled and the result follows. $\square$

52.30. Result. [Jarchow, 1981, 13.5.1] *Inductive limits of ultra-bornological spaces are ultra-bornological.*

Similar to the definition of bornological spaces in 4.1 we define ultra-bornological spaces, see [Jarchow, 1981, 13.1.1]. A *bounded completant set* B in a locally convex vector space E is an absolutely convex bounded set B for which the normed space (E_B, q_B) is complete. A locally convex vector space E is called *ultra-bornological* if the following equivalent conditions are satisfied:

(1) For any locally convex vector space F a linear mapping $T : E \to F$ is continuous if it is bounded on each bounded completant set. It is sufficient to know this for all Banach spaces F.
(2) A seminorm on E is continuous if it is bounded on each bounded completant set.
(3) An absolutely convex subset is a 0-neighborhood if it absorbs each bounded completant set.

52.31. Result. [Jarchow, 1981, 13.1.2] *Every ultra-bornological space is an inductive limit of Banach spaces.*

In fact, $E = \varinjlim_B E_B$ where B runs through all bounded closed absolutely convex sets in E. Compare with the corresponding result 4.2 for bornological spaces.

52.32. Nuclear Operators. A linear operator $T : E \to F$ between Banach spaces is called *nuclear* or *trace class* if it can be written in the form

$$T(x) = \sum_{j=1}^{\infty} \lambda_j \langle x, x_j \rangle y_j,$$

where $x_j \in E'$, $y_j \in F$ with $\|x_j\| \le 1$, $\|y_j\| \le 1$, and $(\lambda_j)_j \in \ell^1$. The *trace* of T is then given by

$$\operatorname{tr}(T) = \sum_{j=1}^{\infty} \lambda_j \langle y_j, x_j \rangle.$$

The operator T is called *strongly nuclear* if $(\lambda_j)_j \in s$ is rapidly decreasing.

52.33. Result. [Jarchow, 1981, 20.2.6] *The dual of the Banach space of all trace class operators on a Hilbert space consists of all bounded operators. The duality is given by* $\langle T, B\rangle = \operatorname{tr}(TB) = \operatorname{tr}(BT)$.

52.34. Result. [Jarchow, 1981, 21.1.7] *Countable inductive limits of strongly nuclear spaces are again strongly nuclear. Products and subspaces of strongly nuclear spaces are strongly nuclear.*

A locally convex space E is called *nuclear* (or *strongly nuclear*) if each absolutely convex 0-neighborhood U contains another one V such that the induced mapping $\widetilde{E_{(V)}} \to \widetilde{E_{(U)}}$ is a nuclear operator (or strongly nuclear operator). A locally convex space is (strongly) nuclear if and only if its completion is it, see [Jarchow, 1981, 21.1.2]. Obviously, a nuclear space is a Schwartz space 52.24 since a nuclear operator is compact. Since nuclear operators factor over Hilbert spaces, see [Jarchow, 1981, 19.7.5], each nuclear space admits a basis of seminorms consisting of Hilbert norms, see [Schaefer, 1971, III.7.3].

52.35. Grothendieck-Pietsch criterion. Consider a directed set $\mathcal{P}$ of non-negative real valued sequences $p = (p_n)$ with the property that for each $n \in \mathbb{N}$ there exists a $p \in \mathcal{P}$ with $p_n > 0$. It defines a complete locally convex space (called *Köthe sequence space*)

$$\Lambda(\mathcal{P}) := \{x = (x_n)_n \in \mathbb{K}^{\mathbb{N}} : p(x) := \sum_n p_n |x_n| < \infty \text{ for all } p \in \mathcal{P}\}$$

with the specified seminorms.

Result. [Jarchow, 1981, 21.8.2] *&* [Treves, 1967, p. 530] *The space* $\Lambda(\mathcal{P})$ *is nuclear if and only if for each* $p \in \mathcal{P}$ *there is a* $q \in \mathcal{P}$ *with*

$$\left(\frac{p_n}{q_n}\right)_n \in \ell^1.$$

The space $\Lambda(\mathcal{P})$ *is strongly nuclear if and only if for each* $p \in \mathcal{P}$ *there is a* $q \in \mathcal{P}$ *with*

$$\left(\frac{p_n}{q_n}\right)_n \in \bigcap_{r>0} \ell^r.$$

52.36. Result. [Jarchow, 1981, 21.8.3.b] $\mathcal{H}(\mathbb{D}^k, \mathbb{C})$ *is strongly nuclear for all* k.

Proof. This is an immediate consequence of the Grothendieck-Pietsch criterion 52.35 by considering the power series expansions in the polycylinder $\mathbb{D}^k$ at 0. The set $\mathcal{P}$ consists of $r(n_1, \dots, n_k) := r^{n_1+\dots+n_k}$ for all $0 < r < 1$. □

52.37. Silva spaces. *A locally convex vector space which is an inductive limit of a sequence of Banach spaces with compact connecting mappings is called a Silva space. A Silva space is ultra-bornological, webbed, complete, and its strong dual is a Fréchet space. The inductive limit describing the Silva space is regular. A Silva space is Baire if and only if it is finite dimensional. The dual space of a nuclear Silva space is nuclear.*

Proof. Let E be a Silva space. That E is ultra-bornological and webbed follows from the permanence properties of ultra-bornological spaces 52.30 and of webbed spaces [5, 5.3.3]. The inductive limit describing E is regular and E is complete by

[Floret, 1971, 7.4 and 7.5]. The dual E' is a Fréchet space since E has a countable base of bounded sets as a regular inductive limit of Banach spaces. If E is nuclear then the dual is also nuclear by [Jarchow, 1981, 21.5.3].

If E has the Baire property, then it is metrizable by [**5**, 5.3.3]. But a metrizable Silva space is finite dimensional by [Floret, 1971, 7.7]. $\square$

53. Appendix: Projective Resolutions of Identity on Banach spaces

One of the main tools for getting results for non-separable Banach spaces is that of projective resolutions of identity. The aim is to construct transfinite sequences of complemented subspaces with separable increment and finally reaching the whole space. This works for Banach spaces with enough projections onto closed subspaces. We will give an account on this, following [Orihuela, Valdivia, 1989]. The results in this appendix are used for the construction of smooth partitions of unity in theorem 16.18 and for obtaining smooth realcompactness in example 19.7

53.1. Definition. Let E be a Banach space, $A \subseteq E$ and $B \subseteq E'$ $\mathbb{Q}$-linear subspaces. Then (A, B) is called *norming pair* if the following two conditions are satisfied:

$$\forall x \in A : \|x\| = \sup\{|\langle x, x^*\rangle| : x^* \in B, \|x^*\| \le 1\}$$
$$\forall x^* \in B : \|x^*\| = \sup\{|\langle x, x^*\rangle| : x \in A, \|x\| \le 1\}.$$

53.2. Proposition. *Let (A, B) be a norming pair on a Banach space E. Then*

(1) *$(\bar{A}, \bar{B})$ is a norming pair.*
(2) *Let $A_0 \subseteq A$, $B_0 \subseteq B$, $\omega \le |A_0| \le \lambda$, and $\omega \le |B_0| \le \lambda$ for some cardinal number λ.*
Then there exists a norming pair $(\tilde{A}, \tilde{B})$ with $A_0 \subseteq \tilde{A} \subseteq A$, $B_0 \subseteq \tilde{B} \subseteq B$, $|\tilde{A}| \le \lambda$ and $|\tilde{B}| \le \lambda$.
(3)
$$x \in A,\ y \in B^o \Rightarrow \|x\| \le \|x + y\|, \text{ in particular } A \cap B^o = \{0\}$$
$$x^* \in A^o,\ y^* \in B \Rightarrow \|y^*\| \le \|y^* + x^*\|, \text{ in particular } A^o \cap B = \{0\}.$$

Proof. (1) Let $x \in \bar{A}$ and $\varepsilon > 0$. Thus there is some $a \in A$ with $\|x - a\| \le \varepsilon$ and we get

$$\begin{aligned}
\|x\| &\le \|x - a\| + \|a\| \le \varepsilon + \sup\{|\langle a, x^*\rangle| : x^* \in B, \|x^*\| \le 1\}\\
&\le \varepsilon + \sup\{|\langle a - x, x^*\rangle| : x^* \in B, \|x^*\| \le 1\}\\
&\qquad + \sup\{|\langle x, x^*\rangle| : x^* \in B, \|x^*\| \le 1\}\\
&\le \varepsilon + \|a - x\| + \sup\{|\langle x, x^*\rangle| : x^* \in B, \|x^*\| \le 1\}\\
&\le 2\varepsilon + \|x\|,
\end{aligned}$$

and for $\varepsilon \to 0$ we get the first condition of a norming pair. The second one is shown analogously.

(2) For every $x \in A$ and $y^* \in B$ choose a countable sets $\psi(x) \subseteq B$ and $\varphi(y^*) \subseteq A$ such that

$$\|x\| = \sup\{|\langle x, x^*\rangle| : x^* \in \psi(x)\} \quad \text{and} \quad \|y^*\| = \sup\{|\langle y, y^*\rangle : y \in \varphi(y^*)\}$$

By recursion on n we construct subsets $A_n \subseteq A$ and $B_n \subseteq B$ with $|A_n| \le \lambda$ and $|B_n| \le \lambda$:

$$\begin{aligned} B_{n+1} &:= \langle B_n\rangle_{\mathbb{Q}} \cup \{\psi(x) : x \in \langle A_n\rangle_{\mathbb{Q}}\} \\ A_{n+1} &:= \langle A_n\rangle_{\mathbb{Q}} \cup \{\varphi(x^*) : x^* \in \langle B_n\rangle_{\mathbb{Q}}\}. \end{aligned}$$

Finally let $\tilde{A} := \bigcup_{n\in\mathbb{N}} A_n$ and $\tilde{B} := \bigcup_{n\in\mathbb{N}} B_n$. Then $(\tilde{A}, \tilde{B})$ is the required norming pair. In fact for $x \in A_n$ we have that

$$\|x\| = \sup\{|\langle x, x^*\rangle| : x \in \psi(x)\} \le \sup\{|\langle x, x^*\rangle| : x \in B_{n+1}\} \le \|x\|$$

Note that $\varphi(\tilde{B}) := \bigcup_{b\in\tilde{B}} \varphi(b) \subseteq \tilde{A}$.

(3) We have

$$\begin{aligned} \|x\| &= \sup\{|\langle x, x^*\rangle| : x^* \in B, \|x^*\| \le 1\} \\ &= \sup\{|\langle x+y, x^*\rangle| : x^* \in B, \|x^*\| \le 1\} \\ &\le \sup\{|\langle x+y, x^*\rangle| : \|x^*\| \le 1\} = \|x+y\| \end{aligned}$$

and analogously for the second inequality. □

53.3. Proposition. *Let (A, B) be a norming pair on a Banach space E consisting of closed subspaces. It is called* conjugate pair *if one of the following equivalent conditions is satisfied.*

1. *There is a projection $P : E \to E$ with image A, kernel B^o and $\|P\| = 1$;*
2. *$E = A + B^o$;*
3. *$\{0\} = A^o \cap \overline{B}^{\sigma(E',E)}$;*
4. *The canonical mapping $A \hookrightarrow E \cong (E', \sigma(E', E))' \to (B, \sigma(B, E))'$ is onto.*

Proof. We have the following commuting diagram:

$$\begin{tikzcd}
B^o \arrow[dr, hook, "\ker\delta"] & & & \\
\{0\} \arrow[u, hook] \arrow[dr, hook] & E \arrow[r, tail, two heads] \arrow[dr, dotted, "\delta"] & (E', \sigma(E', E))' \arrow[d] & E' \\
 & A \arrow[u, hook] \arrow[r, tail, "\delta|_A"'] & (B, \sigma(B, E))' & B \arrow[u, hook]
\end{tikzcd}$$

(1)⇒(2) is obvious.

(2)⇔(3) follows immediately from duality.

(2)⇒(4) Let $z \in (B, \sigma(B, E))'$. By Hahn-Banach there is some $x \in E$ with $x|_B = z$. Let $x = a + b$ with $a \in A$ and $b \in B^o$. Then $a|_B = x|_B = z$.

(4)⇒(1) By (4) the mapping $\delta : A \hookrightarrow E \cong (E', \sigma(E', E))' \to (B, \sigma(B, E))'$ is bijective, since $A \cap B^o = \{0\}$, and hence we may define $P(x) := \delta^{-1}(x|_B)$. Then P is the required norm 1 projection, since $\delta : x \mapsto x|_B$ has norm ≤ 1 and δ_A has norm 1 since (A, B) is norming. □

53.4. Corollary. *Let E be a reflexive Banach space. Then any norming pair (A, B) of closed subspaces is a conjugate pair.*

Proof. In fact we then have

$$A^o \cap \overline{B}^{\sigma(E',E)} = A^o \cap \overline{B}^{\|\,\|} = A^o \cap B = \{0\},$$

since the dual of $(E', \sigma(E', E))$ is E and equals E'' the dual of $(E', \| \quad \|)$. By [Jarchow, 1981, 8.2.5] convex subsets as B have the same closure in these two topologies. □

53.5. Definition. A *projective generator* φ for a Banach space E is a mapping $\varphi : E' \to 2^E$ for which

(1) $\varphi(x^*)$ is a countable subset of $\{x \in E : \|x\| \le 1\}$ for all $x^* \in E'$;
(2) $\|x^*\| = \sup\{|\langle x, x^*\rangle| : x \in \varphi(x^*)\}$;
(3) If (A, B) is norming, with $\varphi(B) := \bigcup_{b \in B} \varphi(b) \subseteq A$, then $(\bar{A}, \bar{B})$ is a conjugate pair.

Note that the first two conditions can be always obtained.

We say that the projection P defined by 53.3 for $(\bar{A}, \bar{B})$ is based on the norming pair (A, B), i.e. $P(E) = \bar{A}$ and $\ker(P) = B^o = \bar{B}^o$.

53.6. Corollary. *Every reflexive Banach space has a projective generator φ.*

Proof. Just choose any φ satisfying 53.5.1 and 53.5.2. Then 53.5.3 is by 53.2.1 and 53.4 automatically satisfied. □

53.7. Theorem. *Let φ be a projective generator for a Banach space E. Let $A_0 \subseteq E$ and $B_0 \subseteq E'$ be infinite sets of cardinality at most λ.*

Then there exists a norm 1 projection P based on a norming pair (A, B) with $A_0 \subseteq A$, $B_0 \subseteq B$, $|A| \le \lambda$, $|B| \le \lambda$ and $\varphi(B) \subseteq A$.

Proof. By 53.2.3 there is a norming pair (A, B) with

$$A_0 \subseteq A, \quad B_0 \subseteq B, \quad |A| \le \lambda, \quad |B| \le \lambda.$$

Note that in the proof of 53.2.3 we used some map φ, and we may take the projective generator for it. Thus we have also $\varphi(B) \subseteq A$. By condition 53.5.3 of the projective generator we thus get that the projection based on (A, B) has the required properties. □

53.8. Proposition. *Every WCD Banach space has a projective generator.*

A Banach space E is called *WCD* (*weakly compactly determined*) if and only if there exists a sequence K_n of weak*-compact subsets of E'' such that for every

$$\forall x \in E \; \forall y \in E'' \setminus E \; \exists n : x \in K_n \text{ and } y \notin K_n.$$

Every WCG Banach space is WCD:
In fact let K be weakly compact (and absolutely convex) such that $\bigcup_{n \in \mathbb{N}} K$ is dense in E. Note that $(E, \sigma(E, E'))$ embeds canonically into $(E'', \sigma(E'', E'))$. Let $K_{n,m} := n\, K + \frac{1}{m}\{x \in E'' : \|x\| \le 1\}$. Then $K_{n,m}$ is weak*-compact, and for any $x \in E$ and $y \in E'' \setminus E$ there exists an $m > 1/\operatorname{dist}(y, E)$ and an n with $\operatorname{dist}(x, n\, K) < \frac{1}{m}$. Hence $x \in K_{n,m}$ and $y \notin E + 1/m\, \{x \in E'' : \|x\| \le 1\} \supseteq K_{n,m}$.
The most important advantage of WCD over WCG Banach spaces are, that they are hereditary with respect to subspaces.
For any finite sequence $n = (n_1, \dots, n_k)$ let

$$C_{n_1,\dots,n_k} := \overline{E \cap K_{n_1} \cap \dots \cap K_{n_k}}^{\sigma(E'', E')}.$$

Then these sets are weak*-compact (since they are contained in K_{n_k}) and if E is not reflexive, then for every $x \in E$ there is a sequence $n : \mathbb{N} \to \mathbb{N}$ such that

$$x \in \bigcap_{k=1}^{\infty} C_{n_1,\dots,n_k} \subseteq E.$$

In fact choose a surjective sequence $n : \mathbb{N} \to \{k : x \in K_k\}$. Then $x \in C_{n_1,\dots,n_k}$ for all k, hence $x \in \bigcap_{k=1}^{\infty} C_{n_1,\dots,n_k}$. If $y \in E'' \setminus E$, then there is some k, such that $y \notin K_{n_k}$ and hence $y \notin C_{n_1,\dots,n_k} \subseteq K_{n_k}$.

Proof of 53.8. Because of 53.6 we may assume that E is not reflexive. For every $x^* \in E'$ we choose a countable set $\varphi(x^*) \subseteq \{x \in E : \|x\| \le 1\}$ such that

$$\|x^*\| = \sup\{|\langle x, x^*\rangle : x \in \varphi(x^*)\} \text{ and}$$
$$\sup\{|\langle x, x^*\rangle| : x \in C_{n_1,\dots,n_k}\} = \sup\{|\langle x, x^*\rangle| : x \in C_{n_1,\dots,n_k} \cap \langle \varphi(x^*)\rangle\}$$

for all finite sequences $(n_1, \dots, n_k)$. We claim that φ is a projective generator: Let (A, B) be a norming pair with $\varphi(B) \subseteq A$. We use 53.3.3 to show that $(\bar{A}, \bar{B})$ is norming. Assume there is some $0 \ne y^* \in A^o \cap \overline{B}^{\sigma(E',E)}$. Thus we can choose $x_0 \in E$ with $|y^*(x_0)| = 1$ and a net $(y_i^*)_i$ in B that converges to y^* in the Mackey topology $\mu(E', E)$ (of uniform convergence on weakly compact subsets of E). In fact this topology on E' has the same dual E as $\sigma(E', E)$ by the Mackey-Arens theorem [Jarchow, 1981, 8.5.5], and hence the same closure of convex sets by [Jarchow, 1981, 8.2.5]. As before we choose a surjective mapping $n : \mathbb{N} \to \{k : x_0 \in K_k\}$. Then

$$x_0 \in C := \bigcap_{k=1}^{\infty} C_{n_1,\dots,n_k} \subseteq E.$$

and C is weakly compact, hence we find an i_0 such that

$$\sup\{|y_{i_0}^*(x) - y^*(x)| : x \in C\} < \tfrac{1}{2}$$

and in particular we have

$$|y_{i_0}^*(x_0)| \ge |y^*(x_0)| - |y_{i_0}^*(x_0) - y^*(x_0)| > 1 - \tfrac{1}{2} = \tfrac{1}{2}.$$

Since the sets forming the intersection are decreasing, C_{n_1} is $\sigma(E'', E')$-compact and

$$W := \{x^{**} \in E'' : |x^{**}(y_{i_0}^* - y^*)| < \tfrac{1}{2}\}$$

is a $\sigma(E'', E')$-open neighborhood of C there is some $k \in \mathbb{N}$ such that $C_{n_1,\dots,n_k} \subseteq W$, i.e.

$$\sup\{|y_{i_0}^*(x) - y^*(x)| : x \in C_{n_1,\dots,n_k}\} \le \tfrac{1}{2}.$$

By the definition of φ there is some $y_0 \in C_{n_1,\dots,n_k} \cap \langle \varphi(y_{i_0}^*)\rangle$ with $|y_{i_0}^*(y_0)| > 1 - \frac{1}{2}$, thus

$$|y^*(y_0)| \ge |y_{i_0}^*(y_0)| - |y_{i_0}^*(y_0) - y^*(y_0)| > \tfrac{1}{2} - \tfrac{1}{2} = 0.$$

Thus $y^*(y_0) \ne 0$ and $y_0 \in \langle \varphi(B)\rangle \subseteq A$, a contradiction. □

Note that if $P \in L(E)$ is a norm-1 projection with closed image A and kernel B^o, then $P^* \in L(E')$ is a norm-1 projection with image $P^*(E) = \ker P^o = B^{oo} = B$ and kernel $\ker P^* = P(E)^o = A^o$. However not all norm-1 projections onto B can be obtained in this way. Hence we consider the dual of proposition 53.3:

53.9. Proposition. *Let (A, B) be a norming pair on a Banach space E consisting of closed subspaces. It is called dual conjugate pair if one of the following equivalent conditions is satisfied.*

(1) *There is a norm-1 projection $P : E' \to E'$ with image B, kernel A^o;*
(2) *$E' = B \oplus A^o$;*

(3) $\{0\} = B^o \cap \overline{A}^{\sigma(E'',E')}$;
(4) *The canonical mapping* $B \hookrightarrow E' \xrightarrow{(\ \ )|A} A'$ *is onto.*

Proof. This follows by applying 53.3 to the norming pair $(B, A) \subseteq (E', E'')$. □

The dual of definition 53.5 is

53.10. Definition. A dual projective generator ψ for a Banach space E' is a mapping $\psi : E \to 2^{E'}$ for which

(1) $\psi(x)$ is a countable subset of $\{x^* \in E' : \|x^*\| \le 1\}$ for all $x \in E$;
(2) $\|x\| = \sup\{|\langle x, x^*\rangle| : x^* \in \psi(x)\}$;
(3) If (A, B) is norming, with $\psi(A) := \bigcup_{a\in A} \psi(a) \subseteq B$, then $(\bar A, \bar B)$ satisfies the condition of 53.9.

Note that the first two conditions can be always obtained.

From 53.7 we get:

53.11. Theorem. *Let ψ be a dual projective generator for a Banach space E. Let $A_0 \subseteq E$ and $B_0 \subseteq E'$ be infinite sets of cardinality at most λ.*
Then there exists a norm 1 projection P in E' with $A_0 \subseteq P^*(E'')$, $B_0 \subseteq P(E')$, $|P^*(E'')| \le \lambda$, $|P(E')| \le \lambda$.

53.12. Proposition. *A Banach space E is Asplund if and only if there exists a dual projective generator on E.*

Note that if P is a norm-1 projection, then so is P^*. But not all norm-1 projections on the dual are of this form.

Proof. ($\Leftarrow$) Let ψ be a dual projective generator for E. Let A_0 be a separable subspace of E. By 53.11 there is a separable subspace A of E and a norm-1 projection P of E' such that $A_0 \supseteq A$, $P(E')$ is separable and isomorphic with A' via the restriction map. Hence A' is separable and also A_0'. By [Stegall, 1975] E is Asplund.

($\Rightarrow$) Consider the $\|\ \ \|$-weak* upper semi-continuous mapping $\phi : X \to 2^{\{x^*:\|x^*\|\le 1\}}$ given by

$$\phi(x) := \{x^* \in E' : \|x^*\| \le 1, \langle x, x^*\rangle = \|x\|\}.$$

By the Jayne-Rogers selection theorem [Jayne, Rogers, 1985], see also [Deville, Godefroy, Zizler, 1993, section I.4] there is a map $f : E \to \{x^* \in E' : \|x^*\| \le 1\}$ with $f(x) \in \phi(x)$ for all $x \in E$ and continuous $f_n : E \to \{x^* : \|x^*\| \le 1\} \subseteq E'$ with $f_n(x) \to f(x)$ in E' for each $x \in E$. One then shows that

$$\psi(x) := \{f(x), f_1(x), \dots\}$$

defines a dual projective generator, see [Orihuela, Valdivia, 1989]. □

53.13. Definition. Projective Resolution of Identity. Let a "long sequence" of continuous projections $P_\alpha \in L(E, E)$ on a Banach space E for all ordinal numbers $\omega \le \alpha \le \operatorname{dens} E$ be given. Recall that $\operatorname{dens}(E)$ is the density of E (a cardinal number, which we identify with the smallest ordinal of same cardinality). Let $E_\alpha := P_\alpha(E)$ and let $R_\alpha := (P_{\alpha+1} - P_\alpha)/(\|P_{\alpha+1} - P_\alpha\|)$ or 0, if $P_{\alpha+1} = P_\alpha$. Then we consider the following properties:

(1) $P_\alpha P_\beta = P_\beta = P_\beta P_\alpha$ for all $\beta \le \alpha$.
(2) $P_{\operatorname{dens} E} = \operatorname{Id}_E$.

(3) $\operatorname{dens} P_\alpha E \le \alpha$ for all α.
(4) $\|P_\alpha\| = 1$ for all α.
(5) $\overline{\bigcup_{\beta<\alpha} P_{\beta+1}E} = P_\alpha E$, or equivalently $\overline{\bigcup_{\beta<\alpha} E_\beta} = E_\alpha$ for every limit ordinal $\alpha \le \operatorname{dens} E$.
(6) For every limit ordinal $\alpha \le \operatorname{dens} E$ we have $P_\alpha(x) = \lim_{\beta>\alpha} P_\beta(x)$, i.e. $\alpha \mapsto P_\alpha(x)$ is continuous.
(7) $E_{\alpha+1}/E_\alpha$ is separable for all $\omega \le \alpha < \operatorname{dens} E$.
(8) $(R_\alpha(x))_\alpha \in c_0([\omega, \operatorname{dens} E])$ for all $x \in E$.
(9) $P_\alpha(x) \in \overline{\langle P_\omega(x) \cup \{R_\beta(x) : \omega \le \beta < \alpha\}\rangle}$.

The family $(P_\alpha)_\alpha$ is called *projective resolution of identity* (*PRI*) if it satisfies (1), (2), (3), (4) and (5).

It is called *separable projective resolution of identity* (*SPRI*) if it satisfies (1), (2), (3), (7), (8) and (9). These are the only properties used in 53.20 and they follow for WCD Banach spaces and for duals of Asplund spaces by 53.15. For $C(K)$ with Valdivia compact K this is not clear, see 53.18 and 53.19. However, we still have 53.21 and in 16.18 we don't use (7), but only (8) and (9) which hold also for PRI, see below.

Remark. Note that from (1) we obtain that $P_\alpha^2 = P_\alpha$ and hence $\|P_\alpha\| \ge 1$, and $E_\alpha := P_\alpha(E)$ is the closed subspace $\{x : P_\alpha(x) = x\}$.

Moreover, $P_\alpha P_\beta = P_\beta = P_\beta P_\alpha$ for $\beta \le \alpha$ is equivalent to $P_\alpha^2 = P_\alpha$, $P_\beta(E) \subseteq P_\alpha(E)$ and $\ker P_\beta \supseteq \ker P_\alpha$.
($\Rightarrow$) $P_\beta x = P_\alpha P_\beta x \in P_\alpha(E)$ and $P_\alpha x = 0$ implies that $P_\beta x = P_\beta P_\alpha x$.
($\Leftarrow$) For $x \in E$ there is some $y \in E$ with $P_\beta x = P_\alpha y$, hence $P_\alpha P_\beta x = P_\alpha P_\alpha y = P_\alpha y = P_\beta x$. And $P_\beta(1 - P_\alpha)x = 0$, since $(1 - P_\alpha)x \in \ker P_\alpha \subseteq \ker P_\beta$.

Note that $E_{\alpha+1}/E_\alpha \cong (P_{\alpha+1} - P_\alpha)(E)$, since $E_\alpha \to E_{\alpha+1}$ has $P_\alpha|_{E_{\alpha+1}}$ as right inverse, and so $E_{\alpha+1}/E_\alpha \cong \ker(P_\alpha|_{E_{\alpha+1}}) = (1 - P_\alpha)P_{\alpha+1}(E) = (P_{\alpha+1} - P_\alpha)(E)$.

(5) $\Leftarrow$ (9), since for $x \in E_\alpha$ we have $x = P_\alpha(x)$ and $E_\omega \cup \{R_\beta(x) : \beta < \alpha\} \subseteq E_\alpha$ for all α.

(3) $\Leftarrow$ (5) & (7) By transfinite induction we get that for successor ordinals $\alpha = \beta + 1$ we have $\operatorname{dens}(E_\alpha) = \operatorname{dens}(E_\beta) + \operatorname{dens}(E_\alpha/E_\beta) = \operatorname{dens}(E_\beta) \le \beta \le \alpha$, since $\operatorname{dens}(E_\alpha/E_\beta) \le \omega$. For limit ordinals it follows from (5), since $\operatorname{dens}(E_\alpha) = \operatorname{dens}(\bigcup_{\beta<\alpha} E_\beta) = \sup\{\operatorname{dens}(E_\beta) : \beta < \alpha\} \le \sup\{\beta : \beta < \alpha\} = \alpha$.

(6) $\Leftarrow$ (4) & (1) & (5) For every limit ordinal $0 < \alpha \le \operatorname{dens} E$ and for all $x \in E$ the net $(P_\beta(x))_{\beta<\alpha}$ converges to $P_\alpha(x)$.
Let first $x \in P_\alpha(E)$ and $\varepsilon > 0$. By (5) there exists a $\gamma < \alpha$ and an $x_\gamma \in P_\gamma(E)$ with $\|x - x_\gamma\| < \varepsilon$. Hence for $\gamma \le \beta < \alpha$ we have by (1) that $P_\beta(x_\gamma) = P_\alpha(x_\gamma)$ and so

$$\begin{aligned}\|P_\alpha(x) - P_\beta(x)\| &= \|P_\alpha(x - x_\gamma)| + \|P_\alpha(x_\gamma) - P_\beta(x_\gamma)| - P_\beta(x_\gamma - x)\|\\ &\le (\|P_\alpha\| + \|P_\beta\|)\,\|x - x_\gamma\| < 2\,\varepsilon.\end{aligned}$$

If $x \in E$ is arbitrary, then $P_\alpha(x) \in P_\alpha(E)$, hence by (1)

$$P_\beta(x) = P_\beta(P_\alpha(x)) \to P_\alpha(P_\alpha(x)) = P_\alpha(x) \text{ for } \beta \nearrow \alpha. \quad \square$$

(8) $\Leftarrow$ (1) & (6) Let $\varepsilon > 0$. Then the set $\{\beta : \beta < \alpha, \|R_\beta(x)\| \ge \varepsilon\}$ is finite, since otherwise there would be an increasing sequence (β_n) such that $\|R_{\beta_n}(x)\| \ge \varepsilon$ and since $\|P_{\alpha+1} - P_\alpha\| = \|(1 - P_\alpha)P_{\alpha+1}\| \ge 1$ also $\|(P_{\beta_n+1} - P_{\beta_n})(x)\| \ge \varepsilon$. Let

$\beta_\infty := \sup_n \beta_n$. Then $\beta_\infty \le \alpha$ is a limit ordinal and $P_{\beta_\infty}(x) = \lim_{\beta<\beta_\infty} P_\beta(x)$ according to (6), a contradiction.

(9) $\Leftarrow$ (6) We prove by transfinite induction that $P_\alpha(x)$ is in the closure of the linear span of $\{R_\beta(x) : \omega \le \beta < \alpha\} \cup P_\omega(x)$.
For $\alpha = \omega$ this is obviously true. Let now $\alpha = \beta + 1$ and assume $P_\beta(x)$ is in the closure of the linear span of $\{R_\gamma(x) : \omega \le \gamma < \beta\} \cup P_\omega(x)$. Since $P_\alpha(x) = P_\beta(x) + \|P_\alpha - P_\beta\|\, R_\beta(x)$ we get that $P_\alpha(x)$ is in the closure of the linear span of $\{R_\gamma(x) : \omega \le \gamma < \alpha\} \cup P_\omega(x)$.
Let now α be a limit ordinal and let $P_\beta(x)$ be in the closure of the linear span of $\{R_\gamma(x) : \omega \le \gamma < \alpha\} \cup P_\omega(x)$ for all $\beta < \alpha$. Then by (6) we get that $P_\alpha(x) = \lim_{\beta<\alpha} P_\beta(x)$ is in this closure as well.

Proposition. *Suppose all complemented subspaces of a Banach space E have PRI then E has a SPRI.*

Proof. We proceed by induction on $\mu := \operatorname{dens} E$. For $\mu = \omega$ nothing is to be shown. Now let $(P_\alpha)_{0\le\alpha\le\mu}$ be a PRI of E. For every $\alpha < \mu$ we have $\alpha + 1 < \mu$ and so $\mu_\alpha := \operatorname{dens}((P_{\alpha+1} - P_\alpha)(E)) \le \operatorname{dens}(P_{\alpha+1}(E)) \le \alpha < \mu$, hence there is a SPRI $(P^\alpha_\beta)_{0\le\beta<\mu_\alpha}$ of $(P_{\alpha+1} - P_\alpha)(E)$. Now consider

$$P_{\alpha,\beta} := P_\alpha + P^\alpha_\beta(P_{\alpha+1} - P_\alpha) = (P_\alpha + P^\alpha_\beta(1 - P_\alpha))P_{\alpha+1}$$

for $\omega \le \alpha < \mu$ and $\omega \le \beta \le \mu_\alpha$ with the lexicographical ordering. This is a well-ordering and since the cardinality of μ^2 is μ and $\mu_\alpha < \mu$ it corresponds to the ordinal segment $[\omega, \mu)$. In fact for any limit ordinal $\alpha > \omega$ we have

$$|[\omega,\alpha)| = \sum_{\omega\le\beta<\alpha} 1 \le \sum_{\omega\le\beta<\alpha} |[\omega,\mu_\alpha)| \le |[\omega,\alpha)|^2 \le |[\omega,\alpha)|.$$

Obviously the $P_{\alpha,\beta}$ are projections that satisfy (1) and (3).
(1) For $P_{\alpha,\beta}$ with the same α this follows from (1) for $P^\alpha_\beta : R_\alpha(E) \to R_\alpha(E)$: In fact

$$\begin{aligned} P_{\alpha,\beta}\, P_{\alpha,\beta'} &:= \Big(P_\alpha + P^\alpha_\beta(P_{\alpha+1} - P_\alpha)\Big)\Big(P_\alpha + P^\alpha_{\beta'}(P_{\alpha+1} - P_\alpha)\Big) \\ &= P_\alpha^2 + P^\alpha_\beta(P_{\alpha+1} - P_\alpha)P_\alpha + P_\alpha P^\alpha_{\beta'}(P_{\alpha+1} - P_\alpha) \\ &\quad + P^\alpha_\beta(P_{\alpha+1} - P_\alpha)P^\alpha_{\beta'}(P_{\alpha+1} - P_\alpha) \\ &= P_\alpha^2 + 0 + 0 + P^\alpha_{\min\{\beta,\beta'\}}(P_{\alpha+1} - P_\alpha) \end{aligned}$$

For different α this follows, since $P_{\alpha_1,\beta}E \subseteq P_{\alpha_1+1}E \subseteq P_{\alpha_2}$ and

$$\begin{gathered} P_\alpha E \subseteq P_{\alpha,\beta} \subseteq P_{\alpha+1} \\ \ker P_\alpha \supseteq \ker P_{\alpha,\beta} = \ker(P_\alpha + P^\alpha_\beta(1 - P_\alpha))P_{\alpha+1} \supseteq \ker P_{\alpha+1} \end{gathered}$$

(3) The density of $P_{\alpha,\beta}E$ is less or equal to $\alpha + 1$.
And clearly they satisfy (7) as well, since $R_{\alpha,\beta} = R^\alpha_\beta(P_{\alpha+1} - P_\alpha)$.
(9) Since this is true for the P_α and the P^β_α it follows for $P_{\alpha,\beta}$ as well.
In fact $P_{\alpha,\beta}(x)$ belongs to the closure of the linear span of $P_\alpha(x)$ and the $R_{\alpha,\beta'} = R^\alpha_{\beta'}(P_{\alpha+1} - P_\alpha)(x)$ for $\beta' < \beta$ by the property of the P^α_β. Furthermore $P_\alpha(x)$ belongs to the closure of the linear span of $R_{\alpha'}(x)$ for $\alpha' < \alpha$ and $P_\omega(x)$ by the property of the P_α and $R_{\alpha'}(x)$ belongs to the closure of the linear span of all $R^{\alpha'}_\beta(R_{\alpha'}x)$ for all $\beta < \operatorname{dens} R_{\alpha'}E$.
(8) For x in the linear span of all $R_{\alpha,\beta}E$ we obviously have that $(R_{\alpha,\beta}(x))_{\alpha,\beta} \in c_0$.

In fact for $x := \sum_{i=1}^n \lambda^i R_{\alpha_i,\beta_i}(x_i)$, we have that $R_{\alpha_i,\beta_i}(x) = \lambda^i R_{\alpha_i,\beta_i}(x_i)$ and $R_{\alpha,\beta}(x) = 0$ for all $(\alpha, \beta) \notin \{(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)\}$.

$$R_\alpha R_\beta = (P_{\alpha+1} - P_\alpha)(P_{\beta+1} - P_\beta) = (1 - P_\alpha)P_{\alpha+1}P_{\beta+1}(1 - P_\beta) = 0,$$

if $\alpha+1 \le \beta$ or $\beta+1 \le \alpha$, since the factors commute. For general x we find by ([9]) a point $\tilde{x}$ in the linear span of the $R_{\alpha,\beta}x$ with $\|x - \tilde{x}\| < \varepsilon$. Then

$$\{(\alpha, \beta) : \|R_{\alpha,\beta}(x)\| \ge \varepsilon\} \subseteq \{(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)\}.$$

Note however that we don't have $\|P_{\alpha,\beta}\| = 1$. □

53.14. Theorem. *Let E be a Banach space with projective generator φ. Then E admits a PRI $(P_\alpha)_\alpha$, where each P_α is based on a norming pair (A_α, B_α) with*

(1) $|A_\alpha| \le \alpha$, $|B_\alpha| \le \alpha$ *for all* α;
(2) $A_\beta \subseteq A_\alpha$ *and* $B_\beta \subseteq B_\alpha$ *for all* $\beta \le \alpha$;
(3) $\bigcup_{\omega \le \beta < \alpha} A_\beta = A_\alpha$ *for all limit ordinals* α;
(4) $\bigcup_{\omega \le \beta < \alpha} B_\beta = B_\alpha$ *for all limit ordinals* α;

Proof. Choose a dense subset $\{x_\alpha : \alpha < \operatorname{dens} E\}$. We construct by transfinite recursion for every ordinal $\alpha \le \operatorname{dens} E$ a norming pair (A_α, B_α) with

$$A_\alpha \supseteq \{x_\beta : \beta < \alpha\}, \quad |A_\alpha| \le \alpha, \quad |B_\alpha| \le \alpha, \quad \varphi(B_\alpha) \subseteq A_\alpha$$
$$A_\beta \subseteq A_\alpha \text{ and } B_\beta \subseteq A_\alpha \text{ for } \beta \le \alpha.$$

For the ordinal ω let $A_0 := \{x_\alpha : \alpha < \omega\}$ and let B_0 be a countable subset of E' such that

$$\|x\| = \sup\{|\langle x, x^*\rangle : x^* \in B_0\} \text{ for all } x \in A_0.$$

By [53.7] there is a norming pair (A_ω, B_ω) with $|A_\omega|, |B_\omega| \le \omega$, $A_\omega \supseteq A_0$, $B_\omega \supseteq B_0$ and $\varphi(B_\omega) \subseteq A_\omega$.

If α is a successor ordinal, i.e. $\alpha = \beta + 1$, then let $A_0 := A_\beta \cup \{x_\beta\}$ and $B_0 := B_\beta$. Again by [53.7] we get a norming pair (A_α, B_α), such that

$$A_0 \subseteq A_\alpha, \quad B_0 \subseteq B_\alpha \subseteq E', \quad |A_\alpha| \le \alpha, \quad |B_\alpha| \le \alpha, \quad \varphi(B_\alpha) \subseteq A_\alpha$$

If α is a limit ordinal, we set

$$A_\alpha := \bigcup_{\beta<\alpha} A_\beta$$
$$B_\alpha := \bigcup_{\beta<\alpha} B_\beta \subseteq E'.$$

Then obviously (A_α, B_α) is a norming pair with $\varphi(B_\alpha) \subseteq A_\alpha$.

Now using the property of the projective generator φ we have that there are norm-1 projections $P_\alpha \in L(E)$ with $P_\alpha(E) = \overline{A_\alpha}$ and $\ker P_\alpha = (\overline{B_\alpha})^o = (B_\alpha)^o$. Hence

[53.13.1] $P_\alpha P_\beta = P_\beta = P_\beta P_\alpha$ for $\beta \le \alpha$

[53.13.3] $\operatorname{dens} P_\alpha E \le \alpha, \quad \operatorname{dens} P_\alpha^*(E')_\sigma \le \alpha,$

[53.13.5] $P_\alpha(E) = \overline{A_\alpha} = \overline{\bigcup_{\beta<\alpha} P_\beta E}$

[53.13.4] $\|P_\alpha\| = 1$

and since $\{x_\alpha : \alpha < \operatorname{dens} E\}$ is dense in E we also have 53.13.2. Furthermore we have that B_α is weak*-dense in $P_\alpha^* E'$. $\square$

53.15. Corollary. *WCD and duals of Asplund spaces have SPRI.* $\square$

53.16. Definition. A compact set K is called *Valdivia compact* if there exists some set Γ with $K \subseteq \mathbb{R}^\Gamma$ and $\{x \in K : \operatorname{carr}(x) \text{ is countable}\}$ being dense in K.

53.17. Lemma. *For a Valdivia compact set $K \subseteq \mathbb{R}^\Gamma$ we consider the set $E := \{x \in \mathbb{R}^\Gamma : \operatorname{carr}(x)$ is countable$\}$. Let μ be the density number of $K \cap E$. Then there exists an increasing long sequence of subsets $\Gamma_\alpha \subseteq \Gamma$ for $\omega \le \alpha \le \mu$ satisfying:*

(i) $|\Gamma_\alpha| \le \alpha$;
(ii) $\bigcup_{\beta<\alpha} \Gamma_\beta = \Gamma_\alpha$ *for limit ordinals* α;
(iii) $\Gamma_\mu = \bigcup cup\{\operatorname{carr}(x) : x \in K\}$;

and such that $K_\alpha := Q_{\Gamma_\alpha}(K) \subseteq K$, where $Q_{\Gamma'} : \mathbb{R}^\Gamma \to \mathbb{R}^{\Gamma'} \hookrightarrow \mathbb{R}^\Gamma$, i.e.

$$Q_{\Gamma'}(x)_\gamma := \begin{cases} x_\gamma & \text{for } \gamma \in \Gamma' \\ 0 & \text{for } \gamma \notin \Gamma \setminus \Gamma' \end{cases}.$$

Thus $K_\alpha \subseteq K$ is a retract via Q_{Γ_α}.

Note that for any Valdivia compact set $K \subseteq \mathbb{R}^\Gamma$ we may always replace Γ by $\bigcup\{\operatorname{carr}(x) : x \in K\} = \bigcup\{\operatorname{carr}(x) : x \in K \cap E\}$, and then (iii) says $\Gamma_\mu = \Gamma$.

Proof. The proof is based on the following claim: Let $\Delta \subseteq \Gamma$ be a infinite subset. Then there exists some subset $\tilde\Delta$ with $\Delta \subseteq \tilde\Delta \subseteq \Gamma$ and $|\Delta| = |\tilde\Delta|$ and $Q_{\tilde\Delta}(K) \subseteq K$. By induction we construct a sequence $\Delta =: \Delta_0 \subseteq \Delta_1 \subseteq \dots \subseteq \Delta_k \subseteq \dots \subseteq \Gamma$ with $|\Delta_k| = |\Delta_0|$ and $Q_{\Delta_k}(\{x \in K \cap E : \operatorname{carr}(x) \subseteq \Delta_{k+1}\})$ being dense in $Q_{\Delta_k}(K)$:
(k+1) Since $K \cap E$ is dense in K, we have that $Q_{\Delta_k}(K \cap E)$ is dense in $Q_{\Delta_k}(K) \subseteq \mathbb{R}^{\Delta_k} \times \{0\} \subseteq \mathbb{R}^\Gamma$. And since the topology of $\mathbb{R}^{\Delta_k}$ has a basis of cardinality $|\Delta_k|$, there is a subset $D \subseteq K \cap E$ with $|D| \le |\Delta_k|$ and $Q_{\Delta_k}(D)$ dense in $Q_{\Delta_k}(K)$. Let $\Delta_{k+1} := \Delta_k \cup \bigcup_{x\in D} \operatorname{carr}(x)$ then $\Delta_{k+1} \supseteq \Delta_k$ and $|\Delta_{k+1}| = |\Delta_k|$. Furthermore $Q_{\Delta_k}(\{x \in K \cap E : \operatorname{carr}(x) \subseteq \Delta_{k+1}\}) \supseteq Q_{\Delta_k}(D)$ is dense in $Q_{\Delta_k}(K)$.
Now $\tilde\Delta := \bigcup_k \Delta_k$ is the required set. In order to show that $Q_{\tilde\Delta}(K) \subseteq K$ let $x \in K$ be arbitrary. Since $Q_{\Delta_k}(x)$ is contained in the closure of $Q_{\Delta_k}(\{x_k \in K \cap E : \operatorname{carr}(x_k) \subseteq \Delta_{k+1}\})$ and hence in the closed set $Q_{\Delta_k}(\{x_k \in K : \operatorname{carr}(x_k) \subseteq \tilde\Gamma\})$. Thus there is an $x_k \in K$ with $\operatorname{carr}(x_k) \subseteq \tilde\Gamma$ and such that x agrees with x_k on Δ_k. Thus $K \ni x_k \to Q_{\tilde\Delta}(x)$, since every finite subset of $\tilde\Delta$ is contained in some Δ_k and outside $\tilde\Delta$ all x_k and $Q_{\tilde\Delta}(x)$ are zero. Since K is closed we get $Q_{\tilde\Delta}(x) \in K$.

Without loss of generality we may assume that $\mu > \omega$. Let $\{x_\alpha : \omega \le \alpha < \mu\}$ be a dense subset of $K \cap E$. Let $\Gamma_\omega := \operatorname{carr}(x_\omega)$. By transfinite induction we define

$$\Gamma_\alpha := \begin{cases} (\Gamma_\beta \cup \operatorname{carr}(x_\beta))^\sim & \text{for } \alpha = \beta + 1, \\ \bigcup_{\beta<\alpha} \Gamma_\beta & \text{for limit ordinals } \alpha. \end{cases}$$

Then the Γ_α satisfy all the requirements. $\square$

53.18. Corollary. *Let K be Valdivia compact. Then $C(K)$ has a PRI.*

Proof. We choose Γ_α as in 53.17 and set $K_\alpha := Q_{\Gamma_\alpha}(K)$. Let $Q_\alpha := Q_{\Gamma_\alpha}|_K$. Then Q_α is a continuous retraction.

$$\begin{tikzcd}
K_\alpha \arrow[rr, "\mathrm{Id}_{K_\alpha}"] \arrow[dr, hook, "\mathrm{incl}"'] & & K_\alpha \\
& K \arrow[ur, "Q_\alpha"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
C(K_\alpha) & & C(K_\alpha) \arrow[ll, "\mathrm{Id}"'] \arrow[dl, hook, "Q_\alpha^*"] \arrow[d] \\
& C(K) \arrow[ul, "\mathrm{incl}^*"] & E_\alpha \arrow[l, hook']
\end{tikzcd}$$

We have $\operatorname{dens}(C(\mathbb{R}^{\Gamma_\alpha})) = |\alpha|$, since we have a base of the topology of this space of that cardinality. Hence $\operatorname{dens}(C(K_\alpha)) \le |\alpha|$. Let $E_\alpha := (Q_\alpha)^*(C(K_\alpha))$. Then E_α is a closed subspace of $C(K)$ and 53.13.3 holds. Furthermore $P_\alpha := Q_\alpha \circ \mathrm{incl}^*_{K_\alpha}$ is a norm-1 projection from $C(K)$ to E_α. The inclusion $\Gamma_\alpha \subseteq \Gamma_\beta$ for $\alpha \le \beta$ implies 53.13.1. To see 53.13.6 and 53.13.5 let $\varepsilon > 0$ and choose a finite covering of K_α by sets

$$U_j := \{x \in \mathbb{R}^{\Gamma_\alpha} : |x_\gamma - x^j_\gamma| < \delta_j \text{ for all } \gamma \in \Delta_j\},$$

where $x^j \in \mathbb{R}^{\Gamma_\alpha}$, $\delta_j > 0$ and $\Delta_j \subseteq \Gamma_\alpha$ is finite and such that for $x', x'' \in U_j \cap K$ we have $|f(x') - f(x'')| < \varepsilon$. Now choose $\alpha_0 < \alpha$ such that $\Gamma_{\alpha_0} \supseteq \Delta_j$ for all of the finitely many j. Since the U_j cover K_α, we have $x \in K_\alpha \cap U_j$ for some j and hence $Q_\beta(x) \in K_\alpha \cap U_j$ for all $\alpha_0 \le \beta < \alpha$. Hence $|f(x) - f(Q_\beta(x))| < \varepsilon$ for all $x \in K_\alpha$ and so $\|P_\alpha(f) - P_\beta(f)\| = \|(1 - P_\beta)P_\alpha(f)\| \le \varepsilon$. Thus we have shown that E has a PRI $(P_\alpha)_\alpha$, with all $E_\alpha \cong C(K_\alpha)$ and $\operatorname{dens}(K_\alpha) \le |\Gamma_\alpha| \le \alpha$. $\square$

53.19. Remark. *The space $C([0,\alpha])$ has a PRI given by*

$$P_\beta(f)(\mu) := \begin{cases} f(\mu) & \text{for } \mu \le \beta \\ f(\beta) & \text{for } \mu \ge \beta \end{cases}.$$

However, there is no PRI on the hyperplane $E := \{f \in C([0,\omega_1]) : f(\omega_1) = 0\}$ of the space $C[0,\omega_1]$. And, in particular, $C[0,\omega_1]$ is not WCD.

Proof. Assume $\{P_\alpha : \omega \le \alpha \le \omega_1\}$ is a PRI on E. Put $\alpha_0 := \omega_0$. We may find $\beta_0 < \omega_1$ with

$$P_{\alpha_0}E \subseteq E_{\beta_0} := \{f \in E : f(\alpha) = 0 \text{ for } \alpha > \beta_0\},$$

because for each f in dense countable subset $D \subseteq P_{\alpha_0}E$ we find a β_f with $f(\alpha) = 0$ for $\alpha \ge \beta_f$. Since E_{β_0} is separable, there is an $\alpha_0 < \alpha_1 < \omega_1$ such that

$$E_{\beta_0} \subseteq P_{\alpha_1}E,$$

in fact $D \subseteq E_{\beta_0}$ is dense and hence for each $f \in D$ and $n \in \mathbb{N}$ there exists an $\alpha_{f,n} < \omega_1$ and $\tilde f \in P_{\alpha_{f,n}}E$ such that $\|f - \tilde f\| \le 1/n$. Then $\alpha_1 := \sup\{\alpha_{f,n} : n \in \mathbb{N}, f \in D\}$ fulfills the requirements.

Now we proceed by induction. Let $\alpha_\infty := \sup_n \alpha_n$ and $\beta_\infty := \sup_n \beta_n$. Then

$$P_{\alpha_\infty}E = \overline{\bigcup_n P_{\alpha_n}E} = F_{\beta_\infty} := \{f \in E : f(\alpha) = 0 \text{ for } \alpha \ge \beta_\infty\}.$$

But F_{β_∞} is not the image of a norm-1 projection: Suppose P were a norm-1 projection on F_{β_∞}. Let $\pi : E \to C(X)$ be the restriction map, where $X := [0, \beta_\infty]$. It is left inverse to the inclusion ι given by $f \mapsto \tilde f$ with $\tilde f(\gamma) = 0$ for $\gamma \ge \beta_\infty$.

Let $\tilde{P} := \pi \circ P \circ \iota \in L(C(X))$. Then $\tilde{P}$ is a norm-1 projection with image $C_{\beta_\infty}(X) := \{f \in C[0,\beta_\infty] : f(\beta_\infty) = 0\}$. Then $C(X) = \ker(\tilde{P}) \oplus C_{\beta_\infty}(X)$. We pick $0 \neq f_0 \in \ker(\tilde{P})$. Since $f_0 \notin \tilde{P}(C(X)) = C_{\beta_\infty}(X) = \ker(\mathrm{ev}_{\beta_\infty})$, we have $f_0(\beta_\infty) \neq 0$, and without loss of generality we may assume that $f_0(\beta_\infty) = 1$. For $f \in C(X)$ we have that $f - \tilde{P}(f) \in \ker \tilde{P}$ and hence there is a $\lambda_f \in \mathbb{R}$ with $f - \tilde{P}(f) = \lambda_f f_0$. In fact evaluating at β_∞ gives $f(\beta_\infty) - 0 = \lambda_f 1$, hence $\tilde{P}(f) = f - f(\beta_\infty) f_0$. Since β_∞ is a limit point, there is for each $\varepsilon > 0$ a $x_\varepsilon < \beta_\infty$ with $f_0(x_\varepsilon) > 1 - \varepsilon$. Now choose $f_\varepsilon \in C(X)$ with $\|f_\varepsilon\| = 1 = -f_\varepsilon(\beta_\infty) = f_\varepsilon(x_\varepsilon)$. Then

$$\begin{aligned}\|Pf_\varepsilon\|_\infty &= \|f_\varepsilon - f_\varepsilon(\beta_\infty) f_0\|_\infty \\ &\geq |f_\varepsilon(x_\varepsilon) - f_\varepsilon(\beta_\infty) f_0(x_\varepsilon)| \\ &\geq 1 + 1(1-\varepsilon) = 2 - \varepsilon.\end{aligned}$$

Hence $\tilde{P} \geq 2$, a contradiction.

Note however that every separable subspace is contained in a 1-complemented separable subspace. □

53.20. Theorem. [Biström, 1993, 3.16] *If E is a realcompact (i.e. non-measurable) Banach space admitting a SPRI, then there is a non-measurable set Γ and a injective continuous linear operator $T : E \to c_0(\Gamma)$.*

Proof. We proof by transfinite induction that for every ordinal α with $\alpha \leq \mu := \mathrm{dens}(E)$ there is a non-measurable set Γ_α and an injective linear operator $T_\alpha : E_\alpha := P_\alpha(E) \to c_0(\Gamma_\alpha)$ with $\|T_\alpha\| \leq 1$.
Note that if E is separable, then there are $x_n^* \in E'$ with $\|x_n^*\| \leq 1$, and which are $\sigma(E', E)$ dense in the unit-ball of E'. Then $T : E \to c_0(\mathbb{N})$, defined by $T(x)_n := \frac{1}{n} x_n^*(x)$, satisfies the requirements: It is obviously a continuous linear mapping into c_0, and it remains to show that it is injective. So let $x \neq 0$. By Hahn-Banach there is a $x^* \in E'$ with $x^*(x) = \|x\|$ and $\|x^*\| \leq 1$. Hence there is some n with $|(x_n^* - x^*)(x)| < \|x\|$ and hence $x_n^*(x) \neq 0$.

In particular we have $T_{\omega_0} : E_{\omega_0} \to c_0(\Gamma_{\omega_0})$.

For successor ordinals $\alpha + 1$ we have $E_{\alpha+1} \cong E_\alpha \times (E_{\alpha+1}/E_\alpha) = E_\alpha \times (P_{\alpha+1} - P_\alpha)(E)$. Let $R_\alpha := (P_{\alpha+1} - P_\alpha)/\|P_{\alpha+1} - P_\alpha\|$, let $F := (P_{\alpha+1} - P_\alpha)(E)$ and let $T : F \to c_0$ be the continuous injection for the, by 53.13.7, separable space F with $\|T\| \leq 1$. Then we define $\Gamma_{\alpha+1} := \Gamma_\alpha \sqcup \mathbb{N}$ and $T_{\alpha+1} : E_{\alpha+1} \to c_0(\Gamma_{\alpha+1})$ by

$$T_{\alpha+1}(x)_\gamma := \begin{cases} T_\alpha(\frac{P_\alpha(x)}{\|P_\alpha\|})_\gamma & \text{for } \gamma \in G_\alpha \\ T(R_\alpha(x))_\gamma & \text{for } \gamma \in \mathbb{N} \end{cases}.$$

Now let α be a limit ordinal. We set

$$\Gamma_\alpha := \Gamma_\omega \sqcup \bigsqcup_{\omega \leq \beta < \alpha} \Gamma_{\beta+1},$$

and define $T_\alpha : E_\alpha := P_\alpha(E) \to c_0(\Gamma_\alpha)$ by

$$T_\alpha(x)_\gamma := \begin{cases} T_\omega(\frac{P_\omega(x)}{\|P_\omega\|}) & \text{for } \gamma \in \Gamma_\omega \\ T_{\beta+1}(R_\beta(x))_\gamma & \text{for } \gamma \in \Gamma_{\beta+1} \end{cases}$$

We show first that $T_\alpha(x) \in c_0(\Gamma_\alpha)$ for all $x \in E$. So let $\varepsilon > 0$. Then the set $\{\beta : \|R_\beta(x)\| \geq \varepsilon,\ \beta < \alpha\}$ is finite by 53.13.8.
Obviously T_α is linear and $\|T_\alpha\| \leq 1$. It is also injective: In fact let $T_\alpha(x) = 0$ for some $x \in E_\alpha$. Then $R_\beta(x) = 0$ for all $\beta < \alpha$ and $P_\omega(x) = 0$, hence by $x = P_\alpha(x) = 0$.

As card(E) is non-measurable, also the smaller cardinal dens(E) is non-measurable. Thus the union Γ_α of non-measurable sets over a non-measurable index set is non-measurable. $\square$

53.21. Corollary. *The WCD Banach spaces and the duals of Asplund spaces continuously and linearly inject into some $c_0(\Gamma)$. The same is true for $C(K)$, where K is Valdivia compact.*

For WCG spaces this is due to [Amir, Lindenstrauss, 1968] and for $C(K)$ with K Valdivia compact it is due to [Argyros, Mercourakis, Negrepontis, 1988.]

Proof. For WCD and duals of Asplund spaces this follows using 53.15. For Valdivia compact spaces K one proceeds by induction on dens(K) and uses the PRI constructed in 53.18. The continuous linear injection $C(K) \to c_0(\Gamma)$ is then given as in 53.20 for $\alpha := \operatorname{dens}(K)$, where T_β exists for $\beta < \alpha$, since $E_\beta \cong C(K_\beta)$ with K_β Valdivia compact and $\operatorname{dens}(K_\beta) \le \beta < \alpha$. $\square$

53.22. Theorem. [Bartle, Graves, 1952] *Let $T : E \to F$ be a bounded linear surjective mapping between Banach spaces. Then there exists a continuous mapping $S : F \to E$ with $T \circ S = \mathrm{Id}$.*

Proof. By the open mapping theorem there is a constant $M_0 > 0$ such that for all $\|y\| \le 1$ there exists an $x \in T^{-1}(y)$ with $\|y\| \le M_0$. In fact there is an M_0 such that $B_{1/M_0} \subseteq T(B_1)$ or equivalently $B_1 \subseteq T(B_{M_0})$. Let $(f_\gamma)_{\gamma\in\Gamma}$ be a continuous partition of unity on $oF := \{y \in F : \|y\| \le 1\}$ with $\operatorname{diam}(\operatorname{supp}(f_\gamma)) \le 1/2$. Choose $x_\gamma \in T^{-1}(\operatorname{carr}(f_\gamma))$ with $\|x_\gamma\| \le M_0$ and for $\|y\| \le 1$ set

$$S_0 y := \sum_{\gamma\in\Gamma} f_\gamma(y) x_\gamma \quad \text{and recursively}$$

$$S_{n+1} y := S_n y + \frac{1}{a_n} S_n(a_n(y - T S_n y)),$$

where $a_n := 2^{2^n}$.

By induction we show that the continuous mappings $S_n : \{y : \|y\| \le 1\} \to E$ satisfy $\|y - TS_n y\| \le 1/a_n$ and $\|S_n y\| \le M_n := M_0 \cdot \prod_{k=0}^{n-1}(1 + 1/a_k)$.

$(n = 0)$ Obviously $\|S_0 y\| \le \sum_\gamma f_\gamma(y)\|x_\gamma\| \le M_0$ and

$$\|y - TS_0 y\| = \Big\|\sum_\gamma f_\gamma(y)(y - Tx_\gamma)\Big\| \le \sum_{\gamma\in\Gamma_y} f_\gamma(y)\,\|y - Tx_\gamma\| \le \frac{1}{2} = a_0,$$

where $\Gamma_y := \{\gamma \in \Gamma : f_\gamma(y) \ne 0\}$.

$(n + 1)$ For $\|y\| \le 1$ and $y_n := a_n(y - TS_n y)$ we have $\|y_n\| \le 1$ by induction hypothesis. Then

$$\|S_{n+1} y\| \le \|S_n y\| + \frac{1}{a_n}\|S_n y_n\| \le M_n + \frac{1}{a_n} M_n = M_{n+1}.$$

Furthermore

$$\begin{aligned}\|y - TS_{n+1} y\| &= \|y - TS_n y - \frac{1}{a_n} TS_n(a_n(y - TS_n y))\| \\ &\le \frac{1}{a_n}\|y_n - TS_n y_n\| \le \frac{1}{a_n} \cdot \frac{1}{a_n} = \frac{1}{a_{n+1}}.\end{aligned}$$

Now (S_n) is Cauchy with respect to uniform convergence on $\{y : \|y\| \le 1\}$. In fact

$$\|S_{n+1} y - S_n y\| \le \frac{1}{a_n}\|S_n(a_n(y - TS_n y))\| \le \frac{M_n}{a_n} \le \frac{M_\infty}{a_n},$$

where $M_\infty := \lim_n M_n$. Thus $S := \lim_n S_n$ is continuous and $\|y - TSy\| = \lim_n \|y - TS_n y\| = 0$, i.e. $TSy = y$. Now $S : F \to E$ defined by $S(y) := \|y\| S(\frac{y}{\|y\|})$ and $S(0) := 0$ is the claimed continuous section. $\square$

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