



These lecture notes are inspired to a large extend by the book

## R.Stöcker/H.Zieschang: Algebraische Topologie, B.G.Teubner, Stuttgart 1988

which I recommend for many of the topics I could not treat in this lecture course, in particular this concerns the homology of products [9, chapter 12], homology with coefficients [9, chapter 10], cohomology [9, chapter 13–15].

As always, I am very thankful for any feedback in the range from simple typing errors up to mathematical incomprehensibilities.

Vienna, 2000.08.01

Andreas Kriegl

Since Simon Hochgerner pointed out, that I forgot to treat the case q = n - 1 - r for r < n - 1 in theorem 10.3, I adopted the proof appropriately.

Vienna, 2000.09.25

Andreas Kriegl

I translated chapter 1 from German to English, converted the whole source from amstex to latex and made some stylistic changes for my lecture course in this summer semester.

Vienna, 2006.02.17

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# 1. Building blocks and homeomorphy

For the first chapter I mainly listed the contents in form of short statements. For details please refer to the book.

#### Ball, sphere and cell

#### Problem of homeomorphy.

When is  $X \cong Y$ ? Either we find a homeomorphism  $f : X \to Y$ , or a topological property, which hold for only one of X and Y, or we cannot decide this question.

### 1.1 Definition of basic building blocks. [9, 1.1.2]

- 1.  $\mathbb{R}$  with the metric given by d(x, y) := |x y|.
- 2.  $I := [0, 1] := \{x \in \mathbb{R} : 0 \le x \le 1\}$ , the unit interval.
- 3.  $\mathbb{R}^n := \prod_n \mathbb{R} = \prod_{i \in n} \mathbb{R} = \prod_{i=0}^{n-1} \mathbb{R} = \{(x_i)_{i=0,\dots,n-1} : x_i \in \mathbb{R}\}$ , with the product topology or, equivalently, with any of the equivalent metrics given by a norm on this vector space.
- 4.  $I^n := \prod_n I = \{(x_i)_{i=0}^{n-1} : 0 \le x_i \le 1 \forall i\} = \{x \in \mathbb{R}^n : \|x (\frac{1}{2}, \dots, \frac{1}{2})\|_{\infty} \le \frac{1}{2}\},$ the *n*-dimensional unit cube, where  $\|x_{-\infty} := \max\{|x_i| : i\}.$
- 5. For subsets  $A \subseteq \mathbb{R}^n$  we denote with  $\dot{A} = \partial_{\mathbb{R}^n} A$  the boundary of A in  $\mathbb{R}^n$ . In particular,  $\dot{I}^n := \partial_{\mathbb{R}^n} I^n = \{(x_i)_i \in I^n : \exists i : x_i \in \{0,1\}\}$ , the boundary of  $I^n$  in  $\mathbb{R}^n$ .
- 6.  $D^n := \{x \in \mathbb{R}^n : ||x||_2 := \sqrt{\sum_{i \in n} (x_i)^2} \le 1\}$ , the *n*-dimensional closed unit ball (with respect to the Euclidean norm). A topological space X is called *n*-BALL iff  $X \cong D^n$ .
- 7.  $\dot{D}^n := \partial_{\mathbb{R}^n} D^n = S^{n-1} := \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ , the n-1-dimensional unit sphere.



A topological space X is called *n*-sphere iff  $X \cong S^n$ .

8.  $\mathring{D}^n := \{x \in \mathbb{R}^n : ||x||_2 < 1\}$ , the interior of the *n*-dimensional unit ball. A topological space X is called *n*-CELL iff  $X \cong \mathring{D}^n$ .

**1.2 Definition.** [9, 1.1.3] An AFFINE HOMEOMORPHISMS is a mapping of the form  $x \mapsto A \cdot x + b$  with an invertible linear A and a fixed vector b.

Hence the ball in  $\mathbb{R}^n$  with center b and radius r is homeomorphic to  $D^n$  and thus is an n-ball.

**1.3 Example.** [9, 1.1.4]  $\mathring{D}^1 \cong \mathbb{R}$ : Use the odd functions  $t \mapsto \tan(\frac{\pi}{2}t)$ , or  $t \mapsto \frac{t}{1-t^2}$  with derivative  $t \mapsto \frac{t^2+1}{(t^2-1)^2} > 0$ , or  $t \mapsto \frac{t}{1-|t|}$  with derivative  $t \mapsto 1/(1-|t|) > 0$  and inverse mapping  $s \mapsto \frac{t}{1+|t|}$ . Note, that any bijective function  $f : [0,1) \to [0,+\infty)$  extends to an odd function  $\tilde{f} : (-1,1) \to \mathbb{R}$  by setting  $\tilde{f}(t) := -f(-t)$  for t < 0. For  $f(t) := \frac{t}{1-t}$  we have  $\tilde{f}(t) = -\frac{-t}{1-(-t)} = \frac{t}{1-|t|}$  and for  $f(t) := \frac{t}{t-t^2}$  we have  $\tilde{f}(t) = -\frac{-t}{1-(-t)^2} = \frac{t}{1-t^2}$ . Note that in both cases f(0) = 0 and  $\exists f'(0) = \lim_{t\to 0+} f'(t)$ , hence  $\tilde{f}$  is a  $C^1$  diffeomorphism. However, in the first case  $\lim_{t\to 0+} f''(t) = 2$  and hence the odd function  $\tilde{f}$  is not  $C^2$ .

**1.4 Example.** [9, 1.1.5]  $\overset{\circ}{D}^n \cong \mathbb{R}^n$ : Use for example  $f: x \mapsto \frac{x}{1-\|x\|} = \frac{x}{\|x\|} \cdot f_1(\|x\|)$  with  $f_1(t) := \frac{t}{1-t}$  and directional derivative  $f'(x)(v) = \frac{1}{1-\|x\|} v + \frac{\langle x|v\rangle}{(1-\|x\|)^2 \|x\|} x \to v$  for  $x \to 0$ .

**1.5 Corollary.** [9, 1.1.6]  $\mathbb{R}^n$  is a cell; products of cells are cells, since  $\mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{n+m}$  by "associativity" of the product.

**1.7 Definition.** A PAIR (X, A) of spaces is a topological space X together with a subspace  $A \subseteq X$ .

A MAPPING  $f : (X, A) \to (Y, B)$  of pairs is a continuous mapping  $f : X \to Y$  with  $f(A) \subseteq B$ . A HOMEOMORPHISM  $f : (X, A) \to X$  (Y, B) of pairs is a mapping of pairs which is a homeomorphism  $f : X \to Y$  and satisfies f(A) = B (and hence induces a homeomorphism  $f|_A : A \to B$ ).  $A \xrightarrow{f|_A} B$ 

**1.8 Definition.** [9, 1.3.2] A mapping  $f : (X, A) \to (Y, B)$  of pairs is called Relative Homeomorphism, iff  $f : X \setminus A \to Y \setminus B$  is a well-defined homeomorphism.

A homeomorphism of pairs is obviously a relative homeomorphism, but not conversely even if  $f|_A : A \to B$  is a homeomorphism: Consider for example  $A := \{-1\},$  $X := A \cup (1, 2], \text{ and } f : t \mapsto t^2 - 2.$ B
Y\B

However, for X and Y compact any homeomorphism  $f: X \setminus \{x_0\} \to Y \setminus \{y_0\}$  extends to a homeomorphism  $\tilde{f}: (X, \{x_0\}) \to (Y, \{y_0\})$  of pairs, since  $X \cong (X \setminus \{x_0\})_{\infty}$ , cf. **1.35**. Note that  $Z_{\infty}$  denotes the 1-point compactification of the locally compact space Z, see [2, 2.2.5].

### **1.9 Example.** [9, 1.1.15]



**1.6 Definition.** A subset  $A \subseteq \mathbb{R}^n$  ist called CONVEX, iff  $x + t(y - x) \in A$  for  $\forall x, y \in A, t \in [0, 1]$ .

**1.10 Theorem.** [9, 1.1.8]  $X \subseteq \mathbb{R}^n$  compact, convex,  $\mathring{X} \neq \emptyset \Rightarrow (X, \dot{X}) \cong (D^n, S^{n-1})$ . In particular, X is a ball,  $\dot{X}$  is a sphere and  $\mathring{X}$  is a cell. If  $X \subseteq \mathbb{R}^n$  is (bounded) compand converted by the second seco

If  $X \subseteq \mathbb{R}^n$  is (bounded,) open and convex and not empty  $\Rightarrow X$  is a cell.

**Proof.** W.l.o.g. let  $0 \in \mathring{X}$  (translate X by  $-x_0$  with  $x_0 \in \mathring{X}$ ). The mapping  $f: \mathring{X} \ni x \mapsto \frac{1}{\|x\|} x \in S^{n-1}$  is bijective, since it keeps rays from 0 invariant and since for  $y \neq 0$  let  $t_0 := \max\{t > 0 : t \ y \in X\}$ . Then  $t \ y \notin X$  for all  $t > t_0$  and  $t \ y \in \mathring{X}$  for all  $0 \le t < t_0$  (consider the cone with an open 0-neighborhood in X as basis and  $t_0 \ y$  as apex), hence  $t_0$  is the unique t > 0 with  $t \ y \in \mathring{X}$ . Since  $\mathring{X}$  is compact f is a homeomorphism and by radial extension we get a continuous bijection

$$D^{n} \setminus \{0\} \cong S^{n-1} \times (0,1] \stackrel{f \times \operatorname{id}}{\cong} \dot{X} \times (0,1] \to X \setminus \{0\},$$
$$x \mapsto \left(\frac{x}{\|x\|}, \|x\|\right) \mapsto \left(f^{-1}\left(\frac{x}{\|x\|}\right), \|x\|\right) \mapsto \|x\| f^{-1}\left(\frac{x}{\|x\|}\right)$$

which extends via  $0 \mapsto 0$  to a continuous bijection of the 1-point compactifications and hence a homeomorphism of pairs  $(D^n, S^{n-1}) \to (X, \dot{X})$ .

The second part follows by applying the first part to  $\overline{X}$ , a compact convex set with non-empty interior X: In order to see this take a point x in the interior of  $\overline{X}$ . So there exists a open neighborhood of x in  $\overline{X}$  and we may assume that this is of the form of an n-simplex (see 3.2) (i.e. a hypertetraeder). Since its vertices are in  $\overline{X}$ we can approximate them by points in X and hence x lies inside this approximating simplex contained in X.

That the boundedness condition can be dropped can be found for a much more general situation in [5, 16.21].

# **1.11 Corollary.** [9, 1.1.9] $I^n$ is a ball and $\dot{I}^n$ is a sphere.

**1.12 Example.** [9, 1.1.10] [9, 1.1.11]  $D^p \times D^q$  is a ball, hence products of balls are balls, and  $\partial(D^p \times D^q) = S^{p-1} \times D^q \cup D^p \times S^{q-1}$  is a sphere:  $D^p \times D^q$  is compact convex, and by exercise (1.1.1A)  $\partial(A \times B) = \partial A \times B \cup A \times \partial B$ . So by [1.10] the result follows.

**1.13 Remark.** [9, 1.1.12] 1.10 is wrong without convexity or compactness assumption: For compactness this is obvious since  $D^n$  ic compact. That, for example, a compact annulus is not a ball will follow from 2.19.

 $\dot{x} + t(x - \dot{x})$ 

**1.14 Example.** [9, 1.1.13]  $S^n = D^n_+ \cup D^n_-$ ,  $D^n_+ \cap D^n_- = S^{n-1} \times \{0\} \cong S^{n-1}$ , where  $D^n_{\pm} := \{(x;t) \in S^n \subseteq \mathbb{R}^n \times \mathbb{R} : \pm t \ge 0\} \cong D^n$  are the southern and northern hemispheres. The stereographic projection  $S^n \setminus \{(0,\ldots,0;1)\} \cong \mathbb{R}^n$  is given by  $(x;t) \mapsto \frac{1}{1-t}x$ .



**1.15 Corollary.** [9, 1.1.14]  $S^n \setminus \{*\}$  is a cell.

**1.16 Example.** [9, 1.1.15.3] For all  $\dot{x} \in S^{n-1}$ :

$$D^n \setminus \{\dot{x}\} \cong \mathbb{R}^{n-1} \times [0, +\infty),$$

via

$$\mathbb{R}^{n-1} \times [0, +\infty) \cong (S^{n-1} \setminus \{\dot{x}\}) \times (0, 1] \cong D^n \setminus \{\dot{x}\},$$
$$(x, t) \mapsto \dot{x} + t(x - \dot{x}).$$



None-homeomorphy of  $X = S^1$  with I follows by counting components of  $X \setminus \{*\}$ .

**1.18 Example.** [9, 1.1.21]  $S^1 \times S^1$  is called torus. It is embeddable into  $\mathbb{R}^3$  by  $(x, y) = (x_1, x_2; y_1, y_2) \mapsto ((R+r y_1)x, r y_2)$  with 0 < r < R. This image is described by the equation  $\{(x, y, z) : (\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2\}$ . Furthermore,  $S^1 \times S^1 \not\cong S^2$  by Jordan's curve theorem, since  $(S^1 \times S^1) \setminus (S^1 \times \{1\})$  is connected.



**1.19 Theorem (Invariance of a domain).** [9, 1.1.16]  $\mathbb{R}^n \supseteq X \cong Y \subseteq \mathbb{R}^n$ , X open in  $\mathbb{R}^n \Rightarrow Y$  open in  $\mathbb{R}^n$ .

We will prove this hard theorem after 10.5.

**1.20 Theorem (Invariance of dimension).** [9, 1.1.17]  $m \neq n \Rightarrow \mathbb{R}^m \ncong \mathbb{R}^n$ ,  $S^m \ncong S^n$ ,  $D^m \ncong D^n$ .

#### **Proof.** Let m < n.

Suppose  $\mathbb{R}^n \cong \mathbb{R}^m$ , then  $\mathbb{R}^n \subseteq \mathbb{R}^n$  is open, but the image  $\mathbb{R}^m \cong \mathbb{R}^m \times \{0\} \subseteq \mathbb{R}^n$  is not, a contradiction to 1.19.

 $S^{m} \cong S^{n} \Rightarrow \mathbb{R}^{m} \cong S^{m} \setminus \{x\} \cong S^{n} \setminus \{y\} \cong \mathbb{R}^{n} \Rightarrow m = n.$  $f: D^{m} \cong D^{n} \Rightarrow \mathring{D}^{n} \cong f^{-1}(\mathring{D}^{n}) \subseteq D^{m} \subseteq \mathbb{R}^{m} \subset \mathbb{R}^{n} \text{ and } f^{-1}(\mathring{D}^{n}) \text{ is not open, a contradiction to } 1.19.$ 

**1.21 Theorem (Invariance of the boundary).** [9, 1.1.18]  $f: D^n \to D^n$  homeomorphism  $\Rightarrow f: (D^n, S^{n-1}) \to (D^n, S^{n-1})$  homeomorphism of pairs.

**Proof.** Let  $\dot{x} \in \dot{D}^n$  with  $y = f(\dot{x}) \notin \dot{D}^n$ . Then  $y \in \mathring{D}^n =: U$  and  $f^{-1}(U)$  is homeomorphic to U but not open, since  $x \in f^{-1}(U) \cap \dot{D}^n$ , a contradiction to 1.19.

**1.22 Definition.** [9, 1.1.19] Let X be an n-ball and  $f: D^n \to X$  a homeomorphism. The BOUNDARY  $\dot{X}$  of X ist defined as the image  $f(\dot{D}^n)$ . This definition makes sense by 1.21.

#### Quotient spaces

**1.23 Definition. Quotient space.** [9, 1.2.1] Cf. [2, 1.2.12]. Let ~ be an equivalence relation on a topological space X. We denote the set of EQUIVALENCE CLASSES  $[x]_{\sim} := \{y \in X : y \sim x\}$  by  $X/\sim$ . The QUOTIENT TOPOLOGY on  $X/\sim$  is the final topology with respect to the mapping  $\pi : X \to X/\sim, x \mapsto [x]_{\sim}$  (i.e. the finest toplogy for which this mapping is continuous, see [2, 1.2.11]).

**1.24 Proposition.** [9, 1.2.2] A subset  $B \subseteq X/\sim$  is open/closed iff  $\pi^{-1}(B)$  is open/closed. The quotient mapping  $\pi$  is continuous and surjective. It is open/closed iff for every open/closed  $A \subseteq X$  the saturated hull  $\pi^{-1}(\pi(A))$  is open/closed.

For a proof see [2, 1.2.12].

The image of the closed subset  $\{(x, y) : x \cdot y = 1, x, y > 0\} \subseteq \mathbb{R}^2$  under the first projection  $\operatorname{pr}_1 : \mathbb{R}^2 \to \mathbb{R}$  is not closed!

**1.25 Definition.** [9, 1.2.9] A mapping  $f : X \to Y$  is called QUOTIENT MAPPING (or final), iff f is surjective continuous and satisfies one of the following equivalent conditions:

- 1. The induced mapping  $X/\sim \to Y$  is a homeomorphism, where  $x_1 \sim x_2 :\Leftrightarrow f(x_1) = f(x_2)$ .
- 2.  $B \subseteq Y$  is open (closed) if  $f^{-1}(B)$  is it.
- 3. A mapping  $g: Y \to Z$  is continuous iff  $g \circ f$  is it.

 $(1\Rightarrow 2) X \to X/\sim$  has this property.  $(2\Rightarrow 3) g^{-1}(W)$  open  $\Leftrightarrow (g \circ f)^{-1}(W) = f^{-1}(g^{-1}W)$  is open.  $(3\Rightarrow 1) X/\sim \to Y$  is continuous by  $(1\Rightarrow 3)$  for  $Y := X/\sim$ . And  $Y \to X/\sim$  is continuous by (3).

1.26 Example. [9, 1.2.3]

- 1.  $I/\sim \cong S^1$ , where  $0 \sim 1$ : The mapping  $t \mapsto e^{2\pi i t}$ ,  $I \to S^1$  factors to homeomorphism  $I/\sim \to S^1$ .
- 2.  $I^2/\sim \cong S^1 \times I$ , where  $(0,t) \sim (1,t)$  for all t.





3.  $I^2/\sim \cong S^1 \times S^1$ , where  $(t,0) \sim (t,1)$  and  $(0,t) \sim (1,t)$  for all t.



**1.27 Proposition. Universal property of**  $X/\sim$ . [9, 1.2.11] [9, 1.2.6] [9, 1.2.5] Let  $f: X \to Y$  be continuous. Then f is compatible with the equivalence relation (i.e.  $x \sim x' \Rightarrow f(x) = f(x')$ ) iff it factors to a continuous mapping  $X/\sim \to Y$ over  $\pi: X \to X/\sim$ . Note that f is compatible with the equivalence relation iff the relation  $f \circ \pi^{-1}$  is a mapping. The factorization  $X/\sim \to Y$  is then given by  $f \circ \pi^{-1}$ .

**Proof.**  $(z, y) \in f \circ \pi^{-1} \Leftrightarrow \exists x \in X : f(x) = y, \pi(x) = z$ . Thus  $f \circ \pi^{-1}$  is a mapping, i.e. y is uniquely determined by z iff  $\pi(x) = \pi(x') \Rightarrow f(x) = f(x')$ .

**1.28 Proposition.** [9, 1.2.4] Functoriality of formation of quotients. Let  $f : X \to Y$  be continuous and compatible with equivalence relations  $\sim_X$  on X and  $\sim_Y$  on Y. Then there is a unique induced continuous mapping  $\tilde{f} : X/\sim_X \to Y/\sim_Y$ .

If f and  $f^{-1}$  are compatible with the equivalence relations and is a homeomorphism, then  $\tilde{f}$  is a homeomorphism.

For a proof see [2, 1.2.11, 1.2.12].

**1.29 Proposition.** [9, 1.2.7] [9, 1.2.12] The restriction of a quotient-mapping to a closed/open saturated set is a quotient-mapping.

Let  $f: X \to Y$  be a quotient mapping,  $B \subseteq Y$  open (or closed),  $A := f^{-1}(B)$ . Then  $f|_A: A \to B$  is a quotient mapping.

For example, the restriction of  $\pi: I \to I/\dot{I}$  to the open set [0,1) is not a quotient mapping.

**Proof.** Let  $U \subseteq B$  with  $(f|_A)^{-1}(U)$  open. Then  $f^{-1}(U) = (f|_A)^{-1}(U)$  is open and hence  $U \subseteq Y$  is open.

**1.30 Corollary.** [9, 1.2.8] A closed/open,  $\forall a \in A, x \in X, x \sim a \Rightarrow x = a, p : X \rightarrow Y$  quotient-mapping  $\Rightarrow p|_A : A \rightarrow p(A) \subseteq Y$  is an embedding.

**Proof.**  $\Rightarrow A = p^{-1}(p(A)) \xrightarrow{1.29} p|_A : A \to p(A)$  is a quotient mapping and injective, hence a homeomorphism.  $\Box$ 

**1.31 Proposition.** [9, 1.2.10] Continuous surjective closed/open mappings are obviously quotient-mappings, but not conversely. Continuous surjective mappings from a compact to a  $T_2$ -space are quotient-mappings, since the image of closed subsets is compact hence closed. f, g quotient mapping  $\Rightarrow g \circ f$  quotient mapping, by 1.25.3.  $g \circ f$  quotient mapping  $\Rightarrow g$  quotient mapping, by 1.25.3.

**1.32** Proposition. Theorem of Whitehead. [9, 1.2.13] Let g be a quotient mapping and X locally compact. Then  $X \times g$  is quotient mapping.

For a counterexample for not locally compact X see [2, 1.2.12]:



Let  $(x_0, z_0) \in W \subseteq X \times Z$  with open  $f^{-1}(W) \subseteq X \times Y$ , where  $f := X \times g$  for  $g : Y \to Z$ . We choose  $y_0 \in g^{-1}(z_0)$  and a compact  $U \in \mathcal{U}(x_0)$  with  $U \times \{y_0\} \subseteq f^{-1}(W)$ . Since  $f^{-1}(W)$  is saturated,  $U \times g^{-1}(g(y)) \subseteq f^{-1}(W)$  provided  $U \times \{y\} \subseteq f^{-1}(W)$ . In particular,  $U \times g^{-1}(z_0) \subseteq f^{-1}(W)$ . Let  $V := \{z \in Z : U \times g^{-1}(z) \subseteq f^{-1}(W)\}$ . Then  $(x_0, z_0) \in U \times V \subseteq W$  and V is open, since  $g^{-1}(V) := \{y \in Y : U \times \{y\} \subseteq f^{-1}(W)\}$  is open (see [2, 2.1.11]).

**1.33 Corollary.** [9, 1.2.14]  $f : X \to X', g : Y \to Y'$  quotient mappings, X, Y' locally compact  $\Rightarrow f \times g$  quotient mapping.

Proof.

$$\begin{array}{c} X \times Y \xrightarrow{f \times Y} X' \times Y \\ \downarrow \\ X \times g \xrightarrow{f \times g} & \downarrow \\ X' \times g \\ X \times Y' \xrightarrow{f \times Y'} X' \times Y' \end{array}$$

# Special cases of quotient mappings

**1.34 Proposition. Collapse of subspace.** [9, 1.3.1] [9, 1.3.3]  $A \subseteq X$  closed  $\Rightarrow \pi : (X, A) \rightarrow (X/A, \{A\})$  is a relative homeomorphism, where  $X/A := X/\sim$  with the equivalence relation generated by  $\forall a, a' \in A : a \sim a'$ . The functorial property

for mappings of pairs is:



**Proof.** That  $\pi: X \setminus A \to X/A \setminus A/A$  is a homeomorphism follows from 1.29. The functorial property follows from 1.27

**1.35 Example.** [9, 1.3.4]  $X/\emptyset \cong X$ ,  $X/\{*\} \cong X$ .  $I/\dot{I} \cong S^1$ ,  $X/A \cong (X \setminus A)_{\infty}$ , provided X compact and  $A \subseteq X$  is closed. In fact, X/A is compact,  $X \setminus A$  is openly embedded into X/A and  $X/A \setminus (X \setminus A)$  is the single point  $A \in X/A$ .

**1.36 Example.** [9, 1.3.5]  $D^n \setminus S^{n-1} = \mathring{D}^n \cong \mathbb{R}^n$  and hence by 1.35  $D^n/S^{n-1} \cong (D^n \setminus S^{n-1})_{\infty} \cong (\mathbb{R}^n)_{\infty} \cong S^n$ . Or, explicitly,

$$x \mapsto \left(t := \|x\|, \frac{x}{\|x\|}\right) \mapsto \left(\sin\left(\pi(1-t)\right)\frac{x}{\|x\|}, \cos\left(\pi(1-t)\right)\right)$$



**1.37 Example.** [9, 1.3.6]  $X \times I$  is called cylinder over X. And  $CX := (X \times I)/(X \times \{0\})$  is called the cone with base X.  $C(S^n) \cong D^{n+1}$ , via  $(x,t) \mapsto tx$ .

**1.38 Example.** [9, 1.3.7] Let  $(X_j, x_j)$  be pointed spaces. The 1-point union is

$$\bigvee_{j \in J} X_j = \bigvee_{j \in J} (X_j, x_j) := \bigsqcup_j X_j / \{x_j : j\}.$$

By 1.24 the projection  $\pi: \bigsqcup_j X_j \to \bigvee_j X_j$  is a closed mapping.

**1.39 Proposition.** [9, 1.3.8]  $X_i$  embeds into  $\bigvee_j X_j$  and  $\bigvee_j X_j$  is union of the images, which have pairwise as intersection the base point.

**Proof.** That the composition  $X_i \hookrightarrow \bigsqcup_j X_j \to \bigvee_j X_j$  is continuous and injective is clear. That it is an embedding follows, since by **1.38** the projection  $\pi$  is also a closed mapping.

**1.40 Proposition.** [9, 1.3.9] Universal and functorial property of the 1-point-union:



**Proof.** This follows from 1.28 and 1.27.

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**1.41 Proposition.** [9, 1.3.10] *Embedding of*  $X_1 \lor \cdots \lor X_n \hookrightarrow X_1 \times \ldots \times X_n$ .



**Proof.** Let  $i_j : X_j \to \prod_{k=1}^n X_k$  be given by  $z \mapsto (x_1, \ldots, x_{j-1}, z, x_j, \ldots, x_n)$ , where the  $x_k$  are the base-points of  $X_k$ . Then  $\bigsqcup_k i_k : \bigsqcup_k X_k \to \prod_k X_k$  factors to the claimed embedding, see exercise (1.7).

**1.42 Example.** [9, 1.3.11] 1.41 is wrong for infinite index sets: The open neighborhoods of the base point in  $\bigvee_j X_j$  are given by  $\bigvee_j U_j$ , where  $U_j$  is an open neighborhood of the base point in  $X_j$ . Hence  $\bigvee X_j$  is not first countable, whereas the product of countable many metrizable spaces  $X_j$  is first countable.

Also countable many circles in  $\mathbb{R}^2$  which intersect only in a single point have as union in  $\mathbb{R}^2$  not their one-point union, since a neighborhood of the single point contains almost all circles completely.



**1.43 Definition.** Gluing. [9, 1.3.12]  $f : X \supseteq A \to Y$  with  $A \subseteq X$  closed.  $Y \cup_f X := Y \sqcup X/\sim$ , where  $a \sim f(a)$  for all  $a \in A$ , is called Y glued with X via f (or along f).



**1.44 Proposition.** [9, 1.3.13] [9, 1.3.14]  $f : X \supseteq A \to Y$  with  $A \subseteq X$  closed.  $\pi|_Y : Y \to Y \cup_f X$  is a closed embedding.  $\pi : (X, A) \to (Y \cup_f X, \pi(Y))$  is a relative homeomorphism.

**Proof.** That  $\pi|_Y : Y \to Y \cup_f X$  is continuous and injective is clear. Now let  $B \subseteq Y$  be closed. Then  $\pi^{-1}(\pi(B)) = B \sqcup f^{-1}(B)$  is closed and hence also  $\pi(B)$ .

That  $\pi: X \setminus A \to Y \cup_f X \setminus Y$  is a homeomorphism follows from 1.29.

**1.45 Proposition.** [9, 1.3.15] Universal property of push-outs  $Y \cup_f X$ :



**Proof.** 1.27.

**1.46 Lemma.** Let  $f_i : X_i \supseteq A_i \to Y$  be given,  $X := X_1 \sqcup X_2$ ,  $A := A_1 \sqcup A_2 \subseteq X$ and  $f := f_1 \sqcup f_2 : X \supseteq A \to Y$ . Then  $Y \cup_f X \cong (Y \cup_{f_1} X_1) \cup_{f_2} X_2$ .

# Proof.



**1.47 Example.** [9, 1.3.16]  $f : X \supseteq A \to Y = \{*\} \Rightarrow Y \cup_f X \cong X/A$ , since X/A satisfies the universal property of the push-out.

 $f: X \supseteq \{*\} \to Y \Rightarrow Y \cup_f X \cong X \lor Y$ , by definition.

 $f: X \supseteq A \to Y$  constant  $\Rightarrow Y \cup_f X \cong X/A \lor Y$ , since we can compose pullbacks:



**1.48 Example.** [9, 1.3.17]  $f: X \supseteq A \to B \subseteq Y$  homeomorphism of closed subsets.  $\Rightarrow Y \cup_f X = \pi(X) \cup \pi(Y)$  with  $\pi(X) \cong X$ ,  $\pi(Y) \cong Y$  and  $\pi(X) \cap \pi(Y) \cong A \cong B$ . This follows from 1.44 since  $Y \cup_f X \cong X \cup_{f^{-1}} Y$ . Note however, that  $Y \cup_f X$ depends not only on  $X \supseteq A$  and  $Y \supseteq B$  but also on the gluing map  $f: A \to B$  as the example  $X = I \times I = Y$  and  $A = B = I \times I$  with  $id \neq f: (x, 1) \mapsto (1 - x, 1)$ ,  $(x, 0) \mapsto (x, 0)$  of a Möbius-strip versus a cylinder shows.



#### **1.49 Proposition.** [9, 1.3.18]



 $\Rightarrow Y \cup_f X \cong Y' \cup_{f'} X'.$ 

**Proof.** By [1.45] we obtain a uniquely determined continuous map  $G \cup F : Y \cup_f X \to Y' \cup_{f'} X'$  with  $(G \cup F) \circ \pi|_X = \pi|_{X'} \circ F$  and  $(G \cup F) \circ \pi|_Y = \pi|_{Y'} \circ F$ . Since  $G^{-1} \circ f' = G^{-1} \circ f' \circ F \circ F|_A^{-1} = G^{-1} \circ G \circ f \circ F|_A^{-1} = f \circ F^{-1}|_{A'}$  we can similarly  $G^{-1} \cup F^{-1} : Y' \cup_{f'} X' \to Y \cup_f X$ . On X and Y (resp. X' and Y') they are inverse to each other, hence define a homeomorphism as required.  $\Box$ 

#### **1.50 Example.** [9, 1.3.19]

- (1)  $Z = X \cup Y$  with X, Y closed.  $\Rightarrow Z = Y \cup_{id} X$ : The canonical mapping  $Y \sqcup X \to Z$  induces a continuous bijective mapping  $Y \cup_{id} X \to Z$ , which is closed and hence a homeomorphism, since  $Y \sqcup X \to Z$  is closed.
- (2)  $Z = X \cup Y$  with X, Y closed,  $A := X \cap Y$ ,  $f : A \to A$  extendable to a homeomorphism of the pair  $(X, A) \Rightarrow Z \cong Y \cup_f X$ : Apply 1.49 to



- (3)  $D^n \cup_f D^n$  for all homeomorphisms  $f : S^{n-1} \to S^{n-1}$ : We can extend f radially to a homeomorphism  $\tilde{f} : D^n \to D^n$  by  $\tilde{f}(x) = ||x|| f(\frac{x}{||x||})$  and can now apply (2).
- (4) Gluing two identical cylinders  $X \times I$  along any homeomorphism  $f : X \times \{0\} \to X \times \{0\}$  yields again the cylinder  $X \times I$ : Since f extends to a homeomorphism  $X \times I \to X \times I$ ,  $(x, t) \mapsto (f(x), t)$  we may apply (2) to obtain  $(X \times I) \cup_f (X \times I) = (X \times I) \cup_{id} (X \times I) \cong X \times I$ .

## Manifolds

**1.51 Definition.** [9, 1.4.1] [9, 1.5.1] An *m*-dimensional MANIFOLD (possibly with boundary) is a topological space X (which we will always require to be Hausdorff and second countable), for which each of its points  $x \in X$  has a neighborhood A which is an *n*-ball, i.e. a homeomorphism  $\varphi : A \cong D^m$  (which we call CHART at x) exists. A point  $x \in X$  is called BOUNDARY POINT iff for some (and by 1.21) any) chart  $\varphi$  at x the point is mapped to  $\varphi(x) \in S^{m-1}$ . The set of all boundary points is called the BOUNDARY of X and denoted by  $\partial X$  or  $\dot{X}$ . A manifold is called CLOSED if it is compact and has empty boundary. Two-dimensional manifolds are called SURFACES.

### **1.57 Examples.** [9, 1.4.4] [9, 1.4.5]

1. 0-manifolds are discrete countable topological spaces.

- 2. The connected 1-manifolds are  $\mathbb{R}$ ,  $S^1$ , I and  $[0, +\infty)$ .
- 3. Quadrics like hyperboloids ( $\cong \mathbb{R}^2 \sqcup \mathbb{R}^2$  or  $\cong S^1 \times \mathbb{R}$ ), paraboloids ( $\cong \mathbb{R}^2$ ), and the cylinder  $S^1 \times \mathbb{R}$  are surfaces.
- 4. Let X be a surface without boundary and  $A \subseteq X$  be a discrete subset. Then  $X \setminus A$  is also a surface without boundary.
- 5.  $D^m$  is a manifold with boundary  $S^{m-1}$ , so  $\mathring{D}^m \cong \mathbb{R}^m$  is a manifold without boundary, and the halfspace  $\mathbb{R}^{m-1} \times [0, +\infty)$  is a manifold with boundary  $\mathbb{R}^{m-1} \times \{0\}$ .

**1.51a Lemma.** Let  $U \subseteq X$  be open in an m-manifold X. Then U is an m-manifold with  $\dot{U} = \dot{X} \cap U$ 

**Proof.** Let  $x \in U$  and  $\varphi : A \xrightarrow{\cong} D^m$  be a chart at  $x \in A$ . Then  $\varphi(U)$  is an open neighborhood of  $\varphi(x)$  in  $D^m$  and hence contains a convex compact neighborhood B which is an m-ball by 1.10. Consequently,  $\varphi : U \supseteq \varphi^{-1}(B) \cong B \subseteq D^m$  is the required chart at x for U. Obviously  $x \in \dot{U} \Leftrightarrow \varphi(x) \in \dot{B} \Leftrightarrow \varphi(x) \in S^{m-1} \Leftrightarrow x \in \dot{X}$ .

**1.52 Proposition.** [9, 1.4.2] [9, 1.5.2] Let  $f : X \to Y$  be a homeomorphism between manifolds. Then  $f(\dot{X}) = \dot{Y}$ .

**Proof.** Let  $x \in X$  and  $\varphi : A \cong D^m$  a chart at x. Then  $\varphi \circ f^{-1} : f(A) \to D^m$  is a chart of Y at f(x) and hence  $x \in \dot{X} \Leftrightarrow f(x) \in \dot{Y}$ .

**1.53 Proposition.** [9, 1.4.3] [9, 1.5.3] Let X be an m-manifold and  $x \in \dot{X}$ . Then there exists a neighborhood U of x in X with  $(U, U \cap \dot{X}, x) \cong (D^{n-1} \times I, D^{n-1} \times \{0\}, (0, 0))$ , an homeomorphism of triples.

**Proof.** By assumption there exists a neighborhood A of x in X and a homeomorphism  $\varphi : A \to D^m$  with  $\varphi(x) \in S^{m-1}$ . Choose an open neighborhood  $W \subseteq A$  of x. Then  $\dot{W} = \dot{X} \cap W$  and the manifold W is homeomorphic to  $\varphi(W) \subseteq D^m$ . Obviously  $\varphi(W)$  contains a neighborhood B of  $\varphi(x)$  homeomorphic to  $D^{m-1} \times I$ , where  $S^{m-1} \cap B$  corresponds to  $D^{m-1} \times \{0\}$ . The set  $U := \varphi^{-1}(B)$  is then the required neighborhood.

**1.54 Corollary.** [9, 1.5.4] The boundary  $\dot{X}$  of a manifold is a manifold without boundary.

**Proof.** By 1.53  $\dot{X}$  is locally homeomorphic to  $D^{n-1} \times \{0\}$  and  $x \in \dot{X}$  corresponds to (0,0) thus is not in the boundary of  $\dot{X}$ .

**1.55 Proposition.** [9, 1.5.7] Let M be an m-dimensional and N an n-dimensional manifold. Then  $M \times N$  is an m+n-dimensional manifold with boundary  $(M \times N)^{\cdot} = \dot{M} \times N \cup_{\mathrm{id}}_{|_{\dot{M} \times \dot{N}}} M \times \dot{N}$ . For a manifold X without boundary (like  $S^1$ ) the cylinder  $X \times I$  is a manifold with boundary  $X \times \{0, 1\}$ .

This way we get examples of 3-manifolds:  $S^2 \times \mathbb{R}$ ,  $S^2 \times I$ , and  $S^2 \times S^1$ .

**Proof.** 1.12 and 1.50.1.

**1.59 Example. Möbius strip.** [9, 1.4.6] The MÖBIUS-STRIP X is defined as  $I \times I / \sim$ , where  $(x, 0) \cong (1 - x, 1)$  for all x. Its boundary is  $(I \times \dot{I}) / \sim \cong S^1$  and hence X is not homeomorphic to the cylinder  $S^1 \times I$ .

An embedding of X into  $\mathbb{R}^3$  is given by factoring

 $(\varphi, r) \mapsto ((2 + (2r - 1)\cos\pi\varphi)\cos 2\pi\varphi, 2 + (2r - 1)\cos\pi\varphi)\sin 2\pi\varphi, (2r - 1)\sin\pi\varphi)$ 

over the quotient.

The Möbius-strip is not orientable which we will make precise later.

**1.60 Proposition.** [9, 1.4.7] [9, 1.5.5] By cutting finitely many disjoint open holes into a manifold one obtains a manifold whose boundary is the union of the boundary of X and the boundaries of the holes. More precisely, let X be an m-manifold and  $f_i: D^m \to X$  embeddings with pairwise disjoint images. Let  $\mathring{D}_i := \{f_i(x) : |x| < \frac{1}{2}\}$ and  $S_i := \{f_i(x) : |x| = \frac{1}{2}\}$ . Then  $X \setminus \bigcup_{i=1}^n \mathring{D}_i$  is an m-manifold with boundary  $\dot{X} \sqcup \bigsqcup_{i=1}^n S_i$ .

The manifold which results by cutting g open holes in the unit-disk  $D^2$  will be denoted  $D_q^2$ .

**Proof.** No point in  $\{f_i(x) : |x| < 1\}$  is a boundary point of X, hence the result follows.

**1.61 Proposition.** [9, 1.4.8] [9, 1.5.6] Let F and F' be two manifolds and R and R' components of the corresponding boundaries and  $g: R \to R'$  a homeomorphism. Then  $F' \cup_g F$  is a manifold in which F and F' are embedded as closed subsets with boundary  $(\dot{F} \setminus R) \cup (\dot{F}' \setminus R')$ .

**Proof.** Let  $A \cong D^m \times I$  and  $A' \cong D^m \times I$  be neighborhoods of  $x \in R$  and  $g(x) \in R'$  with  $\dot{F} \cap A = D^{m-1} \times \{0\}$  and  $\dot{F}' \cap A' = D^{m-1} \times \{0\}$ . W.l.o.g. we may assume that  $g(\dot{F} \cap A) = \dot{F}' \cap A'$ . The image of  $A' \sqcup A$  in  $F' \cup_g F$  is given by gluing  $D^{m-1} \times I \cup D^{m-1} \times I$  along a homeomorphism  $D^{m-1} \times \{0\} \to D^{m-1} \times \{0\}$  and hence by 1.50.3 is homeomorphic to  $D^{m-1} \times I$  where x corresponds to (0,0).  $\Box$ 

**1.62 Example.** [9, 1.4.9]  $S^1 \times S^1$  can be obtained from two copies of  $S^1 \times I$  that way. The same is true for KLEIN'S BOTTLE but with different gluing homeomorphism.





**1.63 Example. Gluing a handle.** [9, 1.4.10] [9, 1.5.8.7] Let X be a surface in which we cut two holes as in 1.60. The surface obtained from X by gluing a handle is  $(X \setminus (\mathring{D}^2 \sqcup \mathring{D}^2)) \cup_f (S^1 \times I)$ , where  $f: S^1 \times I \supseteq S^1 \times \dot{I} \cong S^1 \sqcup S^1 \subseteq D^2 \sqcup D^2$ .

More generally, one can glue handles  $S^{n-1} \times I$  to *n*-manifolds.

**1.64 Example. Connected sum.** [9, 1.4.11] [9, 1.5.8.8] The CONNECTED SUM of two surfaces  $X_1$  and  $X_2$  is given by cutting a whole into each of them and gluing along boundaries of the respective holes.  $X_1 \sharp X_2 := (X_1 \setminus \mathring{D}^2) \cup_f (X_2 \setminus \mathring{D}^2)$ , where  $f: D^2 \supseteq S^1 \cong S^1 \subseteq D^2$ .

More generally, one can define analogously the connected sum of n-manifolds. This however depends essentially on the gluing map.

**1.65 Example. Doubling of a manifold with boundary.** [9, 1.4.12] [9, 1.5.8.9] The DOUBLING OF A MANIFOLD is given by gluing two copied along their boundaries with the identity.  $2X := X \cup_f X$ : where  $f = id : \dot{X} \to \dot{X}$ .

**1.66 Example.** [9, 1.4.13] The compact oriented surfaces  $F_g$  (of genus g) without boundary can be described as:

- 1. boundary of a brezel  $V_g := D_g^2 \times I$  of genus g.
- 2. doubling  $2D_a^2$ .
- 3. connected sum of tori.
- 4. sphere with g handles.





**1.68 Example.** [9, 1.4.15] [9, 1.5.13] The PROJECTIVE PLANE  $\mathbb{P}^2$  is defined as  $(\mathbb{R}^3 \setminus \{0\})/\sim$  with  $x \sim \lambda \cdot x$  für  $\mathbb{R} \ni \lambda \neq 0$ .

More generally, let for  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$  the PROJECTIVE SPACE be defined by  $\mathbb{P}^n_{\mathbb{K}} := (\mathbb{K}^{n+1} \setminus \{0\})/\sim$ , where  $x \sim \lambda x$  for  $0 \neq \lambda \in \mathbb{K}$ . The quotient mapping  $\mathbb{K}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n_{\mathbb{K}}$  is an open mapping, since the saturated hull of an open subset U is the open double-cone with base U and without its apex.

## 1.69 Examples. [9, 1.4.17] [9, 1.4.18]

- 1.  $\mathbb{P}^2 \cong D^2 / \sim$  where  $x \sim -x$  for all  $x \in S^1$ .
- 2.  $\mathbb{P}^n \cong D^n / \sim$  where  $x \sim -x$  for all  $x \in S^{n-1}$ : Consider a hemisphere  $D^n_+ \subseteq S^n$ . Then the open quotient mapping  $S^n \to \mathbb{P}^n$  restricts to a quotient mapping (by 1.31) on the compact set  $D^n_+$  with associated equivalence relation  $x \sim -x$  on  $S^{n-1} \subseteq D^n_+$ . Thus  $\mathbb{P}^n$  is an *n*-manifold.
- 3.  $\mathbb{P}^2$  can be obtained by gluing a disk to a Möbius strip. Consider the closed subsets  $A := \{x \in S^2 : x_2 \leq 0, |x_3| \leq 1/2\}$  and  $B = \{x \in S^3 : x_3 \geq 1/2\}$ . The open quotient mapping induces an homeomorphism on the saturated subset  $B \subseteq D_+^n$ , i.e.  $\pi(B)$  is a 2-Ball. A is mapped to a Möbius-strip by 1.28 and 1.59. Since  $\pi(B) \cup \pi(A) = \mathbb{P}^2$  and  $\pi(B) \cap \pi(A) \cong S^1$  we are done.

**1.71 Proposition.** [9, 1.4.16] [9, 1.5.14] [9, 1.6.6]  $\mathbb{P}^n_{\mathbb{K}}$  is a dn-dimensional connected compact manifold, where  $d := \dim_{\mathbb{R}} \mathbb{K}$ . The mapping  $p : S^{dn-1} \to \mathbb{P}^{n-1}_{\mathbb{K}}$ ,  $x \mapsto [x]$  is a quotient mapping. In particular,  $\mathbb{P}^1_{\mathbb{K}} \cong S^d$ .

**Proof.** Charts  $\mathbb{K}^n \to \mathbb{P}^n_{\mathbb{K}}$ ,  $(x^1, \ldots, x^n) \mapsto [(x^1, \ldots, x^i, 1, x^{i+1}, \ldots, x^n)]$ . The restriction  $\mathbb{K}^{n+1} \supseteq S^{d(n+1)-1} \to \mathbb{P}^n_{\mathbb{K}}$  is a quotient mapping since  $\mathbb{K}^{n+1} \setminus \{0\} \to \mathbb{P}^n_{\mathbb{K}}$  is an open mapping. For  $\mathbb{K} = \mathbb{R}$  it induces the equivalence relation  $x \sim -x$ . In particular  $\mathbb{P}^n_{\mathbb{K}}$  is compact. For n = 1 we have  $\mathbb{P}^1_{\mathbb{K}} \setminus U_1 = \{[(0,1)]\}$ , therefore  $\mathbb{P}^1_{\mathbb{K}} \cong \mathbb{K}_{\infty} \cong S^d$ .

**1.72 Example.** [9, 1.4.19] The none-oriented compact surfaces  $N_g$  (of genus g) without boundary:

- 1. Connected sum of g projective planes.
- 2. Sphere with g Möbius strips glued to it.

Klein's bottle as sum of two Möbius strips, see [4, 9.3].



**1.73 Proposition.** [9, 1.4.20] The none-orientable compact surfaces without boundary as quotient of a 2g-polygon.



**1.86 Theorem.** [9, 1.9.1] Each connected closed surface is homeomorphic to one of the surfaces  $S^2 = F_0, S^1 \times S^1 = F_1, \ldots$  or  $\mathbb{P}^2 = N_1, N_2, \ldots$ 

For a sketch of proof, see [4, 9.4]

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**1.74 Example.** [9, 1.5.9] Union of filled tori  $(D^2 \times S^1) \cup_{id} (S^1 \times D^2) = (D^2 \times D^2)^{\cdot} \cong (D^4)^{\cdot} \cong S^3$  by 1.12. Other point of view:  $S^3 = D^3_+ \cup_{id} D^3_-$  and remove a filled cylinder from  $D_-$  and glue that to  $D_+$  to obtain two tori. With respect to the stereographic projection the torus  $\{(z_1, z_2) \in S^3 \subseteq \mathbb{C}^2 : |z_1| = r_1, |z_2| = r_2\}$  with  $r_1^2 + r_2^2 = 1$  corresponds to the torus with the z-axes as its axes and big radius  $A := 1/r_1 \ge 1$  and small radius  $a := \sqrt{A^2 - 1} = \frac{r_2}{r_1}$ , see [4, 11.6,11.7].



**1.75 Example.** [9, 1.5.10] More generally, let  $f: S^1 \times S^1 \to S^1 \times S^1$  be given by  $f: (z, w) \mapsto (z^a w^b, z^c w^d)$ , where  $a, b, c, d \in \mathbb{Z}$  with  $ad - bc = \pm 1$ .



A meridian  $S^1 \times \{w\}$  on the torus is mapped to a curve  $t \mapsto (e^{2\pi i t}, w) \mapsto (w^b e^{2\pi i a t}, w^d e^{2\pi i c t})$ which winds *a*-times around the axes and *c*-times around the core of  $S^1 \times S^1 \hookrightarrow \mathbb{R}^3$ .

$$M\begin{pmatrix} a & b\\ c & d \end{pmatrix} := (D^2 \times S^1) \cup_f (S^1 \times D^2).$$

In 1.88 together with 1.89 and 1.83 we will indicate that M is often not homeomorphic to  $S^3$ .

**1.76 Example.** [9, 1.5.11] Cf. 1.61. By a HEEGARD DECOMPOSITION of M one understands a representation of M by gluing two handle bodies  $V_g$  (see 1.66.1) of the same genus g along their boundary.

**1.77 Example.** [9, 1.5.12] Cf. 1.67 and 1.73. For relative prime  $1 \le q < p$  let the LENS SPACE be  $L(\frac{q}{p}) := D^3/\sim$ , where  $(\varphi, \theta, 1) \sim (\varphi + 2\pi \frac{q}{p}, -\theta, 1)$  for  $\theta \ge 0$  with respect to spherical coordinates, so the northern hemisphere is identified with the southern one after rotation by  $2\pi \frac{q}{p}$ . The interior of  $D^3$  is mapped homeomorphically

to a 3-cell in  $L(\frac{q}{p})$  by 1.30. The image of points in the open hemispheres have also such neighborhoods (formed by one half in the one part inside the northern hemisphere and one inside the southern). Each *p*-points on the equator obtained by recursively turning by  $2\pi \frac{q}{p}$  get identified. After squeezing  $D^3$  a little in direction of the axes we may view a neighborhood of a point on the equator as a cylinder over a sector of a circle (a piece of cake) where the flat sides lie on the northern and southern hemisphere. In the quotient *p* many of these pieces are glued together along their flat sides thus obtaining again a 3-cell as neighborhood. We will come to this description again in 1.89.



#### Group actions and orbit spaces

**1.78 Definition.** [9, 1.7.3] Group action of a group G on a topological space X is a group-homomorphism  $G \to \text{Homeo}(X)$  into the group of homeomorphisms of X. The ORBIT SPACE is  $X/G := X/\sim = \{Gx : x \in X\}$ , where  $x \sim y : \Leftrightarrow \exists g \in G : y = g \cdot x$ . For this we may without loss of generality assume that G is a subgroup of Homeo(X), since only its image in Homeo(X) is needed.

# 1.79 Examples. [9, 1.7.4]

- 1.  $S^1$  acts on  $\mathbb{C}$  by multiplication  $\Rightarrow \mathbb{C}/S^1 \cong [0, +\infty)$ .
- 2.  $\mathbb{Z}$  acts on  $\mathbb{R}$  by translation  $(k, x) \mapsto k + x \Rightarrow \mathbb{R}/\mathbb{Z} \cong S^1$ ,  $\mathbb{R}^2/\mathbb{Z} \cong S^1 \times \mathbb{R}$ . ATTENTION:  $\mathbb{R}/\mathbb{Z}$  has two meanings.
- 3.  $S^0$  acts on  $S^n$  by reflection (scalar multiplication)  $\Rightarrow S^n/S^0 \cong \mathbb{P}^n$ .

**1.80 Definition.** [9, 1.7.5] G is said to ACT FREELY on X, when no  $g \neq 1$  has a fixed-point on X, i.e.  $gx \neq x$  for all x and  $g \neq 1$ .

**1.81 Theorem.** [9, 1.7.6] Let G act strictly discontinuously on X, i.e. each  $x \in X$  has a neighborhood U with  $gU \cap U \neq \emptyset \Rightarrow g = \text{id.}$  In particular, this is the case, when G is finite and acts freely on a  $T_2$  space X. Then X/G is a closed m-manifold provided X is one.

**Proof.**  $\pi: X \to X/G$  is open and  $U \cong \pi(U)$  is the required neighborhood, since  $\pi^{-1}(\pi(W)) = \bigcup_{g \in G} g W$ . Free actions of finite groups on  $T_2$ -spaces are strictly discontinuous, since for  $x \in X$  and  $g \neq 1$  we find disjoint neighborhoods  $U_g$  of x and  $W_g$  of g x. Then  $U := \bigcap_{g \neq 1} U_g \cap g^{-1}(W_g)$  is the required neighborhood.  $\Box$ 

**1.82 Example.** [9, 1.7.7] Let  $1 be relative prime to <math>q_1, \ldots, q_k \in \mathbb{Z}$ . Then  $E_p := \{g \in \mathbb{C} : g^p = 1\} \cong \mathbb{Z}_p$  acts freely on  $S^{2k-1} \subseteq \mathbb{C}^k$  by  $g \cdot (z_1, \ldots, z_k) \mapsto (g^{q_1}z_1, \ldots, g^{q_k}z_k)$ . The GENERAL LENS SPACE  $L_{2k-1}(p; q_1, \ldots, q_k) := S^{2k-1}/E_p$  of type  $(p; q_1, \ldots, q_k)$  is a closed manifold of dimension 2k - 1. Note that this space depends only on  $q_j \mod p$  and not on  $q_j$  itself, so we may assume  $0 < q_j < p$ .

In particular,  $L_3(p;q,1) \cong L(\frac{q}{p})$ : We may parametrize  $S^3 \subseteq \mathbb{C}^2$  by the quotient mapping  $f: D^2 \times S^1 \twoheadrightarrow S^3$ ,  $(z_1, z_2) \mapsto (z_1, \sqrt{1 - |z_1|^2} z_2)$  and the action of  $E_3 = \langle g \rangle$ , where  $g := e^{2\pi i/p}$ , lifts to the action given by  $g \cdot (z_1, z_2) = (g^q z_1, g z_2)$ . Only the points in  $\{z_1\} \times S^1$  for  $z_1 \in S^1$  get identified by f. A representative subset of  $S^3$ for the action is given by  $\{(z_1, z_2) \in S^3 : |\arg(z_2)| \leq \frac{\pi}{p}\}$ , its preimage in  $D^2 \times S^1$ is homeomorphic to  $D^2 \times I$ , and only points  $(z_1, 0)$  and  $(g^q z_1, 1)$  are in the same orbit. Thus the top  $D^2 \times \{1\}$  and the bottom  $D^2 \times \{0\}$  rotated by  $g^q = e^{2\pi i \frac{q}{p}}$  have to be identified in the orbit space and also the generators  $\{z_1\} \times I$  for  $z_1 \in S^1$ . This gives the description of  $L(\frac{q}{p})$  in [1.77].



One has:

- $L_3(p;q_1,q_2) \cong L_3(p;q_2,q_1)$  via the reflection  $\mathbb{C} \times \mathbb{C} \supseteq S^3 \to S^3 \subseteq \mathbb{C} \times \mathbb{C}$ ,  $(z_1,z_2) \mapsto (z_2,z_1).$
- $L_3(p; q q_1, q q_2) = L_3(p; q_2, q_1)$  for q relative prime to p via the group isomorphism  $g \mapsto g^q$ .
- $L_3(p; -q_1, q_2) \cong L_3(p; q_1, q_2)$  via  $(z_1, z_2) \mapsto (z_1, \overline{z_2})$  and the group isomorphism  $g \mapsto g^{-1} = \overline{g}$ :



**1.83 Theorem.** [9, 1.9.5]  $L(\frac{q}{p}) \cong L(\frac{q'}{p'}) \Leftrightarrow p = p' \text{ and } (q \equiv \pm q' \mod p \text{ or } qq' \equiv \pm 1 \mod p).$ 

**Proof.** (⇐) By 1.82

- $L_3(p;q,1) \cong L_3(p;q',1)$  for  $q' \equiv \pm q \mod p$ .
- $L_3(p;q,1) \cong L_3(p;q',1)$  for  $qq' \equiv \pm 1 \mod p$ , since  $L_3(p;q,1) \cong L_3(p;q'q,q') = L_3(p;\pm 1,q') \cong L_3(p;1,q') \cong L_3(p;q',1)$

 $(\Rightarrow)$  is beyond the algebraic methods of this lecture course.

**1.84 Definition.** [9, 1.7.1] A TOPOLOGICAL GROUP is a topological space together with a group structure, s.t.  $\mu: G \times G \to G$  and inv :  $G \to G$  are continuous.

### 1.85 Examples of topological groups. [9, 1.7.2]

- 1.  $\mathbb{R}^n$  with addition.
- 2.  $S^1 \subseteq \mathbb{C}$  and  $S^3 \subseteq \mathbb{H}$  with multiplication, see [4, ].
- 3.  $G \times H$  for topological groups G and H.
- 4. The general linear group  $GL(n) := GL(n, \mathbb{R}) := \{A \in L(\mathbb{R}^n, \mathbb{R}^n) : \det(A) \neq 0\}$  with composition, see [4, 14.1].
- 6. The special linear group  $SL(n) := \{A \in GL(n) : \det(A) = 1\}$ , see [4, 14.5].
- 5. The orthogonal group  $O(n) := \{A \in GL(n) : A^t \cdot A = id\}$  and the (path-) connected component  $SO(n) := \{T \in O(n) : \det(T) = 1\}$  of the identity in O(n). As topological space  $O(n) \cong SO(n) \times S^0$ . For all this see [4, 14.6].
- 7.  $GL(n, \mathbb{C}) := \{A \in L_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^n) : \det_{\mathbb{C}}(A) \neq 0\}, \text{ see } [4, 14.14].$
- 8. The unitary group  $U(n) := \{A \in GL(n, \mathbb{C}) : A^* \cdot A = \mathrm{id}\}$  with closed subgroup  $SU(n) := \{A \in U(n) : \det_{\mathbb{C}}(A) = 1\}$ , see [4, 14.14]. As topological space  $U(n) \cong SU(n) \times S^1$ , see [?, 1.27]
- 9. In particular  $SO(1) = SU(1) = \{1\}$ ,  $SO(2) \cong U(1) \cong S^1$ ,  $SU(2) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \} = \{ \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} : |a|^2 + |c|^2 = 1 \} \cong S^3$ ,  $SO(3) \cong \mathbb{P}^3$ . For the last homorphism consider the surjective mapping  $f : [0, \pi] \times S^2 \to SO(3)$  given, by associating to an angle  $\varphi \in [0, \pi]$  and an unit-vector  $x \in S^2$  the rotation  $f(\varphi, x)$  by the angle  $\varphi$  around the axes x. This mapping is injective except for f(0, x) = f(0, x') and  $f(\pi, x) = f(\pi, -x)$  for all  $x, x' \in S^2$ . It factors to a surjective mapping  $\tilde{f} : D^3 \to SO(3)$  over the surjective multiplication  $\mu : [0, \pi] \times S^2 \to D^3$ ,  $(\varphi, x) \mapsto \frac{\varphi}{\pi} \cdot x$ , which is injective except for  $\tilde{f}(y) = \tilde{f}(-y)$  for all  $x, x' \in S^2$ . This is exactly the equivalence relation defining  $\mathbb{P}^3 = D^3/\sim$ .

## The problem of homeomorphy

**Remark.** For 3-manifolds one is far from a solution to the classification problem. For n > 3 there can be no algorithm.

**1.87 Theorem.** [9, 1.9.2] Each closed orientable 3-manifold admits a Heegarddecomposition.

Hence in order to solve the classification problem one has to investigate only the homeomorphisms of closed oriented surfaces and determine which gluings give homeomorphic manifolds.

In the following example we study this for the homeomorphisms of the torus considered in 1.75.

**1.88 Example.** [9, 1.9.3] Let  $M = M \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and  $M' = M \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$  with  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and  $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$  in  $SL(2,\mathbb{Z})$ , see 1.75. For  $\alpha, \beta, \gamma, \delta \in S^0$  and  $m, n \in \mathbb{Z}$  consider the homeomorphisms

$$\begin{split} F: D^2 \times S^1 &\to D^2 \times S^1, \quad (z,w) \mapsto (z^{\alpha}w^m, w^{\beta}) \\ G: S^1 \times D^2 &\to S^1 \times D^2, \quad (z,w) \mapsto (z^{\gamma}, z^n w^{\delta}) \end{split}$$

 $\operatorname{If}$ 

$$\begin{pmatrix} \gamma & 0 \\ n & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} \alpha & m \\ 0 & \beta \end{pmatrix},$$

i.e.

 $\gamma a = a'\alpha, \quad \gamma b = a'm + b'\beta, \quad na + \delta c = c'\alpha, \quad nb + \delta d = c'm + d'\beta$ then  $(G|_{S^1 \times S^1}) \circ f = f' \circ (F|_{S^1 \times S^1})$  and thus  $M \cong M'$  by 1.49. Reduction:

$$\begin{aligned} (a \leq 0): \ \alpha := -1, \ \beta := \gamma := \delta := 1, \ m := n := 0 \\ \Rightarrow M \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\cong M \begin{pmatrix} -a & -b \\ c & d \end{pmatrix}, \text{ i.e. w.l.o.g. } a \geq 0. \end{aligned}$$
$$\begin{aligned} (ad - bc = -1): \ \alpha := \beta := \gamma := 1, \ \delta := -1, \ m := n := 0 \\ \Rightarrow M \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\cong M \begin{pmatrix} a & b \\ -c & -d \end{pmatrix}, \text{ i.e. w.l.o.g. } ad - bc = 1. \end{aligned}$$
$$\begin{aligned} (a = 0): \ \Rightarrow bc = -1. \ \alpha := c, \ \beta := b, \ \gamma := 1, \ \delta := 1, \ n := 0, \ m := d \\ \Rightarrow M \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\cong M \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &\cong (D^2 \cup_{\text{id}} D^2) \times S^1 \cong S^2 \times S^1. \end{aligned}$$
$$\begin{aligned} (a = 1): \ \alpha := \delta := a, \ \beta := ad - bc, \ \gamma := 1, \ m := b, \ n := -c \\ \Rightarrow M \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\cong M \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (D^2 \times S^1) \cup_{\text{id}} (S^1 \times D^2) \cong S^3, \ \text{by } 1.74. \end{aligned}$$
$$\begin{aligned} (ad' - b'c = 1): \ \Rightarrow a(d - d') = c(b - b') \ \text{since } ad - bc = 1 \ \text{and } \ \exists m : b - b' = m a, \\ d - d' = m c \ \text{since } gcd(a, c) = 1. \end{aligned}$$

$$\alpha := \beta := \gamma := \delta := 1, n := 0 \Rightarrow M \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cong M \begin{pmatrix} a & b' \\ c & d' \end{pmatrix} =: M(a, c).$$

 $\begin{array}{l} (c':=c-na) \textbf{:} \ \alpha:=\beta:=\gamma:=\delta:=-1, \ m:=0 \Rightarrow M(a,c)\cong M(a,c'), \ \textbf{i.e. w.l.o.g.} \\ 0\leq c< a \ (\text{If } c=0\Rightarrow a=1, \ \text{since } \gcd(a,c)=1\Rightarrow M(a,c)\cong S^3). \end{array}$ 

Thus only the spaces M(a,c) with 0 < c < a and ggT(a,c) = 1 remain.

**1.89 Theorem (Heegard-decomposition via lens spaces).** [9, 1.9.4] For relative prime  $1 \le c < a$  we have  $L(\frac{c}{a}) \cong M(a, c)$ .



**Proof.** We start with  $L(\frac{c}{a}) = D^3 / \sim$  and drill a cylindrical hole into  $D^3$  and glue its top and bottom via  $\sim$  to obtain a filled torus, where collections of a many generators of the cylinder are glued to from a closed curve which winds c-times around the core of the torus (i.e. the axes of the cylinder) and a-times around the axes of the torus. The remaining  $D^3$  with hole is cut into a sectors, each homeomorphic to a piece of a cake, which yield  $D^2 \times I$  after gluing the flat sides (which correspond to points on  $S^2$ ) and groups of a many generators of the cylindrical hole are glued to a circle  $S^1 \times \{t\}$ . After gluing the top and the by  $2\pi \frac{1}{a}$  rotated bottom disc we obtain a second filled torus, where the groups of a many generators of the cylinder form a meridian. In contrast the meridians of the cylindrical hole corresponds to a curve which winds a times around the axes and c times around the core. This is exactly the gluing procedure described in 1.75 for M(a, c).

**1.90 Definition.** [9, 1.9.6] Two embeddings  $f, g: X \to Y$  are called TOPOLOGICAL EQUIVALENT, if there exists a homeomorphism  $h: Y \to Y$  with  $g = h \circ f$ . Each two embeddings  $S^1 \to \mathbb{R}^2$  are by Schönflies's theorem (which is a strong version of Jordan's theorem) equivalent.

**1.91 Definition.** [9, 1.9.7] A KNOT is an embedding  $S^1 \to \mathbb{R}^3 \subseteq S^3$ .

**Remark.** To each knot we may associated the complement of a tubular neighborhood in  $S^3$ . This is a compact connected 3-manifold with a torus as boundary.

By a result of [1] a knot is up to equivalence uniquely determined by the homotopy class of this manifold.

On the other hand, we may consider closed (orientable) surfaces in  $\mathbb{R}^3$  of minimal genus which have the knot as boundary.



Gluing cells

**1.92 Notation.** [9, 1.6.1]  $f: D^n \supseteq S^{n-1} \to X$ . Consider  $X \cup_f D^n$ ,  $p: D^n \sqcup X \to X \cup_f D^n$ ,  $e^n := p(\mathring{D}^n)$ ,  $i := p|_X : X \hookrightarrow X \cup_f D^n =: X \cup e^n$ .



By 1.44  $p: (D^n, S^{n-1}) \to (X \cup e^n, X)$  is a relative homeomorphism and  $i: X \to X \cup e^n$  is a closed embedding.

For  $X T_2$  also  $X \cup e^n$  is  $T_2$ : Points in X can be separated in X by  $U_i$  and the sets  $U_i \cup \{tx : 0 < t < 1, f(x) \in U_i\}$  separate them in  $X \cup e^n$ . When both points are in the open subset  $e_n$ , this is obvious. Otherwise one lies in  $e_n$  and the other in X, so a sphere in  $D^n$  separates them.

Conversely we have:

**1.93 Proposition.** [9, 1.6.2] Let  $Z T_2$ ,  $X \subseteq Z$  closed and  $F : (D^n, S^{n-1}) \to (Z, X)$ a relative homeomorphism.  $\Rightarrow X \cup_f D^n \cong Z$ , where  $f := F|_{S^{n-1}}$ , via  $(F \sqcup i) \circ p^{-1}$ .

**Proof.** We consider



 $j: X \hookrightarrow Z$  is closed and also F, since  $D^n$  is compact and Z is  $T_2$ . Thus g is closed and obviously bijective and continuous, hence a homeomorphism.  $\Box$ 

**1.94 Theorem.** [9, 1.6.3] Let  $f : S^{n-1} \to X$  be continuous and surjective and X $T_2 \Rightarrow p|_{D^n} : D^n \to X \cup_f D^n$  is a quotient mapping.

**Proof.** p is surjective, since f is. Since  $D^n$  is compact and  $X \cup_f D^n$  is  $T_2$ , p is a quotient mapping.

1.95 Examples. [9, 1.6.4]

(1) 
$$f: S^{n-1} \to \{*\} =: X \Rightarrow X \cup_f D^n \stackrel{[1.47]}{\cong} D^n / S^{n-1} \stackrel{[1.36]}{\cong} S^n.$$

- (2)  $f: S^{n-1} \to X \text{ constant} \Rightarrow X \cup_f D^n \cong X \vee (D^n/S^{n-1}) \cong X \vee S^n.$
- (3)  $f = \operatorname{id} : S^{n-1} \to S^{n-1} =: X \Rightarrow X \cup_f D^n \cong D^n$  by 1.94.
- (4)  $f = \text{incl} : S^{n-1} \hookrightarrow D^n =: X \Rightarrow X \cup_f D^n \cong S^n$  by 1.50.2
- (5) [9, 1.6.10] Let  $g_n : S^1 \to S^1$ ,  $z \mapsto z^n$ . Then  $S^1 \cup_{g_0} D^2 \cong S^1 \vee S^1$  by 1.95.2,  $S^1 \cup_{g_1} D^2 \cong D^2$  by 1.95.3,  $S^1 \cup_{g_2} D^2 \cong \mathbb{P}^2$  by 1.69,  $S^1 \cup_{g_k} D^2 \cong S^1 \cup_{g_{-k}} D^2$  by conjugation  $z \mapsto \overline{z}$ .

**1.101 Theorem.** [9, 1.6.9] [9, 1.6.11] Let  $i_j^n : S^1 \hookrightarrow \bigvee_{k=1}^r S^1$ ,  $z \mapsto z^n$  on the  $j^{th}$  summand  $S^1$ , furthermore,  $B_k := \{\exp(\frac{2\pi it}{m}) : k-1 \le t \le k\}$  an arc of length  $\frac{2\pi}{m}$  and  $f_k : B_k \to S^1$ ,  $\exp(\frac{2\pi it}{m}) \mapsto \exp(2\pi i(t-k+1))$ . Finally, let  $i_{j_1}^{n_1} \cdots i_{j_m}^{n_m} : S^1 \to \bigvee^r S^1$  the mapping which coincides on  $B_k$  with  $i_{j_k}^{n_k} \circ f_k$ , i.e. one runs first  $n_1$ -times along the  $j_1$ -th summand  $S^1$ , etc.

For  $g \ge 1$  and  $f := i_1 \cdot i_2 \cdot i_1^{-1} \cdot i_2^{-1} \cdot \dots \cdot i_{2g-1} \cdot i_{2g} \cdot i_{2g-1}^{-1} \cdot i_{2g}^{-1}$  resp.  $f := i_1^2 \cdot i_2^2 \cdot \dots \cdot i_g^2$ we have  $\bigvee^{2g} S^1 \cup_f D^2 \cong F_g$  resp.  $\bigvee^g S^1 \cup_f D^2 \cong N_g$ .



**Proof.** 1.94  $\Rightarrow X_g := \bigvee S^1 \cup_f D^2 \cong D^2/\sim$  where  $x \sim y$  for  $x, y \in S^1 \Leftrightarrow f(x) = f(y)$ . This is precisely the relation from 1.67, resp. 1.73.

**1.97 Proposition.** [9, 1.6.5] [9, 1.6.7] [9, 1.6.8] We have an embedding  $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$  via  $\mathbb{K}^n \cong \mathbb{K}^n \times \{0\} \subseteq \mathbb{K}^{n+1}$ . The mapping

 $F: \mathbb{K}^n \supseteq D^{dn} \to \mathbb{P}^n_{\mathbb{K}}, \quad (x^1, \dots, x^n) \mapsto [(x^1, \dots, x^n, 1 - \|x\|)]$ 

defines a relative homeomorphism  $F : (D^{dn}, S^{dn-1}) \to (\mathbb{P}^n_{\mathbb{K}}, \mathbb{P}^{n-1}_{\mathbb{K}})$ . Thus, by **1.93**,  $\mathbb{P}^n_{\mathbb{K}} = \mathbb{P}^{n-1}_{\mathbb{K}} \cup_{F|_{S^{dn-1}}} D^{dn}$ . Hence we have decompositions into disjoint cells:

 $\mathbb{P}^{n}_{\mathbb{R}} \cong e^{0} \cup e^{1} \cup \cdots \cup e^{n}, \quad \mathbb{P}^{n}_{\mathbb{C}} \cong e^{0} \cup e^{2} \cup \cdots \cup e^{2n}, \text{ and } \mathbb{P}^{n}_{\mathbb{H}} \cong e^{0} \cup e^{4} \cup \cdots \cup e^{4n}$ 

**Proof.** The charts  $\mathbb{K}^n \cong U_{n+1} = \mathbb{P}^n_{\mathbb{K}} \setminus \mathbb{P}^{n-1}_{\mathbb{K}}$ ,  $(x^1, \ldots, x^n) \mapsto [(x^1, \ldots, x^n, 1)]$  were constructed in the proof of 1.71. The mapping  $D^{dn} \setminus S^{dn-1} \to \mathbb{K}^n$ , given by  $x \mapsto (\frac{x_1}{1-\|x\|}, \ldots, \frac{x_n}{1-\|x\|})$ , is a homeomorphism as in 1.4, and thus the composite F is a relative homeomorphism as well. Now use 1.93 and 1.94.

**1.102 Definition. Gluing several cells.** [9, 1.6.12] For continuous mappings  $f_j: D^n \supseteq S^{n-1} \to X$  for  $j \in J$  let

$$X \cup_{(f_j)_j} \bigcup_{j \in J} D^n := X \cup_{\bigsqcup_{j \in J} f_j} \bigsqcup_{j \in J} D^n.$$

1.103 Example. [9, 1.6.13]

(2) 
$$X \cup_{(f_1, f_2)} (D^n \sqcup D^n) \cong (X \cup_{f_1} D^n) \cup_{f_2} D^n$$
, by 1.46.  
(3)  $f_j = \mathrm{id} : S^{n-1} \to S^{n-1} \Rightarrow S^{n-1} \cup_{(f_1, f_2)} (D^n \sqcup D^n) \cong (S^{n-1} \cup e^n) \cup e^n \xrightarrow{1.95.3} D^n \cup e^n \xrightarrow{1.95.4} S^n$ .

(1)  $f_j: S^{n-1} \to \{*\} \Rightarrow X \cup_{(f_j)_j} \bigcup_{j \in J} D^n \cong \bigvee_J S^n$ : By **1.36**  $\lambda: (D^n, S^{n-1}) \to (S^n, \{*\})$  is a relative homeomorphism and hence also  $\bigsqcup_J \lambda = J \times \lambda: (J \times D^n, J \times S^{n-1}) \to (J \times S^n, J \times \{*\})$ . By **1.32** the induced map  $(J \times D^n)/(J \times S^{n-1}) \to (J \times S^n)/(J \times \{*\}) = \bigvee_j S^n$  is a quotient mapping, since J is locally compact as discrete space. Obviously this mapping is bijective, hence a homeomorphism.

#### Inductive limits

**1.104 Definition.** [9, 1.8.1] Let X be a set and  $A_j \subseteq X$  topological spaces with  $X = \bigcup_{j \in J} A_j$  and the trace topology on  $A_j \cap A_k$  induced from  $A_j$  and from  $A_k$  should be identical and the intersection closed. The final topology on X induces on  $A_j$  the given topology, moreover  $A_j \hookrightarrow X$  is a closed embedding: Let B be closed in  $A_j$ , then  $B \cap A_k = B \cap (A_j \cap A_k)$  is closed the topology of  $A_j$  and hence also in that of  $A_k$ , so B is closed in the final topology on X. Conversely, let  $B \subseteq A_j$  be closed in the final topology, then  $B = B \cap A_j = \operatorname{inj}_j^{-1}(B)$  is closed in  $A_j$ .

The canonical mapping  $p := \bigcup_j \operatorname{inj}_j : \bigsqcup_j A_j \to X$  is by definition of the final topology a quotient mapping (it is clearly onto and  $B \subseteq X$  is closed, iff  $\operatorname{inj}_j^{-1}(B) = B \operatorname{cap} A_j$  is closed in  $A_j$ ) and thus we have the corresponding universal property: A mapping  $f : X \to Y$  is continuous, iff  $f|_{A_j} : A_j \to Y$  is continuous for all j.

**1.105 Proposition.** [9, 1.8.3] [9, 1.8.4] Let  $\mathcal{A}$  be a closed (locally) finite covering of X. Then X carries the final topology with respect to  $\mathcal{A}$ .

**Proof.** See [2, 1.2.14.3]: Let  $B \subseteq X$  be such that  $B \cap A \subseteq A$  is closed. In order to show that  $B \subseteq X$  is closed it suffices to prove that  $\overline{\bigcup_{B \in \mathcal{B}} B} = \bigcup_{B \in \mathcal{B}} \overline{B}$  for locally finite families  $\mathcal{B}(:= \{B \cap A : A \in \mathcal{A}\})$ . ( $\supseteq$ ) is obvious. ( $\subseteq$ ) Let  $x \in \overline{\bigcup_{B \in \mathcal{B}} B}$  and U an open neighborhood of x with  $\mathcal{B}_0 := \{B \in \mathcal{B} : B \cap U \neq \emptyset\}$  being finite. Then  $x \notin \overline{\bigcup_{B \in \mathcal{B} \setminus \mathcal{B}_0} B}$  and since

$$x \in \overline{\bigcup_{B \in \mathcal{B}} B} = \overline{\bigcup_{B \in \mathcal{B}_0} B} \cup \overline{\bigcup_{B \in \mathcal{B} \setminus \mathcal{B}_0} B}$$

we have  $x \in \overline{\bigcup_{B \in \mathcal{B}_0} B} = \bigcup_{B \in \mathcal{B}_0} \overline{B} \subseteq \bigcup_{B \in \mathcal{B}} \overline{B}$ .

**1.107 Definition.** [9, 1.8.5] Let  $A_n$  be an increasing sequence of topological spaces, where each  $A_n$  is a closed subspace in  $A_{n+1}$ . Then  $\bigcup_{n \in \mathbb{N}} A_n$  with the final topology is called (INDUCTIVE) LIMIT  $\varinjlim_n A_n$  of the sequence  $(A_n)_n$ .

#### **1.108 Examples.** [9, 1.8.6] [9, 1.8.7]

- 0.  $\mathbb{R}^{\infty} := \varinjlim_{n} \mathbb{R}^{n}$ , the space of finite sequences. Let  $x \in \mathbb{R}^{\infty}$  with  $\varepsilon_{n} > 0$ . Then  $\{y \in \mathbb{R}^{\infty} : |y_{n} - x_{n}| < \varepsilon_{n} \forall n\}$  is an open neighborhood of x in  $\mathbb{R}^{\infty}$ . Conversely, let  $U \subseteq \mathbb{R}^{\infty}$  be an open set containing x. Then there exists an  $\varepsilon_{1} > 0$  with  $K_{1} := \{y_{1} : |y_{1} - x_{1}| \leq \varepsilon_{1}\} \subseteq U \cap \mathbb{R}^{1}$ . Since  $K_{1} \subseteq \mathbb{R}^{1} \subseteq \mathbb{R}^{2}$  is compact, there exists by [2, 2.1.11] an  $\varepsilon_{2} > 0$  with  $K_{2} := \{(y_{1}, y_{2}) : y_{1} \in K_{1}, |y_{2} - x_{2}| \leq \varepsilon_{2}\} \subseteq U \cap \mathbb{R}^{2}$ . Inductively we obtain  $\varepsilon_{n}$  with  $\{y \in \mathbb{R}^{\infty} : |y_{k} - x_{k}| \leq \varepsilon_{k} \forall k\} = \bigcup_{n} K_{n} \subseteq U$ . Thus the sets from above form a basis of the topology. The sets  $\bigcup_{n} \{y \in \mathbb{R}^{n} : |y - x| < \varepsilon_{n}\}$  do not, since for  $\varepsilon_{n} \searrow 0$  they contain none of the neighborhoods from above, since  $(\frac{\delta_{1}}{2}, \ldots, \frac{\delta_{n}}{2}, 0, \ldots)$  is not contained therein for  $\varepsilon_{n} \leq \frac{\delta_{1}}{2}$ .
- 1.  $S^{\infty} := \lim_{n \to \infty} S^n$  is the set of unit vectors in  $\mathbb{R}^{\infty}$ .
- 2.  $\mathbb{P}^{\infty} := \lim_{n \to \infty} \mathbb{P}^n$  is the space of lines through 0 in  $\mathbb{R}^{\infty}$ .
- 3.  $O(\infty) := \lim_{n \to \infty} O(n)$ , where  $GL(n) \hookrightarrow GL(n+1)$  via  $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ .
- 4.  $SO(\infty) := \varinjlim_n SO(n)$
- 5.  $U(\infty) := \lim_{n \to \infty} U(n)$

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6.  $SU(\infty) := \lim_{n \to \infty} SU(n)$ 

## 2. Homotopy

**2.1 Definition.** [9, 2.1.1] A HOMOTOPY is a mapping  $h: I \to C(X, Y)$ , which is continuous as mapping  $\hat{h}: I \times X \to Y$ . Note that this implies, that  $h: I \to C(X, Y)$  is continuous for the compact open topology (a version of the topology of uniform convergence for general topological spaces instead of uniform spaces Y, a subbasis for this topology is given by the sets  $N_{K,U} := \{f \in C(X,Y) : f(K) \subseteq U\}$  with arbitrary compact  $K \subseteq X$  and open  $U \subseteq Y$ ) but not conversely.

Two mappings  $h_j: X \to Y$  for  $j \in \{0, 1\}$  are called HOMOTOPIC (we write  $h_0 \sim h_1$ ) if there exists a homotopy  $h: I \to C(X, Y)$  with  $h(j) = h_j$  for  $j \in \{0, 1\}$ , i.e. a continuous mapping  $H: I \times X \to Y$  with and  $H(j, x) = h_j(x)$  for all  $x \in X$  and  $j \in \{0, 1\}$ .





**2.2 Lemma.** [9, 2.1.2] To be homotopic is an equivalence relation on C(X, Y).

**2.3 Definition.** [9, 2.1.5] The HOMOTOPY CLASS [f] of a mapping  $g \in C(X, Y)$  is  $[f] := \{g \in C(X, Y) : g \text{ is homotopic to } f\}$ . Let  $[X, Y] := \{[f] : f \in C(X, Y)\}$ .

2.4 Lemma. [9, 2.1.3] Homotopy is compatible with the composition.

For  $f : X' \to X$  and  $g : Y \to Y'$  let  $f^* : C(X,Y) \to C(X',Y)$  be defined by  $f^*(k) = k \circ f$  and  $g_* : C(X,Y) \to C(X,Y')$  be defined by  $g_*(k) := g \circ k$ . Finally, let  $C(f,g) := f^* \circ g_* = g_* \circ f^* : C(X,Y) \to C(X',Y'), k \mapsto g \circ k \circ f$ .

$$C(X,Y) \xrightarrow{f^*} C(X',Y)$$

$$g_* \bigvee C(f,g) \qquad \qquad \downarrow g_*$$

$$C(X,Y') \xrightarrow{f^*} C(X',Y')$$

**Proof.** Let  $h : I \to C(X,Y)$  be a homotopy and  $f : X' \to X, g : Y \to Y'$ be continuous. Then  $C(f,g) \circ h := f^* \circ g_* \circ h : I \to C(X',Y')$  is a homotopy  $g \circ h_0 \circ f \sim g \circ h_1 \circ f$ , since  $(C(f,g) \circ h)^{\widehat{}} = g \circ \hat{h} \circ (f \times I)$  is continuous.  $\Box$  **2.5 Definition.** [9, 2.1.4] A mapping  $f : X \to Y$  is called 0-HOMOTOPIC iff it is homotopic to a constant mapping.

A space X is called CONTRACTIBLE, iff  $id_X$  is 0-homotopic.

## 2.6 Remarks. [9, 2.1.6]

- (0) Any two constant mappings into Y are homotopic iff Y is path-connected: In fact a path  $y: I \to Y$  induces a homotopy  $t \mapsto \text{const}_y$ .
- (1)  $[\{*\}, Y]$  is in bijection to the path-components of Y: Homotopy = Path.
- (2) Star-shaped subsets  $A \subseteq \mathbb{R}^n$  are contractible by scalar-multiplication. In particular, this is true for  $A = \mathbb{R}^n$  and for convex subsets  $A \subseteq \mathbb{R}^n$ .
- (4) For a contractible space X there need not exist a homotopy h which keeps  $x_0$  fixed, see the infinite comb 2.40.9.



Contractible spaces are path-connected.

- (5) A composition of a 0-homotopic mapping with any mapping is 0-homotopic: 2.4.
- (6) If Y is contractible then any two mappings  $f_j : X \to Y$  are homotopic, i.e.  $[X,Y] := \{*\} : 2.4$  and 2.2.
- (7) Any continuous none-surjective mapping  $f: X \to S^n$  is 0-homotopic:  $S^n \setminus \{*\} \cong \mathbb{R}^n$  by 1.14, now use 2 and 6.
- (8) If X is contractible and Y is path-connected then again any two mappings  $f_j: X \to Y$  are homotopic, i.e.  $[X, Y] = \{*\}: [5]$  and [2.6.1].
- (9) Any mapping  $f : \mathbb{R}^n \to Y$  is 0-homotopic: 2 and the arguments in 8.

**2.7 Definition.** [9, 2.1.7] [9, 2.1.8] [9, 2.1.10] (1) A HOMOTOPY RELATIVE  $A \subseteq X$  is a homotopy  $h: I \to C(X, Y)$  with  $\operatorname{incl}^* \circ h: I \to C(X, Y) \to C(A, Y)$  constant. Two mappings  $h_j: X \to Y$  are called homotopic relative  $A \subseteq X$ , iff there exists a homotopy  $h: I \to C(X, Y)$  relative A with boundary values  $h(j) = h_j$  for  $j \in \{0, 1\}$ .

(2) A HOMOTOPY OF PAIRS (X, A) and (Y, B) is a homotopy  $h: I \to C(X, Y)$  with  $h(I)(A) \subseteq B$  Two mappings  $h_j: (X, A) \to (Y, B)$  of pairs are called HOMOTOPIC, iff there exists a homotopy (of pairs)  $h: I \to C(X, Y)$  with  $h(I)(A) \subseteq B$  and  $h(j) = h_j$  for  $j \in \{0, 1\}$ . We denote with  $[h_0]$  also this homotopy class and with [(X, A), (Y, B)] the set of all these classes.

(3) A homotopy of pairs with  $A = \{x_0\}$  and  $B = \{y_0\}$  is called BASE-POINT PRESERVING HOMOTOPY. We have  $f \sim g : (X, \{x_0\}) \to (Y, \{y_0\})$  iff  $f \sim g$  relative  $\{x_0\}$ .

**2.10 Example.** [9, 2.1.9] Since *I* is contractible we have  $[I, I] = \{0\}$ . But  $[(I, \dot{I}), (I, \dot{I})] = \{[\text{id}], [t \mapsto 1 - t], [t \mapsto 0], [t \mapsto 1]\}.$ 

**2.11 Lemma.** [9, 2.1.11] Let  $p: X' \to X$  be a quotient mapping and let  $h: I \to C(X, Y)$  be a mapping for which  $p^* \circ h: I \to C(X', Y)$  is a homotopy. Then h is a homotopy.

**Proof.** Note that for quotient-mappings p the induced injective mapping  $p^*$  is in general not an embedding (we may not find compact inverse images). However  $\widehat{p^* \circ h} = \widehat{h} \circ (I \times p)$  and  $I \times p$  is a quotient-mapping by 1.32.

- 2.12 Corollary. [9, 2.1.12]
  - (1) Let  $p: X' \to X$  be a quotient mapping,  $h: I \to C(X', Y)$  be a homotopy and  $h_t \circ p^{-1}: X \to Y$  be a well-defined mapping for all t. Then this defines a homotopy  $I \to C(X, Y)$  as well: This is just a reformulation of [2.11].
  - (3) Let  $f: X \supseteq A \to Y$  be a gluing map and  $h: I \to C(X, Z)$  and  $k: I \to C(Y, Z)$  be homotopies with  $\operatorname{incl}^* \circ h = f^* \circ k$ . Then they induce a homotopy  $I \to C(Y \cup_f X, Z)$ : Apply 1 to  $p: Y \sqcup X \to Y \cup_f X$ .
  - (2) Let  $h: I \to C(X, Y)$  be a homotopy compatible with equivalence relations  $\sim$  on X and on Y, i.e.  $x \sim x' \Rightarrow h(t, x) \sim h(t, x')$ . Then h factors to a homotopy  $I \to C(X/\sim, Y/\sim)$ : Apply 1 to  $(q_Y)_* \circ h: I \to C(X, Y/\sim)$ .
  - (4) Each homotopy  $h: I \to C((X, A), (Y, B))$  of pairs induces a homotopy  $I \to C(X/A, Y/B)$ : 2.
  - (5) Homotopies  $h^j : I \to C((X_j, x_j^0), (Y_j, y_j^0))$  induce a homotopy  $\bigvee_j h^j : I \to C((\bigvee_j X_j, x^0), (\bigvee_j Y_j, y^0)).$

# 2.13 Example. [9, 2.1.13]

- (1) Let  $h_t : (X, I) \to (X, I)$  be given by  $h_t(x, s) := (x, ts)$ . This induces a contraction of  $CX := (X \times I)/(X \times \{0\})$ .
- (2) The contraction of  $D^n = CS^{n-1}$  given by 1 is not compatible with the equivalence relation describing  $D^n/S^{n-1} \cong S^n$ , hence induces no contraction of  $S^n$ . We will see in 2.19 and 9.1, that  $S^n$  is not contractible at all.

#### Homotopy classes for mappings of the circle

**2.14 Definition.** [9, 2.2.1] We consider the (periodic) quotient mapping (and group homomorphism)  $p : \mathbb{R} \to S^1$ ,  $t \mapsto \exp(2\pi i t)$  as well as its restriction  $p|_I : I \to S^1$ .

Each mapping  $\varphi: I \to \mathbb{R}$  factors to a well defined mapping  $\overline{\varphi} := p \circ \varphi \circ p^{-1} : S^1 \to S^1$ iff  $n := \varphi(1) - \varphi(0) \in \mathbb{Z}$ .



Conversely:

2.14

**2.15 Lemma.** [9, 2.2.2] Let  $f: S^1 \to S^1$  be continuous, then there exists a unique continuous  $\varphi: (\mathbb{R}, 0) \to (\mathbb{R}, 0)$  with  $f = f(1) \cdot \overline{\varphi}$ .

$$(\mathbb{R}, 0) \xrightarrow{\varphi} (\mathbb{R}, 0)$$
$$\downarrow^{p} \qquad \qquad \downarrow^{p}$$
$$(S^{1}, 1) \xrightarrow{f(1)^{-1}f} (S^{1}, 1)$$

**Proof.** Replace f by  $f(1)^{-1} \cdot f$ , i.e. w.l.o.g. f(1) = 1. Let  $h := f \circ p$ . Then h is periodic, uniformly continuous and h(0) = 1. So chose  $\delta > 0$  with  $|t - t'| \leq \delta \Rightarrow |h(t) - h(t')| < 2$  and hence  $\frac{h(t)}{h(t')} \neq -1$ . Let  $t_j := j \delta$ . The mapping  $t \mapsto e^{it}$  is a homeomorphism  $(-\pi, \pi) \to S^1 \setminus \{-1\}$  let  $\arg : S^1 \setminus \{-1\} \to (-\pi, \pi) \subseteq \mathbb{R}$  denote its inverse. Then for  $t_j \leq t \leq t_{j+1}$  let

$$\varphi(t) := \frac{1}{2\pi} \left( \arg \frac{h(t_1)}{h(t_0)} + \dots + \arg \frac{h(t)}{h(t_j)} \right),$$

which gives the desired lifting.

This lifting is unique, since the difference of two such liftings has image in the discrete subset  $p^{-1}(1) \subseteq \mathbb{R}$ , and hence is constant (=0).

**2.16 Definition.** [9, 2.2.3] Let  $f: S^1 \to S^1$  be continuous and  $\varphi$  as in 2.15, then deg  $f := \varphi(1) \in \mathbb{Z}$  is called MAPPING DEGREE of f.

**2.17 Theorem.** [9, 2.2.4] deg induces an isomorphism  $[S^1, S^1] \cong \mathbb{Z}$  of semigroups. In more detail:

- (1) The mapping  $g_n : z \mapsto z^n$  from [1.95.5] has degree n.
- (2) Two mappings are homotopic iff they have the same degree.
- (3)  $\deg(f_1 \circ f_2) = \deg(f_1) \cdot \deg(f_2).$

**Proof.** 1 follows since  $\varphi(t) = n \cdot t$ .

2 Let f be a homotopy  $I \to C(S^1, S^1)$ . Then, by 2.15, there exists a lifting  $\varphi: I \to C(\mathbb{R}, \mathbb{R})$  with  $p(\varphi_t(z)) = f_t(1)^{-1} \cdot f_t(p(z))$ . This  $\varphi$  is a homotopy, since we can use for each  $h_t$  the same  $\delta$  in the proof of 2.15. In particular  $\varphi_t(1) \in p^{-1}(1) = \mathbb{Z}$  and hence is constant. So deg $(f_0) = \varphi_0(1) = \varphi_1(1) = \text{deg}(f_1)$ .

Conversely, we define  $\varphi: I \to C(\mathbb{R}, \mathbb{R})$  by  $\varphi_t := (1-t)\varphi_0 + t\varphi_1$ . Then this induces a homotopy  $f: I \to C(S^1, S^1)$ , since  $\varphi_t(1) = \deg(f_0) = \deg(f_1) \in \mathbb{Z}$ .

**3** Let  $n := \deg(f_1)$  and  $m := \deg(f_2)$ . Obviously,  $g_n \circ g_m = g_{nm}$ . By **1** and **2**  $f_1 \sim g_n$  and  $f_2 \sim g_m$ , hence  $f_1 \circ f_2 \sim g_n \circ g_m = g_{nm}$  and thus  $\deg(f_1 \circ f_2) = nm$ .  $\Box$ 

### 2.18 Remarks. [9, 2.2.5]

- (1) deg(id) = 1: id =  $g_1$ ;  $f \sim 0 \Rightarrow deg(f) = 0$ :  $f \sim g_0$ ; deg $(g_{-1} : z \mapsto \overline{z}) = -1$  by 2.17.
- (3) f homeomorphism  $\Rightarrow \deg(f) \in \{\pm 1\}$ , since  $\deg(f)$  is invertible in  $\mathbb{Z}$ .
- (4) incl:  $S^1 \hookrightarrow \mathbb{C} \setminus \{0\}$  is not 0-homotopic, since  $\mathrm{id}_{S^1}$  is not:  $\mathrm{deg}(\mathrm{id}) = 1$  and 2.17. We can use  $[S^n, X]$  to detect holes in X.

(5) The two inclusions of  $S^1 \hookrightarrow S^1 \times S^1$  are not homotopic:  $\operatorname{pr}_1 \circ \operatorname{inc}_1 = \operatorname{id},$  $\operatorname{pr}_1 \circ \operatorname{inc}_2 \sim 0.$ 

**2.19 Lemma.** [9, 2.2.6] S<sup>1</sup> is not contractible.

**Proof.** deg(id) = 1.

**2.20 Definition.** [9, 2.3.1] A subspace  $A \subseteq X$  is called RETRACT iff there exists an  $r: X \to A$  with  $r|_A = id_A$ .

Being a retract is a transitive relation. Retracts in Hausdorff spaces are closed  $(A = \{x \in X : r(x) = x\})$ 

#### **2.21 Lemma.** [9, 2.3.2]

- (1) A subspace  $A \subseteq X$  is a retract of X iff every function  $f : A \to Y$  can be extended to  $\tilde{f} : X \to Y$ .
- (2) Let  $A \subseteq X$  be closed. Then a function  $f : A \to Y$  can be extended to X iff Y is a retract of  $Y \cup_f X$ .

**Proof.** We prove that  $id_A$  can be extended iff any  $f : A \to Y$  can be extended. The extensions  $\tilde{f}$  of  $f : A \to Y$  correspond to retractions  $r = id_Y \cup \tilde{f}$  of  $Y \subseteq Y \cup_f X$ :



**2.22 Lemma.** [9, 2.2.7] There is no retraction of  $D^2$  to  $S^1 \hookrightarrow D^2$ .

**Proof.** Otherwise, let  $r: D^2 \to S^1$  be a retraction to  $\iota: S^1 \hookrightarrow D^2$ . Then id  $= r \circ \iota \sim r \circ 0 = 0$ , a contradiction to 2.17.1.

**2.23 Lemma. Brouwer's fixed point theorem.** [9, 2.2.8] Every continuous mapping  $f: D^2 \to D^2$  has a fixed point.

### Proof.

Assume  $f(x) \neq x$  and let r(x) the unique intersection point of the ray from f(x) to x with  $S^1$ . Then r is a retraction, a contradiction to 2.22.



**2.24 Lemma. Fundamental theorem of algebra.** [9, 2.2.9] Every not-constant polynomial has a root.
**Proof.** Let  $p(x) = a_0 + \cdots + a_{n-1}x^{n-1} + x^n$  be a polynomial without root and  $n \ge 1, s := |a_0| + \cdots + |a_{n-1}| + 1 \ge 1$  and  $z \in S^1$ . Then

$$|p(sz) - (sz)^{n}| \le |a_{0}| + s|a_{1}| + \dots + s^{n-1}|a_{n-1}|$$
  
$$\le s^{n-1}(|a_{0}| + \dots + |a_{n-1}|) < s^{n} = |(sz)^{n}|$$

Hence  $0 \notin \overline{p(sz), (sz)^n}$ . Thus  $z \mapsto s^n z^n, S^1 \to \mathbb{C} \setminus \{0\}$  is homotopic to  $z \mapsto p(sz)$  and consequently 0-homotopic. Hence  $0 \sim g_n : z \mapsto z^n$ , a contradiction to 2.17.

## 2.25 Definition.

[9, 2.2.10] The DEGREE of  $f : S^1 \to \mathbb{R}^2$ with respect to  $z_0 \notin f(S^1)$  is the degree of  $x \mapsto \frac{f(x)-z_0}{|f(x)-z_0|}$  and will be denoted by  $U(f, z_0)$  the TURNING (WINDING) NUMBER of f around  $z_0$ .



**2.26 Lemma.** [9, 2.2.11] If  $z_0$  and  $z_1$  are in the same component of  $\mathbb{C} \setminus f(S^1)$  then  $U(f, z_0) = U(f, z_1)$ .

**Proof.** Let  $t \mapsto z_t$  be a path in  $\mathbb{C} \setminus f(S^1)$ . Then  $t \mapsto (x \mapsto \frac{f(x)-z_t}{|f(x)-z_t|})$  is a homotopy and hence  $U(f, z_0) = U(f, z_1)$ .

**2.27 Lemma.** [9, 2.2.12] There is exactly one unbounded component of  $\mathbb{C} \setminus f(S^1)$  and for z in this component we have U(f, z) = 0.

**Proof.** For x' outside a sufficiently large disk containing  $f(S^1)$  (this is connected and contained in the (unique) unbounded component) the mapping

$$t \mapsto \left( x \mapsto \frac{tf(x) - x'}{|tf(x) - x'|} \right)$$

is a homotopy showing that  $x \mapsto \frac{f(x)-x'}{|f(x)-x'|}$  is 0-homotopic and hence U(f, x') = 0 and  $U(f, _)$  is zero on the unbounded component.

By Jordan's curve theorem there are exactly two components for an embedding  $f: S^1 \to \mathbb{C}$ . And one has  $U(f, z) \in \{\pm 1\}$  for z in the bounded component.

**2.30 Theorem.** [9, 2.3.3] A mapping  $f : X \to Y$  is 0-homotopic iff there exists an extension  $\tilde{f} : CX \to Y$  with  $\tilde{f}|_X = f$ 

**Proof.** We prove that homotopies  $h: X \times I \to Y$  with constant  $h_0$  correspond to extensions  $\tilde{h}_1: CX \to Y$  of  $h_1$ :



**2.28 Theorem of Borsuk and Ulam.** [9, 2.2.13] For every continuous mapping  $f: S^2 \to \mathbb{R}^2$  there is a  $z \in S^2$  with f(z) = f(-z).

**Proof.** Suppose  $f(x) \neq f(-x)$ . Consider  $f_1 : S^2 \to S^1$ ,  $x \mapsto \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$  and  $f_2 : D^2 \to S^1$ ,  $x \mapsto f_1(x, \sqrt{1-|x|^2})$ . Then  $g := f_2|_{S^1} \sim 0$  via  $f_2$ . Since  $f_1$  is odd, so is g. Let  $\varphi$  be the lift of  $g(1)^{-1}g$  and hence  $\varphi(1) = \deg(g)$ . For all t we have  $g(\exp(2\pi i(t+\frac{1}{2}))) = g(-\exp(2\pi i t)) = -g(\exp(2\pi i t))$  and hence

$$\exp(2\pi i\varphi(t+\frac{1}{2})) = g(1)^{-1}g(\exp(2\pi i(t+\frac{1}{2}))) = -g(1)^{-1}g(\exp(2\pi i t))$$
$$= -\exp(2\pi i\varphi(t)) = \exp(2\pi i(\varphi(t)+\frac{1}{2})).$$

Hence  $k := \varphi(t + \frac{1}{2}) - \varphi(t) - \frac{1}{2} \in \mathbb{Z}$  and independent on t. For t = 0 we get  $\varphi(\frac{1}{2}) = k + \frac{1}{2}$  and for  $t = \frac{1}{2}$  we get  $\deg(g) = \varphi(1) = \varphi(\frac{1}{2}) + \frac{1}{2} + k = 2k + 1 \neq 0$ , a contradiction.

**2.29 Ham-Sandwich-Theorem.** [9, 2.2.14] Let  $A_0, A_1, A_2$  be bounded measurable subsets of  $\mathbb{R}^2$ . Then there is a plane which cuts  $A_0, A_1$  and  $A_2$  in exactly equal parts.

**Proof.** We denote the halfspaces with  $H_{a,d} := \{x \in \mathbb{R}^3 : \langle x, a \rangle \leq d\}$  and the volume of the trace of  $A_j$  on this halfspace with  $\mu_j(a,d) := \mu(A_j \cap H_{a,d})$ . Then  $\mu_j : S^2 \times \mathbb{R} \to \mathbb{R}$  is a continuous function with  $\mu_j(-a, -d) + \mu_j(a, d) = \mu(A_j)$ . Let  $d_a$  be the middle point of the closed interval  $I_a := \{d : \mu_0(a, d) = \mu(A_0)/2\}$ . For d in this set we have  $\mu_0(a, d) = \frac{\mu(A_0)}{2} = \mu_0(-a, -d)$  and hence  $d_{-a} = -d_a$ .

Moreover,  $a \mapsto d_a$  is continuous: let  $d_- := \min I_a$  and  $d_+ := \max I_a$ . Then  $\mu_0(a_0, d) < \mu(A_0)/2$  for  $d < d_-$  and by continuity of  $\mu_0$  there exists a  $\delta > 0$  such that  $\mu_0(a, d_- -\varepsilon) < \mu(A_0)/2$  for all  $|a - a_0| < \delta$  and analogously  $\mu_0(a, d_+ +\varepsilon) > \mu(A_0)/2$  for all  $|a - a_0| < \delta$ . Thus  $I_a \subseteq [d_- -\varepsilon, d_+ +\varepsilon]$ . In case  $d_- = d_+$  we get  $|d_a - d_{a_0}| \le \varepsilon$ . Otherwise  $d \mapsto \mu(a_0, d) = \mu(A_0)/2$  is constant on  $[d_-, d_+]$  and thus  $\mu(A_0 \cap (H_{a,d_+} \setminus H_{a,d_-})) = 0$ . Thus we may assume that  $\delta > 0$  is so small, that  $\mu(a, d) = \mu(A_0)/2$  for all  $|a - a_0| < \delta$  and all  $d_- + \varepsilon < d < d_+ - \varepsilon$ . Then again  $|d_a - d_{a_0}| \le \varepsilon$ .

Now let  $f: S^2 \to \mathbb{R}^2$  be given by  $f(a) := (\mu_1(a, d_a), \mu_2(a, d_a))$ . By 2.28 there exists a point  $b \in S^2$  with f(b) = f(-b). Since  $d_{-a} = -d_a$  we have that f(-b) is the volume of  $A_1$  and  $A_2$  on the complement of  $H_{a,d_a}$ .

**2.31 Definition.** [9, 2.3.4] A pair (X, A) is said to have the general HOMOTOPY EXTENSION PROPERTY (HEP) (equiv. is a COFIBRATION) iff A is closed in X and we have



**2.32 Theorem.** [9, 2.3.5] (X, A) has HEP  $\Leftrightarrow X \times \{0\} \cup A \times I$  is a retract of  $X \times I$ .

**Proof.** With  $L := X \times \{0\} \cup A \times I \subseteq X \times I$  we have: (X, A) has HEP;  $: \iff \text{any } f : L \to Y \text{ extends to } X \times I;$  2.21 $L \subseteq X \times I$  is a retract.

#### 2.33 Remarks. [9, 2.3.6]

- (1) The pair  $(D^n, S^{n-1})$  has the HEP: Radial project from the axis at some point above the cylinder is a retraction.
- (2) If (X, A) has HEP then  $(Y \cup_f X, Y)$  has HEP for any  $f : A \to Y$ :



- (3) If Y is obtained from X by gluing cells, then (Y, X) has  $HEP: \leftarrow [1], [2]$ .
- (4) The pair  $(\mathbb{N}_{\infty}, \{\infty\})$  does not have HEP. Otherwise, for  $x \neq \infty$  the map  $t \mapsto r(x, t)$ ,  $I \to L$ , maps  $0 \mapsto (x, 0) \Rightarrow r(\{x\} \times I) \subseteq L \cap (\{x\} \times I) = \{(x, 0)\}$ , but r(x, 1) is near  $r(\infty, 1) = (\infty, 1)$  for x near  $\infty$ .



#### **2.34 Remark.** [9, 2.3.7] Let (X, A) has HEP.

- (1) If  $f \sim g : A \to Y$  and f extends to X then so does g : By Definition of HEP.
- (2) If f : X → Y is 0-homotopic on A, then there exists a mapping g homotopic to f, which is constant on A: Consider the homotopy on A × I and f on X × {0}.
- (3) If  $A = \{x_0\}$  and Y is path-connected, then every mapping  $X \to Y$  is homotopic to a base-point preserving one: Consider f on  $X \times \{0\}$  and a path w on  $\{x_0\} \times I$  between  $f(x_0)$  and  $y_0$ .
- (4) There exists a continuous  $u : X \to I$  with  $A = u^{-1}(0)$ : Define  $u(x) := \sup\{t \operatorname{pr}_2(r(x,t)) : t \in I\}$ . Then  $u : X \to I$  is continuous and  $u(x) = 0 \Leftrightarrow t \leq \operatorname{pr}_2(r(x,t)) \Rightarrow \operatorname{pr}_2(r(x,t)) \geq t > 0$  for t > 0, thus  $r(x,t) \in A \times I$  for t > 0 and hence also  $(x,0) = r(x,0) \in A \times I$ , i.e.  $x \in A$ .
- (5) For closed subsets A of metric spaces Y there exists a function  $u: Y \to I$  as in [4]: Define  $u(y) := d(y, A) = \inf\{d(y, a) : a \in A\}.$

**2.35 Theorem.** [9, 2.3.8] If (X, A) has HEP, then so has  $(X \times I, X \times \dot{I} \cup A \times I)$ .

#### Proof.

We use 2.32 to show that  $X \times I \times I$  has  $L := X \times I \times \{0\} \cup (X \times I \cup A \times I) \times I$  as retract. For this we consider planes E through the axis  $X \times (1/2, 2)$ . For planes intersecting the bottom  $X \times I \times \{0\}$  we take the retraction r of the intersection  $E \cap (X \times I \times I) \cong X \times I$  (via horizontal projection) onto the intersection  $E \cap L \cong X \times \{0\} \cup A \times I$ . For the other planes meeting the sides we take the retraction r of the intersection  $E \cap L \cong X \times \{0\} \cup A \times I$ . For the other planes meeting the sides we take the retraction r of the intersection  $E \cap (X \times I \times I) \cong X \times [0, t]$  (via vertical projection) onto the intersection  $E \cap L \cong X \times \{0\} \cap A \times [0, t]$ . For this we have to use that the retraction  $r : (x, t) \mapsto (r_1(x, t), r_2(x, t))$  given by 2.32 can be chosen such that  $r_2(x, t) \leq t$  by replaceing  $r_2(x, t)$  by min $\{t, r_2(x, t)\}$ .





**2.36 Definition.** [9, 2.4.1] [9, 2.4.2] [9, 2.4.3]

- (1) A HOMOTOPY EQUIVALENCE is a mapping having up to homotopy an inverse. It is enough to assume a homotopy left inverse l and a homotopy right inverse r, i.e.  $[l] \circ [f] = [id]$  and  $[f] \circ [r] = [id]$ , since then  $[f] \circ [l] = [id] \circ [f] \circ [l] = [f] \circ [l] \circ [f] \circ [r] = [f] \circ [id] \circ [r] = [f] \circ [r] = [id]$ . Two spaces are called HOMOTOPY EQUIVALENT (and we write  $\sim$ ) iff there exists a homotopy equivalence.
- (2) A continuous mapping between pairs is called HOMOTOPY EQUIVALENCE OF PAIRS, iff there is a mapping of pairs in the opposite direction which is inverse up to homotopy of pairs.
- (3) A subspace  $A \subseteq X$  is called DEFORMATION RETRACT (DR) iff there is a homotopy  $h_t : X \to X$  with  $h_0 = \operatorname{id}_X$  and  $h_1 : X \to A \subseteq X$  being a retraction to  $A \hookrightarrow X$ .
- (4) The subspace  $A \subseteq X$  is called STRICT DEFORMATION RETRACT (SDR) iff, in addition to 3,  $h_t$  is a homotopy rel. A and there exists a continuous  $u: X \to I$  with  $A = u^{-1}(0)$ .
- (5) A subspace  $A \subseteq X$  is called NEIGHBORHOOD DEFORMATION RETRACT (NDR) iff there exists a continuous  $u: X \to I$  with  $A = u^{-1}(0)$  and a homotopy  $h_t: X \to X$  relative A with  $h_0 = \operatorname{id}_X$  and  $h_1(x) \in A$  for u(x) < 1. Note that the SDRs are exactly the NDRs for which u can be choosen with u(x) < 1 for all  $x \in X$  (replace u by  $\frac{u}{2}$ ).

**2.39 Theorem.** [9, 2.4.4] For (X, A) with HEP the following is equivalent:

- (1)  $A \hookrightarrow X$  is a homotopy-equivalence;
- (2) A is a DR of X;
- (3) A is a SDR of X.

**Proof.**  $(3\Rightarrow 2)$  is obvious.

 $(2\Rightarrow 1)$  is always true. In fact let  $h_t$  be a deformation from  $\mathrm{id}_X$  to a retraction  $h_1: X \to A \subseteq X$ . Then  $h_1$  is a homotopy inverse to  $\iota: A \hookrightarrow X$ , since  $h_1 \circ \iota = \mathrm{id}_A$  and  $\iota \circ h_1 = h_1 \sim h_0 = \mathrm{id}_X$ .

(1 $\Rightarrow$ 2) Let g be a homotopy inverse to  $\iota : A \hookrightarrow X$ . Since  $g \circ \iota \sim \operatorname{id}_A$  and  $g \circ \iota$  extends to  $g : X \to A$  we conclude from 2.34.1 that  $\operatorname{id}_A : A \to A$  has an extension  $r : X \to A \subseteq X$ , i.e. a retraction. Moreover,  $\operatorname{id}_X \sim \iota \circ g = r \circ \iota \circ g \sim r \circ \operatorname{id}_X = r$ .

 $(2\Rightarrow3)$  Let  $h_t: X \to X$  be a deformation from  $h_0 = \operatorname{id}_X$  to a retraction  $h_1 = r: X \to A \subseteq X$  and let  $H_t: W := X \times \dot{I} \cup A \times I \to X$  be given by

$$H_t(x,s) := \begin{cases} h_{st}(r(x)) & \text{für } s = 1 \text{ (the back side)} \\ h_{st}(x) & \text{elsewhere, i.e. for } x \in A \text{ or } s = 0 \text{ (front) or even } t = 1 \text{ (top)} \end{cases}$$

Because of r(x) = x for  $x \in A$  the definition coincides on the intersection. Since the expression for  $H_1$  works on  $X \times I$  and  $(X \times I, W)$  has HEP by 2.35 we can extend  $H_0$  to  $X \times I$  by 2.34.1. This is the required deformation  $id_X \sim r$  rel. A.

Since (X, A) has HEP we have  $A = u^{-1}(0)$  for a  $u: X \to I$  by 2.34.4.

### 2.40 Remarks. [9, 2.4.5]

- (1) X is contractible iff it is homotopy-equivalent to a point: In fact, X is contractible  $\Leftrightarrow$  id<sub>X</sub> ~ const<sub>\*</sub>  $\Leftrightarrow$  {\*}  $\subseteq$  X is a DR  $\Leftrightarrow$  {\*} ~ X.
- (2) Every set which is star-shaped with respect to some point, has this point as SDR. Furthermore, S<sup>n-1</sup> ⊆ ℝ<sup>n</sup> \ {0} is SDR: The radial homotopy from 2.6.2 is the strict deformation.
- (3) Composition of (S)DRs are (S)DRs:

$$h(t,x) := \begin{cases} h^1(2t,x) & \text{for } t \le \frac{1}{2} \\ h^2(h^1(1,x), 2t-1) & \text{for } t \ge \frac{1}{2} \end{cases}$$

and  $u := \max\{u_1, u_2 \circ h_1^1\}.$ 

- (4) If  $\{y\}$  is an (S)DR of Y then so is  $X \times \{y\}$  of  $X \times Y$  and of  $X \vee Y \subseteq X \times Y$ : Use  $h_t(x, y) := (x, h_t(y))$  and u(x, y) := u(y).
- (5a) If (X, A) is an NDR and (Y, B) is an NDR (SDR), then  $(X \times Y, X \times B \cup A \times Y)$  is an NDR (SDR): Let

$$h_t(x,y) := \begin{cases} \left(h_t(x), h_t \frac{u(x)}{u(y)}(y)\right) & \text{for } u(x) \le u(y) \\ \left(h_t \frac{u(y)}{u(x)}(x), h_t(y)\right) & \text{for } u(x) \ge u(y) \end{cases}$$

and  $u(x, y) = \min\{u(x), u(y)\}.$ 

- (6) The complement of a k-dimensional affine subspace of  $\mathbb{R}^n$  has an  $S^{n-k-1}$  as  $SDR: \mathbb{R}^n \setminus \mathbb{R}^k = \mathbb{R}^k \times (\mathbb{R}^{n-k} \setminus \{0\}) \sim \{0\} \times S^{n-k-1}$  by 2, 4 and 3.
- (5)  $X \times \{0\}$  is an SDR of  $X \times I$  and the apex  $X \times \{0\} \in C(X)$  is an SDR of  $CX : \leftarrow 2$ , 4 and 2.12.4.
- (7) The following spaces have  $S^1$  as  $DR: X \times S^1$  for every contractible X and the Möbius strip: By 1 and using  $I \times \{0\} \subseteq I \times [-1, 1]$  for the Möbius strip.
- (8) Every handle-body of genus g has  $S^1 \vee \cdots \vee S^1$  as SDR.



(9) The infinite comb (see 2.6.4) has  $(+\infty, 1)$  as DR but not as SDR.

**2.41 Proposition.** [9, 2.4.6] If A is an NDR (SDR) in X and  $f : A \to Y$  is continuous, then Y is an NDR (SDR) in  $Y \cup_f X$ .





Let  $u: Y \cup_f X \to I$  be given by u(y) := 0 for  $y \in Y$  and u([x]) := u(x) for  $x \in X$ .

**2.42 Corollary.** [9, 2.4.7] If Y is built from X by gluing simultaneously cells, then Y is a SDR in  $Y \setminus P$ , where P is given by picking in every cell a single point.

**Proof.** Use 2.40.2 and 2.41.

**2.43 Example.** [9, 2.4.8] The pointed compact surfaces have  $S^1 \lor \cdots \lor S^1$  as SDR.

**Proof.** By 1.101 they are  $S^1 \vee \cdots \vee S^1 \cup_f (D^2 \setminus \{*\})$ . Now use 2.41.

**2.44 Theorem.** [9, 2.4.9] For a pair (X, A) and  $L := X \times \{0\} \cup A \times I \subseteq X \times I$  the following statements are equivalent:

- (1) (X, A) is NDR;
- (2)  $(X \times I, L)$  is SDR;
- (3) L is a retract of  $X \times I$ ;
- (4) (X, A) has HEP.

## Proof.

 $(1\Rightarrow 2)$  By 2.40.5a, since (X, A) is NDR and  $(I, \{0\})$  is SDR.

- $(2\Rightarrow3)$  Take  $r := h_1$ .
- $(3 \Leftrightarrow 4)$  is 2.32.

 $(3\Rightarrow 1)$  Let  $r = (r_1, r_2)$  be a retraction of  $L \hookrightarrow X \times I$ . Define  $u(x) := \sup\{t - r_2(x,t) : t \in I\}$  and  $h_t(x) := r_1(x,t)$ . Then  $A = u^{-1}(0)$  as in 2.34.4. Furthermore,

 $h_0(x) = r_1(x, 0) = x, h_t(a) = r_1(a, t) = a$  for all  $a \in A$ , and  $u(x) < 1 \Rightarrow r_2(x, 1) > 0 \Rightarrow h_1(x) = r_1(x, 1) \in A$ .

### **2.44a Dependencies for closed subspaces** $A \hookrightarrow X$ .



#### 2.44b Counter-Examples.

Prop.	$L \subseteq E$	$\{(\infty,1)\}\subseteq E$	$\{\infty\}\subseteq\mathbb{N}_\infty$	$S^1 \subseteq D^2$	$\{0\} \subseteq \prod_I I$
SDR	-	—	_	—	_
NDR=HEP	_	—	_	+	_
DR	_	+	_	_	+
Retract	+	+	+	_	+
$A \sim X$	+	+	—	_	+
$A = u^{-1}(0)$	+	+	+	+	_

Here

- $(N_{\infty}, \infty) \cong (\{\frac{1}{n} : 0 \neq n \in \mathbb{N}\} \cup \{0\}, 0),$
- $E := N_{\infty} \times I \cup [0, +\infty] \times \{0\}$  is the infinite comb,
- and  $L := \{\infty\} \times I \cup [0, +\infty] \times \{0\} \subseteq E$ .

**2.45 Definition.** [9, 2.4.10] The MAPPING CYLINDER  $M_f$  of a mapping  $f : X \to Y$  is given by  $Y \cup_f (X \times I)$ , where f is considered as mapping  $X \times \{1\} \cong X \to Y$ .

We have the diagram



where  $f = r \circ i$  and i is a closed embedding with HEP and  $Y \to M_f$  a SDR (along the generators  $X \times I$ ) with retraction r (by 2.40.5 and 2.41).

To see the HEP, construct a retraction  $M_f \times I \to M_f \times \{0\} \cup X \times I$  by projecting radially in the plane  $\{x\} \times I \times I$  from  $\{x\} \times \{1\} \times \{2\}$  and use 2.32.



**2.46 Corollary.** [9, 2.4.12] Two spaces are homotopy equivalent iff there exists a third one which contains both as SDRs.

**Proof.** ( $\Rightarrow$ ) Use the mapping cylinder as third space. Since f is a homotopy equivalence, so is  $i: X \to M_f$  by 2.45 and by the HEP it is a SDR by 2.39.

( $\Leftarrow$ ) Use that SDRs are always homotopy equivalences.

**2.47 Proposition.** [9, 2.4.13] Assume (X, A) has HEP and  $f_j : X \supseteq A \to Y$  are homotopic. Then  $Y \cup_{f_0} X$  and  $Y \cup_{f_1} X$  are homotopy equivalent rel. Y.

**Proof.** Consider the homotopy  $f : A \times I \to Y$  and the space  $Z := Y \cup_f (X \times I)$ . We show that  $Y \cup_{f_i} X$  are SDRs of Z and hence are homotopy equivalent by 2.46:



Where we use that the composite of two push-outs is a push-out, and if the composite of push-out and a commuting square is a push-out then so is the second square, cf.  $\boxed{1.47}$ .

**2.48 Example.** [9, 2.4.14] The dunce hat D, i.e. a triangle with sides  $a, a, a^{-1}$  identified, is contractible: By 1.101, 1.94 and 2.47 we have  $D \cong S^1 \cup_f D^2 \sim S^1 \cup_{\mathrm{id}} D^2 \cong D^2$ .



**2.49 Proposition.** [9, 2.4.15] Let A be contractible and let (X, A) have the HEP. Then the projection  $X \to X/A$  is a homotopy equivalence.

**Proof.** Consider



Then  $\tilde{R}$  given by factoring  $F_1$  is the desired homotopy inverse to  $X \to X/A$ , since  $F_0 = \text{id}$  and  $F_1(A) = \{*\}$ .

# 3. Simplicial Complexes

## **Basic concepts**

**3.1 Remark (Points in general position).** [9, 3.1.1] A finite set of points  $x_0, \ldots, x_q$  in  $\mathbb{R}^n$  is said to be IN GENERAL POSITION if one of the following equivalent conditions is satisfied:

- 1. The affine subspace  $\{\sum_{i} \lambda_{i} x_{i} : \sum_{i} \lambda_{i} = 1\}$  generated by the  $x_{i}$  has dimension q;
- 2. No strict subset of  $\{x_0, \ldots, x_q\}$  generates the same affine subspace;
- 3. The vectors  $x_i x_0$  for i > 0 are linear independent;
- 4. The representation  $\sum_i \lambda_i x_i$  with  $\sum_i \lambda_i = 1$  is unique.

**3.2 Definition (Simplex).** [9, 3.1.2] A SIMPLEX of dimension q (or short: a q-simplex) is the set

$$\sigma := \langle x_0, \dots, x_q \rangle := \left\{ \sum_i \lambda_i x_i : \sum_i \lambda_i = 1, \forall i : \lambda_i > 0 \right\}$$

for points  $\{x_0, \ldots, x_q\}$  in general position. Its closure in  $\mathbb{R}^n$  is the set

$$\bar{\sigma} := \Big\{ \sum_{i} \lambda_i x_i : \sum_{i} \lambda_i = 1, \forall i : \lambda_i \ge 0 \Big\}.$$

The points  $x_i$  are then called the VERTICES of  $\sigma$ . Remark that as extremal points of  $\bar{\sigma}$  they are uniquely determined. The set  $\dot{\sigma} := \bar{\sigma} \setminus \sigma$  is called boundary of  $\sigma$ .



**3.3 Lemma.** [9, 3.1.3] Let  $\sigma$  be a q-simplex. Then  $(\bar{\sigma}, \dot{\sigma}) \cong (D^q, S^{q-1})$ .

**Proof.** Use 1.10 for the affine subspace generated by  $\sigma$ .

3.6

**3.4 Definition (Faces).** [9, 3.1.4] Let  $\sigma$  and  $\tau$  be simplices in  $\mathbb{R}^n$ . Then  $\tau$  is called FACE of  $\sigma$  ( $\tau \leq \sigma$ ) iff the vertices of  $\tau$  form a subset of those of  $\sigma$ .

## **3.5 Remark.** [9, 3.1.5]

- (1) Every q-simplex has  $2^{q+1}$  many faces: In fact this is the number of subsets of  $\{x_0, \ldots, x_q\}$
- (2) The relation of being a face is transitive.
- (3) The closure of a simplex  $\sigma$  is the disjoint union of all its faces  $\overline{\sigma} = \bigcup_{\tau \leq \sigma} \tau$ : Remove all summands  $\lambda_i x_i$  in  $\sum_i \lambda_i x_i$  for which  $\lambda_i = 0$ .

**3.6 Definition (Simplicial Complex).** [9, 3.1.6] A SIMPLICIAL COMPLEX K is a finite set of simplices in some  $\mathbb{R}^n$  with the following properties:

- 1.  $\sigma \in K, \tau \leq \sigma \Rightarrow \tau \in K$ .
- 2.  $\sigma, \tau \in K, \sigma \neq \tau \Rightarrow \sigma \cap \tau = \emptyset$ .

The 0-simplices  $\{x_0\}$  (or their elements  $x_0$ ) are called VERTICES and the 1-simplices are called EDGES of K. The number max $\{\dim \sigma : \sigma \in K\}$  is called DIMENSION of K.

**3.7 Definition (Triangulation).** [9, 3.1.7] For a simplicial complex K the subspace  $|K| := \bigcup_{\sigma \in K} \sigma$  is called the UNDERLYING TOPOLOGICAL SPACE. Every space which is the underlying space of a simplicial complex is called POLYHEDRA. A corresponding simplicial complex is called a TRIANGULATION of the space.

**3.8 Remark.** [9, 3.1.8] By 3.6 we have  $|K| = \bigcup_{\sigma \in K} \bar{\sigma}$ , and  $\bar{\sigma} \cap \bar{\tau}$  is a either empty or the closure of a common face. Every polyhedra is compact and metrizable.

## 3.9 Remarks. [9, 3.1.9]

- 1. Regular polyhedra are triangulations of a 2-sphere.
- 2. There is a triangulation of the Möbius strip by 5 triangles.



- 3. There is a (minimal) triangulation of the projective plane by 10 triangles.
- 4. One can show, that every compact surface, every compact 3-dimensional manifold and every compact differentiable manifold has a triangulation.
- 5. It is not known whether every compact manifold has a triangulation.
- 6. Every ball (and every sphere) has a triangulation given by an *n*-Simplex with all its faces.
- 7. A countable union of circles tangent at some point is not a polyhedra, since it consists of infinite many 1-simplices.

**3.11 Definition (Carrier Simplex).** [9, 3.1.10] For every  $x \in |K|$  exists a unique simplex  $\sigma \in K$  with  $x \in \sigma$ . It is called the CARRIER SIMPLEX of x and denoted  $\operatorname{carr}_{K}(x)$ .

**3.12 Lemma.** [9, 3.1.11] Every point  $x \in |K|$  has a unique representation  $x = \sum_i \lambda_i x_i$ , with  $\sum_i \lambda_i = 1$  and  $\lambda_i > 0$  and vertices  $\{x_i\}$  of K. The  $x_i$  are the vertices of the carrier simplex carr<sub>K</sub>(x) of x.

Conversely, any point  $x = \sum_{o} \lambda_i x_i$ , with  $\sum_i \lambda_i = 1$  and  $\lambda_i > 0$  and  $x_i$  generating a simplex  $\sigma \in K$ , belongs to |K|.

**3.13 Definition.** [9, 3.1.12] A SUBCOMPLEX is a subset  $L \subseteq K$ , that is itself a simplicial complex. This is exactly the case if  $\tau \leq \sigma \in L \Rightarrow \tau \in L$  since (2) is obvious.

**3.14 Lemma.** [9, 3.1.13] A subset  $L \subseteq K$  is a subcomplex iff |L| is closed in |K|.

**Proof.** ( $\Rightarrow$ ) since |L| is compact by 3.8. ( $\Leftarrow$ )  $\tau \leq \sigma \in L \Rightarrow \tau \subseteq \overline{\sigma} \subseteq |L| \Rightarrow \tau \in L$ . Use 3.5.3 and 3.8.

**3.15 Definition (Components of a Complex).** [9, 3.1.14] Two simplices  $\sigma$  and  $\tau$  are called CONNECTIBLE in K iff there are simplices  $\sigma_0 = \sigma, \ldots, \sigma_r = \tau$  with  $\bar{\sigma}_j \cap \bar{\sigma}_{j+1} \neq \emptyset$ . The equivalence classes with respect to being connectible are called the COMPONENTS of K. If there is only one component then K is called connected.

**3.16 Lemma.** [9, 3.1.15] The components of K are subcomplexes and their underlying spaces are the path-components (connected components) of |K|.

**Proof.** Since  $\overline{\sigma}$  is a closed convex subset of some  $\mathbb{R}^n$ , it is path connected and hence the underlying subspace of a component is (path-)connected. Conversely, if two simplices  $\sigma$  and  $\tau$  belong to the same component of the underlying space, then there is a curve c connecting  $\sigma$  with  $\tau$ . This curve meets finitely many simplices  $\sigma_0 = \sigma, \ldots, \sigma_N = \tau$  and we may assume that it meets  $\sigma_i$  before  $\sigma_j$  for i < j. By induction we show that all  $\overline{\sigma_i}$  belong to the same component of K. In fact if  $\sigma_0, \ldots, \sigma_{i-1}$  does so, then let  $t_0 := \min\{t \in [0, 1] : c(t) \in \overline{\sigma_i}\}$ . Then  $c(t) \in \bigcup_{j < i} \sigma_j$ for  $t < t_0$  and hence  $c(t_0) \in \bigcup_{i < j} \overline{\sigma_j} \cap \overline{\sigma_i}$ . Thus  $\overline{\sigma_i}$  is connected with  $\overline{\sigma_j}$  for some j < i.

**3.17 Definition (Simplicial Mapping).** [9, 3.1.16] A mapping  $\varphi : K \to L$  between simplicial complexes is called SIMPLICIAL MAPPING iff

- 1.  $\varphi$  maps vertices to vertices (and we write  $\varphi(\{x\}) =: \{\varphi(x)\});$
- 2. And if  $\sigma$  is generated by vertices  $x_0, \ldots, x_q$  then  $\varphi(\sigma)$  is generated by the vertices  $\varphi(x_0), \ldots, \varphi(x_q)$ , i.e.  $\varphi(\langle x_0, \ldots, x_q \rangle) = \langle \varphi(x_0), \ldots, \varphi(x_q) \rangle$ . Attention: It is not assumed, that the  $\varphi(x_i)$  are pairwise distinct.

**3.18 Lemma.** [9, 3.1.17]

- 1. A simplicial mapping is uniquely determined by its action on the vertices.
- 2. If  $\sigma \leq \tau \in K$  then  $\varphi(\sigma) \leq \varphi(\tau) \in L$ .
- 3.  $\dim(\varphi\sigma) \leq \dim \sigma$ .

**Proof.** This follows immediately, since  $\varphi(\langle x_0, \ldots, x_k \rangle) = \langle \{\varphi(x_i) : 0 \le i \le k\} \rangle$ .  $\Box$ 

**3.19 Definition (Underlying continuous Map).** [9, 3.1.18] Let  $\varphi : K \to L$  be simplicial. Then

$$|\varphi| \left(\sum_i \lambda_i x_i\right) := \sum_i \lambda_i \varphi(x_i) \text{ for } x_i \in K, \ \sum_i \lambda_i = 1 \text{ and } \lambda_i \ge 0$$

defines by 3.12 a continuous map from  $|K| \to |L|$  (which is affine on every closed simplex  $\bar{\sigma}$ ).

**3.20 Remark.** [9, 3.1.19] There are only finitely many simplicial mappings from K to L. For every simplicial map  $\varphi$  the map  $|\varphi|$  is not dimension increasing.

#### **3.21 Lemma.** [9, 3.1.21]

- 1. A map  $\varphi : K \to L$  is an simplicial isomorphism (i.e. has an inverse, which is simplicial) iff it is simplicial and bijective.
- 2. For every simplicial isomorphism  $\varphi$  the mapping  $|\varphi|$  is a homeomorphism.

**Proof.**  $(1, \leftarrow)$  We have to show that the inverse of a bijective simplicial mapping is simplicial.

Let  $\xi = \{x\}$  be a vertex of L and  $\varphi(\sigma) = \gamma$ . We have to show that  $\sigma$  is a 0-simplex. Let  $x_0, \ldots, x_q$  be the vertices of  $\sigma$ . By 3.17.2 the  $\varphi(x_0), \ldots, \varphi(x_q)$  generate the simplex  $\xi = \varphi(\sigma)$  and hence have to be equal to the vertex x of  $\xi$ . Since  $\varphi$  is injective q = 0 and  $\sigma = \{x_0\}$ .

Now let  $\tau = \varphi(\sigma)$  be a simplex in L with vertices  $y_0, \ldots, y_q$ . Let  $x_0, \ldots, x_p$  be the vertices of  $\sigma$ . Since  $\varphi$  is simplicial and injective the images  $\varphi(x_0), \ldots, \varphi(x_p)$  are distinct and generate the simplex  $\varphi(\sigma)$  by 3.17.2 hence are exactly the vertices  $y_0, \ldots, y_q$  of  $\tau$ . Thus p = q and w.l.o.g.  $\varphi(x_j) = y_j$  for all j. So  $\sigma$  is generated by the  $\varphi^{-1}(y_j) = x_j$ .

# Simplicial approximation

**3.22 Definition (Simplicial Approximation).** [9, 3.2.4] Let K and L be two simplicial complexes,  $f : |K| \to |L|$  be continuous. Then a simplicial mapping  $\varphi : K \to L$  is called SIMPLICIAL APPROXIMATION for f iff for all  $x \in |K|$  we have  $|\varphi|(x) \in \overline{\operatorname{carr}_L(f(x))}$ , i.e.  $f(x) \in \sigma \in L \Rightarrow |\varphi|(x) \in \overline{\sigma}$ . This can be expressed shortly by  $\forall \sigma \in L : |\varphi|(f^{-1}(\sigma)) \subseteq \overline{\sigma}$ . In particular, for every  $x \in |K|$  there is then a simplex  $\sigma \in L$  (namely  $\sigma := \operatorname{carr}_L(f(x))$ ) with  $f(x), |\varphi|(x) \in \overline{\sigma}$ . Note that  $|\varphi|(\overline{\sigma}) = \overline{\varphi(\sigma)}$ .

**3.23 Lemma.** [9, 3.2.5] Let  $\varphi$  be a simplicial approximation of f, then  $|\varphi| \sim f$ .

**Proof.** Connect  $|\varphi|(x)$  to f(x) by the segment in carr<sub>L</sub> f(x)

## **3.24 Example.** [9, 3.2.6]

1. Let  $X := |\dot{\sigma}^2|$ . Then  $X \cong S^1$ . If  $\varphi : K \to K$  is simplicial, then either  $\varphi$  is bijective or not surjective, so it has degree in  $\{\pm 1, 0\}$  by 2.18.3 and 2.6.7. Thus every continuous map  $f : X \to X$  with  $|\deg(f)| > 1$  has no simplicial approximation.

2. For  $f: t \mapsto 4t(1-t)$  from  $[0,1] \to [0,1]$  there is no simplicial approximation  $\varphi: K \to K := \{\langle 0 \rangle, \langle 1 \rangle, \langle 0,1 \rangle\}$ : In fact,  $\operatorname{carr}(j) = \{j\}$  for  $j \in \{0,1\}$  and  $\operatorname{carr}(\frac{1}{2}) = \langle 0,1 \rangle$  hence any such  $\varphi$  must satisfy  $\varphi(0) = \varphi(1) = 0$  and hence  $|\varphi|(\frac{1}{2}) = 0 \notin \overline{\{1\}}$ .

In order to get simplicial approximations we have to refine the triangulation of |K|. This can be done with the following barycentric refinement.

**3.25 Definition (Barycentric Refinement).** [9, 3.2.1] The BARYCENTER  $\hat{\sigma}$  of a q-simplex  $\sigma$  with vertices  $x_i$  is given by



For every simplicial complex K the BARYCENTRIC REFINEMENT K' is given by all simplices having as vertices the barycenter of strictly increasing sequences of faces of a simplex in K, i.e.

$$K' := \{ \langle \hat{\sigma}_0, \dots, \hat{\sigma}_q \rangle : \sigma_0 < \dots < \sigma_q \in K \}.$$

**3.26 Theorem.** [9, 3.2.2] For every simplicial complex K the barycentric refinement K' is a simplicial complex of the same dimension d and the same underlying space but with  $\max\{d(\sigma') : \sigma' \in K'\} \leq \frac{d}{d+1}\max\{d(\sigma) : \sigma \in K\}$ . Here  $d(\sigma) := \sup\{|x - y| : x, y \in \sigma\}$  denotes the diameter of  $\sigma$ .

**Proof.** If  $\sigma_0 < \cdots < \sigma_q$ , then the barycenter  $\hat{\sigma}_0, \ldots, \hat{\sigma}_q$  all lie in  $\bar{\sigma}_q$  and are in general position: In fact, let  $\sigma_i = \langle x_0, \ldots, x_{n_i} \rangle$  with  $i \mapsto n_i$  strictly increasing and

$$x = \sum_{i=0}^{q} \lambda_i \hat{\sigma}_i = \sum_i \lambda_i \frac{1}{n_i + 1} \sum_{j=0}^{n_i} x_j = \sum_j x_j \underbrace{\sum_{\substack{i=0\\n_i \ge j}}^{i} \lambda_i \frac{1}{n_i + 1}}_{=:\mu_j} \text{ with } \sum_i \lambda_i = 1.$$

Then

$$\sum_{j} \mu_{j} = \sum_{j} \sum_{\substack{i \\ n_{i} \ge j}} \lambda_{i} \frac{1}{n_{i} + 1} = \sum_{i} \sum_{\substack{j \\ n_{i} \ge j}} \lambda_{i} \frac{1}{n_{i} + 1} = \sum_{i} \lambda_{i} = 1.$$

Since the  $x_i$  are in general position the  $\mu_j$  are uniquely determined and thus also the  $\lambda_i = (n_i + 1) \left( \mu_{n_i} - \sum_{i' < i} \lambda_{i'} \frac{1}{n_{i'} + 1} \right).$ 

We show now by induction on  $q := \dim(\sigma)$  that for  $\sigma \in K$  the set  $\{\sigma' \in K' : \sigma' \subseteq \sigma\}$ is a disjoint partition of  $\sigma$ : For (q = 0) this is obvious. For (q > 0) and  $x \in \sigma \setminus \{\hat{\sigma}\}$  the line through  $\hat{\sigma}$  and x meets  $\dot{\sigma}$  in some point  $y_x$ . By induction hypothesis  $\exists \tau' \in K' :$  $y_x \in \tau'$ . Thus  $y_x$  is a positive convex combination of  $\hat{\tau}_0, \ldots, \hat{\tau}_j$  with  $\tau_0 < \cdots < \tau_j$ . Hence x is a positive convex combination of  $\hat{\tau}_0, \ldots, \hat{\tau}_j$ ,  $\hat{\sigma}$ . Finally, let x', y' be two vertices of some  $\sigma' \in K'$ , i.e.  $x' = \frac{1}{r+1}(x_0 + \cdots + x_r)$  and  $y' = \frac{1}{s+1}(x_0 + \cdots + x_s)$  with  $r < s \le q \le d$  for some simplex  $\sigma = \langle x_0, \ldots, x_q \rangle \in K$ . Then

$$|x' - y'| \le \frac{1}{r+1} \sum_{i} |x_i - y'| \le \max\{|x_i - y'| : i\}$$
$$|x_i - y'| \le \frac{1}{s+1} \sum_{j \ne i} |x_i - x_j| \le \frac{s}{s+1} d(\sigma) \le \frac{d}{d+1} d(\sigma). \quad \Box$$

**3.27 Corollary.** [9, 3.2.3] For every simplicial complex K and every  $\varepsilon > 0$  there is an iterated barycentric refinement  $K^{(q)}$  with  $d(\sigma) < \varepsilon$  for all  $\sigma \in K^{(q)}$ .

**Proof.** 
$$\left(\frac{d}{d+1}\right)^q \to 0 \text{ for } q \to \infty.$$

**3.28 Definition. Star of a Vertex.** [9, 3.2.8] Let  $\xi = \{x\}$  be a vertex of K. Then the STAR of  $\xi$  in K is defined as

$$\operatorname{st}_{K}(\xi) := \bigcup_{\xi \leq \sigma \in K} \sigma = \Big\{ y \in |K| : x \in \overline{\operatorname{carr}_{K}(y)} \Big\},$$

i.e.  $y \in \operatorname{st}_K(\xi) \Leftrightarrow \exists (!)\sigma : \xi \leq \sigma \text{ and } y \in \sigma \Leftrightarrow \xi = \{x\} \leq \operatorname{carr}_K(y) \Leftrightarrow x \in \overline{\operatorname{carr}_K(y)}.$ 



**3.29 Lemma.** [9, 3.2.9] The family of stars of vertices of K form an open covering of |K|. For every open covering there is a refinement by the stars of some iterated barycentric refinement  $K^{(q)}$  of K.

**Proof.** For vertices  $\xi = \{x\}$  of K let  $K_x := \{\sigma \in K : x \text{ is not vertex of } \sigma\}$ . Then  $K_x$  is a subcomplex and hence  $\operatorname{st}_K(\xi) = |K| \setminus |K_x|$  is open.

If  $\sigma \in K$  and x is a vertex of  $\sigma$  then obviously  $\sigma \subseteq \operatorname{st}_K(\{x\})$  and hence the stars form a covering.

By the Lebesgue-covering lemma (see [2, 3.3.3] or [3, 5.1.5]), there is an  $\delta > 0$ such that every set of diameter less then  $\delta$  is contained in some  $U \in \mathcal{U}$ . Choose by  $\boxed{3.26}$  a barycentric refinement  $K^{(q)}$ , such that  $d(\sigma) < \frac{\delta}{2}$  for all  $\sigma \in K^{(q)}$ . For every  $y \in \operatorname{st}_{K^{(q)}}(\{x\})$  we have  $d(y, x) \leq \max\{d(\sigma) : \sigma\}$  hence  $d(\operatorname{st}_{K^{(q)}}(\{x\})) \leq$  $2 \max\{d(\sigma) : \sigma\} < \delta$ , and thus the stars form a refinement of  $\mathcal{U}$ .

**3.30 Corollary. Simplicial Approximation.** [9, 3.2.7] For every continuous map  $f : |K| \to |L|$  there a simplicial approximation  $\varphi : K^{(q)} \to L$  of f for some iterated barycentric refinement  $K^{(q)}$ .

**Proof.** Let q be chosen so large, that by 3.29 the stars of  $K^{(q)}$  form a refinement of the open covering  $\{f^{-1}(\operatorname{st}_L(\eta)) : \eta = \{y\} \in L\}$ . For sake of simplicity

we write K instead of  $K^{(q)}$ . Thus for every vertex  $\xi \in K$  we may choose a vertex  $\varphi(\xi) \in L$  with  $f(\operatorname{st}_K(\xi)) \subseteq \operatorname{st}_L(\varphi(\xi))$ . For  $\sigma \in K$  with vertices  $x_0, \ldots, x_p$  define  $\varphi(\sigma)$  to be the simplex generated by the  $\varphi(x_i)$ . We have to show that this simplex belongs to L. Let  $x \in \sigma$  be any point in  $\sigma$ . Since  $\sigma \subseteq \bigcap_i \operatorname{st}_K(\{x_i\})$  we get  $f(x) \in f(\sigma) \subseteq f(\bigcap_i \operatorname{st}_K(\{x_i\})) \subseteq \bigcap_i f(\operatorname{st}_K(\{x_i\})) \subseteq \bigcap_i \operatorname{st}_L(\varphi(\{x_i\}))$ . Thus  $f(x) \in \operatorname{st}_L(\varphi(\{x_i\}))$ , i.e.  $\varphi(\{x_i\}) \leq \operatorname{carr}_L(f(x)) =: \tau \in L$ , for all *i*. Hence  $|\varphi|(x) \in \varphi(\sigma) := \langle \varphi(x_0), \ldots, \varphi(x_p) \rangle \leq \tau \in L$  and  $\varphi$  is a simplicial approximation of f.

**3.31 Corollary.** [9, 3.2.10] Let X and Y be polyhedra. Then [X, Y] is countable.

#### **3.32 Remark.** [9, 3.2.11]

We obtain a simplicial approximation  $\chi : K' \to K$  of id :  $|K'| \to |K|$  by choosing for every vertex  $\hat{\sigma} \in K'$  a vertex  $\chi(\hat{\sigma})$  of  $\sigma$ . Let  $\hat{\sigma}_0, \ldots, \hat{\sigma}_p$  be the vertices of some simplex  $\sigma' \in K'$  with  $\sigma_0 < \cdots < \sigma_p$  and hence  $\sigma' \subseteq \sigma_p$ . Then the  $\chi(\hat{\sigma}_j)$  are vertices of  $\sigma_j \leq \sigma_p$  and hence they generate a face of  $\sigma_p \in K$ . Thus  $\chi$  extends to a simplicial map.

Let  $x \in \sigma'$ . Then  $|\chi|(x) \in \chi(\sigma') \subseteq \overline{\sigma_p} = \overline{\operatorname{carr}_K(x)}$ , hence  $\chi$  is a simplicial approximation of id.

Let  $\sigma$  be any q-simplex of K. Then there exists a unique simplex  $\sigma' \subseteq \sigma$  which is mapped to  $\sigma$  and all other  $\sigma' \subseteq \sigma$  are mapped to true faces of  $\sigma$ .

**Proof.** We use induction on q. For q = 0 this is obvious, since  $\chi$  is the identity. If q > 0 and  $x := \chi(\hat{\sigma})$  let  $\tau$  be the face of  $\sigma$  opposite to x. By induction hypothesis there is a unique  $\tau' \subseteq \tau$  of K' which is mapped to  $\tau$ . But then the simplex  $\sigma'$  generated by  $\tau'$  and  $\hat{\sigma}$  is the unique simplex mapped to  $\sigma$ : In fact, any simplex contained in  $\sigma$  with vertices  $\hat{\sigma}_0, \ldots, \hat{\sigma}_r$  that is mapped via  $\chi$ to  $\sigma$  has to satisfy  $\sigma_0 < \cdots < \sigma_r \leq \sigma$ . Hence  $r \leq$ dim $(\sigma)$ , and consequently  $r = \dim(\sigma)$  and  $\sigma_r = \sigma$ . Since  $\chi(\hat{\sigma}) = x$  we have that  $\chi(\hat{\sigma}_0), \ldots, \chi(\hat{\sigma}_{r-1})$ generate  $\tau$  and thus  $\tau'$  is the simplex with vertices  $\hat{\sigma}_0, \ldots, \hat{\sigma}_{r-1}$ .



#### Freeing by deformations

**3.33 Proposition.** [9, 3.3.2] Let K be a simplicial complex and  $n > \dim K$ . Then every  $f : |K| \to S^n$  is 0-homotopic. In particular, this is true for  $K := \dot{\sigma}^{k+1}$  with  $n > k = \dim K$ .

**Proof.** By 3.30 there exists a simplicial approximation  $\varphi$  of  $f : |K| \to |\dot{\sigma}^{n+1}|$  for some iterated barycentric subdivision. Then  $|\varphi| : |K| \to |\dot{\sigma}^{n+1}|$  cannot be surjective (since  $n > \dim K$ ) and hence  $f \sim |\varphi|$  is 0-homotopic since  $\dot{\sigma}^{n+1} \setminus \{*\}$  is contractible.

**3.34 Theorem. Freeing of a point.** [9, 3.3.3] Let (K, L) be a simplicial pair and  $e^n$  be an n-cell with  $n > \dim K$ . Then every  $f_0 : (|K|, |L|) \to (e^n, e^n \setminus \{0\})$  is homotopic relative |L| to a mapping  $f_1 : |K| \to e^n \setminus \{0\}$ . **Proof.** We first show this result for  $(|K|, |L|) = (D^k, S^{k-1})$ . By 2.40.6 we have  $e^n \setminus \{0\} \sim S^{n-1}$ . Hence  $f_0|_{S^{k-1}} : S^{k-1} \to e^n \setminus \{0\}$  is 0-homotopic by 3.33. By 2.30 this homotopy gives an extension  $f_1 : D^k = C(S^{k-1}) \to e^n \setminus \{0\}$ . Consider a mapping  $h : (D^k \times I)^{\cdot} \to e^n$  which is this extension on the top, and is  $f_0$  on the bottom and on  $S^{k-1} \times \{t\}$  for all  $t \in I$ . Since  $e^n$  is contractible this mapping h is 0-homotopic by 2.6.6 and hence extends to  $C((D^k \times I)^{\cdot}) \cong D^k \times I$  by 2.30. This extension is the desired homotopy.

For the general case we proceed by induction on the number of cells in  $K \setminus L$ . For K = L the homotopy is constant  $f_0$ . So let  $K \supset L$  and take  $\sigma \in K \setminus L$  of maximal dimension. Then  $M := K \setminus \{\sigma\} \supseteq L$  is a simplicial complex. Obviously  $|M| \cup \overline{\sigma} = |K|$  and  $|M| \cap \overline{\sigma} = \dot{\sigma}$ . Consider the diagram



By induction hypothesis we have the required homotopy (1) relative |L| on |M|. Since  $(\bar{\sigma}, \dot{\sigma})$  has HEP by 2.33.1, we may extend its restriction (1') to  $\dot{\sigma}$  to a homotopy (2) on  $\bar{\sigma}$  with initial value  $f_0$ . The union of these two homotopies (1) and (2) gives a homotopy  $h_t$  rel. |L| indicated by arrow (3) which satisfies  $h_0 = f_0$  and  $h_1(|M|) \subseteq e^n \setminus \{0\}$ . By the special case treated above, there is a homotopy  $g_t$  relative  $\dot{\sigma}$  with  $g_0 = h_1|_{\bar{\sigma}} : (\bar{\sigma}, \dot{\sigma}) \to (e^n, e^n \setminus \{0\})$  and  $g_1(\bar{\sigma}) \subseteq e^n \setminus \{0\}$ . Let  $f_1 := h_1|_{|M|} \cup g_1$ . Then  $f_1(|K|) \subseteq e^n \setminus \{0\}$  and  $f_0 = h_0 \sim h_1 = h_1|_{|M|} \cup g_0 \sim h_1|_{|M|} \cup g_1 = f_1$  relative |L|.

**3.35 Theorem. Freeing of a cell.** [9, 3.3.4] Let Z be obtained from gluing an *n*-cell  $e^n$  to a space Y and k < n. Then every  $f : (D^k, S^{k-1}) \to (Z, Y)$  is homotopic relative  $S^{k-1}$  to a mapping  $f_1 : D^k \to Y$ .



**Proof.** We consider a triangulation K of  $D^k$  such that  $S^{k-1}$  corresponds to a subcomplex L. For  $0 \in e^n \subseteq Z$  we consider the subcomplexes

$$K_0 := \left\{ \sigma \in K : f(\bar{\sigma}) \subseteq Z \setminus \{0\} \right\} = \left\{ \sigma \in K : \bar{\sigma} \subseteq f^{-1}(Z \setminus \{0\}) \right\} \supseteq L \text{ and}$$
$$K_1 := \left\{ \sigma \in K : f(\bar{\sigma}) \subseteq e^n \right\} = \left\{ \sigma \in K : \bar{\sigma} \subseteq f^{-1}(e^n) \right\}$$

By passing to a appropriate iteration (again denoted K) of barycentric subdivisions, we may assume that  $K = K_0 \cup K_1$  by 3.29.

### Now consider the diagram



By 3.34 there exists a mapping (1) homotopic to  $f|_{|K_1|}$  relative  $|K_1 \cap K_0|$ . Gluing the homotopy with the  $f|_{|K_0|}$  gives a homotopy relative  $|K_0|$  to a mapping (2). Composing with the retraction r (homotopic to id relative Y) from 2.42 gives the desired mapping  $f_1 : D^k \to Y$  homotopic to f relative |L|. Remark that the triangle on top, as those above (1) and (2) commute only up to homotopy.

## 4. CW-Spaces

## Basics

**4.1 Definition.** [9, 4.1.3] A CW-COMPLEX is a Hausdorff space X together with a partition  $\mathcal{X}$  into cells e, such that the following properties hold:

- (C1) For every *n*-cell  $e \in \mathcal{X}$  there exists a continuous so-called CHARACTERISTIC MAP  $\chi^e : D^n \to X$ , which restricts to a homeomorphism from  $\overset{\circ}{D}^n$  onto eand which maps  $S^{n-1}$  into the n-1-skeleton  $X^{n-1}$  of X, which is defined to be the union of all cells of dimension less than n in  $\mathcal{X}$ .
- (C2) The closure  $\bar{e}$  of every cell meets only finitely many cells.
- (W) X carries the final topology with respect to  $\bar{e}$  for all cells  $e \in \mathcal{X}$ .

A CW-SPACE is a Hausdorff-space X, which admits a CW-complex  $\mathcal{X}$  (which is called CW-DECOMPOSITION of X).

Remark that if  $\mathcal{X}$  is finite ( $\mathcal{X}$  is then called finite CW-complex), then the conditions (C2) and (W) are automatically satisfied.

If  $X = X^n \neq X^{n-1}$  then the CW-complex is said to be of dimension n. If  $X \neq X^n$  for all n, then it is said to be of infinite dimension.

Remark that, since the image  $\chi(D^n)$  of the *n*-ball under a characteristic map is compact, it coincides with  $\bar{e}$  and  $\chi: D^n \to \bar{e}$  is a quotient mapping. So  $\dot{e} := \bar{e} \setminus e =$  $\chi(D^n) \setminus \chi(\mathring{D}^n) \subseteq \chi(D^n \setminus \mathring{D}^n) = \chi(S^{n-1})$  and conversely  $\chi(S^{n-1}) \subset \chi(D^n) \subseteq \bar{e}$  and  $\chi(S^{n-1}) \subseteq X^{n-1} \subseteq X \setminus e$ , thus  $\dot{e} = \chi(S^{n-1})$  and  $\chi$  is a relative homeomorphism  $(D^n, S^{n-1}) \to (\bar{e}, \dot{e}).$ 



**4.2 Example.** [9, 4.1.4] For every simplicial complex K the underlying space |K| is a finite CW-complex, the cells being the simplices of K and the characteristic maps the inclusions  $\overline{e} \subseteq |K|$ .

The sphere  $S^n$  is a CW-complex with one 0-cell  $e^0$  and one *n*-cell  $e^n$ , in particular the boundary  $\dot{e} = \bar{e} \setminus e$  of an *n*-cell, needn't be a sphere in contrast to the situation for simplicial complexes.

The one point union of spheres is a CW-space with one 0-cell and for each sphere a cell of the same dimension.

 $S^1 \vee S^2$  can be made in a different way into a CW-complex by taking a point  $e^0 \in S^1$  different from the base point. Then  $S^1 = e^0 \cup e^1$  and  $S^1 \vee S^2 = e^0 \cup e^1 \cup e^2$ . But the boundary  $\dot{e}^2$  of the two-cell is not a union of cells.

The compact surfaces of genus g are all CW-complexes with one 0-cell and one 2-cell and 2g 1-cells (in the orientable case) and g 1-cells (in the non-orientable case), see 1.101.

The projective spaces  $\mathbb{P}^n$  are CW-complexes with one cell of each dimension from 0 to n, see 1.97.

**4.3 Definition.** [9, 4.1.5] For a subset  $\mathcal{Y}$  of a CW-decomposition  $\mathcal{X}$  of a space X the underlying space  $Y := \bigcup \{e : e \in \mathcal{Y}\}$  is called CW-SUBSPACE and  $\mathcal{Y}$  is called CW-SUBCOMPLEX, iff  $\mathcal{Y}$  is a CW-decomposition of Y with the trace topology. In this situation (X, Y) is called CW-PAIR.

Let us first characterize finite CW-subcomplexes:

**4.4 Lemma.** Let  $\mathcal{Y}$  be a finite subset of a CW-decomposition  $\mathcal{X}$  of a space X. Then  $\mathcal{Y}$  forms a CW-subcomplex iff  $Y := \bigcup \{e : e \in \mathcal{Y}\}$  is closed. Cf. 3.14.

**Proof.** ( $\Rightarrow$ ) If  $\mathcal{Y}$  is a CW-subcomplex, then for every cell  $e \in \mathcal{Y}$ , there is a characteristic map  $\chi : D^n \twoheadrightarrow \bar{e}^Y$ . Hence  $\bar{e}^Y$  is compact and thus coincides with the closure of e in X, so the finite union  $Y = \bigcup \{\bar{e} : e \in \mathcal{Y}\}$  is closed.

( $\Leftarrow$ ) Since Y is closed the characteristic maps for  $e \in \mathcal{Y} \subseteq \mathcal{X}$  have values in Y and hence are also characteristic maps with respect to  $\mathcal{Y}$ . The other properties are obvious.

**4.5 Lemma.** [9, 4.1.9] Every compact subset of a CW-complex is contained in some finite subcomplex. In particular a CW-complex is compact iff it is finite.

**Proof.** Let X be a CW-complex. We first show by induction on the dimension of X that the closure  $\bar{e}$  of every cell is contained in a finite subcomplex. Assume this is true for all cells of dimension less than n and let e be an n-cell. By (C2) the boundary  $\dot{e}$  meets only finitely many cells, each of dimension less than n. By induction hypotheses each of these cells is contained in some finite subcomplex  $X_i$ . Then union of these complexes is again a complex, by 4.4. If we add e itself to this complex, we get the desired finite complex.

Let now K be compact. For every  $e \in \mathcal{X}$  with  $e \cap K \neq \emptyset$  choose a point  $x_e$  in the intersection. Every subset  $A \subseteq K_0 := \{x_e : e \cap K \neq \emptyset\} \subseteq K$  is closed, since it meets any  $\overline{e}$  only in finitely many points by (C2). Hence  $K_0$  is a discrete compact subset, and hence finite, i.e. K meets only finitely many cells. Since every  $\overline{e}$  is contained in a finite subcomplex, we have that K is contained in the finite union of these subcomplexes.

The last statement of the lemma is now obvious.

**4.6 Corollary.** Every CW-complex carries the final topology with respect to its finite subcomplexes and also with respect to its skeletons.

**Proof.** Since the closure  $\overline{e}$  of every cell e is contained in a finite subcomplex by [4.5] and every finite subcomplex is contained in some skeleton  $X^n$ , these families are confinal to  $\{\overline{e} : e \in \mathcal{X}\}$ . Furthermore, the inclusion of each of its spaces into X is continuous (for the final topology on X induced by the  $\overline{e}$  by property (W)). Hence these families induce the same topology. (Let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  be two families of mappings into a space X, and assume  $\mathcal{F}_2$  is confinal to  $\mathcal{F}_1$ , i.e. for every  $f_1 \in F_1$ there is some  $f_2 \in \mathcal{F}_2$  and a map h such that  $f_1 = f_2 \circ h$ . Let  $X_j$  denote the space X with the final topology induced by  $\mathcal{F}_j$ . Then the identity from  $X_1 \to X_2$ is continuous, since for every  $f_1 \in \mathcal{F}_1$  we can write  $\mathrm{id} \circ f_1 = f_2 \circ h$ )

Now we are able to extend 4.4 to infinite subcomplexes.

**4.7 Proposition.** Let  $\mathcal{X}$  a CW-decomposition of X and let  $\mathcal{Y}$  be a subset of  $\mathcal{X}$  and  $Y := \bigcup \{e : e \in \mathcal{Y}\}$ . Then the following statements are equivalent:

- 1.  $\mathcal{Y}$  is a CW-decomposition of Y with the trace topology;
- 2. Y is closed in X;
- 3. For every cell  $e \in \mathcal{Y}$  we have  $\bar{e} \subseteq Y$ .

**Proof.**  $(2\Rightarrow3)$  is obvious.

 $(1\Rightarrow3)$  follows, since the closure  $\bar{e}^Y$  in Y is compact and hence equals  $\bar{e} := \bar{e}^X$ .

For the converse directions we show first:

(3)  $\Rightarrow$  If  $A \subseteq Y$  has closed trace on  $\bar{e} := \bar{e}^X$  for each  $e \in \mathcal{Y}$ , then A is closed in X. By 4.6 it suffices to show that the trace on every finite CW-subcomplex  $\mathcal{X}_0 \subseteq \mathcal{X}$  is closed. Since there are only finitely many cells  $e_i$  in  $\mathcal{X}_0 \cap \mathcal{Y}$  and for these  $\bar{e}_i \subseteq X_0 \cap Y$  by 4.4 and (3), we get

$$X_0 \cap A = X_0 \cap Y \cap A = \left(\bigcup \bar{e}_i\right) \cap A = \bigcup (\bar{e}_i \cap A),$$

which is closed.

 $(3\Rightarrow 2)$  by taking A = Y in the previous claim.

 $(3\Rightarrow1)$  The previous claim shows the condition (W) for  $\mathcal{Y}$ . The other conditions for being a CW-complex are obvious since  $\bar{e}^X = \bar{e}^Y$ .

**4.8 Corollary.** [9, 4.1.6] Intersections and unions of CW-complexes are CW-complexes. Connected components and topological disjoint unions of CW-complexes are CW-complexes. If  $\mathcal{E} \subseteq \mathcal{X}$  is family of n-cells, then  $X^{n-1} \cup \bigcup \mathcal{E}$  is a CW-complex. Each n-cell e is open in  $X^n$ .

**Proof.** For intersections this follows from  $(1 \Leftrightarrow 2)$  in 4.7. For unions this follows from  $(1 \Leftrightarrow 3)$  in 4.7. That  $X^{n-1} \cup \bigcup \mathcal{E}$  is a CW-complex follows also from  $(1 \Leftrightarrow 3)$  in 4.7. Since  $X^n \setminus e = X^{n-1} \cup \bigcup \{e_1 \neq e : e_1 \text{ an } n\text{-cell in } X^n\}$  is a CW-complex, it is closed by 4.7.2 and hence e is open in  $X^n$ .

The statement on components and topological sums follows, since  $\overline{e}$  is connected and by 4.7 (1 $\Leftrightarrow$ 3).

## **Constructions of CW-spaces**

**4.9 Proposition.** [9, 4.2.9] Let X and Y be two CW-complexes. Then  $X \times Y$  with cells  $e \times f$  for  $e \in \mathcal{X}$  and  $f \in \mathcal{Y}$  satisfies all properties of a CW-complex, with the possible exception of (W). If X or Y is in addition locally compact, then  $X \times Y$  is a CW-complex.

**Proof.** Take the product of the characteristic maps in order to obtain a characteristic map for the product cell.

In order to get the property (W) we have to show that the map  $\bigsqcup_{e,f} \bar{e} \times \bar{f} \to X \times Y$  is a quotient map. Since it can be rewritten as

$$\bigsqcup_{e} \bar{e} \times \bigsqcup_{f} \bar{f} \to \bigsqcup_{e} \bar{e} \times Y \to X \times Y$$

this follows from 1.33 using compactness of  $\bar{e}$  and locally compactness of Y.  $\Box$ 

**4.10 Proposition.** Let (X, A) be a CW-pair. Then  $A \cup X^n$  is obtained from  $A \cup X^{n-1}$  by gluing all n-cells in  $X^n \setminus A$  via the characteristic mappings.

**Proof.** Let  $\mathcal{E}$  be the set of all *n*-cells of  $X \setminus A$  and let characteristic mappings  $\chi^e : D^n \to \overline{e}$  for every  $e \in \mathcal{E}$  be chosen. Let  $\chi := \bigsqcup_{e \in \mathcal{E}} \chi^e : \bigsqcup_{e \in \mathcal{E}} D^n \to \bigcup_{e \in \mathcal{E}} \overline{e} \subseteq X^n$  and  $f := \chi|_{\bigcup_{e \in \mathcal{E}} S^{n-1}}$ . We have to show that the rectangle in



is a push-out. So let  $g^{n-1}: A \cup X^{n-1} \to Z$  and  $g^e: D^n \to Z$  be given, such that  $g^{n-1} \circ \chi^e|_{S^{n-1}} = g^e|_{S^{n-1}}$ . Then  $g: A \cup X^n \to Z$  given by  $g|_{A \cup X^{n-1}} = g^{n-1}$  and  $g|_e = g^e \circ (\chi^e|_{D^n}^{\circ})^{-1}$  is the unique mapping making everything commutative. It is continuous, since on  $\bar{e}$  it equals  $g^{n-1}$  if  $e \subseteq A \cup X^{n-1}$  and composed with the quotient-mapping  $\chi^e: D^n \to \bar{e}$  it equals  $g^e$  for the remaining e.  $\Box$ 

Now we give an inductive description of CW-spaces.

**4.11 Theorem.** [9, 4.2.2] A space X is a CW-complex iff there are spaces  $X^n$ , with  $X^0$  discrete,  $X^n$  is formed from  $X^{n-1}$  by gluing n-cells and X is the limit of the  $X^n$  with respect to the natural inclusions  $X^n \hookrightarrow X^{n+1}$ .

**Proof.** ( $\Rightarrow$ ) We take  $X^n$  to be the *n*-skeleton. Then X carries the final topology with respect to the closed subspaces  $X^n$  and  $X^0$  is discrete by 4.6. Taking  $A := \emptyset$  in 4.10 we get that  $X^n$  can be obtained from  $X^{n-1}$  by gluing all the n-cells via their corresponding characteristic maps restricted to the boundary spheres.

( $\Leftarrow$ ) We first show by induction that  $X^n$  is a CW-complex, with n - 1-skeleton  $X^{n-1}$  and the cells which have been glued to  $X^{n-1}$  as *n*-cells.

For  $X^0$  being a discrete space this is obvious. Since  $X^n$  is obtained from  $X^{n-1}$  by gluing *n*-cells we have

 $X^n$  is obtained from  $X^{n-1}$  by gluing *n*-cells we have that  $X^n$  is Hausdorff by 1.92 and is as set the disjoint union of the closed subspace  $X^{n-1}$ , which is a CW-complex by induction hypothesis, and the homeomorphic image  $\bigcup_e e$  of  $\bigsqcup_e D^n \setminus \bigsqcup_e S^{n-1} = \bigsqcup_e \mathring{D}^n$ .



As characteristic mappings for the *n*-cells e we may use  $p|_{D^n}$ , since it induces a homeomorphism  $\mathring{D}^n \to e$  and it maps  $S^{n-1}$  to  $f(S^{n-1}) \subseteq X^{n-1}$ , which is compact and hence contained in a finite subcomplex of  $X^{n-1}$ . The condition (W) follows, since  $X^n$  carries by construction the final topology with respect to  $X^{n-1}$  and  $p: \bigsqcup D^n \to X^n$ , and  $\bigsqcup D^n$  carries the final topology with respect to the inclusion of the summands  $D^n \subseteq \bigsqcup_e D^n$ .

The inductive limit  $X := \underline{\lim}_n X^n$  now obviously satisfies all axioms of a CWcomplex only Hausdorffness is to be checked. So let x, y be different points in X. We they lie in some  $X^n$  and we find open disjoint neighborhoods  $U^n$  and  $V^n$ in  $X^n$ . We construct open disjoint neighborhoods  $U^k$  and  $V^k$  in  $X^k$  with  $k \ge n$  inductively. In fact, take  $U^k := U^{k-1} \cup p\left(r^{-1}(U^{k-1})\right)$ , where  $r : \bigsqcup D^k \setminus \{0\} \to \bigsqcup X^k$  is the retraction from 2.42. Then  $U^k$  is the image of the open and saturated set  $U^{k-1} \sqcup r^{-1}(U^{k-1}) \subseteq X^{k-1} \bigsqcup D^k$  and hence open, and  $U^k \cap X^{k-1} = U^{k-1}$ . Proceeding the same way with  $V^k$  gives the required disjoint open sets  $U := \bigcup U^k$  and  $V := \bigcup V^k$ .

**Example.** In general gluing a CW-pair to a CW-space does not give a CW-space. Consider for example a surjective map  $f: S^1 \to S^2$ . Then the boundary  $\dot{e} = S^2$  of  $e := (D^2)^o$  is not contained in any 1-dimensional CW-complex.

So we define

**4.12 Definition.** [9, 4.2.4] A continuous map  $f: X \to Y$  between CW-complexes is called CELLULAR iff it maps  $X^n$  into  $Y^n$  for all n.

**4.13 Lemma.** Let  $f : X \supseteq A \to Y$  be given and let  $Y' \subseteq Y$  and  $X' \subseteq X$  be two closed subspaces, such that  $f(A \cap X') \subseteq Y'$ . Then the canonical mapping  $Y' \cup_{f'} X' \to Y \cup_{f} X$  is a closed embedding, where f' := f|A' with  $A' := A \cap X'$ .

#### Proof.

Consider the commutative diagram:



The dashed arrow  $\iota$  exists by the push-out property of the upper left square.

Since  $Y' \cup_{f'} X' = Y' \sqcup (X' \setminus A')$  as sets, we get that  $\iota$  is the inclusion  $Y' \sqcup (X' \setminus A') \subseteq Y \sqcup (X \setminus X \cap A) =$  $Y \sqcup (X \setminus A)$  and hence injective. Now let  $B \subseteq Y' \cup_{f'} X'$  be closed, i.e.  $B = B_1 \sqcup B_2$  with  $B_1 \subseteq Y'$ closed and  $B_2 \subseteq X' \setminus A'$  such that  $p^{-1}(B) = B_2 \cup (f')^{-1}(B_1)$  is closed in X'.

In order to show that  $\iota(B) = B_1 \sqcup B_2 \subseteq Y' \cup (X' \setminus A') \subseteq Y \cup (X \setminus A)$  is closed we only have to show that  $B_2 \cup f^{-1}(B_1)$  is closed in X, which follows from

$$B_2 \cup f^{-1}(B_1) = B_2 \cup \left( (f')^{-1}(B_1) \cup f^{-1}(B_1) \right) = \left( B_2 \cup (f')^{-1}(B_1) \right) \cup f^{-1}(B_1),$$

since  $B_2 \cup (f')^{-1}(B_1) \subseteq X' \subseteq X$  is closed and  $f^{-1}(B_1) \subseteq A \subseteq X$  is closed.  $\Box$ 

**4.14 Theorem.** [9, 4.2.5] Let (X, A) be a CW-pair and  $f : A \to Y$  a cellular mapping into a CW-complex Y. Then  $(Y \cup_f X, Y)$  is a CW-pair with the cells of Y and of  $X \setminus A$  as cells.

**Proof.** We consider the spaces  $Z^n := Y^n \cup_{f_n} X^n$ , where  $f_n := f|_{A^n}$ . Note that  $A^n = A \cap X^n$ . By **4.13** the  $Z^n$  form an increasing sequence of closed subspaces of the Hausdorff space  $Z := Y \cup_f X$ . Obviously  $Z^0$  is discrete and Z carries the final topology induced by all  $Z^n$ . So by **4.11** it remains to show that  $Z^n$  can be obtained from  $Z^{n-1}$  by gluing all *n*-cells of  $Y^n$  and of  $X^n \setminus A^n$ . For this we consider the following commutative diagram:



By 4.10 the following spaces are push-outs of the arrows leading into them:  $Y^n, X^n, Z^n$  and  $A^n \cup X^{n-1}.$ We have  $\operatorname{to}$ show that  $Z^n$ is the push-out of the in- $\text{clusion } Z^{n-1} \ \rightarrow \ \ \,$  $Z^n$ and the union of the two mappings  $\sqcup D^n \to Y^n \to Z^n$  and  $\sqcup D^n \to X^n \to Z^n.$ So let mappings on all the  $D^n$  and on  $Z^{n-1}$ into a space W be given whose composites with the arrows from  $S^{n-1}$  into these spaces are the same.

Then (1), (2), (3), and (4) exist uniquely by the push-out property of the corresponding domains  $Y^n$ ,  $A^n \cup X^{n-1}$ ,  $X^n$  and  $Z^n$ . The map (4) is then the required unique mapping from  $Z^n \to W$ .

**4.15 Corollary.** [9, 4.2.6] Let (X, A) be a CW-pair with  $A \neq \emptyset$ . Then X/A is a CW-space with A as one 0-cell and the image of all cells in  $X \setminus A$ .

**Proof.**  $X/A = \{*\} \cup_f X$  by 1.47, where  $f : A \to \{*\}$  is constant, Now apply 4.14.

**4.16 Corollary.** [9, 4.2.8] Let X be a CW-complex. Then  $X^n/X^{n-1}$  is a join of spheres of dimension n, for each n-cell one.

**Proof.** By 4.15  $X^n/X^{n-1}$  is a CW-space consisting of one 0-cell and all the *n*-cells of X. The characteristic mappings for the *n*-cells into the 0-cell have to be constant and hence  $X^n/X^{n-1} \cong \bigvee_e S^n$  by 1.103.1.

**4.17 Corollary.** [9, 4.2.7] Let  $X_i$  be CW-spaces with base-point  $x_i \in X_i^0$ . Then the join  $\bigvee_i X_i$  is a CW-space.

**Proof.** 
$$\bigvee_i X_i = (\sqcup_i X_i) / \{x_i : i\}$$
 is a CW-space by 4.8 and 4.14.

## Homotopy properties

**4.19 Theorem.** [9, 4.3.2] For every CW-subspace A of a CW-space X we can find a continuous function  $u: X \to I$  s.t.  $A = u^{-1}(0)$  and  $A \hookrightarrow U(A) := u^{-1}(\{t: t < 1\})$  is a SDR. These neighborhoods can be chosen coherently, i.e.  $U(A \cap B) = U(A) \cap U(B)$ . In particular,  $A \hookrightarrow X$  is an NDR.

**Proof.** Let  $X^{-1} := \emptyset$ . By 4.11  $A \cup X^n$  is obtained by glueing the *n*-cells in  $X \setminus A$  to  $A \cup X^{n-1}$ . By 2.42  $A \cup X^{n-1}$  is an SDR in  $A \cup X^n \setminus \bigsqcup_e \{0_e\}$ . Let the

corresponding homotopy be denoted by  $h_t^n$  and the (radial) retraction by  $r^n := h_1^n$ . Note that  $r^n \circ h_t^n = r^n$ .

We first define a function  $u: X \to [0, 1]$  by recursive extension as follows:  $u|_{A \cup X^{-1}} = 0$  and let  $u_n := u|_{A \cup X^n}$  be given by  $u_n|_{A \cup X^{n-1}} = u_{n-1}$  and

$$u_{n}|_{\bar{e}}(\chi^{e}(x)) = \begin{cases} 1 - \|x\| \left(1 - u_{n-1}\left(\chi^{e}\left(\frac{x}{\|x\|}\right)\right)\right) & \text{ für } 0 \neq x \in D^{n} \\ 1 & \text{ für } 0 = 0_{e} \in D^{n} \end{cases}$$

Then  $u_n$  is a well-defined continuous map with  $(u_n)^{-1}(0) = A$  and by 4.6 the same holds for u.

Let  $U(A) := \{x \in X : u(x) < 1\}$  and  $U^n := U(A) \cap (A \cup X^n) = \{x \in A \cup X^n : u_n(x) < 1\}$ . Note that the homotopy  $h_t^n$  on  $A \cup X^n \setminus \bigsqcup_e \{0_e\}$  restricts to a homotopy on  $U^n$ , since with every point  $x \in U^n$  the whole path  $\{h_t^n(x) : t \in I\}$  belongs to  $U^n$ .

By induction on n we construct now homotopies  $H_t^n: U^n \to U^n$ , by

$$H_t^n := \begin{cases} \text{id} & \text{for } t \le \frac{1}{n+1}, \\ h_s^n & \text{for } \frac{1}{n+1} \le t \le \frac{1}{n} \text{ where } s := n \big( t(n+1) - 1 \big) \in [0,1], \\ H_t^{n-1} \circ r^n & \text{for } t \ge \frac{1}{n}. \end{cases}$$

Then  $H_t^n$  is well-defined and  $H_t^n|_{U^{n-1}} = H_t^{n-1}$ , since  $H_t^{n-1} = \text{id}$  for  $t \leq \frac{1}{n}$  and  $r^n|_{A \cap X^{n-1}} = \text{id}$ . The union  $H_t := \bigcup_{n \in \mathbb{N}} H_t^n$  is the required deformation relative A and satisfies also  $H_1 \circ H_t = H_1$  since  $r^n|_{U^{n-1}} = \text{id}$  and hence  $H_1^n \circ H_t^n = H_1^{n-1} \circ r^n \circ H_t^n = H_1^{n-1} \circ r^n = H_1^n$ .

In oder to show that  $A \hookrightarrow X$  is an NDR we consider a new homotopy  $\tilde{h}_t(x) := h_t \max(0,\min(1,2-3u(x)))(x)$ . Then

$$\tilde{h}_t(x) = \begin{cases} x & \text{for } x \in A \text{ or } u(x) \ge \frac{2}{3} \\ h_1(x) & \text{for } t = 1 \text{ and } u(x) \le \frac{1}{3} \end{cases}$$

Thus  $h_t$  extends by id to a homotopy of X and with  $\tilde{u}(x) := \min\{1, 3u(x)\}$  we get the NDR property.

**4.20 Corollary.** [9, 4.3.3] Every point x in a CW-complex X has an open neighborhood, of which it is a SDR.

**Proof.** Let first e be an n-cell. Let  $A := X^n$ . By restricting the homotopy  $H_t$  from [4.19] to  $r^{-1}(e)$  (possible, since  $r \circ H_t = r$ ), where  $r := H_1 : U(A) \to A$  denotes the retraction, we obtain that e is the SDR of a neighborhood. Since every point in a cell e is a SDR of the cell, we obtain the required result by transitivity [2.40.3].  $\Box$ 

**4.21 Theorem. Cellular Approximation.** [9, 4.3.4] For every continuous mapping  $f_0: X \to Y$  between CW-complexes there exists a homotopic cellular mapping. If  $f_0|_A$  is cellular for some CW-subspace A, then the homotopy can be chosen to be relative A.

**Proof.** Again we recursively extend the constant homotopy on A to a homotopy  $h_t^n: A \cup X^n \to Y$  with  $h_1^n$  being cellular. For the induction step we use for each *n*-cell  $e \in X \setminus A$  a characteristic mapping  $\chi: D^n \to \overline{e}$ . By induction hypothesis we get a mapping  $\varphi_0: (D^n \times 0) \cup (S^{n-1} \times I) \to Y$  given by  $f_0 \circ \chi$  on the bottom and  $h_t^{n-1} \circ \chi$  on the mantle  $D^n \times I$  with  $h_1^{n-1} \circ \chi: S^{n-1} \to X^{n-1} \to Y^{n-1} \subseteq Y^n$  Since this domain is a retract in  $D^n \times I$  by 2.33.1 and 2.32 we can extend it to a mapping  $\varphi_0$  on  $D^n \times I$ .

The image  $\varphi_0(D^n \times \{1\})$  is compact and hence contained in a finite CW-complex. Let  $e^{n_1}, \ldots, e^{n_r}$  be the cells of this complex of dimensions  $n_r \geq \cdots \geq n_1 > n$ . Then  $\varphi_0|_{D^n \times \{1\}} : (D^n \times \{1\}, S^{n-1} \times \{1\}) \to (Y^n \cup e^{n_1} \cup \dots \cup e^{n_r}, Y^n) \text{ is well defined. Ap$ plying 3.35 now r-times we can deform  $\varphi_0|_{D^n \times \{1\}}$  successively relative  $S^{n-1} \times \{1\}$ so, that its image avoids  $e^{n_r} \cup \cdots \cup e^{n_1}$ . Let  $\varphi_t$  be the corresponding homotopy. We can extend  $\tilde{\varphi}_1$  via  $\varphi_0$  to a continuous mapping on the boundary  $(D^n \times I)^{\cdot}$ , which is homotopic to  $\varphi_0|_{(D^n \times I)^{\cdot}}$   $S^{n-1}$ relative  $D^n \times \{0\} \cup S^{n-1} \times I$  via  $\varphi_t$ . The pair  $(D^n \times I)$  $I, (D^n \times I)) \cong (D^{n+1}, S^n)$  is a CW-pair and hence has the HEP by 4.19 and  $\varphi_0$  lives on  $D^n \times I$ , so  $\varphi_t$  can be χ extended to  $D^n \times I$  as well by 2.34.1. Now  $\varphi_t$  factors  $h_{*}^{n-1}$  $h_t^n|_{\bar{e}}$ over the quotient mapping  $\chi$  to a homotopy  $h_t^n|_{\bar{e}}$ . The union of the  $h_t^n|_e$  gives the required  $h_t^n$ . 

**4.22 Corollary.** [9, 4.3.5] Let  $f_0, f_1 : X \to Y$  be homotopic and cellular. Then there exists a homotopy  $H : X \times I \to Y$  such that  $H_t(X^n) \subseteq Y^{n+1}$  for all n.

Note that the inclusions of the endpoints in I are homotopic and cellular, but every homotopy has to map that point into the 1-skeleton.

**Proof.** Consider the CW-pair  $(X \times I, X \times \dot{I})$  and the given homotopy  $f: X \times I \to Y$ . Since by assumption its boundary value  $f|_{X \times \dot{I}}$  is cellular, by 4.21 we can find another mapping  $H: X \times I \to Y$ , which is cellular and homotopic to f relative  $X \times \dot{I}$ . Thus H is the required homotopy, since for 0 < t < 1 and every *n*-cell  $e^n$  of X the image  $H(e^n \times \{t\})$  is contained in  $H(e^n \times e^1) \subseteq Y^{n+1}$ .  $\Box$ 

# 5. Fundamental Group

## Basic properties of the fundamental group

**5.1 Definition.** [9, 5.1.1] A path is a continuous mapping  $u : I \to X$ . The CON-CATENATION  $u_0 \cdot u_1$  of two paths  $u_0$  and  $u_1$  is defined by

$$(u_0 \cdot u_1)(t) := \begin{cases} u_0(2t) & \text{for } t \le \frac{1}{2} \\ u_1(2t-1) & \text{for } t \ge \frac{1}{2} \end{cases}.$$

It is continuous provided  $u_0(1) = u_1(0)$ . The INVERSE PATH  $u^{-1} : I \to X$  is given by  $u^{-1}(t) := u(1-t)$ .

Note that concatenation is not associative and the constant path is not a neutral element. The corresponding identities hold only up to reparametrizations.

**5.2 Lemma. Reparametrization.** [9, 5.1.5] Let  $u : I \to X$  be a path and  $f : I \to I$  be the identity on  $\dot{I}$ . Then  $u \sim u \circ f$  rel.  $\dot{I}$ .

**Proof.** A homotopy is given by h(t,s) := u(ts + (1-t)f(s)), see 2.4.

**5.3 Lemma.** [9, 5.1.6]

- 1. Let u, v and w be paths with u(1) = v(0) and v(1) = w(0), then  $(u \cdot v) \cdot w \sim u \cdot (v \cdot w)$  rel. I.
- 2. Let u be path with x := u(0), y := u(1) then  $\text{const}_x \cdot u \sim u \sim u \cdot \text{const}_y$  rel. *I*.
- 3. Let u be a path with x := u(0) and y := u(1). Then  $u \cdot u^{-1} \sim \text{const}_x$  and  $u^{-1} \cdot u \sim \text{const}_y$  rel.  $\dot{I}$ .

**Proof.** In (1) and (2) we only have to reparametrize. In (3) we consider the homotopy, which has constant value on each circle with center  $(\frac{1}{2}, 0)$ .



**5.4 Definition.** [9, 5.1.7] Let  $(X, x_0)$  be a pointed space. Then the FUNDAMENTAL GROUP (or FIRST HOMOTOPY GROUP) is defined by

 $\pi_1(X,x_0):=[(I,\dot{I}),(X,x_0)]\cong[(S^1,\{1\}),(X,\{x_0\})],$ 

where multiplication is given by  $[u] \cdot [w] := [u \cdot w]$ , the neutral element is  $1_{x_0} := [\text{const}_{x_0}]$  and the inverse to [u] is  $[u^{-1}]$ . Both are well-defined by 5.3.

**5.5 Lemma.** [9, 5.1.8] Let  $u: I \to X$  be a path from  $x_0$  to  $x_1$ .

Then  $\operatorname{conj}_{[u]} : \pi_1(X, x_0) \to \pi_1(X, x_1)$  is a group isomorphism, where  $\operatorname{conj}_{[u]} : [v] \mapsto [u^{-1}][v][u] := [u^{-1} \cdot v \cdot u].$ 

**5.6 Lemma.** [9, 5.1.10] Let  $h: I^2 \to I^2$  be like follows:



and a piecewise affine homeomorphism on the interior, e.g.

$$h(t,s) := \begin{cases} (1-2t)(0,0) + 2t \Big( s(0,1) + (1-s)(1,0) \Big) & \text{for } t \le 1/2 \\ (2-2t) \Big( s(0,1) + (1-s)(1,0) \Big) + (2t-1)(1,1) & \text{for } t \ge 1/2 \end{cases}$$

For continuous  $f: (I^2)^{\cdot} \to X$  and  $u_j(s) := f(s, j)$  resp.  $v_j(t) := f(j, t)$  its values on the 4 edges the following statements are equivalent

- 1. There exists a continuous extension of f to  $I^2$ ;
- 2. f is 0-homotopic;
- 3. There exists a continuous extension of  $f \circ h$  to  $I^2$ ;
- 4.  $u_o \cdot v_1 \sim v_0 \cdot u_1$  rel. I.

## Proof.



**5.7 Corollary.** Let X be a topological group (monoid) then  $\pi_1(X, 1)$  is abelian, where 1 denotes the neutral element.

**Proof.** Consider the map  $(t, s) \mapsto u(t) \cdot v(s)$ .

**5.8 Proposition.** [9, 5.1.12] Let  $V : \pi_1(X, x_0) = [(S^1, 1), (X, x_0)] \to [S^1, X]$  be the mapping forgetting the base-points. Then

- 1. [u] is in the image of V iff u(1) can be connected by a path with  $x_0$ .
- 2. V is surjective iff X is path connected.
- 3.  $V(\alpha) = V(\beta)$  iff there exists a  $\gamma \in \pi_1(X, x_0)$  with  $\beta = \gamma^{-1} \cdot \alpha \cdot \gamma$ .
- 4. V is injective iff  $\pi_1(X, x_0)$  is abelian.
- 5. The 'kernel'  $V^{-1}([const_{x_0}])$  of V is trivial.

Warning: Since V is not a group-homomorphism, 5 does not contradict 4.

**Proof.** (1) [u] is in the image of V if u is homotopic to a base point preserving closed path. The homotopy at the base-point gives a path connecting u(1) with  $x_0$ . Conversely any path v from c(1) to  $x_0$  can be used to give a homotopy between u and the base point preserving path  $v^{-1} \cdot u \cdot v$ , cf. 2.34.3, since  $(S^1, 1)$  has HEP by 4.19.

 $(1\Rightarrow 2)$  is obvious.

(3) Let  $\alpha = [u]$  and  $\beta = [v]$ . Then  $V(\alpha) = V(\beta)$  iff u is homotopic to v.

( $\Rightarrow$ ) Let *h* be such a homotopy and w(t) := h(t, 1). Then by 5.6 (1 $\Rightarrow$ 4) we have  $w \cdot v \sim u \cdot w$  rel. *İ*, i.e.  $\gamma \cdot \beta = \alpha \cdot \gamma$  and hence  $\beta = \gamma^{-1} \cdot \alpha \cdot \gamma$  for  $\gamma := [w]$ . ( $\Leftarrow$ ) Let  $\beta = \gamma^{-1} \cdot \alpha \cdot \gamma$  and  $\gamma = [w]$ . Then  $\gamma \cdot \beta = \alpha \cdot \gamma$  and hence  $w \cdot u \sim v \cdot w$  rel. *İ*. Then by 5.6 (1 $\Leftarrow$ 4) we have  $u \sim v$ , i.e.  $V(\alpha) = V(\beta)$ .

 $(3\Rightarrow 4)$ 

 $(\Rightarrow)$  Let  $\alpha, \gamma \in \pi_1(X)$  and  $\beta := \gamma^{-1} \cdot \alpha \cdot \gamma$ . By (3) we have  $V(\alpha) = V(\beta)$  and since V is assumed to be injective we get  $\alpha = \beta$ , i.e.  $\gamma \cdot \alpha = \alpha \cdot \gamma$ .

( $\Leftarrow$ ) Conversely, if  $V(\alpha) = V(\beta)$ , then by (3) there exists a  $\gamma$  with  $\beta = \gamma^{-1} \cdot \alpha \cdot \gamma$ and  $\gamma^{-1} \cdot \alpha \cdot \gamma = \alpha$ , provided  $\alpha$  and  $\gamma$  commute.

 $(3\Rightarrow5)$  Let  $V(\alpha) = [\text{const}_{x_0}] = V(\text{const}_{x_0})$ . By (3) there exists a  $\gamma$  with  $\alpha = \gamma^{-1} \cdot [\text{const}_{x_0}] \cdot \gamma = \gamma^{-1} \cdot \gamma = 1$ .

**5.9 Corollary.** [9, 5.1.13] Let X be path connected. Then the following statements are equivalent:

- 1.  $\pi_1(X, x_0) \cong 1$  for some (any)  $x_0 \in X$ , i.e. every  $u: S^1 \to X$  is 0-homotopic rel.  $\dot{I}$  by 5.3;
- 2.  $[S^1, X] = \{0\}, i.e. every u : S^1 \to X \text{ is } 0\text{-homotopic};$
- 3. Any two paths which agree on the endpoints are homotopic rel. I.

A path connected space satisfying these equivalent conditions is called SIMPLY CONNECTED.

**Proof.**  $(1\Rightarrow 2)$  since  $V: \pi^1(X, x_0) \to [S^1, X]$  is onto by 5.8.2

(2 $\Rightarrow$ 3) From 5.6 (2 $\Rightarrow$ 4) with  $v_j := \text{const}_{x_j}$  follows  $u \cdot \text{const}_{x_0} \sim \text{const}_{x_1} \cdot v$  rel.  $\dot{I}$  and hence  $u \sim v$  rel.  $\dot{I}$  by 5.3.2.

 $(3 \Rightarrow 1)$  is obvious, since  $u \sim \text{const}_{x_0}$ 

**Corollary.** Let X be contractible, then X is simply connected.

**Proof.** By 2.6.6 we get that  $[S^1, X] = \{0\}$  provided X is contractible.

**5.10 Example.** [9, 5.1.9] Let X be a CW-complex without 1-cells, e.g.  $X = S^n$  for n > 1. Then  $\pi_1(X, x_0) = \{1\}$  for all  $x_0 \in X^0$ .

In fact every  $u: (I, \dot{I}) \to (X, x_0)$  is by 4.21 homotopic rel.  $\dot{I}$  to a cellular mapping v, i.e.  $v(I) \subseteq X^1 = X^0$ , hence v is constant.

Note that such an X is path connected iff it has exactly one 0-cell. ( $\Rightarrow$ ) Let  $x_0$  and  $x_1$  be two 0-cells and u be a path between them. By 4.21 u is homotopic to a cellular and hence constant path rel.  $\dot{I}$ , since X has no 1-cells. Thus

## $x_0 = x_1.$

( $\Leftarrow$ ) Since cells are path-connected each point in  $X^n$  can be connected with some point in  $X^{n-1}$  and by induction with the unique point in  $X^0$ .

**5.11 Definition.** [9, 5.1.15] Every  $f : (X, x_0) \to (Y, y_0)$  induces a group homomorphism  $\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0)$  given by  $\pi_1(f)[u] := [f \circ u]$ . Just use that  $u \sim v \Rightarrow f \circ u \sim f \circ v$  and  $f \circ (u \cdot v) = (f \circ u) \cdot (f \circ v)$  to get well-definedness and the homomorphic-property.

**5.12 Corollary.** [9, 5.1.16]  $\pi_1$  is a functor from the category of pointed topological spaces to that of groups, i.e. it preserves identities and commutativity of diagrams.

**Proof.** trivial

5.16

**5.13 Proposition.** [9, 5.1.18]  $\pi_1$  is homotopy invariant.

More precisely: If  $f \sim g$  rel.  $x_0$  then  $\pi_1(f) = \pi_1(g)$ . If  $f \sim g$  then  $\pi_1(g) = \operatorname{conj}_{[u]} \circ \pi_1(f)$ , where u is the path given by the homotopy at  $x_0$ . If  $f: X \to Y$  is a homotopy equivalence then  $\pi_1(f): \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$  is an isomorphism.

**Proof.** If  $f \sim g$  rel.  $x_0$  and  $[v] \in \pi_1(X, x_0)$  then  $f \circ v \sim g \circ v$  rel.  $\dot{I}$ , i.e.  $\pi_1(f)[v] = \pi_1(g)[v]$ .

If h is a free homotopy from f to g, then  $w(t) := h(t, x_0)$  defines a path from  $f(x_0)$  to  $g(x_0)$ . And applying 5.6 (1 $\Rightarrow$ 4) to  $(s, t) \mapsto h(t, v(s))$  we get  $(f \circ v) \cdot w \sim w \cdot (g \circ v)$  rel.  $\dot{I}$ , and hence  $[f \circ v] \cdot [w] = [(f \circ v) \cdot w] = [w \cdot (g \circ v)] = [w] \cdot [g \circ v]$ , i.e.  $\pi_1(g)[v] = [g \circ v] = [w]^{-1} \cdot [f \circ v] \cdot [w] = [w]^{-1} \cdot \pi_1(f)[v] \cdot [w] = (\operatorname{conj}_{[w]} \circ \pi_1(f))([v])$ .

Let now  $f: X \to Y$  be a homotopy equivalence with homotopy inverse  $g: Y \to X$ . Then up to conjugation  $\pi_1(f)$  and  $\pi_1(g)$  are inverse to each other.

## The fundamental group of the circle

**5.15 Proposition.** [9, 5.2.2] The composition deg  $\circ V : \pi_1(S^1, 1) \to [S^1, S^1] \to \mathbb{Z}$  is a group isomorphism.

**Proof.** By 2.17 we have that deg is a bijection. By 5.8 V is surjective since  $S^1$  is path-connected. By 5.7 and 5.8 it is also injective since  $S^1$  is a topological group.

Remains to show that the composite is a group-homomorphism: Recall that deg([u]) is given by evaluating a lifting  $u : (S^1, 1) \to (S^1, 1)$  to a curve  $\tilde{u} : \mathbb{R} \to \mathbb{R}$  with  $\tilde{u}(0) = 0$  and  $\exp(2\pi i \tilde{u}(t)) = u(\exp(2\pi i t))$  at 1. Given  $u, v \in \pi_1(S^1, 1)$  with lifts  $\tilde{u}$  and  $\tilde{v}$ , then the lift of  $u \cdot v$  is given by

$$t \mapsto \begin{cases} \tilde{u}(2t) & \text{for } t \leq \frac{1}{2} \\ \tilde{u}(1) + \tilde{v}(2t-1) & \text{for } t \geq \frac{1}{2}. \end{cases} \square$$

**5.16 Corollary.** [9, 5.2.4]  $\pi(X, x_0) \cong \mathbb{Z}$  for every space X which contains  $S^1$  as DR. In particular this is true for  $\mathbb{C} \setminus \{0\}$ , the Möbius strip, a full torus and the complement of a line in  $\mathbb{R}^3$ .

#### Constructions from group theory

**5.17 Definition.** [9, 5.3.1] We will denote with 1 the NEUTRAL ELEMENT in a given group.

A SUBGROUP of a group G is a subset  $H \subseteq G$ , which is with the restricted group operations itself a group, i.e.  $h_1, h_2 \in H \Rightarrow h_1h_2 \in H, h_1^{-1} \in H, 1 \in H$ .

The SUBGROUP  $\langle X \rangle_{SG}$  generated by a subset  $X \subseteq G$  is defined to be the smallest subgroup of G containing X, i.e.

$$\langle X \rangle_{SG} := \bigcap \{ H : X \subseteq H \le G \} = \Big\{ x_1^{\varepsilon_1} \cdot \dots \cdot x_n^{\varepsilon_n} : x_j \in X, \varepsilon_j \in \{\pm 1\} \Big\}.$$

Given an equivalence relation  $\sim$  on G we can form the quotient set  $G/\sim$  and have the natural mapping  $\pi : G \to G/\sim$ . In order that  $G/\sim$  carries a group structure, for which  $\pi$  is a homomorphism, i.e.  $\pi(x \cdot y) = \pi(x) \cdot \pi(y)$ , we need precisely that  $\sim$ is a CONGRUENCE RELATION, i.e.  $x_1 \sim x_2, y_1 \sim y_2 \Rightarrow x_1^{-1} \sim x_2^{-1}, x_1 \cdot y_1 \sim x_2 \sim y_2$ .

Then  $H := \{x : x \sim e\} = \pi^{-1}(e)$  is a NORMAL SUBGROUP (we write  $H \triangleleft G$ ), i.e. is a subgroup such that  $g \in G$ ,  $h \in H \Rightarrow g^{-1}hg \in H$ . And conversely, for normal subgroups  $H \triangleleft G$  we have that  $x \sim x \cdot h$  for all  $x \in G$  and  $h \in H$  defines a congruence relation  $\sim$  and  $G/H := G/\sim = \{gH : g \in G\}$ . This shows, that normal subgroups are exactly the kernels of group homomorphisms. Every surjective group morphism  $p : G \to G_1$  is up to an isomorphism  $G \to G/\ker p$ .

The NORMAL SUBGROUP  $\langle X \rangle_{NG}$  generated by a subset  $X \subseteq G$  is defined to be the smallest normal subgroup of G containing X, i.e.

$$\langle X \rangle_{NG} := \bigcap \{ H : X \subseteq H \lhd G \} = \Big\{ g_1^{-1} y_1 g_1 \cdots g_n^{-1} y_n g_n : g_j \in G, y_j \in \langle X \rangle_{SG} \Big\}.$$

**5.18 Definition.** Let  $G_i$  be groups. Then the PRODUCT  $\prod_i G_i$  of is defined to be the solution of the following universal problem:



A concrete realization of  $\prod_i G_i$  is the cartesian product with the component-wise group operations.

**5.19 Definition.** Let  $G_i$  be groups. Then the COPRODUCT (FREE PRODUCT)  $\coprod_i G_i$  of is defined to be the solution of the following universal problem:



**Remark.** [9, 5.3.3] A concrete realization of  $\coprod_i G_i$  is constructed as follows. Take the set X of all finite sequences of elements of the disjoint union  $\bigsqcup_i G_i$ . With concatenation of sequences X becomes a monoid, where the empty sequence is the neutral element. Every  $G_i$  is injectively mapped into X by mapping g to the sequence with the single entry g. However this injection is not multiplicative and X is not a group. So we consider the congruence relation generated by  $(g, h) \sim (gh)$  if g, h belong to the same group and  $(1_i) \sim \emptyset$  for the neutral element  $1_i$  of any group  $G_i$ . Then  $X/\sim$  is a group and the composite  $G_i \to X \to X/\sim$  is the required group homomorphism and this object satisfies the universal property of the coproduct.

In every equivalence class of  $X/\sim$  we find a unique representative of the form  $(g_1, \ldots, g_n)$ , with  $g_j \in G_{i_j} \setminus \{1\}$  and  $i_j \neq i_{j+1}$ . Since  $(g_1, \ldots, g_n)$  is just the product of the images of  $g_i \in G_i$  we may write this also as  $g_1 \cdots g_n$ .

**5.20 Definition.** [9, 5.7.8] Let H,  $G_1$ ,  $G_2$  be groups and  $f_j : H \to G_j$  group homomorphisms. Then the PUSH-OUT  $G_1 \coprod_H G_2$  of  $(f_1, f_2)$  is a solution of the following universal problem:



It can be constructed as follows:

$$G_1 \coprod_H G_2 := (G_1 \coprod G_2)/N, \text{ where } N := \left\langle f_1(h) \cdot f_2(h)^{-1} : h \in H \right\rangle_{NT}$$

and where  $g_j$  is given by composing the inclusion  $G_j \to G_1 \coprod G_2$  with the natural quotient mapping  $G_1 \coprod G_2 \to (G_1 \coprod G_2)/N$ .

**5.21 Definition.** [9, 5.6.3] Let G be a group. Then the ABELIZATION  ${}^{ab}G$  of G is an Abelian group being solution of the following universal problem:



where A is an arbitrary Abelian group.

A realization of  ${}^{ab}G$  is given by G/G', where the COMMUTATOR SUBGROUP G' denotes the normalizer generated by all COMMUTATORS  $[g,h] := ghg^{-1}h^{-1}$ . Remark that  $G' = \{[g_1, h_1] \cdots [g_n, h_n] : g_j, h_j \in G\}$ .

**Remark.** From general categorical results we conclude that the product (and more general limits) in the category of Abelian groups is the product (limit) formed in that of all groups. And abelization of a coproduct (more generally a colimit) is just the coproduct (colimit) of the abelizations formed in the category of Abelian groups.

**5.22 Definition.** [9, 5.3.7] Let  $G_i$  be abelian groups. Then the COPRODUCT (DI-RECT SUM)  $^{ab} \coprod_i G_i$  of is defined to be the solution of the following universal problem:



where H is an arbitrary Abelian group.

**Remark.** A concrete realization of  ${}^{ab} \coprod_i G_i$  is given by those elements of  $\prod_i G_i$ , for which almost all coordinates are equal to the neutral element.

**5.23 Definition.** [9, 5.5.3] Let X be a set. Then the FREE GROUP  $\mathcal{F}(X)$  is the universal solution to



where the arrows starting at X are just mappings and  $\tilde{f}$  is a group homomorphism.

**Remark.** [9, 5.5.2] One has  $\mathcal{F}(X) \cong \mathcal{F}(\bigsqcup_{x \in X} \{x\}) \cong \coprod_{x \in X} F(\{x\})$  by a general categorical argument, and  $\mathcal{F}(\{*\}) \cong \mathbb{Z}$ , as is easily seen.

**5.24 Definition.** Let X be a set. Then the FREE ABELIAN GROUP  ${}^{ab}\mathcal{F}(X)$  is the universal solution to



where the arrows starting at X are just mappings and  $\tilde{f}$  is a group homomorphism.

**Remark.** By a general categorical argument we have  ${}^{ab}(\mathcal{F}(X)) \cong {}^{ab}\mathcal{F}(X)$ . And  ${}^{ab}\mathcal{F}(X) \cong {}^{ab}\coprod_x \mathcal{F}(\{x\}) \cong {}^{ab}\coprod_x \mathbb{Z}$ , which are just the finite sequences in  $\mathbb{Z}^X$ .

**5.25 Definition.** [9, 5.6.1] Given a set X and a subset  $R \subseteq \mathcal{F}(X)$  we define

$$\langle X:R\rangle := \mathcal{F}(X)/\langle R\rangle_{NT}$$

to be the GROUP WITH GENERATORS X AND DEFINING RELATIONS R. If  $\langle X : R \rangle \cong$  G, then  $\langle X : R \rangle$  is called REPRESENTATION of the group G.

**5.26 Examples.** One has  $\mathcal{F}(X) := \langle X : \emptyset \rangle$  and  $\mathbb{Z}_n := \langle x : x^n \rangle$ . More generally,  $\coprod_j \langle X_j : R_j \rangle = \langle \bigsqcup X_j : \bigcup_j R_j \rangle$ . Moreover  ${}^{ab} \langle X : R \rangle = \langle X : R \cup \{ [x, y] : x, y \in X \} \rangle$ 

5.27 Remark. [9, 5.8.1] Obviously we have:

- 1.  $\langle X : R \rangle \cong \langle X : R \cup \{r'\} \rangle$  for  $r' \in \langle R \rangle_{NT}$ .
- $2. \ \langle X:R\rangle \cong \langle X\cup\{a\}:R\cup\{a^{-1}w\}\rangle \text{ for } a\notin X \text{ and } w\in \mathcal{F}(X).$

These operations are called Tietze operations.

**5.28 Theorem.** [9, 5.8.2] Two finite representations  $\langle X : R \rangle$  and  $\langle Y : S \rangle$  describe isomorphic groups iff there is a finite sequence of Tietze operations converting one description into the other.

For a proof see [9].

**Remark.** The word problem for finitely presented groups is the problem to determine whether two elements  $w, w' \in \mathcal{F}(X)$  define the same element of  $\langle X : R \rangle$ , or equivalently whether  $w^{-1}w' \in \langle R \rangle_{NT}$ .

The isomorphy problem is to determine whether two finite group representations describe isomorphic groups.

It has been shown that both problems have no algorithmic solution.

#### Group descriptions of CW-spaces

**5.29 Proposition.** [9, 5.2.6] For pointed spaces  $(X_i, x_i)$  we have the following isomorphism  $\pi_1(\prod_i X_i, (x_i)_i) \cong \prod_i \pi_1(X_i, x_i)$ .

**Proof.** Obvious, since  $[(Y, y), (\prod_i X_i, (x_i)_i)] \cong \prod_i [(Y, y), (X_i, x_i)]$ , by composition with the coordinate projections, and since the multiplication of paths in  $\prod_i X_i$  is given component-wise.

**5.30 Proposition.** [9, 5.1.21] Let  $X_0$  be a path component of X and let  $x_0 \in X_0$ . Then the inclusion of  $X_0 \subseteq X$  induces an isomorphism  $\pi_1(X_0, x_0) \cong \pi_1(X, x_0)$ .

**Proof.** Since  $S^1$  and  $S^1 \times I$  is path connected, the paths and the homotopies have values in  $X_0$ .

**5.31 Proposition.** Let  $X_{\alpha}$  be subspaces of X such that every compact set is contained in some  $X_{\alpha}$ . And for any two of these subspaces there is a third one containing both. Let  $x_0 \in X_{\alpha}$  for all  $\alpha$ . Then  $\pi_1(X, x_0)$  is the INDUCTIVE LIMIT of all  $\pi_1(X_{\alpha}, x_0)$ .

**Proof.** Let G be any group and  $f_{\alpha}: \pi_1(X_{\alpha}) \to G$  be group-homomorphisms, such that for every inclusion  $i: X_{\alpha} \subseteq X_{\beta}$  we have  $f_{\beta} \circ \pi_1(i) = f_{\alpha}$ . We have to find a unique group-homomorphism  $f: \pi_1(X) \to G$ , which satisfies  $f \circ \pi_1(i) = f_{\alpha}$  for all inclusions  $i: X_{\alpha} \to X$ . Since every closed curve w in X is contained in some  $X_{\alpha}$ , we have to define  $f([w]_X) := f_{\alpha}([w]_{X_{\alpha}})$ . We only have to show that f is well-defined: So let  $[w_1]_X = [w_2]_X$  for curves  $w_1$  in  $X_{\alpha_1}$  and  $w_2 \in X_{\alpha_2}$ . The image of the homotopy  $w_1 \sim w_2$  is contained in some  $X_{\alpha}$ , which we may assume to contain  $X_{\alpha_1}$  and  $X_{\alpha_2}$ . Thus  $f_{\alpha_1}([w_1]_{X_{\alpha_1}}) = f_{\alpha}([w_1]_{X_{\alpha}}) = f_{\alpha}([w_2]_{X_{\alpha}}) = f_{\alpha_2}([w_2]_{X_{\alpha_2}})$ .  $\Box$ 

5.33 Theorem von Seifert und van Kampen. [9, 5.3.11]

Let X be covered by two open path connected subsets  $U_1$  and  $U_2$  such that  $U_1 \cap U_2$ is path connected and let  $x_0 \in U_1 \cap U_2$ . Then

$$\pi_1(U_1 \cap U_2, x_0) \xrightarrow{i_*^2} \pi_1(U_2, x_0)$$

$$\downarrow^{i_*} \qquad \qquad j_*^2 \downarrow$$

$$\pi_1(U_1, x_0) \xrightarrow{j_*^1} \pi_1(X, x_0)$$

is a push-out, where all arrows are induced by the corresponding inclusions.

**Proof.** Let  $G_j := \pi_1(U_j, x_0)$  für  $j \in \{1, 2\}$ ,  $G_0 := \pi_1(U_1 \cap U_2, x_0)$ ,  $G := \pi_1(U_1 \cup U_2, x_0) = \pi_1(X, x_0)$  and  $\bar{G} := (G_1 \coprod G_2)/N$  with  $g_i : G_i \to \bar{G}$  the push-out, where N is the normal subgroup generated by  $\{i_*^1([u]) \cdot i_*^2([u])^{-1} : [u] \in G_0\}$ . By

the universal property of the push-out there exists a unique group-homomorphism  $\varphi: \bar{G} \to G$  and we only have to show that it is bijective.

**Surjectivity:** Let  $[w] \in \pi_1(X)$ . By the Lebesgue-covering lemma applied to [0, 1] we may take n sufficiently large such that for each  $0 \leq i < n$  we have  $w[t_i, t_{i+1}] \subseteq U_{\varepsilon_i}$  for some  $\varepsilon_i \in \{1, 2\}$  and  $t_i := \frac{i}{n}$ . Let  $w_j$  be the restriction of w to  $[t_j, t_{j+1}]$  and let  $v_i$  be a path from  $x_0$  to  $w(t_i)$  in  $U_{\varepsilon_i} \cap U_{\varepsilon_{i-1}}$ . We may take  $v_0$  and  $v_n$  to be constant  $x_0$ . Let  $u_i := v_i \cdot w_i \cdot v_{i+1}^{-1}$ . Then  $u_i$  is a closed path in  $U_{\varepsilon_i}$  and  $w \sim u_0 \cdot \ldots \cdot u_{n-1}$  in X rel.  $\dot{I}$ . Let  $\bar{g}_i := g_{\varepsilon_i}([u]_{U_{\varepsilon_i}}) \in \bar{G}$ .



Hence

$$[w]_X = [u_0]_X \cdot \ldots \cdot [u_{n-1}]_X = j_*^{\varepsilon_0}([u_0]_{U_{\varepsilon_0}}) \cdot \ldots \cdot j_*^{\varepsilon_{n-1}}([u_{n-1}]_{U_{\varepsilon_{n-1}}})$$
$$= \varphi(g_1) \cdot \ldots \cdot \varphi(g_{n-1}) = \varphi(g_1 \cdot \ldots \cdot g_{n-1}) \in \varphi(\bar{G}).$$

**Injectivity:** Let  $z \in \overline{G} = (G_1 \coprod G_2)/N$  with  $\varphi(z) = 1 = [\text{const}_{x_0}] \in G$ . Then we find closed paths  $u_i$  in  $U_{\varepsilon_i}$  for certain  $\varepsilon_i \in \{1, 2\}$  with  $z = g_{\varepsilon_1}([u_1]_{U_{\varepsilon_1}}) \cdot \ldots \cdot g_{\varepsilon_n}([u_n]_{U_{\varepsilon_n}})$ . Since

$$[\operatorname{const}_{x_0}]_X = \varphi(z) = \varphi\Big(g_{\varepsilon_1}([u_1]) \cdot \ldots \cdot g_{\varepsilon_n}([u_n])\Big)$$
$$= \varphi\Big(g_{\varepsilon_1}([u_1])\Big) \cdot \ldots \cdot \varphi\Big(g_{\varepsilon_n}([u_n])\Big) = [u_1]_X \cdot \ldots \cdot [u_n]_X = [u_1 \cdot \ldots \cdot u_n]_X$$

there is a homotopy  $H: I \times I \to X$  relative I between  $u_1 \cdots u_n$  and  $\operatorname{const}_{x_0}$ . We partition  $I \times I$  into squares Q, such that  $H(Q) \subseteq U_{\varepsilon_Q}$  for certain  $\varepsilon_Q \in \{1, 2\}$ . We may assume that the resulting partition on the bottom edge  $I \times \{0\} \cong I$  is finer than  $0 < \frac{1}{n} < \frac{2}{n} < \ldots < \frac{n}{n} = 1$ . For every vertex k of this partition we choose a curve  $v_k$  connecting  $x_0$  with H(k). If  $H(k) \in U_j$  then we may assume that  $v_k(I) \subseteq U_j$ . If  $H(k) = x_0$ , we may assume that  $v_k$  is constant. For every edge c of such a square Q we define the curve closed curve  $u_c := v_{c(0)} \cdot (H \circ c) \cdot v_{c(1)}^{-1}$  through  $x_0$ . Since  $u_c$  is contained in some  $U_j$  we may consider  $[u_c]_{U_j}$  and its image  $\bar{c} := g_j([u_c]_{U_j}) \in \bar{G}$ . This is well defined, since if  $u_c$  is contained in  $U_1 \cap U_2$  then  $[u_c]_{U_1 \cap U_2}$  is mapped to  $[u_c]_{U_j} \in G_j$  for  $i \in \{1, 2\}$  and further on to the same element  $\bar{c}$  in the push-out  $\bar{G}$ .

Let now Q be such a square with edges d, r, u, l. Then  $d \cdot r \sim l \cdot u$  rel.  $\dot{I}$  in Q, hence  $u_d \cdot u_r \sim u_l \cdot u_u$  rel.  $\dot{I}$  in  $U_{\varepsilon_Q}$ , i.e.  $[u_d] \cdot [u_r] = [u_l] \cdot [u_u]$  in  $G_{\varepsilon_Q}$  and thus  $\bar{d} \cdot \bar{r} = g_{\varepsilon_Q}([u_d]) \cdot g_{\varepsilon_Q}([u_r]) = g_{\varepsilon_Q}([u_l] \cdot [u_u]) = \bar{l} \cdot \bar{u}$  in  $\bar{G}$ .

Multiplying in  $\overline{G}$  all these equations resulting from one row, gives that the product corresponding to the top line equals in  $\overline{G}$  that corresponding to the bottom line, since the inner vertical parts cancel, and those at the boundary are 1. Since the top row represents 1, we get that the same is true for the bottom one. But  $u_i$  is homotopic in  $U_{\varepsilon_i}$  rel.  $\dot{I}$  to the concatenation of the corresponding  $u_c$  in the bottom row, i.e.  $[u_i]_{U_{\varepsilon_i}} = \prod_{c \subseteq [\frac{i-1}{n}, \frac{i}{n}] \times \{0\}} [u_c]_{U_{\varepsilon_i}}$  in  $G_{\varepsilon_i}$ . Thus  $z = \prod_i g_{\varepsilon_i}([u_i]_{U_{\varepsilon_i}}) = \prod_{c \subseteq [0,1] \times \{0\}} g_{\varepsilon_i}([u_c]_{U_{\varepsilon_i}}) = \prod_c \bar{c} = 1$  in  $\bar{G}$ .

**5.34 Corollary.** [9, 5.3.9] [9, 5.3.12] Let  $X = U_1 \cup U_2$  be as in 5.33.

- 1. If  $U_1 \cap U_2$  is simply connected, then  $\pi_1(U_1 \cup U_2) \cong \pi_1(U_1) \amalg \pi_1(U_2)$ .
- 2. If  $U_1$  and  $U_2$  are simply connected, then  $U_1 \cup U_2$  is simply connected.
- 3. If  $U_2$  is simply connected, then  $\operatorname{incl}_* : \pi_1(U_1) \to \pi_1(X)$  in the push-out square is an epimorphism and its kernel is generated by the image of  $\operatorname{incl}_* : \pi_1(U_1 \cap U_2) \to \pi_1(U_1)$ .
- 4. If  $U_2$  and  $U_1 \cap U_2$  are simply connected, then  $\pi_1(U_1) \cong \pi_1(U_1 \cup U_2)$ .

# Proof.

- 1 In this situation  $N = \{1\}$  and hence  $G_1 \prod G_2$  is the push-out.
- 2 Here  $G_1 \coprod G_2 = \{1\} \coprod \{1\} = \{1\}$  and hence also the push-out.
- 3 In this situation  $G_1 \coprod G_2 = G_1 \coprod \{1\} \cong G_1$  and N is the normal subgroup generated by the image of  $G_0$  in  $G_1$ .
- 4 Here we have  $N = \{1\}$  and hence the push-out is isomorphic to  $G_1$ .

**5.35 Theorem.** [9, 5.4.8] Let a CW-complex X be the union of two connected CW-subcomplexes A and B. Let  $x_0 \in A \cap B$  and  $A \cap B$  be connected. Then  $\pi_1$  maps the push-out square to a push-out.

**Proof.** By 4.19 we may choose open neighborhoods U(A), U(B) and  $U(A \cap B) = U(A) \cap U(B)$  which contain A, B and  $A \cap B$  as DRs. Then application of 5.33 and of 5.13 gives the result.



**5.36 Proposition.** [9, 5.4.9] Let A and B be (connected) CW-complexes. Then  $\pi_1(A \lor B, x_0) \cong \pi_1(A) \coprod \pi_1(B)$ .

**Proof.** Since  $A \cap B$  in  $A \vee B$  is  $\{x_0\}$  and hence simply connected this follows from 5.35 and 5.34.

**5.37 Example.** We have  $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} \coprod \mathbb{Z}$ . However, for spaces being not CW-spaces in general  $\pi_1(A \vee B) \neq \pi_1(A) \coprod \pi_1(B)$ : Take for example for A and B the
subset of  $\mathbb{R}^2$  formed by infinite many circles tangent at the base point. The closed curve which passed through all those circles alternatingly can not be expressed as finite product of words in  $\pi_1(A)$  and  $\pi_1(B)$ .

**5.38 Proposition.** [9, 5.5.9] Let  $X_j$  be a CW-complex with base-point  $x_j \in X_j^0$ . Then  $\pi_1(\bigvee X_j) \cong \coprod_j \pi_1(X_j)$ . In particular we have  $\pi_1(\bigvee_J S^1) \cong \coprod_J \mathbb{Z} \cong \mathcal{F}(J)$ , where the free generators of  $\pi_1(\bigvee_j S^1)$  are just the inclusions  $\operatorname{inj}_j : S^1 \to \bigvee_J S^1$ .

**Proof.** This follows from 5.36 by induction and by 5.31, since every compact subset is by 4.5 contained in a finite subcomplex of the CW-complex of  $\bigvee_{j \in J} X_j$  given by 4.17.

**5.39 Corollary.** [9, 5.4.1] [9, 5.4.2] Let Y be path connected with  $y_0 \in Y$  and  $f: S^{n-1} \to Y$  be continuous. Then the inclusion  $Y \subseteq Y \cup_f e^n$  induces an isomorphism  $\pi_1(Y, y_0) \to \pi_1(Y \cup_f e^n, y_0)$  if  $n \ge 3$  and an epimorphism if n = 2. In the later case the kernel is the normal subgroup generated by  $[v][f][v^{-1}]$ , where v is a path from  $y_0$  to f(1). So

$$\pi_1(Y \cup_f e^n) \cong \pi_1(Y) / \langle \operatorname{conj}_{[v][f]} \rangle_{NT}$$

One could say that by gluing  $e^2$  to Y the element  $[f] \in \pi_1(Y)$  gets killed.



Then V and  $U \cap V = e^n \setminus \{0\} \sim S^{n-1}$  are simply connected for  $n \geq 3$ , by 5.10. Thus the inclusion  $U \subseteq Y \cup_f e^n$  induces an isomorphism by 5.34.4. Since Y is a DR of U by 2.42 the inclusion of  $Y \to U$  induces an isomorphism by 5.13.

Now for n = 2. Again V is simply connected, but  $U \cap V \sim S^1$  and hence  $\pi_1(U \cap V, y_0)$  is the infinite cyclic group generated by the image of a circle of radius say 1/2. This path is homotopic to  $[v][f][v^{-1}]$  in  $Y \cup_f e^2$ , hence everything follows by 5.34.3.  $\Box$ 

# **5.40 Example.** [9, 5.4.4] We have $\pi_1(S^1 \cup_{z^n} e^2) \cong \mathbb{Z}_n$ .

In particular,  $\pi_1(\mathbb{P}^2) = \pi_1(S^1 \cup_{z^2} e^2) \cong \mathbb{Z}_2$ . This can be easily visualized: The top semi-circle  $\alpha$  in  $D^2$  has as  $\alpha^2$  the full circle, which is contractible to 0. Equally,  $\mathbb{P}^2$  is obtained by glueing a 2-cell to the boundary of a Möbius strip and the generator  $\alpha \in \pi_1(\mathbb{P}^2)$  is just the middle line on the Möbius strip. Its square is homotopic to the boundary of the Möbius strip which is contractible in the disk.



**5.42 Corollary.** [9, 5.4.3] [9, 5.4.6] Let X be a CW-complex and  $x_0 \in X^0$ . Then  $X^2 \hookrightarrow X$  induces an isomorphism  $\pi_1(X^2, x_0) \cong \pi_1(X, x_0)$  and  $X^1 \hookrightarrow X$  an epimorphism  $\pi_1(X^1, x_0) \twoheadrightarrow \pi_1(X, x_0)$  with the normal subgroup generated by  $\operatorname{conj}_{[v_e]}[\chi^e|_{S^1}]$  as kernel, where  $v_e$  is a path joining  $x_0$  and  $\chi^e(1)$  in Y and e runs through all 1-cells in the connected component of  $x_0$  in X.

**Proof.** If X is a finite CW-complex then this follows from 5.39 by induction. By 4.5 any compact subset of X is contained in a finite subcomplex  $X_0$  hence  $\pi_1(X, x_0)$  is the inductive limit of the  $\pi(X_0, x_0)$  for the finite subcomplexes  $X_0$  containing  $x_0$  by 5.31, hence the result hold in general.

**5.43 Example.** [9, 5.4.7] Since  $\mathbb{P}^n = \mathbb{P}^2 \cup e^3 \cup \cdots \cup e^n$  we have  $\pi_1(\mathbb{P}^n) \cong \pi_1(\mathbb{P}^2) \cong \mathbb{Z}_2$ .

**5.32 Definition.** [9, 5.5.11] A CW-complex X with  $X = X^1$  is called a GRAPH. A graph is called TREE if it is simply connected.

**5.44 Lemma.** [9, 5.5.12] A connected graph is a tree iff it is contractible.

**Proof.** ( $\Rightarrow$ ) Let  $X^0$  be the 0-skeleton of a tree X. And let  $x_0 \in X^0$  be fixed. Every  $x \in X^0$  can be connected by a path with  $x_0$ , which gives a homotopy  $X^0 \to X$ . By [4.19] it can be extended to a homotopy  $h_t : X \to X$  with  $h_0 = \operatorname{id}_X$  and  $h_1(X^0) = \{x_0\}$ . Let  $e \subseteq X$  be a 1-cell with characteristic map  $\chi_e : I \cong D^1 \to X$ . Then  $[h_1 \circ \chi_e] \in \pi_1(X, x_0) = \{1\}$ , hence there is a homotopy  $h_t^e : (I, \dot{I}) \to (X, \{x_0\})$  with  $h_0^e = h_1 \circ \chi_e$  and  $h_1^e(I) = \{x_0\}$ . Let  $k_t^e : X^0 \cup e \to X$  be defined by  $k_t^e(X^0) = \{x_0\}$  and  $k_t^e = h_t^e \circ \chi_e^{-1}$  on e. Taking the union of all  $k_t^e$  gives a homotopy  $k_t : X^1 \to X$  between  $h_1$  and a constant map.

**5.45 Lemma.** [9, 5.5.13] Every connected graph X contains a maximal tree. Any maximal tree in X contains all vertices of X.

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**Proof.** Let  $\mathcal{M}$  be the set of trees of X ordered by inclusion. Since the union of any linear ordered subset of  $\mathcal{M}$  is a tree (use 4.5), we get by Zorns lemma a maximal tree  $Y \subseteq X$ .

Let Y be a maximal tree and suppose that there is some  $x_0 \in X^0 \setminus Y^0$ . Let  $w : I \to X$  be a path connecting  $x_0$  and Y. Let  $t_1$  be minimal in  $w^{-1}(Y)$  (hence  $w(t_1) \in Y^0$ ) and  $t_0 < t_1$  be maximal in  $w^{-1}(X^0 \setminus Y^0)$ . Then  $w([t_0, t_1])$  is the closure of a 1-cell e and  $Y \cup \overline{e}$  is a larger tree, since Y is a SDR of  $Y \cup \overline{e}$  by deformation along  $w|_{[t_0, t_1]}$ .  $\Box$ 



**5.47 Corollary.** [9, 5.5.17] Every connected CW-space is homotopy equivalent to a CW-complex with just one 0-cell.

**Proof.** By 2.49 we have that  $X \to X/Y$  with a maximal tree Y as constructed in 5.45 is a homotopy equivalence since Y is contractible by 5.44 and  $(X^1, Y)$ has the HEP by 4.19.

**5.46 Proposition.** [9, 5.5.14] Let X be a connected graph and  $x_0 \in X^0$ . Let  $Y \subseteq X$  be a maximal tree. For every 0-cell x choose a path  $v_x$  in Y connecting  $x_0$  with x. And for every 1-cell  $e \subseteq X^1 \setminus Y$  with characteristic mapping  $\chi^e : I \cong D^1 \to X^1$  let  $s(e) := [v_{\chi^e(0)}][\chi^e][v_{\chi^e(1)}]^{-1} \in \pi_1(X, x_0)$ . Then

$$s: \mathcal{F}(\{e: e \text{ is } 1\text{-cell in } X^1 \setminus Y\}) \xrightarrow{\cong} \pi_1(X, x_0),$$

*i.e.*  $\pi_1(X, x_0)$  is the free group generated by  $\{s(e) : e \text{ is } 1\text{-cell in } X^1 \setminus Y\}$ .

**Proof.** The quotient mapping  $p: X^1 \to X^1/Y$  is a homotopy-equivalence onto a CW-space with just one 0-cell Y. By 4.15  $X^1/Y \cong \bigvee_e S^1$ , where e runs through the 1-cells in  $X^1 \setminus Y$ , see also 4.16. Thus  $\pi_1(X, x_0) \cong \pi_1(X^1/Y, y_0) = \mathcal{F}(\{e : e \text{ is 1-cell in } X^1 \setminus Y\})$  by 5.38. The inverse of this isomorphism is given by  $e \mapsto [v_{\chi^e(0)} \cdot \chi^e \cdot v_{\chi^e(1)}^{-1}] = s(e).$ 

**5.48 Corollary.** [9, 5.5.16] Let X be a connected graph with  $d_0$  vertices and  $d_1$  edges. Then  $\pi_1(X)$  is a free group of  $1 - d_0 + d_1$  generators.

**Proof.** By induction we show that for all  $1 \leq n \leq d_0$  there are trees  $Y_n \subseteq X$  with n vertices and n-1 edges: Let  $Y_n$  for  $n < d_0$  be given and choose a point  $x_0 \in X^0 \setminus Y_n$  and a path w connecting  $x_0$  with  $Y_n$ . Then proceed as in the proof of 5.45 to find an edge  $w([t_0, t_1])$  connecting a vertex outside  $Y_n$  with one in  $Y_n$ . Now  $Y_{n+1} = Y_n \cup w([t_0, t_1])$  is the required tree with one more vertex and one more edge.

By 5.46 the result follows, since there are  $d_1 - (d_0 - 1)$  many 1-cells not in  $Y_{d_1}$ .  $\Box$ 

**5.49 Theorem.** [9, 5.6.4] Let X be a CW-complex with maximal tree Y. Let generators  $s(e^1)$  be constructed for every  $e^1 \in X^1 \setminus Y^1$  as in 5.46. For every 2-cell

 $e^2 \in X^2$  define  $r(e^2) := [u \cdot \chi_{e^2}]_{S^1} \cdot u^{-1}] \in \pi_1(X^1, x_0)$ , where u is a path from  $x_0$  to  $\chi_{e^2}(1)$  in  $X^1$  and  $\chi_{e^2} : D^2 \to e^2$  a characteristic mapping. Then

$$\pi_1(X, x_0) \cong \left\langle \left\{ s(e^1) : e^1 \text{ is } 1\text{-cell in } X^1 \setminus Y^1 \right\} : \left\{ r(e^2) : e^2 \in X^2 \right\} \right\rangle.$$

**Proof.** By 5.42 the mapping  $\pi_1(X^1, x_0) \to \pi_1(X^2, x_0) \cong \pi_1(X, x_0)$  induced by  $X^1 \hookrightarrow X^2 \hookrightarrow X$  is surjective and its kernel is the normal subgroup generated by  $r(e^2) = [u \cdot \chi_{e^2}|_{S^1} \cdot u^{-1}] \in \pi_1(X^1) \cong \mathcal{F}(\{s(e^1) : e^1 \text{ is } 1\text{-cell in } X^1 \setminus Y^1\}).$ 

**5.50 Remark.** [9, 5.6.5] For every group  $G = \langle S : R \rangle$  there is a 2-dimensional CW-complex X denoted CW(S : R) with  $\pi_1(X) \cong G$ .

**Proof.** Let  $X^1 := \bigvee_S S^1$ . Every  $r \in R \subseteq \mathcal{F}(S) \cong \pi_1(X^1)$  is the homotopy class of a curve mapping  $f_r : S^1 \to X^1$ . Thus glue a 2-cell to  $X^1$  via this mapping. I.e.  $X = CW(S:R) := X^1 \bigcup_f (\bigsqcup_{r \in R} e^2)$ , where  $f := \bigsqcup_{r \in R} f_r$ .  $\Box$ 

Note that this construction depends on the choice of  $f_r \in [r]$ . However different choices give rise to homotopy equivalent spaces by 2.47.

**5.51 Proposition.** [9, 5.8.6] Every connected CW-complex of dimension less or equal to 2 is homotopy equivalent to CW(S:R) for some representation  $\langle S:R \rangle$  of its fundamental group.

**Proof.** Choose a maximal tree  $Y \subseteq X^1$ . Then by the proof of [5.46] we have that X is homotopy equivalent to X/Y, which has as 1-skeleton  $\bigvee_S S^1$ . For every 2-cell e we choose a characteristic map  $\chi^e$ . Thus  $X/Y = (\bigvee_S S^1) \cup_{\bigsqcup_e \chi^e|_{S^1}} \bigsqcup_e D^2$ . By 2.34.3 we can deform  $\chi^e|_{S^1}$  to a base point preserving map  $f^e: S^1 \to X^1$ . Hence by 2.47 X/Y is homotopy equivalent to  $CW\langle S: \{f^e: e\}\rangle$ .

**Remark.** Note that this does not solve the isomorphy problem for 2-dimensional CW-complexes, since although two such spaces X and X' with isomorphic fundamental group are homotopy equivalent to CW(S, R) and CW(S, R') for representations  $\langle S : R \rangle \cong \langle S' : R' \rangle$  of the homotopy group, the space CW(S : R) and CW(S' : R') need not be homotopy equivalent, e.g.  $\pi_1(S^2) = \{1\} = \pi_1(\{*\})$  but  $S^2$  is not homotopy equivalent to a point  $\{*\}$  by 9.1 and 2.40.1.

The following lemma shows exactly how the homotopy type might change while passing to other representations of the same group.

**5.52 Lemma.** [9, 5.8.7] We have  $CW(S : R \cup \{r\}) \sim CW(S : R) \lor S^2$  for  $r \in \langle R \rangle_{NT} \setminus R$  and  $CW(S \cup \{s\} : R \cup \{s^{-1}w\}) \sim CW(S : R)$  for  $s \notin S$  and  $w \in \mathcal{F}(X)$ .

This shows that  $CW(\langle S: R \rangle) := CW(S: R)$  would not be well-defined.

**Proof.** If X = CW(S : R) and  $Y = CW(S : R \cup \{r\})$  with  $r \in \langle R \rangle_{NG}$ . Then  $Y = X \cup_f e^2$ , where  $f : S^1 \to \bigvee_S S^1 = X^1 \subseteq X$  is such that  $[f] = r \in \pi_1(\bigvee_S S^1) = \mathcal{F}(S)$ . Since  $r \in \langle R \rangle_{NG}$ , we have that  $[f]_X = 1 \in \pi_1(X) = \pi_1(\bigvee_S S^1)/\langle R \rangle_{NG}$ , hence  $f \sim 0$  in X. Thus  $Y = X \cup_f e^2 \sim X \cup_0 e^2 = X \vee S^2$  by 2.36.3.

If X = CW(S : R) and  $Y = CW(S \cup \{s\} : R \cup \{s^{-1}w\})$ . Then  $Y = (X \vee S^1) \cup_f e^2$ , where  $f = \sigma^{-1} \cdot \omega$  for the inclusion  $\sigma : S^1 \to X \vee S^1$  and  $w = [\omega] \in \pi_1(X) \cong \mathcal{F}(S)$ . Thus  $Y = X \cup_{f|_{S_{-}^1}} D^2$  and since  $S_{-}^1 \subseteq D^2$  is a SDR we have that X is also a SDR in Y, by 2.41.

### 5.53 Example. [9, 5.7.1]

The fundamental group of the orientable compact surface of genus  $g \geq 0$  is

$$\langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g : [\alpha_1, \beta_1] \cdot \ldots \cdot [\alpha_g, \beta_g] \rangle.$$

That of the non-orientable compact surface of genus  $g \ge 1$  is

 $\langle \alpha_1, \ldots, \alpha_g : \alpha_1^2 \cdot \ldots \cdot \alpha_g^2 \rangle.$ 

**Proof.** By 1.101 these surfaces are obtained by gluing one 2-cell e to a join  $\bigvee S^1$  of 2g, respectively g, many  $S^1$  and the gluing map is given by  $i_1 \cdot i_2 \cdot i_1^{-1} \cdot i_2^{-1} \cdot \ldots$  and  $i_1^2 \cdot \ldots \cdot i_g^2$ , so the homotopy class of the characteristic mapping  $\chi^e|_{S^1}$  is  $[\alpha_1, \beta_1] \cdot \ldots \cdot [\alpha_g, \beta_g]$  and  $\alpha_1^2 \cdot \ldots \cdot \alpha_g^2$ , respectively.

**5.54 Corollary.** [9, 5.7.2] None of the spaces in 5.53 are homotopy equivalent.

**Proof.** The abelization of the fundamental groups are  $\mathbb{Z}^{2g}$  and  $\mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$ . In fact

$${}^{ab}\langle\alpha_1,\beta_1,\ldots,\alpha_g,\beta_g:[\alpha_1,\beta_1]\cdots [\alpha_g,\beta_g]\rangle = \\ = \langle\alpha_1,\beta_1,\ldots,\alpha_g,\beta_g:[\alpha_1,\beta_1]\cdots [\alpha_g,\beta_g],[\alpha_i,\alpha_j],[\beta_i,\beta_j],[\alpha_i,\beta_j]\rangle \\ \underbrace{5.27.1}_{=====} \langle\alpha_1,\beta_1,\ldots,\alpha_g,\beta_g:[\alpha_i,\alpha_j],[\beta_i,\beta_j],[\alpha_i,\beta_j]\rangle \\ = {}^{ab}\langle\alpha_1,\beta_1,\ldots,\alpha_g,\beta_g:\emptyset\rangle \\ = {}^{ab}\mathcal{F}(\alpha_1,\beta_1,\ldots,\alpha_g,\beta_g) = \mathbb{Z}^{2g}$$

and

$${}^{ab}\langle \alpha_1, \dots, \alpha_g : \alpha_1^2 \cdots \alpha_g^2 \rangle = {}^{ab}\langle \alpha_1, \dots, \alpha_g : (\alpha_1 \cdots \alpha_g)^2 \rangle$$

$$\underbrace{5.27.2}_{ab} \langle \alpha_1, \dots, \alpha_g, \alpha : \alpha^2, \alpha^{-1}\alpha_1 \dots \alpha_g \rangle$$

$$\underbrace{5.27.2}_{ab} \langle \alpha_1, \dots, \alpha_{g-1}, \alpha : \alpha^2 \rangle$$

$$= {}^{ab}(\langle \alpha_1, \dots, \alpha_{g-1} : \emptyset \rangle \amalg \langle \alpha : \alpha^2 \rangle)$$

$$= \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2. \quad \Box$$

Geometric interpretations are the following:

 $S^2$  is simply connected by 5.10 hence  $\pi_1$  has no generator and no relation.

 $S^1 \times S^1$  is a torus. By 5.29 the generators  $\alpha$  and  $\beta$  of  $\pi_1$  are given by  $S^1 \times \{1\}$  and  $\{1\} \times S^1$ , which are a meridian and an equator in the 3-dimensional picture. This can be also seen by gluing the 4 edges of a square as  $\alpha\beta\alpha^{-1}\beta^{-1}$ . The relation  $\alpha\beta = \beta\alpha$  is seen geometrically by taking as homotopy the closed curves given by running through some arc on the equator, then the meridian at that position and then the rest of the equator.

The oriented surface of genus g is obtained by cutting 2g holes into the sphere and gluing g cylinders to these holes. Let  $x_0$  be one point on the sphere not contained in the holes. As generators  $\alpha_j$  we may take curves through  $x_0$  along some generator  $\{x\} \times I$  of the cylinder and as  $\beta_i$  loops around one boundary component  $S^1 \times \{0\}$ of the cylinder. Then  $\alpha_i \beta_i \alpha_i^{-1}$  describes the loop around the other component and  $\alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}$  is a loop around both holes. The product of all these loops is a loop with all holes lying on one side and hence homotopic to a point, cf. 2.40.8 We have discussed the generator  $\alpha$  and the relation  $\alpha^2 \sim 1$  on  $\mathbb{P}^2$  in 5.40.

The non-orientable surface of genus g is obtained from a sphere by cutting g holes and gluing g Möbius-strips. The generators  $\alpha_j$  are just conjugates of the middle lines on the Möbius strips. Their squares are homotopic to the boundary circles. And hence the product of all  $\alpha_i^2$  is homotopic to a loop around all holes, which is in turn homotopic to a point.

This shows that beside the sphere, the torus and the projective plane these fundamental groups are not abelian.

# 6. Coverings

We take up the method leading to the calculation  $\pi_1(S^1) \cong \mathbb{Z}$  in 5.15. Basic ingredient was the lifting property of the mapping  $t \mapsto \exp(2\pi i t)$ ,  $\mathbb{R} \twoheadrightarrow S^1$ , see 2.17. Its main property can be stated abstractly as follows:

**6.1 Definition.** Coverings. [9, 6.1.1] A COVERING MAP  $p: Y \to X$  is a surjective continuous map, such that every  $x \in X$  has an open neighborhood  $U \subseteq X$ for which  $p|_{p^{-1}(U)}: p^{-1}(U) \to U$  is up to an homeomorphism just the projection  $\operatorname{pr}: \bigsqcup_J U \to U$  for some set  $J \neq \emptyset$ , i.e.



The images of the summands Uin  $p^{-1}(U) \subseteq Y$  are called the called FIBERS, X is called BASE, and Y TOTAL SPACE.

**Remark.** Let G be any group acting on Y. In 1.78 we considered the orbit space  $Y/G := Y/\sim$ , where  $y \sim y' : \Leftrightarrow \exists g \in G: y' = g \cdot y$  with the quotient topology and the corresponding quotient mapping  $\pi: Y \to Y/G$ . Let us assume that this is a covering, i.e. for every  $y \in Y$  there has to exist an open neighborhood  $U \subseteq Y/G$ such that  $\pi^{-1}(U)$  is a disjoint union of open subsets  $\tilde{U}$  homeomorphic via  $\pi$  to U. So  $U = \pi(\tilde{U})$  and  $\pi^{-1}(U) = \pi^{-1}(\pi(\tilde{U})) = G(\tilde{U})$ . Thus we would like that  $g(\tilde{U}) \cap g'(\tilde{U}) = \emptyset$  for all  $g \neq g'$ . In 1.81 we called a group action satisfying this condition ACT STRICTLY DISCONTINUOUS, i.e. every  $y \in Y$  has a neighborhood V such that  $g(V) \cap V = \emptyset$  for all  $g \neq e$ .

**6.15 Lemma.** Let G be a group acting strictly discontinuous on Y. Then the quotient mapping  $q: Y \to Y/G$  is a covering map.

**Proof.** Since G acts strictly discontinuous we find for each  $y \in Y$  a neighborhood V with  $g \cdot V \cap V \neq \emptyset \Rightarrow g = 1$ . Thus  $q|_V : V \rightarrow q(V) =: U$  is bijective and  $q^{-1}(U) = G \cdot V = \bigsqcup_{g \in G} g \cdot V$  is open in Y and hence U is open in Y/G. 

# 6.2 Example.

- 1. Let  $Y := \{(\sin(2\pi t), \cos(2\pi t), t) : t \in \mathbb{R}\} \cong \mathbb{R}$  and  $p = \operatorname{pr}_{1,2} : Y \to S^1 \subseteq \mathbb{R}^2$ . Then p is a covering map. Use 6.15 for  $S^1 = \mathbb{R}/\mathbb{Z}$ .
- 2. The map  $z \mapsto z^n : S^1 \to S^1$  is an *n*-fold covering map. Use 6.15 for  $S^1 = S^1 / \mathbb{Z}_n.$
- 3. The map  $S^n \to \mathbb{P}^n$  is a two-fold covering map. Use 6.15 for  $\mathbb{P}^n = S^n/\mathbb{Z}_2$ , see 1.68 and 1.71.
- 4. Let  $p_1 : Y_1 \to X_1$  and  $p_2 : Y_2 \to X_2$  be two covering maps, then so is  $p_1 \times p_2 : Y_1 \times Y_2 \to X_1 \times X_2$ . Examples:  $\mathbb{R}^2 \to S^1 \times S^1$ ,  $\mathbb{R}^2 \to \mathbb{R} \times S^1$  and  $\mathbb{R} \times S^1 \to S^1 \times S^1$ .
- 5. There is a twofold covering map from  $I \times S^1$  to the Möbius strip. Use 6.15 for the action of  $\mathbb{Z}_2$  on  $[-1,1] \times S^1$  given by  $(t,\varphi) \mapsto (-t,\varphi+\pi)$ , see exercise (1.15).

- 6. The torus is a two fold covering of Klein's bottle. Use 6.15 for the action of  $\mathbb{Z}_2$  on  $S^1 \times S^1$  given by  $(\varphi, \psi) \mapsto (-\varphi, \psi + \pi)$ , see exercise (1.17.3).
- 7.  $\mathbb{Z}_p$  acts transitively on  $S^{2k-1}$  and the orbit space is the lens space (see 1.82), so we get a covering  $S^{2k-1} \to L(p; q_1, \ldots, q_k)$ .

**6.3 Lemma.** [9, 6.1.3] Let  $p: Y \to X$  be a covering. Then

- 1. The fibers are discrete in Y.
- 2. Every open subset of a trivialising set is trivialising.
- 3. Let  $A \subseteq X$ . Then  $p|_{p^{-1}(A)} : p^{-1}(A) \to A$  is a covering map.
- 4. If  $B \subseteq Y$  is connected and  $p(B) \subseteq U$  for some trivialising set U, then B is contained in some leaf.
- 5. The mapping p is a surjective open local homeomorphism and hence a quotient mapping.

**Proof.** (1) Points in the fiber are separated by the leaves.

(2) and (3) Take the restriction of the diagram above.

 $(\underline{4})$  *B* is covered by the leaves. Since each leaf is open, so is the trace on *B*. Since *B* is connected only one leaf may hit *B*, thus *B* is contained in this leaf.

(5) Obviously the projection is a local homeomorphism. Hence it is open and a quotient mapping.

**Lemma.** Let X and  $Y \neq \emptyset$  be path connected, locally path connected and Hausdorff and Y compact. Then every local homeomorphism  $f: Y \to X$  is a covering.

**Proof.** Since f is a local homeomorphism, the fibers  $f^{-1}(x)$  are discrete and closed and hence finite since Y is compact.

Let us show next that f is surjective. In fact the image is open in X, since f is a local homeomorphism. It is closed, since Y is compact and X is Hausdorff. Since X is assumed to be (path-)connected and  $Y \neq \emptyset$  it has to be all of X.

Let  $x \in X$ . Choose pairwise disjoint neighborhoods  $V_y$  for each  $y \in f^{-1}(x)$  which are mapped homeomorphically onto corresponding neighborhood of x. By taking the inverse images of the (finite) intersection  $U := \bigcap_{y \in f^{-1}(x)} f(V_y)$  in the  $V_y$  we may assume that the image is the same neighborhood U for all  $y \in f^{-1}(x)$ . Hence U is trivializing with leaves  $V_y$  and thus  $p: Y \to X$  is a covering.  $\Box$ 

### Example.

Not every surjective local homeomorphism is a covering map.

Take for example an open interval  $I \subset \mathbb{R}$  of length more than  $2\pi$ . Then the restriction  $I \to S^1$  of the covering from 6.2.1 is not a covering.



#### 6.10 Definition. Homomorphisms of coverings.

Let  $p': Y \twoheadrightarrow X'$  and  $p: Y \twoheadrightarrow X$  be two coverings with the same total space Y. A HOMOMORPHISM f of these coverings is a map  $f: X' \to X$  such that the diagram to the right commutates.



Note that such an f exists, iff p factors over p', i.e. the fibers of p' are contained in fibers of p. If such an f exists it is uniquely determined since p' is onto. So we get a category Cov<sup>Y</sup> (a quasi-ordering) of all coverings with total space Y.

Conversely, let  $p' : Y' \to X$  and  $p : Y \to X$  be two coverings with the same base space X. A HOMOMORPHISM f of these coverings is a fiber respecting map  $f : Y' \to Y$ , i.e. the diagram on the right commutates.



We denote the set of all homomorphisms from  $p': Y' \to X$  to  $p: Y \to X$  by  $\operatorname{Hom}_X(p',p)$ . So we get a category  $\operatorname{Cov}_X$  of all coverings with base space X.

Remark that a homomorphism f is nothing else but a lift of  $p': Y' \to X$  along  $p: Y \to X$ . The automorphisms f, i.e. invertible homomorphisms  $p \to p$ , are also called COVERING TRANSFORMATIONS or DECKTRANSFORMATIONS, and we write  $\operatorname{Aut}(p)$  for the group formed by them.

### 6.4a Remark. Unique lifts along covering maps exist locally.

Let  $p: (Y, y_0) \to (X, x_0)$  be a covering and  $g: (Z, z_0) \to (X, x_0)$ . Take a trivialising neighborhood U of  $x_0$  and let  $\tilde{U}$  be the leaf of p over U which contains  $y_0$ . Then  $(p|_{\tilde{U}}): \tilde{U} \to U$  is a homeomorphism and hence  $(p|_{\tilde{U}})^{-1} \circ g: Z \supseteq g^{-1}(U) \to \tilde{U} \subseteq Y$ is a continuous local lift of g.



**6.4 Lemma. Uniqueness of lifts.** [9, 6.2.4] Let  $p: Y \to X$  be a covering map and  $g: Z \to X$  be continuous, where Z is connected. Then any two lifts of g, which coincide in one point are equal. In particular, if g is constant so are its lifts.

**Proof.** Let  $g^1, g^2$  be two lifts of g. Then the set of points  $\{z \in Z : g^1(z) = g^2(z)\}$  is clopen. In fact if  $U^j$  is the leaf over U containing of  $g^j(z)$ , then  $g^j = (p|_{U^j})^{-1} \circ g$  on the neighborhood  $(g^1)^{-1}(U^1) \cap (g^2)^{-1}(U^2)$  of z. Hence  $g^1 = g^2$  locally around z iff  $g^1(z) = g^2(z)$ .

### 6.22 Lemma.

Let X locally path connected and let  $q: Z \to Y$  and  $p: Y \to X$  be given.

Then the following statements hold:



- 0. If p and  $p \circ q$  are coverings and Y is (path-)connected, then q is onto.
- 1. If p and  $p \circ q$  are coverings and q is onto, then q is a covering.
- 2. If p and q are coverings and X is locally simply-connected, then  $p \circ q$  is a covering.
- 3. If q and  $p \circ q$  are coverings, then p is a covering.

**Proof.** (0) We claim that the image of q is clopen in Y and hence coincides with the connected space Y. For this we consider all leaves V over path-connected open subsets  $U \subseteq X$ , which are trivializing for p and  $q \circ p$ . It suffices to show that if such a leaf V meets the image q(W) of a leaf W over U for  $p \circ q$  then it is contained in q(W). So let  $w_0 \in W$  be such that  $q(w_0) \in V$ . Since V has to be path-connected as well, we may connect  $q(w_0)$  with any  $v \in V$  by a curve c in V. The curve  $p \circ c$  has a lift  $\tilde{c} = (p \circ q|_W)^{-1} \circ p \circ c$  starting at  $w_0 \in (p \circ q)^{-1}(p(c(0)))$  with values in W. By the uniqueness of local lifts  $q \circ \tilde{c}$  coincides with c and hence  $v = c(1) = q(\tilde{c}(1)) \in q(W)$ .

(1) Take a path connected set  $U \subseteq X$  being trivialising for  $p \circ q$  and p. Every leaf W of  $p \circ q$  over U is mapped by q into some leaf V of p over U: In fact, since the leafs are homeomorphic to U, they are path-connected as well, hence q(W) is completely contained in a leaf V of p over  $U = (p \circ q)(W)$  by **6.3.4**. Thus  $q^{-1}(V)$  is the topological disjoint union of all leafs W of  $p \circ q$  over U, which meet  $q^{-1}(U)$ . Moreover,  $q|_W = (p|_V)^{-1} \circ p|_V \circ q|_W = (p|_V)^{-1} \circ (p \circ q)|_W$  is a homeomorphism  $W \cong U \cong V$ .

(2) Let p and q be coverings, with X locally simply connected. Then the leafs  $V_j$  of p over a simply connected neighborhood U are again simply connected, hence are trivialising neighborhoods of q as will be shown in 6.20a. Hence  $(p \circ q)^{-1}(U) = q^{-1}(p^{-1}(U)) = q^{-1}(\bigsqcup_j V_j) = \bigsqcup_j q^{-1}(V_j)$  and  $q^{-1}(V_j) \cong \bigsqcup_{J_j} V_j$ . Thus  $(p \circ q)|_{\tilde{V}_j} = p|_{V_j} \circ q|_{\tilde{V}_j}$  is a homeomorphism  $\tilde{V}_j \cong V_j \cong U$  for every leaf  $\tilde{V}_j$  over  $V_j$ . Hence  $p \circ q$  is a covering as well.

(3) Let  $p \circ q$  and q be coverings. We claim that p is a covering. Let U be pathconnected and trivialising for  $p \circ q$  and  $\tilde{U}$  a leaf of  $p \circ q$  over U. Since q is an open mapping,  $V := q(\tilde{U})$  is open in Y. Since  $(p \circ q)|_{\tilde{U}}$  is an embedding the same is true for  $q|_{\tilde{U}}$ . Thus  $q|_{\tilde{U}}$  is a homeomorphism  $\tilde{U} \cong V$  and consequently also  $p|_V =$  $(p \circ q)|_{\tilde{U}} \circ (q|_{\tilde{U}})^{-1} : V \to \tilde{U} \to U$ . We claim that  $q(\tilde{U})$  is a path-component of  $p^{-1}(U)$ and hence these sets form a disjoint partition of  $p^{-1}(U)$ : Let  $z_0 \in \tilde{U}$  be chosen and let c be a continuous curve in  $p^{-1}(U)$  from  $q(z_0)$  to some point  $y \in p^{-1}(U)$ . By 6.4a we have a lift  $\tilde{c} := ((p \circ q)|_{\tilde{U}})^{-1} \circ (p \circ c)$  into  $\tilde{U}$  of  $p \circ c$  with initial value  $z_0$ . Then c and  $q \circ \tilde{c}$  are two lifts of  $p \circ c$  with initial value  $q(z_0)$  hence coincide and thus  $q(\tilde{c}(1)) = c(1) = y$ .

**6.18a The category**  $\operatorname{Cov}_{\operatorname{norm}}^{Y}$ . We try to get a description of the category  $\operatorname{Cov}^{Y}$  of coverings with fixed total space Y. For every group G acting strictly discontinuous on Y (and w.l.o.g. we may assume that  $G \subseteq \text{Homeo}(Y)$ ) we get a covering  $\pi: Y \to Y/G$  by 6.15.

Can we recover G from the covering  $\pi: Y \to Y/G$ ?

Yes: If Y is path connected and locally path connected then  $\operatorname{Aut}(\pi) = G$ : Obviously,  $G \subseteq \operatorname{Aut}(\pi)$ . Conversely, let  $\Phi \in \operatorname{Aut}(\pi)$ , i.e.  $\pi(y) = \pi(\Phi(y))$  for all  $y \in Y$ . Choose  $y_0 \in Y$ , then there is some  $g_0 \in G$  with  $g_0 \cdot y_0 = \Phi(y_0)$  since G acts transitively on the fibers of  $\pi$ . Since the two mappings  $\Phi$  and  $g_0$  cover the identity (i.e. are lifts of  $\pi$  along  $\pi$ ) and coincide on  $y_0$  they are equal by  $\boxed{6.4}$ .

Note, that if  $G' \leq G$  is a subgroup then  $\pi : Y \to Y/G$  factors over  $\pi' : Y \to Y/G'$  to a unique mapping  $f : Y/G' \to Y/G$ , i.e. a homomorphism  $\pi' \to \pi$ . So we get a functor  $\operatorname{Act}_{str.dis.}(Y) \to \operatorname{Cov}^Y$  from the partially ordered set (hence category)  $\operatorname{Act}_{str.dis.}(Y)$  of subgroups of Homeo(Y) for which the action on Y is strictly discontinuous.

Is this functor DENSE, i.e. is every covering mapping  $p: Y \to X$  up to isomorphy in the image of this functor? For this we have to find a subgroup  $G \leq \text{Homeo}(Y)$ for which the action on Y is strictly continuous and such that  $p \cong (\pi: Y \to Y/G)$ . The natural candidate is G := Aut(p).

Obviously the action of  $\operatorname{Aut}(p)$  on Y is strictly discontinuous, since for g in  $\operatorname{Aut}(p)$  we have that  $g(\tilde{U}) \cap \tilde{U} \neq \emptyset$  implies that there exists some  $y \in \tilde{U}$  with  $g(y) \in \tilde{U}$ . From p(g(y)) = p(y) and since  $p|_{\tilde{U}} : \tilde{U} \to U$  is injective we conclude that g(y) = y, but then  $g = \operatorname{id}$  by 6.4.

Since every  $g \in \operatorname{Aut}(p)$  is fiber preserving, we have that p is constant on the  $\operatorname{Aut}(p)$ -orbits and hence p factors to a quotient mapping  $Y/\operatorname{Aut}(p) \to X$ , by 6.22.3:



This mapping is injective (and hence an isomorphism) iff every two points in the same fiber of p are in the same orbit under  $\operatorname{Aut}(p)$ , i.e. iff  $\operatorname{Aut}(p)$  acts transitive on the fibers of p (such coverings p are called NORMAL). Note that for a group G acting strictly discontinuous on Y the covering  $\pi : Y \to Y/G$  is obviously normal. Let  $\operatorname{Cov}_{\operatorname{norm}}^Y$  denotes the category of normal coverings with total space Y. Then we have:

**6.18 Theorem.** [9, 6.5.3] For path-connected and locally path connected Y we have an equivalence of categories

$$\operatorname{Cov}_{norm}^{Y} \sim \operatorname{Act}_{str.dis.}(Y),$$

*i.e.* there exists a functor in the opposite direction and the compositions of these two are up to natural isomorphisms the identity.

**Proof.** The functor  $\operatorname{Act}_{str.dis.}(Y) \to \operatorname{Cov}_{\operatorname{norm}}^{Y}$  is given by  $\operatorname{Homeo}(Y) \ge G \mapsto (\pi : Y \to Y/G)$  and if  $G' \le G$  then  $\pi : Y \to Y/G$  factors over  $\pi' : Y \to Y/G'$  to a unique mapping  $f : Y/G' \to Y/G$ , i.e. a homomorphism  $\pi' \to \pi$ .

Conversely, every homomorphism  $f : \pi' \to \pi$  has to be the unique factorization of  $\pi : Y \to Y/G$  and it induces an inclusion  $\operatorname{Aut}(\pi') \subseteq \operatorname{Aut}(\pi)$ , since  $\Phi \in \operatorname{Aut}(\pi') \Rightarrow \pi' \circ \Phi = \pi' \Rightarrow \pi \circ \Phi = f \circ \pi' \circ \Phi = f \circ \pi' = \pi$ , i.e.  $\Phi \in \operatorname{Aut}(\pi)$ . Thus the functor is full and faithfull.

It is a general categorical result, that a full and faithful functor which is up to isomorphisms surjective on objects is an equivalence. In fact an inverse is given by selecting for every object in the range category an inverse image up to an isomorphism and by the full and faithfulness this can be extended to a functor.

We have shown above that the functor is dense, hence it gives the desired equivalence of categories.  $\hfill\square$ 

We now try to desribe the category  $\text{Cov}_X$  of coverings with base X in algebraic terms. Since the homomorphisms  $p' \to p$  are lifts of p' along p we have to study liftings along coverings in more detail.

**6.5 Theorem. Lifting of curves.** [9, 6.2.2] [9, 6.2.5] Let  $p: Y \to X$  be a covering. Every path  $w: I \to X$  has a unique lift  ${}^{y}\tilde{w}$  with  ${}^{y}\tilde{w}(0) = y$  for given  $y \in p^{-1}(w(0))$ . Paths homotopic relative their initial value have homotopic lifts.

In particular we have an action of  $\pi_1(X, x_0)$  on  $p^{-1}(x_0)$  given by  $[u] : y \mapsto {}^y \tilde{u}(1)$ , *i.e.* the end-point of the lift of u, which starts bei y.

The total space Y is path connected iff X is path connected and this action is transitive, i.e. for all  $y_1, y_2 \in p^{-1}(x_0)$  there exists a  $g \in \pi_1(X, x_0)$  with  $y_1 \cdot g = y_2$  (equivalently: there exists a  $y_0 \in p^{-1}(x_0)$  with  $y_0 \cdot \pi_1(X, x_0) = p^{-1}(x_0)$ ).

**Proof.** By 6.4 we have to show existence of a lift. By considering a path w as a homotopy being constant in the second factor, it is enough to show that homotopies  $h: I \times I \to X$  can be lifted.

For this choose a partition of  $I^2$  into squares  $Q_{i,j}$ , such that  $h(Q_{i,j})$  is contained in a trivialising neighborhood  $U_{i,j}$  of X. Now construct inductively a lift  $\tilde{h}^1$  along  $\bigcup_i Q_{i,1}$ , by taking the leaf  $\tilde{U}_{i,1}$  over the trivialising neighborhood of  $Q_{i,1}$  which contains the image under  $\tilde{h}$  of the right bottom corner of  $Q_{i-1,1}$  and hence also of the right side edge of  $Q_{i-1,1}$  (by  $\boxed{6.3.4}$ ). Then  $\tilde{h}|_{Q_{i,1}}$  can be defined as  $(p|_{U_{i,1}})^{-1} \circ h|_{Q_{i,1}}$ . Now proceed by induction in the same way to obtain lifts  $\tilde{h}^j$  for all stripes  $\bigcup_i Q_{i,j}$ . By induction we can show that the lifts agree on the horizontal lines: In fact the image of h on a horizontal edge is contained in the intersection of the trivialising sets containing the image of the square above and below. And since the lifts  $\tilde{h}^j$  and  $\tilde{h}^{j-1}$  are contained in the respective leaves, and thus in the leaf over the intersection, they have to be equal. We call the unique homotopy  $y_0 \tilde{h}$ .

Now suppose h is a homotopy rel. I between two paths  $w_0$  and  $w_1$  from  $x_0$  to  $x_1$ and let  $y_0 \in p^{-1}(x_0)$ . The homotopy  $y_0 \tilde{h}$  has as boundary values lifts  $\tilde{w}_0$  and  $\tilde{w}_1$ with  $\tilde{w}_0(0) = y_0$ . Since  $s \mapsto y_0 \tilde{h}(0, s)$  is a lift of the constant path  $x_0$ , it has to be constant, hence  $\tilde{w}_1(0) = y_0$ . So these are the unique lifts of  $w_j$  with initial value  $y_0$ . Since  $s \mapsto h(1, s)$  is a lift of the constant path  $x_1$ , it is constant, i.e.  $\tilde{h}$  is a homotopy rel. I.

The lifting property gives us a mapping  $\pi_1(X, x_0)$  to the mappings of  $p^{-1}(x_0)$  by setting  $[u](y) := {}^y \tilde{u}(1)$ . This is well defined, since curves u homotopic relative  $\dot{I}$  have lifts  ${}^y \tilde{u}$  homotopic relative  $\dot{I}$  and hence have the same end point.

Composition law: The lift of  $y_0 \widetilde{u \cdot w}$  is  $y_0 \widetilde{u} \cdot y_1 \widetilde{v}$ , where  $y_1 := y_0 \widetilde{u}(1)$ .

Moreover we have  $[u \cdot v](y) = \stackrel{y u \cdot v}{(1)} = (\stackrel{y u}{u} \cdot \stackrel{y_1}{v} \tilde{v})(1) = \stackrel{y_1}{v} \tilde{v}(1) = [v](y_1) = [v]([u](y))$ , where  $y_1 = \stackrel{y}{u} \tilde{u}(1) = [u](y)$ . Hence, we consider this mapping as a **right** action, i.e. we write  $y \cdot [u]$  for [u](y). Then we have  $y \cdot ([u] \cdot [v]) = (y \cdot [u]) \cdot [v]$ . Thus [u] acts on  $p^{-1}(x_0)$  as bijection.

In particular, [u] acts on  $p^{-1}(x_0)$  as bijection.

Now the statement on path-connectedness:

If Y is path connected then so is the surjective continuous image X. Furthermore a curve v connecting  $y_1, y_2 \in p^{-1}(x_0)$  has a closed curve  $u := p \circ v$  as image and  $v = {}^{y_1}\tilde{u}$ , so  $y_1 \cdot [u] = y_2$ , i.e. the action is transitive.

Conversely let  $y_1 \in Y$  be arbitrary. Since X is path connected we have a curve u connecting  $p(y_1)$  with  $x_0$ . Its lift  $y_1 \tilde{u}$  connects  $y_1$  with  $y := y_1 \tilde{u}(1) \in p^{-1}(x_0)$ . Since  $\pi_1(X, x_0)$  acts transitive on  $p^{-1}(x_0)$  there is a  $[u'] \in \pi_1(X, x_0)$  with  $y \cdot [u'] = y_0$ , i.e. the curve  $y \tilde{u}'$  connects y with  $y_0$ .

**6.6 Corollary.** [9, 6.3.5] Let X be path-connected. Then the fibers of any covering  $p: Y \to X$  can be mapped bijectively onto one another by lifting a curve connecting the foot points.

**Proof.** Let  $F_0 := p^{-1}(x_0)$ ,  $F_1 := p^{-1}(x_1)$  and let u be a path from  $x_0$  to  $x_1$  then  $y \mapsto {}^y \tilde{u}(1)$  defines a mapping  $F_0 \to F_1$  and  $y \mapsto {}^y \widetilde{u^{-1}}(1)$  a mapping  $F_1 \to F_0$  and these mappings are inverse to each other, since the lift of the curve  $u \cdot u^{-1} \sim 0$  is 0-homotopic rel.  $\dot{I}$  and hence closed.

**6.20a Corollary.** Let X be simply connected and  $p: Y \to X$  be a path connected covering. Then p is a homeomorphism. In particular every simply connected open subset on the base space of a covering is a trivialising neighborhood.

**Proof.** Since  $\pi_1(X, x_0) = \{1\}$  acts transitively on the fiber  $p^{-1}(x_0)$  by 6.5, the fiber has to be single pointed, hence p is injective and thus a homeomorphism.  $\Box$ 

**6.7 General lifting theorem.** [9, 6.2.6] Let Z be path connected and locally path connected. Let  $p: Y \to X$  be a covering and  $g: Z \to X$  continuous. Let  $x_0 \in X$ ,  $y_0 \in Y$  and  $z_0 \in Z$  be base points and all maps base point preserving. Then g has a base point preserving lift  $\tilde{g}$  iff  $\operatorname{im}(\pi_1(g)) \subseteq \operatorname{im}(\pi_1(p))$ .

**Proof.** ( $\Rightarrow$ ) If  $g = p \circ \tilde{g}$  then  $\operatorname{im}(\pi_1(g)) = \operatorname{im}(\pi_1(p) \circ \pi_1(g)) \subseteq \operatorname{im}(\pi_1(p))$ .

 $(\Leftarrow)$  Let  $z \in Z$  be arbitrary. Since Z is path-connected we may choose a path w from  $z_0$  to z and take the lift  $y_0 \widetilde{g \circ w}$  and define  $\widetilde{g}(z) := y_0 \widetilde{g \circ w}(1)$ .

First we have to show that this definition is independent from the choice of w. So let  $w_j$  be two paths from  $z_0$  to z. Then  $g \circ (w_0 \cdot w_1^{-1}) = (g \circ w_0) \cdot (g \circ w_1)^{-1}$  is a closed path through  $x_0$ , hence by assumption there exists a closed path v through  $y_0$  with  $p \circ v \sim (g \circ w_0) \cdot (g \circ w_1)^{-1}$  rel.  $\dot{I}$  and hence  $(p \circ v) \cdot (g \circ w_1) \sim (g \circ w_0)$  rel.  $\dot{I}$ . Thus  ${}^{y_0} \widehat{g \circ w_0}(1) = {}^{y_0} ((p \circ v) \cdot (g \circ w_1))^{\sim}(1) = ({}^{y_0} \widehat{g \circ w_1})(1) = {}^{y_0} \widehat{g \circ w_1}(1)$ .

Remains to show that  $\tilde{g}$  is continuous. Let  $z \in Z$  be fixed and let  $\tilde{U}$  be a small leaf over a trivialising neighborhood U of g(z) containing  $\tilde{g}(z)$ . Let W be a pathconnected neighborhood of z with  $g(W) \subseteq U$  and let w be a path from  $z_0$  to z. Then for every  $z' \in W$  we can choose a path  $w_{z'}$  in W from z to z'. Hence  $\tilde{g}(z') = {}^{y_0}(g \circ (w \cdot w_{z'}))^{\sim}(1) = ({}^{y_0} \widetilde{g \circ w} \cdot \tilde{g}^{(z)} \widetilde{g \circ w_{z'}})(1) = {}^{\tilde{g}(z)} \widetilde{g \circ w_{z'}}(1)$ . But since  $g \circ w_{z'}$  is contained in the trivialising neighborhood U and  $\tilde{U}$  is the leaf over Ucontaining the lift  $\tilde{g}(z)$ , we have that  ${}^{\tilde{g}(z)} \widetilde{g \circ w_{z'}} = (p|_{\tilde{U}})^{-1} \circ g \circ w_{z'}$ , and hence  $\tilde{g}(z') = ((p|_{\tilde{U}})^{-1} \circ g)(z')$  and thus is continuous.  $\Box$ 

Thus it is important to determine the image of  $\pi_1(p): \pi_1(Y, y_0) \to \pi_1(X, x_0).$ 

**6.8 Proposition.** [9, 6.3.1] Let  $p : (Y, y_0) \to (X, x_0)$  be a covering. Then the induced map  $\pi_1(p) : \pi_1(Y, y_0) \to \pi_1(X, x_0)$  is injective and its image is formed by those  $[w] \in \pi_1(X, x_0)$  for which for (some) any representative w the lift  $y_0 \tilde{w}$  is closed, i.e. by those  $g \in \pi_1(X, x_0) =: G$  which act trivial on  $y_0$ , i.e.  $y_0 \cdot g = y_0$ . This is the so called ISOTROPY SUBGROUP  $G_{y_0}$  of G at  $y_0$  with respect to the action of G on  $p^{-1}(x_0)$ .

$$\pi_1(p): \pi_1(Y, y_0) \cong \pi_1(X, x_0)_{y_0}.$$

**Proof.** Injectivity: Let  $[v] \in \pi_1(Y, y_0)$  be such that  $1 = [p \circ v]$ , i.e.  $p \circ v \sim \text{const}_{x_0}$ . By **6.5** we may lift the homotopy. Since the lift of  $\text{const}_{x_0}$  is just  $\text{const}_{y_0}$  we have [v] = 1.

If some u has a closed lift v, then  $\pi_1(p)[v] = [p \circ v] = [u]$ , hence  $[u] \in \operatorname{im}(\pi_1(p))$ . Conversely let  $[u] \in \operatorname{im} \pi_1(p)$ . Then there exists a closed curve v through  $y_0$  with  $[p \circ v] = \pi_1(p)[v] = [u]$ , hence  $u \sim p \circ v$  rel.  $\dot{I}$ , and so  $y_0 \tilde{u} \sim y_0 \tilde{p} \circ v = v$  rel.  $\dot{I}$ , thus  $y_0 \tilde{u}$  is closed as well.

In view of 6.5 we study now abstractly given transitive (right) actions of a group G on sets (i.e. discrete spaces) F.

**6.9 Lemma. Transitive actions.** Let G act transitively on F (and F') from the right. A G-EQUIVARIANT MAPPING or G-HOMOMORPHISM  $\varphi$  is a mapping  $\varphi : F \to F'$ , which satisfies  $\varphi(y \cdot g) = \varphi(y) \cdot g$ . We write  $\operatorname{Hom}_G(F, F')$  for the set of all G-homomorphisms  $F \to F'$  and  $G_y := \{g \in G : y \cdot g = y\}$  for the isotropy subgroup of  $y \in F$ . Then

- 1. We have  $G_{y \cdot g} = g^{-1}G_yg$ .
- 2.  $\{G_y : y \in F\}$  is a conjugacy class of subgroups of G, i.e. an equivalence class of subgroups of H with respect to the relation of being conjugate.
- 3. Let H be a subgroup of G. Then the set  $G/H := \{Hg : g \in G\}$  of right classes carries a unique (transitive) right G-module structure, such that the canonical projection  $\pi : G \twoheadrightarrow G/H$ ,  $g \mapsto Hg$  is G-equivariant.
- 5. For  $y \in F$  the mapping  $G \twoheadrightarrow F$  given by  $g \mapsto y \cdot g$  factors to a G-isomorphism  $G/G_y \xrightarrow{\simeq} F$ .
- 6. For  $\varphi \in \operatorname{Hom}_G(F, F')$  we have  $G_y \subseteq G_{\varphi(y)}$ . Conversely if  $y_0 \in F$  and  $y_1 \in F'$  satisfy  $G_{y_0} \subseteq G_{y_1}$ , then there is a unique  $\varphi \in \operatorname{Hom}_G(F, F')$  with  $\varphi(y_0) = y_1$ .
- 9.  $F \cong_G F' \Leftrightarrow \{G_y : y \in F\} = \{G_{y'} : y' \in F'\} \Leftrightarrow \{G_y : y \in F\} \cap \{G_{y'} : y' \in F'\} \neq \emptyset.$

Note, that we refrain from writing the quotient G/H correctly as  $H \setminus G$ .

**Proof.** (1) We have  $G_{y \cdot g} = g^{-1}G_yg$ , since  $h \in G_{y \cdot g} \Leftrightarrow y \cdot g \cdot h = y \cdot g \Leftrightarrow y \cdot (ghg^{-1}) = y$ , i.e.  $ghg^{-1} \in G_y$ .

(2) Since G acts transitively,  $\{G_y : y \in F\} = \{g^{-1}G_yg = G_{y_0 \cdot g} : g \in G\}$  is a conjugacy class by (1).

(3) The only possible action on G/H such that  $\pi$  is *G*-equivariant is given by  $Hg \cdot g' = \pi(g) \cdot g' := \pi(g \cdot g') = \pi(gg') = Hgg'$ . That the so defined action makes sense, follows from  $Hg_1 = Hg_2 \Rightarrow (Hg_1) \cdot g = Hg_1g = Hg_2g = (Hg_2) \cdot g$ .

(5) Consider  $\operatorname{ev}_y : G \twoheadrightarrow y \cdot G$  given by  $g \mapsto y \cdot g$ . This mapping has image  $y \cdot G = F$ , since G acts transitively. Furthermore g' and g have the same image  $y \cdot g' = y \cdot g$  iff  $g'g^{-1} \in G_y$ , so  $\operatorname{ev}_y$  factors to a G-isomorphism  $G/G_y \to F$ .

(6) We have  $G_y = \{g : y \cdot g = y\} \subseteq \{g : \varphi(y) \cdot g = \varphi(y \cdot g) = \varphi(y)\} = G_{\varphi(y)}$ . Conversely let  $G_{y_0} \subseteq G_{y_1}$  and  $y \in F$ . Since G acts transitively there exists a  $g \in G$  with  $y = y_0 \cdot g$ . Define  $\varphi(y) = \varphi(y_0 \cdot g) := \varphi(y_0) \cdot g = y_1 \cdot g$ . This definition makes sense, since  $y_0 \cdot g' = y_0 \cdot g$  implies  $g'g^{-1} \in G_{y_0} \subseteq G_{y_1}$  and hence  $y_1 \cdot g' = y_1 \cdot g$ . By construction  $\varphi$  is G-equivariant.

(9) (1 $\Rightarrow$ 2) Let  $\varphi: F \to F'$  be a *G*-equivariant isomorphism. Then  $G_y \subseteq G_{\varphi(y)} \subseteq G_{\varphi^{-1}(\varphi(y))} = G_y$  by (6).

(1 $\Leftarrow$ 3) By assumption there are  $y \in F$  and  $y' \in F'$  with  $G_y = G_{y'}$  hence  $F \cong_G G/G_y = G/G_{y'} \cong_G F'$  by 5.

# **6.9a The category** Subgr(G).

We use 6.9.3 for associateing to each subgroup  $H \leq G$  the transitive action of G on G/H. In order to extend this to a full and faithfull functor, we have to define the morphisms  $H \to H'$  appropriately:

Let  $\varphi \in \operatorname{Hom}_G(G/H, G/H')$  and  $y_0 := H \in G/H$ . By 6.9.6  $\varphi$  is uniquely determined by  $y_1 := \varphi(y_0) =: H'g \in G/H'$  with  $H = G_{y_0} \subseteq G_{y_1} = g^{-1}H'g$  by 6.9.1. So we define

$$\operatorname{Hom}(H, H') := \{g : gH \subseteq H'g\}/H',$$

where H' acts on  $\{g : gH \subseteq H'g\}$  by multiplication from the left, since  $gH \subseteq H'g$ and  $h' \in H'$  implies  $h'gH \subseteq h'H'g = H'h'g = H'h'g$ .

Then the set Subgr(G) of subgroups  $H \leq G$  and  $H''g' \circ H'g := H''g'g$  as composition of these morphisms forms a category:

The composition  $H''g' \circ H'g := H''g'g$  is well-defined, since  $gH \subseteq H'g$  and  $g'H' \subseteq H''g' \Rightarrow g'gH \subseteq g'H'g \subseteq H''g'g$  and since  $H''(h''g')(h'g) = H''g'h'g = H''\bar{h}''g'g = H''g'g$  for  $\bar{h}'' := g'h'(g')^{-1} \in g'H'(g')^{-1} \subseteq H''$ . The identity on H is given by H = H 1.

**Theorem.** We have an equivalence  $\operatorname{Act}_{tr}(G) \sim \operatorname{Subgr}(G)$  of categories.

**Proof.** The functor  $\operatorname{Subgr}(G) \to \operatorname{Act}_{\operatorname{tr}}(G)$  is given on morphisms by:

$$\operatorname{Hom}(H, H') \ni H'g_0 \mapsto (H g \mapsto H' g_0 g) \in \operatorname{Hom}_G(G/H, G/H').$$

This is well-defined, since  $H g = H \bar{g} \Rightarrow g_0 \bar{g} (g_0 g)^{-1} = g_0 \bar{g} g^{-1} g_0^{-1} \in g_0 H g_0^{-1} \subseteq H'$  $\Rightarrow H' g_0 g = H' g_0 \bar{g}$  and since  $H' (h' g_0) g = H' g_0 g$  for  $h' \in H'$ .

Functorality:  $H \mapsto \operatorname{id}_{G/H}$  and the composition  $H''g_2 \circ H'g_1 := H''g_2g_1$  is mapped to  $H g \mapsto H' g_1g \mapsto H''g_2g_1g$ .

The functor is faithfull:  $\forall g : Hg \mapsto H'g_0g = H'\bar{g}_0g \Rightarrow H'g_0 = H'\bar{g}_0 \in \operatorname{Hom}(H, H').$ 

The functor is full by what we have shown above.

The functor is dense by 6.9.5

**6.9b Corollary.** [9, 6.3.3] Let G act transitively on F from the right. With  $\operatorname{Aut}_G(F)$  we denote the group of all G-equivariant isomorphisms  $F \to F$ . For a subgroup H of G one denotes with  $\operatorname{Norm}_G(H) := \{g \in G : H = g^{-1}Hg\}$ , the smallest subgroup of G, which contains H as normal subgroup. Then

$$\operatorname{Aut}_G(F) \cong \operatorname{Norm}_G(G_{y_0})/G_{y_0}$$

**Proof.** By [6.9a] we have  $\operatorname{Hom}_G(F, F') \cong \operatorname{Hom}(H, H') := \{g : gH \subseteq H'g\}/H'$ , where  $H := G_g$  and  $H' := G_{g'}$  are isotropy subgroups of G for the action on F and F'. Moreover,  $H'g \in \operatorname{Hom}(H, H')$  is an isomorphism  $\Leftrightarrow \exists Hg' \in \operatorname{Hom}(H', H)$  with  $H = Hg' \circ H'g = Hg'g$  and  $H' = H'g \circ Hg' = H'gg' \Leftrightarrow \exists g' \in G$  with  $g'H' \subseteq Hg'$ ,  $g'g \in H$ , and  $gg' \in H' \Leftrightarrow \exists g' \in G$  with  $H \subseteq g^{-1}H'g \subseteq (g'g)^{-1}Hg'g = H$ ,  $g'H' \subseteq Hg', g'g \in H$ , and  $gg' \in H' \Leftrightarrow H = g^{-1}H'g$  (and  $g' := g^{-1}$ )

Thus  $\operatorname{Aut}_G(F) \cong \operatorname{Aut}(H) = \operatorname{Norm}_G(H)/H$  by 6.9a.

**6.9d Corollary.** We have a bijection between isomorphy classes of transitive right actions of G and conjugacy classes of subgroups of G.

**Proof.** By the proof of 6.9b we have that  $H'g \in \text{Hom}(H, H')$  is an isomorphism, iff  $H = g^{-1}H'g$ , i.e. H and H' belong to the same conjugacy class. Now the result follows from 6.9a.

**6.9e Corollary.** Let  $p: Y \to X$  be a covering with path connected Y and  $x_0 \in X$ . The images  $\pi_1(p)(\pi_1(Y,y))$  for  $y \in p^{-1}(x_0)$  form a conjugacy class of subgroups in  $\pi_1(X, x_0)$ .

This class is called the CHARACTERISTIC CONJUGACY CLASS of the covering p.

**Proof.** By 6.8  $\pi_1(p)(\pi_1(X, y)) = G_y$  for  $G := \pi_1(X, x_0)$  and  $y \in F := p^{-1}(x_0)$ , and by 6.9.2  $\{G_y : y \in F\}$  is a conjugacy class of subgroups of G.

**6.9c Corollary.** For transitiv actions of G on F the following statements are equivalent:

- 1.  $G_y$  is normal in G for some (all)  $y \in F$ ;
- 2.  $G_y = G_{y'}$  for all  $y, y' \in F$ ;
- The induced action of G / ∩<sub>y∈F</sub> G<sub>y</sub> is free,
   i.e. if g ∈ G has a fixed point y ∈ F then it acts as identity on F;
- 4.  $\operatorname{Aut}_G(F)$  acts transitive on F.

For 3 note that  $\bigcap_{y \in F} G_y$  is the kernel of the action  $G \to \operatorname{Bij}(F)$ .

**Proof.** (1  $\Rightarrow$  4) If  $G_{y_0}$  is normal, then Norm<sub>G</sub>( $G_{y_0}$ ) = G and hence Aut<sub>G</sub>(F)  $\cong$   $G/G_{y_0}$  by 6.9b which obviously acts transitive, since G does.

 $(4 \Rightarrow 3)$  Let  $y_0 \cdot g = y_0$  and  $y \in F$ . Since  $\operatorname{Aut}_G(F)$  acts transitive there is an automorphism  $\varphi$  with  $y = \varphi(y_0) = \varphi(y_0 \cdot g) = \varphi(y_0) \cdot g = y \cdot g$ .

 $(3 \Rightarrow 2)$  Let  $g \in G_y$ , i.e. y is a fixed point of g. Hence g acts as identity, so  $g \in G_{y'}$  for all  $y' \in F$ .

$$(2 \Rightarrow 1)$$
 is obvious, since  $G_y = G_{y \cdot g} = g^{-1} \cdot G_y \cdot g$  by 6.9.1.

Let us now show that  $\operatorname{Cov}_X^{\operatorname{pc}} \to \operatorname{Act}_{\operatorname{tr}}(G)$  is full and faithful:

**6.11 Proposition.** Let X be locally path connected. Let  $p: Y \to X$  and  $p': Y' \to X$  be two path-connected coverings with typical fibers  $F := p^{-1}(x_0)$  and  $F' := (p')^{-1}(x_0)$  and  $G := \pi_1(X, x_0)$ . Then  $\operatorname{Hom}_X(p, p') \cong \operatorname{Hom}_G(F, F')$  via  $\Phi \mapsto \Phi|_F$ .

**Proof.** The mapping  $\Phi \mapsto \Phi|_F$  is well-defined, i.e.  $\Phi|_F$  is a *G*-homomorphism, since  $\Phi(y \cdot [u]) = (\Phi \circ {}^y \tilde{u})(1) = {}^{\Phi(y)} \tilde{u}(1) = \Phi(y) \cdot [u].$ 

It is injective, since  $\Phi_1|_F = \Phi_2|_F$  implies  $\Phi_1(y_0) = \Phi_2(y_0)$  and hence  $\Phi_1 = \Phi_2$ , by the uniqueness of lifts of p proved in 6.5.

Surjectivity: Let  $\varphi : F \to F'$  be *G*-equivariant. As  $\Phi : Y \to Y'$  we take the lift of  $p : Y \to X$  which maps  $y_0 \in F$  to  $\varphi(y_0) \in F'$ . This lift exists by 6.7, since  $\pi_1(p)(\pi_1(Y, y_0)) = G_{y_0} \subseteq G_{\varphi(y_0)} = \pi_1(p')(\pi_1(Y', \varphi(y_0)))$  and with X also Y is locally path-connected. By 6.9.6  $\Phi|_F = \varphi$ , since both are *G*-equivariant and coincide on  $y_0$ .

**6.24 Corollary.** [9, 6.3.4] Two path-connected coverings of a locally path-connected space are isomorphic, iff their conjugacy classes are the same.

**Proof.** 
$$p \cong p' \xleftarrow{6.11} F \cong_G F' \xleftarrow{6.9d} \{G_y : y \in F\} = \{G_{y'} : y' \in F'\}.$$

**6.12 Corollary.** [9, 6.5.5] Let Y be path connected and X be locally path connected. For any covering map  $p: Y \to X$  we have

$$\operatorname{Aut}(p) \cong \operatorname{Aut}_{\pi_1(X,x_0)}(p^{-1}(x_0)) \cong \operatorname{Norm}(\pi_1(p)(\pi_1(Y,y_0)))/\pi_1(p)(\pi_1(Y,y_0)).$$

The inverse of this isomorphism is given by mapping  $[u] \in \text{Norm}(\pi_1(p)(\pi_1(Y, y_0)))$ to the unique covering transformation f which maps  $y_0$  to  $y_0 \tilde{u}$ .

**Proof.** Since the elements of Aut are just the isomorphisms of an object with itself, this follows directly from 6.11, 6.9b and 6.8.

**6.17 Corollary. Normal coverings.** [9, 6.5.8] For path-connected coverings  $p: Y \to X$  of locally connected spaces X the following conditions are equivalent:

- 1.  $\pi_1(p)(\pi_1(Y, y))$  is normal for (some) all y in the fiber over  $x_0$ ;
- 2. The characteristic conjugacy class of the covering consists of a single group;
- 3. If one lift of a closed path through  $x_0$  is closed, then so are all lifts;
- The covering p is normal, i.e. the group Aut(p) acts transitive on the fiber over x<sub>0</sub>.

In particular this is true if  $\pi_1(X)$  is abelian or the covering is 2-fold or  $\pi_1(Y) = \{1\}$ .

**Proof.** Let  $G := \pi_1(X, x_0)$  and  $F := p^{-1}(x_0)$ . By **6.8**  $\pi_1(p)(\pi_1(Y, y)) = G_y$  and by **6.9e** the characteristic conjugacy class is  $\{G_y : y \in F\}$ ; the lift with initial value y of a closed curve u through  $x_0$  is closed iff y is a fixed point of [u] acting on F; and the group of covering transformations is  $\operatorname{Aut}(p) \cong \operatorname{Aut}_G(F)$  by **6.11**. So the result follows from **6.9c**.

### 6.26 Example. [9, 6.1.5]

Since every subgroup of an abelian group is normal and also any subgroup of index two, the simplest non-normal covering could best be found among the 3-fold coverings of  $S^1 \vee S^1$ . There is a three-fold covering of  $S^1 \vee S^1$ , which is not normal.

**Proof.** Let  $\{y_0, y_1, y_2\}$  be the fiber over  $x_0$ , let a and b denote parametrizations of the two factors  $S^1$  in  $S^1 \vee S^1$  and let  $a_0, a_1, a_2$  be the leafs over a and  $b_0, b_1, b_2$  be the leafs over b. Let  $b_i$  be from  $y_{i+1}$  to  $y_{i+2} \pmod{3}$ . Let  $a_0$  be a closed path at  $y_0$  and  $a_1$  and  $a_2$  connect  $y_1$  and  $y_2$  in opposite directions.



So a has closed as well as none closed lifts.

**6.13 Corollary.** [9, 6.5.6] If  $p: Y \to X$  is a convering with Y simply connected and X locally path connected, then  $\operatorname{Aut}(p) \cong \pi_1(X, x_0)$ .

**Proof.** In this situation  $\pi_1(Y, y_0) = \{1\}$  and hence  $\operatorname{Norm}_G(G_{y_0}) = G$  and so we have  $\operatorname{Aut}_G(F) \cong \operatorname{Norm}_G(G_{y_0})/\{1\} \cong G$ .

This can be used to calculate  $\pi_1(X, x_0)$  by finding a covering  $p : \tilde{X} \to X$  with simply connected total space  $\tilde{X}$  (see 6.20) and then determine its automorphism group.

# 6.16 Examples of the fundamental group of orbit spaces. [9, 5.7.5]

We can apply 6.13 to the examples in 6.2. In particular, we have  $\mathbb{Z}$  as group of covering transformations of  $\mathbb{R} \to S^1$  and  $\mathbb{Z}_2$  as group of covering transformations of  $S^n \to \mathbb{P}^n$  for n > 1. Furthermore, the homotopy group of  $L(\frac{q}{p}) \cong S^3/\mathbb{Z}_p$  from 1.82 is  $\mathbb{Z}_p$  and that of  $M\begin{pmatrix}a & b\\ c & d\end{pmatrix} \cong L(\frac{c}{q})$  from 1.75 (see 1.89) is  $\mathbb{Z}_{|a|}$ 

#### Example. Orbits spaces need not be Hausdorff.

Consider the ordinary differential equation

$$\frac{dx}{dt} = \cos^2 x, \quad \frac{dy}{dt} = \sin x$$

Since this vector field is bounded, the solutions exist globally and we get a smooth function  $\varphi$ :  $\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$  associating to each  $t \in \mathbb{R}$  and  $(x, y) \in \mathbb{R}^2$  the solution with value (x, y) at 0 at time t. If the initial value satisfies  $\cos^2 x = 0$  then the solution is  $y(t) = y(0) + t \cdot \sin x$ . Otherwise we have  $\frac{dy}{dx} = \frac{\sin x}{\cos^2 x} = \frac{d}{dx} \frac{1}{\cos x}$ , hence it has to be contained in  $\{(y, x) : y(x) = \frac{1}{\cos x}\}$ . Moreover the time it takes from  $x = x_0$  to  $x = x_1$  is given by  $t(x_1) - t(x_0) = \int_{x_0}^{x_1} \frac{dt}{dx} = \int_{x_0}^{x_1} \frac{1}{\cos^2 x} dx = \tan x |_{x=x_0}^{x_1}$ .



Note that the orbit space  $\mathbb{R}^2/\mathbb{R}$  is **not Hausdorff** (and  $\mathbb{R}^2/\mathbb{Z}$  as well). It consists of a countable union  $\bigsqcup_{\mathbb{Z}} \mathbb{R}$  of  $\mathbb{R}'s$  together with the points  $\pi/2 + \pi \cdot \mathbb{Z}$ . A neighborhood basis of  $\pi/2 + k\pi$  is given by end-interval of the two neighboring  $\mathbb{R}'s$ . However,  $\mathbb{Z}$  acts strictly discontinuous on  $\mathbb{R}^2$ .

We may also form the space  $X := ([-\pi/2, \pi/2] \times \mathbb{R})/\sim$ , where  $(-\pi/2, -t) \sim (\pi/2, t)$ . Since the action of  $\mathbb{R}$  is compatible with this equivalence relation  $\mathbb{R}$  acts fixed-point free on this Möbius strip X as well. The **orbits** of the discrete subgroup  $\mathbb{Z} \subseteq \mathbb{R}$  are obviously **closed** subsets. However, the **action is not strictly discontinuous**, since for any neighborhood of  $[(\pi/2, 0)]$  some translate by  $t \in \mathbb{Z}$  meets it again.

## 6.20 Maximal Covering.

We aim to show that the functor  $\operatorname{Cov}_X^{\operatorname{pc}} \to \operatorname{Act}_{\operatorname{tr}}(G)$ , where  $G := \pi_1(X, x_0)$ , is an equivalence of categories. In view of 6.11 it remains to show its denseness.

For this we search for the "maximal" elements (i.e. the initial objects in categorical language) first. For transitive actions of G the maximal object is G with the right multiplication on itself, since for every action of G on some F we have G-equivariant mappings  $ev_y : G \to F, g \mapsto y \cdot g$ , for  $y \in F$  by 6.9.6.

The corresponding maximal path-connected covering  $p: \tilde{X} \to X$  should thus have as typical fiber  $p^{-1}(x_0) = G$  and the action of  $G = \pi_1(X, x_0)$  on it should be given by right multiplication. In particular, we must have  $G_y = \{1\}$  for all  $y \in G$ . Choose a base-point  $y_0 \in \tilde{X}$  with  $x_0 = p(y_0)$ . Since  $\pi_1(p) : \pi_1(\tilde{X}, y_0) \to G_{y_0} = \{1\}$  is an isomorphism by **6.8**, we have that  $\tilde{X}$  should be simply connected.

For every point  $y \in \tilde{X}$ we find a path  $v_y$  from  $y_0$  to y and since  $\tilde{X}$  is simply connected the homotopy class  $[v_y]$  rel. I is well defined. Let  $\sim$  denote temporarily the relation of being 'homotopic relative I'.



Thus  $y \mapsto [v_y]$  gives bijection  $\tilde{X} \cong C((I,0), (\tilde{X}, y_0))/\sim$  with inverse  $ev_1 : v(1) \leftrightarrow [v]$ . By the lifting property 6.5, these homotopy classes correspond bijectively to homotopy classes of paths starting at  $x_0$ .

Let U be a path-connected neighborhood of  $x_1 \in X$ . We calculate  $\operatorname{ev}_1^{-1}(U)$ . Note that  $\operatorname{ev}_1^{-1}(x_1) = \{[v] : v \text{ is a path in } X \text{ from } x_0 \text{ to } x_1\}$  and in particular  $\operatorname{ev}_1^{-1}(x_0) = \pi_1(X, x_0)$ .

$$\begin{aligned} \operatorname{ev}_{1}^{-1}(U) &= \{ [w] : w(1) \in U \} & (\text{now use } w \sim w \cdot u^{-1} \cdot u, \text{ then}) \\ &= \{ [v] \cdot [u] : v(0) = x_{0}, v(1) = x_{1}, u(0) = x_{1}, u(I) \subseteq U \} \\ &= \{ [v] \cdot [u] : [v] \in \operatorname{ev}_{1}^{-1}(x_{1}), u(0) = x_{1}, u(I) \subseteq U \} \\ &= \bigcup \Big\{ [^{v]} \tilde{U} : [v] \in \operatorname{ev}_{1}^{-1}(x_{1}) \Big\}, \text{ with } {}^{[v]} \tilde{U} := \{ [v] \cdot [u] : u(0) = x_{1}, u(I) \subseteq U \}. \end{aligned}$$

Since U is path-connected the mapping  $\operatorname{ev}_1|_{[v]\tilde{U}} : {}^{[v]}\tilde{U} \to U$  is onto. In order that it is injective, we need that  $u_0(1) = u_1(1) \Rightarrow [u_0] = [u_1]$ , i.e. every closed curve in U through  $x_1$  should be 0-homotopic in X. A space X which has a neighborhood of sets with this property is called SEMI-LOCALLY SIMPLY CONNECTED. Note that the closed curves are assumed to be local (i.e. contained in U), whereas the homotopy may leave U. Since any subset of such a set U has the same property, we get for a locally connected semi-locally simply connected space a neighborhood-basis of connected sets U with this property. Note that  $[v_1]\tilde{U} \cap [v_2]\tilde{U} \neq \emptyset$  iff there exist curves  $u_i$  with  $[v_1] \cdot [u_1] = [v_2] \cdot [u_2]$ , where  $u_i$  are curves in U from  $x_1$  to the same endpoint. Hence  $[u_1] = [u_2]$  by the semi-local simple connectedness and thus  $[v_1] = [v_2]$ .

For a path-connected, locally path connected and semi-locally simply connected space X we thus define  $\tilde{X}$  to be the set  $C((I,0),(X,x_0))/\sim$  and  $p_1: \tilde{X} \to X$  by  $p_1([u]) := \operatorname{ev}_1(u) = u(1)$ . Since for every U as above we want  $[^{[u]}\tilde{U}$  to be a leaf over U, we declare those sets to be open in  $\tilde{X}$ . In order that these sets form the basis of a topology we have to consider two such neighborhoods  $U_0$  and  $U_1$  and  $y \in {}^{y_0}\tilde{U}_0 \cap {}^{y_1}\tilde{U}_1$ . Then  $p_1(y) \in U_0 \cap U_1$  and hence we can find such a neighborhood  $U \subseteq U_0 \cap U_1$  of  $p_1(y)$ . Then  $y \in {}^y\tilde{U}$  and  ${}^y\tilde{U} \subseteq {}^{y_0}\tilde{U}_0 \cap {}^{y_1}\tilde{U}_1$ . Obviously we have that  $p_1|_{y\tilde{U}}: {}^y\tilde{U} \to U$  is a homeomorphism, and hence  $p_1: \tilde{X} \to X$  is a covering map.

Note that for any path u starting at  $x_0$  we have that  $t \mapsto [u_t]$  is the lift along  $p_1$  with starting value  $[\text{const}_{x_0}] =: y_0$ , where  $u_t(s) := u(ts)$ . Thus  $\tilde{X}$  is path-connected.

Finally  $\tilde{X}$  is simply connected: Let v be a closed curve in  $\tilde{X}$  through  $y_0$ . Then  $u := p_1 \circ v$  is a closed curve through  $x_0$  and  $v(t) = [s \mapsto u(ts)]$ , since both sides are lifts of u with starting point  $y_0$ . Hence  $[\operatorname{const}_{x_0}] = y_0 = v(0) = v(1) = [u]$ . Since homotopies can be lifted, we have  $\operatorname{const}_{y_0} \sim v$  rel.  $\dot{I}$ .

#### Theorem. Universal covering. [9, 6.6.2]

Let X be path-connected, locally path-connected and semi-locally simply-connected. Then there exists a path-connected, simple-connected covering  $p_1: \tilde{X} \twoheadrightarrow X$ . Every simply connected path-connected covering of X covers any other path-connected covering.

**Proof.** We have just shown the first part. The other one follows, since we can lift the projection of any simple connected covering by 6.7 and the lift is a covering by 6.22.1.

**6.21 Denseness of**  $\operatorname{Cov}_X^{\mathbf{pc}} \to \operatorname{Act}_{\mathbf{tr}}(G)$ . Let us return to the question of surjectivity of  $\operatorname{Cov}_X^{\mathrm{pc}} \to \operatorname{Act}_{\mathrm{tr}}(G)$ , where  $G := \pi_1(X, x_0)$ . Let G act transitively on F. By  $6.9.5 \ \mathrm{F} \cong G/H$ , where  $H := G_y$  is any isotropy subgroup of this action. By 6.13we have  $\operatorname{Aut}(p_1) \cong \pi_1(X, x_0) =: G$  and it acts strictly discontinuous on  $\tilde{X}$  by what we have shown in 6.18a. By construction in 6.19 the typical fiber  $p_1^{-1}(x_0) = G$ and the action of G on it is given by right multiplication. Thus the subgroup Hacts also strictly discontinuously on  $\tilde{X}$  and hence  $\tilde{X} \to \tilde{X}/H =: Y$  is a covering by 6.15. Furthermore the mapping  $p_1 : \tilde{X} \to X = \tilde{X}/G$  factors over  $\tilde{X} \to Y$  to give some  $p : Y \to X$  which is a covering by 6.22.3.

It remains to show that the action corresponding to the covering  $p: \tilde{X}/H =: Y \to X$  is isomorphic to H. The standard fiber of this covering is  $p_1^{-1}(x_0)/H = G/H \cong_G F$ . The action of G on  $p^{-1}(x_0) = G/H$  is obviously given by factoring the action of G on  $p_1^{-1}(x_0) = G$  (by right multiplication) over the canonical quotient mapping  $\pi: G \to G/H$  and is thus up to the isomorphism  $F \cong_G G/H$  the given action on F. Thus we have found the desired covering  $Y \to X$ , and hence obtain the following:

**6.23 Theorem.** [9, 6.6.3] Let X be path-connected, locally path-connected and semilocally simply connected. Then we have an equivalence between the category of pathconnected coverings of X and transitive actions of  $G := \pi_1(X, x_0)$ .

 $\operatorname{Cov}_X^{pc} \sim \operatorname{Act}_{tr}(G).$ 

**Proof.** By 6.11 the functor is full and faithful and by 6.21 it is dense, hence induces an equivalence.

# 6.25a The category $\operatorname{Cov}_X^{\mathbf{pc}}$ is not quasi-ordered.

I.e., we give an example that for two coverings  $p: Y \to X$  and  $p': Y' \to X$  there may be more than one element in  $\text{Hom}_X(p, p')$  up to isomorphy.

By [6.23] it is enough to consider the corresponding question for transitive Gactions. For this we will consider subgroups  $H \leq H' \leq G$  for which  $\operatorname{Norm}_G(H) = H$ and  $\operatorname{Norm}_G(H') = H'$  and for which a  $g \notin H'$  exists with  $gHg^{-1} \subseteq H'$ . In particular,  $\operatorname{Aut}_G(H) = \{1\}$ ,  $\operatorname{Aut}_G(H') = \{1\}$ , and  $H' \neq H'g \in \operatorname{Hom}(H, H')$ . By 6.9a this gives the corresponding result for transitive actions of G.

Remains to show that H, H', G and g can be found. So let F be finite,  $G := \operatorname{Bij}(F)$ and let  $\{F_j : j \in J\}$  be a partition of F in disjoint subsets of different non-zero cardinality. Then  $H := \{\varphi \in G : \forall j \in J : \varphi(F_j) = F_j\}$  is a subgroup with Norm<sub>G</sub>(H) = H: In fact, let  $g \in G$  be such that  $gHg^{-1} \subseteq H$  and assume  $g \notin H$ , i.e. there is some j with  $g(F_j) \neq F_j$ . Since  $|g(F_j)| = |F_j|$  there has to exist a  $j \in J$ and  $y_1, y_2 \in F_j$  such that  $g(y_1)$  and  $g(y_2)$  are in different sets  $F_{j_1}$  and  $F_{j_2}$ . In fact take j with  $|F_j|$  maximal and  $g(F_j) \neq F_j$ , then  $g(F_j) = F_i$  so  $g(F_j)$  can not meet  $F_i$ . Now take  $h \in H$  given by exchanging just  $y_1$  and  $y_2$ . Then  $ghg^{-1}$  maps  $g(y_1)$  to  $g(y_2)$ , and hence  $F_{j_1}$  is not invariant, so  $ghg^{-1} \notin H$ .

If  $F = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$  and  $F_1 = \{1\}$ ,  $F_2 = \{2, 3\}$ ,  $F_3 = \{4, 5, 6\}$  and  $F_4 = \{7, 8, 9, 10\}$ . Let H be given by the partition  $\{F_1, F_2, F_3, F_4\}$  and H' be given by  $\{F_1 \cup F_2, F_3 \cup F_4\}$  and let  $g := (1, 4)(2, 5)(3, 6) \notin H'$ . Then  $gHg^{-1} \subseteq H'$ , since  $g^{-1}(F_1 \cup F_2) = F_3$ ,  $g^{-1}(F_3) = F_1 \cup F_2$  and  $g^{-1}(F_4) = F_4$ , hence  $ghg^{-1}(F_3) = gh(F_1 \cup F_2) = g(F_1 \cup F_2) = F_3$ ,  $ghg^{-1}(F_1 \cup F_2) = gh(F_3) = g(F_3) = F_1 \cup F_2$  and  $ghg^{-1}(F_4) = F_4$ .

By 5.50 the group G can then be realized as fundamental group of a 2-dimensional CW-complex X.

**Example.** Let  $p: Y \to X$  and  $p': Y' \to X$  be two coverings. Then there may exist homomorphisms in  $\operatorname{Hom}_X(p, p')$  and  $\operatorname{Hom}_X(p', p)$  without  $p \cong p'$ . In fact we can translate this to transitive actions. So we need subgroups  $H \leq G$  and  $H' \leq G$  which are not conjugate, but such that H is contained in some conjugate.

 $H' \leq G$  which are not conjugate, but such that H is contained in some conjugate  $g^{-1}H'g$  of H' and conversely. Then  $G/H \to G/(g^{-1}H'g) \cong G/H'$  is G-equivariant as is  $G/H' \to G/((g')^{-1}Hg') \cong G/H$ , but G/H is not isomorphic to G/H'. In [6, p.187] the existence of such groups is shown.

**6.25 Example. Threefold coverings.** [9, 6.7.3] We now try to identify all 3-fold coverings of  $S^1 \vee S^1$  and also those of the torus  $S^1 \times S^1$  and Klein's bottle. For G we have in these cases  $\langle \{\alpha, \beta\} : \emptyset \rangle$ ,  $\langle \{\alpha, \beta\} : \{\alpha\beta = \beta\alpha\} \rangle$ , and  $\langle \{\alpha, \beta\} : \{\alpha^2\beta^2 = 1\} \rangle$ .

First we have to determine all transitive actions of  $\langle \{\alpha, \beta\} : \emptyset \rangle$  on  $\{0, 1, 2\}$ , i.e. group-homomorphisms from the free group with two generators  $\alpha$  and  $\beta$  into that group of permutations of  $\{0, 1, 2\}$ . We write such permutations in cycle notation, i.e. these are

 $\{(0), (01), (02), (12), (012), (021)\}.$ 

Where (0) has order 1, (012) and (021) have order 3 and the rest order 2. Up to symmetry we may assume that the image a of  $\alpha$  has order less or equal to the image b of  $\beta$ . Note, that two actions on  $\{0, 1, 2\}$  are isomorphic if there exists a

permutation which conjugates the generators for one action onto those of the other one.

If  $\operatorname{ord} a = 1$ , i.e. a = (0) then  $\operatorname{ord} b$  has to be 3 (otherwise the resulting action is not transitive) and the two possible choices are conjugate via (01).

If ord a = 2, then ord b can be 2, but b has to be different from a (otherwise not transitive) and any two choices  $\{a, b\}$  and  $\{a', b'\}$  are conjugate via the common element  $c \in \{a, b\} \cap \{a', b'\}$ ; or b can have order 3, and again the choices of b are conjugate by a, and that of a are conjugate by b or  $b^{-1}$ .

If ord a = 3 = ord b, they can be either the same or different.

So we get representatives for all transitive actions with (-) + indicating (none-)normality:

a	b	$S^1 \vee S^1$	$S^1 \times S^1$	Kleins bottle		
$\bigcirc (0)$	(012)	+	+	∄		
(012)	(0)	+	+	∄		
(01)	(02)	_	∄	—		
(01)	(012)	-	∄	∄		
(012)	(01)	_	∄	∄		
(012)	(012)	+	+	∄		
(012)	(021)	+	+	+		

Note, that the action is normal iff every  $g \in G$  acts either fixed-point free or is the identity by **6.9c.3**. Thus at least the generators a and b have be of order 3 or 1. This excludes the 3 actions in the middle. All other cases are normal, because there the group generated by a and b is  $\{(0), (012), (021)\}$  and only the identity (0) has fixed points.

The last two columns are determined by checking  $a^2b^2 = 1$  and ab = ba.

**6.27 Proposition.** [9, 6.8.1] Let  $p : Y \to X$  be a covering. Then the following statements are true:

- 1. If X is a CW-complex then so is Y. The cells of Y are the path-components (leafs) of  $p^{-1}(e)$  for all cells e of X.
- 2. If X is a manifold so is Y.
- 3. If X is a topological group, so is Y.

**Proof.** (1) Let e be a cell of X. Since e is locally path-connected so is  $p^{-1}(e)$  and every component of  $p^{-1}(e)$  is homeomorphic to e via the projection, since the restriction of the projection is a covering map and e is simply connected. Since  $D^n$  is simply connected we may lift a characteristic map to a characteristic map  $\tilde{\chi}^e$  of the lifted cell  $\tilde{e}$  by 6.7. One can show that the properties (C) and (W) are satisfied.

(2) We may take the chart domains to be trivialising sets in X. The leafs can then be used as chart domains of Y.

(3) The group structures  $X \times X \to X$  and  $X \to X$  can be lifted to mappings  $Y \times Y \to Y$  and  $Y \to Y$ : In fact chose  $1 \in p^{-1}(1)$ . Then  $\pi_1(\mu \circ (p \times p))([u_1], [u_2]) = [(p \circ u_1) \cdot (p \circ u_2)] = \pi_1(p)[u_1 \cdot u_2]$  by 5.7. Thus  $\mu \circ (p \times p)$  has a unique lift to  $\tilde{\mu}: Y \times Y \to Y$  by 6.7. Similarly  $\pi_1(\nu \circ p)([u]) = [p \circ u]^{-1} = \pi_1(p)[u]^{-1}$ .

**6.28 Theorem.** [9, 6.9.1] Every subgroup H of a free group G is free. If H has finite index k in G, then rank $(H) = (\operatorname{rank}(G) - 1) \cdot k + 1$ . In particular, there exist subgroups of any finite rank in the free group of rank 2.

**Proof.** Let G be a free group and H a subgroup of G. By 5.38 G is the fundamental group of a join X of 1-spheres. Since X has a universal covering  $\tilde{X} \to X$  by 6.20, there exists also a covering  $Y \to X$  with isotropy subgroup H. By 6.27 Y is a graph, and hence its homotopy group H is a free group by 5.46.

If *H* has finite index *k* in *G*, then  $\operatorname{rank}(H) = (\operatorname{rank}(G) - 1) \cdot k + 1$  by 5.48, since the fiber  $F \cong G/H$  by the proof of 6.21 and hence *Y* has *k*-times as many cells of fixed dimension as *X*.

Let  $G := \langle \{a, b\} : \emptyset \rangle$  and  $k \ge 1$ . Then there exists a unique surjective homomorphism  $\varphi : G \to \mathbb{Z}_k$  with  $\varphi(a) = 1$  and  $\varphi(b) = 0$ . Thus  $H := \ker \varphi$  has index k in G and hence rank H = (2-1)k + 1 = k + 1.  $\Box$ 

# 7. Simplicial Homology

Since it is difficult to calculate within non Abelian groups we try to associate abelian groups to a topological space. Certainly we could take  ${}^{ab}\pi_1(X)$ , but in order to calculate this we can hardly avoid the non-commutative group  $\pi_1(X)$  as intermediate step. So we have to find a more direct approach. We start with the most explicitly describable spaces, i.e. the simplicial complexes K. By **3.30** there is to each closed curve  $|\dot{\Delta}| = S^1 \rightarrow |K|$  a homotopic simplicial approximation c from some barycentric refinement of  $\dot{\Delta}$  to K. Note that any barycentric refinement of  $\dot{\Delta}$  is just a finite sequence of adjacent edges. If we want to get rid of commutativity we should consider c as formal linear combination  $\sum_{\sigma} n_{\sigma} \cdot \sigma$  with integer coefficients  $n_{\sigma}$  of oriented edges  $\sigma$  in K (we dropped those edges with same start and end point). That c is a closed (and connected) curve corresponds to the assumption that every vertex occurs equally often as start and as end point. So we can associate to such a linear combination  $\sum_{\sigma} n_{\sigma} \cdot \sigma := \sum_{\sigma} n_{\sigma} \cdot \partial \sigma$ , where  $\partial \sigma$  is just  $x_1 - x_0$ , when  $\sigma$  is the edge from  $x_0$  to  $x_1$ . Thus  $c := \sum_{\sigma} n_{\sigma} \cdot \sigma$  is closed iff  $\partial c = 0$ .

Next we should reformulate what it means that c is 0-homotopic, i.e. there exists an extension  $\tilde{c} : |\Delta| = D^2 \rightarrow |K|$ . Again by **3.30** we may assume that  $\tilde{c}$  is simplicial from some barycentric refinement of  $\Delta$ . The image of  $\tilde{c}$  can be viewed as 2-chain, i.e. formal linear combination  $\sum n_{\sigma} \cdot \sigma$  with integer coefficients  $n_{\sigma}$  of ordered 2-simplices  $\sigma$  of K. Note that an orientation of a triangle induces (or even is) a coherent orientation on the boundary edges. That  $\tilde{c}$  is an extension of c means that the edges of these simplices, which do not belong to c, occur as often with one orientation as with the other. And those which do belong to c occur exactly that often more with that orientation than with the other. So we can define the boundary  $\partial(\sum_{\sigma} n_{\sigma} \cdot \sigma)$  of a linear combination of 2-simplices as  $\sum_{\sigma} n_{\sigma} \cdot \partial \sigma$ , where  $\partial \sigma = \langle x_0, x_1 \rangle + \langle x_1, x_2 \rangle + \langle x_2, x_0 \rangle$ , when  $\sigma$  is the triangle with vertices  $x_0, x_1, x_2$  in that ordering. Then c is 0-homotopic iff there exists a 2-chain with boundary c. We call such a chain c EXACT or 0-HOMOLOGUE. The difference between closed and exact 1-chains is an obstruction to simply connectedness of |K|. At the same time this easily generalizes to k-chains:

### Homology groups

**7.1 Definition.** [9, 7.1.1] [9, 7.1.4] An ORIENTATION OF A *q*-SIMPLEX (with q > 0) is an equivalence class of linear orderings of the vertices, where two such orderings are equivalent iff they can be transformed into each other by an even permutation. So if a *q*-simplex  $\sigma$  has vertices  $x_0, \ldots, x_q$  then an orientation is fixed by specifying an ordering  $x_{\sigma(0)} < \cdots < x_{\sigma(q)}$  and two such orderings  $\sigma$  and  $\sigma'$  describe the same oriented simplex iff sign $(\sigma' \circ \sigma^{-1}) = +1$ . We will denote the corresponding ordered simplex with  $\langle x_{\sigma(0)}, \ldots, x_{\sigma(q)} \rangle$ . Let  $\sigma^{-1}$  denote the simplex with the same vertices as  $\sigma$  but the opposite orientation.



The  $q^{th}$ -chain group

 $C_0(K) := {}^{ab} \langle \{ \sigma : \sigma \text{ is } 0 \text{-simplex in } K \} \rangle$ 

 $C_q(K) := {}^{ab} \langle \{ \sigma : \sigma \text{ is ordered } q \text{-simplex in } K \} : \{ \sigma^{-1} + \sigma : \sigma \text{ is ordered } q \text{-simplex in } K \} \rangle$ 

is the free abelian group with all ordered q-simplices as generators modulo the relation  $\sigma + \sigma^{-1} = 0$ .

**7.2 Lemma.** [9, 7.1.5] By picking an ordering of each simplex we get an unnatural isomorphism from  $C_q(K)$  to the free abelian group with the unordered q-simplices as generators.

**Proof.** We consider the map which associates to each ordered simplex either the unordered simplex, if the ordering is the selected one, or the negative of the unordered simplex, otherwise. This induces a surjective group-homomorphisms  $O := {}^{ab}\mathcal{F}(\{\sigma : \sigma \text{ is ordered } q \text{-simplex in} K\}) \to U := {}^{ab}\mathcal{F}(\{\sigma : \sigma \text{ is unordered } q \text{-simplex in} K\})$ . It factors over  $C_q(K)$ , since  $\sigma + \sigma^{-1}$  is mapped to 0. The induced surjective homomorphism  $C_q(X) \to U$  is injective, since  $g := \sum_{\sigma} n_{\sigma} \cdot \sigma + n_{\sigma^{-1}} \cdot \sigma^{-1} \in O$  is mapped to  $\sum_{\sigma} (n_{\sigma} - n_{\sigma^{-1}}) \cdot \sigma$  (where  $\sigma$  runs through the unordered simplices with the picked ordering) and this vanishes only if  $n_{\sigma} = n_{\sigma^{-1}}$ , i.e. if the image of g in  $C_q(K)$  is 0.

Note that

$$\partial \langle x_0, x_1 \rangle = x_1 - x_0 = \langle x_0, x_1 \rangle + \langle x_0, x_1' \rangle^{-1};$$
  

$$\partial \langle x_0, x_1, x_2 \rangle = \langle x_0, x_1 \rangle + \langle x_1, x_2 \rangle + \langle x_2, x_0 \rangle$$
  

$$= \langle x_0, x_1, \overline{x_2} \rangle + \langle \overline{x_0}, x_1, x_2 \rangle + \langle x_0, \overline{x_1}, x_2 \rangle^{-1}$$
  

$$= \langle \overline{x_0}, x_1, x_2 \rangle + \langle x_0, \overline{x_1}, x_2 \rangle^{-1} + \langle x_0, x_1, \overline{x_2} \rangle,$$

where  $x_i$  indicates that  $x_i$  has to be left out. Let  $\sigma$  be the tetrahedron with the natural orientation  $x_0 < x_1 < x_2 < x_3$ . Its faces should have orientation  $\langle x_1, x_2, x_3 \rangle$ ,  $\langle x_0, x_2, x_3 \rangle^{-1}$ ,  $\langle x_0, x_1, x_3 \rangle$  and  $\langle x_0, x_1, x_2 \rangle^{-1}$ .

This leads to the generalized definition:

**7.3 Definition.** [9, 7.1.2] [9, 7.1.6] The ORDERING of the face  $\sigma'$  opposite to the vertex  $x_j$  in  $\sigma = \langle x_0, \ldots, x_q \rangle$  should be given by  $\sigma' := \langle x_0, \ldots, x_{j-1}, \overline{x_j}, x_{j+1}, \ldots, x_q \rangle^{(-1)^j}$ . Let us show that this definition makes sense. So let  $\sigma$  be a permutation of  $\{0, \ldots, q\}$  and  $\sigma'$  be the face opposite to j. Then  $\langle x_{\sigma(0)}, \ldots, x_{\sigma(q)} \rangle = \langle x_0, \ldots, x_q \rangle^{\operatorname{sign} \sigma}$  and we have to show that

$$\langle x_{\sigma(0)}, \dots, \overline{x_j}, \dots, x_{\sigma(q)} \rangle^{(-1)^i} = \langle x_0, \dots, x_{j-1}, \overline{x_j}, x_{j+1}, \dots, x_q \rangle^{(-1)^j \operatorname{sign} \sigma}$$

where *i* is the position of *j* in  $\sigma(0), \ldots, \sigma(q)$ , i.e.  $i = \sigma^{-1}(j)$ . Without loss of generality let  $i \leq j$  (otherwise consider  $\sigma^{-1}$  instead). Consider the permutations of

 $\{0, \ldots, q\}$  given by the table

0	 i-1	i	 j-1	j	j+1	 q
0	 i-1	i+1	 j	i	j + 1	 q
$\sigma(0)$	 $\sigma(i-1)$	$\sigma(i+1)$	 $\sigma(j)$	$\sigma(i)$	$\sigma(j+1)$	 $\sigma(q)$

The first one is the cyclic permutation  $(i, i+1, \ldots, j-1, j)$ , hence has sign  $(-1)^{j-i}$ , the second one is  $\sigma$ , and the composite leaves  $j = \sigma(i)$  invariant, has sign  $(-1)^{j-i} \cdot \text{sign } \sigma$ , and induces the identity

$$\langle x_{\sigma(0)}, \dots, \overline{x_j}, \dots, x_{\sigma(q)} \rangle = \langle x_0, \dots, x_{j-1}, \overline{x_j}, x_{j+1}, \dots, x_q \rangle^{(-1)^{i-j} \operatorname{sign} \sigma}$$

Now we define the BOUNDARY OF AN ORIENTED q-SIMPLEX  $\sigma = \langle x_0, \ldots, x_q \rangle$  (for q > 0) to be

$$\partial \sigma = \sum_{j=0}^{q} (-1)^j \langle x_0, \dots, x_{j-1}, \overline{x_j}, x_{j+1}, \dots, x_q \rangle.$$

For  $q \leq 0$  one puts  $\partial \sigma := 0$ . Extended by linearity and factorization over  $\sigma^{-1} \sim -\sigma$ we obtain linear mappings  $\partial := \partial_q : C_q(K) \to C_{q-1}(K)$ . For  $0 > q \in \mathbb{Z}$  one puts  $C_q(K) := \{0\}$  and  $\partial_q := 0 : C_q(K) \to C_{q-1}(K)$ .

**7.4 Definition.** [9, 7.1.7] [9, 7.1.8] With  $Z_q(K) := \operatorname{Ker}(\partial_q)$  we denote the set of CLOSED *q*-CHAINS. With  $B_q(K) := \operatorname{Im}(\partial_{q+1})$  we denote the set of EXACT (or 0-HOMOLOGOUS) *q*-chains. Two *q* chains are called HOMOLOGOUS iff their difference is exact.

If q = 0 then  $C_q(K) = Z_q(K)$ . If  $q = \dim(K)$  then  $B_q(K) = \{0\}$ .

**7.6 Theorem.** [9, 7.1.9]  $0 = \partial^2 = \partial_q \circ \partial_{q+1}$  and hence  $B_q \subseteq Z_q$ .

**Proof.** Let  $\sigma = \langle x_0, \ldots, x_{q+1} \rangle$  with  $q \ge 1$ . Then

$$\partial \partial \sigma = \partial \sum_{j=0}^{q+1} (-1)^j \langle x_0, \dots, \overline{x_j}, \dots, x_{q+1} \rangle$$
  
=  $\sum_{j=0}^{q+1} (-1)^j \left( \sum_{i=0}^{j-1} (-1)^i \langle x_0, \dots, \overline{x_i}, \dots, \overline{x_j}, \dots, x_{q+1} \rangle + \sum_{i=j+1}^{q+1} (-1)^{i-1} \langle x_0, \dots, \overline{x_j}, \dots, \overline{x_i}, \dots, x_{q+1} \rangle \right)$   
=  $\sum_{i < j} ((-1)^{i+j} - (-1)^{j+i}) \langle x_0, \dots, \overline{x_i}, \dots, \overline{x_j}, \dots, x_{q+1} \rangle$   
=  $0$ 

**7.7 Definition.** [9, 8.3.1] A CHAIN COMPLEX is a family  $(C_q)_{q\in\mathbb{Z}}$  of Abelian groups together with group-homomorphisms  $\partial_q : C_q \to C_{q-1}$  which satisfy  $\partial_q \circ \partial_{q+1} = 0$ . Equally, we may consider  $C := \bigoplus_{q\in\mathbb{Z}} C_q$ , which is a  $\mathbb{Z}$ -graded Abelian group and  $\partial := \bigoplus_{q\in\mathbb{Z}} \partial_q$ , which is a graded group homomorphism  $C \to C$  of degree -1 and satisfies  $\partial^2 = 0$ .

**7.8 Definition.** [9, 7.1.10] For a chain complex  $(C, \partial)$  we define its HOMOLO-GY  $H(C, \partial) := \ker \partial / \operatorname{im} \partial$ . This is a  $\mathbb{Z}$ -graded abelian group with  $H(C, \partial) = \bigoplus_{q \in \mathbb{Z}} H_q(C, \partial)$ , where  $H_q(C, \partial) := \ker \partial_q / \operatorname{im} \partial_{q+1}$ . The group  $H_q(K) := Z_q(K)/B_q(K)$  is called the q-th HOMOLOGY GROUP of K.

### Examples and exact sequences

**7.9 Example.** [9, 7.2.1] We consider the following simplicial complex K formed by one triangle  $\sigma_2$  with vertices  $x_0$ ,  $x_1$ ,  $x_2$  and edges  $\sigma_1^0$ ,  $\sigma_1^1$ ,  $\sigma_1^2$  one further point  $x_3$  connected by 1-simplices  $\sigma_1^3$  and  $\sigma_1^4$  with  $x_1$  and with  $x_2$ .



Since  $\partial c_2 = m(\sigma_1^0 + \sigma_1^1 + \sigma_1^2) \neq 0$  for  $m \neq 0$  the only closed 2-cycle is 0, hence  $H_2(K) = 0$ .

The boundary  $\partial c_1 = (b_1 - b_2) x_0 + (b_2 - b_0 + b_3) x_1 + (b_0 - b_1 - b_4) x_2 + (b_4 - b_3) x_3$ vanishes, iff  $b_2 = b_1$ ,  $b_4 = b_3$  and  $b_0 = b_1 + b_3$ . So  $Z_1(K)$  is formed by  $c_1 = b_1(\sigma_1^0 + \sigma_1^1 + \sigma_1^2) + b_3(\sigma_1^0 + \sigma_1^3 + \sigma_1^4)$  and hence  $z_1 := \sigma_1^0 + \sigma_1^1 + \sigma_1^2$  and  $z'_1 := \sigma_1^0 + \sigma_1^3 + \sigma_1^4$ form a basis with  $\partial c_2 = m z_1$ . So  $B_1(K) = \{m z_1 : m \in \mathbb{Z}\}$  und  $H_1(K) \cong \mathbb{Z}$ .

For the determination of  $H_0(K)$  see 7.12.

**7.10 Remark.** [9, 7.2.2] We have  $H_q(K) = 0$  for q < 0 and  $q > \dim K$ . Furthermore,  $H_{\dim K}(K) = C_{\dim K}(K)$  (by 7.5) is a free abelian group as subgroup of  $C_{\dim K}(K)$ .

**7.11 Lemma.** [9, 7.2.3] If  $K_1, \ldots, K_m$  are the connected components of K, then  $C_q(K) \cong \bigoplus_{j \le m} C_q(K_j)$  and  $H_q(K) \cong \bigoplus_{j \le m} H_q(K_j)$ .

**7.12 Lemma.** [9, 7.2.4]  $H_0(K)$  is a free abelian group. Generators are given by choosing in each component one point.

Proof.

Because of [7.11] we may assume that K is connected. Let  $\varepsilon : C_0(K) \to \mathbb{Z}$  be the linear map given by  $x \mapsto 1$  for all vertices  $x \in K$ . Obviously  $\varepsilon$  is surjective. Remains to show that its kernel is  $B_0(K)$ . Every two vertices  $x_0$  and  $x_1$  are homologous, since there is a 1-chain connecting  $x_0$  with  $x_1$ . Thus  $c := \sum_x n_x \cdot x$  is homologous to  $(\sum_x n_x) \cdot x_0 = \varepsilon(c) \cdot x_0$  and hence  $\operatorname{Ker}(\varepsilon) \subseteq B_0$ . Conversely let  $c = \partial(\sum_\sigma n_\sigma \cdot \sigma) = \sum_\sigma n_\sigma \cdot \partial\sigma$ . Since  $\varepsilon(\partial \langle x_0, x_1 \rangle) = \varepsilon(x_1 - x_0) = 0$  we have the converse inclusion.  $\Box$ 

**7.13 Example.** [9, 7.2.10] The homology of the cylinder  $X := S^1 \times I$ . Note that  $S^1 \times I \sim S^1$  and hence we would expect  $H_2(X) = 0$  and  $H_1(X) = {}^{ab}\pi_1(S^1) = \mathbb{Z}$ .

Let us show that this is in fact true. We consider the triangulation given by 6 triangles. We will show in a later section that the homology does not depend on the triangulation. We orient the triangles in the natural way.

 $H_2(X)$ : Let  $z_2 = \sum_{\dim \sigma=2} n_{\sigma} \cdot \sigma \in Z_2(X) = H_2(X)$ , i.e.  $\partial z_2 = 0$ . Since those edges, which join the inner boundary with the outer one belong to exactly two 2-simplices, the coefficients of these two simplices have to be equal. So  $n := n_{\sigma}$  is independent on  $\sigma$ . However  $\partial(\sum_{\sigma} \sigma)$  is the difference of the outer boundary and the inner one, hence not zero, and so  $z_2 = n(\sum_{\sigma} \sigma)$  is a cycle only if n = 0, i.e.  $H_2(X) = \{0\}$ .

 $H_1(X)$ : Let  $[z_1] \in H_1(X)$ , i.e.  $z_1 = \sum_{\dim \sigma=2} n_{\sigma} \cdot \sigma \in Z_1(X)$  with  $\partial z_1 = 0$ . Since we may replace  $z_1$  by a homologous chain, it is enough to consider linear combinations of a subset of edges, such that for each triangle at least 2 edges belong to this subset. In particular we can use the 6 interior edges. Since each vertex is a boundary point of exactly two of these edges the corresponding coefficients have to be equal (if we orient them coherently). Thus  $z_1$  is homologous to a multiple of the sum  $c_1$  of these 6 edges. Hence  $H_1(X)$  is generated by  $[c_1]$ . The only multiple of  $c_1$ , which is a boundary, is 0, since the boundary of  $\sum_{\dim \sigma=2} n_{\sigma} \cdot \sigma$  contains  $n_{\sigma} \cdot \sigma_1$ , where  $\sigma_1$  is the edge of  $\sigma$ , which is not an interior one.

**7.14 Example.** [9, 7.2.14] The homology of the projective plane  $X := \mathbb{P}^2$ . We use the triangulation of  $\mathbb{P}^2$  by 10 triangles described in 3.9.2. And we take the obvious orientation of all triangles. Note however that on the "boundary edges" these orientations are not coherent.

 $H^2(X)$ : Let  $z_2 = \sum_{\dim \sigma=2} n_{\sigma} \cdot \sigma \in Z_2(X) = H_2(X)$ , i.e.  $\partial z_2 = 0$ . Since those edges, which belong to the "interior" in the drawing belong to exactly two 2-simplices, the coefficient of these two simplices have to be equal. So  $n := n_{\sigma}$  is independent on  $\sigma$ . However  $\partial(\sum_{\sigma} \sigma)$  is twice the sum a + b + c of three edges along which we have to glue, and hence is not zero. So  $z_2 = n(\sum_{\sigma} \sigma)$  is a cycle only if n = 0, i.e.  $H_2(X) = \{0\}$ .

 $H_1(X)$ : Let  $[z_1] \in H_1(X)$ , i.e.  $z_1 = \sum_{\dim \sigma = 1} n_{\sigma} \cdot \sigma \in Z_1(X)$  with  $\partial z_1 = 0$ . Now we may replace  $z_1$  by a homologous chain using all edges except the 3 inner most ones and the 3 edges normal to the "boundary". Now consider the vertices on the inner most triangle. Since for each such point exactly two of the remaining edges have it as a boundary point, they have to have the same coefficient, and hence may be replaced by the corresponding "boundary" parts. So  $z_1$  is seen to be homologous to a sum of "boundary" edges. But another argument of the same kind shows that they must occur with the same coefficient. Hence  $H_1(X)$  is generated by a + b + c. As we have show above 2(a + b + c) is the boundary of the sum over all triangles. Whereas a + b + c is not a boundary of some 2-chain  $\sum_{\sigma} n_{\sigma} \cdot \sigma$ , since as before such a chain must have all coefficients equal to say n and hence its boundary is 2n(a + b + c). Thus  $H_1(\mathbb{P}^2) = \mathbb{Z}_2$ , which is no big surprise, since  $\pi_1(\mathbb{P}^2) = \mathbb{Z}_2$ .

**7.15 Definition.** [9, 8.2.1] A sequence  $A \xrightarrow{f} B \xrightarrow{g} C$  of abelian groups is called EXACT at B iff ker  $g = \operatorname{im} f$ . A finite (or infinite) sequence of groups  $C_q$  and group homomorphisms  $f_q : C_{q+1} \to C_q$  is called exact if it is exact at all (but the end) points.

7.16 Remark. [9, 8.2.2]

- 1 A sequence  $0 \to A \xrightarrow{f} B$  is exact iff f is injective.
- 2 A sequence  $A \xrightarrow{f} B \to 0$  is exact iff f is surjective.

- 3 A sequence  $0 \to A \xrightarrow{f} B \to 0$  is exact iff f is bijective.
- 4 Let  $A_{q+1} \xrightarrow{f_{q+1}} A_q \xrightarrow{f_q} A_{q-1}Eastf_{q-1}A_{q-2}$  be exact. Then the following statements are equivalent:
  - $f_{q+1}$  is onto;
  - $f_q = 0;$
  - $f_{q-1}$  is injective.

**7.17 Lemma.** Let  $0 \to C_0 \to C_1 \to \cdots \to C_n \to 0$  be an exact sequence of finitely generated free abelian groups. Then  $\sum_{q=0}^{n} (-1)^q \operatorname{rank} C_q = 0$ .

**Proof.** For an  $\mathbb{Z}$ -linear map (i.e. abelian group homomorphism) f we have rank(ker f)+ rank(im f) = rank(dom f) by the pendent to the classical formula from linear algebra. Thus taking the alternating sum of all rank(dom  $f_q$ ) gives a telescoping one and hence evaluates to 0.

**7.18 Proposition.** [9, 7.2.5] Let K be a one dimensional connected simplicial complex. Then  $H_1(K)$  is a free abelian group with  $1 - \alpha_0 + \alpha_1$  many generators, where  $\alpha_i$  are the number of *i*-simplices.

Compare with 5.48.

**Proof.** Consider the sequence



It is exact by definition and the vertical arrow is an isomorphism by 7.12 and hence we get by 7.17 the equation  $0 = \operatorname{rank}(H_1) - \alpha_1 + \alpha_0 - 1$ 

**7.19 Definition.** [9, 7.2.6] Let K be a simplicial complex in  $\mathbb{R}^n$ . Let  $p \in \mathbb{R}^n$  be not contained in the affine subspace generated by all  $\sigma \in K$ . Let  $p \star \langle x_0, \ldots, x_q \rangle := \langle p, x_0, \ldots, x_q \rangle$ . Let  $p \star K := K \cup \{p \star \sigma : \sigma \in K\} \cup \{p\}$ . It is called the CONE over K with vertex p and is obviously a simplicial complex. Note that we can extend  $p \star (\_)$  to a linear mapping  $C_q(K) \to C_q(p \star K)$ .

**7.20 Proposition.** [9, 7.2.7] *He have*  $H_q(p \star K) = \{0\}$  *for all*  $q \neq 0$ .

**Proof.** Let c be a q-chain of K. Then

$$\partial(p \star c) = \begin{cases} c - \varepsilon(c)p & \text{if } q = 0\\ c - p \star \partial c & \text{otherwise.} \end{cases}$$

Note that this shows that any q-chain c (with q > 0) is homologous to  $p \star \partial c$ . In order to show this we may assume that  $c = \langle x_0, \ldots, x_q \rangle$ . For q = 0 we have  $\partial(p \star c) = \partial \langle p, x_0 \rangle = x_0 - p = c - \varepsilon(c)p$ . For q > 0 we get

$$\partial(p \star c) = \partial \langle p, x_0, \dots, x_q \rangle$$
  
=  $\langle x_0, \dots, x_q \rangle - \sum_{i=0}^q (-1)^i \langle p, x_0, \dots, x_i, \dots, x_q \rangle = c - p \star \partial c.$ 

Now let  $c \in Z_q(p \star K)$  for q > 0. We have to show that it is a boundary. Clearly c is a combination of simplices of the form  $\langle x_0, \ldots, x_q \rangle$  and  $\langle p, x_0, \ldots, x_{q-1} \rangle$ , i.e.  $c = c_q + p \star c_{q-1}$  with  $c_q \in C_q(K)$  and  $c_{q-1} \in C_{q-1}(K)$ . Hence  $c = c_q + p \star c_{q-1} = \partial(p \star c_q) + p \star \partial c_q + p \star c_{q-1}$ . So  $p \star (\partial c_q + c_{q-1}) \in Z_q$ . But, again by the equation above, the boundary of such a cone vanishes only if  $\partial c_q + c_{q-1} = 0$ , hence c is a boundary.

**7.21 Corollary.** [9, 7.2.8] For an n-simplex  $\sigma_n$  let  $K(\sigma_n) := \{\tau : \tau \leq \sigma_n\}$ . Then  $K(\sigma_n)$  is a connected simplicial complex of dimension n with  $|K(\sigma_n)|$  being an n-ball. We have  $H_q(K(\sigma_n)) = 0$  for  $q \neq 0$ .

**Proof.** Let 
$$\sigma_n = \langle x_0, \dots, x_n \rangle$$
 and  $\sigma_{n-1} = \langle x_1, \dots, x_n \rangle$  then  $K(\sigma_n) = x_0 \star K(\sigma_{n-1})$ .

**7.22** Proposition. [9, 7.2.9] For an n + 1-simplex  $\sigma_{n+1}$  let  $K(\dot{\sigma}_{n+1}) := \{\tau : \tau < \sigma_{n+1}\}$ . Then  $K(\dot{\sigma}_{n+1})$  is a connected simplicial complex of dimension n with  $|K(\dot{\sigma}_{n+1})|$  being an n-sphere and we have

$$H_q(K(\dot{\sigma}_{n+1})) \cong \begin{cases} \mathbb{Z} & \text{for } q \in \{0, n\} \\ 0 & \text{otherwise.} \end{cases}$$

A generator of  $H_n(K(\dot{\sigma}_{n+1}))$  is  $\partial \sigma_{n+1} := \sum_{j=0}^{n+1} (-1)^j \langle x_0, \dots, \overline{x_j}, \dots, x_{n+1} \rangle$ .

**Proof.** Let  $K := K(\dot{\sigma}_{n+1})$  and  $L := K(\sigma_{n+1})$ . Then  $L \setminus K = \{\sigma_{n+1}\}$  and we have

By 7.21 the top row is exact (for q > 0). Thus we have exactness in the bottom row for all 0 < q < n. By exactness the arrow  $\langle \sigma_{n+1} \rangle \cong C_{n+1}(L) \xrightarrow{\partial} C_n(L)$  is injective, and  $H_n(K) = Z_n(K) = Z_n(L) = \partial(C_{n+1}(L)) \cong C_{n+1}(L) = \mathbb{Z}$ .

We will show later that if  $|K| \sim |L|$  then  $H_q(K) \cong H_q(L)$  for all  $q \in \mathbb{Z}$ , hence it makes sense to speak about the homology groups of a polyhedra.

#### 7.23 5'Lemma. [9, 8.2.3] Let

$$\begin{array}{c|c} A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} A_4 \xrightarrow{\varphi_4} A_5 \\ f_1 \middle| \cong & f_2 \middle| \cong & f_3 \middle| & f_4 \middle| \cong & f_5 \middle| \cong \\ B_1 \xrightarrow{\psi_1} B_2 \xrightarrow{\psi_2} B_3 \xrightarrow{\psi_3} B_4 \xrightarrow{\psi_4} B_5 \end{array}$$

be a commutative diagram with exact horizontal rows. If all but the middle vertical arrow are isomorphisms so is the middle one.

**Proof.** ( $f_3$  is injective)

$$\begin{split} f_3a_3 &= 0 \Rightarrow 0 = \psi_3 f_3 a_3 = f_4 \varphi_3 a_3 \\ \stackrel{f_4 \text{ inj.}}{\Rightarrow} \varphi_3 a_3 &= 0 \\ &\stackrel{\text{exact at } A_3}{\Rightarrow} \exists a_2 : a_3 = \varphi_2 a_2 \\ &\Rightarrow 0 = f_3 a_3 = f_3 \varphi_2 a_2 = \psi_2 f_2 a_2 \\ \stackrel{\text{exact at } B_2}{\Rightarrow} \exists b_1 : f_2 a_2 = \psi_1 b_1 \\ \stackrel{f_1 \text{ surj.}}{\Rightarrow} \exists a_1 : b_1 = f_1 a_1 \\ &\Rightarrow f_2 a_2 = \psi_1 f_1 a_1 = f_2 \varphi_1 a_1 \\ \stackrel{f_2 \text{ inj.}}{\Rightarrow} a_2 = \varphi_1 a_1 \\ \stackrel{\text{exact at } A_2}{\Rightarrow} a_3 = \varphi_2 a_2 = \varphi_2 \varphi_1 a_1 = 0 \\ a_1 \xrightarrow{\varphi_1}{\Rightarrow} a_2 \xrightarrow{\varphi_2}{\Rightarrow} a_3 \xrightarrow{\varphi_3}{\phi_3} 0 \\ f_1 \downarrow \cong f_2 \downarrow \cong f_3 \downarrow f_4 \downarrow \cong \\ b_1 \xrightarrow{\psi_1}{\Rightarrow} f_2(a_2) \xrightarrow{\psi_2}{\Rightarrow} 0 \xrightarrow{\psi_3}{\phi_3} 0 \end{split}$$

 $(f_3 \text{ is onto})$  $f_4 \text{ surj.}$ 

$$b_{3} \stackrel{f_{4} \text{ surj.}}{\Rightarrow} \exists a_{4} : f_{4}a_{4} = \psi_{3}b_{3}$$

$$\stackrel{\text{exact at } B_{4}}{\Rightarrow} f_{5}\varphi_{4}a_{4} = \psi_{4}f_{4}a_{4} = \psi_{4}\psi_{3}b_{3} = 0$$

$$\stackrel{f_{5} \text{ inj.}}{\Rightarrow} \varphi_{4}a_{4} = 0$$

$$\stackrel{\text{exact bei } A_{4}}{\Rightarrow} \exists a_{3} : a_{4} = \varphi_{3}a_{3}$$

$$\Rightarrow \psi_{3}f_{3}a_{3} = f_{4}\varphi_{3}a_{3} = f_{4}a_{4} = \psi_{3}b_{3}$$

$$\stackrel{\text{exact at } B_{3}}{\Rightarrow} \exists b_{2} : b_{3} - f_{3}a_{3} = \varphi_{2}b_{2}$$

$$\stackrel{f_{2} \text{ surj.}}{\Rightarrow} \exists a_{2} : b_{2} = f_{2}a_{2}$$

$$\Rightarrow b_{3} = f_{3}a_{3} + \psi_{2}b_{2} = f_{3}a_{3} + \psi_{2}f_{2}a_{2} = f_{3}(a_{3} + \varphi_{2}a_{2})$$

$$\bullet \qquad a_{2} \stackrel{\varphi_{2}}{\longrightarrow} a_{3} \stackrel{\varphi_{3}}{\longrightarrow} a_{4} \stackrel{\varphi_{4}}{\longrightarrow} \varphi_{4}a_{4}$$

$$\stackrel{f_{2}}{=} f_{3} \stackrel{\psi_{2}}{\longrightarrow} b_{3} \stackrel{\psi_{3}}{\longrightarrow} \psi_{3}b_{3} \stackrel{\psi_{4}}{\longrightarrow} 0$$

**Remark.** An exact sequence of the from

$$0 \to A \to B \to C \to 0$$

is called SHORT EXACT.

1. We have that the top line in the diagram

$$\begin{array}{c|c} A_{q+1} & \xrightarrow{f_{q+1}} & A_q & \xrightarrow{f_q} & A_{q-1} \\ & & & & & \\ & & & & & \\ & & & & & \\ 0 & \longrightarrow & f_{q+1}(A_{q+1}) & \longrightarrow & A_q & \longrightarrow & f_q(A_q) & \longrightarrow & 0 \end{array}$$

is exact at  ${\cal A}_q$  iff the bottom row is short exact.

2. Up to an isomorphism we have the following description of short exact sequences:



- 3. The sequence  $0 \to A \to A \oplus C \to C \to 0$  is short exact.
- 4. The sequence  $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}_m \to 0$  is short exact.

**7.24 Lemma.** [9, 8.2.4] For a short exact sequence  $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$  the following statements are equivalent:

- 1. There is an isomorphism  $\varphi : A \oplus C \to B$  such that the diagram below is commutative;
- 2. g has a right inverse  $\rho$ ;
- 3. f has a left inverse  $\lambda$ .



Under these equivalent conditions the sequence is called SPLITTING.

**Proof.**  $(1\Rightarrow 2)$  The morphism  $\rho := \varphi \circ \operatorname{inj}_2 : c \mapsto \varphi(0, c)$  is right inverse to g.

 $(2\Rightarrow3)$  The morphism  $\operatorname{id}_B - \rho \circ g$  has image in  $\operatorname{ker}(g)$ , hence factors to a morphism  $\lambda : B \to A$  over f. Hence  $f \circ \lambda \circ f = (\operatorname{id}_B - \rho \circ g) \circ f = f - 0 = f \circ \operatorname{id}$  and so  $\lambda \circ f = \operatorname{id}$ .

 $(3\Rightarrow1)$  Define  $\psi := (\lambda, g) : B \to A \oplus C$ . Then  $\psi$  makes the diagram commutative  $(\operatorname{pr}_2 \circ \psi = g \text{ and } \psi \circ f = (\operatorname{id}_A, 0) = \operatorname{inj}_1)$  and therefore is an isomorphism by 7.23.

**7.25 Example.** [9, 8.2.5] The sequence  $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}_m \to 0$  does not split. In fact, every  $a \in \mathbb{Z}_m$  has order  $\operatorname{ord}(a) \leq m < \infty$  but all  $b \in \mathbb{Z}$  have order  $\operatorname{ord}(b) = \infty$ .

**7.26 Remark.** If C is free abelian then any short exact sequence  $0 \to A \to B \to C \to 0$  splits.

If  $0 \to A \to B \to C \to 0$  is exact and A and C is finitely generated then so is B. In fact, the generators of A together with inverse images of those of C generate B.

**7.27 Definition.** [9, 8.3.4] Let  $(C, \partial)$  and  $(C', \partial')$  be two chain complexes. A CHAIN MAPPING is a family of homomorphisms  $f_q : C_q \to C'_q$  which commutes with  $\partial$ , i.e.  $\partial'_q \circ f_q = f_{q-1} \circ \partial_q$ .

**7.28 Definition.** [9, 7.3.1] Let  $\varphi : K \to L$  be a simplicial map between simplicial complexes. Define group homomorphisms  $C_q(\varphi) : C_q(K) \to C_q(L)$  by  $C_q(\varphi) := 0$ 

for q < 0 or  $q > \dim K$  and by  $C_q(\varphi)(\langle x_0, \ldots, x_q \rangle) := \langle \varphi(x_0), \ldots, \varphi(x_q) \rangle$  if  $\varphi$  is injective on  $\{x_0, \ldots, x_q\}$  and  $C_q(\varphi)(\langle x_0, \ldots, x_q \rangle) := 0$  otherwise.

**7.29 Proposition.** [9, 7.3.2] For every simplicial map  $\varphi : K \to L$  the induced map  $(C_q(\varphi))_{q \in \mathbb{Z}}$  is a chain mapping.

**Proof.** We have to show that  $\partial_q(C_q(\varphi)(\sigma)) = C_{q-1}(\varphi)(\partial_q \sigma)$  for every q-simplex  $\sigma = \langle x_0, \ldots, x_q \rangle$ . If all vertices  $\varphi(x_j)$  are distinct or are at least two pairs identical this is obvious. So we may assume that exactly two are the same. By reordering we may assume  $\varphi(x_0) = \varphi(x_1)$ . Then  $C_q(\varphi)(\sigma) = 0$  and hence also  $\partial C_q(\varphi)(\sigma)$ . On the other hand  $\partial \sigma = \langle x_1, \ldots, x_q \rangle - \langle x_0, x_2, \ldots, x_q \rangle + \sum_{j=2}^q (-1)^j \langle x_0, \ldots, \overline{x_j}, \ldots, x_q \rangle$ . The first two simplices have the same image under  $C_{q-1}(\varphi)$ . The sum is mapped to 0, since  $\varphi(x_0) = \varphi(x_1)$ .

**7.30 Lemma.** [9, 8.3.5] The chain maps form a category. Any chain map f induces homomorphisms  $H_q(f): H_q(C) \to H_q(C')$ .

**Proof.** The first statement is obvious.

Since  $f \circ \partial = \partial \circ f$  we have that  $f(Z_q) \subseteq Z'_q$  and  $f(B_q) \subseteq B'_q$  and hence  $H_q(f) : H_q(C) \to H_q(C')$  makes sense,

**7.31 Theorem.** [9, 8.3.8] Let  $0 \to C' \xrightarrow{f} C \xrightarrow{g} C'' \to 0$  be a short exact sequence of chain mappings. Then we obtain a long exact sequence in homology:

$$\dots \xrightarrow{\partial_*} H_q(C') \xrightarrow{H_q(f)} H_q(C) \xrightarrow{H_q(g)} H_q(C'') \xrightarrow{\partial_*} H_{q-1}(C') \xrightarrow{H_{q-1}(f)} \dots$$

In particular we can apply this to a chain complex C and a chain subcomplex C'and C'' := C/C', since  $\partial$  factors over  $\partial'' : C'' \to C''$ , via  $\partial''(c+C') := \partial c + C'$ .

**Proof.** Consider

We first show that it is possible to choose elements in the corresponding inverse images and then we will show that the resulting class does not depend on any of the choices.

So let  $z''_q \in C''_q$  be a cycle, i.e.  $\partial z''_q = 0$ . Since g is onto we find  $x_q \in C_q$  with  $gx_q = z''_q$ . Since  $g\partial x_q = \partial gx_q = \partial z''_q = 0$ , we find  $x'_{q-1} \in C'_{q-1}$  with  $fx'_{q-1} = \partial x_q$ . And hence  $x'_{q-1} \in f^{-1}\partial g^{-1}z''_q$ . Furthermore  $f\partial x'_{q-1} = \partial fx'_{q-1} = \partial \partial x_q = 0$ . Since f is injective we get  $\partial x'_{q-1} = 0$  and hence we may form the class  $\partial_*[z''_q] := [x'_{q-1}]$ .

$$\begin{array}{c|c} x_q & \xrightarrow{g} z''_q \\ & \partial \\ & \partial \\ x'_{q-1} & \xrightarrow{f} \partial x_q & \xrightarrow{g} 0 \\ \partial \\ \partial \\ \partial \\ x'_{q-1} & \xrightarrow{f} 0 \end{array}$$

Now the independency from all choices, So let  $[z''_q] = [\bar{z}''_q]$ , i.e.  $\exists x''_{q+1} : \partial x''_{q+1} = z''_q - \bar{z}''_q$ . Choose  $x_q, \bar{x}_q \in C_q$  as before, so that  $gx_q = x''_q$  and  $g\bar{x}_q = \bar{x}''_q$ . Also as before choose  $x'_{q-1}, \bar{x}'_{q-1} \in C'_{q-1}$  with  $fx'_{q-1} = \partial x_q$  and  $f\bar{x}'_{q-1} = \partial \bar{x}_q$ . We have to show that  $[x'_{q-1}] = [\bar{x}'_{q-1}]$ . So choose  $x_{q+1} \in C_{q+1}$  with  $gx_{q+1} = x''_{q+1}$ . Then  $g\partial x_{q+1} = \partial gx_{q+1} = \partial x''_{q+1} = z''_q - \bar{z}''_q = g(x_q - \bar{x}_q)$ , hence there exists an  $x'_q \in C_q$  with  $fx'_q = \partial x_{q+1} - x_q + \bar{x}_q$ . And  $f\partial x'_q = \partial fx'_q = \partial(\partial x_{q+1} - x_q + \bar{x}_q) = 0 - \partial x_q + \partial \bar{x}_q = -f(x'_{q-1} - \bar{x}'_{q-1})$ . Since f is injective we have  $x'_{q-1} = \bar{x}'_{q-1} + \partial x'_q$ , i.e.  $[x'_{q-1}] = [\bar{x}'_{q-1}]$ .

Exactness at  $H_q(C')$ :  $(\subseteq) f_*\partial_*[z''] = [ff^{-1}\partial g^{-1}z''] = [\partial g^{-1}z''] = 0.$   $(\supseteq)$  Let  $\partial z' = 0$  and  $0 = f_*[z'] = [fz']$ , i.e.  $\exists x: \partial x = fz'$ . Then x'' := gx satisfies  $\partial x'' = \partial gx = g\partial x = gfz' = 0$  and  $\partial_*[x''] = [f^{-1}\partial g^{-1}gx] = [f^{-1}\partial x] = [z'].$ 

Exactness at  $H_q(C)$ :  $(\subseteq)$  since  $g \circ f = 0$ .  $(\supseteq)$  Let  $\partial z = 0$  with  $0 = g_*[z] = [gz]$ , i.e.  $\exists x'': \partial x'' = gz$ . Then  $\exists x: gx = x''$ . Hence  $gz = \partial x'' = \partial gx = g\partial x \Rightarrow \exists x': fx' = z - \partial x \Rightarrow f\partial x' = \partial fx' = \partial(z - \partial x) = 0$  $\Rightarrow \partial x' = 0$  and  $f_*[x'] = [fx'] = [z - \partial x] = [z]$ .

Exactness at  $H_q(C'')$ :  $(\subseteq)$  We have  $\partial_*g_*[z] = [f^{-1}\partial g^{-1}gz] = [f^{-1}\partial z] = [f^{-1}0] = 0.$   $(\supseteq)$  Let  $\partial z'' = 0$  and  $0 = \partial_*[z'']$ , i.e.  $\exists x': \partial x' = z'$ , where  $z' \in f^{-1}\partial g^{-1}z''$ , i.e.  $\exists x: gx = z''$  and  $fz' = \partial x$ . Then  $\partial(x - fx') = fz' - f(\partial x') = 0$  and g(x - fx') = z'' - 0, i.e.  $g_*[x - fx'] = [z'']$ .

## **Relative homology**

**7.32 Definition.** [9, 7.4.1] Let  $K_0 \subseteq K$  be a simplicial subcomplex. Then  $C(K_0)$  is a chain subcomplex of C(K) and hence we may form the chain complex  $C(K, K_0)$ given by  $C_q(K, K_0) := C_q(K)/C_q(K_0)$ . Note that we can identify this so-called RELATIVE CHAIN GROUP with the free abelian group generated by all q-simplices in  $K \setminus K_0$ . The boundary operator is given by taking the boundary of  $\sum_{\sigma} k_{\sigma} \cdot \sigma$  in C(K), but deleting all summands of simplices in  $C(K_0)$ . The q-th homology group of  $C(K, K_0)$  will be denoted by  $H_q(K, K_0)$  and is call the RELATIVE HOMOLOGY of K with respect to  $K_0$ . Using the short exact sequence  $0 \to C(K_0) \to C(K) \to$  $C(K, K_0) \to 0$  we get a long exact sequence in homology by [7.31].

7.33 Remark. [9, 7.4.2]

- 1. If  $K_0 = K$  then  $C_q(K, K_0) = C_q(\emptyset) = \{0\}$  and hence  $H_q(K, K) = \{0\}$ .
- 2. If  $K_0 = \emptyset$  then  $C_q(K, K_0) = C_q(K)$  and hence  $H_q(K, \emptyset) = H_q(K)$ .

- 3. If K is connected and  $K \supseteq K_0 \neq \emptyset$ , then  $H_0(K, K_0) = \{0\}$ . In fact let  $z \in C_0(K, K_0)$ , i.e.  $z = \sum_{x \in K \setminus K_0} k_x \cdot x$ . Let  $x_0 \in K_0$  be chosen fixed. Since K is connected we find for every  $x \in K$  a 1-chain c with boundary  $\partial c = x x_0$ , hence  $z \sim \varepsilon(z) \cdot x_0 = 0$  in  $C_0(K, K_0)$ .
- 4. Note that in [7.22] we calculated the relative chain complex  $C_q(K, L)$ , where  $K := K(\sigma_n)$  and  $L := K(\dot{\sigma}_n)$  and obtained  $C_q(K, L) = \{0\}$  for  $q \neq n$  and  $C_n(K, L) = \langle \sigma_n \rangle \cong \mathbb{Z}$ . Hence  $H_q(K, L) \cong \{0\}$  for  $q \neq 0$  and  $H_n(K, L) \cong \mathbb{Z}$ .

**7.34 Example.** [9, 7.4.7] Let M be the Möbius strip with boundary  $\partial M$ . We have a triangulation of M in 5 triangles as in 3.9.2. Since  $\partial M$  is a 1-sphere  $H_1(\partial M) \cong \mathbb{Z}$  by 7.22, where a generator is given by the 1-cycle r formed by the 5-edges of the boundary.

Furthermore  $H_1(M) \cong \mathbb{Z}$ , where a generator is given by the sum *m* of the remaining edges: In fact every triangle has two of these edges, so it suffices to consider linear combinations of these edges. Since every vertex belongs to exactly two of these edges, the coefficients have to be equal.

If a combination of triangles has a multiple of m as boundary, their coefficients have to be 0, cf. 7.13.

Now consider the following fragment of the long exact homology sequence:

Since  $H_0(\partial M) \cong \mathbb{Z} \cong H_0(M)$  by  $\boxed{7.12}$ , where a generator is given by any point  $x_0$  in  $\partial M \subseteq M$ , we have that the rightmost arrow is a bijection, so the one to the left is 0 and hence the previous one is onto. Remains to calculate the image of  $\langle [r] \rangle = H_1(\partial M) \to H_1(M) = \langle [m] \rangle$ . For this we consider the sum over all triangles (alternating oriented). It has boundary 2m - r and hence [r] is mapped to 2[m]. Thus  $H_1(M, \partial M) \cong \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$ .

**7.35 Proposition.** [9, 8.3.11] Let (C, C') and (D, D') be pairs of chain complexes, C'' = C/C', D'' = D/D' and  $f : (C, C') \to (D, D')$  be a chain mapping of pairs. This induces a homomorphism which intertwines with the long exact homology sequences.

$$\cdots \xrightarrow{\partial_*} H_q(C') \xrightarrow{H_q(i)} H_q(C) \xrightarrow{H_q(p)} H_q(C'') \xrightarrow{\partial_*} H_{q-1}(C'') \xrightarrow{H_{q-1}(g)} \cdots$$

$$f_* \downarrow \qquad f_* \downarrow$$

**Proof.** The commutativity of all but the rectangle involving  $\partial_*$  is obvious. For this remaining one let  $z'' \in C''$  be a cycle. We have to show that  $\partial_* f_*[z''] = f_*\partial_*[z'']$ . So let  $z' \in i^{-1}\partial p^{-1}z''$ , i.e.  $iz' = \partial x$  for some x with px = z''. Then  $f_*\partial_*[z''] = [fz']$  and we have to show that  $j(fz') \in \partial q^{-1}fz''$ , which follows from  $jfz' = fiz' = f\partial x = \partial fx$  and q(fx) = f(px) = fz''.



**7.36 Corollary.** [9, 7.4.6] Proposition 7.35 applies in particular to a simplicial mapping  $\varphi : (K, K_0) \to (L, L_0)$  of pairs.

**7.37 Excision Theorem.** [9, 7.4.9] Let K be the union of two subcomplexes  $K_0$  and  $K_1$ . Then  $(K_1, K_0 \cap K_1) \to (K, K_0)$  induces an isomorphism  $H(K_1, K_0 \cap K_1) \to H(K_1 \cup K_0, K_0)$ .

**Proof.** Note that we have

$$K_1 \setminus (K_0 \cap K_1) = K_1 \setminus K_0 = (K_0 \cup K_1) \setminus K_0$$

and also

This is gives even an isomorphism on the level of chain complexes, as follows from the commutativity of the diagram.  $\hfill \Box$ 

If  $U := K \setminus K_1 = K_0 \setminus (K_0 \cap K_1)$  then  $K_1 = K \setminus U$  and  $K_0 \cap K_1 = K_0 \setminus U$ , hence the isomorphism of [7.37] reads  $H(K \setminus U, K_0 \setminus U) \cong H(K, K_0)$ . Conversely, if  $(K, K_0)$  is a pair of simplicial complexes and  $U \subseteq K_0$  is such that  $K_1 := K \setminus U$  is a simplicial complex, then we get:

**7.38 Corollary.** [9, 7.4.8] Let  $K_0 \subseteq K$  be a pair of simplicial complexes and  $U \subseteq K_0$  a set such that  $\forall \tau \in U : \tau < \sigma \Rightarrow \sigma \in U$ . Then  $K_1 := K \setminus U$  and  $K_0 \cap K_1 := K_0 \setminus U$  are simplicial complexes and  $H(K, K_0) \cong H(K \setminus U, K_0 \setminus U)$ .  $\Box$
## 8. Singular Homology

### Basics

**8.1 Definition.** [9, 9.1.1] The STANDARD (CLOSED!) q-SIMPLEX  $\Delta_q$  is the simplex spanned by the standard unit vectors  $e_j \in \mathbb{R}^{q+1}$  for  $0 \leq j \leq q$ . So

$$\Delta_q := \Big\{ (\lambda_0, \dots, \lambda_q) : 0 \le \lambda_j \le 1 : \sum_j \lambda_j = 1 \Big\}.$$

**8.2 Definition.** [9, 9.1.2] For  $q \ge 1$  and  $0 \le j \le q$  let the FACE-MAP  $\delta_{q-1}^j : \Delta_{q-1} \to \Delta_q$  be the unique affine map, which maps  $e_i$  to  $e_i$  for i < j and to  $e_{i+1}$  for i > j, i.e.

$$e_0, \ldots, e_{q-1} \mapsto e_0, \ldots, \overline{e_j}, \ldots, e_q.$$

**8.3 Lemma.** [9, 9.1.3] For  $q \ge 2$  and  $0 \le k < j \le q$  we have  $\delta_{q-1}^j \circ \delta_{q-2}^k = \delta_{q-1}^k \circ \delta_{q-2}^{j-1}$ .

**Proof.** The mapping on the left side has the following effect on the edges:

$$e_0, \ldots, e_{q-2} \mapsto e_0, \ldots, \overline{e_k}, \ldots, e_{q-1} \mapsto e_0, \ldots, \overline{e_k}, \ldots, \overline{e_j}, \ldots, e_q$$

And on the right side:

$$e_0, \dots, e_{q-2} \mapsto e_0, \dots, \boxed{e_{j-1}}, \dots, e_{q-1} \mapsto e_0, \dots, \boxed{e_k}, \dots, \boxed{e_j}, \dots, e_q \quad \Box$$

**8.4 Definition.** [9, 9.1.4] Let X be a topological space. A SINGULAR q-SIMPLEX is a continuous map  $\sigma : \Delta_q \to X$ . The q-th SINGULAR CHAIN GROUP  $S_q(X)$  is the free abelian group generated by all singular q-simplices, i.e.

$$S_q(X) := {}^{\mathrm{ab}}\mathcal{F}(C(\Delta_q, X))$$

Its elements are called SINGULAR  $q\text{-}\mathrm{CHAINS}.$  The boundary operator  $\partial$  is the linear extension of

$$\partial: \sigma \mapsto \sum_{j=0}^q (-1)^j \sigma \circ \delta^j.$$

By 8.3 the groups  $S_q(X)$  together with  $\partial$  from a chain complex S(X):

$$\begin{aligned} \partial \partial \sigma &= \partial \Big( \sum_{j=0}^{q} (-1)^{j} \sigma \circ \delta^{j} \Big) \\ &= \sum_{j=0}^{q} (-1)^{j} \sum_{k=0}^{q-1} (-1)^{k} \sigma \circ \delta^{j} \circ \delta^{k} \\ &= \sum_{0 \leq k < j \leq q} (-1)^{j+k} \sigma \circ \delta^{j} \circ \delta^{k} + \sum_{0 \leq j \leq k < q} (-1)^{j+k} \sigma \circ \delta^{j} \circ \delta^{k} \\ &\stackrel{\boxed{8.3}}{=} \sum_{0 \leq k < j \leq q} (-1)^{j+k} \sigma \circ \delta^{k} \circ \delta^{j-1} + \sum_{0 \leq j < k \leq q} (-1)^{j+k-1} \sigma \circ \delta^{j} \circ \delta^{k-1} \\ &= \sum_{0 \leq k < j \leq q} (-1)^{j+k} \sigma \circ \delta^{k} \circ \delta^{j-1} + \sum_{0 \leq k < j \leq q} (-1)^{k+j-1} \sigma \circ \delta^{k} \circ \delta^{j-1} = 0. \quad \Box \end{aligned}$$

The q-th SINGULAR HOMOLOGY GROUP  $H_q(X)$  is defined to be  $H_q(S(X))$ . The elements of  $B_q(X) := B_q(S(X))$  are called (SINGULAR) q-BOUNDARIES and those of  $Z_q(X) := Z_q(S(X))$  are called (SINGULAR) q-CYCLES.

Remark that singular 0-simplices can be identified with the points in X and singular 1-simplices with paths in X.

**8.5 Definition.** [9, 9.1.6] [9, 9.1.8] [9, 9.1.9] Let  $f : X \to Y$  be continuous. Then f induces a chain-mapping  $f_* := S(f) : S(X) \to S(Y)$  and hence group-homomorphisms  $f_* := H_q(f) : H_q(X) \to H_q(Y)$ .

$$\partial \Big( S(f)(\sigma) \Big) = \partial (f \circ \sigma) = \sum_{j=0}^{q} (-1)^j f \circ \sigma \circ \delta^j = S(f) \Big( \sum_{j=0}^{q} (-1)^j \sigma \circ \delta^j \Big) = S(f) \Big( \partial (\sigma) \Big).$$

So  $H_q$  is a functor from continuous maps between topological spaces into group homomorphisms between abelian groups.

**8.6 Remark.** [9, 9.1.7] The identity  $id_{\Delta_q} : \Delta_q \to \Delta_q$  is a singular q-simplex of  $\Delta_q$ , which we will denote again by  $\Delta_q$ . If  $\sigma$  is a singular q-simplex in X, then  $S(\sigma)(\Delta_q) = \sigma$ . We will make use of this several times in order to construct natural transformations, by defining them first for the standard simplex, see 8.22 and 8.30.

**8.7 Theorem.** [9, 9.1.10] Let  $X = \{*\}$  be a single point. Then  $H_q(X) = \{0\}$  for  $q \neq 0$  and  $H_0(X) = S_0(X) \cong \mathbb{Z}$ .

A space X is called ACYCLIC iff it is path-connected and  $H_q(X) = \{0\}$  for  $q \neq 0$ .

**Proof.** The only singular q-simplex is the constant mapping. Its boundary is  $\partial \sigma_q = (\sum_{i=0}^{q} (-1)^j)\sigma_{q-1}$ . For even q > 0 we have that  $Z_q(X) = \{0\}$  for odd q we have that  $B_q(X) = Z_q(X)$ , hence in both cases  $H_q(X) = \{0\}$ . For q = 0 we have  $B_0(X) = \{0\}$  and  $Z_0(X) = S_0(X) \cong \mathbb{Z}$ .

**8.8 Corollary.** [9, 9.1.11] Let  $f: X \to Y$  be constant then  $H_q(f) = 0$  for  $q \neq 0$ .

**Proof.** Obvious, since f factors over a single point.

**8.9 Proposition.** [9, 9.1.12] Let  $X_j$  be the path components of X. Then the inclusions of  $X_j \to X$  induce an isomorphism  $\bigoplus_j H_q(X_j) \to H_q(X)$ ; cf. 7.11.

**Proof.** This follows as [7.11]: Let  $\sigma$  be a singular simplex of X. Then  $\sigma$  is completely contained in some  $X_j$ , hence the induced map  $\bigoplus_j S_q(X_j) \to S_q(X)$  is onto. Conversely this linear map is injective, since the chains in the various  $X_j$  have disjoint images. Thus we have a bijection  $\bigoplus_j S_q(X_j) \cong S_q(X)$ , which induces an isomorphism of homology groups.

**8.10 Proposition.** [9, 9.1.13] Let X be a topological space. Then  $H_0(X)$  is a free abelian group with generators given by choosing one point in each path-component; cf. 7.12.

**Proof.** Because of 8.9 we may assume that X is path-connected. The mapping  $\varepsilon: S_0(X) \to \mathbb{Z}, \sum_{\sigma} n_{\sigma} \cdot \sigma \mapsto \sum_{\sigma} n_{\sigma}$  is onto and as in 7.12 its kernel is just  $B_0(X)$ , so  $\varepsilon$  induces an isomorphism  $H_0(X) \cong \mathbb{Z}$ ; cf. 7.12.

**8.11 Corollary.** [9, 9.1.14] Let X and Y be path-connected. Then every continuous mapping  $f: X \to Y$  induces an isomorphism  $H_0(f): H_0(X) \to H_0(Y)$ .

**Proof.** Obvious since the generator is mapped to a generator.

**8.12 Definition.** [9, 9.1.15] Let  $A \subseteq \mathbb{R}^n$  be convex and  $p \in A$  be fixed. For a singular q-simplex  $\sigma : \Delta_q \to A$  we define the CONE  $p \star \sigma : \Delta_{q+1} \to A$  by

$$(p \star \sigma)((1-t)e^0 + t\delta^0(x)) := (1-t)p + t\sigma(x) \text{ for } t \in [0,1] \text{ and } x \in \Delta_q.$$

For a q-chain  $c = \sum_{\sigma} n_{\sigma} \cdot \sigma$  we extend it by linearity

$$p \star c := \sum_{\sigma} n_{\sigma} \cdot (p \star \sigma)$$

and obtain a homomorphism  $S_q(A) \to S_{q+1}(A)$ ; cf. 7.19.

**8.13 Lemma.** [9, 9.1.16] Let  $c \in S_q(A)$  then

$$\partial(p \star c) = c - \varepsilon(c) p \text{ for } q = 0,$$
  
$$\partial(p \star c) = c - p \star \partial c \text{ for } q > 0,$$

where  $\varepsilon \left( \sum_{x} n_x \cdot x \right) = \sum_{x} n_x$ ; cf. 7.20.

**Proof.** It is enough to show this for singular simplices  $c = \sigma_q$ . For q = 0 we have that  $p \star \sigma : \Delta_1 \to X$  is a path from p to  $\sigma$  hence  $\partial(p \star \sigma) = \sigma - p = \sigma - \varepsilon(\sigma)p$ . For q > 0 we have  $(p \star \sigma) \circ \delta^0 = \sigma$  and  $(p \star \sigma) \circ \delta^i = p \star (\sigma \circ \delta^{i-1})$  for i > 0, hence  $\partial(p \star \sigma) = \sigma - p \star \partial \sigma$ .

**8.14 Corollary.** [9, 9.1.18] Let  $A \subseteq \mathbb{R}^n$  be convex. Then A is acyclic; cf. 7.20 & 7.21.

**Proof.** Let  $p \in A$  and z be a q-cycle for q > 0. Then  $z = \partial(p \star z)$  by [8.13] and hence  $Z_q(A) = B_q(A)$ , i.e.  $H_q(A) \cong \{0\}$ .

### **Relative homology**

**8.15 Definition.** [9, 9.2.1] Let (X, A) be a pair of spaces. Then we get a pair of chain complexes (S(X), S(A)) and hence a short exact sequence

$$0 \to S(A) \to S(X) \to S(X, A) \to 0,$$

where  $S_q(X, A) := S_q(X)/S_q(A)$ . Its elements are called RELATIVE SINGULAR *q*-CHAINS. But unlike **7.32** we can not identify them with formal linear combinations of simplices in  $X \setminus A$ .

8.16 Remark. [9, 9.2.3] But as in 7.32 we get a long exact sequence in homology  $\dots \to H_q(A) \to H_q(X) \to H_q(X, A) \to H_{q-1}(A) \to \dots,$ 

where  $H_q(X, A) := H_q(S(X, A))$ . Note that  $z \in S_q(X)$  with  $\partial z \in S_q(A)$  describe the classes  $[z + S_q(A)] \in H_q(X, A)$ .

For a continuous mapping of pairs  $(A, X) \to (B, Y)$  we get a homology ladder as in 7.35.

8.17 Remark. [9, 9.2.2] As in 7.33 we get

- 1.  $H_q(X,X) = \{0\},\$
- 2.  $H_q(X, \emptyset) \cong H_q(X)$ , and
- 3.  $H_0(X, A) = \{0\}$  for path connected X and  $A \neq \emptyset$ .

8.18 Remark. [9, 9.2.4] Using the long exact homology sequence

$$\dots H_{q+1}(X,A) \to H_q(A) \to H_q(X) \to H_q(X,A) \to H_{q-1}(A) \to \dots,$$

we obtain:

- 1. Let  $A \subseteq X$  be such that  $H_q(A) \to H_q(X)$  is injective for all q. Then we get short exact sequences  $0 \to H_q(A) \to H_q(X) \to H_q(X, A) \to 0$ , where the right most arrow is 0, since the next one in the exact homology sequence of the pair is assumed to be injective.
- 2. Let  $A \subseteq X$  be a retract. Then by functorality  $H_q(A) \to H_q(X)$  is a retract and hence by (1) we have (splitting) short exact sequences.
- 3. Let  $x_0 \in X$ . The constant mapping  $X \to \{x_0\}$  is a retraction, hence  $H_q(X) \cong H_q(\{x_0\}) \oplus H_q(X, \{x_0\})$  by 2. By 8.7 we have that  $H_q(\{x_0\}) = \{0\}$  for  $q \neq 0$  and  $H_0(\{x_0\}) = \mathbb{Z}$ , hence  $H_q(X, \{x_0\}) \cong H_q(X)$  for q > 0 and  $0 \to \mathbb{Z} \to H_0(X) \to H_0(X, \{x_0\}) \to 0$  is splitting exact.
- 4. Let  $f : (X, A) \to (Y, B)$  be such that  $f_* : H_q(A) \to H_q(B)$  and  $f_* : H_q(X) \to H_q(Y)$  are isomorphisms for all q. Then the same is true for  $f_* : H_q(X, A) \to H_q(Y, B)$  by the 5'Lemma.

**8.19 Theorem. Exact Homology Sequence of a Triple.** [9, 9.2.5] Let  $B \subseteq A \subseteq X$ . Then we get a long exact homology sequence

$$\cdots \to H_{q+1}(X,A) \xrightarrow{\partial_*} H_q(A,B) \to H_q(X,B) \to H_q(X,A) \to \dots$$

The boundary operator  $\partial_*$  can also be described by  $[z]_{(X,A)} \mapsto [\partial z]_{(A,B)}$  for  $z \in S_q(X)$  or as composition  $H_{q+1}(X,A) \xrightarrow{\partial_*} H_q(A) \to H_q(A,B)$ .

**Proof.** We have a short sequence

$$0 \to S(A, B) \to S(X, B) \to S(X, A) \to 0.$$

given by



Hence the bottom row is exact at S(X, A) and at S(A, B): In fact for  $\dot{a} \in S(A, B)$  let the image in S(X, B) be 0. Then  $a = b \in S(B)$  and hence  $\dot{a} = 0$  in S(A, B).



It is also exact at S(X, B), since for  $\dot{x} \in S(X, B)$  which is mapped to 0 in S(X, A) the image  $x \in S(X)$  is an  $a \in S(A)$  and hence satisfies  $\dot{a}$  is mapped to  $\dot{x}$ .



So this short exact sequence induces a long exact sequence in homology. The boundary operator maps by construction  $\boxed{7.31}$  the class [z + S(A)] with  $\partial z \in S(A)$  to  $[\partial z + S(B)]$ . This is precisely the image of value of the boundary operator  $[\partial z]$  for the pair (X, A) under the natural map  $H(A) \to H(A, B)$ .

**8.20 Exercise.** Let X be path-connected and  $A \subseteq X$ . Then  $H_0(X, A) = 0$ .

#### Homotopy theorem

We are now going to prove that homotopic mappings induce the same mappings in homology. For this we consider first a homotopy, which is as free and as natural as possible, i.e. the homotopy given by  $\operatorname{inj}_t : X \to X \times I$ ,  $x \mapsto (x,t)$ . We have to show that  $\operatorname{inj}_0$  and  $\operatorname{inj}_1$  induce the same mapping in homology. So the images of a cycle should differ only by a boundary. Let  $\sigma : \Delta_q \to X$  be a singular simplex. Then we may consider the cylinder  $\sigma(\Delta_q) \times I$  over  $\sigma(\Delta_q)$ . It seems clear, that we can triangulate  $\Delta_q \times I$ . The image of the corresponding chain  $c_{q+1}$  under  $\sigma \times I$  gives then a q+1-chain in  $X \times I$ , whose boundary consists of the parts  $\sigma \times \{1\} = \operatorname{inj}_1 \circ \sigma$  and  $\sigma \times \{0\} = \operatorname{inj}_0 \circ \sigma$  and a triangulation of  $(\sigma \times I)_* \partial c_q$ . Note that it would havew been easier here, if we had defined the singular homology by using squares instead of triangles, since it is not so clear how to describe an explicit triangulation of  $\Delta_q \times I$ , in fact we will show the existence of  $c_{q+1}$  by induction in the following lemma.

We make use of the following

**8.21 Definition.** [9, 8.4.6] Let  $R, S : \mathcal{X} \to \mathcal{Y}$  be two functors. A NATURAL TRANS-FORMATION  $\varphi : R \to S$  is a family  $\varphi_X : R(X) \to S(X)$  of  $\mathcal{Y}$ -morphisms for every object  $X \in \mathcal{X}$  such that for every  $\mathcal{X}$ -morphism  $f : X \to X'$  the following diagram commutes:

$$\begin{array}{c|c} R(X) \xrightarrow{\varphi_X} S(X) \\ R(f) & S(f) \\ R(X') \xrightarrow{\varphi_{X'}} S(X') \end{array}$$

**8.22 Lemma.** [9, 9.3.7] Let  $\varphi_0, \varphi_1 : S(\_) \to S(\_\times I)$  be two natural transformations and assume furthermore that  $H_0(\varphi_0) = H_0(\varphi_1) : H_0(\{*\}) \to H_0(\{*\} \times I)$ . Then  $\varphi_0$ and  $\varphi_1$  are chain homotopic, i.e. there exists natural homomorphisms  $\mathcal{Z} = (\mathcal{Z}_q)_q$ with  $\mathcal{Z}_q : S_q(X) \to S_{q+1}(X \times I)$  and  $\partial \mathcal{Z}_q + \mathcal{Z}_{q-1}\partial = \varphi_1 - \varphi_0$  on  $S_q(X)$ .

## **Proof.** We construct $\mathcal{Z}_q$ by induction on q:

For q < 0 let  $Z_q := 0$ . Now let  $Z_j$  for all j < q be already constructed. Consider the natural transformation  $\varphi := \varphi_1 - \varphi_0$ . We first treat the case  $X := \Delta_q$ . So we have to find  $Z_q(\Delta_q) =: c_{q+1} \in S_q(\Delta_q \times I)$  with  $\partial c_{q+1} = \varphi \Delta_q - Z_{q-1} \partial \Delta_q$ . For q = 0this follows from the assumption  $[\varphi(\Delta_0)] = 0 \in H_0(\Delta_0 \times I)$ . For q > 0 we can use that  $S(\Delta_q \times I)$  is acyclic (since  $\Delta_q \times I$  is a convex subset of  $\mathbb{R}^{q+2}$ ) by 8.14. So we only have to show that the right side is a cycle. In fact by induction hypothesis (applied to  $\partial \Delta_q$ ) we have

$$\partial \Big(\varphi \Delta_q - \mathcal{Z}_{q-1} \partial \Delta_q \Big) = \varphi \partial \Delta_q - \Big(\varphi - \mathcal{Z}_{q-2} \partial \Big) \partial \Delta_q = \varphi \partial \Delta_q - \Big(\varphi \partial \Delta_q - \mathcal{Z}_{q-2} \partial \partial \Delta_q \Big) = 0.$$

Now we extend  $\mathcal{Z}_q : S_q(X) \to S_{q+1}(X \times I)$  by naturality to the case of a general X: I.e. for  $\sigma : \Delta_q \to X$  we define  $\mathcal{Z}_q(\sigma) := S_{q+1}(\sigma \times I)(c_{q+1})$ .

Then  $\mathcal{Z}_q$  is in fact natural, since  $S_q(f \times I)\mathcal{Z}_q(\sigma) = S_{q+1}(f \times I)S_{q+1}(\sigma \times I)c_{q+1}$  and  $\mathcal{Z}_qS_q(f)(\sigma) = \mathcal{Z}_q(f\sigma) = S_q(f\sigma \times I)c_{q+1}$  and  $(f \times I) \circ (\sigma \times I) = (f \circ \sigma) \times I$ .

$$\begin{split} S_q(Y) & \xrightarrow{\mathcal{Z}_q} S_q(Y \times I) \\ & \uparrow f_* & \uparrow (f \times I)_* \\ S_q(X) & \xrightarrow{\mathcal{Z}_q} S_q(X \times I) \\ & \uparrow \sigma_* & \uparrow (\sigma \times I)_* \\ S_q(\Delta_q) & \xrightarrow{\mathcal{Z}_q} S_q(\Delta_q \times I) \end{split}$$

Furthermore  $\mathcal{Z}_q$  is also a chain-homotopy, since

$$\begin{aligned} \partial \mathcal{Z}_q(\sigma) &= \partial S_q(\sigma \times I)(c_{q+1}) = S_q(\sigma \times I)\partial c_{q+1} = S_q(\sigma \times I)(\varphi \Delta_q - \mathcal{Z}_{q-1}\partial \Delta_q) \\ &= \varphi S_q(\sigma)(\Delta_q) - \mathcal{Z}_{q-1}\partial S_q(\sigma)(\Delta_q) = \varphi(\sigma) - \mathcal{Z}_{q-1}\partial(\sigma). \quad \Box \end{aligned}$$

**8.23 Definition.** [9, 8.3.12] [9, 8.3.15] Two chain mappings  $\varphi, \psi : C \to C'$  are called (CHAIN) HOMOTOPIC and we write  $\psi \sim \varphi$  if there are homomorphisms  $\mathcal{Z} : C_q \to C'_{q+1}$  such that  $\psi - \varphi = \partial \mathcal{Z} + \mathcal{Z}\partial$ .

**8.24 Proposition.** [9, 8.3.13] Let  $\varphi \sim \psi : C \rightarrow C'$  then  $H(\varphi) = H(\psi) : H(C) \rightarrow H(C')$ .

**Proof.** Let  $[c] \in H(C)$ , i.e.  $\partial c = 0$  then  $H(\psi)[c] - H(\varphi)[c] = [(\psi - \varphi)c] = [\mathcal{Z}\partial c + \partial \mathcal{Z}c] = [\partial \mathcal{Z}c] = 0.$ 

**8.25 Proposition.** [9, 8.3.14] Chain homotopies are compatible with compositions.

**Proof.** Clearly, for  $\varphi \sim \psi$  we have  $\chi \circ \varphi \sim \chi \circ \psi$  and  $\varphi \circ \chi \sim \psi \circ \chi$  and being homotopic is transitive.

**8.26 Theorem.** [9, 9.3.1] Let  $f \sim g : (X, A) \to (Y, B)$ . Then  $f_* = g_* : H_q(X, A) \to H_q(Y, B)$ .

**Proof.** By 8.22 we have that the chain mappings induced by the inclusions  $\operatorname{inj}_j : X \to X \times I$  are chain homotopic to each other for  $j \in \{0, 1\}$  by a chain homotopy Z. Let h be a homotopy between f and g, i.e.  $f = h \circ \operatorname{inj}_0$  and  $g = h \circ \operatorname{inj}_1$ . By 8.25 we have  $S(f) \sim S(g) : S(X) \to S(Y)$  and also the restrictions  $S(f) \sim S(g) : S(A) \to S(B)$ , since the constructed homotopy is natural. Thus  $S(f) \sim S(g) : S(X, A) \to S(X, B)$ . By 8.24 we have that  $H(f) = H(g) : H(X, A) \to H(X, B)$ .

**8.27 Corollary.** [9, 9.3.2] Let  $f \sim g : X \to Y$ . Then  $f_* = g_* : H_q(X) \to H_q(Y)$ .

**Proof.** Obvious, since  $H_q(X, \emptyset) \cong H_q(X)$ .

**8.28 Corollary.** [9, 9.3.3] Let  $f : X \to Y$  be a homotopy equivalence. Then  $f_* : H_q(X) \to H_q(Y)$  is an isomorphism for all q. In particular all contractible spaces are acyclic.

**Proof.** Obvious by functoriality and 8.27 since an inverse g up to homotopy induces an inverse H(g) of H(f).

8.29 Corollary. [9, 9.3.4] [9, 9.3.5] [9, 9.3.6]

- 1. Let  $A \subseteq X$  be a DR. Then  $H_q(A) \to H_q(X)$  is an isomorphism and hence  $H_q(X, A) = \{0\}$  for all q.
- 2. Let  $B \subseteq A \subseteq X$  and A be a DR of X. Then  $H_q(A, B) \to H_q(X, B)$  is an isomorphism.
- 3. Let  $B \subseteq A \subseteq X$  and B be a DR of A. Then  $H_q(X,B) \to H_q(X,A)$  is an isomorphism.

**Proof.** The first part follows as special case from 8.28 and from the long exact homology sequence of a pair, 8.15. The other two cases then follow by using the long exact homology sequence of a triple, 8.19.

#### Excision theorem

In order to prove the general excision theorem we need barycentric refinement for singular simplices, since a singular simplex in X need neither be contained in S(U) nor in S(V) for a given covering  $\{U, V\}$  of X.

**8.30 Definition.** [9, 9.4.1] For the standard q-simplex  $\Delta_q$  we define the BARYCEN-TRIC CHAIN  $B(\Delta_q) \in S_q(\Delta_q)$  recursively by

$$B(\Delta_0) := \Delta_0 = \{e_0\}$$
  
$$B(\Delta_q) := \widehat{\Delta_q} \star \sum_{j=0}^q (-1)^j S(\delta^j) (B(\Delta_{q-1})) \text{ for } q \ge 1,$$

where  $\widehat{\Delta}_q$  denotes the barycenter  $\frac{1}{q+1} \sum_{j=0}^{q} e^j$ .

Now we define in a natural way  $B(\sigma) := B(S(\sigma)(\Delta_q)) = S(\sigma)B(\Delta_q)$  for  $\sigma : \Delta_q \to X$  and extend it linearly to  $B : S_q(X) \to S_q(X)$  by setting  $B(\sum_{\sigma} n_{\sigma} \cdot \sigma) := \sum_{\sigma} n_{\sigma}B(\sigma)$ .

**8.31 Proposition.** [9, 9.4.2] The barycentric refinement is a natural chain mapping  $B: S(\_) \to S(\_)$  with  $B \sim id$ .

**Proof.** Let us first show naturality: So let  $f: X \to Y$  be continuous. Then

$$(f_*B)\sigma = (f_*\sigma_*B)\Delta_q = (f \circ \sigma)_*B\Delta_q = B(f \circ \sigma) = Bf_*\sigma.$$

Next we prove that it is a chain mapping, i.e.  $\partial B = B\partial$ . On  $S_q(X)$  with  $q \leq 0$  this is obvious. Note that the formula for  $B(\Delta_q)$  can be rewritten as

$$B\Delta_q = \Delta_q \star B\partial\Delta_q.$$

so we use induction for q > 0:

$$\partial B\sigma = \partial \sigma_* B\Delta_q = \sigma_* \partial B\Delta_q = \sigma_* \partial \left(\widehat{\Delta_q} \star B\partial \Delta_q\right)$$

$$\underbrace{\overset{8.13}{=\!\!=\!\!=}} \sigma_* \left( B\partial \Delta_q - \widehat{\Delta_q} \star \partial B\partial \Delta_q \right) \underbrace{\overset{\text{I.Hyp.}}{=\!\!=\!\!=}} B\sigma_* \partial (\Delta_q) - \sigma_* \left(\widehat{\Delta_q} \star B\partial \partial (\Delta_q)\right)$$

$$= B\partial \sigma_* (\Delta_q) - 0 = B\partial \sigma.$$

Finally we prove the existence of a chain homotopy  $\mathrm{id} \sim B : S \to S$ . Let  $i : X \to X \times I$  be given by  $x \mapsto (x,0)$  and  $p : X \times I \to X$  given by  $(x,t) \mapsto x$  then  $S(p) \circ S(i) = \mathrm{id}$ . By 8.22 we have a chain homotopy  $S(i) \sim S(i) \circ B$ . Composing with S(p) gives a chain homotopy  $\mathrm{id} = S(p) \circ S(i) \sim S(p) \circ S(i) \circ B = B$  by 8.25.

**8.32 Corollary.** [9, 9.4.3] Let  $Y \subseteq X$  then  $B_* = id : H(X, Y) \to H(X, Y)$ .

By iteration we get the corresponding results for  $B^r := B \stackrel{r \text{ times}}{\circ} \dots \circ B$ .

**Proof.** Let  $\alpha \in H_q(X, Y)$  be given, i.e.  $\alpha = [z + S_q(Y)]$  for a  $z \in S_q(X)$  with  $\partial z \in S_{q-1}(Y)$ . By 8.31  $B \sim \text{id.}$  Let  $\mathcal{Z}$  be a corresponding natural chain homotopy, then  $Bz - z = \partial \mathcal{Z}_q z - \mathcal{Z}_{q-1} \partial z \in \partial \mathcal{Z}_q z + S_q(Y)$ , i.e. Bz is homologous to z relative Y and, furthermore,  $\partial Bz \in \partial z + 0 + \partial \mathcal{Z}_{q-1} \partial z \in S_{q-1}(Y)$ , so Bz is a cycle relative Y, i.e.  $\alpha = [z + S_q(Y)] = [Bz + S_q(Y)] = B(\alpha)$ .

**8.33 Lemma.** [9, 9.4.4] Let X be the union of two open subsets U and V. Then for every  $c \in S_q(X)$  there is an r > 0 with  $B^r c \in S_q(U) + S_q(V) \subseteq S_q(X)$ .

**Proof.** It is enough to show this for c being a singular simplex  $\sigma : \Delta_q \to X$ . The sets  $\sigma^{-1}(U)$  and  $\sigma^{-1}(V)$  form an open covering of  $\Delta_q$ . Let  $\lambda$  be the Lebesgue number for this covering, i.e. all subsets of  $\Delta_q$  of diameter less than  $\lambda$  belong to one of the two sets. Since  $B^r(\Delta_q)$  is a finite linear combination of singular simplices, whose image are closed simplices of the r-th barycentric refinement of  $K := \{\tau : \tau \leq \Delta_q\}$ , we have by 3.27 that each summand of  $B^r(\Delta_q)$  has image in  $\sigma^{-1}(U)$  or in  $\sigma^{-1}(V)$ . Hence  $B^r(\sigma) = B^r S(\sigma)(\Delta_q) = S(\sigma)B^r(\Delta_q)$  is a combination of terms in  $S_q(U)$  and in  $S_q(V)$ .

#### 8.34 Excision theorem. [9, 9.4.5]

Let  $X_j \subseteq X$  for  $j \in \{0,1\}$  such that the interiors  $X_j^o$  cover X. Then the inclusion  $i_* : (X_2, X_2 \cap X_1) \to (X_2 \cup X_1, X_1)$  induces isomorphisms  $H_q(X_2, X_2 \cap X_1) \to H_q(X_2 \cup X_1, X_1)$  for all q. In particular this applies to  $X_1 := Y \subseteq X$  and  $X_2 := X \setminus Z$  for subsets Z and Y satisfying  $\overline{Z} \subseteq Y^o$  and so gives isomorphisms  $H_q(X \setminus Z, Y \setminus Z) \to H_q(X, Y)$ . **Proof.** We have to show that  $i_*: H_q(X_2, X_2 \cap X_1) \to H_q(X_2 \cup X_1, X_1)$  is bijective.

*i*<sub>\*</sub> is onto: Let  $\beta \in H_q(X_2 \cup X_1, X_1)$ , i.e.  $\beta = [z + S_q(X_1)]$  for some  $z \in S_q(X)$ with  $\partial z \in S_q(X_1)$ . By **8.33** there exists an r > 0 and  $u_j \in S_q(X_j^o)$  such that  $z \sim B^r z = u_1 + u_2 \sim u_2$  relative  $X_1$  by **8.32**. We have  $\partial u_2 \in S_{q-1}(X_2)$  and  $\partial u_2 = \partial B^r z - \partial u_1 = B^r \partial z - \partial u_1 \in S_{q-1}(X_1)$ , hence  $\partial u_2 \in S_{q-1}(X_1 \cap X_2)$ . So  $\alpha := [u_2 + S_q(X_2 \cap X_1)] \in H_q(X_2, X_2 \cap X_1)$  and it is mapped by  $i_*$  to  $\beta$ .

 $\begin{array}{l} i_* \text{ is injective: Let } \alpha \in H_q(X_2, X_2 \cap X_1) \text{ be such that } i_*\alpha = 0. \text{ Then } \alpha = [x_2 + S_q(X_2 \cap X_1)] \text{ for some } x_2 \in S_q(X_2). \text{ Since } i_*(\alpha) = 0 \text{ we have a } (q+1)\text{-chain } c \text{ in } X \text{ and a } q\text{-chain } x_1 \text{ in } X_1 \text{ with } \partial c = x_2 + x_1. \text{ Again by } \boxed{8.33} \text{ there is an } r > 0 \text{ such that } B^r c = u_1 + u_2 \text{ with } u_j \in S_q(X_j^o). \text{ Hence } \partial u_1 + \partial u_2 = \partial B^r c = B^r \partial c = B^r(x_2 + x_1). \text{ So } a := B^r x_2 - \partial u_2 = \partial u_1 - B^r x_1 \text{ is a chain in } X_1 \cap X_2 \text{ and } x_2 \sim B^r x_2 = \partial u_2 + a \sim 0 \text{ relative } X_1 \text{ by } \boxed{8.32}, \text{ i.e. } \alpha = [x_2 + S_q(X_2 \cap X_1)] = [\partial u_2 + a + S_q(X_2 \cap X_1)] = 0. \end{array}$ 

The alternate description is valid, since the interiors of  $X_1 := Y$  and  $X_2 := X \setminus Z$ cover X iff  $Y^o = X_1^o \supseteq X \setminus X_2^o = X \setminus (X \setminus \overline{Z}) = \overline{Z}$ . Obviously  $Y \setminus Z = X_1 \cap X_2$ .  $\Box$ 

**8.35 Corollary.** [9, 9.4.6] [9, 9.4.7] Let (X, A) be a CW-pair. Then the quotient map  $p: (X, A) \to (X/A, A/A)$  induces an isomorphism in homology for all q and hence  $H_q(X, A) \cong H_q(X/A)$  for all  $q \neq 0$ .

**Proof.** By 4.19 we have an open neighborhood U of A in X, of which A is a SDR. Let  $p: X \to X/A =: Y$  be the quotient mapping and let  $V := p(U) \subseteq X/A =: Y$ and  $y := A/A \in X/A$ . Since U is saturated also  $V \subseteq Y$  is open and  $p(A) = \{y\}$  a SDR in V. Now consider

$$\begin{array}{c|c} H_q(X,A) & \xrightarrow{\cong} & H_q(X,U) \lessdot \stackrel{\cong}{\longrightarrow} & H_q(X \setminus A, U \setminus A) \\ & p_* & & & \\ p_* & & & \\ H_q(Y,\{y\}) & \xrightarrow{\cong} & H_q(Y,V) \end{Vmatrix} \stackrel{\cong}{\longleftarrow} & H_q(Y \setminus \{y\}, V \setminus \{y\}) \end{array}$$

By 1.34 we have that  $p: (X, A) \to (Y, \{y\})$  is a relative homeomorphism, so the vertical arrow on the right side is induced by an isomorphism of pairs and hence is an isomorphism. The horizontal arrows on the right side are isomorphisms by the excision theorem 8.34. Hence the vertical arrow in the middle is an isomorphism. By 8.29 the horizontal arrows on the left are isomorphisms, hence also the vertical arrow on the left.

By 8.18.3 we have finally that 
$$H_q(Y, \{y\}) \cong H_q(Y)$$
 for  $q > 0$ .

**8.37 Proposition.** [9, 9.4.8] Let  $f : (X, A) \to (Y, B)$  be a relative homeomorphism of CW-pairs. Assume furthermore that  $X \setminus A$  contains only finitely many cells or  $f : X \to Y$  is a quotient mapping. Then  $f_* : H_q(X, A) \to H_q(Y, B)$  is an isomorphism for all q.

8.39

**Proof.** By 1.34 we have an induced continuous bijective mapping  $X/A \to Y/B$  making the following diagrams commute:

That this bijection is a homeomorphism follows in case  $X \setminus A$  has only finitely many cells since then X/A is compact, and in the case where  $f : X \to Y$  is a quotient map then so is  $X \to Y \to Y/B$  and hence also  $X/A \to Y/B$ . Both X/A and Y/B are CW-complexes by 4.15 and by 8.18.2 the bottom arrow on the right is an isomorphism. By 8.35 the vertical arrows on the right are isomorphisms as well, so the same has to be true for the top one.

**8.38 Proposition.** [9, 9.4.9] Let  $X_j$  be CW-complexes which 0-cells  $x_j \in X_j$  as base-points. Then we have natural isomorphisms  ${}^{ab} \coprod_j H_q(X_j) \to H_q(\bigvee_j X_j)$  for  $q \neq 0$ .

**Proof.** We have  $\bigvee_j X_j := \bigsqcup_j X_j / A$  where  $A := \{x_j : j \in J\}$ . Hence  $H_q(A) = 0$  for  $q \neq 0$  and so

$$H_q\left(\bigvee_j X_j\right) \stackrel{\textcircled{\textbf{8.36}}}{\cong} H_q\left(\bigsqcup_j X_j, A\right) \stackrel{A \text{``acyclic''}}{\cong} H_q\left(\bigsqcup_j X_j\right) \stackrel{\textcircled{\textbf{8.9}}}{\cong} {}^{ab} \coprod_j H_q(X_j). \quad \Box$$

**8.39 Proposition.** [9, 9.4.10] Let  $X = X_1 \cup X_2$ , where  $X_j \subseteq X$  is open. Then there is a long exact sequence (the so called MAYER-VIETORIS SEQUENCE)

$$\cdots \to H_q(X_1 \cap X_2) \to H_q(X_1) \oplus H_q(X_2) \to H_q(X) \to H_{q-1}(X_1 \cap X_2) \to \dots$$

**Proof.** Let S := S(X),  $S_1 := S(X_1) \subseteq S(X)$  and  $S_2 := S(X_2) \subseteq S(X)$ . Then  $S(X_1 \cap X_2) = S_1 \cap S_2$ . Let  $S_1 + S_2$  be the chain complex which has the subgroup of S generated by  $S_1$  and  $S_2$  in every dimension. We claim that the following short sequence

$$0 \to S_1/(S_1 \cap S_2) \to S/S_2 \to S/(S_1 + S_2) \to 0$$

is exact. In fact by the first isomorphy theorem we have  $S_1/(S_1 \cap S_2) \cong (S_1+S_2)/S_2$ and hence the inclusion  $S_1 + S_2 \subseteq S$  induces an injection  $S_1/(S_1 \cap S_2) \to S/S_2$ . The quotient of it is by the second isomorphy theorem  $(S/S_2)/((S_1 + S_2)/S_2) \cong S/(S_1 + S_2)$ , which proves the claim. By 8.34 we have that the inclusion  $(S_1, S_1 \cap S_2) \hookrightarrow (S, S_2)$  induces an isomorphism  $H(S_1/(S_1 \cap S_2)) =: H(X_1, X_1 \cap X_2) \to H(X_1 \cup X_2, X_2) := H(S/S_2)$ . Hence the long exact homology sequence 7.31 gives  $H(S/(S_1 + S_2)) = 0$ .

If we consider now the short exact sequence

$$0 \to S_1 + S_2 \to S \to S/(S_1 + S_2) \to 0$$

we deduce from the long exact homology sequence that  $H(S_1 + S_2) \to H(S)$  is an isomorphism.

Now consider the sequence

$$0 \to S_1 \cap S_2 \to S_1 \oplus S_2 \to S_1 + S_2 \to 0,$$

where the inclusion is given by  $c \mapsto (c, -c)$  and the projection by  $(c_1, c_2) \mapsto c_1 + c_2$ . This is obviously short exact, since  $(c_1, c_2)$  is mapped to 0 iff  $c_1 + c_2 = 0$ , i.e.  $c := c_1 = -c_2 \in S_1 \cap S_2$  is mapped to  $(c_1, c_2)$ .

So we get a long exact sequence in homology, where we may replace  $H(S_1 + S_2)$  by H(S) =: H(X) by what we said above. Note that the boundary operator is given by  $[z] \mapsto [\partial z_1]$ , where  $B^r z = z_1 + z_2$ .

#### 8.40 Remark. [9, 9.4.12]

(1) It is enough to assume in 8.39 that there are neighborhoods of  $X_1$  and  $X_2$  which have  $X_1$  and  $X_2$  and their intersection has  $X_1 \cap X_2$  as DRs. In particular this applies to *CW*-subspaces  $X_i$  of a *CW*-complex X by 4.19.

(2) Let  $X_1 \cap X_2$  be acyclic. Then the Mayer-Vietoris sequence gives  $H_q(X) \cong H_q(X_1) \oplus H_q(X_2)$  for  $q \neq 0$ . In fact only the case q = 1 needs some argument: We have the exact sequence

$$0 = H_1(X_1 \cap X_2) \longrightarrow H_1(X_1) \oplus H_1(X_2) \longrightarrow H_1(X)$$
$$\mathbb{Z} = H_0(X_1 \cap X_2) \longrightarrow H_0(X_1) \oplus H_0(X_2) \longrightarrow H_0(X) \longrightarrow 0$$

and the mapping  $H_0(X_1 \cap X_2) \to H_0(X_1) \oplus H_0(X_2)$  is injective, since the generator is mapped to a generator of  $H_0(X_1)$  and of  $H_0(X_2)$ .

(3) Let  $X_1$  and  $X_2$  be acyclic, then we have  $H_q(X_1 \cap X_2) \cong H_{q+1}(X)$  for q > 0 and furthermore  $H_1(X)$  is free abelian and

gives  $H_1(X) \cong \mathbb{Z}^{k-1}$  via the rank formula rank(ker f) + rank(im f) = rank(dom f), where we used that  $X = X_1 \cup X_2$  is connected being the union of two connected sets.

(4) Consider the covering  $S^n = D^n_+ \cup D^n_-$ . By (1) we get a long exact Mayer-Vietoris sequence. And since  $D^n_+$  and  $D^n_-$  are contractible, they are acyclic. So  $H_q(S^n) \cong H_{q-1}(D^n_+ \cap D^n_-) = H_{q-1}(S^{n-1})$  for q, n > 0. Inductively we hence

8.42

get  $H_q(S^n) \cong H_{q-n}(S^0) = \{0\}$  for q > n, since  $S^0$  is discrete and  $H_q(S^n) \cong H_1(S^{n-q+1}) = \{0\}$  for 0 < q < n, since

and  $H_n(S^n) \cong H_1(S^1) \cong \mathbb{Z}$ , since

#### Homology of balls, spheres and their complements

**8.41 Proposition.** [9, 9.5.1] Let  $n \ge 0$  then

$$H_q(\Delta_n, \dot{\Delta}_n) \cong \begin{cases} \mathbb{Z} & \text{for } q = n \\ 0 & \text{otherwise} \end{cases}$$

The generator in  $H_n(\Delta_n, \dot{\Delta}_n)$  will be denoted  $[\Delta_n]$  and is given by the relative homology class of the singular simplex  $id_{\Delta_n} : \Delta_n \to \Delta_n$ .

**Proof.** We proof this by induction on n.  $(n = 0) H_q(\Delta_0, \dot{\Delta}_0) = H_q(\{1\}, \emptyset) \xrightarrow[8.17.2]{8.17.2} H_q(\{*\}).$ 

(n > 0) We consider  $\Delta_{n-1}$  as face opposite to  $e_n$  in  $\Delta_n$  and let  $A_n := \dot{\Delta}_n \setminus \Delta_{n-1}$ . Since  $A_n$  is a DR of  $\Delta_n$ , we conclude from the homology-sequence (a, 19) of the triple  $A_n \subseteq \dot{\Delta}_n \subseteq \Delta_n$  that  $H_q(\Delta_n, \dot{\Delta}_n) \cong H_{q-1}(\dot{\Delta}_n, A_n)$ . Since  $\Delta_{n-1} \setminus \dot{\Delta}_{n-1} = \dot{\Delta}_n \setminus A_n$  we get from (a, 37) or (a, 36) that the inclusion induces an isomorphism  $H_{q-1}(\Delta_{n-1}, \dot{\Delta}_{n-1}) \cong H_{q-1}(\dot{\Delta}_n, A_n)$ . Hence  $H_q(\Delta_n, \dot{\Delta}_n) \cong H_{q-1}(\Delta_{n-1}, \dot{\Delta}_{n-1})$  and by recursion we finally arrive in case  $q \ge n$  at  $H_{q-n}(\Delta_0, \dot{\Delta}_0)$  which we calculated above, and in case q < n at  $H_0(\Delta_{n-q}, \dot{\Delta}_{n-q}) = 0$  by (a, 16), since  $\Delta_{n-q}$  is connected and  $\dot{\Delta}_{n-q} \neq \emptyset$ .

Let  $[\Delta_n]$  denote the relative homology class in  $H_n(\Delta_n, \dot{\Delta}_n)$  of  $\mathrm{id}_{\Delta_n} : \Delta_n \to \Delta_n$ . Then its image in  $H_{n-1}(\dot{\Delta}_n, A_n)$  ist given by  $[\partial \mathrm{id}_{\Delta_n} + S_n(A_n)]$  which equals the image  $[\mathrm{id}_{\Delta_{n-1}} + s_n(A_n)]$  of  $[\Delta_{n-1}] \in H_{n-1}(\Delta_{n-1}; \dot{\Delta}_{n-1})$ . Obviously  $[\Delta_0]$  is the generator of  $H_0(\Delta_0, \dot{\Delta}_0) = H_0(\{1\})$ .

**8.42 Corollary.** [9, 9.5.2] For  $n \ge 0$  we have

$$H_q(D^n, S^{n-1}) \cong \begin{cases} \mathbb{Z} & \text{for } q = n \\ 0 & \text{otherwise} \end{cases}$$

We denote the canonical generator by  $[D^n]$ . It is given by the relative homology class of a homeomorphism  $\Delta_n \to D^n$ .

**8.43 Corollary.** [9, 9.5.3] For n > 0 we have

$$H_q(S^n) \cong \begin{cases} \mathbb{Z} & \text{for } q = n \text{ or } q = 0\\ 0 & \text{otherwise} \end{cases}$$

We denote the canonical generator by  $[S^n]$ . It is given by  $[S^n] = \partial_*([D^n]) = [\partial D^n]$ . So this gives a different proof from  $\boxed{8.40.4}$ 

**Proof.** Consider the homology sequence of the pair  $S^n \subseteq D^{n+1}$ :

**8.44 Corollary.** [9, 9.5.6] By <u>8.38</u> we have  $H_q(\bigvee_j S^n) = 0$  for  $q \notin \{0, n\}$  and  $H_n(\bigvee_j S^n) \cong {}^{ab} \coprod_j \mathbb{Z}$  and the generators are  $(\operatorname{inj}_j)_*[S^n]$ .

**1.20** Proposition. Let  $m \neq n$  then  $\mathbb{R}^m \ncong \mathbb{R}^n$  and  $S^m \ncong S^n$ .

We have "proved" this by applying the theorem 1.19 of the invariance of domains.

**Proof of 1.20** for  $\mathbb{R}^n$  and  $S^n$ . Let  $m \neq n$  and m > 0. Then  $H_m(S^m) \cong \mathbb{Z}$  but  $H_m(S^n) = \{0\}$ , so  $S^m \not\sim S^n$ . Assume  $\mathbb{R}^n \cong \mathbb{R}^m$  then  $S^{m-1} \sim \mathbb{R}^m \setminus \{0\} \cong \mathbb{R}^n \setminus \{0\} \sim S^{n-1}$ , hence m = n.

**9.1 Proposition.** [9, 11.1.1]  $S^n$  is not contractible and is not a retract in  $D^{n+1}$ 

**Proof.** Since  $H_n(S^n) \cong \mathbb{Z} \not\cong \{0\} = H_n(\{*\})$  the first statement is clear. And the second follows, since retracts of contractible spaces are contractible. In fact let  $h_t : X \to X$  be a contraction and let  $i : A \to X$  have a left inverse  $p : X \to A$ . Then  $p \circ h_t \circ i : A \to A$  is a contraction of A.

**9.2 Corollary. Brouwers fixed point theorem.** [9, 11.1.2] *Every continuous* map  $f: D^n \to D^n$  has a fixed point.

**Proof.** Otherwise we can define a retraction as in 2.23.

**10.1 Proposition.** [9, 11.7.1] Let  $B \subseteq S^n$  be a ball. Then  $S^n \setminus B$  is acyclic.

**Proof.** Induction on  $r := \dim B$ .

(r=0) Then B is a point and hence  $S^n \setminus B \cong \mathbb{R}^n$  is contractible and thus acyclic. (r+1) Let  $z \in Z_q(S^n \setminus B)$  for q > 0 and  $z := x - y \in Z_0(S^n \setminus B)$  for q = 0 with  $x, y \in S^n \setminus B$ . We have to show that  $\exists b \in S_{q+1}(S^n \setminus B)$  with  $\partial b = z$ .

Consider a homeomorphisms  $f: I^{r+1} = I^r \times I \cong B$ . Then  $B_t := f(I^r \times \{t\})$  is an *r*-ball. Thus by induction hypothesis there are  $b_t \in S_{q+1}(S^n \setminus B_t)$  with  $\partial b_t = z$ considered as element in  $S_q(S^n \setminus B_t) \leftarrow S_q(S^n \setminus B)$ . Since the image of  $b_t$  is disjoint to  $B_t$ , we can choose an open neighborhood  $V_t$  of t such that  $I^r \times V_t \subseteq f^{-1}(S^n \setminus \operatorname{Im}(b_t))$ . Using compactness we find a partition of  $0 = t_0 < t_1 < \cdots < t_N = 1$  of I into finitely many intervals  $I_j := [t_{j-1}, t_j]$  such that for each j there exists a t with  $I_j \subset V_t$ . Let  $b_j := b_t \in S_{q+1}(Y_j)$  where  $Y_j$  is the open subset  $S^n \setminus f(I^r \times I_j)$ . Now let  $X_j := \bigcap_{i < j} Y_i$ . Then  $X_j \cup Y_j = \bigcap_{i < j} Y_i \cup Y_j = Y_{j-1} \cup Y_j = S^n \setminus f(I^r \times \{t_j\})$ and  $X_j \cap Y_j = X_{j+1}$ .

We now show by induction on j that [z] = 0 in  $H_q(X_j)$ . For (j = 1) nothing is to be shown, since  $X_1 = S^n$ . For (j + 1) we apply the Mayer-Vietoris sequence 8.39 to the open sets  $X_j$  and  $Y_j$ :

$$S^{n} \setminus f(I^{r} \times \{t_{j}\}) \xrightarrow{X_{j+1}} H_{q+1}(X_{j} \cup Y_{j}) \longrightarrow H_{q}(X_{j} \cap Y_{j}) \longrightarrow H_{q}(X_{j}) \oplus H_{q}(Y_{j})$$
  
ind. on  $r \parallel 0$ 

The image of  $[z] \in H_q(X_{j+1})$  in  $H_q(X_j) \oplus H_q(Y_j)$  is zero, since the first component is  $[z] = 0 \in H_q(X_j)$  by induction hypothesis on j, and the second component  $[z] = [\partial b_j] = 0 \in H_q(Y_j)$ . Since the space on the left side is zero, the arrow on the right is injective we get that  $[z] = 0 \in H_q(X_{j+1})$ .

Since  $X_j = S^n \setminus B$  finally, we are done.

**10.2 Theorem.** [9, 11.7.4] Let  $S \subseteq S^n$  be an r-sphere with  $0 \le r \le n-1$  and  $n \ge 2$ . Then

$$H_q(S^n \setminus S) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } q = 0 \text{ and } r = n - 1 \\ \mathbb{Z} & \text{for } q \in \{0, n - 1 - r\} \text{ and } r < n - 1 \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** Induction on r.

(r=0) Then  $S \cong \{-1,+1\}$  and  $S^n \setminus S \sim S^{n-1}$ , so the result follows from 8.40 or 8.43.

(r > 0) We have  $S^r = D^r_- \cup D^r_+$  and  $B_{\pm} := f(D^r_{\pm})$  are *r*-balls and  $S' := f(S^{r-1})$  an (r-1)-sphere. By 10.1  $S^n \setminus B_{\pm}$  are acyclic and since  $S^n \setminus S' = (S^n \setminus B_+) \cup (S^n \setminus B_-)$  and  $S^n \setminus S = (S^n \setminus B_+) \cap (S^n \setminus B_-)$  we get by 8.40.3 that  $H_q(S^n \setminus S) \cong H_{q+1}(S^n \setminus S')$  for q > 0 and  $H_0(S^n \setminus S) \cong H_1(S^n \setminus S') \oplus \mathbb{Z}$ . By recursion we finally arrive at  $H_{q+r}(S^n \setminus \{\pm 1\}) = H_{q+r}(S^{n-1})$ , which we treated before.

**10.3 Proposition.** [9, 11.7.2] [9, 11.7.5] Let  $n \ge 2$ . If  $B \subseteq \mathbb{R}^n$  is a ball, then

$$H_q(\mathbb{R}^n \setminus B) = \begin{cases} \mathbb{Z} & \text{for } q \in \{0, n-1\} \\ 0 & \text{otherwise.} \end{cases}$$

If  $S \subseteq \mathbb{R}^n$  is an r-sphere with  $0 \leq r \leq n-1$ , then

$$H_q(\mathbb{R}^n \setminus S) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } (q = n - 1, r = 0) \text{ or } (q = 0, r = n - 1) \\ \mathbb{Z} & \text{for } (q = n - 1, r \neq 0) \text{ or } (q \in \{0, n - 1 - r\}, r \neq n - 1) \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** Let  $A \subseteq \mathbb{R}^n \cong S^n \setminus \{P_+\} \subset S^n$  be compact. The long exact homology sequence 8.16 of the pair  $(S^n \setminus A, \mathbb{R}^n \setminus A)$  gives

 $\rightarrow H_{q+1}(S^n \setminus A, \mathbb{R}^n \setminus A) \xrightarrow{\partial_*} H_q(\mathbb{R}^n \setminus A) \rightarrow H_q(S^n \setminus A) \rightarrow H_q(S^n \setminus A, \mathbb{R}^n \setminus A) \rightarrow$ By the excision theorem 8.34 applied to  $A \subseteq \mathbb{R}^n \subseteq S^n$  we get  $H_q(S^n \setminus A, \mathbb{R}^n \setminus A) \cong$  $H_q(S^n, \mathbb{R}^n)$ , which is isomorphic by 8.29 to  $H_q(S^n, \{*\})$ , since  $\mathbb{R}^n$  is contractible.

This homology equals for q > 0 by 8.18.3 that of  $H_q(S^n)$  and is 0 for q = 0 by 8.20 since  $S^n$  is path-connected. So

$$H_q(S^n, \{*\}) = \begin{cases} \mathbb{Z} & \text{for } q = n \\ 0 & \text{otherwise} \end{cases}$$

The long exact sequence from above thus is

$$\to H_{q+1}(S^n, \{*\}) \xrightarrow{\partial_*} H_q(\mathbb{R}^n \setminus A) \to H_q(S^n \setminus A) \to H_q(S^n, \{*\}) \to$$

In particular,  $H_q(\mathbb{R}^n \setminus A) \cong H_q(S^n \setminus A)$  for  $q \notin \{n-1, n\}$  and near q = n-1 it is for A a sphere or ball:

$$0 \to H_n(\mathbb{R}^n \setminus A) \to 0 \to \mathbb{Z} \to H_{n-1}(\mathbb{R}^n \setminus A) \to H_{n-1}(S^n \setminus A) \to 0,$$

This gives  $H_n(\mathbb{R}^n \setminus A) = 0$  and  $H_{n-1}(\mathbb{R}^n \setminus A) \cong \mathbb{Z} \oplus H_{n-1}(S^n \setminus A)$ , from which the claimed result follows.

**10.4 Corollary (Jordan's separation theorem generalized).** [9, 11.7.6] [9, 11.7.7] Let  $X \in \{\mathbb{R}^n, S^n\}$ . For any r-sphere S with r < n-1 we have that  $X \setminus S$  is connected (i.e. we cannot cut X into two pieces along such a sphere).

If S is an n-1-sphere then  $X \setminus S$  has two components, both of which have S as boundary. If  $X = S^n$  then the components are acyclic.

**Proof.** For spheres of dimension r < n-1 the result follows from 10.2 and 10.3 since  $H_0(X \setminus S) \cong \mathbb{Z}$  in these cases.

If S is a sphere of dimension n-1, then  $H_0(X \setminus S) \cong \mathbb{Z}^2$  by 10.2 and 10.3. Hence  $X \setminus S$  has two components, say U and V.

That for  $X = S^n$  the components are acyclic follows from  $H_q(U) \oplus H_q(V) \cong H_q(X \setminus S) = \{0\}$  for  $q \neq 0$ .

 $(\dot{U} \subseteq S)$  In fact  $\dot{U} \cap U = \emptyset$ , since U is open and thus  $\dot{U} = \overline{U} \setminus U^o = \overline{U} \setminus U$ . From  $U \subseteq \sim V$  we get  $\overline{U} \subseteq \overline{V} = \sim V$  since V is open and hence  $U \cap V = \emptyset$ . So  $\dot{U} = \overline{U} \setminus U \subseteq (X \setminus V) \setminus U = X \setminus (U \cup V) = S$ .

 $(S \subseteq \dot{U})$  Let  $x \in S$  and W be a neighborhood of  $x \in X$ . Choose n-1-balls B and B' with  $S = B \cup B'$  and such that  $x \in B \subseteq W$ . Let c be a path in  $\mathbb{R}^n$  from U to V, which avoids  $B' \subseteq S$  (this is possible by 10.3) since  $X \setminus B'$  is path connected). Let  $t_0 := \sup\{t : c(t) \in U\}$ . Hence  $y := c(t_0) \in \overline{U} \setminus U = \dot{U} \subseteq S = B \cup B'$ . Hence  $y \in B \subseteq W$  and so  $W \cap \dot{U}$  contains y and is not empty, hence  $x \in \dot{U}$ .

**10.5 Remark.** [9, 11.7.8] For dimension 2 we have Schönflies's theorem (see [7, §9]), i.e. for every Jordan curve in  $S^2$ , i.e. injective continuous mapping  $c: S^1 \to S^2$  there exists a homeomorphism  $f: S^2 \cong S^2$  with  $f|_{S^1} = c$ . Thus up to a homeomorphism a Jordan-curve looks like the equator  $S^1 \subseteq S^2$ .

In dimension greater than 2, Alexanders horned sphere is a counterexample: One component of the complement is not simply connected. This gives at the same time an example of an open subset of  $S^3$ , which is homologically trivial (i.e. acyclic) but not homotopy-theoretical  $(\pi_1(U) \neq 0)$ .



**1.19** Corollary (Invariance of the domain). Let  $X, Y \subseteq \mathbb{R}^n$  be homeomorphic. If X is open then so is Y.

**Proof.** Take  $x \in X$  and  $y := f(x) \in Y$ . By assumption there is a ball  $B := \{z : |z - x| \leq r\} \subseteq X$ . Let  $S := \partial B$ . Then  $\mathbb{R}^n \setminus f(S) = (\mathbb{R}^n \setminus f(B)) \cup (f(B) \setminus f(S))$ . The first part is connected by 10.1 and the second one coincides with  $f(B \setminus S) \cong B \setminus S = \overset{o}{D^n}$  and hence is connected as well. Thus they are the path components of  $\mathbb{R}^n \setminus f(S)$  and hence open. Since the set on the right side is a neighborhood of y in Y, we have that Y is open.  $\Box$ 

**10.6 Exercise.** Let  $\nu : S^1 \to S^1 \vee S^1$  be the closed path, which first runs once through one and then through the other factor as in 5.14. Then  $\nu_* : H_1(S^1) \to H_1(S^1 \vee S^1)$  is given by  $\pi_1(\{S^1\}_+) = (i_1)_*(\{S^1\}_+) + (i_2)_*(\{S^1\}_+)$ .

## Cellular homology

**10.7 Proposition.** [9, 9.6.1] Let X be a CW-complex. Then  $H_p(X^q, X^{q-1}) = 0$  for  $p \neq q$ .

**Proof.** For q < 0 this is clear. For q = 0 we have  $H_p(X^q, X^{q-1}) = H_p(X^0, \emptyset) = 0$  by 8.15, 8.7 and 8.9.

So let q > 0. For p = 0 we have  $H_0(X^{q-1}) \twoheadrightarrow H_0(X^q) \xrightarrow{0} H_0(X^q, X^{q-1}) \to 0$  and the first mapping is onto (since  $X^q$  has less components). So the next arrow is 0.

Now let  $p \neq 0$ . By 8.36 we have  $H_p(X^q, X^{q-1}) \cong H_p(X^q/X^{q-1})$  and so the result follows from 8.44, since  $X^q/X^{q-1} \cong \bigvee S^q$ .

**10.8 Corollary.** [9, 9.6.2] The inclusions induce an epimorphism  $H_q(X^q) \to H_q(X)$ and an isomorphism  $H_q(X^{q+1}) \to H_q(X)$ .

**Proof.** By 10.7 and

$$H_{p+1}(X^q, X^{q-1}) \to H_p(X^{q-1}) \to H_p(X^q) \to H_p(X^q, X^{q-1})$$

the first arrow in sequence

 $H_q(X^q) \to H_q(X^{q+1}) \to \dots \to H_q(X^p) \to \dots \to H_q(X)$ 

is onto and the others are isomorphisms. So we have the result for finite CWcomplexes. In the general case we use that every singular simplex lies in some

 $X^p$ , hence all mappings are surjective. Similar one shows injectivity, since  $[z] = 0 \in H_q(X)$  implies  $z = \partial c$  for some  $c \in S_q(X) = \bigcup_j S_q(X^j)$ , hence  $[z] = 0 \in H_q(X^j)$ .

**10.9 Corollary.** [9, 9.6.3] Let X be a CW-space without q-cells. Then  $H_q(X) = 0$ . In particular  $H_q(X) = 0$  for  $q > \dim X$ .

**Proof.** From the homology sequence

$$H_{q+1}(X^p, X^{p-1}) \to H_q(X^{p-1}) \to H_q(X^p) \to H_q(X^p, X^{p-1})$$

for q > p and 10.7 we deduce  $H_q(X^{q-1}) \cong \ldots \cong H_q(X^{-1}) = 0$ . By assumption  $X^q = X^{q-1}$  and hence  $H_q(X^q, X^{q-1}) = 0$  so we get the surjectivity of  $H_q(X^{q-1}) \to H_q(X^q)$  and thus  $H_q(X^q) = 0$  as well. Now the result follows since by 10.8  $H_q(X^q) \twoheadrightarrow H_q(X)$  is onto.

**10.10 Definition.** [9, 9.6.4] The *q*-th CELLULAR CHAIN GROUP of a CW-complex X is defined as

$$C_q(X) := H_q(X^q, X^{q-1}),$$

and its elements are called CELLULAR q-CHAINS. For every q-cell e in X with characteristic map  $\chi^e : (D^q, S^{q-1}) \to (X^q, X^{q-1})$  we define a so-called orientation  $\chi^e_*([D^q]) \in C_q(X)$  as the image of  $\chi^e_* : H_q(D^q, S^{q-1}) \cong \mathbb{Z} \to H_q(X^q, X^{q-1})$ , where  $[D^q]$  denotes the generator in  $H_q(D^q, S^{q-1})$  induced from a homeomorphism  $\Delta^q \to D^q$ .

For every cell there are exactly two orientations, which differ only by their sign. And  $C_q(X)$  is a free abelian group generated by a selection of orientations for each *q*-cell.

**Proof.** Let  $\chi_1$  and  $\chi_2$  be two characteristic mappings for e. We can consider them as rel. homeomorphisms  $\chi_j : (D^q, S^{q-1}) \to (X^{q-1} \cup e, X^{q-1})$ . By **8.37** these factorizations induce isomorphisms. Hence  $H_q(\chi_1)[D^q] = \pm H_q(\chi_2)[D^q]$ , since the generator  $[D^q]$  has to be mapped to a generator of  $H_q(D^q, S^{q-1})$ , and the only ones are  $\pm [D^q]$ .

Obviously  $C_0(X) = H_0(X^0)$  is free abelian.

For q > 0 the projection  $p: (X^q, X^{q-1}) \to (Y, \{y_0\}) := (X^q/X^{q-1}, X^{q-1}/X^{q-1})$ induces by 8.35 an isomorphism  $p_*: C_q(X) \to H_q(Y, y_0)$ . Since Y is a join of q-spheres we have that  $p_*\chi^e_*[D^q]$  form a basis in the free abelian group  $H_q(Y, y_0)$ , as follows from 8.44. In fact consider the following commutative diagram:

where the vertical arrows are rel. homeomorphisms and hence induce isomorphisms in homology

and the bottom arrow maps the generator  $[S^q] \in H_q(S^q) \cong H_q(S^q, \{*\})$  to one of the generators in  $H_q(X^q/X^{q-1}) \cong H_q(X^q/X^{q-1}, \{*\})$ .  $\Box$ 

**10.11 Definition.** [9, 9.6.6] Using the long exact sequences for the pairs  $(X^{q+1}, X^q)$  and  $(X^q, X^{q-1})$  we have

$$C_{q+1}(X)$$

$$\|$$

$$\cdots \longrightarrow H_{q+1}(X^{q+1}, X^{q}) \xrightarrow{\partial_{*}} H_{q}(X^{q}) \longrightarrow H_{q}(X^{q+1}) \longrightarrow \cdots$$

$$\|$$

$$\cdots \longrightarrow H_{q}(X^{q-1}) \longrightarrow H_{q}(X^{q}) \xrightarrow{j_{*}} H_{q}(X^{q}, X^{q-1}) \longrightarrow \cdots$$

$$\|$$

$$C_{q}(X)$$

Let  $\partial := j_* \circ \partial_* : C_{q+1}(X) \to C_q(X)$ . We have  $\partial^2 = 0$  by the exactness of the second sequence at  $H_q(X^q, X^{q-1})$  and thus we obtain a chain complex. Its homology H(C(X)) is called CELLULAR HOMOLOGY of the CW-complex X. For any q + 1-cell e with characteristic map  $\chi^e$  we get

 $\partial(\chi_*^e[D^{q+1}]) = j_*\partial_*\chi_*^e[D^{q+1}] = j_*(\chi^e|_{S^q})_*\partial_*[D^{q+1}] = j_*(\chi^e|_{S^q})_*[\partial D^{q+1}] = j_*(\chi^e|_{S^q})_*[S^q],$  by the homology ladder

$$\begin{array}{c|c} H_{q+1}(D^{q+1}, S^q) \xrightarrow{\chi^*_*} H_{q+1}(X^{q+1}, X^q) = & C_{q+1}(X) \\ & & & & \\ \partial_* \bigg| & & \partial_* \bigg| & & \partial_* \bigg| \\ & & & & & \\ H_q(S^q) \xrightarrow{(\chi^e | S^q)_*} H_q(X^q) \xrightarrow{j_*} C_q(X). \end{array}$$

#### Singular versus cellular homology

**10.12 Proposition.** [9, 9.6.9] [9, 9.6.11] The homomorphism  $j_* : H_q(X^q) \to H_q(X^q, X^{q-1})$  is injective and maps onto the q-th cellular cycles. The map  $i_* : H_q(X^q) \to H_q(X)$  is onto and its kernel is mapped by  $j_*$  onto the q-th cellular boundaries.

Thus one obtains isomorphisms

$$j_*: H_q(C(X)) \xrightarrow{\cong} H_q(X),$$

which are natural for cellular mappings.

**Proof.** From the exact sequence  $0 \xrightarrow{10.9} H_q(X^{q-1}) \to H_q(X^q) \xrightarrow{j_*} H_q(X^q, X^{q-1}) =: C_q(X)$  we deduce that  $j_*$  is injective and hence  $\operatorname{Ker}(\partial) = \operatorname{Ker}(j_*\partial_*) = \operatorname{Ker}(\partial_*) = \operatorname{Im}(j_*)$ .

From the exact homology sequence 8.16 of the pair  $(X, X^{q+1})$ 

$$H_{q+1}(X^{q+1}) \xrightarrow{\longrightarrow} H_{q+1}(X) \xrightarrow{0} H_{q+1}(X, X^{q+1}) \xrightarrow{0} H_q(X^{q+1}) \xrightarrow{\cong} H_q(X)$$

$$\|$$

$$0$$

we get  $H_{q+1}(X, X^{q+1}) = 0$ . By the exact homology sequence 8.19 for the triple  $X^q \subseteq X^{q+1} \subseteq X$ 

$$H_{q+1}(X^{q+1}, X^q) \longrightarrow H_{q+1}(X, X^q) \longrightarrow H_{q+1}(X, X^{q+1})$$

we get that  $H_{q+1}(X^{q+1}, X^q) \to H_{q+1}(X, X^q)$  is onto. The q-th cellular boundary is the image of the top row in

$$\begin{array}{cccc} H_{q+1}(X^{q+1}, X^q) \xrightarrow{\partial_*} & H_q(X^q) \xrightarrow{j_*} & H_q(X^q, X^{q-1}) \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ H_{q+1}(X, X^q) \xrightarrow{\partial_*} & H_q(X^q) \xrightarrow{i_*} & H_q(X) \end{array}$$

Since the rectangle commutes by naturality of  $\partial_*$  and since Im  $\partial_* = \operatorname{Ker} i_*$  we get

$$\operatorname{Im}(\partial) = \operatorname{Im}(j_*\partial_*) = j_*(\operatorname{Im}\partial_*) = j_*(\operatorname{Ker} i_*).$$

Hence the q-th cellular boundaries are the image of Ker  $i_*$  under  $j_*$ . Now we get the desired natural isomorphism



**10.13 Proposition.** [9, 9.6.10] For  $q \ge 1$  we have that in the short exact sequence  $0 \to \operatorname{Ker}(i_*) \to H_q(X^q) \xrightarrow{i_*} H_q(X) \to 0$ 

 $H_q(X^q)$  is free abelian and  $\operatorname{Ker}(i_*)$  is generated by  $H_q(\chi^e)[S^q]$ , where  $\chi^e: S^q \to X^q$  is a chosen gluing map for any q + 1-cell e in X.

Proof.



By 10.12 we have that  $H_q(X^q) \cong \operatorname{Ker} \partial_q \subseteq C_q(X)$  and hence is free abelian. Furthermore  $H_q(X^{q+1}) \cong H_q(X)$  by 10.8, and hence the kernel of  $i_*$ :  $H_q(X^q) \to H_q(X)$  equals the kernel of  $H_q(X^q) \to H_q(X^{q+1}) \cong H_q(X)$ , and equals by the homology sequence of the pair  $(X, X^q)$  the image of  $\partial_* : C_{q+1}(X) :=$  $H_{q+1}(X^{q+1}, X^q) \to H_q(X^q)$ . By 10.10 we have that  $C_{q+1}(X)$  is the free abelian group generated by  $\chi^e_*[D^{q+1}]$ , where  $\chi^e: (D^{q+1}, S^q) \to (X^{q+1}, X^q)$  are chosen characteristic maps for all q + 1-cells e in X. By 10.11 (see 8.15) we have that  $\partial_*(\chi^e_*[D^{q+1}]) = [\partial \chi^e[D^{q+1}]] = \chi^e_*[S^q]$ .

10.14 Proposition. [9, 9.9.10] For the projective spaces we have

$$H_q(\mathbb{P}^n(\mathbb{C})) \cong \begin{cases} \mathbb{Z} & \text{for } q = 0, 2, \dots, 2n \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_q(\mathbb{P}^n(\mathbb{H})) \cong \begin{cases} \mathbb{Z} & \text{for } q = 0, 4, \dots, 4n \\ 0 & \text{otherwise} \end{cases}$$

**Proof.** Since there are no-cells in all but the dimensions divisible by 2 (or 4) the boundary operator of the cellular homology is 0 (since either domain or codomain is zero) and hence the homology coincides with the cellular chain complex.  $\Box$ 

## Simplicial versus singular homology

We are now going to show that the singular homology of a singular complex K is naturally isomorphic to the homology of the associated CW-space |K|. The idea behind this isomorphism is very easy: To a given simplex  $\sigma = \langle x_0, \ldots, x_q \rangle \in K$  one associates the affine singular simplex  $\bar{\sigma} : \Delta_q \to |K|$ , which maps  $e_j \to x_j$  for all  $0 \le j \le q$ . We will show that this induces a map  $H_q(K) \to H_q(|K|), [\sigma] \mapsto [\sigma]$ . In order that it is well defined we have to show that an even permutation of the vertices does not change the homology class of  $\bar{\sigma}$ . We do this in the following

**10.15 Lemma.** [9, 9.7.1] Let  $\tau$  be a permutation of  $\{0, \ldots, q\}$ . Then  $\tau$  induces a affine mapping  $\tau : (\Delta_q, \dot{\Delta}_q) \to (\Delta_q, \dot{\Delta}_q)$ , with  $H_q(\tau)[\Delta_q] = \operatorname{sign}(\tau)[\Delta_q] \in H_q(\Delta_q, \dot{\Delta}_q)$ .

**Proof.** Since any permutation is a product of transpositions, we may assume that  $\tau$  is a transposition, say (0, 1). Let an affine  $\sigma : \Delta_{q+1} \to \Delta_q$  be defined by  $e_0; e_1, e_2 \ldots \mapsto e_1; e_0, e_1, \ldots$ . The boundary of this singular q + 1-simplex in  $\Delta_q$  is  $\partial \sigma = \sigma \circ \delta^0 + \sum_{i \notin \{0,2\}} (-1)^i \sigma \circ \delta^i + \sigma \circ \delta^2 = \mathrm{id}_{\Delta_q} + c + \tau$  for  $c := \sum_{i \notin \{0,2\}} (-1)^i \sigma \circ \delta^i \in S_q(\dot{\Delta}_q)$ . Hence  $H(\tau)[\Delta_q] = -[\tau] = -[\Delta_q]$ .

Although this lemma shows that the mapping  $H_q(K) \to H_q(|K|)$  is well-defined, it is not so obvious to show that it will be an isomorphism, since there are a lot more singular simplices in |K| then just the simplices of K. So we will make a little detour via the cellular homology.

**10.16 Definition.** [9, 9.7.2] Let  $\sigma = \langle x_0, \ldots, x_q \rangle$  be an oriented q-simplex in a simplicial complex K. This induces an affine mapping  $\bar{\sigma} : (\Delta_q, \dot{\Delta}_q) \to (|K|^q, |K|^{q-1})$ , which can be considered as characteristic mapping for  $\sigma \subseteq |K|$ . Note however that  $\bar{\sigma}$  depends on the chosen ordering of the vertices. Hence we get a mapping

$$\Phi: C_q(K) \to C_q(|K|) = H_q(|K|^q, |K|^{q-1}), \quad \Phi(\sigma) := \bar{\sigma}_*[\Delta_q] = [\bar{\sigma}].$$

This is well-defined (i.e. depends no longer on the ordering but only on the orientation) by 10.15. Note that we used an identification 7.2 of  $C_q(K)$  with the free abelian group generated by the simplices with some fixed orientation.

**10.17 Theorem.** [9, 9.7.3] The mapping  $\Phi$  defines a natural isomorphism  $C(\_) \rightarrow C(|\_|)$ .

**Proof.** That  $\Phi_K : C(K) \to C(|K|)$  is an isomorphism is clear, since the free generators  $\sigma$  7.2 are mapped to the free generators  $[\bar{\sigma}]$  10.10.

It is natural for simplicial mappings  $\psi : K \to L$ . In fact take a simplex  $\sigma = \langle x_0, \ldots, x_q \rangle \in K$ . If  $\psi$  is injective on the vertices  $x_j$  of  $\sigma$ , then

$$\Phi\psi\sigma = \Phi\langle\psi(x_0),\ldots,\psi(x_q)\rangle = [\overline{\langle\psi(x_0),\ldots,\psi(x_q)\rangle}] = [|\psi|\circ\bar{\sigma}] = |\psi|_*\Phi\sigma.$$

In the other case  $\psi \sigma = 0$ , hence  $\Phi \psi \sigma = 0$  and  $|\psi|_* \Phi \sigma = |\psi|_* [\bar{\sigma}] = [|\psi| \circ \bar{\sigma}]$ , but  $|\psi| \circ \bar{\sigma}$  has values in  $|L|^{q-1}$ , hence  $[|\psi| \circ \bar{\sigma}] = 0 \in H_q(|L|^q, |L|^{q-1})$ .

Let us show that it is a chain mapping. For  $\sigma = \langle x_0, \ldots, x_q \rangle$  we have

$$\partial \Phi \sigma = j_* \partial_* [\bar{\sigma}] = j_* [\partial \bar{\sigma}] = \left[ \partial \bar{\sigma} \right] = \left[ \sum_j (-1)^j \bar{\sigma} \circ \delta^j \right] \quad \text{and}$$
$$\Phi \partial \sigma = \Phi \left( \sum_j (-1)^j \langle x_0, \dots, \overline{x_j}, \dots, x_q \rangle \right) = \left[ \sum_j (-1)^j \bar{\sigma} \circ \delta^j \right]$$
$$\Phi \partial$$

So  $\partial \Phi = \Phi \partial$ .

**10.18 Corollary.** [9, 9.7.4] Let K be a simplicial complex. Then we have natural isomorphisms  $H_q(K) \xrightarrow{\Phi_*} H_q(C(|K|)) \rightarrow H_q(|K|)$ , from the simplicial to the cellular and further on to the singular homology.

**Proof.** This follows by composing the isomorphisms in 10.17 and 10.12.

Let us now come back to the description of the isomorphism  $H(K) \cong H(|K|)$ indicated in the introduction to this section.

**10.19 Proposition.** [9, 9.7.7] The isomorphism  $H(K) \cong H(|K|)$  between simplicial and singular homology can be described as follows: Choose a linear ordering of the vertices of K, and then map a simplex  $\sigma = \langle x_0, \ldots, x_q \rangle$  with  $x_0 < \cdots < x_q$  to  $\overline{\sigma}$ , which is just  $\sigma$  considered as map  $\Delta_q \to |K|$ .

**Proof.** We consider the following commutative diagram



and take  $\alpha \in H_q(K)$ . It can be represented by a simplicial cycle  $z := \sum_{\sigma} n_{\sigma} \sigma \in Z_q(K) \subseteq C_q(K)$ . On the other hand we can consider the singular q-chain  $\bar{z} := \sum_{\sigma} n_{\sigma} \bar{\sigma} \in S_q(|K|)$ . It is a cycle, since  $\partial \bar{z} = \sum_{\sigma} n_{\sigma} \partial \bar{\sigma} \stackrel{!}{=} \sum_{\sigma} n_{\sigma} \overline{\partial \sigma} = \overline{\partial}(\sum_{\sigma} n_{\sigma} \sigma) = \overline{\partial z} = \bar{0} = 0$ . Since the image of  $\bar{\sigma}$  is the closure of the simplex  $\sigma$  it is contained in the q-skeleton  $|K|^q$ , and hence we may consider  $\beta := [\bar{z}] \in H_q(|K|^q)$ . Note that  $j_*(\beta) = j_*[\sum_{\sigma} n_{\sigma} \bar{\sigma}] = \sum_{\sigma} n_{\sigma} \bar{\sigma}_*[\Delta_q] = \sum_{\sigma} n_{\sigma} \Phi(\sigma) = \Phi(z)$ . Thus the composition of isomorphisms  $H_q(K) \xrightarrow{\Phi_*} H_q(C(|K|)) \to H_q(|K|)$  maps  $\alpha = [z] \mapsto [\Phi(z)] \mapsto i_*j_*^{-1}(\Phi(z)) = i_*[\beta] = [\bar{z}]$ .

#### Fundamental group versus first homology group

**10.20 Proposition.** [9, 9.8.1] We have a natural homomorphism  $h_1 : \pi_1(X, x_0) \to H_1(X)$  given by  $[\varphi] \mapsto \varphi_*[S^1] = [\varphi]$ . For the last equality we consider  $\varphi : (S^1, 1) \to (X, x_0)$  either as singular chain  $\dot{\Delta}_2 \cong S^1 \to X$  or as singular simplex  $\Delta_1 \to S^1 \to X$ .

If X is path-connected then this homomorphism is surjective and its kernel is just the commutator subgroup. Thus  $H_1(X)$  is just the abelization of  $\pi_1(X, x_0)$ .

**Proof.** That h is natural is clear. Let us show that it is a homomorphism. So let two closed curves  $\varphi, \psi : (S^1, 1) \to (X, x_0)$  be given.

We need a formula for the concatenation of paths considered as mappings  $(S^1, 1) \rightarrow (X, x_0)$ : The corresponding paths  $I \rightarrow S^1$  are obtained by composing with  $t \mapsto e^{2\pi i t}$ , hence  $\varphi \cdot \psi$  is given by  $(\varphi, \psi) \circ \nu : (S^1, 1) \rightarrow (S^1, 1) \vee (S^1, 1) \rightarrow (X, x_0)$ , where  $\nu : S^1 \rightarrow S^1 \vee S^1$  is given by  $t \mapsto (e^{2\pi i 2t}, 1) \in S^1 \vee S^1 \subseteq S^1 \times S^1$  for  $2t \leq 1$  and  $t \mapsto (1, e^{2\pi i (2t-1)}) \in S^1 \vee S^1$  for  $2t \geq 1$ .

We also need a formula for  $\nu_* : H_1(S^1) \to H_1(S^1 \vee S^1)$ : Consider the relative homeomorphism  $\sigma : (\Delta_1, \dot{\Delta}_1) \to (S^1, 1)$  given by  $(1 - t)e_0 + te_1 \mapsto e^{2\pi i t}$ . It induces an isomorphism  $H_1(\Delta_1, \dot{\Delta}_1) \to H_1(S^1, 1) \cong H_1(S^1)$ , which maps the generator  $[\Delta_1]$ to  $[S^1]$ . Now take the barycentric refinement  $B\sigma$  of  $\sigma$ . We have  $\nu_*[S^1] = \nu_*[\sigma] =$  $\nu_*[B\sigma] = [\operatorname{inj}_1 \circ \sigma] + [\operatorname{inj}_2 \circ \sigma] = [S^1] \oplus [S^1] \in H_1(S^1 \vee S^1) = H_1(S^1) \oplus H_1(S^1)$ . Thus we have

$$h_{1}([\varphi][\psi]) \stackrel{\fbox{5.14}}{=} h_{1}([(\varphi,\psi)\circ\nu]) = ((\varphi,\psi)\circ\nu)_{*}[S^{1}] = (\varphi,\psi)_{*}\nu_{*}[S^{1}]$$

$$\stackrel{\fbox{10.6}}{=} (\varphi,\psi)_{*}\left([S^{1}]\oplus[S^{1}]\right) = \varphi_{*}[S^{1}] + \psi_{*}[S^{1}]$$

$$= h_{1}[\varphi] + h_{1}[\psi].$$

Although the theorem is valid for arbitrary path-connected topological spaces, see [8, IV.3.8], we give the proof only for connected CW-complexes X. Since  $\pi_1$  and  $H_1$  do not depend on cells of dimension greater then 2 by 5.42 and 10.9, we may assume dim  $X \leq 2$ . The theorem is invariant under homotopy equivalences, hence we may assume by 5.47 and 2.49 that X has exactly one 0-cell and that this cell is  $x_0$ . So  $X^1$  is a one point union of 1-cells and X is obtained by gluing 2-cells e via maps  $f^e: S^1 \to X^1$ . By 2.34 and 2.47 we may assume that  $f^e(1) = x_0$ . Now consider the commutative diagram



By 5.49 the top  $i_*$  is onto and its kernel N is the normal subgroup generated by the  $[f^e]$ . By 10.13 the bottom  $i_*$  is onto and its kernel U is the subgroup generated by  $(f^e)_*[S^1]$ . By 5.24 we know that the abelization of a free group is the free abelian group and by 5.38 and 8.44 the two spaces in the middle are free resp. free abelian, with the corresponding generators. So we have that the result is true for  $X^1$ . Furthermore  $h_1(N) = U$ , since the generators of N are mapped to those of U. By diagram chasing the general result follows: Let  $G := \pi_1(X^1, x_0)$ . The map  $h_1 : \pi_1(X, x_0) \to H_1(X)$  is obviously surjective and its kernel is given by all gN for which  $h_1(gN) = h_1(g)U = 0$ , i.e.  $h_1(g) \in U$ . By surjectivity of  $h_1 : N \to U$  we have an  $n \in N$  with  $h_1(n) = h_1(g)$ , i.e.  $gn^{-1} \in G'$ . So  $gN \in (G/N)'$ . The converse inclusion is clear, since  $H_1(X)$  is abelian.

**10.21 Corollary.** [9, 9.8.2] For the closed orientable surface X of genus g we have  $H_1(X) \cong \mathbb{Z}^{2g}$  for the non-orientable one we have  $H_1(X) \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$  for the projective spaces we have  $H_1(\mathbb{P}^n) \cong \mathbb{Z}_2$  for  $2 \le n \le \infty$ .

**Proof.** Use the formulas given in 5.53, 5.54 and 5.43.

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**10.23 Proposition.** [9, 9.9.2] Let  $f : S^1 \to S^1$  be continuous of degree k. Then  $f_*: H_1(S^1) \to H_1(S^1)$  is given by  $[S^1] \mapsto k \cdot [S^1]$ .

**Proof.** We know that f acts by multiplication in homotopy and using the naturality of  $h_1$  gives the same result for homology.

$$\mathbb{Z} \xrightarrow{\cong} \pi_1(S^1, 1) \xrightarrow{\cong} H_1(S^1)$$

$$k \cdot \bigvee_{k \to 1} \begin{array}{c} \pi_1(f) & H_1(f) \\ & & H_1(f) \\ & & \mathbb{Z} \xrightarrow{\cong} \pi_1(S^1, 1) \xrightarrow{\cong} H_1(S^1) \end{array}$$

For a direct proof see [9, 9.5.5] and 2.17

**10.24 Proposition.** [9, 9.9.9] We have for the homology of the closed orientable surface of genus g:

$$H_q(X) \cong \begin{cases} \mathbb{Z} & \text{for } q = 0, 2 \\ \mathbb{Z}^{2g} & \text{for } q = 1 \\ 0 & \text{otherwise} \end{cases}$$

and for the non-orientable ones:

$$H_q(X) \cong \begin{cases} \mathbb{Z} & \text{for } q = 0\\ \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2 & \text{for } q = 1\\ 0 & \text{otherwise} \end{cases}$$

**Proof.** We calculate the cellular homology. Recall that in both cases X can be described as the CW-complex obtained by gluing one 2-cell e to a join of circles  $S^1$  along a map  $f: S^1 \to \bigvee^k S^1$  of the form  $i_{j_1}^{n_1} \cdots i_{j_m}^{n_m}$ . Thus the non-vanishing cellular chain groups are  $C_0(X) \cong \mathbb{Z}$ ,  $C_1(X) \cong \mathbb{R}^k$  and  $C_2(X) \cong \mathbb{Z}$  with generators given by the base-point 1, the 1-cells  $e_j^1$  and the 2-cell  $e^2$ . As in the proof of 10.20 and using 10.23 one shows that  $f_*[S^1] = n_1 \cdot [\bar{e}_{j_1}^1] + \cdots + n_m \cdot [\bar{e}_{j_m}^1]$ . Hence  $\partial_2(e^2) = j_*(\chi^e|_{S^1})[S^1] = n_1 e_{j_1}^1 + \cdots + n_m e_{j_m}^1$ , whereas  $\partial_1 = 0$ . In case of an oriented closed surface X of genus g we thus have  $\partial_2 e^2 = e_1^1 + e_2^1 - e_1^1 - e_2^1 + \cdots = 0$ , hence  $H_q(X) = H_q(C(X)) = C_q(X)$  is as claimed. In case of a non-orientable surfaces X of genus g we have  $\partial_2 e^2 = \partial(\chi^{e^2}_*[D^2]) =$ 

 $\begin{array}{l} \hline 10.11 \\ \hline 10.11$ 

10.25 Proposition. [9, 9.9.14] For the projective spaces we have

$$H_q(\mathbb{P}^n(\mathbb{R})) \cong \begin{cases} \mathbb{Z} & \text{for } q = 0 \text{ or } q = n \equiv 1 \mod 2\\ \mathbb{Z}_2 & \text{for } 0 < q < n \text{ with } q \equiv 1 \mod 2\\ 0 & \text{otherwise} \end{cases}$$

**Proof.** The idea is to consider a CW-decomposition of  $S^n$  compatible with the equivalence relation  $x \sim -x$ , which gives  $\mathbb{P}^n = S^n/\sim$ . For this we consider the spheres  $S^0 \subset S^1 \subset \cdots \subset S^n$  and the cells  $\{x \in S^q : \pm x_{q+1} > 0\}$  with characteristic map  $f_{\pm}^q : x \mapsto (x, \pm \sqrt{1-|x|^2})$ . They form a cell decomposition of  $S^n$  and hence  $e_{\pm}^q := (f_{\pm}^q)_*[D^q]$  is a basis in  $C_q(S^n)$ . We have the reflection  $r : D^q \to D^q$ ,  $x \mapsto -x$  and may consider it as mapping  $r : (S^q, S^{q-1}) \to (S^q, S^{q-1})$  to obtain an homomorphism  $r_* : C_q(S^n) \to C_q(S^n)$ .

We claim  $r_*e^q_+ = (-1)^q e^q_-$ : Note that  $r_*[D^q] = (-1)^q [D^q]$  which is obvious for q = 1 and follows by induction for  $q \ge 2$ . Since  $r \circ f^q_+ = f^q_- \circ r$  we thus get  $r_*e^q_+ = r_*(f^q_+)_*[D^q] = (f^q_-)_*r_*[D^q] = (-1)^q e^q_-$ .

Next we claim that  $\partial e_{\pm}^{q+1} = \pm (e_{\pm}^q - e_{\pm}^q) = \partial e_{\pm}^{q+1}$ : Since  $f_{\pm}^{q+1}|S^q|$  id we get  $\partial e_{\pm}^{q+1} = j_*[S^q]$  using 10.11. Now consider

$$\begin{array}{c|c} H_q(S^{q-1}) \longrightarrow H_q(S^q) \xrightarrow{j_*} H_q(S^q, S^{q-1}) \xrightarrow{\partial_*} H_{q-1}(S^{q-1}) \longrightarrow H_{q-1}(S^q) \\ \\ \parallel & \parallel & \parallel & \parallel \\ 0 & \mathbb{Z} & {}^{ab} \langle \{e_{\pm}^q\} \rangle & \mathbb{Z} & 0 \end{array}$$

So  $\partial_* \neq 0$  since it is onto and in particular applied to the generators  $e^q_{\pm}$  we have  $\partial_* e^q_+ = \partial_* e^q_- \neq 0$ . So Ker  $\partial_* \cong \mathbb{Z}$  is generated by  $e^q_+ - e^q_-$ , but it coincides with the image of  $j_*$  and hence is generated by  $j_*[S^q]$ . Thus  $j_*[S^q] = \pm (e^q_+ - e^q_-)$ .

Now  $\mathbb{P}^n$  is a *CW*-complex with cells  $e^n = p(e_{\pm}^q)$  and with characteristic mappings  $p \circ f_{\pm}^q : D^q \to \mathbb{P}^q$ . Hence the generators of  $C_q(\mathbb{P}^n)$  are given by  $(p \circ f_{\pm}^q)_*[D^q]$  again denoted  $e^q$ . Since  $p \circ r = p$  we have by the first claim that  $p_*(e_{\pm}^q) = (-1)^q p_*(r_*e_{\pm}^q) = (-1)^q p_*(e_{\pm}^q) = (-1)^q e^q$ . For  $0 < q \le n$  we get by the second claim that

$$\partial e^{q} = \partial p_{*}(e^{q}_{+}) = p_{*}\partial(e^{q}_{+}) = \pm p_{*}(e^{q-1}_{+} - e^{q-1}_{-})$$
$$= \pm (1 - (-1)^{q-1})e^{q-1} = \begin{cases} 0 & \text{for odd } q \\ \pm 2e^{q-1} & \text{for even } q \end{cases}$$

For even q with  $0 < q \leq n$  we have no non-trivial cycle in  $C_q(\mathbb{P}^n)$ , since  $\partial e^q = \pm 2e^{q-1}$ . For odd  $0 < q \leq n$  we have that  $e^q$  is a cycle and  $2e^q = \pm \partial e^{q+1}$  is a boundary for q < n. So the claimed homology follows.

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