Fréchet Spaces

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$$
\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & E \arrow[r, hook] \arrow[d, equal] & s^{\mathbb{N}} \arrow[r, two heads] \arrow[u] & s^{\mathbb{N}}/E \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & E \arrow[r, hook] & s^{\mathbb{N}} \times_{s^{\mathbb{N}}/E} \tilde{Q} \arrow[r, two heads] \arrow[u, two heads] & \tilde{Q} \arrow[r] \arrow[u, two heads] & 0 \\
 & & s \arrow[u, hook] \arrow[r, equal] & s \arrow[u, hook] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
$$

This is the script for my lecture course during the summer semester 2016. It can be downloaded at http://www.mat.univie.ac.at/∼kriegl/Skripten/2016SS.pdf Many of the proofs are taken from Meise and Vogt's book [**MV92**] and I will give detailed references to it, but also to Jarchow's book [**Jar81**].

As prerequiste the user is assumed to be familiar with basic functional analysis (for Banach spaces) and the basics of locally convex theory as presented in lecture courses on higher functional analysis. I will refer to my script [**Kri14**] for these results.

The main focus is on Fréchet spaces and additional topological properties for them. Leading examples of Fréchet spaces will be the Köthe sequences spaces and in particular the power series spaces with the space s of rapidly decreasing sequences as most relevant member. We will have to consider several of these properties also for general locally convex spaces, in particular, since the strong dual of Fréchet spaces is rarely Fréchet.

Our discussion will start with properties of locally convex spaces which are preserved by the formation of inductive or projective limits. And we will then consider what is inherited by the strong dual. Then we consider how properties of continuous linear maps translate into properties of the adjoint mappings using short exact sequences. And we will introduce topological properties which garantee the splitting of such sequences. These and further properties will also play a role in determining situations where continuous linear mappings are locally bounded and for characterizing the subspaces and the quotients of s.

I will put online a detailed list of the treated sections at the end of the semester under http://www.mat.univie.ac.at/∼kriegl/LVA-2016-SS.html.

Obviously the attentive reader will find misprints and even errors. Thus I kindly ask to inform me about such - future generations of students will appreciate the corrections.

Andreas Kriegl, Vienna in February 2016

Contents

1. Basics on Fréchet spaces

In this section we describe ((reduced) projective) limits of locally convex spaces, recall some basic facts on Fréchet spaces and introduce Köthe sequence spaces $\lambda^p(A)$ and in particular power series spaces $\lambda^p_r(A)$ as important examples.

1.1 Locally convex spaces
(See [**Kri14**, 1.4.4], [**Jar81**, 6.5 p.108], [**MV92**, 22 p.230]).
Let us recall that a LOCALLY CONVEX SPACE E is a linear space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ together with a compatible topology (i.e. addition $E \times E \to E$ and scalar multiplication $\mathbb{K} \times E \to E$ are continuous) and which has a 0-neighborhood basis consisting of (absolutely) convex sets. Equivalently, the topology can be described by a set $\mathcal{P}$ of SEMINORMS (i.e. subadditiv and positive homogeneous functions $p : E \to \mathbb{R}$). The correspondance is given by using the unit-balls $\{x : p(x) \le 1\}$ of the seminorms as 0-neighborhood subbasis and conversely considering the MINKOWSKI-FUNCTIONALS p_U (see 1.3) for U in a 0-neighborhood basis consisting of absolutely convex sets.

As usual we will require the topology to be Hausdorff or, equivalently, that the seminorms separate points, i.e. $\bigcap_{p\in\mathcal{P}} p^{-1}(0) = \{0\}$. We will abbreviate these spaces by LCS.

1.2 Limits of lcs (See [**MV92**, 24 p.257], [**Jar81**, 2.6 p.37]).
Let $\mathcal{F} : (J, \succ) \to \underline{lcs}$ be a functor from a partially ordered set (or even a small category) into the category of locally convex spaces, i.e. for every (object) $j \in J$ we are given an lcs $\mathcal{F}(j)$ and for every (morphism) $j \succ j'$ a continuous linear mapping $\mathcal{F}(j \succ j') : \mathcal{F}(j) \to \mathcal{F}(j')$ satisfying $\mathcal{F}(j' \succ j'') \circ \mathcal{F}(j \succ j') = \mathcal{F}(j \succ j' \succ j'')$.

Then the (INVERSE) LIMIT of $\mathcal{F}$ is the lcs

$$\lim \mathcal{F} := \Big\{ x = (x_j)_{j\in J} \in \prod_{j\in J} \mathcal{F}(j) : \mathcal{F}(j \succ j')(x_j) = x_{j'} \text{ for all } j \succ j' \Big\}$$

with the topology induced from the product topology, i.e. the initial topology induced by the projections $\mathrm{pr}_j : \lim \mathcal{F} \subseteq \prod_{j\in J} \mathcal{F}(j) \to F(j)$ for $j \in J$.

We call the limit a PROJECTIVE LIMIT (and we write $\varprojlim \mathcal{F}$ instead of $\lim \mathcal{F}$), iff $(J, \succ)$ is directed, i.e. $\forall j_1, j_2 \in J\ \exists j \in J : j \succ j_1, j_2$.

If $J' \subseteq J$ is initial in J, i.e. $\forall j \in J\ \exists j' \in J : j' \succ j$, then $\lim \mathcal{F}|_J \cong \lim \mathcal{F}$: In fact, the isomorphism is given by restricting the canonical projection $\prod_{j\in J} \mathcal{F}(j) \to \prod_{j'\in J'} \mathcal{F}(j')$ to the subspaces formed by the projective limits, see [**Kri08**, 3.13].

A projective limit is called REDUCED, iff all projections $\mathrm{pr}_j : \varprojlim \mathcal{F} \to \mathcal{F}(j)$ have dense image. By replacing $\mathcal{F}(j)$ with the closure $\bar{\mathcal{F}}(j)$ of the image of $\mathrm{pr}_j(\varprojlim \mathcal{F})$ in $\mathcal{F}(j)$ we get that $\varprojlim \mathcal{F}$ equals $\varprojlim \bar{\mathcal{F}}$, which is a reduced projective limit. Note that $\bar{\mathcal{F}}(j \succ j')$ is then a well defined continuous linear mapping with dense image.

As closed subspace in the product the limit of complete lcs is complete. Recall, that an lcs E is COMPLETE iff every CAUCHY-NET (i.e. $x : (I, \succ) \to E$ satisfying $\forall p \in \mathcal{P}\ \forall \varepsilon > 0\ \exists i \in I\ \forall i', i'' \succ i$: $p(x_{i'} - x_{i''}) < \varepsilon$) converges in E.

1.3 Complete lcs as limits of Banach spaces

(See [**MV92**, 24.5 p.260],[**Kri14**, 3.3.4]).
For absolutely convex $A \subseteq E$ the MINKOWSKI-FUNCTIONAL p_A is defined by

$$p_A(x) := \inf\{\lambda > 0 : x \in \lambda A\}.$$

Note that $\{x : p_A(x) < \infty\}$ is the linear span $\langle A\rangle_{\mathrm{vs}} = \bigcup_{n\in\mathbb{N}} n\,A$ of A, which coincides with E iff A is absorbing. The Minkowski-functional is a seminorm on this subspace with kernel $\bigcap_{\lambda>0} \lambda\,A$. If A is bounded (in each direction), then this kernel is $\{0\}$. By E_A we denote the resulting quotient space of $\langle A\rangle_{\mathrm{vs}}$, normed by the norm induced by p_A. If p is a seminorm on E and $A := \{x : p(x) < 1\}$ its unit ball, then we write E_p instead of $E_A = E/\ker(p)$. If A is absorbing we denote the canonical quotient map $\iota_A : E \twoheadrightarrow E_A$ and if A is bounded we denote the canonical inclusion $\iota^A : E_A \rightarrowtail E$.

Every lcs E is a dense subspace of a (projective) limit of Banach spaces: For every seminorm p we consider the completion $\tilde{E}_p$ of the space $E_p := E/\ker p$, normed by the uniquely determined seminorm $\tilde{p} : E_p \to \mathbb{R}$ with $p = \tilde{p}\circ\mathrm{pr} : E \to E_p \to \mathbb{R}$. Then E embedds topologically into $\prod_p E_p \hookrightarrow \prod_p \tilde{E}_p$ and in fact has dense image in the (projective) limit (see [**Kri08**, 3.46]) where the connecting mappings $\iota^p_{p'} : E_p \to E_{p'}$ for $p \geq p'$ (and hence $\ker p \subseteq \ker p'$) are given by $x + \ker p \mapsto x + \ker p'$: In fact, let $y \in \varprojlim_p \tilde{E}_p$ and $U = \prod_p U_p$ be a neighborhood of y in $\prod_p \tilde{E}_p$, i.e. $U_p = \tilde{E}_p$ for all but finitely many $p_1, \dots, p_n$. Choose a $p_0 \succ p_1, \dots, p_n$ and a neighborhood W of y_{p_0} in $\tilde{E}_{p_0}$ such that $\iota^{p_0}_{p_i}(W) \subseteq U_{p_i}$ for $1 \leq i \leq n$. Let $x \in E$ be such that $\iota_{p_0}(x) \in W$. Then $\iota_{p_i}(x) = \iota^{p_0}_{p_i}(\iota_{p_0}(x)) \subseteq U_{p_i}$ for $1 \leq i \leq n$, i.e. $\iota(x) \in U$.

Thus, if E is complete, then it coincides with this limit. The limit is a (reduced) projective one, since the set of seminorms of E can be assumed to be directed, i.e. for each two seminorms p_1 and p_2 we may assume that $\max\{p_1, p_2\}$ is a seminorm as well.

1.4 Lemma. Metrizable lcs

(See [**Kri14**, 3.5.2], [**MV92**, 25.1 p.276], [**Jar81**, 2.8.1 p.40]).
Let E be an lcs.

1. *E has a countable 0-neighborhood basis.*
$\Leftrightarrow$ 2. *E has a countable basis of seminorms.*
$\Leftrightarrow$ 3. *The topology of E can be described by a translation invariant metric.*

Proof. (3$\Rightarrow$1) The set $\{x : d(x, 0) < \frac{1}{n}\}$ form a countable 0-neighborhood basis.
(1$\Rightarrow$2) Take the Minkowski functionals of the 0-neighborhoods in the basis.
(2$\Rightarrow$3) Embed $E \hookrightarrow \prod_{n\in\mathbb{N}} E_{p_n}$, a product of normed spaces. Then $d(x, y) := \sum_n \frac{1}{2^n} \frac{\|x_n - y_n\|}{1 + \|x_n - y_n\|}$ gives the required metric. □

1.5 Definition. Fréchet spaces

(See [**Kri14**, 2.2.1], [**Jar81**, 6.5.3 p.109], [**MV92**, 25.1 p.276]).
A FRÉCHET SPACE ((F) for short) is a locally convex space, which satisfies the equivalent conditions of 1.4 and is (sequentially) complete (equivalently, the translation invariant metric of 1.4.3 is complete).

Fréchet spaces are Baire spaces, hence the CLOSED GRAPH THEOREM (cf. [**Kri14**, 4.3.1], [**MV92**, 24.31 p.270], [**Jar81**, 5.4.1 p.92]) and the OPEN MAPPING THEOREM (cf. [**Kri14**, 4.3.5], [**MV92**, 24.30 p.270], [**Jar81**, 5.5.2 p.95]) hold for linear maps between Fréchet spaces.

1.6 Remark. Equivalence of bases of seminorms.
Two sets $\mathcal{P}$ and $\mathcal{P}'$ of seminorms on a vector space E describe the same locally convex space, iff each seminorm of one set is dominated by finitely many seminorms of the other set (i.e. $\forall p' \in \mathcal{P}'\ \exists n \in \mathbb{N}\ \exists p_1, \dots, p_n \in \mathcal{P}$: $p' \leq \sum_{i=1}^n p_i$, and conversely).

Thus for Fréchet spaces we may assume that we have an increasing sequence of seminorms p_n as basis: In fact, we may replace a given countable set $\{p_n : n \in \mathbb{N}\}$ of seminorms by $\{p'_n := \sum_{i \leq n} p_i : n \in \mathbb{N}\}$.

1.7 Lemma. Stability of Fréchet spaces (See [**MV92**, 25.3 p.277])**.**
Closed subspaces of Fréchet spaces are Fréchet and quotients of Fréchet spaces by closed subspaces are Fréchet. Limits of countable many Fréchet spaces are Fréchet. The Fréchet spaces are exactly the (projective) limits of sequences of Banach spaces.

Proof. The trace of the countable 0-neighborhoodbasis (or countable many seminorms) is a 0-neighborhoodbasis (are the generating seminorms) of the subspace.

The QUOTIENT SEMINORMS $\tilde{q}(x+F) := \inf\{y \in F : q(x+y)\}$ are a basis of seminorms on the quotient, see [**Kri14**, 3.3.3]. And since Cauchy-sequences can be lifted along the quotient mapping (see [**Kri14**, 3.5.3]) the quotient is (sequentially-)complete as well.

We obviously get a countable basis of seminorms for the product of countable many Fréchet spaces, and since limits of complete spaces are complete, such a limit is a Fréchet space. □

1.8 Examples of Fréchet spaces.

1. $\boldsymbol{\ell^p}$, $\boldsymbol{c_0}$: Every Banach space (in particular, ℓ^p for $1 \leq p \leq \infty$ and c_0) is a Fréchet space.
2. $\boldsymbol{\mathbb{K}^{\mathbb{N}}}$: The space $\mathbb{K}^{\mathbb{N}}$ of all sequences is a Fréchet space with respect to the product topology, i.e. the pointwise(=coordinatewise) convergence. It is the limit of $\mathbb{K}^n$ for $n \in \mathbb{N}$, in fact $E_{p_n} = \mathbb{K}^n$, when $E := \mathbb{K}^{\mathbb{N}}$ and $p_n(x) := \sum_{i<n} |x_i|$.
3. $\boldsymbol{C(X)}$: Let $C(X,\mathbb{K})$ be the space of continuous functions on a topological space X supplied with the topology of uniform convergence on the compact subsets $K \subseteq X$, i.e. induced by the seminorms $p_K : f \mapsto \|f|_K\|_\infty$. In order for $C(X,\mathbb{K})$ to be complete we need, that a function, with continuous restrictions on all compact subsets, is continuous. This is the case, when X is a KELLEY-SPACE (i.e. carries the final topology with respect to its compact subsets). Then $C(X,\mathbb{K}) = \varprojlim_K C(K,\mathbb{K})$, since $C(X,\mathbb{K})_{p_K} = C(X,\mathbb{K})/\{f : f|_K = 0\} = C(K,\mathbb{K})$. If X has a countable basis for the compact sets, then $C(X,\mathbb{K})$ is metrizable.
4. $\boldsymbol{H(U)}$: If $U \subseteq \mathbb{C}^n$ is open, then the space $H(U)$ of holomorphic functions on U is a closed subspace of $C(U,\mathbb{C})$, hence Fréchet.
5. $\boldsymbol{C^\infty(X)}$: Let X be an open subset of some $\mathbb{R}^n$ or a smooth finite dimensional paracompact connected manifold. Then the space $C^\infty(X,\mathbb{K})$ of smooth functions on X is a Fréchet space with the topology of uniform convergence of each derivative separately on compact subsets (contained in some chart).

6. $\mathcal{S}$: The space $\mathcal{S}$ of RAPIDLY DECREASING FUNCTIONS on $\mathbb{R}^n$ is a Fréchet space, where the seminorms are given by $\sup\{(1+\|x\|)^j \|f^{(k)}(x)\| : x \in \mathbb{R}^n\}$ for $j, k \in \mathbb{N}$.
7. $C_{\mathcal{W}}(U)$: Spaces of weighted continuous functions. Let X be a Kelley-space and $\mathcal{W}$ a (countable) set of non-negative UPPER SEMI-CONTINUOUS (i.e. $w^{-1}([\alpha,\infty))$ is closed for all $\alpha \in \mathbb{R}$) functions $w : X \to \mathbb{R}$. Then $C_{\mathcal{W}}(U,\mathbb{K}) := \{f \in C(U,\mathbb{K}) : w \cdot f \text{ is bounded for each } w \in \mathcal{W}\}$, cf. [Sch12, 4.3 p.76]. A particular case is 3, where $\mathcal{W} = \{\chi_K : K \subseteq U \text{ is compact}\}$, cf. [Sch12, 4.4 p.76].
8. $H_{\mathcal{W}}(U)$: Spaces of weighted holomorphic functions. Let $U \subseteq \mathbb{C}$ be open and $\mathcal{W}$ be as in 7. Then $H_{\mathcal{W}}(U) := H(U,\mathbb{C}) \cap C_{\mathcal{W}}(U,\mathbb{C})$. [Sch12, 4.3 p.76]
9. $C^{(M)}(U)$: Let $U \subseteq \mathbb{R}^n$ be open and M_k be a sequence of positive real numbers. The space of DENJOY-CARLEMAN FUNCTIONS on U of BEURLING TYPE is
$$C^{(M)}(U,\mathbb{K}) := \Big\{ f \in C^\infty(U,\mathbb{K}) : \|f\|_{K,\rho} := \sup\Big\{ \frac{\|f^{(j)}(x)\|}{j!\, M_j\, \rho^j} : j \in \mathbb{N}, x \in K \Big\} < \infty \text{ for all compact } K \subseteq U \text{ and } \rho > 0 \Big\}.$$

1.9 Definition. Köthe sequence spaces
(See [MV92, 27 p.307], [Jar81, 1.7.E p.27]).
Let A be a set of $\mathbb{R}$-valued sequences, which satisfies $\forall n \in \mathbb{N}\ \exists a \in A$: $a_n \neq 0$.

Then for $1 \le q \le \infty$ the KÖTHE SEQUENCE SPACE $\lambda^q(A)$ is defined as
$$\lambda^q(A) := \{x \in \mathbb{K}^{\mathbb{N}} : \forall a \in A : a \cdot x \in \ell^q\}$$
with the seminorms given by $x \mapsto \|x\|_a := \|a \cdot x\|_{\ell^q}$. Moreover,
$$c_0(A) := \{x \in \lambda^\infty(A) : \forall a \in A : x \cdot a \in c_0\}$$
as subspace of $\lambda^\infty(A)$.

1.10 Remark.

1. We may (and will always) assume that all $a \in A$ are $\mathbb{R}_+ : \{t \in \mathbb{R}; T \ge 0\}$-valued, since obviously $\lambda^p(A) = \lambda^p(|A|)$, where $|A| := \{j \mapsto |a_j| : a \in A\}$.
2. We may (and will always) assume that A is directed, i.e.
$$\forall a, b \in A\ \exists c \in A\ \forall n \in \mathbb{N} : c_n \ge \max\{a_n, b_n\} :$$
Otherwise, let $\tilde{A} := \{\sum_{a \in \tilde{a}} a : \tilde{a} \subseteq A \text{ finite}\}$. Then $\tilde{A} \supseteq A$ is directed and $\|x\|_{\tilde{a}} := \|\sum_{a\in\tilde{a}} a \cdot x\|_{\ell^p} \le \sum_{a\in\tilde{a}} \|a \cdot x\|_{\ell^p} =: \sum_{a\in\tilde{a}} \|x\|_a$. Now apply 1.6.
3. If A is countable, we may replace A by an increasing sequence $\{\tilde{a}_n : n \in \mathbb{N}\}$: In fact, let $A = \{a_n : n \in \mathbb{N}\}$ and $\tilde{a}_n := \sum_{k \le n} a_k$. Then $\|a_n \cdot x\|_p \le \|\tilde{a}_n \cdot x\|_p = \|\sum_{k\le n} a_k \cdot x\|_p \le \sum_{k \le n} \|a_k \cdot x\|_p$, cf. 1.6.

1.11 Lemma. Köthe sequence spaces as limits (See [MV92, 27.2 p.307]).
The Köthe sequence space $\lambda^q(A)$ is isomorphic to $\lim \mathcal{F}$, where the functor $\mathcal{F}$ on $(A, \ge)$ is given by $\mathcal{F}(a) := \ell^q$ and $\mathcal{F}(a \ge a') : \mathcal{F}(a) \to \mathcal{F}(a')$ is given by $x \mapsto \frac{a'}{a} x$, where
$$\frac{a'}{a} : n \mapsto \begin{cases} \frac{a'_n}{a_n} & \text{for } a_n \neq 0, \\ 0 & \text{for } a_n = 0 \text{ (and hence } a'_n = 0). \end{cases}$$

In particular, if A is countable, then $\lambda^q(A)$ and $c_0(A)$ are Fréchet spaces (See [**MV92**, 27.1 p.307]).

Proof. The isomorphism is given by $\lambda^q(A) \ni x \mapsto (a \cdot x)_{a\in A}$ with inverse mapping $\lim \mathcal{F} \ni y \mapsto x := (\frac{1}{a(n)_n} y_n^{a(n)})_{n\in\mathbb{N}}$, where the $a(n) \in A$ are choosen such that $a(n)_n > 0$.
($\circlearrowleft$) Let $y = (y^a)_{a\in A} \in \lim \mathcal{F} \subseteq \prod_{a\in A} \ell^q$ be given. For $b \in A$ and $n \in \mathbb{N}$ let $c(n) \geq \max\{a(n), b\}$. Then $y^b = \frac{b}{c(n)} y^{c(n)}$ and $y^{a(n)} = \frac{a(n)}{c(n)} y^{c(n)}$, thus (since $c(n)_n \geq a(n)_n > 0$):

$$y_n^b = \frac{b_n}{c(n)_n} y_n^{c(n)} = \frac{b_n}{c(n)_n} \frac{c(n)_n}{a(n)_n} y_n^{a(n)} = \frac{b_n}{a(n)_n} y_n^{a(n)} = (b \cdot x)_n. \quad \square$$

1.12 Convention. Calculating with ∞.
Put $\infty \geq x \,\forall x$, $0 + \infty := \infty$, $0 \cdot \infty := 0$ and extend $+$ and $\cdot$ by monotonicity and commutativity to mappings $[0, \infty] \times [0, \infty] \to [0, \infty]$. Then

$$\infty \geq x + \infty \geq 0 + \infty = \infty \Rightarrow \forall x \geq 0 : x + \infty = \infty,$$
$$\forall 0 < x, y < \infty : x \cdot \infty \geq x \cdot \frac{y}{x} = y \Rightarrow \forall x > 0 : x \cdot \infty = \infty,$$

and then $+$ and $\cdot$ are associative and distributiv.

Let $1/0 := \infty$, $1/\infty := 0$ and $x/y := x \cdot \frac{1}{y}$. Then

$$x/0 := x \cdot \frac{1}{0} = x \cdot \infty := \begin{cases} 0 & \text{for } x = 0 \\ \infty & \text{for } 0 < x \leq \infty \end{cases} \text{ and } x/\infty := x \cdot \frac{1}{\infty} = x \cdot 0 = 0.$$

1.13 Remark. Köthe sequence spaces as reduced projective limits.
Let $E = \lambda^p(A)$ (resp. $E = c_0(A)$) and $\ell = \ell^p$ (resp. $\ell = c_0$). For $a = (a_k)_k \in A$ the mapping $x \mapsto a \cdot x$, $E \to \ell$ and hence $x \mapsto \|a \cdot x\|_{\ell^p} =: \|x\|_a$ has kernel

$$\ker \|\text{-}\|_a = \{x \in E : x|_{N_a} = 0\}, \text{ where } N_a := \operatorname{carr} a := \{k : a_k \neq 0\}.$$

By assumption on Köthe sequence spaces $\mathbb{N} = \bigcup_{a\in A} N_a$. Define the Banach spaces

$$\ell^p(a) := \{x \in \mathbb{R}^{N_a} : \|a \cdot x\|_{\ell^p} < \infty\} \cong \ell^p(\operatorname{carr} a) := \{x \in \ell^p : \operatorname{carr} x \subseteq \operatorname{carr} a\}$$
$$c_0(a) := \{x \in \ell^\infty(a) \subseteq \mathbb{R}^{N_a} \subseteq \mathbb{R}^{\mathbb{N}} : \lim a \cdot x = 0\}$$

Obviously, the coproduct $\mathbb{R}^{(N_a)}$ is dense in $\ell^p(a)$ for $p < \infty$ and hence also $\mathbb{R}^{(\mathbb{N})} \subseteq E$, since $\mathbb{R}^{(\mathbb{N})} \subseteq E/\ker \|\text{-}\|_a \subseteq \ell^p(a)$. $\Rightarrow E_a := (E/\ker \|\text{-}\|_a)^\sim \cong \ell^p(a)$ for $1 \leq p < \infty$ (resp. $E_a \cong c_0(a)$). By completeness $E = \varprojlim_a E_a$.

$$\begin{tikzcd}
 & E_a \cong \ell^p(a) \arrow[r, "\cdot a", "\cong"'] \arrow[dd, dotted, two heads, "\text{-}|_{\operatorname{carr} a'}"] & \ell^p(\operatorname{carr} a) \arrow[r, hook] \arrow[dd, "\cdot \frac{a'}{a}"] & \ell^p \arrow[dd, "\cdot \frac{a'}{a}"] \\
E = \lambda^p(A) \arrow[ur, "\text{-}|_{\operatorname{carr} a}"] \arrow[dr, "\text{-}|_{\operatorname{carr} a'}"'] & & & \\
 & E_{a'} \cong \ell^p(a') \arrow[r, "\cdot a'", "\cong"'] & \ell^p(\operatorname{carr} a') \arrow[r, hook] & \ell^p
\end{tikzcd}$$

For $p = \infty$ however, we only get $c_0(a) \subseteq E_a \subseteq \ell^\infty(a)$ and not necessarily $E_a = \ell^\infty(a)$, e.g. for $E := s$, see . Nevertheless

$$\varprojlim_a E_a = \lambda^\infty(A) = \bigcap_a \ell^\infty(a) = \varprojlim_a \ell^\infty(a),$$

but the projective limit on the right side is not reduced!

1.14 Definition. Power series space (See [**MV92**, 29 p.337])**.**
A particular case of Köthe sequence spaces is, when $A = A_{\alpha,r} := \{j \mapsto e^{t\alpha_j} : t < r\}$ for some $r \in \mathbb{R}$ and a fixed sequence $(\alpha_j)_j$ increasing monotone towards $+\infty$. Then $\lambda^q_r(\alpha) := \lambda^q(A_{\alpha,r})$ is called POWER SERIES SPACE (of FINITE TYPE if $r < +\infty$ and of INFINITE TYPE if $r = +\infty$). Note that for $r < \infty$ the mapping $\Phi : \lambda^q_r(\alpha) \to \lambda^q_0(\alpha)$, $x \mapsto (e^{r\,\alpha_j} x_j)_j$ is an isomorphism, since $\|\Phi x\|_t = \|x\|_{t+r}$, see 1.26.1.

1.15 Examples of Köthe sequence spaces (See [**MV92**, 29.4 p.339])**.**

1. If A is a singleton, then $\lambda^p(A) \cong \ell^p$ and $c_0(A) \cong c_0$.
2. Let $A := \{e_n : n \in \mathbb{N}\}$, where e_n are the standard unit vectors in $\mathbb{R}^\mathbb{N}$. Then $\lambda^\infty(A) = \lambda^p(A) = c_0(A) = \mathbb{R}^\mathbb{N}$ for all $p \in [1, \infty]$. Note that we can equally take $\{\chi_F = \max\{e_k : k \in F\} : F \subseteq \mathbb{N} \text{ is finite}\}$ instead of A.
3. Let $A = \mathbb{R}^\mathbb{N}$ be the set of all real sequences $(a_k)_k$. Then $\lambda^\infty(A) = \mathbb{K}^{(\mathbb{N})} := \coprod_{j\in\mathbb{N}} \mathbb{K}$ (cf. [**Kri14**, 3.6.1]): Suppose there is an $x \in \lambda^\infty(A)$ with $\operatorname{carr}(x)$ being not finite. Now define $a \in A$ as $a_k := k/|x_k|$, which should be (say) 1 if $x_k = 0$. Then $|(a\cdot x)_k| = k$ for all $k \in \operatorname{carr} x$, hence is not bounded. A basis of seminorms on the coproduct is given by $x \mapsto \sum_k |a_k\, x_k| \leq 2\sup\{|2^k a_k\, x_k| : k \in \mathbb{N}\}$, with $a_k \in \mathbb{R}$. This space is not Fréchet!
4. Let A be the set of all polynomials. Then $s := \lambda^\infty(A)$ is the SPACE OF FAST FALLING SEQUENCES. We get the same space if we use the subset $\{n \mapsto n^k : k \in \mathbb{N}\} \subseteq A$ or better $\{n \mapsto (1+n)^k : k \in \mathbb{N}\}$ instead of A, since this sequence is increasing. Note that we should put $0^0 := 1$ (otherwise, the first set will not satisfy the requirements for a Köthe sequence space) but then the set is not linearly ordered (since $0^0 > 0^k$ for $k > 0$). Let $p : x \mapsto \sum_{k\leq d} a_k\, x^k$. Then $\|\text{-}\|_p \leq \sum_{k\leq d} a_k \|\text{-}\|_k$.
   Moreover, $\|n \mapsto (1+n)^k x_n\|_{\ell^1} \leq \|n \mapsto (1+n)^{k+2} x_n\|_{\ell^\infty} \cdot \sum_n \frac{1}{(1+n)^2}$, hence $s = \lambda^p(A) = c_0(A)$ for all $1 \leq p \leq \infty$.
   The space s is the power series space $\lambda_\infty(\alpha)$ for $\alpha(n) := \ln(1+n)$.
5. If $A = \{n \mapsto r^n : r > 0\} = \{n \mapsto e^{s\,n} : s \in \mathbb{R}\}$ then $\lambda^\infty(A) = \lambda^1(A) = H(\mathbb{C})$, the SPACE OF ENTIRE FUNCTIONS. It is the power series space $\lambda_\infty(\alpha)$ for $\alpha(n) := n$ (See [**MV92**, 29.4.2 p.340]).
   $$x \mapsto \left(z \mapsto \sum_{n=0}^\infty x_n z^n\right)$$
   In fact, the power series $\sum_n a_n z^n$ converges for all $|z| < R$ iff $\{a_n r^n : n \in \mathbb{N}\}$ is bounded (equivalently, absolutely summable) for all $r < R$.
6. If $A = \{n \mapsto r^n : 0 < r < 1\} = \{n \mapsto e^{s\,n} : s < 0\}$ then $\lambda^\infty(A) = \lambda^1(A) = H(\mathbb{D})$, the SPACE OF HOLOMORPHIC FUNCTIONS ON THE UNIT DISK [**MV92**, 29.4.3 p.340].
   $$x \mapsto \left(z \mapsto \sum_{n=0}^\infty x_n z^n\right)$$
   It is the power series space $\lambda_0(\alpha)$ for $\alpha(n) := n$ (See [**MV92**, 29.4.2 p.340])

7. For $1 \le p < \infty$ and $\frac{1}{q} + \frac{1}{p} = 1$ we have $\lambda^1(\ell^p) = (\ell^q, \sigma(\ell^q, \ell^p)$ as lcs:
(⊇) By the Hölder inequality $\|x\|_y^{\lambda^1} = \|x \cdot y\|_{\ell^1} \le \|x\|_{\ell^q} \cdot \|y\|_{\ell^p} < \infty$ for all $y \in \ell^p$ and $x \in \ell^q$.
(⊆) Let $x \in \mathbb{K}^{\mathbb{N}}$ be such that $\|x \cdot y\|_1 < \infty$ for all $y \in \ell^p$. Then the linear map $y \mapsto x \cdot y$, $\ell^p \to \ell^1$ has closed graph and thus is continuous. Consequently, $y \mapsto \sum_n x_n \cdot y_n$ is a continuous linear functional, hence $x \in (\ell^p)^* = \ell^q$ (see [**Kri14**, 5.3.1]).
$\lambda^1(\ell^\infty) = (\ell^1, \sigma(\ell^1, \ell^\infty))$: For (⊆) choose $y = 1$.
$\lambda^1(c_0) = (\ell^1, \sigma(\ell^1, c_0))$: Suppose $x \in \lambda^1(c_0) \setminus \ell^1$, choose $k \mapsto n_k$ strictly increasing with $\sum_{j=n_k+1}^{n_{k+1}} |x_j| \ge k$ and $y_j := \frac{1}{k}$ for $n_n < j \le n_{k+1}$. Then $\|x \cdot y\|_{\ell^1} \ge \sum_k 1 = \infty$.

1.16 Proposition. Function spaces isomorphic to s
(See [**MV92**, 29.5 p.340]).
The following spaces are isomorphic to s: $C^\infty_{2\pi}(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$, $C^\infty_{[a,b]}(\mathbb{R})$, and $C^\infty([a,b])$.

Proof. (1) $\boldsymbol{C^\infty_{2\pi}(\mathbb{R}) \cong s}$ via Fourier-coefficients $f \mapsto \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-ikt}\, dt\Big)_{k\in\mathbb{Z}}$, cf. [**Kri07b**, 5.4.5] and 1.26.3: Let $c_k(f) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-ikt}\, dt$. Then $c_k(f') = ik\, c_k(f)$ by [**Kri07b**, 5.4.4 p.101] or [**Kri06**, 9.3.5], $f \in L^1 \Rightarrow (c_k(f))_{k\in\mathbb{Z}} \in c_0$ and $c \in \ell^1 \Rightarrow \sum_k c_k \exp_k$ converges absolute in C by Riemann-Lebesgue [**Kri07b**, 5.4.1 p.95], [**Kri06**, 9.3.6]. Note, that that s is taken with index set $\mathbb{Z}$ instead of $\mathbb{N}$, but see 1.26.3.

(2) $\boldsymbol{\mathcal{S}(\mathbb{R}) \cong s}$ (See [**MV92**, 29.5.2 p.341]):
Let $\rho(t) := e^{-t^2}$ and consider the Hilbert space completion $L^2_\rho(\mathbb{R})$ of the space of polynomials with respect to the inner product $\langle f|g\rangle_\rho := \int_{\mathbb{R}} f(t)\overline{g(t)}\ \rho(t)\, dt$. Obviously $L^2_\rho(\mathbb{R}) \cong L^2(\mathbb{R})$ via $f \mapsto \sqrt{\rho}\, f$. Gram-Schmidt orthonormalization applied to the monomials $t \mapsto t^n$ gives an orthonormal basis $(\frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n)_{n\in\mathbb{N}}$, where H_n are the Hermite polynomials (cf. [**Kri07b**, 6.3.9 p.118]), which can also be obtained recursively $H_0 := 1$, $H_{n+1}(t) := 2t\, H_n(t) - 2n\, H_{n-1}(t)$:
From the recursion we get $H_n' = 2n\, H_{n-1}$ by induction. In fact $H_0' = 0$, $H_1(t) = 2t$, $H_1' = 2H_0$, and hence

$$\begin{aligned} H_{n+1}' &= \big(2 \,\mathrm{id}\, H_n - 2n\, H_{n-1}\big)' = 2\, H_n + 2 \,\mathrm{id}\, H_n' - 2n\, H_{n-1}' \\ &= 2\, H_n + 4n \,\mathrm{id}\, H_{n-1} - 4n(n-1)\, H_{n-2} \\ &= 2\, H_n + 2n \cdot (2 \,\mathrm{id}\, H_{n-1} - 2(n-1)\, H_{n-2}) = 2(n+1)\, H_n \end{aligned}$$

Moreover, $H_n = (-1)^n \frac{\rho^{(n)}}{\rho}$ since

$$\begin{aligned} H_{n+1} &= 2 \,\mathrm{id}\, H_n - H_n' = 2 \,\mathrm{id}\, (-1)^n \frac{\rho^{(n)}}{\rho} - \Big((-1)^n \frac{\rho^{(n)}}{\rho}\Big)' \\ &= (-1)^n \Big(2 \,\mathrm{id}\, \frac{\rho^{(n)}}{\rho} - \frac{\rho\, \rho^{(n+1)} - (-2 \,\mathrm{id}\, \rho)\, \rho^{(n)}}{\rho^2}\Big) = (-1)^{n+1} \frac{\rho^{(n+1)}}{\rho}. \end{aligned}$$

By induction we get for $m \ge n$:

$$\begin{aligned} \langle H_{m+1}|H_n\rangle_\rho &= \int \rho\, H_{m+1}\, H_n = \int \rho\, (2 \,\mathrm{id}\, H_m - H_m')\, H_n = \int (-\rho'\, H_m - \rho\, H_m')\, H_n \\ &= \int \rho\, H_m\, H_n' - \int (\rho\, H_m\, H_n)' = 2n \int \rho\, H_m\, H_{n-1} = 0. \end{aligned}$$

Finally, $\int \rho = \sqrt{\pi}$ and again by induction

$$\begin{aligned}\|H_n\|_\rho^2 = \int \rho\, H_n^2 &= (-1)^n \int H_n\, \rho^{(n)} \overset{\text{part.int.}}{=\!=\!=} (-1)^{n-1} \int H_n'\, \rho^{(n-1)} \\ &= \int 2n\, H_{n-1}\, (-1)^{n-1} \rho^{(n-1)} = 2n\, \|H_{n-1}\|_\rho^2 \\ &= 2n\, \sqrt{\pi}\, 2^{n-1}(n-1)! = 2^n\, n!\, \sqrt{\pi}.\end{aligned}$$

Thus the corresponding HERMITE FUNCTIONS $h_n := \frac{\sqrt{\rho}}{\sqrt{2^n\, n!\, \sqrt{\pi}}} H_n$ form an orthonormal basis of $L^2(\mathbb{R})$. For $A_\pm : \mathcal{S} \to \mathcal{S}$, defined by $f \mapsto \mathrm{id} \cdot f \mp f'$, we have:

$$\begin{aligned} A_-(h_n) := h_n' + \mathrm{id}\; h_n &= \frac{(\sqrt{\rho}\, H_n)' - \mathrm{id}\, \sqrt{\rho}\, H_n}{\sqrt{2^n\, n!\, \sqrt{\pi}}} \\ &= \frac{1}{\sqrt{2^n\, n!\, \sqrt{\pi}}} \sqrt{\rho}\, H_n' = \frac{2n}{\sqrt{2^n\, n!\, \sqrt{\pi}}} \sqrt{\rho}\, H_{n-1} \\ &= \frac{\sqrt{2n}}{\sqrt{2^{n-1}\, (n-1)!\, \sqrt{\pi}}} \sqrt{\rho}\, H_{n-1} = \sqrt{2n}\, h_{n-1} \\ \Rightarrow A_+^m f &\overset{\text{onb}}{=\!=\!=} \sum_{n=0}^\infty \langle A_+^m f | h_n\rangle\, h_n \overset{\text{part.int.}}{=\!=\!=} \sum_{n=0}^\infty \langle f | A_-^m h_n\rangle\, h_n \\ &= \sum_{n \geq m} 2^{m/2} \sqrt{n(n-1)\dots(n-m+1)}\, \langle f | h_{n-m}\rangle\, h_n \end{aligned}$$

$$\overset{\text{onb}}{\Longrightarrow} |\langle A_+^m f | h_{n+m}\rangle|^2 = 2^m (n+m)(n+m-1)\dots(n+1) |\langle f|h_n\rangle|^2$$

$$\Rightarrow \sum_{n=0}^\infty n^m |\langle f|h_n\rangle|^2 \leq 2^{-m} \sum_{n=0}^\infty |\langle A_+^m f | h_{n+m}\rangle|^2 \leq 2^{-m} \|A_+^m f\|_{L^2(\mathbb{R})}^2 < \infty,$$

hence $\mathcal{S} \to s$, $f \mapsto (\langle f|h_n\rangle)_{n\geq 0}$ is continuous and obviously injective.
It is also onto: Let $a \in s$. Then $\sum_n a_n h_n$ converges in $\mathcal{S}$, since

$$-h_k' + \mathrm{id} \cdot h_k = A_+^1(h_k) = \sum_{n\geq 1} 2^{1/2} \sqrt{n}\, \langle h_k | h_{n-1}\rangle\, h_n = \sqrt{2(k+1)}\, h_{k+1}$$

$$\Rightarrow h_n' = \sqrt{\frac{n}{2}} h_{n-1} - \sqrt{\frac{n+1}{2}} h_{n+1} \quad \text{and} \quad \mathrm{id} \cdot h_n = \sqrt{\frac{n}{2}} h_{n-1} + \sqrt{\frac{n+1}{2}} h_{n+1}.$$

(3) $\boldsymbol{C^\infty_{[a,b]}(\mathbb{R}) := \{f \in C^\infty(\mathbb{R}) : f(x) = 0\ \forall x \notin [a,b]\} \cong s}$
(See [**MV92**, 29.5.3 p.342]):
W.l.o.g. $-a = b = \pi/2$.

$\Phi : S(\mathbb{R}) \to C^\infty_{[-\pi/2,\pi/2]}(\mathbb{R})$, $\Phi(f)(t) := f(\tan(t))\ \forall |t| < \pi/2$ is an iso, since

$$\Phi(f)^{(p)} = \sum_{j=1}^p \frac{\tilde g_{j,p}}{\cos^{j+p}} f^{(j)} \circ \tan \ \text{ with } \tilde g_{j,p} \in C^\infty_{2\pi}(\mathbb{R}, \mathbb{R})$$

$$|\tan(t)^k f^{(j)}(\tan(t))| \leq \sup_{x\in\mathbb{R}} |x^k f^{(j)}(x)| =: C_{k,j} < \infty\ \forall |t| < \pi/2$$

Since $\tan(x) \sim \dfrac{1}{\cos(x)}$ for x near $\pm\pi/2$ we get $|\Phi(f)^{(p)}(t)| \to 0$ for $t \to \pm\pi/2$.

And the inverse mapping is given by $f \mapsto f \circ \arctan$ using analogous arguments: $\arctan'(s) = \frac{1}{1+s^2} \Rightarrow \arctan^{(n)}(s) = \frac{q_n(s)}{(1+s^2)^n}$ with $\deg(q_n) \leq n-1$. Thus

$$t^k\, \Phi^{-1}(f)^{(p)}(t) = \sum_{j=1}^p \frac{t^k\, g_{j,p}(t)}{(1+t^2)^n} f^{(j)}(\arctan(t)) \text{ with } \deg(g_{j,p}) \leq n-1,$$

and

$$\begin{aligned} t^{n+k-1} f^{(j)}(\arctan(t)) &= \tan\Big(\pm(\pi/2-s)\Big)^{n+k-1} f^{(j)}\Big(\pm(\pi/2-s)\Big) \\ &= (\pm\cot(s))^{n+k-1} f^{(j)}\Big(\pm(\pi/2-s)\Big) \\ &= \left(\frac{\pm s\cos(s)}{\sin(s)}\right)^{n+k-1} \frac{f^{(j)}(\pm(\pi/2-s))}{s^{n+k-1}} \to 0 \text{ for } s \searrow 0. \end{aligned}$$

Now the result follows since $S(\mathbb{R}) \cong s$ by 2.

(4) $\boldsymbol{C^\infty([a,b]) \cong s}$ (See [**MV92**, 29.5.4 p.343]):
W.l.o.g. $-a = b = 1$.

$$\Phi : f \mapsto f \circ \cos,\ C^\infty([-1,1]) \to C^\infty_{2\pi,\text{even}} \cong s$$

is continuous and injective. It is also onto, since

$$f := g \circ \arccos \in C([-1,1]) \cap C^\infty(]-1,1[),\ f' = -\frac{g'}{\sin} \circ \arccos,\ \text{and } \frac{g'}{\sin} \in C^\infty_{2\pi,\text{even}}.$$

Note that via Fourier-coefficents $C^\infty_{2\pi,\text{even}} = \{f \in C^\infty_{2\pi} : f(x) = f(-x)\} = \{f \in C^\infty_{2\pi} : c_n(f) = c_{-n}(f)\} \cong s$, via $\big(x \mapsto \sum_{n\ge 0} a_n \cos(n\,x)\big) \leftarrow\!\shortmid (a_n)_n$. Thus $s \to C^\infty([-1,1])$ is given by $a \mapsto \sum_{n\in\mathbb{N}} a_n \cos(n \arccos t) = \sum_{n\in\mathbb{N}} a_n T_n$, where $T_n : t \mapsto \cos(n \arccos t)$ are the Tschebyscheff(=Chebyshev) polynomials. □

1.17 Definition. Schauder-basis and absolute basis.
A sequence $(e_j)_{j\in\mathbb{N}}$ is called Schauder-basis in [**MV92**, Def. in 24.27 p.322] (or called topological basis in [**Jar81**, 14.2 p.292]) of the lcs E, if

$$\forall x \in E\, \exists!\, \xi = (\xi_j(x))_j \in \mathbb{K}^\mathbb{N} : x = \sum_j \xi_j(x)\, e_j.$$

The mappings $x \mapsto \xi_j(x)$ are then linear.
Obviously, the standard basis $(e_j)_{j\in\mathbb{N}}$ is a Schauder-basis in $\lambda^p(A)$ for any A:

$$\Big\| a \cdot \Big(x - \sum_{j=0}^n x_j\, e_j\Big)\Big\|_{\ell^p} = \Big\|\sum_{j>n} a_j\, x_j\, e_j\Big\|_{\ell^p} = \Big(\sum_{j>n} |a_j\, x_j|^p\Big)^{1/p} \to 0.$$

A Schauder-basis is called absolute basis (See [**MV92**, Def. in 24.27 p.322], [**Jar81**, 14.7.6 p.314]), iff

$$\forall p\, \exists p'\, \exists C > 0\, \forall x : \sum_j |\xi_j(x)|\, p(e_j) \le C\, p'(x)$$

The standard basis is an absolute basis in $\lambda^p(A)$ iff $\lambda^p(A) = \lambda^1(A)$:

$$\begin{aligned} \forall a\, \exists a'\, \exists C > 0\, \forall x : \sum_j |x_j|\, \underbrace{\|e_j \cdot a\|_{\ell^p}}_{=|a_j|} &\le C\, \|x \cdot a'\|_{\ell^p}, \\ \text{i.e. } \|x \cdot a\|_{\ell^p} \le \|x \cdot a\|_{\ell^1} &\le C\, \|x \cdot a'\|_{\ell^p}. \end{aligned}$$

1.18 Lemma on (F) with Schauder-basis (See [**MV92**, 28.10 p.331])**.**
Let F be a Fréchet-space with Schauder-basis $(e_j)_j$ and corresponding coefficient functionals ξ_j. Then

$$\forall p\, \exists p'\, \exists C\, \forall x : \sup_{k\in\mathbb{N}} p\Big(\sum_{j\le k} \xi_j(x) e_j\Big) \le C\, p'(x).$$

Proof. Let $(\|\text{-}\|_n)_n$ be an increasing basis of seminorms of F. We consider new seminorms $\tilde{p}_n(x) := \sup_{k\in\mathbb{N}} \big\|\sum_{j=1}^k \xi_j(x)\, e_j\big\|_n$. Obviously, $\|\text{-}\|_n \le \tilde{p}_n$ since $\sum_{j\le k} \xi_j(x) e_j$ converges to x, thus the metrizable locally convex topology τ induced by the seminorms $\tilde{p}_n$ is finer than the given one. In order to apply the open mapping theorem it is enough to show completeness of τ: Let $(x^m)_m$ be a Cauchy-sequence for τ. We have

$$\big|\xi_k(x^{m'}) - \xi_k(x^{m''})\big|\, \|e_k\|_n \le \Big\|\sum_{j=1}^{k} \xi_j(x^{m'} - x^{m''})\, e_j\Big\|_n + \Big\|\sum_{j=1}^{k-1} \xi_j(x^{m'} - x^{m''})\, e_j\Big\|_n$$
$$\le 2\,\tilde{p}_n(x^{m'} - x^{m''}).$$

Since $\forall k\, \exists n : \|e_k\|_n > 0$ the sequence $(\xi_k(x^m))_m$ is Cauchy in $\mathbb{K}$, let x_k^∞ be its limit. Since $(x^m)_m$ is Cauchy, we have

$$\forall n\, \forall \varepsilon > 0\, \exists m\, \forall m', m'' \ge m\, \forall k :$$
$$\varepsilon \ge \tilde{p}_n(x^{m'} - x^{m''}) \ge \Big\|\sum_{j=1}^{k} \xi_j(x^{m'})\, e_j - \sum_{j=1}^{k} \xi_j(x^{m''})\, e_j\Big\|_n.$$

With $m'' \to \infty$ we obtain

$$\Big\|\sum_{j=1}^{k} \xi_j(x^{m'})\, e_j - \sum_{j=1}^{k} x_j^\infty\, e_j\Big\|_n \le \varepsilon.$$

Thus

$$\forall k, p : \Big\|\sum_{j=k+1}^{k+p} x_j^\infty\, e_j\Big\|_n \le 2\varepsilon + \Big\|\sum_{j=k+1}^{k+p} \xi_j(x^m)\, e_j\Big\|_n.$$

Since $\sum_j \xi_j(x^m)\, e_j$ converges in E, the sequence $\sum_j x_j^\infty\, e_j$ is Cauchy, hence converges to some $x^\infty := \sum_{j=0}^\infty x_j^\infty\, e_j \in E$ with $\xi_j(x^\infty) = x_j^\infty$, since (e_j) is a Schauder-basis. By the inequality above, we have that $x^{m'} \to x^\infty$ with respect to τ. □

1.19 Corollary. Schauder-bases in (F) have continuous coefficients
(See [**MV92**, 28.11 p.332]).
Let F be a Fréchet-space with Schauder-basis $(e_j)_j$ and corresponding coefficient functionals ξ_j. Then $\forall p\, \exists p'\, \exists C > 0\, \forall x\, \forall j : |\xi_j(x)|\, p(e_j) \le C\, p'(x)$. □

In [**Jar81**, 14.2 p.292] a Schauder-basis is defined as a topologogical basis for which the coefficient functionals are continuous.

1.20 $H(\mathbb{D}_R)$ has $(z^k)_{k\in\mathbb{N}}$ as absolute basis (See [**MV92**, 27.27 p.323]).
Let $\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$ be the disk with radius $0 < R \le \infty$. Taylor development $f(z) = \sum_k \frac{f^{(k)}(0)}{k!} z^k$ shows that $(z \mapsto z^k)_{k\in\mathbb{N}}$ is a Schauder-basis of $H(\mathbb{D}_R)$. This is even an absolute basis: $\|f\|_r := \sup\{|f(z)| : |z| \le r\}$ for $r < R$ is a basis of seminorms and $\forall f \in H(\mathbb{D}_R)\, \forall r < r' < R$:

$$\sum_j \Big|\frac{f^{(j)}(0)}{j!}\Big|\, \|z^j\|_r \overset{[\textbf{Kri11}, 3.30]}{=\!=\!=} \sum_j \Big|\frac{1}{2\pi i}\int_{|z|=r'} \frac{f(z)}{z^{j+1}}\, dz\Big|\, r^j = \sum_{j=0}^\infty \Big(\frac{r}{r'}\Big)^j\, \|f\|_{r'}.$$

1.21 The Fréchet spaces with absolute basis are the spaces $\lambda^1(A)$
(See [**MV92**, 27.26 p.323], [**Jar81**, 14.7.8 p.314]).
For Fréchet space E we have: $\exists A$ countable: $E \cong \lambda^1(A) \Leftrightarrow E$ has an absolute basis.

Proof. ($\Rightarrow$) The standard basis $(e_j)_{j\in\mathbb{N}}$ is obviously an absolute basis of $\lambda^1(A)$.

($\Leftarrow$) (See [**MV92**, 27.25 p.322]) Let $(e_j)_j$ be an absolute basis of E and consider the Köthe matrix $A := (j \mapsto \|e_j\|_p)_{p\in\mathbb{N}}$. Then $\xi : E \to \mathbb{K}^{\mathbb{N}}$, $x \mapsto (\xi_j(x))_j$ is linear. $(e_j)_j$ absolute basis $\Rightarrow$

$$\forall p\; \exists p'\; \exists C\; \forall x : \sum_j |\xi_j(x)|\, \|e_j\|_p \le C\|x\|_{p'}$$

$\Rightarrow \xi(x) \in \lambda^1(A)$ and $\xi : E \to \lambda^1(A)$ continuous and injective.
Claim: ξ is onto $\lambda^1(A)$:

$$y = (y_j)_j \in \lambda^1(A) \Rightarrow \Big\| \sum_{j=n+1}^{n+k} y_j e_j \Big\|_p \le \sum_{j=n+1}^{n+k} |y_j|\, \|e_j\|_p$$

$\xRightarrow{y\,\in\,\lambda^1(A)} n \mapsto \sum_{j\le n} y_j e_j$ Cauchy in $E \Rightarrow$ converges to $x := \sum_j y_j e_j$ with $\xi(x) = y$, i.e. ξ onto. $\xRightarrow{\text{open map.thm.}} \xi$ is isomorphism. $\square$

1.22 Dual space of $\lambda^p(A)$ (See [**MV92**, 27.11 p.313])**.**
Let $\lambda := \lambda^p(A)$ with $1 \le p < \infty$ or $\lambda := c_0(A)$. Then

$$x^* \mapsto (x^*(e_j))_{j\in\mathbb{N}},\; \lambda^* \to \lambda^1(\lambda) := \Big\{ y \in \mathbb{K}^{\mathbb{N}} : \forall x \in \lambda : \sum_j |x_j y_j| < \infty \Big\}$$

is linear and injective. If A is countable it is even bijective.

Proof.
($\rightarrowtail$) $y \in \lambda^*$:

$$\forall x \in \lambda : x = \sum_{j=0}^{\infty} x_j e_j \Rightarrow\; y(x) = y\Big(\sum_{j=0}^{\infty} x_j\, e_j\Big) = \sum_{j=0}^{\infty} x_j\, \underbrace{y(e_j)}_{=:y_j}.$$

$$\|\varepsilon\|_\infty \le 1 \Rightarrow \varepsilon \cdot x \in \lambda, \text{ hence } \sum_j x_j\, y_j \text{ converges absolutely.}$$

($\twoheadrightarrow$) $y \in \lambda^1(\lambda) \Rightarrow y^n := \chi_{\{1,\dots,n\}} \cdot y \in \lambda^*$ and

$$\lim_{n\to\infty} y^n(x) = \lim_{n\to\infty} \sum_{j\le n} x_j y_j = \sum_{j=0}^{\infty} x_j y_j =: y(x)\; \forall x \in \lambda$$

$U_n := \{x \in \lambda : |y^n(x)| \le 1\} \Rightarrow U := \bigcap_{n\in\mathbb{N}} U_n$ is barrel (see 2.1), $y \in U^o$, λ barrelled by 2.3 $\Rightarrow U$ 0-nbhd, hence $y \in \lambda^*$. $\square$

Counter-example.
Let $A = c_0$. By 1.15.7 we have $\lambda := \lambda^1(A) = (\ell^1, \sigma(\ell^1, A))$, and hence $\lambda^* = A = c_0$, whereas $\lambda^1(\lambda) = \lambda^1(\ell^1) = \ell^\infty$.

1.23 Minkowski-functionals on polars in the dual.
For any subset $A \subseteq E$ we have the POLAR

$$A^o := \{x^* \in E^* : |x^*(x)| \le 1\; \forall x \in A\}.$$

The Minkowski-functional p_{A^o} (on the linear span of A^o) is given by

$$p_{A^o}(x^*) := \inf\Big\{\lambda > 0 : x^* \in \lambda\, A^o\Big\} = \inf\Big\{\lambda > 0 : \Big|\frac{x^*}{\lambda}(x)\Big| \le 1\; \forall x \in A\Big\}$$
$$= \inf\Big\{\lambda > 0 : |x^*(x)| \le \lambda\; \forall x \in A\Big\} = \sup\Big\{|x^*(x)| : x \in A\Big\} = \|x^*|_A\|_\infty.$$

In the particular case, where $A = U \subseteq E$ is a 0-neighborhood, the polar U^o is bounded in the strong dual E^*. In fact, the STRONG TOPOLOGY is that of uniform convergence on the bounded sets $B \subseteq U$, i.e. given by the seminorms $\|\_|_B\|_\infty$. Since B is bounded, it is contained in $K \cdot U$ for some $K > 0$, hence $\|x^*|_B\|_\infty := \sup\{|x^*(x)| : x \in B\} \le \sup\{|x^*(K\,u)| : u \in U\} \le K \sup\{|x^*(x)| : x \in U\}$, which is at most K for $x^* \in U^o$.

1.24 Minkowski-functionals for polars of 0-nbhds in $\lambda^p(A)$ (See [MV92, 27.12 p.313]).

Let $\lambda := \lambda^p(A)$ for $1 \le p < \infty$ or $\lambda := c_0(A)$. For $a \in A$ let $U_a := \{x \in \lambda : \|a \cdot x\|_{\ell^p} < 1\}$ and $\|\_\|_a^* := p_{(U_a)^o} = \|\_|_{U_a}\|_\infty$ with unit-ball $(U_a)^o$. Then

$$\|y\|_a^* = \Big\|\frac{y}{a}\Big\|_{\ell^q} \text{ for } \frac{1}{p} + \frac{1}{q} = 1 \text{ or } q = 1 \text{ in case } \lambda = c_0(A).$$

Proof. Let first $1 < p < \infty$ and $y \in \lambda^*$. We assume first, that $\operatorname{carr} y \subseteq \operatorname{carr} a$. Then

$$\|y\|_a^* \overset{\boxed{1.23}}{=\!=} \|y|_{U_a}\|_\infty = \sup_{x \in U_a} |y(x)| \overset{\boxed{1.22}}{=\!=} \sup_{x \in U_a} \Big|\sum_{j \in \mathbb{N}} x_j y_j\Big| = \sup_{x \in U_a} \Big|\sum_{j \in \operatorname{carr} a} x_j y_j\Big|$$
$$= \sup\Big\{\Big|\sum_{j \in \operatorname{carr} a} x_j a_j \cdot \frac{y_j}{a_j}\Big| : \|(x_j a_j)_{j \in \operatorname{carr} a}\|_{\ell^p} \le 1\Big\} \overset{\ell^q = (\ell^p)^*}{=\!=\!=} \Big\|\frac{y}{a}\Big\|_{\ell^q},$$

and for $\operatorname{carr} y \not\subseteq \operatorname{carr} a$ we get ∞ on both sides.
Analogous for $\lambda = \lambda^1(A)$ and $\lambda = c_0(A)$. □

1.25 Theorem. Equality of $\lambda^p(A)$ for various p (See [MV92, 27.16 p.315]).

1. *$\exists\, 1 \le p \ne p' \le \infty$: $\lambda^p(A) = \lambda^{p'}(A)$ as lcs;*

$\Leftrightarrow$ 2. *$\forall\, 1 \le p \ne p' \le \infty$: $\lambda^p(A) = \lambda^{p'}(A)$ as lcs;*

$\Leftrightarrow$ 3. *$\forall a \in A\ \exists a' \in A$: $\frac{a}{a'} \in \ell^1$.*

If A is countable, then it is enough to assume equality in 1 and 2 only as sets.

Proof. If A is countable, and $p < p' \Rightarrow \lambda^p(A) \to \lambda^{p'}(A)$ continuous injective $\xrightarrow{\text{open map.thm.}} \lambda^p(A) = \lambda^{p'}(A)$ as Fréchet spaces in 1 and 2.

(3 ⇒ 2) Since $\lambda^p(A) \to \lambda^{p'}(A)$ injects continuously for $1 \le p \le p' \le \infty$, we have to show that $\lambda^\infty(A)$ injects continuously in $\lambda^1(A)$. Let $a \in A$. $\exists a'$ satisfying 3. ⇒

$$\forall x \in \lambda^\infty(A) : \|x\|_a^{\lambda^1(A)} := \sum_j |x_j a_j| = \sum_j \Big|x_j a'_j \frac{a_j}{a'_j}\Big| \le \|x\|_{a'}^{\lambda^\infty(A)} \Big\|\frac{a}{a'}\Big\|_{\ell^1}.$$

(2 ⇒ 1) is trivial.

(1 ⇒ 3) For $p' = \infty$ we get $\lambda^\infty(A) = c_0(A)$, since $\ell^p \subseteq c_0 \subseteq \ell^\infty$.

$$\Rightarrow \forall a\ \exists a' \ge a\ \exists C > 0 : \|\_\|_a^{\lambda^p(A)} \le C\, \|\_\|_{a'}^{\lambda^{p'}(A)}$$
$$\xRightarrow{\text{[Kri14, 1.3.3,1.3.7]}} \forall a\ \exists a' \ge a\ \exists C > 0 : U_{a'}^{p'} := \{x : \|x\|_{a'}^{\lambda^{p'}} \le 1\} \subseteq C\, U_a^p$$
$$\xRightarrow{\boxed{1.24}} \forall y \big(\in \langle (U_a^p)^o \rangle_{\text{lin.sp}}\big) : \|y\|_{a',p'}^* \le C \|y\|_{a,p}^*.$$

$1/q+1/p := 1$; $1/q'+1/p' := 1$. $\forall \eta \in \ell^q$, $\|\eta\|_{\ell^q} \leq 1$

$$\xRightarrow{\text{Hölder, } \boxed{1.22}} \eta \cdot a \in (U_a^p)^o \subseteq C\,(U_{a'}^{p'})^o, \text{ since } \|a \cdot x\|_{\ell^p} \leq 1\, \forall x \in U_a^p.$$

$$\xRightarrow{\boxed{1.24}} \Big(\sum_j |\eta_j|^{q'} \big(\frac{a_j}{a'_j}\big)^{q'}\Big)^{1/q'} = \|\eta \cdot a|_{U_{a'}^{p'}}\|_\infty \leq C$$

$$\xRightarrow{\xi := \eta^{q'}} \sup\Big\{\sum_j |\xi_j| \big(\frac{a_j}{a'_j}\big)^{q'} : \xi \in \ell^{q/q'}, \|\xi\|_{\ell^{q/q'}} = (\|\eta\|_{\ell^q})^{q'} \leq 1\Big\} \leq C^{q'}$$

$t := \frac{q}{q-q'}$, i.e. $\frac{1}{t}+\frac{q'}{q} = 1$, $(\ell^{q/q'})^* = \ell^t$

$$\xRightarrow{\boxed{1.15.7}} \Big(\frac{a}{a'}\Big)^{q'} \in \ell^t \text{ and } \sum_j \Big(\frac{a_j}{a'_j}\Big)^{q'\,t} = \bigg(\Big\|\Big(\frac{a}{a'}\Big)^{q'}\Big\|_{\ell^t}\bigg)^t \leq C^{q'\,t}.$$

$\Rightarrow \exists d := q't \geq 1$ $\forall a$ $\exists a'$ $\sum_j (a_j/a'_j)^d < \infty$. W.l.o.g. $d \in \mathbb{N}$. Let $a^{(0)} = a$ and choose $a^{(1)}, a^{(2)}, \dots, a^{(d)}$ recursively with $\sum_j (a_j^{(k)}/a_j^{(k+1)})^d < \infty$ for $0 \leq k < d$.

$$\xRightarrow{\text{Hölder inductive}} \sum_j \frac{a_j^{(0)}}{a_j^{(d)}} = \sum_j \prod_{k=0}^{d-1} \frac{a_j^{(k)}}{a_j^{(k+1)}} < \infty.$$

□

1.26 Proposition. Equalities for power series spaces
(See [**MV92**, Aufgabe 1+2 p.323]).

1. *Let* $0 < \alpha = (\alpha_n)_n \nearrow \infty$, $R \in [0,\infty)$, $p \in [1,\infty]$. *Then* $\lambda_R^p(\alpha) \cong \lambda_0^p(\alpha)$.
2. *Let* $R \in \{0,\infty\}$. *Then* $\lambda_R^1(\alpha) = \lambda_R^1(\beta) \Leftrightarrow \exists C \geq 1$: $\frac{1}{C}\alpha \leq \beta \leq C\alpha$.
3. $\sup\{\frac{\alpha_{2j+1}}{\alpha_j}\} < \infty \Rightarrow \lambda_R^p(\alpha) \times \lambda_R^p(\alpha) \cong \lambda_R^p(\alpha)$ *for* $R \in \{0,+\infty\}$ *and* $p \in [1,\infty]$. *In particular,* $s \times s \cong s$ *and* $s(\mathbb{Z}) \cong s(\mathbb{N})$.
4. $\lambda_\infty^1(\alpha) = \lambda_\infty^p(\alpha)$ $\forall p \in [1,\infty] \Leftrightarrow \exists r < 1$: $\sum_j r^{\alpha_j} < \infty \Leftrightarrow \sup\{\frac{\ln j}{\alpha_j} : j\} < \infty$.
5. $\lambda_0^1(\alpha) = \lambda_0^p(\alpha)$ $\forall p \in [1,\infty] \Leftrightarrow \forall r < 1$: $\sum_j r^{\alpha_j} < \infty \Leftrightarrow \lim_{j\to\infty} \frac{\ln j}{\alpha_j} = 0$.

Proof. $\boxed{1}$ $\varphi : \lambda_R^p(\alpha) \to \lambda_0^p(\alpha)$, $x \mapsto (e^{R\alpha_j} x_j)_{j\in\mathbb{N}}$ is an isomorphism, since $e^{r\alpha_j} e^{R\alpha_j} x_j = e^{(R+r)\alpha_j} x_j$ for $r < 0$ ($\Leftrightarrow R + r < R$).

$\boxed{2}$ ($\Leftarrow$) $\frac{1}{C}\alpha \leq \beta \leq C\alpha \Rightarrow \|(e^{\frac{1}{C}\,r\,\alpha_j} x_j)_j\|_{\ell^p} \leq \|(e^{r\,\beta_j} x_j)_j\|_{\ell^p} \leq \|(e^{C\,r\,\alpha_j} x_j)_j\|_{\ell^p}$ and $\{\frac{1}{C}r : r < R\} = \{r : r < R\} = \{C\,r : r < R\}$ for $R = 0$ and similarly for $R = \infty$.

($\Rightarrow$) Let $\gamma_j := \max\{\alpha_j, \beta_j\}$ for $R = \infty$, resp. $\gamma_j := \min\{\alpha_j, \beta_j\}$ for $R = 0$. Then $\lambda_R^1(\gamma) \subseteq \lambda_R^1(\alpha) \cap \lambda_R^1(\beta)$ and the inclusion is continuous, since $|x_j e^{r\alpha_j}| \leq |x_j e^{r\gamma_j}|$ for all $0 < r < R = +\infty$ resp. all $r < R = 0$. Moreover, $\lambda_R^1(\gamma) = \lambda_R^1(\alpha) \cap \lambda_R^1(\beta)$, since $x \in \lambda_R^1(\alpha) \cap \lambda_R^1(\beta) \Rightarrow \ \forall r < R$:

$$\infty > \sum_j |x_j e^{r\alpha_j}| + \sum_j |x_j e^{r\beta_j}| \geq \sum_{j,\alpha_j=\gamma_j} |x_j e^{r\alpha_j}| + \sum_{j,\beta_j=\gamma_j} |x_j e^{r\beta_j}| \geq \sum_j |x_j e^{r\gamma_j}|.$$

By the open-mapping theorem $\lambda^1_R(\alpha) = \lambda^1_R(\gamma) = \lambda^1_R(\beta)$ as Fréchet spaces.

$$\xRightarrow{\boxed{1.24}} \forall r\ \exists s > r\ \exists C > 0 : \|\_\|_r^{\lambda^1_R(\alpha)} \le C\,\|\_\|_s^{\lambda^1_R(\beta)}$$

$$\xRightarrow{[\textbf{Kri14}, 1.3.3, 1.3.7]} \forall r\ \exists s > r\ \exists C > 0 : U^\beta_s \subseteq C\,U^\alpha_r$$

$$\Rightarrow \forall y(\in \langle (U^\alpha_r)^o\rangle_{\text{lin.sp}}) : \|y\|^*_{s,\beta} \le C\|y\|^*_{r,\alpha}$$

$$\Rightarrow a_r \in (U^\alpha_r)^o \subseteq C\,(U^\beta_s)^o, \text{ since } \forall x \in U^\alpha_r : 1 \ge \|x\|_r := \|a_r \cdot x\|_{\ell^1}$$

$$\Rightarrow C \ge \|a_r\|^*_{s,\beta} \overset{\boxed{1.24}}{=\!=} \sup\Big\{\frac{a_{j,r}}{b_{j,s}} = e^{r\,\alpha_j - s\,\beta_j} : j\Big\}$$

$$\Rightarrow \sup\{r\,\alpha_j - s\,\beta_j : j\} \le \ln C$$

$$\Rightarrow \begin{cases} \frac{\alpha_j}{\beta_j} \le \frac{1}{r}\Big(\frac{\ln C}{\beta_j} + s\Big) \le C' & \text{in case } r > 0,\\ \frac{\alpha_j}{\beta_j} \ge \frac{1}{-r}\Big(-\frac{\ln C}{\beta_j} + (-s)\Big) \ge C' > 0 & \text{in case } r < 0.\end{cases}$$

$\boxed{3}$ Let $\Phi : \lambda^p_R(\alpha) \to \lambda^p_R(\alpha) \times \lambda^p_R(\alpha)$ be given by

$$\Phi(x) := (x^{\text{even}}, x^{\text{odd}}) := ((x_{2n})_{n\in\mathbb{N}}, (x_{2n+1})_{n\in\mathbb{N}}).$$

$(r > 0)$ Then $\|x^\sigma\|_r = \|j \mapsto e^{\alpha_j r} x_{2j+\sigma}\|_{\ell^p} \le \|j \mapsto e^{\alpha_{2j+\sigma} r} x_{2j+\sigma}\|_{\ell^p} \le \|x\|_r$ and

$$\|x\|_r = \|j \mapsto e^{\alpha_j r} x_j\|_{\ell^p} = \|j \mapsto e^{\alpha_{2j} r} x_{2j}\|_{\ell^p} + \|j \mapsto e^{\alpha_{2j+1} r} x_{2j+1}\|_{\ell^p}$$
$$\le \|j \mapsto e^{\alpha_j r'} x_{2j}\|_{\ell^p} + \|j \mapsto e^{\alpha_j r''} x_{2j+1}\|_{\ell^p} = \|x^{\text{even}}\|_{r'} + \|x^{\text{odd}}\|_{r''},$$

where $R > r' > r \sup \frac{\alpha_{2j}}{\alpha_j}$ and $R > r'' > r \sup \frac{\alpha_{2j+1}}{\alpha_j} > r \sup \frac{\alpha_{2j}}{\alpha_j}$.

$\boxed{4}$ $\lambda^1_\infty(\alpha) = \lambda^p_\infty(\alpha) \overset{\boxed{1.25}}{\Longleftrightarrow} \forall r\ \exists s(> r) : \sum_j (e^{r-s})^{\alpha_j} = \sum_j \frac{a_{j,r}}{a_{j,s}} < \infty \Leftrightarrow$ $\Leftrightarrow \exists q = e^{r-s} < 1 : \sum_j q^{\alpha_j} < \infty \Leftrightarrow \exists \delta > 0 : \delta \ln j \le \alpha_j$:

$(\Leftarrow)$ $q^{\alpha_j} \le e^{(r-s)\delta \ln j} = j^{\delta(r-s)} \le j^{-2}$, provided $s > r + \frac{2}{\delta}$.

$(\Rightarrow)$ $\frac{\ln j}{\alpha_j}$ unbounded $\Rightarrow \forall n\ \exists j_n : \frac{\ln j_n}{\alpha_{j_n}} \ge n$, w.l.o.g. $j_{n+1} \ge 2j_n \ge 8$. Then for $q = e^{-x}$ with $x > 0$ we have

$$\sum_j q^{\alpha_j} = \sum_n \sum_{j=j_{n-1}+1}^{j_n} e^{-x\,\alpha_j} \ge \sum_n \underbrace{(j_n - j_{n-1})}_{\ge j_n/2} e^{-x\,\alpha_{j_n}}$$
$$\ge \sum_n e^{\ln(\frac{j_n}{2}) - \frac{x}{n}\ \ln(j_n)} \ge \sum_{n \ge 2x} 1,$$

since $\ln(\frac{j_n}{2}) \ge \ln(j_n^{x/n})$, or equivalently $j_n^{1-x/n} \ge 4^{1/2} = 2$.

$\boxed{5}$ $\lambda^1_0(\alpha) = \lambda^p_0(\alpha) \overset{\boxed{1.25}}{\Longleftrightarrow} \forall r < 0\ \exists (r <) s < 0 : \sum_j (e^{r-s})^{\alpha_j} = \sum_j \frac{a_{j,r}}{a_{j,s}} < \infty \Leftrightarrow$ $\forall q = e^{r-s} < 1 : \sum_j q^{\alpha_j} < \infty \Leftrightarrow \lim_{j\to\infty} \frac{\ln j}{\alpha_j} = 0$:

$(\Leftarrow)$ $\lim_{j\to\infty} \frac{\ln j}{\alpha_j} = 0 \Rightarrow \forall x\ \exists N\ \forall j \ge N : \frac{\ln j}{\alpha_j} < \frac{x}{2} \Rightarrow$

$$\sum_{j\ge N} e^{-x\,\alpha_j} \le \sum_{j \ge N} e^{-x\,\frac{2}{x}\ln j} = \sum_{j\ge N} \frac{1}{j^2} < \infty$$

$(\Rightarrow)$ $\lim_{j\to\infty} \frac{\ln j}{\alpha_j} \ne 0 \Rightarrow \exists \delta > 0\ \forall n\ \exists j_n : \frac{\ln j_n}{\alpha_{j_n}} \ge \delta$, w.l.o.g. $j_{n+1} \ge 2j_n \ge 8$. Then for $q := e^{-x}$ with $x := \frac{\delta}{2}$ we have

$$\sum_j q^{\alpha_j} = \sum_n \sum_{j=j_{n-1}+1}^{j_n} e^{-x\,\alpha_j} \ge \sum_n \underbrace{(j_n - j_{n-1})}_{\ge j_n/2} e^{-x\,\alpha_{j_n}} \ge \sum_n e^{\ln(\frac{j_n}{2}) - \frac{x}{\delta}\ \ln(j_n)} \ge \sum_n 1,$$

since $\ln(\frac{j_n}{2}) \geq \ln(j_n^{1/2})$, or equivalently $j_n^{1-1/2} \geq 4^{1/2} = 2$. $\square$

2. Colimit closed (coreflective) subcategories

In this section we describe ((reduced) inductive) colimits of locally convex spaces. And we consider the classes of (ultra-)bornological and (infra-)barrelled spaces, all of which are invariant under the formation of colimits. We give descriptions of Köthe sequence spaces as colimits of Banach spaces.

Barrelled and bornological spaces

2.1 Definition. Bornological and barrelled spaces.
An lcs is BORNOLOGICAL (cf. [**Kri14**, 2.1.7], [**MV92**, 24.9 p.262], [**Jar81**, 13.1 p.272]) if BOUNDED LINEAR MAPPINGS (i.e. being bounded on bounded sets) on it are continuous, or equivalently, every BORNIVOROUS (i.e. absorbing each bounded set) absolutely convex subset is a 0-neighborhood (See [**MV92**, 24.10 p.263]).

An lcs is ULTRABORNOLOGICAL (See [**Kri14**, 5.4.20], [**MV92**, 24.14 p.264], [**Jar81**, 13.1 p.272]) if all linear maps on it, which are bounded on the BANACH-DISKS (i.e. absolutely convex bounded sets B for which E_B is complete), are continuous, or equivalently, every absolutely convex subset, which absorbs all Banach-disks, is a 0-neighborhood.

An lcs is called BARRELLED (german: TONNELLIERT) (See [**Kri14**, 4.2.1], [**Jar81**, 11.1 p.219], [**MV92**, Def. in 23.19 p.252]) if every BARREL (german: TONNE) (i.e. closed absolutely-convex absorbing subset) is a 0-neighborhood, equivalently, the uniform boundedness theorem holds (cf. [**Kri14**, 4.2.2]).

An lcs is called INFRA-BARRELLED (german: QUASI-TONNELIERT, INFRA-TONNELIERT) (See [**Kri14**, 5.4.20], [**Jar81**, 11.1 p.219], [**MV92**, Def. in 23.19 p.252]) if every BORNIVOROUS BARREL is a 0-neighborhood, equivalently, E embeds topologically into the bidual (See [**Kri14**, 5.4.20]).

Since obviously "bornivorous⇒absorbs Banach disks" and barrels absorb Banach-disks by the Banach-Mackey-Theorem (See [**Kri07b**, 7.4.18]) we have the following implications (See [**MV92**, 24.12 p.263], [**MV92**, 24.15 p.264]):

$$\begin{tikzcd}
 & \text{bornivorous barrel} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{barrel} \arrow[dr, Rightarrow] & & \text{bornivorous} \arrow[dl, Rightarrow] \\
 & \text{absorbs Banach-disks} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \text{infra-barrelled} & \\
\text{barrelled} \arrow[ur, Rightarrow] & & \text{bornological} \arrow[ul, Rightarrow] \\
 & \text{ultrabornological} \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] &
\end{tikzcd}$$

For sequentially complete or at least LOCALLY-COMPLETE lcs (i.e. the Banach-disks form a basis for the bornology) the implications from left to right can clearly be inverted (See [**MV92**, 23.20 p.252],[**MV92**, 23.21 p.253]).

2.2 Lemma. (See [**Kri14**, 2.1.6], [**Jar81**, 10.1.4 p.197]).
In any metrizable lcs every convergent sequence is Mackey-convergent.

A sequence $(x_n)_{n\in\mathbb{N}}$ in an lcs is called MACKEY CONVERGENT towards x_∞ iff there exists a sequence $\lambda_n \to \infty$ in $\mathbb{R}$ with $\{\lambda_n(x_n - x_\infty) : n \in \mathbb{N}\}$ being bounded.

Proof. Let $(p_k)_{k\in\mathbb{N}}$ be a basis of seminorms. Since for each k the sequence $\mu_n^{(k)} := p_k(x_n - x_\infty) \to 0$ for $n \to \infty$ we find another sequence $0 \neq \mu_n^\infty \to 0$ with $\{\mu_n^k/\mu_n^\infty : n \in \mathbb{N}\}$ bounded for each k (See [**Kri14**, 2.1.6]). Then $\lambda_n := 1/\mu_k^\infty$ has the required property. □

2.3 Corollary (See [**MV92**, 23.23 p.253], [**Kri14**, 4.1.11], [**Kri14**, 4.2.4])**.**
Fréchet spaces are ultrabornological, hence bornological, barrelled and infrabarrelled.

Proof. Metrizable lcs are bornological (See [**Kri14**, 2.1.7], [**MV92**, 24.13 p.264]), since any bounded linear mapping f on them is (sequentially) continuous: Let $x_n \to x_\infty$, then by 2.2 there are $\lambda_n \to \infty$ with $n \mapsto f(\lambda_n(x_n - x_\infty)) = \lambda_n(f(x_n) - f(x_\infty))$ bounded, hence $f(x_n) \to f(x_\infty)$. Completeness implies now that the space is even ultrabornological. □

2.4 Colimits.
Let $\mathcal{F} : \mathcal{J} \to \underline{lcs}$ be a functor from a partially ordered set $(J, \succ)^{\mathrm{op}} = (J, \prec)$ or even from a small category $\mathcal{J}$ into that of locally convex spaces. The COLIMIT $\operatorname{colim} \mathcal{F}$ of $\mathcal{F}$ (See [**Kri08**, 3.25]) is then given as quotient of the COPRODUCT (DIRECT SUM, cf. [**Kri14**, 3.6.1])

$$\coprod_j \mathcal{F}(j) := \Big\{x \in \prod_j \mathcal{F}(j) : x_j = 0 \text{ for all but finitely many } j\Big\}$$

with the final locally convex structure with respect to the inclusions $E_j \hookrightarrow \coprod_j \mathcal{F}(j)$ (whose continuous seminorms are those which restricted to each summand $\mathcal{F}(j)$ are seminorms of $\mathcal{F}(j)$), where we factor out the congruence relation generated $x^{(j)} \sim (\mathcal{F}(f)(x))^{(j')}$ for every $j \prec j'$ (morphism $f : j \to j'$ in $\mathcal{J}$), where $x^{(j)}$ denotes the point with j-th coordinate $x \in \mathcal{F}(j)$ and all other coordinates equal to 0. Since the topology on this quotient need not be Hausdorff, one has to factor out the closure of $\{0\}$ in addition, i.e. the intersection of the kernels of all its seminorms.

In the particular case, where $\mathcal{J}^{\mathrm{op}} = (J, \succ)$ is directed, the first (not necessarily Hausdorff) quotient is given by $\bigsqcup_j \mathcal{F}(j)/\sim$, where $x_1 \in \mathcal{F}(j_1)$ is equivalent to $x_2 \in \mathcal{F}(j_2)$ iff for some $j \succ j_1, j_2$: $\mathcal{F}(j_1 \prec j)(x_1) = \mathcal{F}(j_2 \prec j)(x_2)$. In this case the colimit is also called INDUCTIVE LIMIT (See [**Jar81**, 4.5 p.82]) and denoted $\varinjlim \mathcal{F}$.

An inductive limits is called REDUCED, iff all $\iota_j : \mathcal{F}(j) \to \varinjlim \mathcal{F}$ are injective. By replacing $\mathcal{F}(j)$ with the image $\tilde{\mathcal{F}}(j)$ of ι_j in $\varinjlim \mathcal{F}$ supplied with its quotient structure, we get that $\varinjlim \mathcal{F}$ equals $\varinjlim \tilde{\mathcal{F}}$, which is a reduced inductive limit (See [**Jar81**, 4.5.2 p.82]). Note for this that $\tilde{\mathcal{F}}(j \prec j')$ is then a well defined injective continuous linear mapping.

An even more restricted situation is, when $J = (\mathbb{N}, \leq)$, i.e. we have an inductive limit of a sequence of spaces (the STEPS of the limit). The inductive limit of a sequence of Fréchet-spaces (a so-called (LF)-SPACE) is almost never a Fréchet space (See [**Kri14**, 4.1.13]): STRICT INDUCTIVE LIMITS of sequences (i.e. E_n is a closed topological subspace in E_{n+1} for each n), which are not finally constant, can not be Baire spaces and hence are not Fréchet; And, more generally, by [**Jar81**, 12.4.4 p.259] a metrizable space with a countable base of bornology has to be normed, in particular this is valid for (locally) complete (LB)-SPACES (See [**Flo73**, 5.5 p.73]), i.e. inductive limits of a sequence of Banach spaces. Even more generally, if all F_n and $F_\infty := \varinjlim_n F_n$ (hence $F_\infty = \bigcup_{n\in\mathbb{N}} \iota_n(F_n)$) are Fréchet, then by Grothendieck's

factorization theorem $\boxed{2.6}$ $F_\infty \subseteq \iota_n(F_n)$ for some n.
Furthermore, it is not true in general that (LB)-spaces are complete and REGULAR (See [**Mak63**, Beispiel 2]), i.e. bounded sets are contained and bounded in some step, or, stronger, converging sequences (resp. compact subsets) are converging (resp. compact) in some step.

2.5 Stability under colimits.
Colimits of bornological spaces E_j are again bornological (See [**Jar81**, 13.1.5 p.273], [**MV92**, 24.16 p.264]), since bounded linear mappings on $\operatorname{colim}_j E_j$ are bounded mappings on each E_j and hence continuous on E_j, and by the universal property of the limit also continuous on $\operatorname{colim}_j E_j$.

By definition any bornological space E is the inductive limit of the spaces E_B, where B runs through the bounded (closed) absolutely convex subsets. Thus the bornological spaces are exactly the colimits of normed spaces.

The same argument works for ultrabornological instead of bornological, since the continuous images $f(B)$ of Banach disks B are again Banach disks: $E_B \to F_{f(B)}$ is a quotient mapping, since

$$\begin{aligned}
p_{f(B)}(f(x)) &= \inf\big\{\lambda > 0 : f(x) \in \lambda \cdot f(B) = f(\lambda\, B),\ \text{i.e. } \exists b \in B : f(x - \lambda\, b) = 0\big\}\\
&= \inf\big\{\lambda > 0 : \exists b \in B\, \exists z \in \ker f : \lambda\, b - x = z\big\}\\
&= \inf\big\{\lambda > 0 : \exists z \in \ker f\, \exists b \in B : x + z = \lambda\, b\big\}\\
&= \inf\Big\{\inf\big\{\lambda > 0 : \exists b \in B : x + z = \lambda\, b\big\} : z \in \ker f\Big\}\\
&= \inf\big\{p_B(x+z) : z \in \ker f\big\} = \widetilde{p_B}(f(x))
\end{aligned}$$

is the quotient norm (See [**Kri14**, 4.3.6]).

Furthermore, (infra-)barrelled spaces are stable under colimits (See [**Jar81**, 11.3.1.c p.223]): For quotients this follows since inverse images of barrels are barrels and of bornivorous sets are bornivorous. For coproducts it can be found in [**Jar81**, 8.8.10 p.168]

2.6 Grothendiecks factorization theorem (See [**MV92**, 24.33 p.271])**.**
Let F be an lcs, let E and E_n for $n \in \mathbb{N}$ be Fréchet spaces, $f_n \in L(E_n, F)$ and $f \in L(E, F)$ continuous linear mappings. If $f(E) \subseteq \bigcup_{n\in\mathbb{N}} f_n(E_n)$ then there exists an $m \in \mathbb{N}$ with $f(E) \subseteq f_m(E_m)$. If, in addition, f_m is injective, then there exists an $\tilde f \in L(E, E_m)$ with $f = f_m \circ \tilde f$.

$$\begin{tikzcd}
E \arrow[r, "f"] \arrow[d, dotted, "\tilde f"'] \arrow[dr, dotted] & \bigcup_n f_n(E_n) \arrow[r, hook] & F \\
\exists E_m \arrow[r, "\exists f_m"'] & f_m(E_m) \arrow[u, hook] &
\end{tikzcd}$$

Proof. Let $G_n := \{(x,y) \in E \times E_n : f(x) = f_n(y)\} = \operatorname{graph}(f_n^{-1} \circ f)$ be the pull-back of f and f_n, a closed linear subspace of the Fréchet space $E \times E_n$. Then $\operatorname{pr}_1(G_n) = \{x \in E : \exists y : f(x) = f_n(y)\} = f^{-1}(f_n(E_n))$ and hence $\bigcup_n \operatorname{pr}_1(G_n) = f^{-1}(\bigcup_n f_n(E_n)) = E$. By the theorem of Baire (see [**Kri14**, 4.1.11]), there exists an m such that $\operatorname{pr}_1(G_m)$ is not meagre, hence by the open mapping theorem (see [**Kri14**, 4.3.6]) $\operatorname{pr}_1 : G_m \to E$ is onto, i.e. $f(E) = f(f^{-1}(f_m(E_m))) \subseteq f_m(E_m)$.

If, in addition, f_m is injective, then $\tilde f := f_m^{-1} \circ f : E \to E_m$ is a well-defined linear mapping with closed graph G_m, hence is continuous by the closed graph theorem (see [**Kri14**, 4.3.1]). $\square$

2.7 Lemma (See [**MV92**, 24.34 p.272])**.**
Let an lcs E carry the final structure with respect to countable many continuous linear mappings $f_n : E_n \to E$ for Fréchet spaces E_n with $\bigcup_n f_n(E_n) = E$.
Then E is the (reduced) inductive limit of a sequence of Fréchet spaces.

Proof. We construct a strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$ with $\bigcup_{j\le n_k} f_j(E_j) \subseteq f_{n_{k+1}}(E_{n_{k+1}})$. For the Fréchet space $F := \coprod_{j\le n_k} E_j$ consider the continuous linear map $f : (x_j)_{j\le n_k} \mapsto \sum_{j\le n_k} f_j(x_j)$, $F \to E$. By 2.6 there exists an n_{k+1} such that $\bigcup_{j\le n_k} f_j(E_j) = f(F) \subseteq f_{n_{k+1}}(E_{n_{k+1}})$. Let $\tilde E_k := E_{n_k}/\ker f_{n_k} \rightarrowtail\!\!\!\!\twoheadrightarrow f_{n_k}(E_{n_k}) \hookrightarrow E$. The mapping $E_j \to \tilde E_{k+1}$ for $j \le n_k$ has closed graph, hence is continuous by the closed graph theorem, and thus also $\tilde E_k \to \tilde E_{k+1}$. The inductive limit structure on E of the increasing sequence of Fréchet spaces $\tilde E_k$ is finer than the given one since $\tilde E_k \rightarrowtail E$ is continuous. Because of $f_j(E_j) \subseteq \tilde E_{n_k}$ for $j \le n_k$ it is also coarser. $\square$

2.8 Corollary. All representations of an (LF) space are equivalent (See [**MV92**, 24.35 p.273])**.**
Let E be the reduced inductive limit of two sequences of Fréchet spaces $(E_n^{(i)})_{n\in\mathbb{N}}$ for $i \in \{0,1\}$. Then $\forall n \in \mathbb{N}\ \exists k \in \mathbb{N} : E_n^{(0)}$ embeds continuously into $E_k^{(1)}$ (and similarly $E_n^{(1)}$ into $E_k^{(0)}$). $\square$

2.9 Elements in $\boldsymbol{\lambda^\infty(A)}$ (See [**MV92**, 27.4 p.308])**.**

1. *$b \in \lambda^\infty(A) \Leftrightarrow \forall a \in A\, \exists C_a > 0$: $|b_j| \le \inf_{a\in A} C_a/a_j$.*
2. *If A is countable, then $\exists b \in \lambda^\infty(A)\, \forall j : b_j > 0$.*
3. *If A is countable, then $\forall b \in \lambda^\infty(A)\ \exists b' \in \lambda^\infty(A)\ \forall j : 0 \ne b'_j \ge |b_j|$.*

Proof. (1) $b \in \lambda^\infty(A) \Leftrightarrow \forall a$: $b \cdot a$ bounded (by $C_a > 0$), i.e. $\forall j$: $|b_j| \le C_a/a_j \Leftrightarrow \forall j$: $|b_j| \le \inf_a C_a/a_j$.

(2) Let $A := \{a^{(k)} : k \in \mathbb{N}\}$ with $k \mapsto a^{(k)}$ increasing. For each $k \in \mathbb{N}$ choose $C_k > k \max\{1, a_0^{(k)}, \dots, a_k^{(k)}\}$. $\Rightarrow C_k/a_j^{(k)} \ge k$ for all $k \ge j$. $\Rightarrow b_j := \inf_k C_k/a_j^{(k)} = \min_k C_k/a_j^{(k)} > 0$ and $b \in \lambda^\infty(A)$.

(3) By (2) there is a $b' \in \lambda^\infty(A)$ with $b'_j > 0$ forall j. For $b \in \lambda^\infty(A)$ also $b'' : j \mapsto \max\{|b_j|, b'_j\}$ is in $\lambda^\infty(A)$ and satisfies $b'' \ge |b|$ and $\forall j : b''_j \ge b'_j > 0$. $\square$

2.10 Bounded sets in $\boldsymbol{\lambda^p(A)}$ (See [**MV92**, 27.5,27.6 p.309])**.**
For $1 \le p \le \infty$ the sets $B_b^p := \{x : \|x/b\|_{\ell^p} \le 1\}$ for $b \in \lambda^\infty(A)$ form a basis of the bornology of $\lambda^p(A)$ if $p = \infty$ or if A is countable.
The sets $B_b^o := B_b^\infty \cap c_0(A)$ for $b \in \lambda^\infty(A)$ form a basis of the bornology of $c_0(A)$.

Proof. $b \in \lambda^\infty(A) \Rightarrow B_b^p \subseteq \lambda^p(A)$ bounded, since $x \in B_b^p \Rightarrow \operatorname{carr} x \subseteq \operatorname{carr} b$ and $\forall a \in A$: $\|x\|_a = \|x \cdot a\|_{\ell^p} = \|x \cdot \frac{1}{b} \cdot b \cdot a\|_{\ell^p} \le \|x/b\|_{\ell^p} \cdot \|b \cdot a\|_\infty \le 1 \cdot \|b \cdot a\|_\infty$.

Conversely, let $B \subseteq \lambda^\infty(A)$ be bounded, i.e. $\forall a \in A\, \exists C_a > 0\, \forall x \in B$: $\|x{\cdot}a\|_{\ell^\infty} \le C_a$. Let $b_j := \inf\{\frac{C_a}{a_j} : a \in A\}$, which is $< \infty$, since $a_j > 0$ for some a. Then $b \in \lambda^\infty(A)$, since $|b_j\, a_j| \le C_a$ for all $a \in A$ and $j \in \mathbb{N}$. Furthermore, since $|x_j\, \frac{a_j}{C_a}| \le 1$ for all $a \in A$ and $j \in \mathbb{N}$, we get $|x_j\, \frac{1}{b_j}| = \sup_{a\in A} |x_j\, \frac{a_j}{C_a}| \le 1$, i.e. $\|x \cdot \frac{1}{b}\|_{\ell^\infty} \le 1$ for all

$x \in B$.
Since $c_0(A)$ is a subspace of $\lambda^\infty(A)$, this works for $c_0(A)$ as well.

Now for $1 \le p < \infty$ and $A = \{a^{(k)} : k \in \mathbb{N}\}$ countable: Let $B \subseteq \lambda^p(A)$ be bounded, i.e. $\forall k\, \exists C_k > 0\, \forall x \in B : \|x\|_k \le C_k \xRightarrow{\boxed{2.9.1}} b := \inf_k 2^{k+1} C_k / a^{(k)} \in \lambda^\infty(A)$.

$$\frac{1}{b_j} = \sup_k \frac{a_j^{(k)}}{2^{k+1} C_k} \le \sum_k \frac{a_j^{(k)}}{2^{k+1} C_k} \Rightarrow$$

$$\|x/b\|_{\ell^p} \le \Big\| \sum_k \frac{x a^{(k)}}{2^{k+1} C_k} \Big\|_{\ell^p} \le \sum_k \frac{\|x a^{(k)}\|_{\ell^p}}{2^{k+1} C_k} \le \sum_k \frac{\|x\|_k}{2^{k+1} C_k} \le \sum_k \frac{1}{2^{k+1}} = 1$$

$\Rightarrow x \in B_b^p$, i.e. $B \subseteq B_b^p$. $\square$

2.11 Counter-example.
Let $A := \ell^p$ for $1 \le p < \infty$. Then $\lambda^\infty(A) := \{x \in \mathbb{K}^\mathbb{N} : \forall y \in \ell^p : \|x \cdot y\|_{\ell^\infty} < \infty\}$ is the linear space ℓ^∞:
($\supseteq$) $x \in \ell^\infty$, $y \in \ell^p \subseteq \ell^\infty \Rightarrow \|x \cdot y\|_{\ell^\infty} \le \|x\|_{\ell^\infty} \cdot \|y\|_{\ell^\infty}$.
($\subseteq$) Suppose $x \in \lambda^\infty(A)$ is unbounded $\Rightarrow \exists j_n$ (W.l.o.g. strictly increasing) with $|x_{j_n}| \ge n\, 2^n$.

$$y_j := \begin{cases} 2^{-n} & \text{for } j = j_n \\ 0 & \text{otherwise} \end{cases}.$$

Then $\|y\|_{\ell^p} \le \|y\|_{\ell^1} = \sum_n \frac{1}{2^n} < \infty$, but $\|x \cdot y\|_{\ell^\infty} \ge |x_{j_n}\, y_{j_n}| \ge n$, i.e. $x \notin \lambda^\infty(A)$.

Note that $\lambda^r(A) = \ell^\infty$ as linear spaces for all $p \le r \le \infty$:
$\ell^\infty \subseteq \lambda^p(A) \subseteq \lambda^r(A) \subseteq \lambda^\infty(A) = \ell^\infty$, since $\|x \cdot y\|_{\ell^p} \le \|x\|_{\ell^\infty} \cdot \|y\|_{\ell^p}$.

Now let $s_0 := 0$ and recursively $s_{n+1} := s_n + n$ and put

$$x_j^{(n)} := \begin{cases} 1 & \text{for } s_n \le j < s_{n+1} \\ 0 & \text{otherwise} \end{cases}$$

and $B := \{x^{(n)} : n \in \mathbb{N}\}$. Then *$B$ is bounded in $\lambda^p(A)$*, since

$$\|x^{(n)} \cdot y\|_{\ell^p} = \Big(\sum_{j=s_n}^{s_{n+1}-1} |y_j|^p \Big)^{1/p} \le \|y\|_{\ell^p}.$$

However, *there is no $b \in \lambda^\infty(A) = \ell^\infty$ such that $\|x \cdot \frac{1}{b}\|_{\ell^p} \le 1$ for all $x \in B$*:
In fact, let $\beta := \|b\|_{\ell^\infty} < \infty$ then

$$\Big\| x^{(n)} \cdot \frac{1}{b} \Big\|_{\ell^p} \ge \|x^{(n)}\|_{\ell^p} \frac{1}{\beta} = \frac{n^{1/p}}{\beta} \to \infty.$$

2.12 $\lambda^p(A)$ as colimit of (uncountable many) ℓ^p's for countable A
(See [**MV92**, 27.7 p.309]).
There exists a basis $\mathcal{B}$ for the bornology of $\lambda^p(A)$ with $\lambda^p(A)_B \cong \ell^p$ for all $B \in \mathcal{B}$.

$$\forall b \in \lambda^\infty(A) : \ell^p(\operatorname{carr} b) \xrightarrow[\cong]{\cdot b} \langle B_b^p \rangle = (\lambda^p(A))_{B_b^p} \subseteq \lambda^p(A)$$

Proof. $\boxed{2.9.3} \Rightarrow \forall b \in \lambda^\infty(A)\ \exists b' \in \lambda^\infty(A)\ \forall j$: $0 \ne b'_j \ge |b_j| \Rightarrow B_b^p \subseteq B_{b'}^p$ and $\lambda^p(A)_{B_{b'}^p} \cong \ell^p$. $\square$

3. Limit closed (reflective) subcategories

In the following sections we consider classes of locally convex spaces which are invariant under the formation of limits, i.e. various completeness conditions, semi-reflexivity, Montel spaces, Schwartz spaces, and nuclear spaces. And we characterize those Köthe sequence spaces having these properties.

Completeness, compactness and (DN)

In this section we consider various completness conditions. And we discuss (pre-)compact subsets and operators, since they are relevant for the classes to follow. We introduce the property (DN) which allows to differentiate between power series spaces of finite and of infinite type.

3.1 Completeness.
For lcs E we consider the following completeness conditions:

- E is called COMPLETE iff evevry Cauchy net (or Cauchy filter) converges.
- E is called QUASI COMPLETE iff every closed bounded subset is complete.
- E is called SEQUENTIALLY COMPLETE iff Cauchy sequences converge.
- E is called LOCALLY COMPLETE (or MACKEY-COMPLETE) iff E_B is a Banach space for every closed absolutely convex bounded subset $B \subseteq E$.

One obviously has the implications:

complete $\Rightarrow$ quasi-complete $\Rightarrow$ sequentially complete $\Rightarrow$ locally complete.

For metrizable spaces all 4 conditions are equivalent (See [**Kri14**, 2.2.2]). Each of these completeness properties is inherited by closed subspaces ([**Kri14**, 3.1.4]), products ([**Kri14**, 3.2.1]), and coproducts ([**Kri14**, 3.6.1]) (See [**Jar81**, 3.2.5 p.59], [**Jar81**, 3.2.6 p.59], [**Jar81**, 6.6.7 p.111]).

The completion (i.e. reflector) of any lcs E is given by the space of all linear functionals on E^*, whose restrictions to equicontinuous subsets are $\sigma(E^*, E)$-continuous, supplied with the topology of uniform convergence on the equicontinuous subsets, see [**Kri14**, 5.5.7].

3.2 Precompact sets.
A subset $K \subseteq E$ in an lcs is called PRECOMPACT iff

$$\forall U\, \exists F \subseteq E \text{ finite} : K \subseteq F + U = \bigcup_{y \in F} (y + U).$$

This is exactly the case, when K is relatively compact in the completion of E (See [**Kri07a**, 6.2]). The precompact subsets of a product of lcs's are those whose projections to the factors are precompact; The precompact subsets of a coproduct of lcs's are those whose projections to the summands are precompact and are almost always $\{0\}$ (See [**Kri07a**, 6.3]).

3.3 Mackey-Arens Theorem
(See [**Kri14**, 5.4.15], [**Jar81**, 8.5.5 p.158], [**MV92**, 23.8 p.247]).
The finest topology compatible with a dual pairing (E, F) is the Mackey-topology $\mu(E, F)$, i.e. the topology of uniform convergence on $\sigma(F, E)$-compact absolutely convex subsets of F.

3.4 Alaŏglu-Bourbaki Theorem
(See [**Kri14**, 5.4.12], [**Jar81**, 8.5.2 p.157], [**MV92**, 23.5 p.245]).
Each equicontinuous set is relatively compact with respect to $\tau_{pc}(E^*, E)$, the topology of uniform convergence on precompact subsets, or equivalently, with respect to $\sigma(E^*, E)$.

Proof of the equivalence (See [**Jar81**, 8.5.1.b p.156]). Let $U \subseteq E$ be a 0-neighborhood, $x^* \in U^o$, and $A \subseteq E$ be precompact, i.e. $x^* + A^o$ a typical neighborhood of x^* with respect to $\tau_{\text{pc}}(E^*, E)$. Thus there is a finite set $F \subseteq E$ with $3A \subseteq F+U$, hence $(x^*+F^o)\cap U^o \subseteq x^*+A^o$, since for all $y^* \in U^0$ with $y^*-x^* \in F^o$ and $a \in A$ exist $y \in F$ and $u \in U$ with $3a = y + u$ and hence

$$\begin{aligned}\big|(y^* - x^*)(a)\big| &= \frac{1}{3}\Big|(y^* - x^*)(y+u)\Big| \le \frac{1}{3}\Big(\big|(y^* - x^*)(y)\big| + \big|y^*(u)\big| + \big|x^*(u)\big|\Big)\\ &\le \frac{1}{3}(1+1+1) = 1, \text{ i.e. } y^* - x^* \in A^o. \quad \square\end{aligned}$$

3.5 Proposition (See [**Jar81**, 8.5.3 p.157]).
E separable $\Rightarrow$ equicontinuous subsets are $\sigma(E^*, E)$-metrizable.

Proof. Let $D := \{x_j : j \in \mathbb{N}\} \subseteq E$ be dense and let E_0 be the linear span of D. Then $\sigma(E^*, E_0)$ is Hausdorff. Let W be a 0-nbhd. for $\sigma(E^*, E_0)$, i.e. $\exists y_i = \sum_{j=1}^\infty \lambda_j^i x_j \in E_0$ with $\lambda_j^i = 0$ for almost all j, say $j \le m$, and $\{y_1, \dots, y_n\}^o \subseteq W$. For $\max\{\sum_j |\lambda_j^i| : i\} < \lambda \in \mathbb{Q}$ we have $\frac{1}{\lambda} \cdot \{x_1, \dots, x_m\}^o \subseteq \{y_1, \dots, y_n\}^o \subseteq W$. Thus $\sigma(E^*, E_0)$ is metrizable and coincides with $\sigma(E^*, E)$ on equicontinuous sets: In fact, $E_0 \subseteq E \Rightarrow \sigma(E^*, E) \to \sigma(E^*, E_0)$ is continuous. Conversely, let U be a 0-nbhd in E, $x^* \in U^o$, $\varepsilon > 0$, and $x_i \in E$. Choose $\tilde{x}_i \in E_0$ with $\tilde{x}_i - x_i \in \frac{\varepsilon}{3}U$. For $y^* \in U^o \cap (x^* + \frac{\varepsilon}{3}\{\tilde{x}_1, \dots, \tilde{x}_k\}^o)$ we have:

$$|(y^* - x^*)(x_i)| \le |y^*(x_i - \tilde{x}_i)| + |x^*(x_i - \tilde{x}_i)| + |(y^* - x^*)(\tilde{x}_i)| \le 3\frac{\varepsilon}{3},$$

i.e. $y^* \in U^o \cap (x^* + \varepsilon\{x_1, \dots, x_k\}^o)$. $\square$

3.6 Lemma. Compact subsets of Fréchet spaces
(See [**Kri07b**, 6.4.3 p.119], [**Jar81**, 10.1.1 p.196]).
A subset of a Fréchet space is precompact (equivalently, relatively compact) if and only if it is contained in the closed convex hull of some 0-sequence.

3.7 Definition. Compact operator.
A linear operator between Banach spaces is called (weakly) compact if the image of the unit ball is (weakly) relatively compact.

A linear operator between Hilbert spaces is compact iff it can be approximated by finite dimensional operators with respect to the operator norm, see [**Kri07b**, 6.4.8].

3.8 Lemma. Orthogonal representation of compact operators
(See [**Kri07a**, 5.3], [**Jar81**, 20.1.2 p.452]).
An operator T between Hilbert spaces is compact iff there are orthonormal sequences e_n and f_n and $\lambda_n \to 0$ such that $Tx = \sum_n \lambda_n \langle e_n, x\rangle f_n$.

Proof. ($\Leftarrow$) If T has such a representation, then the finite sums define finite dimensional operators which converge to T.

($\Rightarrow$) Since any compact $T : E \to F$ induces a compact injective operator $T : (\ker T)^\perp \to \overline{T(E)}$ with dense image, we may assume that T is injective. Now we consider the positive compact operator T^*T. Its eigenvalues are all non-zero, since $T^*Tx = 0$ implies $\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle = 0$. By [**Kri07b**, 6.5.4] there is an orthonormal sequences of Eigen-vectors e_n with Eigen-value $0 \neq \lambda_n^2 \to 0$ such that $T^*Tx = \sum_n \lambda_n^2 \langle e_n, x\rangle e_n$. Let $f_n := \frac{1}{\lambda_n} Te_n$. Then a simple direct calculation shows that the f_n are orthonormal. Note that $x = \sum_n \langle e_n, x\rangle e_n$. Otherwise the compact positive operator T^*T restricted to the orthogonal complement $\{e_k : k\}^\perp$ would have a unit Eigen-vector e with positive Eigen-value λ. Which is impossible by definition of the e_k. So we obtain $Tx = \sum_n \langle e_n, x\rangle \lambda_n f_n$.

Another way to prove this is to use the polar decomposition $T = U\,|T|$, see [**Kri14**, 7.24], where U is a partial isometry and $|T|$ a positive and also compact operator. The spectral theorem for $|T|$ gives an orthonormal family e_n and $\lambda \in c_0$, such that $Tx = \sum_k \lambda_k \langle e_k, x\rangle e_k$. Applying U to this equation, shows that we may take $f_k := Ue_k$. $\square$

3.9 Corollary (See [**Kri07a**, 5.4], [**Jar81**, 20.1.3 p.453]).
An operator T between Hilbert spaces is compact iff $\langle Te_n, f_n\rangle \to 0$ holds for all orthonormal sequences e_n and f_n.

Proof. ($\Rightarrow$) Since $|\langle Te_n, f_n\rangle \leq \|Te_n\| \cdot \|f_n\| = \|Te_n\|$ it is enough to show that $Te_n \to 0$. Since e_n converges weakly to 0 (in fact $\langle x, e_n\rangle$ is even quadratic summable) we conclude that Te_n converges to 0 weakly. Since e_n is contained in the unit-ball and T is compact, every subsequence of Te_n has a subsequence, which is convergent. And the limit has to be 0, since this is true for the weak topology. But from this it easily follows that $Te_n \to 0$.

($\Leftarrow$) Given $\varepsilon > 0$ we choose maximal orthonormal sequences $(e_i)_{i\in I}$ and $(f_i)_{i\in I}$ such that $|\langle Te_i, f_i\rangle| \geq \varepsilon$. By assumption I must be finite. We consider the orthonormal projections $P := \sum_{i\in I} e_i \otimes e_i$ and $Q := \sum_{i\in I} f_i \otimes f_i$. For the composition with the ortho-projections on the complement we obtain $(1-Q)T(1-P) = T - (TP + QT - QTP) =: T - S$. Hence S is a finite dimensional operator and we claim that $\|T-S\| \leq \varepsilon$. Suppose this were not true. Then there is an x with $\|(T-S)x\| > \varepsilon\,\|x\|$ and hence an y such that $|\langle T(1-P)x, (1-Q)y\rangle| = |\langle (T-S)x, y\rangle| > \varepsilon\,\|x\|\,\|y\|$. Let $e_0 := (1-P)x$ and $f_0 := (1-Q)y$. Obviously $e_0, f_0 \neq 0$ and hence we may assume without loss of generality that $\|e_0\| = 1 = \|f_0\|$ and hence $\|x\| \geq 1$ and $\|y\| \geq 1$. Since $e_0 \in (1-P)(E) \subseteq P(E)^\perp = \{e_i : i \in I\}^\perp$ and $f_0 \in (1-Q)(F) \subseteq \{f_i : i \in I\}^\perp$ we get a contradiction to the maximality of I. $\square$

3.10 Compact diagonal operators between ℓ^p's (See [**MV92**, 27.8 p.309]).
Let $\ell := \ell^p$ with $1 \leq p < \infty$ or $c_0 \subseteq \ell \subseteq \ell^\infty$ invariant under multiplication with ℓ^∞. Let $D : \ell \to \ell$ be a diagonal-operator with coeffcients $d \in \ell^\infty$.

(1) *D is compact*

$\Leftrightarrow$ (2) *$d \in c_0$*

$\Leftrightarrow$ (3) *D is weakly-compact in case $\ell = \ell^1$.*

Proof. ($\boxed{1} \Rightarrow \boxed{2}$) Let

$$T_\varepsilon : \ell \to \ell, \quad T_\varepsilon(x)_j := \begin{cases} x_j/d_j & \text{for } |d_j| \geq \varepsilon, \\ 0 & \text{elsewhere.} \end{cases}$$

$\Rightarrow P_\varepsilon := D \circ T_\varepsilon$ is a compact projection $\Rightarrow P_\varepsilon(\ell) = \ker(1 - P_\varepsilon) = \{x \in \ell : \operatorname{carr} x \subseteq \{j : |d_j| \geq \varepsilon\}\}$ is finite dimensional (by [**Kri14**, 3.4.5]) $\Rightarrow \{j : |d_j| \geq \varepsilon\}$ is finite.

(2$\Rightarrow$1) $P_n : x \mapsto x \cdot \chi_{\{1,\dots,n\}} \Rightarrow \|D - D \circ P_n\| \leq \sup\{|d_j| : j > n\} \to 0$, since $d \in c_0$, i.e. D is compact as limit of fin.dim. operators.

(1$\Rightarrow$3) is trivial

(3$\Rightarrow$2) $P_\varepsilon := D \circ T_\varepsilon$ is weakly-compact. Suppose $\{j : |d_j| \geq \varepsilon\}$ infinite $\Rightarrow P_\varepsilon(\ell) \cong \ell^1$ and the closed unit disk in ℓ^1 is weakly compact $\Rightarrow \ell^1$ reflexive (see 3.17), a contradiction. $\square$

3.11 Approximation numbers for diagonal operators on ℓ^2.
Let $D : \ell^2 \to \ell^2$ be a diagonal-operator with coefficients $d \in \ell^\infty$ with $|d_i| \searrow 0$. Then its APPROXIMATION NUMBERS *are*

$$a_n(D) := \inf\Big\{\|D - T\| : \dim T(\ell^2) \leq n\Big\} = d_n.$$

Proof (See [**MV92**, Aufgabe 16.(3) p.392])**.** Note that $\|D\| = \|d\|_{\ell^\infty} = \sup\{|d_i| : i \in \mathbb{N}\}$ since $\|D(x)\| = \|d \cdot x\|_{\ell^2} \leq \|d\|_{\ell^\infty} \cdot \|x\|_{\ell^2}$ and $D(e^{(k)}) = d_k\, e^{(k)}$.

Thus $a_n(D) \leq \|D_n\| = \sup\{|d_k| : k \geq n\}$, where D_n is the diagonal operator with entries $d \cdot \chi_{[n,\infty)}$ with $\dim((D - D_n)(\ell^2)) = n$. Conversely, let $\dim T(\ell^2) \leq n$. Then $\exists y = \sum_{i=0}^n y_i e_i$ with $\|y\|_{\ell^2} = 1$ and $T(y) = 0$. Thus $\|D - T\|_{\ell^2} \geq \|(D - T)y\|_{\ell^2} = \|Dy\|_{\ell^2} = (\sum_{i=0}^n |d_i y_i|^2)^{1/2} \geq \min\{|d_i| : i \leq n\}\, \|y\|_{\ell^2} = |d_n|$. $\square$

3.12 Proposition. Equality $\boldsymbol{\lambda_r(\alpha) = \lambda_r(\beta)}$ (See [**MV92**, 29.1 p.338])**.**
For $r \in \{0, +\infty\}$ let $\lambda_r := \lambda_r^2$.

(1) $\lambda_r(\alpha) \cong \lambda_r(\beta)$;

$\Leftrightarrow$ (2) $\lambda_r(\alpha) = \lambda_r(\beta)$ *as lcs;*

$\Leftrightarrow$ (3) $\lambda_r(\alpha) = \lambda_r(\beta)$ *as sets;*

$\Leftrightarrow$ (4) $\exists C > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0$*:* $\frac{1}{C}\alpha_n \leq \beta_n \leq C\alpha_n$*.*

Proof. $(4 \Rightarrow 3)$ is obvious.

$(3 \Rightarrow 2)$ apply the closed graph theorem using that convergence in λ_r implies coordinatewise convergence.

$(2 \Rightarrow 1)$ is obvious.

$(1 \Rightarrow 4)$ Let $\Lambda_s^\alpha := \lambda_r(\alpha)_s := \lambda_r(\alpha)/\ker\|\_\|_s$ for $s < r$.

$A : \lambda_r(\alpha) \to \lambda_r(\beta)$ iso, $B := A^{-1} \Rightarrow$

$\Rightarrow \forall t < r\ \exists s < r\ \exists C > 0 : \|Ax\|_t \leq C\|x\|_s$,

$\forall s < s' < r\ \exists t < t' < r\ \exists D > 0 : \|By\|_{s'} \leq D\|y\|_{t'}$

$\Rightarrow \exists \tilde{A} \in L(\Lambda_s^\alpha, \Lambda_t^\beta),\ \tilde{B} \in L(\Lambda_{t'}^\beta, \Lambda_{s'}^\alpha) : \iota_t^{t'} = \tilde{A} \circ \iota_s^{s'} \circ \tilde{B} : \Lambda_{t'}^\beta \to \Lambda_{s'}^\alpha \to \Lambda_s^\alpha \to \Lambda_t^\beta$

$\Rightarrow \iota_s^{s'}$ compact by 3.10 and $a_n(\iota_t^{t'}) = e^{(t-t')\beta_n}$, $a_n(\iota_s^{s'}) = e^{(s-s')\alpha_n}$ by 3.11.

Obviously $a_n(\iota_t^{t'}) \leq \|\tilde{A}\| a_n(\iota_s^{s'}) \|\tilde{B}\|$ cf. 4.168

$\Rightarrow \beta_n \leq C\alpha_n + D$ for $C := \dfrac{s' - s}{t' - t}$, $D := \dfrac{\log(\|\tilde{A}\|\|\tilde{B}\|)}{t' - t}$

$$\begin{tikzcd}
& \lambda_r(\beta) \arrow[rr, "B"] \arrow[dl] & & \lambda_r(\alpha) \arrow[rr, "A"] \arrow[dl] \arrow[dr] & & \lambda_r(\beta) \arrow[dr] & \\
\Lambda^\beta_{t'} \arrow[rr, dotted, "\tilde{B}"] \arrow[rrrrrr, bend right, "\iota^{t'}_t"'] & & \Lambda^\alpha_{s'} \arrow[rr, "\iota^{s'}_s"] & & \Lambda^\beta_s \arrow[rr, dotted, "\tilde{A}"] & & \Lambda^\beta_t
\end{tikzcd}$$

□

3.13 Definition. Dominating norm (DN).
Let $\|\_\|_k$ be a monotone increasing basis of seminorms for the Fréchet space E. Then E is said to have property (DN) iff

$$\exists q\ \forall p\ \exists p'\ \exists C\ \forall x : \|x\|_p^2 \le C\|x\|_q\|x\|_{p'}$$

It follows that $\|\_\|_q$ is a norm, a so-called **d**ominating **n**orm.

3.14 Inheritance properties of (DN) (See [**MV92**, 29.2 p.339])**.**

1. *(DN) is topological invariant.*
2. *(DN) is inherited by closed subspaces.*
3. *$\lambda_\infty(\alpha)$ has (DN).*

Proof. (1) and (2) are obvious.

(3) For $t_0 < t_1 < t_2$ let $p := \frac{t_2-t_0}{t_2-t_1}$, $q := \frac{t_2-t_0}{t_1-t_0}$, $f_k := (e^{t_0\alpha_k}|x_k|)^{2/p}$ and $g_k := (e^{t_2\alpha_k}|x_k|)^{2/q} \xRightarrow{\text{Hölder}} (\|x\|_{t_1})^2 = \|f\,g\|_1 \le \|x\|_{t_0}^{2/p}\|x\|_{t_2}^{2/q} \Rightarrow (\|x\|_k)^2 \le \|x\|_0\|x\|_{2k}$. □

3.15 Corollary (See [**MV92**, 29.3 p.339])**.**
$\lambda_0(\alpha) \not\cong \lambda_\infty(\beta)$ for all $\alpha, \beta \nearrow \infty$.

Proof. By 3.14 $\lambda_\infty(\beta)$ has (DN). Indirectly, suppose $\lambda_0(\alpha)$ has (DN), i.e.

$$\exists\tau < 0\,\forall t < 0\,\exists T < 0\,\exists C > 0 : \|x\|_t^2 \le C\|x\|_\tau\|x\|_T.$$

$x := e_j \Rightarrow e^{2t\alpha_j} \le C\,e^{\tau\alpha_j + T\alpha_j} \le C\,e^{\tau\alpha_j} \Rightarrow 2t \le \frac{1}{\alpha_j}\ln(C) + \tau$, $\lim_j \alpha_j = +\infty \Rightarrow t \le \tau$, a contradiction. □

Reflexive spaces

3.16 Definition. Reflexive spaces.
An lcs is called SEMI-REFLEXIVE iff the canonical mapping $\delta : E \to E^{**}$ is onto.
An lcs is called REFLEXIVE iff the canonical mapping $\delta : E \to E^{**}$ is an isomorphism of lcs (See [**MV92**, Def. nach 23.17 p.251], [**Kri14**, 5.4.21], [**Jar81**, 11.4 p.227]).

Reflexive spaces are stable under products, coproduct and regular reduced inductive limits. Semi-reflexive space are in addition stable under closed subspaces (See [**Jar81**, 11.4.5 p.228]).

3.17 Characterizing semi-reflexivity
(See [**MV92**, 23.18 p.251], [**Kri14**, 5.4.22], [**Jar81**, 11.4.1 p.227])**.**
An lcs is semi-reflexive iff every bounded subset is relatively weakly compact.

Corollary (See [**Jar81**, 11.4.6 p.229])**.**
Semi-reflexive spaces are quasi-complete.

Proof. Let (x_i) be a Cauchy-net in a closed bounded $B \subseteq E$. Then (x_i) is Cauchy for the weak topology and since B is weakly compact (x_i) converges weakly to some x_∞. Let U be a closed absolutely convex 0-neighborhood. Thus $x_i - x_{i'} \in U$ finally, and since U is also weakly-closed ([**Kri14**, 5.4.8]) $x_i - x_\infty \in U$ finally. □

3.18 Characterizing reflexivity
(See [**MV92**, 23.22 p.253], [**Kri14**, 5.4.23])**,** [**Jar81**, 11.4.2 p.228]**.**
An lcs is reflexive iff it is semi-reflexive and (infra-)barrelled.

3.19 Corollary. Characterizing reflexive Fréchet spaces
(See [**MV92**, 23.24 p.253])**.**
A Fréchet space is reflexive iff every bounded subset is weakly relatively compact.

Proof. Since every (F) space is (infra-)barrelled by 2.3 the result follows from 3.18. □

3.20 $\lambda^p(A)$ ist reflexive for $1 < p < \infty$ (See [**MV92**, 27.3 p.307])**.**

Proof. ℓ^p reflexive $\xRightarrow{1.13,\,3.16}$ λ^p reflexive. □

Montel spaces

3.21 Definition. Montel spaces.
An lcs is called SEMI-MONTEL SPACE (See [**MV92**, Def. in 24.23 p.267], [**Kri07a**, 4.47,4.48 p.104]) iff all its bounded subsets are relatively compact.

An lcs is called MONTEL SPACE (denoted (M) for short) (See [**MV92**, Def. in 24.23 p.267], [**Kri07a**, 4.47,4.48 p.104]) iff it is semi-Montel and infra-barrelled.

3.22 Montel spaces are reflexiv
(See [**MV92**, 24.24 p.267], [**Jar81**, 11.5.1 p.230])**.**
(Semi-)Montel spaces E are (semi-)reflexive and their $\sigma(E, E^*)$-convergent sequences are convergent.

Proof. By definition bounded sets in semi-Montel spaces E are relatively compact hence also relatively compact for the weak topology. Thus E is semi-reflexive by 3.17. Since Montel spaces are infra-barrelled by definition, they are reflexive and barrelled by 3.18. Weakly convergent sequences are bounded, hence are relatively compact for semi-Montel spaces, so the weak topology coincides with the given one on this closure. □

3.23 Inheritance properties of Montel spaces (See [**Jar81**, 11.5.4 p.230])**.**
Obviously closed subspaces, products and coproducts of semi-Montel spaces are semi-Montel. Since barrelledness is inherited by products and coproducts (see 2.5) the same is true for the Montel property. The only normable (Semi-)Montel spaces are the finite dimensional ones, see [**Kri14**, 3.4.5].

3.24 Proposition (See [**Jar81**, 9.3.7 p.179])**.**
$\gamma(E^*, E) = \tau_c(E^*, \tilde{E})$, where $\gamma(E^*, E)$ is the finest locally convex topology on

$E^* = \tilde{E}^*$ into which all polars U^o for 0-nbhds U in E (or the completion $\tilde{E}$) with their compact topology continuously embed and $\tau_c(E^*, \tilde{E})$ is the topology of uniform convergence on compact subsets of the completion $\tilde{E}$.

Proof. Since for 0-nbhds U in E (or $\tilde{E}$) the polar U^o is $\sigma(E^*, \tilde{E})$ compact and even $\tau_{pc}(E^*, \tilde{E}) = \tau_c(E^*, \tilde{E})$ compact by 3.4, we have $\gamma \geq \tau_c(E^*, \tilde{E}) \geq \sigma(E^*, \tilde{E})$. By Grothendieck's completion result (See [Kri14, 5.5.7]) $\tilde{E} = (E^*, \gamma)^*$, hence γ is compatible with the duality $(E^*, \tilde{E})$, i.e. coincides with the topology of uniform convergence on the closed equicontinuous subsets in $(E^*, \gamma)^* = \tilde{E}$ (see [Kri14, 5.4.11]). Let $\mathcal{C}$ be set of these subsets. All of them are compact for $\tau_{pc}(\tilde{E}, (E^*, \gamma))$ by 3.4. The identity $(\tilde{E}, \tau_{pc}(\tilde{E}, (E^*, \gamma))) \to \tilde{E}$ is continuous, since $\tilde{E}$ carries the topology of uniform convergence on the equicontinuous subsets (polars U^o) in $\tilde{E}^* = E^*$ and polars U^o are γ-compact by definition. Thus the sets in $\mathcal{C}$ are compact in $\tilde{E}$. Hence $\tau_c(E^*, \tilde{E}) \geq \gamma$. □

3.25 Proposition (See [Jar81, 11.5.2 p.230]).
Semi-Montel ⇔ quasi-complete and equicontinuous subsets are relatively $\beta(E^*, E)$-compact.

Proof. (⇒) semi-Montel ⇒ quasi-complete, $\beta(E^*, E) = \tau_{pc}(E^*, E)$ ⇒ equicontinuous sets are relatively $\beta(E^*, E)$-compact by the Alaŏglu-Bourbaki Theorem 3.4.

(⇐) By 3.24 and assumption we have $\tau_c(E^*, \tilde{E}) = \gamma(E^*, E) \geq \beta(E^*, E)$. Let $\circ$ (resp. $\bullet$) denote the polarization with respect to the duality (E, E^*) (resp. $(\tilde{E}, E^*)$) then for each bounded $B \subseteq E$ there exists a compact $K \subseteq \tilde{E}$ with $K^\bullet \subseteq B^\circ$ and hence $(K^\bullet)_\bullet \supseteq (B^\circ)_\bullet \supseteq B$. Since the closed absolutely convex hull $(K^\bullet)_\bullet$ of (pre)compact sets K is precompact (see the proof of [Kri07b, 6.4.3]) also B is precompact and by quasi-completeness relatively compact. □

3.26 Proposition (See [Jar81, 11.5.4.f p.230]).
Duals of (M)-spaces are (M).

Proof. E^* semi-Montel: $B \subseteq E^*$ bounded ⇒ B equicontinuous by the uniform boundedness theorem, since E is barrelled by 3.22 ⇒ B relatively compact for $\tau_{pc}(E^*, E) = \beta(E^*, E)$ by the Alaŏglu-Bourbaki Theorem 3.4. Since duals of reflexive spaces are reflexive they are (infra-)barrelled. □

3.27 Proposition (See [Jar81, 11.6.2 p.231]).
A Fréchet space is Montel ((FM) for short) iff it is separable and $\sigma(E^*, E)$-convergent sequences are $\beta(E^*, E)$-convergent.

Proof. (⇒) E (FM) ⇒ E^* (M), by 3.26 ⇒ $\sigma(E^*, E)$-convergent sequences are convergent, by 3.22.
$\{U_n : n \in \mathbb{N}\}$ abs.convex, closed 0-nbhd. basis. We show that E_{U_n} is separable, otherwise $\exists \varepsilon > 0\, \exists A_1 \subseteq U_1$ uncountable with $q_{U_1}(x - x') \geq \varepsilon$ for all $x \neq x' \in A_1$. U_2 is absorbing ⇒ $\exists k_2$: $A_1 \cap k_2 U_2$ uncountable ⇒ … ⇒ $\exists k_n, A_n$: $A_n \subset A_{n-1} \cap k_n U_n$ uncountable. Choose $x_n \in A_n \setminus A_{n+1}$. Then $B := \{x_n : n \in \mathbb{N}\}$ is bounded ⇒ B is relatively compact ⇒ ∃ converging subsequence $(x_{n_i})_i$, a contradiction.

(⇐) U 0-nbhd $\xRightarrow{3.5}$ U^o is $\sigma(E^*, E)$-metrizable ⇒ $(U^o, \sigma(E^*, E)) \to (E^*, \beta(E^*, E))$ is continuous ⇒ U^o is $\beta(E^*, E)$-compact ⇒ E semi-Montel, by 3.25. □

3.28 Theorem of Dieudonné-Gommes characterizing Montel for $\lambda^p(A)$ (See [**MV92**, 27.9 p.310]).
Let $A = \{a^{(n)} : n \in \mathbb{N}\}$ be countable. Then

(1) $\exists\, 1 \le p \le \infty$: $\lambda^p(A)$ *(M)*.

$\Leftrightarrow$ (2) $\forall\, 1 \le p \le \infty$: $\lambda^p(A)$ *(M)*.

$\Leftrightarrow$ (3) $\lambda^\infty(A) = c_0(A)$.

$\Leftrightarrow$ (4) $\lambda^1(A)$ *is reflexiv.*

$\Leftrightarrow$ (5) $\forall\, 1 \le p \le \infty$: *not* $\exists$ *normed* ∞*-dim. top.-lin. subspace in* $\lambda^p(A)$.

$\Leftrightarrow$ (6) $\forall$ *infinite* $J \subseteq \mathbb{N}$ $\forall n$ $\exists k$: $\inf_{j\in J} a_j^{(n)}/a_j^{(k)} = 0$.

Proof. ($\boxed{4} \Rightarrow \boxed{3}$) $b \in \lambda^\infty(A) \xRightarrow{\boxed{2.9.3}}$ W.l.o.g. $b_j > 0$ for all $j \xRightarrow{\boxed{2.10}} B := \{x : \|x/b\|_{\ell^1} \le 1\}$ is bounded in $E := \lambda^1(A) \xRightarrow{\boxed{3.17}} B$ is weakly relatively compact in E $\Rightarrow \forall k : \iota_k \circ \iota^B : E_B \rightarrowtail E \twoheadrightarrow E_k$ weakly compact. Define (for $\ell := \ell^1$)

$$R : \ell \to E_B, \quad x \mapsto b \cdot x,$$
$$S : E_k \to \ell, \quad [x] \mapsto x \cdot a^{(k)}, \text{ and}$$
$$D : \ell \to \ell, \quad x \mapsto b \cdot a^{(k)} \cdot x.$$

R and S are isometries (by $\boxed{2.12}$ and $\boxed{1.13}$), $D = S \circ \iota_k \circ \iota^B \circ R : \ell^1 \to \ell^1$ weakly compact $\Rightarrow \lim b \cdot a^{(k)} = 0$, by $\boxed{3.10.3}$, i.e. $b \in c_0(A)$.

($\boxed{3} \Rightarrow \boxed{2}$) By $\boxed{3.18}$ and $\boxed{2.3}$ we have to show that bounded sets B in $\lambda^p(A)$ are relatively compact. W.l.o.g. $B = B_b^p$ with $b \in \lambda^\infty(A)$ by $\boxed{2.10}$. $\lambda^\infty(A) = c_0(A) \Rightarrow$ D from above (with $\ell := \ell^p$ for $p < \infty$ and $\ell := c_0$ for $p = \infty$) is compact by $\boxed{3.10}$ $\Rightarrow \forall k : \iota_k \circ \iota^B$ compact $\Rightarrow B$ relatively compact (cf. the proof of $\boxed{3.31}$).

($\boxed{2} \Rightarrow \boxed{4}$) By $\boxed{3.22}$ Montel spaces are reflexive.

($\boxed{1} \Rightarrow \boxed{3}$) As in ($\boxed{4} \Rightarrow \boxed{3}$) let $b \in \lambda^\infty(A)$ with $b_j > 0$ for all j. Then the bounded set $B := \{x : \|x/b\|_{\ell^p} \le 1\}$ is relatively compact in $E := \lambda^p(A)$ by ($\boxed{1}$). $\Rightarrow$ $\forall k : \iota_k \circ \iota^B : E_B \rightarrowtail E \twoheadrightarrow E_k$, $x \mapsto [x]$, is compact, where $\ell := \ell^p$ for $p < \infty$ and $\ell := S(E_k) \subseteq \ell^\infty$ for $p = \infty$. R and S are isometries (by $\boxed{2.12}$ and $\boxed{1.13}$), $D = S \circ \iota_k \circ \iota^B \circ R$ is compact $\Rightarrow \lim b \cdot a^{(k)} = 0$ by $\boxed{3.10}$, i.e. $b \in c_0(A)$.

($\boxed{2} \Rightarrow \boxed{1}$) trivial.

($\boxed{2} \Rightarrow \boxed{5}$) since normed Montel spaces are finite dimensional by $\boxed{3.23}$ and $\boxed{4.171}$.

($\boxed{5} \Rightarrow \boxed{6}$) Suppose $J \subseteq \mathbb{N}$, $\exists n$ $\forall m \ge n$: $\inf_{j\in J} a_j^{(n)}/a_j^{(m)} > 0$. $\Rightarrow$ On $E_0 := \{x \in \lambda^p(A) : \operatorname{carr} x \subseteq J\}$ the topology induced by $\|\text{\_}\|_n$ coincides with that of $\lambda^p(A) \Rightarrow$ E_0 finite dimensional by ($\boxed{5}$) $\Rightarrow J$ finite.

($\boxed{6} \Rightarrow \boxed{3}$) Indirect, suppose $\exists b \in \lambda^\infty(A) \setminus c_0(A) \Rightarrow \forall k$ $\exists C_k > 0$ $\forall j$: $|b_j| a_j^{(k)} \le C_k$ and $\exists n$ $\exists$ infinite $J \subseteq \mathbb{N}$ $\exists \varepsilon > 0$ $\forall j \in J$: $|b_j| a_j^{(n)} \ge \varepsilon$. $\Rightarrow \forall j \in J$: $a_j^{(k)} \le C_k/|b_j| \le C_k a_j^{(n)}/\varepsilon$, i.e. $\inf\{a_j^{(n)}/a_j^{(k)} : j \in J\} \ge \varepsilon/C_k$, a contradiction. $\square$

Schwartz spaces

3.29 Definition. Schwartz spaces.
An lcs E is called Schwartz space ((S) for short)(See [**MV92**, Def. in 24.16 p.265], [**Kri07a**, 6.4],[**Jar81**, 10.4 p.201]) iff for each absolutely convex 0-neighborhood U there exists a 0-neighborhood $V \subseteq U$ such that $\iota_U^V : E_V \to E_U$ is precompact, i.e. for each $\varepsilon > 0$ exists a finite subset $F = \{x_1, \dots, x_n\} \subseteq V$ with $V \subseteq F + \varepsilon U = \bigcup_j (x_j + \varepsilon U)$.

3.30 Lemma (See [**MV92**, 24.17 p.265], [**Kri07a**, 6.7], [**Jar81**, 17.1.7 p.370])**.**
An lcs is Schwartz iff for every continuous linear $T : E \to F$ into a normed space F there exists a 0-neighborhood in E with precompact image in F.

Proof. ($\Leftarrow$) For absolutely convex 0-neighborhoods U consider $\iota_U : E \to E_U$. By assumption there exists a 0-neighborhood V such that $\iota_U(V) \subseteq E_U$ is precompact.

($\Rightarrow$) The set $U := T^{-1}(\{x \in F : \|x\| < 1\})$ is an absolutely convex 0-neighborhood, hence there exists a V such that $\iota_U^V : E_V \to E_U$ is precompact, so the image $T(V) = \tilde{T}(\iota_U^V(\iota_V(V)))$ is precompact, where $\tilde{T}$ is the continuous factorization of T over $\iota_U : E \to E_U$. $\square$

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "\iota_V"'] \arrow[dr, "\iota_U"] & F \\ E_V \arrow[r, two heads, "\iota_U^V"'] & E_U \arrow[u, dotted, "\tilde{T}"'] \end{tikzcd}$$

3.31 Quasi-complete Schwartz implies semi-Montel
(See [**MV92**, 24.19 p.265], [**Jar81**, 10.4.3 p.202])**.**
A Schwartz space is semi-Montel iff it is quasi-complete. (See [**Kri07a**, 6.6])

Proof. ($\Leftarrow$) Let $B \subseteq E$ be bounded. For every U exists by definition a V with $E_V \to E_U$ precompact. In particular, $\iota_U(B)$ is precompact in E_U (since V absorbs B) and hence relatively compact in the completion $\widetilde{E_U}$, see 3.2. Since $\tilde{E}$ is complete it is closed in $\prod_U \widetilde{E_U}$ and hence $B \subseteq \prod_U \overline{\iota_U(B)}$ is by Tychonoffs theorem relatively compact in $\tilde{E}$, i.e. $B \subseteq E$ is precompact. For the converse use 3.25. $\square$

3.32 Inheritance properties of Schwartz spaces
(See [**Kri07a**, 6.21], [**Jar81**, 21.1.7 p.481], resp. [**Jar81**, 21.2.3 p.483])**.**
Closed subspaces, products, quotients, and countable coproducts of Schwartz spaces are Schwartz. This will be shown jointly for nuclear spaces in 3.73.

3.33 Proposition. (See [**Jar81**, 10.4.1 p.201], [**MV92**, 24.22 p.267])**.**
An lcs E is Schwartz $\Leftrightarrow \forall U\, \exists V \subseteq U : U^o \subseteq E^*_{V^o}$ compact, i.e. $E^*_{U^o} \to E^*_{V^o}$ is a compact operator.

Proof. Let U and V be absolutely convex 0-nbhds with $V \subseteq U$.

$U^o \subseteq E^*_{V^o} = (E_V)^*$ is (pre)compact;

$\Leftrightarrow$ $(\iota_U^V)^* = \iota_{V^o}^{U^o}$ is $\tau_{\mathrm{pc}}((E_U)^*, E_U)$-$\beta((E_V)^*, E_V)$-continuous;
($\Leftarrow$) U^o is $\tau_{\mathrm{pc}}((E_U)^*, E_U)$-compact by the Alaoğlu-Bourbaki theorem 3.4, hence its image in $(E_V)^*$ is $\beta((E_V)^*, E_V)$-compact.
($\Rightarrow$) Since $(\iota_U^V)^*$ is $\sigma((E_U)^*, E_U)$-$\sigma((E_V)^*, E_V)$-continuous and on U^o the topologies $\tau_{\mathrm{pc}}((E_U)^*, E_U)$ and $\sigma((E_U)^*, E_U)$ coincide by 3.4 and similarly on its image $(\iota_U^V)^*(U^o)$ the topologies $\beta((E_V)^*, E_V)$ and $\sigma((E_V)^*, E_V)$ coincide by assumption.

$\Leftrightarrow$ $\exists A \subseteq E_U$ precompact: $(\iota_U^V)^*(A^\bullet) \subseteq \iota_V(V)^o$, i.e. $(\iota_U)^*(A^\bullet) \subseteq V^o$, where $\bullet$ denotes the polar with respect to $(E^*_{U^o}, E_U)$;

$\Leftrightarrow$ $\exists A \subseteq E_U$ precompact: $\iota_U(V) \subseteq (A^\bullet)_\bullet$, see [Jar81, 8.6.2.b p.161];
$\Leftrightarrow$ $\iota_U(V) \subseteq E_U$ precompact, by the bipolar theorem. □

3.34 Proposition (See [Jar81, 11.6.3 p.232]).
A Fréchet space is Schwartz space ((FS) for short) iff it is separable and $\sigma(E^*, E)$-convergent sequences are equicontinuously convergent, i.e. uniformly convergent on some 0-neighborhood V, or, with other words, convergent in the normed space $E^*_{V^o}$.

Proof. ($\Rightarrow$) (FS) $\Rightarrow$ (M) $\Rightarrow$ separable, by 3.31, 2.3, and 3.27.
Let (u_n) be $\sigma(E^*, E)$-convergent $\Rightarrow U := \{u_n : n \in \mathbb{N}\}_o \subseteq E$ is a barrel, hence a 0-nbhd $\Rightarrow \exists V$ 0-nbhd. with $(\{u_n : n \in \mathbb{N}\}_o)^o$ compact in $E^*_{V^o} = (E_V)^*$, by 3.33 $\Rightarrow u_n$ converges equicontinuously.

($\Leftarrow$) Let U 0-nbhd. and $U \supseteq U_k \supseteq U_{k+1}$ be a 0-nbhd. basis. By 3.33 we have to show $\exists k : U^o \subseteq E^*_{U_k^o}$ is compact. Since U^o is $\sigma(E^*, E)$-metrizable, by 3.5, it suffices to show that there exists some k such that $\sigma(E^*, E)$-converging sequences in U^o converge in $F_k := E^*_{U_k^o}$. Otherwise, $\forall k\, \exists (u_n^k)_n$ convergent in U^o (towards 0) but not convergent in F_k for $n \to \infty$. Let $B_0 \supseteq B_1 \supseteq \dots$ be a countable 0-nbhd. basis for the metrizable topology $\sigma(E^*, E)$ on U^o. Let $m_n := \min\{m : u_i^k \in B_n\ \forall k \le n\, \forall i > m\}$, then $u^1_{m_n+1}, \dots, u^1_{m_{n+1}}; \dots; u^n_{m_n+1}, \dots, u^n_{m_{n+1}} \in B_n$. These blocks together give a weak 0-sequence (u_n) in U^o, not convergent in F_k, (i.e. not equicontinuously convergent), since $(u_n^k)_{n > m_k}$ is a subsequence, a contradiction. □

3.35 Theorem. Characterizing Schwartz for $\lambda^p(A)$
(See [MV92, 27.10 p.312]).
Let $A = \{a^{(k)} : k \in \mathbb{N}\}$ be countable.

(1) $\exists\, 1 \le p \le \infty$: $\lambda^p(A)$ (S)
$\Leftrightarrow$ (2) $\forall\, 1 \le p \le \infty$: $\lambda^p(A)$ (S)
$\Leftrightarrow$ (3) $\forall k\ \exists m \ge k$: $\lim_{j\to\infty} a_j^{(k)}/a_j^{(m)} = 0$.

Proof. Let $E = \lambda^p(A)$ for $1 \le p \le \infty$, $E_k \cong \ell^p(a_k)$ for $1 \le p < \infty$ (cf. 1.13), w.l.o.g. $\operatorname{carr} a_k = \mathbb{N}$. For $m \ge k$ define

$$D : x \mapsto x \cdot a_k/a_m, \quad \ell^p \to \ell^p$$
$$A_m : x \mapsto x \cdot a_m, \quad E_m \to \ell^p$$

A_m is isometry and $D = A_k \circ \iota_m^k \circ A_m^{-1}$. If $p = \infty$ replace ℓ^p by $\ell := A_m(E_m)$.

(1$\Rightarrow$3) $\Rightarrow \forall k\ \exists m \ge k$: $\iota_m^k : E_m \to E_k$ compact $\Rightarrow D = A_k \circ \iota_m^k \circ A_m^{-1}$ compact $\xRightarrow{3.10} \lim a^{(k)}/a^{(m)} = 0$.

(3$\Rightarrow$2) Let $1 \le p \le \infty$. $\forall k\ \exists m \ge k$ satisfying (3) $\xRightarrow{3.10} D = A_k \circ \iota_m^k \circ A_m^{-1}$ compact $\Rightarrow \iota_m^k$ compact $\Rightarrow$ (2).

(2$\Rightarrow$1) trivial. □

3.36 Example of (FM), but not Schwartz (See [MV92, 27.21 p.319]).

$$A := \{a^{(k)} : k \in \mathbb{N}\} \text{ with } a_{i,j}^{(k)} := \begin{cases} (ki)^k & \text{for } j < k \\ k^j & \text{for } j \ge k \end{cases} \Rightarrow \lambda^p(A) \text{ is (F), (M), not (S)}$$

Proof. $m > k$, $j > m \Rightarrow a_{i,j}^{(k)}/a_{i,j}^{(m)} = (k/m)^j \xRightarrow{3.35} \lambda^p(A)$ not Schwartz.

Let $I \subseteq \mathbb{N}^2$, n fixed. $\forall k \geq n$: $\inf_{(i,j)\in I} a^{(n)}_{i,j}/a^{(k)}_{i,j} =: \varepsilon_k > 0$.
Claim: I is finite:
$k := n+1$, $j \geq n+1 \Rightarrow$

$$\varepsilon_{n+1} \leq a^{(n)}_{i,j}/a^{(n+1)}_{i,j} = \left(\frac{n}{n+1}\right)^j$$

$\Rightarrow \exists j_0$: $I \subseteq \mathbb{N} \times \{1, \dots, j_0\}$. Let $1 \leq j \leq j_0$, $k > \max\{j, n\}$, $(i,j) \in I \Rightarrow$

$$\varepsilon_k \leq \begin{cases} a^{(n)}_{i,j}/a^{(k)}_{i,j} = \frac{(ni)^n}{(ki)^k} = \frac{n^n}{k^k} i^{n-k} & \text{for } j < n (< k) \\ \\ a^{(n)}_{i,j}/a^{(k)}_{i,j} = \frac{n^j}{(ki)^k} & \text{for } (k >) j \geq n \end{cases}$$

$\Rightarrow I \cap (\mathbb{N} \times \{j\})$ is finite.
$\xRightarrow{\boxed{3.28.6}} \lambda^p(A)$ Montel (for all $1 \leq p \leq \infty$). $\square$

Tensor products

In this section we introduce the projective tensor product as universal solution for linearizing bilinear continuous maps and the injective tensor product as subspace of te space of all bounded linear (or bilinear) operators. of locally convex spaces in order to define nuclearity. Nuclear spaces are then deinied as those locally convex spaces, where these to tensor product functors coincide. And we use these tensor products to obtain descriptions for various types of vector valued summable sequences.

3.37 Definition. Projective tensor product (See [**Kri07a**, 3.3 p.53]).
The ALGEBRAIC TENSOR PRODUCT $E \otimes F$ of two linear spaces E and F is the universal solution for turning bilinear mappings into linear ones, i.e. there exists a bilinear mapping $\otimes : E \times F \to E \otimes F$ such that

$$\begin{tikzcd} E \times F \arrow[r, "\otimes"] \arrow[dr, "\forall f \text{ bilinear}"'] & E \otimes F \arrow[d, dotted, "\exists! \tilde{f} \text{ linear}"] \\ & \forall G \end{tikzcd}$$

The linear space $E \otimes F$ can be obtained as subspace of $L(E, F; \mathbb{K})^*$ (the dual of the bilinear forms) generated by the image of $\otimes : E \times F \to E \otimes F \subseteq L(E, F; \mathbb{K})^*$ given by $(x, y) \mapsto \mathrm{ev}_{(x,y)}$ (See [**Kri07a**, 3.1 p.50]).

For locally convex spaces the solution of the corresponding universal problem for (bi)linear continuous mappings is called PROJECTIVE TENSOR PRODUCT $E \otimes_\pi F$, it is the linear spaces $E \otimes F$ supplied with the finest locally convex topology for which $\otimes : E \times F \to E \otimes F$ is continuous. This topology exists since the union of locally convex topologies is locally convex and $E \times F \to E \otimes F$ is continuous for the weak topology on $E \otimes F$ generated by those linear functionals which correspond to continuous bi-linear functionals on $E \times F$. It has the universal property, since the inverse image of a locally convex topology under a linear mapping $\tilde{T}$ is again a locally convex topology, such that $\otimes$ is continuous, provided the associated bilinear mapping T is continuous.

The space $E \otimes_\pi F$ is Hausdorff, since the set $E^* \times F^*$ separates points in $E \otimes F$: Let $0 \neq z = \sum_k x_k \otimes y_k$ be given. By replacing linear dependent x_k by the corresponding linear combinations and using bilinearity of $\otimes$, we may assume that

the x_k are linearly independent. Now choose $x^* \in E^*$ and $y^* \in F^*$ be such that $x^*(x_k) = \delta_{1,k}$ and $y^*(y_1) = 1$. Then $(x^* \otimes y^*)(z) = 1 \neq 0$.

We denote the SPACE OF CONTINUOUS LINEAR MAPPINGS from E to F by $\mathcal{L}(E, F)$, and the SPACE OF CONTINUOUS MULTI-LINEAR MAPPINGS by $\mathcal{L}(E_1, \dots, E_n; F)$.

The COMPLETION of $E \otimes_\pi F$ will be denoted $E \hat{\otimes}_\pi F$.

Since a bilinear mapping is continuous iff it is so at 0, a 0-neighborhood basis in $E \otimes_\pi F$ is given by all those absolutely convex sets, for which the inverse image under $\otimes$ is a 0-neighborhood in $E \times F$. A basis is thus given by the absolutely convex hulls denoted $U \otimes V$ of the images of $U \times V$ under $\otimes$, where U resp. V runs through a 0-neighborhood basis of E resp. F. We only have to show that these sets $U \otimes V$ are absorbing (see [**Jar81**, 6.5.3 p.108]). So let $z = \sum_{k \leq K} x_k \otimes y_k \in E \otimes F$ be arbitrary. Then there are $a_k > 0$ and $b_k > 0$ such that $x_k \in a_k U$ and $y_k \in b_k V$ and hence $z = \sum_k a_k\, b_k \frac{x_k}{a_k} \otimes \frac{y_k}{b_k} \in (\sum_k a_k\, b_k) \cdot \langle U \otimes V \rangle_{\text{abs.conv.}}$. Consequently, the Minkowski-functionals $p_{U \otimes V}$ form a base of the seminorms of $E \otimes_\pi F$ and we will denote them by $\pi_{U,V}$. In terms of the Minkowski-functionals p_U and p_V of U and V we obtain that $z \in (\sum_k p_U(x_k)\, p_V(y_k))\, U \otimes V$ for any $z = \sum_k x_k \otimes y_k$ since $x_k \in p_U(x_k) \cdot U$ for closed U, and thus $p_{U \otimes V}(z) \leq \inf\{\sum_k p_U(x_k)\, p_V(y_k) : z = \sum_k x_k \otimes y_k\}$. We now show the converse:

3.38 Proposition. Seminorms of the projective tensor product
(See [**Kri07a**, 3.4 p.53], [**Jar81**, 15.1.1 p.324]).

$$p_{U \otimes V}(z) = \inf\Big\{\sum_k p_U(x_k) \cdot p_V(y_k) : z = \sum_k x_k \otimes y_k\Big\}.$$

Proof. Let $z \in \lambda \cdot (U \otimes V)$ with $\lambda > 0$. Then $z = \lambda \sum \lambda_k\, (u_k \otimes v_k)$ with $u_k \in U$, $v_k \in V$ and $\sum_k |\lambda_k| = 1$. Hence $z = \sum x_k \otimes v_k$, where $x_k = \lambda \lambda_k u_k$, and $\sum_k p_U(x_k) \cdot p_V(v_k) \leq \sum \lambda |\lambda_k| = \lambda$. Taking the infimum of all λ shows that $p_{U \otimes V}(z)$ is greater or equal to the infimum on the right side. □

3.39 Theorem. Compact subsets of the projective tensor product
(See [**Kri07a**, 3.21 p.61] and [**Jar81**, 15.6.3 p.336]).

Compact subsets of the completed projective tensor product $E \hat{\otimes}_\pi F$ for metrizable spaces E and F are contained in the closed absolutely convex hull of a tensor product of precompact sets in E and F.

Proof. Every compact set K in the Fréchet space $E \hat{\otimes}_\pi F$ is contained in the closed absolutely convex hull of a 0-sequence $(z_k)_k$ in $E \hat{\otimes}_\pi F$ by 3.6. For this 0-sequence we can choose k_n strictly increasing, such that $z_k \in U_n \otimes V_n$ for all $k \geq k_n$, where $(U_n)_n$ and $(V_n)_n$ are countable 0-neighborhood bases of the topology of E and F. For $k_n \leq k < k_{n+1}$ we can choose finite (disjoint) sets $N_k \subseteq \mathbb{N}$ and $\sum_{j \in N_k} |\lambda_j| = 1$, $x_j \in U_n$ and $y_j \in V_n$ such that $z_k = \sum_{j \in N_k} \lambda_j\, x_j \otimes y_j$. Let $A := \{x_j : j \in \bigsqcup_k N_k\}$ and $B := \{y_j : j \in \bigsqcup_k N_k\}$. These are formed by two sequences converging to 0, and hence are precompact. Furthermore, each $z \in K$ can be written as

$$z = \sum_{k=0}^{\infty} \mu_k z_k = \sum_k \sum_{j \in N_k} \mu_k \lambda_j\, x_j \otimes y_j$$

with $\sum_k |\mu_k| \leq 1$ and hence $\sum_k \sum_{j \in N_k} |\mu_k \lambda_j| = \sum_k |\mu_k| \sum_{j \in N_k} |\lambda_j| \leq 1$. From this it easily follows that the series on the right hand side converges (even Mackey) and hence z is contained in the closed absolutely convex hull of $A \otimes B$. □

3.40 Corollary. Elements of the completed tensor product as limits (See [**Kri07a**, 3.22 p.61], [**Jar81**, 15.6.4 p.337]).
For metrizable E and F every $z \in E\hat{\otimes}_\pi F$ has a representation of the form $z = \sum_n \lambda_n\, x_n \otimes y_n$, where $\lambda \in \ell^1$ and x and y are bounded (or even 0-)sequences.

Since for every $\lambda \in \ell^1$ there exists a $\rho \in c_0$ and $\mu \in \ell^1$ with $\lambda_n = \rho_n^2 \mu_n$ it is enough to find bounded sequences x_n and y_n.

Proof. In the previous proof we have just shown that $z = \sum_j \mu_{k_j} \lambda_j\, x_j \otimes y_j$. □

3.41 Definition. Summable vector valued sequences.
For lcs F we consider the following spaces of (somehow summable) series in F:

- $\ell^1\{F\} := \ell^1(\mathbb{N}, F) := \{f \in F^{\mathbb{N}} : \forall p \text{ SN of } F : \tilde{p}(f) := \sum_{j=0}^{\infty} p(f_j) < \infty\}$, the space of ABSOLUTELY SUMMABLE SEQUENCES in F (called ABSOLUTELY CAUCHY SEQUENCES in [**Jar81**, 15.7.5 p.341]).
  Recall the Reordering Theorem of Riemann [**Kri05**, 2.5.18].
- $\ell^1\langle F\rangle$, the space of UNCONDITIONALLY CAUCHY SUMMABLE SEQUENCES $(x_j)_{j\in\mathbb{N}}$ in F (see [**Jar81**, 14.6.1 p.305]), i.e. for which the net $F \mapsto \sum_{j\in F} x_j$, where F runs through the finite subsets of $\mathbb{N}$ ordered by inclusion, is Cauchy:
  ($\Leftarrow$) Let σ be a permutation of $\mathbb{N}$. For any U we find a finite $F_0 \subseteq \mathbb{N}$ such that $\sum_{k\in F_2} x_k - \sum_{k\in F_1} x_k \in U$ for all finite $F_1, F_2 \supseteq F_0$. Let $n_0 := \max \sigma^{-1}(F_0)$, hence $\sigma^{-1}(F_0) \subseteq \{n : n \le n_0\}$. Then, for all $n_2 \ge n_1 > n_0$, we have
  $$\sum_{n=n_1}^{n_2} x_{\sigma(n)} = \sum_{n\le n_2} x_{\sigma(n)} - \sum_{n<n_1} x_{\sigma(n)} = \sum_{k\in F_2} x_k - \sum_{k\in F_1} x_k \in U, \text{ where}$$
  $F_2 := \sigma(\{n : n \le n_2\}) \supseteq F_1 := \sigma(\{n : n < n_1\}) \supseteq \sigma(\{n : n \le n_0\}) \supseteq F_0$.

  ($\Rightarrow$) Otherwise, $\exists U\, \forall F$ finite $\exists F'$ finite : $F' \cap F = \emptyset$ and $\sum_{n\in F'} x_n \notin 2U$ ($\exists F_1, F_2 \supseteq F : 4U \not\ni \sum_{n\in F_2} x_n - \sum_{n\in F_1} x_n = \sum_{n\in F_2\setminus F_1} x_n - \sum_{n\in F_1\setminus F_2} x_n$, now take $F' := F_2 \setminus F_1$ or $F' := F_1 \setminus F_2$). Since $\sum_n x_n$ is Cauchy, there is some n_0 such that $\sum_{n=n_1}^{n_2} x_n \in U$ for all $n_2 \ge n_1 \ge n_0$. Let $F_0 := \{n \in \mathbb{N} : n \le n_0\}$ and F_0' a corresponding set. We construct n_k, F_k, and $F_k' \ne \emptyset$ recursively as $n_{k+1} := \max F_k'$, $F_{k+1} := \{n : n \le n_{k+1}\} \supseteq F_k' \cup F_k$. Let $F_k'' := F_{k+1} \setminus (F_k \sqcup F_k')$. Then
  $$\sum_{n\in F_k''} x_n = \sum_{n\in F_{k+1}} x_n - \sum_{n\in F_k} x_n - \sum_{n\in F_k'} x_n = \sum_{n\in F_{k+1}\setminus F_k} x_n - \sum_{n\in F_k'} x_n,$$
  where $F_{k+1} \setminus F_k = \{n : n_k < n \le n_{k+1}\}$ with $n_{k+1} \ge n_k \ge n_0$, so $\sum_{n\in F_{k+1}\setminus F_k} x_n \in U$, wheras $\sum_{n\in F_k'} x_n \notin 2U$, hence $\sum_{n\in F_k''} x_n \notin U$. The elements in the sequence $F_0, F_0', F_0'', F_1', F_1'', \dots$ define a permutation σ of $\mathbb{N}$ for which $\sum_n x_{\sigma(n)}$ is not Cauchy.
- $\ell^1[F] := L(c_0, F)$, the space of SCALARLY ABSOLUTELY SUMMABLE SEQUENCES in F (See [**Kri07a**, 4.9] and [**Jar81**, 19.4.3 p.427]): Since the standard unit vectors e_k generate a dense subspace in c_0 every $f \in L(c_0, F)$ is uniquely determined by its values $f_k := f(e_k)$. Moreover, f is continuous=bounded iff $\{(y^* \circ f)(x) = \sum_{j\in\mathbb{N}} x_j\, y^*(f_j) : x \in c_0, \|x\|_\infty \le 1\}$ is bounded for each $y^* \in F^*$, i.e. $\{(x_j\, y^*(f_j))_j : x \in c_0, \|x\|_\infty \le 1\}$ is bounded in ℓ^1, i.e. $(y^*(f_j))_j \in \lambda^1(c_0) = \ell^1$ by 1.15.7, i.e. $(f_j)_j$ is scalarly absolutely summable.

This can be extended $1 < q < \infty$:

- $\ell^q\{F\} := \ell^q(\mathbb{N},F) := \{f \in F^{\mathbb{N}} : \forall p : \tilde{p}(f) := \big(\sum_{j=0}^\infty p(f_j)^q\big)^{1/q} < \infty\}$, the space of ABSOLUTELY q-SUMMABLE SEQUENCES in F (See [Jar81, 19.4 p.425]).
- $\ell^q[F] = L(\ell^p, F)$, the space of SCALARLY ABSOLUTELY q-SUMMABLE SEQUENCES in F, where $\frac{1}{p}+\frac{1}{q}=1$ (See [Jar81, 19.4.1 p.426] and [Jar81, 19.4.3 p.427]): Since the standard unit vectors e_k generate a dense subspace in ℓ^p every $f \in L(\ell^p, F)$ is uniquely determined by its values $f_k := f(e_k)$. Moreover, f is continuous=bounded iff $\{(y^* \circ f)(x) = \sum_{j\in\mathbb{N}} x_j\, y^*(f_j) : x \in \ell^p, \|x\|_p \le 1\}$ is bounded for each $y^* \in F^*$, i.e. $\{(x_j\, y^*(f_j))_j : x \in \ell^p, \|x\|_p \le 1\}$ is bounded in ℓ^1, i.e. $(y^*(f_j))_j \in \lambda^1(\ell^p) = \ell^q$ by 1.15.7, i.e. $(f_j)_j$ is scalarly absolutely q-summable.

3.42 Lemma. Description of $\ell^1\{F\}$ as tensor product
(See [Kri07a, 4.12], [Jar81, 15.7.6 p.341]).
For lcs F we have a dense topological embedding $\ell^1 \otimes_\pi F \hookrightarrow\!\!\!\!\to \ell^1(\mathbb{N},F)$.
Thus $\ell^1 \hat{\otimes}_\pi F \cong \ell^1(\mathbb{N},F)$ for complete F, where $\hat{\otimes}_\pi$ denotes the completion of the projective tensor product.

Proof. We first show that the natural mapping $\ell^1_c \otimes_\pi F \to \ell^1_c(\mathbb{N},F)$, $x \otimes y \mapsto (x_j y)_{j\in\mathbb{N}}$, is an isomorphism, where ℓ^1_c is the dense subspace in ℓ^1 of finite sequences and $\ell^1_c(\mathbb{N},F)$ the analogous subspace in $\ell^1(\mathbb{N},F)$. Since $\mathbb{R}^k \otimes_\pi F \cong F^k$ we have a bijection. Let $z = \sum_k x^{(k)} \otimes y^{(k)} \in \ell^1_c \otimes F$ and p be a seminorm of F. For the corresponding norm $\tilde{p}$ of $\ell^1_c(\mathbb{N},F)$ we have

$$\tilde{p}(z) := \sum_j p(z_j) = \sum_j p\left(\sum_k x_j^{(k)}\, y^{(k)}\right) \le \sum_j \sum_k |x_j^{(k)}|\, p(y^{(k)}) \le$$
$$\le \sum_k \sum_j |x_j^{(k)}|\, p(y^{(k)}) = \sum_k \|x^{(k)}\|_{\ell^1} \cdot p(y^{(k)}),$$

Taking the infimum of the right side over all representations of z shows that $\tilde{p} \le p^\pi$, where p^π is projective tensor norm formed from $\|\_\|_{\ell^1}$ and p, see 3.38.
Conversely each $z = (z_j)_j \in \ell^1_c(\mathbb{N},F)$ can be written as image of the finite sum $\sum_j e_j \otimes z_j$, where e_j denotes the standard unit vector in ℓ^1. Thus we have for the tensor norm p^π that

$$p^\pi(z) \le \sum_j \|e_j\|_{\ell^1} \cdot p(z_j) = \sum_j p(z_j) = \tilde{p}(z)$$

which shows the converse relation.

Now, since $\ell^1_c(\mathbb{N},F)$ is dense in $\ell^1(\mathbb{N},F)$ and the latter space is complete for complete F (as can be shown analogously to the case $\ell^1(\mathbb{N},\mathbb{R})$), we have the desired isomorphism:

$$\ell^1(\mathbb{N},F) = \widehat{\ell^1_c(\mathbb{N},F)} \cong \ell^1_c \hat{\otimes}_\pi F \cong \ell^1 \hat{\otimes}_\pi F.$$

Here we used that the dense emnbedding $\ell^1_c \hookrightarrow \ell^1$ induces a dense embedding $\ell^1_c \otimes_\pi F \hookrightarrow \ell^1 \otimes_\pi F$, see [Kri07a, 3.19,3.20] or [Jar81, 15.2.3,15.2.4 p.327]. □

3.43 Lemma. The seminorms of $\ell^1[F]$ (See [Kri07a, 4.32], [Jar81, 19.4.3a p.427]).
The structure on $\ell^q[F]$ induced from $L((\ell^q)^*, F)$ for $1 < q < \infty$ (resp. from $L(c_0, F)$ for $q = 1$) is given by the seminorms

$$\tilde{p}(f) := \sup\Big\{\Big(\sum_{n=1}^\infty |y^*(f_n)|^q\Big)^{1/q} : |y^*| \le p\Big\},$$

where p runs through all continuous seminorms of F.

Proof. Let p be a continuous seminorm on F and $V := \{y \in F : p(y) \le 1\}$. As in 3.45 we use that $p(y) = \sup\{|y^*(y)| : y^* \in V^o\}$. Thus we can calculate the seminorm p_∞ on $L((\ell^q)^*, F)$ associated to p as follows, where B denotes the closed unit-ball in $\ell := (\ell^q)^*$ (resp. c_0 for $q = 1$) and $\iota : \ell^q[F] \to L(\ell, F)$, $\iota(f)(\lambda) := \sum_k f_k \lambda_k$, the canonical bijection:

$$\begin{aligned}\tilde{p}(f) := p_\infty(\iota(f)) &:= \sup\{p(\iota(f)(\lambda)) : \lambda \in B\}\\ &= \sup\Big\{\Big|y^*\Big(\sum_{k=1}^\infty f_k\lambda_k\Big)\Big| : \lambda \in B, y^* \in V^o\Big\}\\ &\le \sup\Big\{\underbrace{\|\lambda\|_{(\ell^q)^*}}_{\le 1}\ \|(y^*(f_k))_k\|_{\ell^q} : \lambda \in B, y^* \in V^o\Big\}\\ &\le \sup\Big\{\Big(\sum_{k=1}^\infty |y^*(f_k)|^q\Big)^{1/q} : |y^*| \le p\Big\}\end{aligned}$$

Conversely, let $f \in \ell^q[F]$ and $|y^*| \le p$. Then for $\varepsilon > 0$ there exists an n such that $\big(\sum_{k>n} |y^*(f_k)|^q\big)^{1/q} < \varepsilon$. Let $\lambda_k\, y^*(f_k) := |y^*(f_k)|$ for $k \le n$ and $\lambda_k = 0$ otherwise. Then $\lambda \in B$ and

$$\sum_{k=1}^\infty |y^*(f_k)|^q = \sum_{k\le n}(\lambda_k\, y^*(f_k))^q + \sum_{k>n}|y^*(f_k)|^q \le \sum_{k=0}^\infty(\lambda_k\, y^*(f_k))^q + \varepsilon^q \le p_\infty(\iota(f))^q + \varepsilon^q.$$

Hence we have also the converse relation. □

3.44 Definition. Injective tensor product
(See [**Kri07a**, 4.21 p.93], [**Jar81**, 16.1 p.344]).
We consider the bilinear mapping

$$E \times F \to L(E^*, F), \text{ given by } (x, y) \mapsto \big(x^* \mapsto x^*(x)y\big).$$

It is well-defined, since $\mathrm{ev}_x : E^* \to \mathbb{R}$ is bounded. In fact $\mathrm{ev}_x : E^* \to \mathbb{R}$ is even continuous for the weak topology $\sigma(E^*, E)$ and hence also for the topology $\beta(E^*, E)$ of uniform convergence on bounded sets. This induces a linear map

$$E \otimes F \to L(E^*, F), \text{ given by } x \otimes y \mapsto (x^* \mapsto x^*(x)y).$$

We claim that this mapping is injective. In fact take $\sum_i x_i \otimes y_i \in E \otimes F$ with x_i linearly independent. By Hahn-Banach we can find continuous linear functionals x_i^* with $x_i^*(x_j) = \delta_{i,j}$. Assume that the image of $\sum_i x_i \otimes y_i$ is 0 in $L(E^*, F)$. Since it has value y_i on x_i^*, we have that $y_i = 0$ for all i and hence $\sum_i x_i \otimes y_i = 0$.

We define the INJECTIVE TENSOR PRODUCT (also called ε-TENSOR PRODUCT in [**Tre67**]) $E \otimes_\varepsilon F$ to be the algebraic tensor product with the locally convex topology induced by the injective inclusion into $L(E^*, F)$, where $L(E^*, F)$ is supplied with the topology of uniform convergence on equicontinuous subsets of E^*. Since this topology on $L(E^*, F)$ is obviously Hausdorff, the same is true for $E \otimes_\varepsilon F$.

Note that, since F topologically embeds into the space $(F^*)'$ of bounded (with respect to the equicontinous subsets of E^*) linear functionals on E^* by [**Kri14**, 5.4.11], the structure of $E \otimes_\varepsilon F$ is also initial with respect to $E \otimes F \to L(E^*, F) \to L(E^*, (F^*)') \cong L(E^*, F^*; \mathbb{R})$, $x \otimes y \mapsto \big((x^*, y^*) \mapsto x^*(x) \cdot y^*(y)\big)$, which gives a more symmetric form and consequently $E \otimes_\varepsilon F \cong F \otimes_\varepsilon E$. Since the seminorms of

$L(E^*, F^*; \mathbb{R})$ are given by suprema on $U^o \times V^o$, where U and V are 0-neighborhoods, we have for the corresponding seminorm $\varepsilon_{U,V}$ on $E \otimes_\varepsilon F$:

$$\varepsilon_{U,V}\Big(\sum_k x_k \otimes y_k\Big) := \sup\Big\{\Big|\sum_k x^*(x_k)\, y^*(y_k)\Big| : x^* \in U^o, y^* \in V^o\Big\}$$

3.45 Corollary. Seminorms of the injective tensor product
(See [**Kri07a**, 4.22 p.94], [**Jar81**, 16.1 p.344]).

A defining family of seminorms on $E \otimes_\varepsilon F$ is given by

$$\varepsilon_{U,V} : \sum_i x_i \otimes y_i \mapsto \sup\Big\{\Big|\sum_i x^*(x_i)\, y^*(y_i)\Big| : x^* \in U^o, y^* \in V^o\Big\},$$

where U and V run through the 0-neighborhoods of E and F. The injective tensor product $E \otimes_\varepsilon F$ is metrizable (resp. normable) if E and F are. □

Let us show next, that the canonical bilinear mapping $E \times F \to L(E^*, F)$ is continuous, which implies that the identity $E \otimes_\pi F \to E \otimes_\varepsilon F$ is continuous:
In fact, take an equicontinuous set $\mathcal{E} \subseteq E^*$, i.e. $\mathcal{E}$ is contained in the polar U^o of a 0-neighborhood U, and take furthermore an absolutely convex 0-neighborhood $V \subseteq F$. Then $U \times V$ is mapped into the typical 0-nbhd. $\{T : T(\mathcal{E}) \subseteq V\}$, since $(x \otimes y)(x^*) = x^*(x)\, y \in \{\lambda : |\lambda| \leq 1\} \cdot V \subseteq V$ for $x^* \in \mathcal{E} \subseteq U^o$.

3.46 Corollary (See [**Kri07a**, 4.23 p.94], [**Jar81**, 16.1.3 p.345]).
$E \otimes_\pi F \to E \otimes_\varepsilon F$ is continuous.

Proof. In the diagram

$$\begin{tikzcd} E \otimes_\pi F \arrow[r, dotted] & E \otimes_\varepsilon F \arrow[d, hook] \\ E \times F \arrow[u, "\otimes"] \arrow[r] & L(E^*, F) \end{tikzcd}$$

continuity of the bilinear map at the bottom implies continuity of the top arrow. □

3.47 Definition (See [**Kri07a**, 4.24 p.94], [**Jar81**, 16.1.4 p.345]).
An lcs E is called NUCLEAR ((N) for short) iff $E \otimes_\pi F = E \otimes_\varepsilon F$ for all lcs F.

3.48 Corollary (See [**Kri07a**, 4.26]).
The space $E' \otimes_\varepsilon F$ embeds into $L(E, F)$.

Proof. In fact, since obviously $E' \otimes_\varepsilon F \cong F \otimes_\varepsilon E'$, it embeds into $L(F^*, E') \cong L(E, (F^*)')$ via $x^* \otimes y \mapsto (x \mapsto (y^* \mapsto x^*(x)\, y^*(y)))$. This embedding factors over the embedding $\delta_* : L(E, F) \hookrightarrow L(E, (F^*)')$, by $x^* \otimes y \mapsto (x \mapsto x^*(x) y)$. Hence this map $E' \otimes_\varepsilon F \to L(E, F)$ is an embedding.

$$\begin{tikzcd} E' \otimes_\varepsilon F \arrow[r, hook] \arrow[dr, hook, dotted] & L(E, (F^*)') \\ & L(E, F) \arrow[u, hook, "\delta_*"'] \end{tikzcd} \qquad \begin{tikzcd} x^* \otimes y \arrow[r, mapsto] \arrow[dr, mapsto] & \big(x \mapsto (y^* \mapsto x^*(x)\, y^*(y))\big) \\ & (x \mapsto x^*(x) y) \arrow[u, mapsto, "\delta_*"'] \end{tikzcd}$$ □

3.49 Lemma. Completeness of $\ell^1\langle F\rangle$
(See [**Kri07a**, 4.33], [**Jar81**, 16.5.1 p.358]).
The subspace $\ell^1\langle F\rangle$ of $\ell^1[F]$ is closed. For complete F both spaces are complete.

Hence we will always consider the initial structure on $\ell^1\langle F\rangle$ induced from $\ell^1[F]$.

Proof. In order to show that $\ell^1\langle F\rangle$ is closed in $\ell^1[F]$, take $x = (x_k)_k \in \ell^1[F]$ in the closure of $\ell^1\langle F\rangle$. We have to show that the net $K \mapsto \sum_{k\in K} x_k$ is Cauchy, where K runs through the finite subsets of $\mathbb{N}$. So let p be a seminorm of F and $\varepsilon > 0$. By the assumption we can find a $y \in \ell^1\langle F\rangle$ with $\tilde{p}(x-y) \le \varepsilon$. Thus the net $\sum_{k\in K} y_k$ is Cauchy in F, i.e. there is a finite $K_0 \subseteq \mathbb{N}$ such that $p\big(\sum_{k\in K_2} y_k - \sum_{k\in K_1} y_k\big) \le \varepsilon$ for all $K_1, K_2 \supseteq K_0$. Hence for $K_0 \subseteq K_1 \subset K_2$ we have:

$$\begin{aligned}
p\Big(\sum_{k\in K_2} x_k - \sum_{k\in K_1} x_k\Big) &= p\Big(\sum_{k\in K_2\setminus K_1} x_k\Big) \le p\Big(\sum_{k\in K_2\setminus K_1} (x_k - y_k)\Big) + p\Big(\sum_{k\in K_2\setminus K_1} y_k\Big) \\
&\le \sup\Big\{\Big|y^*\Big(\sum_{k\in K_2\setminus K_1} (x_k-y_k)\Big)\Big| : |y^*| \le p\Big\} + p\Big(\sum_{k\in K_2\setminus K_1} y_k\Big) \\
&\le \sup\Big\{\sum_{k=0}^{\infty} |y^*(x_k-y_k)| : |y^*| \le p\Big\} + p\Big(\sum_{k\in K_2\setminus K_1} y_k\Big) \\
&\xlongequal{\boxed{3.43}} \tilde{p}(x-y) + p\Big(\sum_{k\in K_2} y_k - \sum_{k\in K_1} y_k\Big) \le \varepsilon + \varepsilon,
\end{aligned}$$

which shows that $K \mapsto \sum_{k\in K} x_k$ is a Cauchy-net.

Since $\ell^1[F] \cong L(c_0, F)$, it is complete for complete F. □

3.50 Theorem. Description of $\ell^1\langle F\rangle$ as tensor product
(See [**Kri07a**, 4.34], [**Jar81**, 16.5.2 p.359]).
For lcs F we have a dense topological embedding $\ell^1 \otimes_\varepsilon F \hookrightarrow\!\!\!\!\rightarrow \ell^1\langle F\rangle$.
Thus $\ell^1 \widehat{\otimes}_\varepsilon F \cong \ell^1\langle F\rangle$ for complete F, where $\hat{\otimes}_\varepsilon$ denotes the completion of the injective tensor product.

Proof. By $\boxed{3.48}$ we have that $\ell^1 \otimes_\varepsilon F \cong c_0' \otimes_\varepsilon F$ embeds into $L(c_0, F)$, the space of scalarly absolutely summable sequences. Obviously $\lambda \otimes y \in \ell^1 \otimes F$ is contained in $\ell^1\{F\} \subseteq \ell^1\langle F\rangle$. We show that $\ell^1_c \otimes F = \mathbb{K}^{(\mathbb{N})} \otimes F \cong F^{(\mathbb{N})}$ is dense in $\ell^1\langle F\rangle$ with respect to the structure inherited from $\ell^1[F]$. So let $x \in \ell^1\langle F\rangle$ and consider $x^n := x|_{[0,\dots,n-1]} \in F^n \subseteq F^{(\mathbb{N})} \subseteq \ell^1[F]$. We claim that $x^n \to x$ in $\ell^1[F]$: Let p be a continuous seminorm on F. Since $K \mapsto \sum_{k\in K} x_k$ is Cauchy, we have for $\mathbb{K} = \mathbb{R}$:

$$\begin{aligned}
\tilde{p}(x-x^n) &\xlongequal{\boxed{3.43}} \sup\Big\{\sum_{k\ge n} |y^*(x_k)| : |y^*| \le p\Big\} = \sup\Big\{\sum_{k=n}^{m} |y^*(x_k)| : |y^*| \le p,\, m \ge n\Big\} \\
&= \sup\Big\{\Big|\sum_{\substack{m\ge k\ge n\\ y^*(x_k)>0}} y^*(x_k)\Big| + \Big|\sum_{\substack{m\ge k\ge n\\ y^*(x_k)<0}} y^*(x_k)\Big| : |y^*| \le p,\, m \ge n\Big\} \\
&\le \sup\Big\{\Big|y^*\Big(\sum_{\substack{m\ge k\ge n\\ y^*(x_k)>0}} x_k\Big)\Big| : |y^*| \le p,\, m \ge n\Big\} + \sup\Big\{\Big|y^*\Big(\sum_{\substack{m\ge k\ge n\\ y^*(x_k)<0}} x_k\Big)\Big| : \dots\Big\} \\
&\le 2 \sup\Big\{p\Big(\sum_{k\in K'} x_k\Big) : K' \text{ finite},\, K' \cap [0, n-1] = \emptyset\Big\} \le 2\varepsilon
\end{aligned}$$

for n sufficiently large. In the complex case we have to make a more involved estimation for $\sum_{k>n} |y^*(x_k)|$. Let $P := \{z \in \mathbb{C} : \Re z > 0 \text{ and } -\Re z < \Im z \le \Re z\}$. For every $z \ne 0$ there is a unique $j \in \{0,1,2,3\}$ with $i^j z \in P$. Then $|z| \le 2\Re(i^j z) \le 2|z|$. Thus we can split the sum into 4 parts corresponding to $j \in \{0,1,2,3\}$, where

$i^j y^*(x_k) \in P$. For each subsum we have

$$\sum_{\substack{m\ge k\ge n\\ i^j\,y^*(x_k)\in P}} |y^*(x_k)| = \sum_{\substack{m\ge k\ge n\\ i^j\,y^*(x_k)\in P}} 2\Re(i^j\,y^*(x_k)) = 2\,\Re\Big(i^j\,y^*\Big(\sum_{\substack{m\ge k\ge n\\ i^j\,y^*(x_k)\in P}} x_k\Big)\Big)$$
$$\le 2\,\Big|y^*\Big(\sum_{\substack{m\ge k\ge n\\ i^j\,y^*(x_k)\in P}} x_k\Big)\Big| \le 2\,p\Big(\sum_{\substack{m\ge k\ge n\\ i^j\,y^*(x_k)\in P}} x_k\Big) \le 2\,\varepsilon$$

Thus we have $\tilde p(x - x^n) \le 8\,\varepsilon$.

Since $\ell^1\langle F\rangle$ is complete for complete F by 3.49, the result follows. □

Operator ideals

In order to give several equivalent descriptions of nuclear spaces in terms of the connecting morphism in their projective representation, we introduce the ideals of approximable, of nuclear, and of summing operators between Banach spaces and prove the most relevant relations between them.
In this section **all lcs are assumed to be Banach spaces**!

3.51 Definition. Several operator ideals.
For $1 \le p < \infty$ define the following classes of operators between Banach spaces:

- $\mathcal{A}_p$, the class of p-APPROXIMABLE OPERATORS (See [**Kri07a**, Def. before 5.26 p.128]), i.e. those for which the approximation numbers $(a_n(T)) \in \ell^p$, see 3.11. WARNING: This class is denoted $\mathcal{S}_p$ (for Schatten-class) in [**Jar81**, 19.8 p.440] and [**MV92**, 16.6 p.143]!
- $\mathcal{N}_p$, the class of p-NUCLEAR OPERATORS, i.e. those which have a representation of the form $T = \sum_{n=0}^\infty x_n^* \otimes y_n$ with $(x_n^*) \in \ell^p\{E^*\}$ and $(y_n) \in \ell^q\{F\}$, where $\frac{1}{p} + \frac{1}{q} = 1$, see [**Kri07a**, 5.9].
- $\mathcal{S}_p$, the class of p-SUMMING OPERATORS, i.e. those with $T_*(\ell^p[E]) \subseteq \ell^p\{F\}$, see [**Kri07a**, 5.18]. These classes are denotes $\mathcal{P}_p$ in [**Jar81**, 19.5 p.428].

In the case $p = 1$ we suppress the "1-" from these definitions. In particular, T is a nuclear operator, iff there exists $a_j \in E^*$ and $b_j \in F$ with $\sum_j \|a_j\|\,\|b_j\| < \infty$ and

$$T(x) = \sum_{j\in\mathbb{N}} a_j(x)\,b_j \text{ for all } x.$$

All these classes are operator ideals, since for $A, B \in L$ they are closed under $T \mapsto A \circ T \circ B$. For approximable this follows from $a_{n+m}(R \circ S) \le a_n(R)\cdot a_m(S)$, see 3.53 below, for the others from $S_*(\ell^p\{E\}) \subseteq \ell^p(\{F\})$ and $S_*(\ell^p[E]) \subseteq \ell^p[F]$ (since $\ell^q(\mathbb{N}, \_)$ and $L(\ell, \_)$ are obviously functorial).

3.52 Lemma (See [**Jar81**, 17.3.3 p.377])**.**
The space $\mathcal{N}_1(E,F)$ of nuclear operators is the image of $E^*\hat\otimes_\pi F$ in $L(E,F)$.

Proof. By 3.40 the elements of $E^*\hat\otimes_\pi F$ are those of the from $\sum_n \lambda_n x_n^* \otimes y_n$ with x^*, y bounded sequences and $\lambda \in \ell^1$. □

3.53 Proposition (See [**Kri07a**, 5.29], [**Jar81**, 19.10.1 p.445])**.**
Let $0 < p, q, r < \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.
Then $\mathcal{A}_q \circ \mathcal{A}_p \subseteq \mathcal{A}_r$. In particular, we will use $\mathcal{A}_2 \circ \mathcal{A}_2 \subseteq \mathcal{A}_1$ and $(\mathcal{A}_1)^n \subseteq \mathcal{A}_{1/n}$.

Proof. We have $a_{n+m}(S\circ T)\le a_n(S)\,a_m(T)$:
In fact, let $R := S_0\circ T + (S-S_0)\circ T_0$ for n- resp. m-dimensional S_0 resp. T_0, then $a_{n+m}(S\circ T)\le \|S\circ T - R\| = \|(S-S_0)\circ(T-T_0)\|\le \|S-S_0\|\cdot\|T-T_0\|$.

Using the Hölder inequality for $\frac{r}{p}+\frac{r}{q}=1$ we obtain:

$$\Big(\sum_n a_n(S\circ T)^r\Big)^{1/r} \le 2^{1/r}\Big(\sum_n a_{2n}(S\circ T)^r\Big)^{1/r} \le 2^{1/r}\Big(\sum_n a_n(S)^r\cdot a_n(T)^r\Big)^{1/r}$$
$$\le 2^{1/r}\Big(\sum_n a_n(S)^p\Big)^{1/p}\cdot\Big(\sum_n a_n(T)^q\Big)^{1/q} \qquad \square$$

3.54 Proposition (See [**Kri07a**, 5.30], [**Jar81**, 20.2.3 p.454])**.**
An linear operator $T : E\to F$ between Hilbert spaces is p-approximable provided $(\langle Te_n, f_n\rangle)_n\in\ell^p$ for all orthonormal sequences e_n and f_n.

It can be shown that the converse is valid as well, see [**Jar81**, 20.2.3 p.454].

Proof. By 3.9 we conclude that T is compact and hence admits by 3.8 a representation $Tx=\sum_n\lambda_n\langle e_n,x\rangle f_n$ with $\lambda_n\to 0$ and orthonormal sequences e_n and f_n. Since $\lambda_n=\langle Te_n,f_n\rangle$ we have that $(\lambda_n)_n\in\ell^p$. By applying a permutation and putting signs to f_n we may assume that $0<\lambda_{n+1}\le\lambda_n$ Let $T_n(x):=\sum_{k<n}\lambda_k\langle e_k,x\rangle f_k$. Then

$$a_n(T)\le\|T-T_n\| = \sup\Big\{\Big\|\sum_{k\ge n}\lambda_k\langle e_k,x\rangle f_k\Big\| : \|x\|\le 1\Big\}$$
$$=\sup\Big\{\Big(\sum_{k\ge n}\lambda_k^2|\langle e_k,x\rangle|^2\Big)^{1/2} : \|x\|\le 1\Big\}\le\lambda_n,$$

hence $T\in\mathcal{A}_p$. $\square$

3.55 Auerbach's Lemma (See [**Kri07a**, 5.26], [**Jar81**, 14.1.7 p.291])**.**
Let E be a finite dimensional Banach space. Then there are unit vectors $x_i\in E$ and $x_i^*\in E^*$ with $x_i^*(x_j)=\delta_{i,j}$ for $1\le i,j\le\dim E$.

Proof. Let $e_1,\dots,e_n$ be an algebraic basis of E. For the weakly compact unit ball K of E^* we consider the continuous map $f: K^n\to\mathbb{K}$, $(x_1^*,\dots,x_n^*)\mapsto|\det(x_j^*(e_i))|$. Let $(x_1^*,\dots,x_n^*)$ be a point where it attains its maximum. Since the e_i are linearly independent this maximum is positive. Hence there is a unique solution with $x_j\in E$ of the equations

$$\sum_j x_j^*(e_i)x_j = e_i \text{ for } 1\le i\le n.$$

Applying any x_k^* to this equation, yields the equations

$$\sum_j x_j^*(e_i)\,x_k^*(x_j) = x_k^*(e_i) \text{ for } 1\le i\le n.$$

whose unique solution is $x_j^*(x_i)=\delta_{i,j}$.

$$f(x_1^*,\dots,x_n^*)\cdot\big|\det(y_j^*(x_i))\big| = \Big|\det(x_j^*(e_i))\cdot\det(y_j^*(x_i))\Big|$$
$$=\Big|\det\Big(\sum_k x_k^*(e_i)\,y_j^*(x_k)\Big)\Big| = \big|\det(y_j^*(e_i))\big|$$
$$= f(y_1^*,\dots,y_n^*)\le f(x_1^*,\dots,x_n^*) \text{ for all } y_i^*\in K.$$

Thus $|\det(y_j^*(x_i))| \le 1$. Choosing $y_j^* = x_j^*$ for all $j \ne k$ shows that $|y_k^*(x_k)| \le 1$ and hence $\|x_k\| \le 1$. From $1 = x_j^*(x_j) \le \|x_j^*\| \, \|x_j\|$ we conclude that $\|x_j\| = 1 = \|x_j^*\|$. $\square$

3.56 Lemma. (See [**Kri07a**, 5.27], [**Jar81**, 19.8.4 p.441]).
Let $T \in \mathcal{L}(E,F)$ be such that $\dim T(E) = k < \infty$. Then T can be written as $T = \sum_{j=1}^k \lambda_j \, x_j^* \otimes y_j$ with $\|x_j^*\| \le 1$ and $\|y_j\| \le 1$ and $0 < \lambda_j \le \|T\|$.

Proof. We may assume that T is onto. By 3.55 we have a biorthogonal sequence y_j and y_j^* for F. Let $\lambda_j := \|T^* y_j^*\|$. Then $0 < \lambda_j \le \|T^*\| = \|T\|$ and $x_j^* := \frac{1}{\lambda_j} T^* y_j^* \in oE^*$. So we have $Tx = \sum_j y_j^*(Tx)\, y_j = \sum_j \lambda_j \, x_j^*(x)\, y_j$. $\square$

3.57 Corollary (See [**Kri07a**, 5.28], [**Jar81**, 19.8.6 p.442]).
We have $\mathcal{A}_1 \subseteq \mathcal{N}_1$.

Proof. See [**Jar81**, 19.8.5 p.442]. Let $T \in \mathcal{A}_1(E,F)$. We have to show that it can be written as $T = \sum_n \lambda_n \, x_n^* \otimes y_n$ with $x_n^* \in oE^*$, $y_n \in oF$ and $\lambda \in \ell^1$.
Let $\varepsilon > 0$. Choose T_n with $\dim T_n(E) \le 2^n$ and $\|T - T_n\| \le (1+\varepsilon)\, a_{2^n}(T)$. Let $D_n := T_{n+1} - T_n$. Then $d_n := \dim D_n(E) \le 3 \cdot 2^n$ and since $a_n(T) \to 0$ we have $\|T - T_n\| \to 0$, hence $T = \sum_{n=0}^\infty D_n$. By 3.56 we have $T = \sum_{n=0}^\infty \sum_{j=1}^{d_n} \lambda_{n,j} \, x_{n,j}^* \otimes y_{n,j}$, with $x_{n,j}^* \in oE^*$, $y_{n,j} \in oF$ and $0 \le \lambda_{n,j} \le \|D_n\|$. We estimate as follows

$$\begin{aligned}
\sum_n \sum_{j=1}^{d_n} \lambda_{n,j} &\le \sum_n d_n \, \|D_n\| \le 3 \sum_n 2^n (\|T_{n+1} - T\| + \|T_n - T\|) \\
&\le 3 \cdot \sum_n 2^n (1+\varepsilon) \big(a_{2^{n+1}}(T) + a_{2^n}(T)\big) \\
&\le 3 \cdot \sum_n 2^{n+1} (1+\varepsilon)\, a_{2^n}(T) \le 2^2\, 3\, (1+\varepsilon) \sum_n 2^{n-1}\, a_{2^n}(T) \\
&\le 2^2\, 3\, (1+\varepsilon) \sum_n a_n(T) \qquad (\text{since } a_n(T) \text{ is decreasing})
\end{aligned}$$

to conclude that $(\lambda_{n,j})_{n,j} \in \ell^1$. $\square$

3.58 Lemma (See [**MV92**, 28.14 p.334], [**Jar81**, 21.6.1 p.496]).
Diagonal operators on ℓ^p (for $1 \le p < \infty$) are nuclear iff they have ℓ^1 coefficients.

Cf. 3.10 and 3.11.

Proof. ($\Leftarrow$) obvious, since $D = \sum_n d_n \operatorname{ev}_n \otimes e_n$ with $\|e_n\|_{\ell^p} = 1 = \|\operatorname{ev}_n\|_{\ell^q}$
($\Rightarrow$) Let $a^{(n)} \in \ell^q = (\ell^p)^*$, $b^{(n)} \in \ell^p$ with $\sum_n \|a^{(n)}\|_{\ell^q} \cdot \|b^{(n)}\|_{\ell^p} < \infty$ and $D(x) = \sum_n a^{(n)}(x)\, b^{(n)}$ for all $x \in \ell^p$. With $x := e_k$ we get $d_k = D(e_k)_k = \sum_n a_k^{(n)} \cdot b_k^{(n)}$ and hence $\|d\|_{\ell^1} = \sum_k |d_k| \le \sum_{k,n} |a_k^{(n)}| \cdot |b_k^{(n)}| \le \sum_n \|a^{(n)}\|_{\ell^q} \cdot \|b^{(n)}\|_{\ell^p} < \infty$ by the Hölder-inequality. $\square$

We will apply this to the connecting mappings $\iota_n^k : \lambda_k \to \lambda_n$ for the Köthe-sequence spaces $\lambda = \lambda^p(A)$ with $1 \le p \le \infty$. Only the case $p = \infty$ needs special attention (see 1.13): Let the diagonal operator $D := \iota_n^k : \lambda_k \to \lambda_n$ be nuclear. Then $D|_{c_0} : c_0 \hookrightarrow \lambda_k \to \lambda_n \hookrightarrow \ell^\infty$ is nuclear, so $a^{(n)} \in \ell^1 = (c_0)^*$ and $b^{(n)} \in \ell^\infty$ and hence the same proof as above for $p = 1$ shows that the diagonal d of D is absolutely summable.

3.59 Proposition. Factorization property of $\mathcal{N}$
(See [**Kri07a**, 5.6 p.119], [**Jar81**, 17.3.2 p.377]).

A map $T : E \to F$ between Banach spaces is nuclear iff there are continuous linear operators $S : E \to \ell^\infty$ and $R : \ell^1 \to F$ such that T factors as diagonal operator $D : \ell^\infty \to \ell^1$ with diagonal $d \in \ell^1$, i.e.

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, dotted, "S"'] & F \\ \ell^\infty \arrow[r, dotted, "D"'] & \ell^1 \arrow[u, dotted, "R"'] \end{tikzcd}$$

Proof. ($\Rightarrow$) Let T be represented by $\sum_k d_k\, x_k^* \otimes y_k$ with $\|x_k^*\|_{E^*} \le 1$, $\|y_k\|_F \le 1$ and $d \in \ell^1$. Then $S(x) := (x_k^*(x))_k$ and $R((\mu_k)_k) := \sum_k \mu_k\, y_k$ define linear operators $S : E \to \ell^\infty$ and $R : \ell^1 \to F$ of norm ≤ 1 and $T = R \circ D \circ S$, where $D : \ell^\infty \to \ell^1$ denotes the diagonal operator, with diagonal $(d_k)_k$.

($\Leftarrow$) Since the nuclear operators form an ideal, it is enough to show that such diagonal operators $D : (\mu_k)_k \mapsto (d_k \mu_k)_k$, $D : \ell^\infty \to \ell^1$ are nuclear, which is clear since they can be represented by $\sum_k d_k x_k^* \otimes y_k$, where $x_k^* := e_k \in \ell^1 \subseteq (\ell^\infty)^*$ and $y_k := e_k \in \ell^1$. □

3.60 Lemma. $\mathcal{N} \subseteq \mathcal{K}$ (See [**Kri07a**, 5.7], [**Jar81**, 17.3.4 p.379])**.**
Every nuclear operator is compact.

Proof. Let T be a nuclear mapping. Since the compact mappings form an ideal, we may assume by 3.59 that T is a diagonal-operator $\ell^\infty \to \ell^1$ with absolutely summable diagonal $(\lambda_k)_k$. Such an operator is compact, since the finite sub-sums $\sum_{k \le n} \lambda_k\, e_k \otimes e_k$ define finite dimensional operators, which converge to T uniformly on the unit-ball of ℓ^∞. □

3.61 Lemma (See [**Kri07a**, 5.19], [**Jar81**, 19.5.1 p.428])**.**
Every p-summing operator induces a continuous linear map from $\ell^p[E] \to \ell^p\{F\}$. Thus we may consider the space $\mathcal{S}_p(E, F)$ of p-summing operators as normed subspace of the space $L(\ell^p[E], \ell^p\{F\})$.

Here we consider the space $\ell^p\{F\}$ supplied with the norm

$$\|(y_k)_k\|_\pi := \Big(\sum_k \|y_k\|_F^p\Big)^{1/p}.$$

As in 3.42 one can show that $\ell^p\{F\}$ is complete (see [**Jar81**, 19.4.1 p.426]). For $p > 1$ it is however not isomorphic to $\ell^p \hat{\otimes}_\pi F$. Otherwise we would obtain for $E = \ell^p$, that $\ell^p \hat{\otimes}_\pi \ell^p = \ell^p\{\ell^p\} = \ell^p(\mathbb{N} \times \mathbb{N})$, which is not the case..

On $\ell^p[E]$ we consider the operator norm of $L(\ell^q, E)$ (see 3.43):

$$\|(x_k)_k\|_\varepsilon := \sup\Big\{ \Big(\sum_k |x^*(x_k)|^p\Big)^{1/p} : x^* \in E^*, \|x^*\| \le 1 \Big\}.$$

It is obvious, that the inclusion $\ell^p\{E\} \to \ell^p[E]$ is a contraction (i.e. has norm ≤ 1).

Proof. Let $T : E \to F$ be a p-summing operator. We will apply the closed graph theorem to $T_* : \ell^p[E] \to \ell^p\{E\}$, $(x_n)_{n=1}^\infty \mapsto (T(x_n))_{n=1}^\infty$, so consider $x^{(k)} \to x$ in $\ell^p[E]$ with $T_*(x^{(k)}) \to y$ in $\ell^p\{F\}$. Since obviously $\|T_*(z)\|_\varepsilon \le \|T\| \cdot \|z\|_\varepsilon$ with respect to the operator norms $\|\text{-}\|_\varepsilon$, we get $\|y - T_*(x)\|_\varepsilon \le \|y - T_*(x^{(k)})\|_\varepsilon + \|T_*(x^{(k)} - x)\|_\varepsilon \le \|y - T_*(x^{(k)})\|_\pi + \|T\|\, \|x^{(k)} - x\|_\varepsilon \to 0$, and hence $T_*(x) = y$. □

3.62 Corollary (See [**Kri07a**, 5.20], [**Jar81**, 19.5.2 p.428]).
An operator $T : E \to F$ is p-summing iff there exists a $R > 0$ such that

$$\|(Tx^{(k)})_k\|_\pi := \Big(\sum_k \|Tx^{(k)}\|^p\Big)^{1/p} \le R\cdot \sup_{\|x^*\|\le 1}\Big(\sum_k |x^*(x^{(k)})|^p\Big)^{1/p} =: R\cdot\|(x^{(k)})_k\|_\varepsilon.$$

for all finite sequences $(x^{(k)})_k$. The smallest such R is the norm of $T_* : \ell^p[E] \to \ell^p\{F\}$, and is also denoted $\|T\|_{\mathcal{S}_p}$. Consequently, $\mathcal{N}_1 \subseteq \mathcal{S}_1$.

Proof.
($\Rightarrow$) By 3.61 we have that T_* is continuous, and hence the required property holds with $R := \|T_*\|$ and all (even the infinite) sequences in $\ell^p[E]$.

($\Leftarrow$) For $x = (x^{(k)})_k \in \ell^p[E]$ we have $\|(Tx^{(k)})_k\|_\pi = \sup_m\big(\sum_{k=1}^m \|Tx^{(k)}\|^p\big)^{1/p} \le R\cdot\|(x^{(k)})_{k\le m}\|_\varepsilon \le R\cdot\|(x^{(k)})_k\|_\varepsilon < \infty$ and hence $(Tx^{(k)})_k \in \ell^p\{F\}$.

($\mathcal{N}_1 \subseteq \mathcal{S}_1$) by 3.59, since any diagonal operator $D : \ell^\infty \to \ell^1$ with diagonal $d \in \ell^1$ is 1-summing:

$$\begin{aligned}\sum_{k=1}^m \|Dx^{(k)}\|_{\ell^1} &= \sum_k\sum_j |d_j\, x_j^{(k)}| = \sum_j |d_j| \sum_k |x_j^{(k)}| \le \|d\|_{\ell^1} \sup_j \sum_k |\operatorname{ev}_j(x^{(k)})| \\ &\le \|d\|_{\ell^1} \sup\Big\{\sum_k |x^*(x^{(k)})| : x^* \in (\ell^\infty)^*, \|x^*\| \le 1\Big\}. \qquad\square\end{aligned}$$

3.63 Proposition (See [**Kri07a**, 5.21], [**Jar81**, 19.5.4 p.430]).
For $p \le q$ we have $\mathcal{S}_p \subseteq \mathcal{S}_q$.

Also $\mathcal{N}_p \subseteq \mathcal{N}_q$ can be shown under the same assumption, see [**Jar81**, 19.7.5 p.437].

Proof. Let $T \in \mathcal{S}_p$ and let $r \ge 0$ be given by $\frac{1}{r} + \frac{1}{q} = \frac{1}{p}$. Let $\lambda_k := \|Tx_k\|^{q/r}$. Then $\|Tx_k\| = \lambda_k^{r/q}$ and hence $\|T(\lambda_k x_k)\|^p = \|\lambda_k T(x_k)\|^p = \lambda_k^p \cdot \|Tx_k\|^p = \|Tx_k\|^{p(\frac{q}{r}+1)} = \|Tx_k\|^q$ and so the Hölder's inequality (cf. the proof of 3.53) shows that

$$\begin{aligned}\Big(\sum_k \|Tx_k\|^q\Big)^{1/p} &= \Big(\sum_k \|T(\lambda_k x_k)\|^p\Big)^{1/p} = \big\|(T(\lambda_k x_k))_k\big\|_\pi \\ &\le \|T\|_{\mathcal{S}_p}\cdot\big\|(\lambda_k x_k)_k\big\|_\varepsilon = \|T\|_{\mathcal{S}_p}\cdot \sup_{\|x^*\|\le 1}\Big(\sum_k |x^*(\lambda_k^p x_k)|^p\Big)^{1/p} \\ &\le \|T\|_{\mathcal{S}_p}\cdot\Big(\sum_k \lambda_k^r\Big)^{1/r}\cdot \sup_{\|x^*\|\le 1}\Big(\sum_k |x^*(x_k)|^q\Big)^{1/q} \\ &\le \|T\|_{\mathcal{S}_p}\cdot\Big(\sum_k \|Tx_k\|^q\Big)^{1/r}\cdot \sup_{\|x^*\|\le 1}\Big(\sum_k |x^*(x_k)|^q\Big)^{1/q}\end{aligned}$$

Dividing by $\Big(\sum_k \|Tx_k\|^q\Big)^{1/r} = \Big(\sum_k \|Tx_k\|^q\Big)^{1/p-1/q}$ gives

$$\Big(\sum_k \|Tx_k\|^q\Big)^{1/q} \le \|T\|_{\mathcal{S}_p}\cdot \sup_{\|x^*\|\le 1}\Big(|x^*(x_k)|^q\Big)^{1/q}.$$

Thus $T \in \mathcal{S}_q$ by 3.62. $\square$

3.64 Lemma. Summing via measures
(See [**Kri07a**, 5.22], [**Jar81**, 19.6.1 p.431]).

An operator T is p-summing iff there exists some PROBABILITY MEASURE *μ on the compact unit ball oE^* and an $M > 0$ such that*

$$\|Tx\| \leq M \cdot \Big(\int_{oE^*} |x^*(x)|^p \, d\mu(x^*)\Big)^{1/p}.$$

Proof. Note that the right hand side is nothing else but $M \cdot \|\delta(x)\|_p$, where $\delta : E \rightarrowtail C(oE^*)$.

($\Leftarrow$) If μ is a probability measure (i.e. $\mu(oE^*) = 1$) with that property, then

$$\sum_k \|Tx_k\|^p \leq M^p \int_{oE^*} \sum_k |x^*(x_k)|^p \, d\mu(x^*) \leq M^p \cdot \sup\Big\{\sum_k |x^*(x_k)|^p : x^* \in oE^*\Big\}.$$

So $T \in \mathcal{S}_p$ by $\boxed{3.62}$.

($\Rightarrow$) Let $T \in \mathcal{S}_p(E, F)$. For every finite sequence $x = (x_1, \dots, x_m)$ in E let $f_x \in C(oE^*)$ be defined by

$$f_x(x^*) := \|T\|^p_{\mathcal{S}_p} \cdot \sum_i |x^*(x_i)|^p - \sum_i \|Tx_i\|^p = \sum_i \Big(\|T\|^p_{\mathcal{S}_p} \cdot |x^*(x_i)|^p - \|Tx_i\|^p\Big).$$

The set $B := \{f_x : x \in E^{(\mathbb{N})}\}$ is convex in $C(oE^*)$. In fact let x and y be two finite sequences in E and $\lambda + \mu = 1$ with $\lambda \geq 0$ and $\mu \geq 0$. Let z be the sequence obtained by appending $\mu^{1/p} y$ to $\lambda^{1/p} x$. Then

$$\begin{aligned}(\lambda f_x + \mu f_y)(x^*) &= \sum_i \lambda\Big(\|T\|^p_{\mathcal{S}_p} |x^*(x_i)|^p - \|Tx_i\|^p\Big) + \\ &\qquad + \sum_j \mu\Big(\|T\|^p_{\mathcal{S}_p} |x^*(y_j)|^p - \|Ty_j\|^p\Big) \\ &= \sum_i (\|T\|^p_{\mathcal{S}_p}) |x^*(\lambda^{1/p} x_i)|^p - \|T(\lambda^{1/p} x_i)\|^p + \\ &\qquad + \sum_j (\|T\|^p_{\mathcal{S}_p}) |x^*(\mu^{1/p} y_j)|^p - \|T(\mu^{1/p} y_j)\|^p \\ &= \sum_k (\|T\|^p_{\mathcal{S}_p}) |x^*(z_k)|^p - \|T(z_k)\|^p = f_z(x^*).\end{aligned}$$

By $\boxed{3.62}$ we have that $\sup_{x^* \in oE^*} f_x(x^*) \geq 0$. Thus the open set $A := \{f \in C(oE^*) : \sup_{x^* \in oE^*} f(x) < 0\}$ is disjoint from B. So by the consequence [**Kri07b**, 7.2.1] of Hahn-Banach there exists a regular Borel measure μ on oE^* and a constant α such that $\langle \mu, f\rangle < \alpha \leq \langle \mu, g\rangle$ for all $f \in A$ and $g \in B$. Since $0 \in B$ we have $\alpha \leq 0$. Since A contains the constant negative functions we have $\alpha = 0$ and $\mu(oE^*) > 0$. Without loss of generality we may assume $\|\mu\| = 1$. Hence for every $x \in E$ we have

$$0 \leq \langle \mu, f_x\rangle = \int_{oE^*} \Big(\|T\|^p_{\mathcal{S}_p} |x^*(x)|^p - \|Tx\|^p\Big) \, d\mu(x^*)$$

and thus $\|Tx\|^p \leq \|T\|^p_{\mathcal{S}_p} \cdot \int_{oE^*} |x^*(x)|^p \, d\mu(x^*)$. ☐

3.65 Theorem. Factorization of absolutely 2-summing operators
(See [**Kri07a**, 5.24], [**Jar81**, 19.6.4 p.433]).
The operators T in $\mathcal{S}_2$ are characterized by the existence of a compact space K and a measure μ on K such that we have the following factorization:

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, dotted] & F \\ C(K) \arrow[r, hook, "i"] & \mathcal{L}^2(\mu) \arrow[u, dotted] \end{tikzcd}$$

Proof. ($\Leftarrow$) It is enough to show that the canonical mapping $\iota : C(K) \to \mathcal{L}^2(\mu)$ is absolutely 2-summing. So let δ_x be the point measure at x. Then for finitely many $f_k \in C(K)$ we have

$$\begin{aligned}\sum_k \|\iota(f_k)\|^2_{\ell^2} &= \int_K \sum_k |f_k(x)|^2 \, d\mu(x) = \int_K \sum_k |\delta_x(f_k)|^2 \, d\mu(x) \\ &\le \mu(K) \cdot \sup\Big\{ \sum_k |\nu(f_k)|^2 : \nu \in C(K)^*, \|\nu\| \le 1 \Big\},\end{aligned}$$

hence the natural mapping ι belongs to $\mathcal{S}_2$ by $\boxed{3.62}$.

($\Rightarrow$) By $\boxed{3.64}$ there is some probability measure $\mu \in \mathcal{M}(oE^*)$ such that

$$\|Tx\| \le M \cdot \Big(\int_{oE^*} |x^*(x)|^2 \, d\mu(x^*) \Big)^{1/2}.$$

The map $\delta : E \to C(oE^*)$, $x \mapsto \mathrm{ev}_x$, is isometric. Now consider the diagram

$$\begin{tikzcd}
E \arrow[rr, "T"] \arrow[dr, dotted, "S"'] \arrow[dd, hook, "\delta"'] & & F \\
& H \arrow[r, "1"] \arrow[ur, dotted, "R"] \arrow[dr, hook] & H \arrow[u, dotted, "R"'] \\
C(oE^*) \arrow[rr, "\iota"] & & \mathcal{L}^2(\mu) \arrow[u, dotted, "P"']
\end{tikzcd}$$

where H denotes the closure of the image of $\iota \circ \delta$ in $\mathcal{L}^2(\mu)$. The operator T factorizes via a continuous linear operator $R : H \to F$, since $\|Tx\| \le M \cdot \|\iota(\delta(x))\|_{\ell^2}$ for some $M > 0$. Using the ortho-projection $P : \mathcal{L}^2(\mu) \to H$ we get the factorization $R \circ P \circ (\iota \circ \delta) = R \circ \iota \circ \delta = R \circ S = T$. $\square$

3.66 Proposition (See [**Kri07a**, 5.31], [**Jar81**, 20.5.1 p.467])**.**
For Hilbert spaces we have $\mathcal{S}_2 \subseteq \mathcal{A}_2$.

Proof.

For orthonormal families e_k and f_k and $T \in \mathcal{S}_2$ we have by $\boxed{3.61}$

$$\Big(\sum_k \|Te_k\|^2 \Big)^{1/2} \le \|T\|_{\mathcal{S}_2} \cdot \sup_{\|x\| \le 1} \Big(\sum_k |\langle x, e_k \rangle|^2 \Big)^{1/2} \le \|T\|_{\mathcal{S}_2}.$$

And by the Cauchy-Schwarz inequality $|\langle Te_k, f_k \rangle| \le \|Te_k\| \cdot \|f_k\| = \|Te_k\|$ we get $\sum_k |\langle Te_k, f_k \rangle|^2 \le \sum_k \|Te_k\|^2 < \|T\|^2_{\mathcal{S}_2} < \infty$, hence $T \in \mathcal{A}_2$ by $\boxed{3.54}$. $\square$

3.67 Overview. One has the following inclusions for $1 < p < q < \infty$:

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, hook, "\boxed{3.57}"] \arrow[d, hook, "\text{obvious}"'] & \mathcal{N}_1 \arrow[r, hook, "\boxed{3.62}"] \arrow[d, hook, "{[\text{Jar81, 19.7.5 p.437}]}"'] & \mathcal{S}_1 \arrow[d, hook, "\boxed{3.63}"] \\
\mathcal{A}_p \arrow[d, hook, "\text{obvious}"'] & \mathcal{N}_p \arrow[r, hook, "{[\text{Jar81, 19.7.8 p.438}]}"] \arrow[d, hook, "{[\text{Jar81, 19.7.5 p.437}]}"'] & \mathcal{S}_p \arrow[d, hook, "\boxed{3.63}"] \\
\mathcal{A}_q & \mathcal{N}_q \arrow[r, hook, "{[\text{Jar81, 19.7.8 p.438}]}"] & \mathcal{S}_q
\end{tikzcd}$$

For Hilbert spaces one has the following results for $1 < p < \infty$:

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, equal, "{[\text{Jar81}, 20.2.5 \text{ p.456}]}"] \arrow[d, hook] & \mathcal{N}_1 \arrow[r, hook] \arrow[d, hook] & \mathcal{S}_1 \arrow[d, equal, "{[\text{Jar81}, 20.5.1 \text{ p.467}]}"] \\
\mathcal{A}_2 \arrow[r, equal, "{[\text{Jar81}, 20.5.1 \text{ p.467}]}"] \arrow[d, hook] & \mathcal{N}_p \arrow[r, equal, "{[\text{Jar81}, 20.5.1 \text{ p.467}]}"] \arrow[d, equal, "{[\text{Jar81}, 20.5.1 \text{ p.467}]}"'] & \mathcal{S}_p \arrow[d, hook] \\
\mathcal{A}_\infty & \mathcal{N}_\infty \arrow[r, hook] & \mathcal{S}_\infty
\end{tikzcd}$$

Nuclear spaces

In this section we characterize nuclear spaces in several ways and we prove their inheritance properties We show that the nuclear (Fréchet) spaces are exactly the (closed) subspaces of products of (countable many) copies of s.

3.68 Definition.
A linear mapping $T : E \to F$ between lcs is called NUCLEAR OPERATOR (See [**Jar81**, 17.3 p.376], [**Kri07a**, 5.6]) iff there exist $\{a_n : n \in \mathbb{N}\} \subseteq E^*$ equicontinuous, B a Banach disk, $b_n \in B$, and $\lambda \in \ell^1$ with

$$Tx = \sum_{n=1}^{\infty} \lambda_n \, a_n(x) \, b_n \text{ for all } x \in E.$$

This is exactly the case, iff there is an absolutely convex 0-neighborhood $U \subseteq E$ and a Banach disk $B \subseteq F$, such that T factors over a nuclear mapping $\tilde{T} : \widehat{E_U} \to F_B$, i.e.

$$\begin{tikzcd}
E \arrow[r, "T"] \arrow[d] & F \\
\widetilde{E_U} \arrow[r, dotted, "\tilde{T}"'] & F_B \arrow[u]
\end{tikzcd}$$

The nuclear mappings form an ideal: For composition from the left side with some R replace b_n by $R(b_n)$, and from the right side replace a_n by $a_n \circ R = R^*(a_n)$ (Note that $a_n \in U^o \Rightarrow R^*(a_n) \in (R^{-1}(U))^o$).

3.69 Proposition (See [**Jar81**, 17.3.8 p.380])**.**
Let $T : E \to F$ be nuclear and G any lcs. Then $T \otimes G : E \otimes_\varepsilon G \to F \otimes_\pi G$ is continuous.

Note, that as for any bifunctor we denote with $T \otimes G$ the morphism $T \otimes \mathrm{id}_G$.

Proof. We may represent $T = \sum_n \lambda_n \, a_n \otimes b_n$ with $a_n \in U^o$ for some 0-nbhd. U and $b_n \in B$, a Banach-disk. Let $V \subseteq F$ and $W \subseteq G$ be 0-nbhds and let

$\rho := \sup\{q_V(b) : b \in B\}$. For $w = \sum_{j=1}^k x_j \otimes z_j \in E \otimes G$ we get

$$\begin{aligned}(T \otimes G)(w) &= \sum_{j=1}^{k} T(x_j) \otimes z_j = \sum_{j,n} \lambda_n\, a_n(x_j)\, b_n \otimes z_j \\ &= \sum_n \lambda_n\, b_n \otimes \sum_j a_n(x_j)\, z_j \text{ and hence}\end{aligned}$$

$$\begin{aligned}\pi_{V,W}\Big((T \otimes G)(w)\Big) &= \pi_{V,W}\Big(\sum_{n=1}^{\infty} \lambda_n\, b_n \otimes \Big(\sum_{j=1}^{k} a_n(x_j) z_j\Big)\Big) \\ &\le \sum_{n=1}^{\infty} |\lambda_n|\, q_V(b_n)\, \sup\Big\{\Big|\sum_{j=1}^{k} a_n(x_j) z^*(z_j)\Big| : z^* \in W^o\Big\} \\ &\le \rho \sum_{n=1}^{\infty} |\lambda_n|\, \sup\Big\{\Big|\sum_{j=1}^{k} x^*(x_j) z^*(z_j)\Big| : x^* \in U^o, z^* \in W^o\Big\} \\ &\le \rho\, \|\lambda\|_{\ell^1}\, \varepsilon_{U,W}(w). \quad \square\end{aligned}$$

3.70 Theorem. Characterizing nuclear spaces in multiple ways (See [**Kri07a**, 6.17], [**Jar81**, 21.2.1 p.482]).

Let $1 \le p < \infty$. Then

1. *E is nuclear;*
2. $\Leftrightarrow$ *$E \otimes_\pi F = E \otimes_\varepsilon F$ for every Banach space F;*
3. $\Leftrightarrow$ *$E \otimes_\pi \ell^1 = E \otimes_\varepsilon \ell^1$;*
4. $\Leftrightarrow$ *$\ell^1\{E\} = \ell^1\langle E\rangle$ topologically;*
5. $\Leftrightarrow$ *$\ell^1\{E\} = \ell^1[E]$ topologically;*
6. $\Leftrightarrow$ *The connecting maps of the projective representation can be chosen absolutely summing (or $\mathcal{S}_p$);*
7. $\Leftrightarrow$ *The connecting maps of the projective representation can be chosen nuclear (or $\mathcal{N}_p$);*
8. $\Leftrightarrow$ *The connecting maps of the projective representation can be chosen 1-approximable (or $\mathcal{A}_p$);*
9. $\Leftrightarrow$ *Every continuous linear map into a Banach space is nuclear.*

Proof. We give the proof for $1 \le p \le 2$ only. For the general case one needs in addition that $\mathcal{S}_p \circ \mathcal{S}_q \subseteq \mathcal{S}_r$ (see [**Jar81**, 19.10.3 p.446]) and $\mathcal{N}_1 \subseteq \mathcal{N}_p \subseteq \mathcal{S}_p$ (see [**Jar81**, 19.7.5 p.437] and [**Jar81**, 19.7.8 p.438]).

(1 $\Rightarrow$ 2 $\Rightarrow$ 3) and (5 $\Rightarrow$ 4 $\Rightarrow$ 3) are obvious by 3.42 and 3.50.

(3 $\Rightarrow$ 6) From (3) we obtain that $\ell^1\langle\tilde{E}\rangle \cong \ell^1\{\tilde{E}\}$. Thus for every $U \subseteq E$ there exists a $V \subseteq E$ and a $\delta > 0$ such that $\pi_U \le \delta\, \varepsilon_V$, where

$$\pi_U\big((x_k)_k\big) := \sum_k p_U(x_k)$$

is the semi-norm associated to U on $\ell^1 \hat{\otimes}_\pi E \cong \ell^1\{E\}$, see 3.42, and where

$$\varepsilon_V((x_k)_k) := \sup\Big\{\sum_k |y^*(x_k)| : y^* \in V^o\Big\}$$

is the semi-norm associated to U on $L(c_0, F) = \ell^1[E]$ and hence on the subspace $\ell^1\langle E\rangle \cong \ell^1 \hat{\otimes}_\varepsilon E$, see 3.50. From this it follows by 3.62 that the connecting map is absolutely summing.

($\boxed{6}\Rightarrow\boxed{5}$) For every U we can find by assumption a V such that the connecting map $\iota_U^V : E_V \to E_U$ is absolutely summing. Hence if $(x_k)_k \in \ell^1[E]$, then the images are in $\ell^1[\tilde{E}_V]$ and hence in $\ell^1\{\tilde{E}_U\}$. Moreover, by $\boxed{3.62}$

$$\pi_U\big((x_k)_k\big) := \sum_{k=1}^n p_U(x_k) = \sum_{k=1}^n \big\|\iota_U^V(\iota_V(x_k))\big\|_U \leq$$
$$\leq \|\iota_U^V\|_{\mathcal{S}_1} \cdot \sup\Big\{\sum_k |x^*(x_k)| : x^* \in V^o\Big\} =: \|\iota_U^V\|_{\mathcal{S}_1} \cdot \varepsilon_V((x_k)_k),$$

Since U was arbitrary we have ($\boxed{5}$).

($\boxed{7}\Rightarrow\boxed{1}$) By assumption for every U there exists a U' such that the connecting map $\iota_U^{U'}$ is nuclear. By $\boxed{3.69}$ we have that $\iota_U^{U'} \otimes \tilde{F}_V : \tilde{E}_{U'} \otimes_\varepsilon \tilde{F}_V \to \tilde{E}_U \otimes_\pi \tilde{F}_V$ is continuous. Thus $\pi_{U,V} \leq c \cdot \varepsilon_{U',V}$ for some $c > 0$, i.e. $E \otimes_\varepsilon F = E \otimes_\pi F$. Recall the corresponding norms on $E_U \otimes_\pi E_V$ and on $E_U \otimes_\varepsilon E_V$:

$$\pi_{U,V}(z) := \inf\Big\{\sum_k p_U(x_k)\, p_V(y_k) : z = \sum_k x_k \otimes y_k\Big\} \text{ and}$$
$$\varepsilon_{U',V}\Big(\sum_k x_k \otimes y_k\Big) := \sup\Big\{\Big|\sum_k x^*(x_k)\, y^*(y_k)\Big| : x^* \in (U')^o, y^* \in V^o\Big\}$$

($\boxed{6} \Leftrightarrow \boxed{7} \Leftrightarrow \boxed{8}$) Now let us show that for all mentioned ideals it is the same to assume that the connecting mappings belong to them.
In fact, we have $\mathcal{A}_1 \subseteq \mathcal{N}_1 \subseteq \mathcal{S}_1 \subseteq \mathcal{S}_p \subseteq \mathcal{S}_2$ for $1 \leq p \leq 2$ by $\boxed{3.57}$, $\boxed{3.62}$,$\boxed{3.63}$. The composite of three $\mathcal{S}_2$ maps belongs to $\mathcal{A}_2$, since the following diagram shows that it factors over a map between Hilbert spaces (see $\boxed{3.65}$) of class $\mathcal{S}_2 \subseteq \mathcal{A}_2$ (by $\boxed{3.66}$):

$$\begin{tikzcd}
E_3 \arrow[rr, "\mathcal{S}_2"] \arrow[dr] & & E_2 \arrow[r, "\mathcal{S}_2"] & E_1 \arrow[rr, "\mathcal{S}_2"] \arrow[dr] & & E_0 \\
& \mathcal{L}^2(\mu_3) \arrow[ur] \arrow[rrr, dotted, "\mathcal{S}_2\subseteq\mathcal{A}_2"] & & & \mathcal{L}^2(\mu_1) \arrow[ur] &
\end{tikzcd}$$

Since $(\mathcal{A}_2)^2 \subseteq \mathcal{A}_1$ by $\boxed{3.53}$ we have that $(\mathcal{S}_2)^6 \subseteq \mathcal{A}_1$. Now choose for a given seminorm p successively $p_6 \geq p_5 \geq \cdots \geq p_1 \geq p$ such that the connecting maps all belong to $\mathcal{S}_2$. Then the connecting mapping $\widetilde{E_{p_6}} \to \widetilde{E_p}$ belongs to $\mathcal{A}_1$.

($\boxed{7}\Leftrightarrow\boxed{9}$) Recall that a map $T : E \to F$ with values in a Banach space is called nuclear (see $\boxed{3.68}$ and $\boxed{3.59}$), iff it factors over a nuclear map $T_1 : E_1 \to F$ on some Banach space E_1. In fact, for E_1 we may choose $\widetilde{E_U}$ for some 0-neighborhood U. Now we can proceed as for the corresponding result $\boxed{3.30}$ for compact mappings and Schwartz spaces.

$$\begin{tikzcd}
E \arrow[r, "T"] \arrow[d, "\iota_V"'] \arrow[dr, "\iota_U"] & F \\
\tilde{E}_V \arrow[r, "\iota_U^V"'] & \tilde{E}_U \arrow[u, "\tilde{T}"']
\end{tikzcd}$$

□

3.71 Characterizing nuclear (F) spaces via summable sequences
(See [**Kri07a**, 6.18], [**Jar81**, 21.2.4 p.483]).
A Fréchet space is nuclear iff $\ell^1\{E\} = \ell^1[E]$ (or $\ell^1\{E\} = \ell^1\langle E\rangle$) holds algebraically.

Proof. Since $\ell^1\{E\}$ and $\ell^1\langle E\rangle$ are Fréchet spaces it follows from the closed graph theorem that the identity is a homeomorphism. □

3.72 Corollary. Nuclear spaces have a basis of Hilbert seminorms
(See [**MV92**, 28.1 p.325], [**Kri07a**, 6.19.1]).

Proof. By what we have shown in the proof of (6⇒8) in 3.70 using 3.65 every natural mapping $E \to E_p$ factors over some Hilbert-space H. Taking the norm q of the Hilbert-space, we get a continuous seminorm $E \to H \xrightarrow{q} \mathbb{R}$, which dominates p. □

3.73 Inheritance properties for nuclear and Schwartz spaces
(See [**Kri07a**, 6.21], [**Jar81**, 21.2.3 p.500], [**Jar81**, 21.1.7 p.481]).
Both the nuclear and the Schwartz spaces are stable with respect to:

1. *products,* (See [**MV92**, 28.7 p.328], [**Kri07a**, 6.21], [**Jar81**, 21.1.3 p.479], [**Jar81**, 21.1.4 p.480])
2. *subspaces,* (See [**MV92**, 28.6 p.328], [**Kri07a**, 6.21], [**Jar81**, 21.1.5 p.481], [**Jar81**, 21.1.6 p.481])
3. *countable coproducts,* (See [**MV92**, 28.7 p.328], [**Kri07a**, 6.21], [**Jar81**, 21.1.3 p.479], [**Jar81**, 21.1.4 p.480])
4. *quotients,* (See [**MV92**, 28.6 p.328], [**Kri07a**, 6.21], [**Jar81**, 21.1.5 p.481], [**Jar81**, 21.1.6 p.481])
5. *completions,* (See [**Jar81**, 21.1.2 p.481])
6. *projective tensor products,* (See [**Jar81**, 15.6.5 p.337])

Proof.

(1) A typical seminorm on $E := \prod_i E_i$ is of the form $p : x \mapsto \max_{i\in A} p_i(x_i)$, where A is finite and p_i are seminorms on E_i. Obviously $\widetilde{E_p} = \prod_{i\in A} \widetilde{(E_i)_{p_i}}$. For every p_i we can find a seminorm $q_i \geq p_i$ such that the canonical mapping $\widetilde{(E_i)_{q_i}} \to \widetilde{(E_i)_{p_i}}$ is precompact/nuclear. Then the canonical mapping $\prod_{i\in A} \widetilde{(E_i)_{q_i}} \to \prod_{i\in A} \widetilde{(E_i)_{p_i}}$ is precompact/nuclear, in fact a finite product $\prod_{i\in A} T_i$ can be written as $\sum_{i\in A} \mathrm{inj}_i \circ T_i \circ \mathrm{pr}_i$ and hence belongs to the considered ideal. Thus we may use $q := \max_{i\in A} q_i$ as the required seminorm.

(2) First for Schwartz spaces. Let E be a subspace of F. The seminorms on E are the restrictions of seminorms p on F. Let $q \geq p$ be a seminorm such that $F_q \to F_p$ is precompact. Since $E_{p|_E} \to F_p$ is an embedding ($\ker p|_E = \ker p \cap E$) we have the diagram:

$$\begin{tikzcd} E_{q|_E} \arrow[r] \arrow[d, hook] & E_{p|_E} \arrow[d, hook] \\ F_q \arrow[r] & F_p \end{tikzcd}$$

Since the bottom arrow is precompact, the same is true for the top arrow.

Now for nuclear spaces. The corresponding proof will not work for nuclear mappings, but for absolutely summing mappings, since the ideal $\mathcal{S}_1$ is obviously injective, i.e. if $T : E \to F_1 \hookrightarrow F$ belongs to $\mathcal{S}_1$ and F_1 is a closed subspace of F, then $T : E \to F_1$ belongs to $\mathcal{S}_1$, since $\ell^1\{F_1\} = \ell^1\{F\} \cap F_1^{\mathbb{N}}$.

(3) First for Schwartz spaces. Recall that a basis of seminorms on a countable coproduct $E = \coprod_k E_k$ is given by $\sup_k p_k$, where the p_k run through the seminorms of E_k and $\sup_k p_k : (x_k)_k \mapsto \sup_k p_k(x_k)$. By assumption we can find seminorms $q_k \geq p_k$ such that the connecting map $T_k : (E_k)_{q_k} \to (E_k)_{p_k}$ is precompact. Furthermore we may assume that its norm is less than $\frac{1}{2^k}$, by replacing q_k with $2^k \|T_k\| q_k$. Now the following diagram shows that we get a natural bijection $\coprod_k (E_k)_{p_k} \cong$

$(\coprod_k E_k)_{\sup_k p_k}$ which is an isometry if we supply $\coprod_k (E_k)_{p_k}$ with the norm $(x_k)_k \mapsto \sup\{p_k(x_k) : k\}$, and analogously for the q_k.

$$\begin{tikzcd}
\ker(\sup_k p_k) \arrow[rrr, equal] \arrow[d, hook] & & & \coprod_k \ker(p_k) \arrow[d, hook] \\
\coprod_k E_k \arrow[rrr, equal] \arrow[dd] \arrow[dr, "\sup_k p_k"'] & & & \coprod_k E_k \arrow[dd] \arrow[dl, "\coprod_k p_k"] \\
& \mathbb{R} & \coprod_k \mathbb{R} \arrow[l, "\sup"'] & \\
(\coprod_k E_k)_{\sup_k p_k} \arrow[ur] \arrow[rrr, leftrightarrow, "\cong"] & & & \coprod_k (E_k)_{p_k} \arrow[ul]
\end{tikzcd}$$

Up to these isometries the connecting map is nothing else but

$$T := \coprod_k T_k : \coprod_k (E_k)_{q_k} \to \coprod_k (E_k)_{p_k}.$$

Since the finite subsums $\coprod_{k\le n} T_k$ converge uniformly to $\coprod_k T_k$ on the unit-ball with respect to $p = \sup_k p_k$ and are precompact operators by the result on products, hence so is the infinite sum.

Now for nuclear spaces. We proceed as before using that the connecting mappings T_k can be chosen of the form $T_k = \sum_j (\lambda_k)_j (x_k^*)_j \otimes (y_k)_j$ with $\lambda_k \in \ell^1$ and sequences $x_k^* \in o((E_k)_{q_k})^o$ and $y_k \in o((E_k)_{p_k})$. By replacing q_k by $\|\lambda_k\|_1\, 2^k q_k$, we have that $(\lambda_1, \lambda_2, \dots) \in \ell^1$ and hence the connecting mapping T admits the representation $\sum_{k,j} (\lambda_k)_j (x_k^*)_j \otimes (y_k)_j$, where $(x_k^*)_j$ can be extended to the corresponding space, since $(E_k)_{q_k}$ embeds isometrically into it.

($\boxed{4}$) First for Schwartz spaces. Let $F := E/N$, where N is a closed subspace and let $\pi : E \to F$ denote the quotient mapping. Let $\tilde p$ be a seminorm on F. By assumption there exists a seminorm q on E with $q \ge \tilde p \circ \pi$ and such that $E_q \to F_{\tilde p\circ\pi}$ is precompact. Let $\tilde q$ be the corresponding quotient semi-norm on F, see [**Kri07b**, 4.3.3]. Then $q \ge \tilde q \circ \pi \ge \tilde p \circ \pi$. Now the following diagram shows that we get a natural isometry $E_{\tilde p\circ\pi} \cong E\tilde p$ and similarly for $\tilde q$.

$$\begin{tikzcd}
\ker\pi \arrow[r, hook, dotted] \arrow[d, equal] & \ker(\tilde p\circ\pi) \arrow[r, equal] \arrow[d, hook] & \pi^{-1}(\ker\tilde p) \arrow[r, two heads] & \ker\tilde p \arrow[d, hook] \\
N \arrow[r, hook] & E \arrow[rr, two heads, "\pi"] \arrow[dr, "\tilde p\circ\pi"] \arrow[dd] & & F \arrow[dl, "\tilde p"'] \arrow[dd] \\
& & \mathbb{R} & \\
& E_{\tilde p\circ\pi} \arrow[ur, dotted, "\tilde p\circ\pi"] \arrow[rr, dotted, two heads, "\cong"] & & F_{\tilde p} \arrow[ul, dotted, "\tilde p"']
\end{tikzcd}$$

Another argumentation for the same result would be an application of the isomorphy-theorem $F/\ker\tilde p \cong (E/N)/(\ker(\tilde p\circ\pi)/N) \cong E/\ker(\tilde p\circ\pi)$.

Hence we have the diagram:

$$\begin{tikzcd}
& F_{\tilde q} \arrow[r] & F_{\tilde p} \\
E_q \arrow[ur] \arrow[r, two heads] & E_{\tilde q\circ\pi} \arrow[u, "\cong"] \arrow[r, two heads] & E_{\tilde p\circ\pi}. \arrow[u, "\cong"]
\end{tikzcd}$$

Note that connecting morphisms are always quotient maps, since the projections $E \to E_q$ are. So the diagonal arrow is open, since it is up to the vertical isomorphism the connecting map $E_q \to E_{\tilde q\circ\pi}$. Hence the image of the unit ball in E_q is a 0-nbhd in $F_{\tilde q}$ whose image is precompact in $E_{\tilde p\circ\pi} \cong F_{\tilde p}$.

Now in order that this proof works also for nuclear spaces, we can use the following: It is enough to consider the situation, where $E \twoheadrightarrow E_1 \to F$ is nuclear, $E \twoheadrightarrow E_1$ is a quotient map and E a Hilbert space (by 3.72). But then the sequence $E_2 \rightarrowtail E \twoheadrightarrow E_1$ splits, where E_2 is the kernel of the quotient map $E \twoheadrightarrow E_1$, and hence $E_1 \to F$ can be written as $E_1 \hookrightarrow E \twoheadrightarrow E_1 \to F$ and thus is nuclear.

(5) Use that $\widetilde{E_q} = \widetilde{E}_{\tilde q}$, where $\tilde q$ denotes the unique extension of q to a seminorm on $\tilde E$.

$$\begin{tikzcd}
\ker q \arrow[r, equal] \arrow[d, hook] & E\cap\ker\tilde q \arrow[r, hook] & \ker\tilde q \arrow[d, hook] \\
E \arrow[rr, hook, two heads] \arrow[dr, "q"] \arrow[dd] & & \tilde E \arrow[dl, "\tilde q"'] \arrow[dd] \\
& \mathbb{R} & \\
E_q \arrow[ur, dotted, "q"] \arrow[rr, hook, two heads, dotted] & & \tilde E_{\tilde q} \arrow[ul, dotted, "\tilde q"']
\end{tikzcd}$$

(6) First for Schwartz spaces. Recall that the typical 0-neighborhoods of $E \otimes_\pi F$ are the absolutely convex hulls $U_1 \otimes U_2$ of $\{u_1 \otimes u_2 : u_1 \in U_1, u_2 \in U_2\}$, where the U_i are absolutely convex 0-neighborhoods in E_i. By assumption there are 0-neighborhoods $V_i \subseteq U_i$ in E_i such that for every $0 < \varepsilon \le 1$ there is a finite set B_i such that $V_i \subseteq B_i + \varepsilon\, U_i$. Taking intersection with U_i shows that $V_i \subseteq (B_i + \varepsilon\, U_i) \cap U_i \subseteq (B_i \cap 2U_i + \varepsilon\, U_i)$. In fact $b + \varepsilon\, u \in U_i$ implies that $b \in U_i - \varepsilon\, u \subseteq U_i - U_i \subseteq 2U_i$. Thus we may assume that $B_i \subseteq 2U_i$. Now we have that

$$\begin{aligned} V_1 \otimes V_2 &\subseteq B_1 \otimes B_2 + \varepsilon\, B_1 \otimes U_2 + \varepsilon\, U_1 \otimes B_2 + \varepsilon^2\, U_1 \otimes U_2 \\ &\subseteq B_1 \otimes B_2 + (2\varepsilon + 2\varepsilon + \varepsilon^2)\, U_1 \otimes U_2. \end{aligned}$$

So let $V := \frac{1}{5}\, V_1 \otimes V_2 \subseteq \frac{1}{4+\varepsilon}\, V_1 \otimes V_2$ and $B := \frac{1}{4+\varepsilon}\, B_1 \otimes B_2$. Then $V \subseteq B + \varepsilon\, U$. Since B is the absolutely convex hull of a finite set, it is precompact, hence we can find a finite set B_0 such that $B \subseteq B_0 + \varepsilon\, U$, and so $V \subseteq B_0 + 2\varepsilon U$.

For nuclear spaces E and F we take an arbitrary lcs G and calculate as follows:

$$\begin{aligned} (E \otimes_\pi F) \otimes_\varepsilon G &\overset{E \text{ nucl.}}{\cong} (E \otimes_\varepsilon F) \otimes_\varepsilon G \cong E \otimes_\varepsilon (F \otimes_\varepsilon G) \\ &\overset{F \text{ nucl.}}{\cong} E \otimes_\varepsilon (F \otimes_\pi G) \overset{E \text{ nucl.}}{\cong} E \otimes_\pi (F \otimes_\pi G) \cong (E \otimes_\pi F) \otimes_\pi G. \quad \square \end{aligned}$$

3.74 Nuclearity of $\lambda^p(A)$ (See [**MV92**, 28.16 p.335])**.**
Let $A = \{a^{(k)} : k \in \mathbb{N}\}$ be countable. Then

1. $\exists p \in [1, \infty]$: $\lambda^p(A)$ *(N);*
$\Leftrightarrow$ 2. $\forall p \in [1, \infty]$: $\lambda^p(A)$ *(N);*
$\Leftrightarrow$ 3. $c_0(A)$ *(N);*
$\Leftrightarrow$ 4. $\exists p, q : 1 \le p < q \le \infty$ *und* $\lambda^p(A) = \lambda^q(A)$*;*
$\Leftrightarrow$ 5. $\forall p, q : 1 \le p < q \le \infty \Rightarrow \lambda^p(A) = \lambda^q(A)$*;*
$\Leftrightarrow$ 6. $\forall k\, \exists m \ge k$: $\|a^{(k)}/a^{(m)}\|_1 < \infty$.

Proof. (2⇒3) $\lambda^\infty(A)$ (N) $\Rightarrow$ $\lambda^\infty(A)$ is (S) by 3.60 and hence (M) by 3.31 $\Rightarrow \lambda^\infty(A) = c_0(A)$ by 3.28.

(1⇒6) and (3⇒6) follows for $p < \infty$ from 3.58 for the diagonal operators $D \cong \iota_k^m$ on ℓ^p resp. c_0, hence also for $\lambda^\infty(A) = c_0(A)$.

(6⇒2) follows from 3.58 for $p < \infty$. By 3.59 the diagonal operator $\ell^\infty \to \ell^1$ with diagonal $d = a^{(k)}/a^{(m)} \in \ell^1$ is nuclear and hence its composite $\lambda_m \hookrightarrow \ell^\infty \to \ell^1 \to \ell^\infty$ is nuclear and thus absolutely summming, and so also the connecting homomorphism $\lambda_m \to \lambda_k$ is absolutely summing.

([2]⇒[1]) obvious.

([4]⇔[5]⇔[6]) follows from [1.25] □

One can show the following:

3.75 Theorem of Dynin-Mityagin
(See [**MV92**, 28.12 p.332], [**Jar81**, 21.10.1 p.510]).
For nuclear Fréchet spaces ((NF) for short) each Schauder-basis is absolute (recall [1.17]).

3.76 Corollary (See [**MV92**, 28.13 p.334]).
Each (NF) space with Schauder-basis $(e_j)_j$ is isomorphic to $\lambda^1(A)$, where

$$A := \{j \mapsto \|e_j\|_k : k \in \mathbb{N}\}.$$

Proof. This is a direct consequence of [3.75] and [1.21]. □

It was open for a long time whether all (NF) spaces have a Schauder-basis. The first counter-example was given in [**MZ74**], see [**Jar81**, 21.10.9 p.516] for a simpler counter-example. Rather recently it was shown in [**DV00**] that the complete ultra-bornological nuclear space $C^\omega(\mathbb{R},\mathbb{R})$ of real-analytic functions does not have a Schauder-basis.

3.77 Theorem of Grothendieck-Pietsch
(See [**MV92**, 28.15 p.334] ([**Jar81**, 21.6.2 p.497])).
A nuclearity criterium for (F) with Schauder-basis $(e_j)_{j\in\mathbb{N}}$ is:

$$\forall k\,\exists m \geq k : \sum_j \frac{\|e_j\|_k}{\|e_j\|_m} < \infty.$$

Proof. ($\Rightarrow$) Let E be (NF) with a Schauder-basis $(e_j)_j$. Then $E \cong \lambda^1(\{a^{(k)} : k \in \mathbb{N}\})$ with $a^{(k)}_j := \|e_j\|_k$ by [3.76] and thus the claimed condition is satisfied by [3.74].

($\Leftarrow$) For any continuous seminorm p choose p' with $\sum_j \frac{p(e_j)}{p'(e_j)} < \infty$. By [1.19] there exists a p'' and $C > 0$ such that

$$\forall x\,\forall j : |\xi_j(x)|\, p'(e_j) \leq C\, p''(x),$$

where ξ_j are the coefficient functionals. Then ξ_j factors (for $p'(e_j) \neq 0$) over $\iota_{p''} : E \twoheadrightarrow E_{p''}$ to a $\tilde{\xi}_j \in (E_{p''})^*$. Thus $D : E_{p''} \to E_p$, $x \mapsto \sum_{j=0}^\infty \tilde{\xi}_j(x)\, \iota_p(e_j)$, is a nuclear mapping, since

$$\sum_{j=0}^\infty \|\tilde{\xi}_j\|\, \|\iota_p(e_j)\| = \sum_{j=0}^\infty \sup\{|\xi_j(x)| : p''(x) \leq 1\}\, p(e_j) \leq C \sum_{j=0}^\infty \frac{p(e_j)}{p'(e_j)} < \infty.$$

Thus the connecting mapping $\iota_p^{p''}$ is nuclear, since it equals D:

$$(D \circ \iota_{p''})(x) = \sum_{j=0}^\infty \tilde{\xi}_j\big(\iota_{p''}(x)\big)\, \iota_p(e_j) = \sum_{j=0}^\infty \xi_j(x)\, \iota_p(e_j)$$
$$= \iota_p\Big(\sum_{j=0}^\infty \xi_j(x)\, e_j\Big) = (\iota_p^{p''} \circ \iota_{p''})(x). \quad \square$$

3.78 Nuclearity of power series spaces $\lambda_R(\alpha)$
(See [**MV92**, 29.6 p.344], [**Jar81**, 21.6.3 p.497]).

1. *$\lambda_\infty(\alpha)$ is nuclear* $\Leftrightarrow \sup_n \frac{\log(n)}{\alpha_n} < \infty$.
2. *$\lambda_0(\alpha)$ is nuclear* $\Leftrightarrow \lim_n \frac{\log(n)}{\alpha_n} = 0$.

Proof. ($\boxed{1}$) $\lambda_\infty(\alpha)$ nuclear $\overset{\boxed{3.74}}{\Longleftrightarrow} \exists t > 0 : C := \sum_n e^{-t\alpha_n} < \infty$

$$(\Rightarrow) \quad \Rightarrow\; n e^{-t\alpha_n} \le \sum_{j=1}^n e^{-t\alpha_j} \le C \Rightarrow \frac{\log(n)}{\alpha_n} \le \frac{\log(C)}{\alpha_n} + t \le \frac{\log(C)}{\alpha_0} + t =: D.$$

$$(\Leftarrow) \quad \sup_n \frac{\log(n)}{\alpha_n} \le D \Rightarrow e^{-D\alpha_n} \le \frac{1}{n} \Rightarrow \sum_n e^{-2D\alpha_n} < \infty$$

($\boxed{2}$)

$$\lambda_0(\alpha) \text{ is nuclear} \overset{\boxed{3.74}}{\Longleftrightarrow} \forall t > 0 : \sum_n e^{-t\alpha_n} < \infty, \text{ now proceed as in } (\boxed{1}). \quad \square$$

Example (See [**Jar81**, 21.6.4 p.498]).
$s = \lambda_\infty(\log(n))$ is nuclear and hence also the function spaces in $\boxed{1.16}$;
$\lambda_0(\log(n))$ is not nuclear, but Schwartz by $\boxed{3.35}$.

3.79 Lemma (See [**MV92**, 29.7 p.344]).
Let E be (N), p a continuous Hilbert SN, and $U := \{x : p(x) \le 1\}$ its unit-ball. Then there exists a fast-falling ONB $(e_n)_{n\in\mathbb{N}}$ of $E^*_{U^o}$, i.e.

$$\forall k\, \exists V : \{n^k e_n : n \in \mathbb{N}\} \subseteq V^o.$$

Proof.

$$E\ (N) \overset{\boxed{3.70.8},\boxed{3.53}}{\Longrightarrow} \forall k > 0\, \exists p_k \ge p, \text{ cont. Hilbert SN} : \iota_p^{p_k} \in \mathcal{A}_{1/k}(E_{p_k}, E_p)$$

$$\text{As in } \boxed{3.33}\text{:} \quad (E_p)^* \cong E^*_{U^o},\ (E_{p_k})^* \cong E^*_{U_k^o} \text{ with } U_k := \{x : p_k(x) \le 1\}$$

$$\overset{\boxed{3.54}}{\Longrightarrow} \iota_k := (\iota_p^{p_k})^* : E^*_{U^o} \rightarrowtail E^*_{U_k^o},\ \iota_k \in \mathcal{A}_{1/k}(E^*_{U^o}, E^*_{U_k^o})$$

$$y = \iota_k(y) = \sum_j a_j^{(k)} \langle y, e_j^{(k)} \rangle f_j^{(k)} \text{ with}$$

$$(e_j^{(k)})_j \text{ ON in } E^*_{U^o},\ (f_j^{(k)})_j \text{ ON in } E^*_{U_k^o},$$

$$(a_j^{(k)})_j \downarrow,\ C^{1/k} := \sum_j (a_j^{(k)})^{1/k} < \infty.$$

$$\Rightarrow m(a_m^{(k)})^{1/k} \le \sum_{j=1}^m (a_j^{(k)})^{1/k} \le C^{1/k}, \text{ d.h. } a_m^{(k)} \le C/m^k$$

$$\text{und } (e_j^{(k)})_j \text{ ONB in } E^*_{U^o}, \text{ da } \iota_k \text{ inj.}$$

Let $(\tilde{e}_n)_n$ be the diagonal-enumeration of $(e_i^{(j)})_{i,j}$, drop recursively those which are linearly dependent on ealier ones, and apply Gram-Schmidt to obtain an ONB

$(e_n)_n$ in $E^*_{U^o}$. Let $n \geq (2k)^2$.

$$\forall j < \sqrt{n} - k : \sum_{i \leq j+k} i = \frac{1}{2}(j+k)(j+k+1) < \frac{n+\sqrt{n}}{2} \leq n$$

$$\Rightarrow \forall j < \sqrt{n} - k : e_n \perp e_j^{(k)}, \text{ since then } \tilde{e}_n \text{ lies on a diagonal below } (j,k)$$

$$\Rightarrow j_n := \lceil \sqrt{n} - k \rceil \geq \frac{\sqrt{n}}{2} \text{ (since } k \leq \frac{\sqrt{n}}{2}\text{) and } \forall j < j_n : e_n \perp e_j^{(k)}$$

$$\Rightarrow \sum_{j=j_n}^{\infty} \langle e_n, e_j^{(k)} \rangle \, e_j^{(k)} = e_n = \iota_k(e_n) = \sum_{j=j_n}^{\infty} a_j^{(k)} \, \langle e_n, e_j^{(k)} \rangle \, f_j^{(k)}$$

$$\Rightarrow \|e_n\|^2_{U_k^o} = \sum_{j=j_n}^{\infty} |a_j^{(k)} \, \langle e_n, e_j^{(k)} \rangle|^2 \leq |a_{j_n}^{(k)}|^2 \sum_{j=j_n}^{\infty} |\langle e_n, e_j^{(k)} \rangle|^2 \leq$$

$$\leq \frac{C^2}{(j_n)^{2k}} \|e_n\|^2_{U^o} \leq \frac{(2^k C)^2}{n^k} =: \frac{C_k}{n^k} \text{ for all large } n$$

$$\Rightarrow (e_n)_n \text{ fast falling in } E^*. \quad \square$$

3.80 Theorem of Komura-Komura
(See [**MV92**, 29.8 p.346], [**Jar81**, 21.7.1 p.500]).
Let E be an lcs: E is (N) $\Leftrightarrow \exists I$: $E \hookrightarrow s^I := \prod_{i \in I} s$.

Proof. ($\Rightarrow$)

$$E \text{ (N)} \Rightarrow \exists (p_i)_{i \in I} \text{ basis of Hilbert SN, let } U_i := \{x : p_i(x) \leq 1\}$$

$$\xRightarrow{\boxed{3.79}} \forall i \; \exists (e_n^i)_n \text{ fast falling ONB in } E^*_{U_i^o}$$

$$\Rightarrow \forall k \; \exists V_k : \{n^k e_n^i : n\} \subseteq (V_k)^o, \text{ i.e. } \forall x \in V_k : \sup_n |n^k e_n^i(x)| \leq 1$$

$$\Rightarrow f_i : E \to s, \; x \mapsto (e_n^i(x))_n, \text{ is continuous}$$

$$\Rightarrow f := (f_i)_{i \in I} : E \to s^I \text{ is continuous.}$$

$$\forall x \in E : \mathrm{ev}_x = \delta_{\iota_i(x)} \text{ is continuous on the Hilbert space } E^*_{U_i^o} = (E_{U_i})^*$$

$$\xRightarrow{[\textbf{Kri07b}, 6.2.9]} \exists x^* \in E^*_{U_i^o} \; \forall y^* \in E^*_{U_i^o} : \langle x^*, y^* \rangle = \mathrm{ev}_x(y^*) = y^*(x)$$

$$\Rightarrow \|f_i(x)\|_0^2 := \sum_n |e_n^i(x)|^2 = \sum_n |\langle x^*, e_n^i \rangle|^2 = \|x^*\|^2_{E^*_{U_i^o}} = \|\mathrm{ev}_x\|^2 = p_i(x)^2$$

$$\Rightarrow f \text{ is an embedding onto } f(E) \subseteq s^I.$$

($\Leftarrow$) s (N), $E \hookrightarrow s^I \xRightarrow{\boxed{3.73.1}, \boxed{3.73.2}} E$ (N). $\square$

3.81 Nuclear Fréchet spaces (See [**MV92**, 29.9 p.346], [**Jar81**, 21.7.3 p.502]).
E is (NF) $\Leftrightarrow$ E is isomorphic to a closed linear subspace of $s^{\mathbb{N}}$.

Proof. ($\Leftarrow$) $s^{\mathbb{N}}$ is (NF) by $\boxed{3.78}$ and $\boxed{3.73.1}$, thus also E by $\boxed{3.73.2}$.
($\Rightarrow$) For the (NF) space E exists a countable basis $\mathcal{P}$ of Hilbert SN and by the proof of $\boxed{3.80}$ E embedds into $s^{\mathcal{P}}$. $\square$

3.82 Remark.
Note that we have the following implications under the assumption on the bottom

of the arrow:

$$\text{nuclear} \xRightarrow{\boxed{3.60}} \text{Schwartz} \xRightarrow[\text{q.-compl.}]{\boxed{3.31}} \text{s.-Montel} \xRightarrow{\boxed{3.22}} \text{s.-reflexive} \xRightarrow{\boxed{3.17}} \text{q.-complete.}$$

The converse does not hold even for Fréchet spaces:

$$\text{nuclear} \not\xLeftarrow{\boxed{3.78}} \text{Schwartz} \not\xLeftarrow{\boxed{3.36}} \text{Montel} \not\xLeftarrow{\ell^2} \text{reflexive} \not\xLeftarrow{c_0,\ \ell^1} \text{complete.}$$

4. Duality

Spaces of (linear) functions

In this section we discuss how the hom-functor behaves on (co-)limits.

Let X be a set and $\mathcal{B}$ be a BORNOLOGY on X, i.e. a set of subsets of X containing all single pointed subsets and with any two sets a set containing their union. For lcs F let $\ell^\infty(X,F)$ be the linear space of all mappings $f : X \to F$, which are bounded on each $B \in \mathcal{B}$. For continuous seminorms p of F and $B \in \mathcal{B}$ let p^B be the seminorm on $\ell^\infty(X,F)$ definied by $p^B(f) := \sup p(f(B))$. These seminorms describe the Hausdorff topology of uniform convergence on the sets $B \in \mathcal{B}$. Obviously, $\ell^\infty(B,F)$ is as complete as F is for each $B \in \mathcal{B}$ and hence the same is true for the projective limit $\ell^\infty(X,F) \cong \varprojlim_{B\in\mathcal{B}} \ell^\infty(B,F)$, see [**Kri07a**, 2.28]. Note that the lcs $\ell^\infty(X,F)$ will not change, iff we add all subsets of sets in $\mathcal{B}$ to $\mathcal{B}$.

The space $L(E,F)$ of bounded linear mappings $E \to F$ between lcs (or even from a convex bornological space into an lcs) is closed in $\ell^\infty(E,F)$, where $\mathcal{B}$ are the bounded subsets of E, and hence has the same completeness properties as F. In particular, E' is always complete with respect to $\beta(E',E)$. Note that a CONVEX BORNOLOGICAL SPACE (cbs for short) is a linear space together with a bornology, which is closed under formation of absolutely convex hulls and multiplication with (say) 2. We will always assume that cbs are SEPARATED, i.e. $\{0\}$ is the only bounded linear subspace. The von Neumann bornology of all bounded sets of an lcs E describes a cbs ${}^bE$ and conversely to any cbs F we may associate the finest locally convex topology ${}^tF$ for which the sets in the given bornology are bounded, i.e. with the corresponding bornivorous absolutely convex subsets a 0-nbhd basis. See [**Gac04**] for more on this concept.

For the space $\mathcal{L}(E,F)$ of continuous linear mappings and for the particular case E^* these completeness inheritance properties are not valid. However, if E is bornological then $\mathcal{L}(E,F) = L(E,F)$ and $E^* = E'$.

More generally the question arrises, whether $\mathcal{L}(\_,F)$ (or in particular $(\_)^*$) transforms inductive limits into projective ones. By the universal property algebraically the dual of a colimit is the limit of the duals: The continuous linear mappings on a colimit $E := \operatorname{colim}_j E_j$ correspond uniquely to the families of morphisms $f_j : E_j \to F$ with $(\iota^j_{j'})^*(f_j) = f_{j'} \circ \iota^j_{j'} = f_j$ for all $j \prec j'$, i.e. which are compatible with respect to the connecting morphisms $\iota^j_{j'} : E_j \to E_{j'}$. These are the elements in the limit of the $\mathcal{L}(E_j,F)$ with connecting mappings $(\iota^j_{j'})^* = \mathcal{L}(\iota^j_{j'},F) : \mathcal{L}(E_{j'},F) \to \mathcal{L}(E_j,F)$. However, this linear (continuous) bijection $\mathcal{L}(\operatorname{colim}_j E_j,F) \to \lim_j \mathcal{L}(E_j,F)$ is not to be expected an homeomorphism, since for a typical 0-nbhd. B^o in $E^* = \mathcal{L}(E,\mathbb{K})$ with $B \subseteq E$ bounded, we would have to find 0-nbhds. B^o_j in E^*_j with $B^o \supseteq E^* \cap \prod_{j\in J} B^o_j$ and such that $B^o_j = E^*_j$ for almost all j. This is possible, if E is a regular inductive limit (i.e. $\operatorname{colim}_j E_j$

formed in cbs), since then $B = \iota^j(B_j)$ with $B_j \subseteq E_j$ bounded for some j and hence $B^o \supseteq ((\iota^j)^*)^{-1}(B_j^o)$, but not in general.
Note, that the representation $E = \varinjlim_B E_B$ of a bornological space is such a regular inductive limit.
Note furthermore, that if $(\_)^*$ is supplied with the bornology of equicontinuous sets, then $(\operatorname{colim}_j E_j)^* = \lim_j E_j^*$ in *cbs*: In fact, let U^o be a typical bounded set in E^*, i.e. $U \subseteq E := \operatorname{colim} E_j$ a 0-nbhd. Then each $U_j := (\iota_j)^{-1}(U) \subseteq E_j$ is a 0-nbhd and the image of U^o in $\lim_j E_j^* \subseteq \prod_j E_j^*$ is contained in the bounded set $\prod_j (U_j)^o$, since $|(\iota_j)^*(u^*)(u_j)| = |u^*(\iota_j(u_j))| \le 1$ for $u^* \in U$ and $u_j \in U_j$.

We want to consider inheritance with respect to L or $\mathcal{L}$. If $E \neq \{0\}$, then F is a topological direct summand in $L(E,F)$ and in $\mathcal{L}(E,F)$: In fact, let $0 \neq x \in E$ and $x^* \in E^*$ with $x^*(x) = 1$. Then $\iota : \mathbb{K} \to E$, $\lambda \mapsto \lambda \cdot x$ has $x^* : E \to \mathbb{K}$ as left-inverse, and hence $L(x^*, F) : F \cong L(\mathbb{K}, F) \to L(E,F)$ has $L(\iota, F) : L(E,F) \to L(\mathbb{K}, F) \cong F$ as left-inverse and the same works for $\mathcal{L}$. And similary, $F^* = \mathcal{L}(F, \mathbb{K})$ is a topological direct summand in $\mathcal{L}(F,E)$, via $\mathcal{L}(F,\iota)$ with left-inverse $\mathcal{L}(F, x^*)$ and the same way $F' = L(F,\mathbb{K})$ is a topological direct summand in $L(F,E)$. Thus in order to show some topological property for $\mathcal{L}(E,F)$ it is reasonable to assume the property for F and for E^*. Consequently a first step in answering this question is to consider inheritance with respect to $(\_)^*$.

Completeness of dual spaces

In this section we consider completeness conditions for the (strong) dual and we introduce the classes of infra-c_0-barrelled and of c_0-barrelled space in this connection.

Recall the Banach Steinhaus Theorem [**Kri07b**, 5.2.6], by which $\mathcal{L}(E,F)$ is sequentially complete if E is barrelled and F is sequentially complete:
Let $(f_n)_n$ be a Cauchy-sequence in $\mathcal{L}(E,F)$. Then $(f_n)_{n\in\mathbb{N}}$ is Cauchy pointwise, hence pointwise convergent to some function $f_\infty : E \to F$, which is continuous by the Banach Steinhaus Theorem. For each bounded $B \subseteq E$ and closed absolutely convex 0-nbhd $U \subseteq F$ there exists an n with $(f_{n'} - f_{n''})(B) \subseteq U$ for $n', n'' \ge n$. Taking for each $x \in B$ the pointwise limit for $n'' \to \infty$ yields $(f_{n'} - f_\infty)(x) \in U$. Thus $f_n \to f_\infty$ in $\mathcal{L}(E,F)$.

4.1 Example of a non-complete dual space.
Let F be a barrelled non-complete space (in [**Val71**] even normed bornological barrelled spaces are constructed, which are not ultra-bornological and hence not even locally complete). Let $E := (F^*, \sigma(F^*, F))$. Thus $F = E^*$ and by the barrelledness of F the $\sigma(F^*, F)$-bounded subsets are the equicontinuous ones. Hence the topology $\beta(E^*, E)$ coincides with the topology of uniform convergence on equicontinuous sets and hence with the given non-complete topology of F.

If E is infra-barrelled, then the dual E^* is at least locally complete: In fact, under this assumption the $\beta(E^*, E)$-bounded sets are equicontinuous and $E^*_{U^o} = (E_U)^*$ is complete as dual of a normed space. In order to improve this result, we need the following characterization:

4.2 Proposition (See [**Jar81**, 10.2.4 p.198], [**Woz13**, 2.39 p.21])**.**
For any lcs E we have:

1. *E is locally complete;*

$\Leftrightarrow$ 2. *The absolutely convex hull of every Mackey-0-sequence is relatively compact;*

$\Leftrightarrow$ 3. *The absolutely convex hull of every $\sigma(E,E^*)$-0-sequence is relatively compact in $(E,\sigma(E,E^*))$.*

Proof. ($\boxed{1}\Rightarrow\boxed{3}$) Let $x_n \to 0$ in $\sigma(E,E^*)$. Then $\{x_n : n\in\mathbb{N}\}$ is (weakly-)bounded and by locally completeness bounded in some closed Banach disk B. Thus $T:\ell^1\to E$, $\lambda\mapsto\sum_{n=0}^\infty \lambda_n x_n$, is well-defined and maps the unit ball $o\ell^1$ onto

$$A_0 := \Big\{\sum_{n=0}^\infty \lambda_n x_n : \|\lambda\|_{\ell^1}\le 1\Big\}\subseteq B.$$

It is $\sigma(\ell^1,c_0)$-$\sigma(E,E^*)$-continuous, since $x^*\circ T = (x^*(x_n))_{n\in\mathbb{N}}\in c_0\subseteq(\ell^1)^*$ for each $x^*\in E^*$. Since $o\ell^1$ is $\sigma(\ell^1,c_0)$-compact, its image A_0 is $\sigma(E,E^*)$-compact, absolute convex, and contains the x_n. Hence their absolutely convex hull is relatively compact for $\sigma(E,E^*)$.

($\boxed{3}\Rightarrow\boxed{2}$) Let x_n be a Mackey-0-sequence. By $\boxed{3}$ the $\sigma(E,E^*)$-closure C of the absolutely convex hull of $\{x_n : n\in\mathbb{N}\}$ is $\sigma(E,E^*)$-compact and hence $\sigma(E,E^*)$-complete. Since closed absolutely convex sets in E are $\sigma(E,E^*)$-closed, C is even complete in E by the next lemma $\boxed{4.3}$. Since $\{x_n : n\in\mathbb{N}\}\cup\{0\}$ is compact, its closed absolutely convex hull is precompact (by the proof of [**Kri07b**, 6.4.3]) and thus compact by completeness of C.

($\boxed{2}\Rightarrow\boxed{1}$) Suppose there is a closed absolutely convex bounded set B, such that E_B is not complete. Choose $\tilde{x}\in\widetilde{E_B}\setminus E_B$ and iteratively construct a sequence $(x_i)_{i\in\mathbb{N}}$ in E_B such that

$$\Big\|\tilde{x}-\sum_{i=1}^n x_i\Big\|_B \le \frac{1}{3^{n+2}}$$

and hence $\tilde{x}=\sum_{i=1}^\infty x_i$. Now let $y_n := 2^n x_n\in E_B$ and observe that

$$\|y_n\|_B \le 2^n\Big(\Big\|\tilde{x}-\sum_{i=1}^n x_i\Big\|_B + \Big\|\tilde{x}-\sum_{i=1}^{n-1} x_i\Big\|_B\Big)\le\Big(\frac{2}{3}\Big)^{n+1}\to 0.$$

Hence $\tilde{x}=\sum_{n=1}^\infty 2^{-n}y_n$ is in the closure of the absolutely convex hull of the (Mackey-)0-sequence (y_n) in the Banach space $\widetilde{E_B}$. Consider the initial topology τ' with respect to the inclusion $\iota: E_B\rightarrowtail E$. Since B is closed in E, it is closed for τ', thus $(E_B,\|\_\|_B)$ has a basis of τ'-closed sets. By the lemma $\boxed{4.4}$ below the extension $\tilde{\iota}:\widetilde{E_B}\rightarrowtail\tilde{E}$ is injective. Since $\tilde{\iota}(\tilde{x})=\sum_{n=1}^\infty 2^{-n}y_n$ is in the (by $\boxed{2}$) compact closure of the absolutely convex hull of $\{y_n : n\in\mathbb{N}\}\subseteq B$ in E, we get $\tilde{x}\in E\cap B\subseteq E_B$, a contradiction. □

4.3 Lemma (See [**Jar81**, 3.2.4 p.59])**.**
Let $\tau\ge\tau'$ be two lc-topologies on a vector space E and assume that (E,τ) has a 0-nbhd basis $\mathcal{U}$ consisting of τ'-closed subsets.
If $(x_i)_i$ is τ-Cauchy net in E, which converges to x_∞ with respect to τ', then it does so with respect to τ.
Thus, if a subset of E is (sequentially) complete for τ', then it is also for τ.

Proof. Cf. the proof of the corollary to $\boxed{3.17}$: Let $(x_i)_i$ be a τ-Cauchy net, which is τ'-convergent to x_∞, and let $U\in\mathcal{U}$. Thus there exists an i such that $x_{i'}-x_{i''}\in U$ for all $i',i''\succ i$. For fixed i', the net $i''\mapsto x_{i'}-x_{i''}\in U$ is τ-Cauchy and τ'-convergent to $x_{i'}-x_\infty$. Since U is τ'-closed we get $x_{i'}-x_\infty\in U$, i.e. $(x_i)_i$ is τ-convergent to x_∞. □

4.4 Lemma (See [**Jar81**, 3.4.5 p.63])**.**
Let (E,τ) be an lcs, $T\in\mathcal{L}(E,F)$ be injective and $\tau'\le\tau$ be the initial topology

on E with respect to T. If (E,τ) has a 0-nbhd-basis of τ'-closed subsets then the extension $\tilde{T}:\tilde{E}\to\tilde{F}$ to the completions is injective.

Proof. Let $\tilde{x}\in\ker\tilde{T}\subseteq\tilde{E}$. Thus there exists a net $(x_i)_i$ in E convergent to $\tilde{x}$ in $\tilde{E}$ and hence $T(x_i)=\tilde{T}(x_i)\to\tilde{T}(\tilde{x})=0$. Thus $x_i\to 0$ with respect to τ' and then $\tilde{x}=\tau\text{-}\lim_{i\to\infty}x_i=0$ by 4.3. □

4.5 Proposition (See [**Jar81**, 12.1.4 p.250], [**Woz13**, 2.64 p.30]).
The dual E^* of an lcs E is locally complete iff $(E,\mu(E,E^*))$ is INFRA-c_0-BARRELLED, *i.e. every 0-sequence in $(E^*,\beta(E^*,E))$ is equicontinuous.*
Thus for infra-c_0-barrelled spaces the dual is barrelled iff it is infra-barrelled, and it is ultra-bornological iff it is bornological.

Furthermore, an lcs E is called c_0-BARRELLED iff every 0-sequence in $(E^*,\sigma(E^*,E))$ is equicontinuous, see [**Jar81**, 12.1 p.249].

Proof. ($\Rightarrow$) Let $x_n^*\to 0$ in $(E^*,\beta(E^*,E))$ and hence in $(E^*,\sigma(E^*,E))$. So their closed absolutely convex hull K is $\sigma(E^*,E)$-compact by 4.2 (1$\Rightarrow$3). Thus K_o is a 0-nbhd of $\mu(E,E^*)$ and $x_n^*\in K\subseteq(K_o)^o$.

($\Leftarrow$) By 4.2 (2$\Rightarrow$1) it is enough to show that for any Mackey-0-sequence (x_n^*) in E^* its absolutely convex hull A is relatively compact w.r.t. $\beta(E^*,E)$. Any such sequence is equicontinuous (by the infra-c_0-barrelledness), hence A is relatively compact for $\sigma(E^*,E)$ and thus the closure of A is complete. Since $\beta(E^*,E)$ has a 0-nbhd basis of $\sigma(E^*,E)$-closed sets (B^o), it is also $\beta(E^*,E)$-complete by 4.3. Since $\{x_n^*:n\in\mathbb{N}\}\cup\{0\}$ is $\beta(E^*,E)$-compact, we get that the closed absolutely convex hull of the sequence is precompact and hence compact w.r.t $\beta(E^*,E)$.. □

4.6 Proposition (See [**Jar81**, 11.2.4 p.222]).
E infra-barrelled $\Rightarrow$ E^* is quasi-complete.

Proof. Let $B\subseteq E^*_\beta$ be bounded. Since E is infra-barrelled, B is equicontinuous, i.e. $B\subseteq U^o$ for some 0-neighborhood. The polar U^o is $\sigma(E^*,E)$-compact by 3.4, hence $\sigma(E^*,E)$-complete and therefore also $\beta(E^*,E)$-complete by 4.3, since $\beta(E^*,E)$ has a basis of $\sigma(E^*,E)$-closed subsets (B^o). □

Barrelledness and bornologicity of dual spaces

In this section we give conditions that garantee barrelledness or ultra-bornologicity of the strong dual. For this we show that the (appropriate) duality functor preserves reduced projective limits and products. We introduce the classes of (infra-)countably-barrelled spaces and discuss their relationship to the other barrelledness conditions.

If order to show that E^* is bornological, we have to represent E^* as inductive limit of normed spaces. So it is reasonable to assume that E is representable as projective limit of normed spaces. Because of $E^*=\tilde{E}^*$ (at least bornologically) it is no big restriction to assume that E is complete and hence $E=\varprojlim_U\tilde{E}_U$, a reduced projective limit of Banach spaces. Remains to check, whether

$$E^*=(\varprojlim_U\tilde{E}_U)^*\stackrel{?}{=}\varinjlim_U(E_U)^*.$$

For any functor $\mathcal{F}$ we have a natural morphism $\mathcal{F}(\lim X_i) \to \lim \mathcal{F}(X_i)$ by the universal property of the right side.

$$\begin{tikzcd}
X_i \arrow[rr, "X_\alpha"] & & X_j \\
& \lim_k X_k \arrow[ul, "\mathrm{pr}_i"] \arrow[ur, "\mathrm{pr}_j"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}(X_i) \arrow[rr, "\mathcal{F}(X_\alpha)"] & & \mathcal{F}(X_j) \\
& \lim_k \mathcal{F}(X_k) \arrow[ul, "\mathrm{pr}_i"'] \arrow[ur, "\mathrm{pr}_j"] & \\
& \mathcal{F}(\lim_k X_k) \arrow[uul, bend left, "\mathcal{F}(\mathrm{pr}_i)"] \arrow[uur, bend right, "\mathcal{F}(\mathrm{pr}_j)"'] \arrow[u, dotted, "!"] &
\end{tikzcd}$$

4.7 Lemma. Reduced projective limits (See [**Kri07a**, 3.25])**.**

Let $\varprojlim_i X_i$ be a reduced projective limit, and $f_i : X_i \to Y_i$ be continuous linear mappings with dense image which intertwine with all connecting mappings. Then the canonical mapping $\varprojlim_i f_i$ has dense image.

$$\begin{tikzcd}
X_i \arrow[rrr, two heads, "f_i"] & & & Y_i \\
& \varprojlim_i X_i \arrow[ul, "\mathrm{pr}_i"] \arrow[dl, "\mathrm{pr}_{i'}"'] \arrow[r, dotted, two heads, "\varprojlim_i f_i"] & \varprojlim_i Y_i \arrow[ur, "\mathrm{pr}_i"'] \arrow[dr, "\mathrm{pr}_{i'}"] & \\
X_{i'} \arrow[uu] \arrow[rrr, two heads, "f_{i'}"] & & & Y_{i'} \arrow[uu]
\end{tikzcd}$$

Proof. Let $z \in \varprojlim_i Y_i$ be given. Take an arbitrary 0-neighborhood $\mathrm{pr}_i^{-1}(2\,U_i)$. Since f_i has dense image we may find an $x_i \in X_i$ with $f_i(x_i) - \mathrm{pr}_i(z) \in U_i$. Since the first limit is reduced we can find an $x \in E$ with $\mathrm{pr}_i(x) - x_i \in f_i^{-1}(U_i)$. But then

$$\mathrm{pr}_i\Big(\varprojlim_i f_i(x) - z\Big) = (f_i \circ \mathrm{pr}_i)(x) - f_i(x_i) + f_i(x_i) - \mathrm{pr}_i(z) \in 2U_i,$$

i.e. $\varprojlim_i f_i$ has dense image. $\square$

4.8 Lemma. The dual of products (See [**Kri07a**, 3.26])**.**
The functor $(\_)^* : \underline{lcs} \to \underline{cbs}^{op}$ preserves products, where E^* is considered with the bornology of equicontinuous sets.

Here $\underline{cbs}$ denotes the category of convex bornological spaces with those linear mappings, which map bounded sets to bounded sets, as morphisms.

Proof. By the general argument above we have a mapping $\coprod_i E_i^* \to (\prod_i E_i)^*$, where $\coprod_i E_i^*$ denotes the coproduct in $\underline{cbs}$ and hence the product in $\underline{cbs}^{op}$. Since $\prod_i E_i$ obviously separates points in $\coprod_i E_i^*$ this mapping is injective. Let us show that it is a bornological quotient map, i.e. bounded sets in the image are images of bounded sets. This implies that it is a bornological isomorphism. So let $(\prod_i U_i)^o$ be a typical bounded:=equicontinuous subset of $(\prod_i E_i)^*$, i.e. the U_i are 0-neighborhoods of E_i and $U_i = E_i$ for all $i \notin J$, where J is a finite subset of I. Let $T \in (\prod_i U_i)^o$. Then $T(x) = 0$ for all $x = (x_i)_i$ with $x_j = 0$ for all $j \in J$ (use that every multiple of such an x belongs to $\prod_i U_i$). Let $T_i := T \circ \mathrm{inj}_i \in U_i^o \subseteq E_i^*$ for all i. Then $T = \sum_{j \in J} T_j \in \coprod_{j \in J} U_j^o$ and $\coprod_{j \in J} U_j^o$ is bounded in $\coprod_i E_i^*$. $\square$

4.9 Lemma. The dual of reduced projective limits (See [**Kri07a**, 3.27])**.**
The functor $(\_)^* : \underline{lcs} \to \underline{cbs}^{op}$ preserves reduced projective limits, where E^* is again considered with the bornology of equicontinuous sets.

Proof. So let $E := \varprojlim_i E_i$ be a reduced projective limit. As in the proof of 4.8 we have a natural mapping $\varinjlim_i E_i^* \to (\varprojlim E_i)^*$. Since all projections $\mathrm{pr}_i : E \twoheadrightarrow E_i$

have dense image the dual cone $\mathrm{pr}_i^* : E_i^* \rightarrowtail E^*$ consists of injective mappings only. Let $x^* \in E^*$ be given. Then there has to exist an i and a 0-neighborhood $U_i \subseteq E_i$ with $x^*(\mathrm{pr}_i^{-1}(U_i)) \subseteq \mathbb{D} := \{\lambda \in \mathbb{K} : |\lambda| \leq 1\}$. In particular $x^*(\ker \mathrm{pr}_i |_E) = 0$ and hence there exists a linear $x_i^* : \mathrm{pr}_i(E) \to \mathbb{R}$ with $x^* = x_i^* \circ \mathrm{pr}_i = \mathrm{pr}_i^*(x_i^*)$. Since $x_i^*(U_i \cap \mathrm{pr}_i(E)) = x^*(\mathrm{pr}_i^{-1}(U_i)) \subseteq \mathbb{D}$ we may extend x_i^* to a continuous functional in $U_i^o \subseteq (E_i)^*$ on the closure E_i of $\mathrm{pr}_i(E)$. Thus the union of all images $\mathrm{pr}_i^*((E_i)^*)$ is E^*. Moreover the same argument shows that every bounded:=equicontinuous set $(\mathrm{pr}_i^{-1}(U_i))^o$ is the image of the bounded set U_i^o under pr_i^*. From this it is clear that $(\varprojlim_i E_i)^*$ is the injective limit in <u>*cbs*</u>, since any family of bounded linear mappings $T_i : E_i^* \to F$ that commute with the connecting morphisms can be extended to a bounded linear mapping $T : E^* = \bigcup_i \mathrm{pr}_i^*(E_i^*) \to F$. □

In particular, $\tilde{E} = \varprojlim_U \widetilde{E_U}$ is a reduced projective limit, so $(\tilde{E})^* = \varinjlim_U (E_U)^* = \varinjlim_U E_{U^o}^*$ as convex bornological space (with respect to the equicontinuous bornology). But this does not imply that it is true for the strong topology and this topology on $(\tilde{E})^*$ need not be bornological.

What about infra-barrelledness of E^*?
Let $V \subseteq E^*$ be a bornivorous barrel, so V absorbs every bounded set in E^* and, in particular, the polars U^o of (closed absolutely convex) 0-nbhds U in E. From $K \cdot V \supseteq U^o$ we conclude, that $V_o \subseteq K \cdot (U^o)_o = K \cdot U$, i.e. V_o is bounded, and thus $(V_o)^o$ is a 0-neighborhood, but not necessarily (contained in) V, since $\beta(E^*, E)$ need not be compatible with duality (E^*, E).

4.10 Proposition (See [**Tre67**, p373], [**Kri07a**, 4.47])**.**
The strong dual of any semi-reflexive space is barrelled.

An lcs E is sometimes called DISTINGUISHED iff E^* barrelled, see [**Jar81**, 13.4.5 p.280].

Proof. Let V be a barrel in E_β^*. Since E is semi-reflexive the strong topology is compatible with the duality, and hence V is also closed for the weak-topology $\sigma(E^*, E)$ by [**Kri07b**, 7.4.8]. We show that the polar V_o is a bounded subset of E (which implies that $V = (V_o)^o$ is a 0-neighborhood in E_β^*). For this it is enough to show that V_o is bounded in $\sigma(E, E^*)$: Since V is assumed to be absorbing, we find for every $x^* \in E^*$ a $\lambda > 0$ with $x^* \in \lambda V$. Thus $|x^*(V_o)| \leq \lambda$. □

4.11 Proposition (See [**MV92**, 24.23 p.267], [**Woz13**, 3.52 p.56])**.**
The strong dual of any complete Schwartz space is ultra-bornological.

Proof. Let E be a complete Schwartz space. By 3.31 it is semi-Montel, hence $\beta(E^*, E) = \tau_c(E^*, E) = \tau_{pc}(E^*, E)$, by completeness. By the theorem 3.4 of Alaoğlu-Bourbaki U^o is $\sigma(E^*, E)$-compact (and even $\tau_{pc}(E^*, E)$-compact) for all 0-nbhds U and therefore by [**Kri07b**, 7.4.17] is a Banach disk. The inclusions $\iota_{U^o} : E_{U^o}^* \rightarrowtail (E^*, \tau_c(E^*, E))$ are bounded=continuous and therefore $\eta \geq \tau_c(E^*, E) \geq \sigma(E^*, E)$, where η denotes the ultra-bornological final locally convex topology on E^* generated by these mappings.

To see the converse $\tau_c(E^*, E) \geq \eta$, we choose 0-nbhds $V \subseteq U$ such that U^o is compact in $E_{V^o}^*$ (by 3.33). By continuity of $\iota_{V^o} : E_{V^o}^* \rightarrowtail (E^*, \eta)$ the polar U^o is compact in (E^*, η) and therefore $\mathrm{id} : (U^o, \eta) \to (U^o, \sigma(E^*, E))$ is a homeomorphism, i.e. $\sigma(E^*, E) = \eta$ on U^o, and, since $\gamma(E^*, E)$ is the finest such locally convex topology, $\gamma(E^*, E) \geq \eta$ and $\gamma(E^*, E) = \tau_c(E^*, \tilde{E}) = \tau_c(E^*, E)$ by 3.24 and completeness. □

4.12 Definition (See [**Jar81**, 12.2 p.251]).
An lcs E is called (INFRA-)COUNTABLY-BARRELLED (or (QUASI-)$\aleph_0$-BARRELLED) iff every countable intersection of closed absolutely convex 0-nbhds is a 0-nbhd provided it is (bornivorous) absorbing.

By the following proposition we get:
(infra-)barrelled $\Rightarrow$ (infra-)countably-barrelled $\Rightarrow$ (infra-)c_0-barrelled,

4.13 Proposition (See [**Jar81**, 12.2.1 p.252]).
Let E be an lcs. Then

1. *E is (infra-)countably-barrelled;*
$\Leftrightarrow$ 2. *For any lcs F every (β)σ-bounded sequence in $\mathcal{L}(E,F)$ is equicontinuous;*
$\Leftrightarrow$ 3. *For any Banach space F every (β)σ-bounded sequence in $\mathcal{L}(E,F)$ is equicontinuous.*

Here β denotes the topology of uniform convergence on each bounded set and σ that of pointwise convergence.

Proof. (1 $\Rightarrow$ 2) Let $T_n \in \mathcal{L}(E,F)$ be a sequence as considered in 2. Let V be a closed absolutely convex 0-nbhd in F. For every finite (resp. bounded) set $B \subseteq E$ there exists a $\rho > 0$ such that $\bigcup_{n\in\mathbb{N}} T_n(B) \subseteq \rho V$. Thus $U := \bigcap_{n\in\mathbb{N}} T_n^{-1}(V)$ is an absorbing (resp. bornivorous) absolutely convex set, hence a 0-nbhd by 1. Since $T_n(U) \subseteq V$ for all n, we get that $\{T_n : n \in \mathbb{N}\}$ is equicontinuous.

(2 $\Rightarrow$ 3) is trivial.

(3 $\Rightarrow$ 1) Let $U = \bigcap_{n\in\mathbb{N}} U_n$ be absorbing (resp. bornivorous) with U_n absolutely convex closed 0-nbhds. Let $F := \{x \in E^{\mathbb{N}} : x \text{ is finally constant}\}$. The subset $V := F \cap \prod_{n\in\mathbb{N}} U_n$ is absolutely convex and absorbing (since U absorbs the finite set $\{x_j : j \in \mathbb{N}\}$ for $x \in F$) in F. Let $T_n : E \to F$ be given by $x \mapsto (x_i)_{i\in\mathbb{N}}$ with $x_i := x$ for $i \le n$ and $x_i := 0$ for $i > n$. Then $T_n(\bigcap_{i\le n} U_i) \subseteq V$ and hence $\tilde{T}_n = \iota_V \circ T_n : E \to F \twoheadrightarrow \widetilde{F_V}$ is continuous. Since U is absorbing (resp. bornivorous), the set $\{\tilde{T}_n : n \in \mathbb{N}\}$ is σ(resp. β)-bounded ($B \subseteq \lambda U \Rightarrow T_n(B) \subseteq \lambda V$), hence equicontinuous by 3, so there exists a 0-nbhd $W \subseteq E$ with $2\tilde{T}_n(W) \subseteq \overline{\iota_V(V)} \cap F_V = \iota_V(V)$ for all n. Thus $\forall w \in W\ \exists v \in V : 2T_n(w) - v \in \ker p_V = \bigcap_{\lambda>0} \lambda V \subseteq V$ and, in particular, $2w = 2(\mathrm{pr}_n(T_n(w))) \in \mathrm{pr}_n(v) + \mathrm{pr}_n(V) \subseteq 2U_n$, i.e. $W \subseteq U_n$ for all $n \in \mathbb{N}$, hence $W \subseteq U$ and we are done. □

4.14 Lemma (See [**Jar81**, 12.2.2 p.252]).
Every locally complete infra-countably-barrelled lcs is countably-barrelled.
Every locally complete infra-c_0-barrelled lcs is c_0-barrelled.

Proof. Let $V = \bigcap_{n\in\mathbb{N}} V_n$ be absorbing as required in the definition 4.12. So V is a barrel, hence absorbs Banach-disks by the Banach-Mackey-Theorem (See [**Kri07b**, 7.4.18]). Since in locally complete lcs every closed bounded absolutely convex set is a Banach-disk, V is even bornivorous, hence a 0-nbhd by infra-countably-barrelledness.

The same proof works for (infra-)c_0-barrelledness with $V_n := \{x_n^*\}_o$ for a given 0-sequence x_n^* in E_σ^*, cf. the proof of 1$\Rightarrow$2 in 4.13. □

Remark.
We have shown the following implications, where the dotted ones are valid under

the assumption of c^∞-completeness:

$$\begin{tikzcd}
\text{ulta-bornological} \arrow[d, Rightarrow] \arrow[dr, Rightarrow, shift left] & \\
\text{barrelled} \arrow[d, Rightarrow] \arrow[dr, Rightarrow, shift left] & \text{bornological} \arrow[ul, Rightarrow, dotted, shift left] \arrow[d, Rightarrow] \\
\text{countably-barrelled} \arrow[d, Rightarrow] \arrow[dr, Rightarrow, shift left] & \text{infra-barrelled} \arrow[ul, Rightarrow, dotted, shift left] \arrow[d, Rightarrow] \\
c_0\text{-barrelled} \arrow[dr, Rightarrow, shift left] & \text{infra-countably-barrelled} \arrow[ul, Rightarrow, dotted, shift left] \arrow[d, Rightarrow] \\
 & \text{infra-}c_0\text{-barrelled} \arrow[ul, Rightarrow, dotted, shift left]
\end{tikzcd}$$

Duals of Fréchet spaces

In this section we describe the property (DF), which the strong duals of Fréchet spaces have, and which garantees in turn that their strong dual is Fréchet.

4.15 Lemma (See [**MV92**, 25.6 p.279], [**Jar81**, 12.2.4 p.253])**.**
Let E be metrizable. Then E^* is countably-barrelled.

Proof. Let $(U_n)_n$ be a 0-nbhd-basis of E. Then $U_n^o \subseteq E^*$ is bounded. Let V_n be closed absolutely convex 0-nbhds in E^* and $V_\infty := \bigcap_{n\in\mathbb{N}} V_n$ be bornivorous.

Recursively we will find $\rho_i > 0$ and $B_i \subseteq E$ bounded such that

$$B_i^o \subseteq V_i \text{ and } \rho_i U_i^o \subseteq \frac{1}{2^{i+2}} V_\infty \cap B_j^o \text{ for all } i, j \leq n.$$

For $(n = 0)$ take a bounded set $B_0 \subseteq E$ such that $B_0^o \subseteq V_0$ and find a $\rho_0 > 0$ with $\rho_0 U_0^o \subseteq \frac{1}{4} V_\infty \cap B_0^o$. For the induction step choose $\rho_n U_n^o \subseteq \frac{1}{2^{n+2}} V_\infty \cap \bigcap_{i<n} B_i^o$. The set $K := \sum_{i\leq n} \rho_i U_i^o$ is absolutely convex, $\sigma(E^*, E)$-compact, and contained in $\sum_{i\leq n} \frac{1}{2^{i+2}} V_\infty \subseteq \frac{1}{2} V_n$. Let $V' \subseteq \frac{1}{2} V_n$ be a $\sigma(E^*, E)$-closed absolutely convex 0-nbhd in E^*. Then $B_n := (V' + K)_o$ is bounded and $B_n^o = V' + K \subseteq V_n$ by the bipolar theorem.

Thus $W := \bigcap_n B_n^o \subseteq E^*$ satisfies $W = (W_o)^o$ and absorbs each U_i^o, hence W_o is bounded and thus $(W_o)^o = W \subseteq V$ is a 0-nbhd. in E^*. This shows infra-countably-barrelledness. Since E^* is complete, countably-barrelledness follows by 4.14. □

4.16 Proposition (See [**MV92**, 25.12 p.281], [**Jar81**, 13,4, p.280])**.**
Let E be a metrizable lcs. Then

1. *E^* is ultra-bornological;*

⇔ 2. *E^* is bornological;*

⇔ 3. *E^* is barrelled;*

⇔ 4. *E^* is infra-barrelled.*

We will give an example (of a non-distinguished $\lambda^1(A)$) in 4.25 for which these equivalent conditions are not satisfied.
In [**Jar81**, 13.4.2 p.279] it is shown that for metrizable E the bornologification $\beta(E^*, E)_{\text{born}}$ of $\beta(E^*, E)$ is $\beta(E^*, E^{**})$.

Proof. ($\boxed{1} \Rightarrow \boxed{2} \Rightarrow \boxed{4}$) are obvious.

($\boxed{4} \Rightarrow \boxed{3}$) and ($\boxed{2} \Rightarrow \boxed{1}$) since $E^* = E'$ is complete.

($\boxed{3} \Rightarrow \boxed{2}$) Let V be absolutely convex and bornivorous in E^*. Thus for every bounded=equicontinuous set U_n^o (where the U_n from a 0-nbhd basis in E) there exists a $\lambda_n > 0$ with $V \supseteq 2\lambda_n U_n^o$. Let U be the absolutely convex hull of $\bigcup_{n\in\mathbb{N}} \lambda_n U_n^o$. Then $U \subseteq \frac{1}{2}V$. The absolutely convex hull A_k of $\bigcup_{j\le n} \lambda_j U_j^o$ is $\sigma(E^*, E)$-compact (by exercise 34 to [**Kri07a**]), hence closed in $\beta(E^*, E) \geq \sigma(E^*, E)$.

We claim that $\bar{U} \subseteq V$: Let $x_0^* \in E^* \setminus V \subseteq E^* \setminus 2U$. Since A_n is closed there exists a 0-nbhd $V_n \subseteq E^*$ with $(x_0^* + V_n) \cap 2A_n = \emptyset$. Let $W := \bigcap_{n\in\mathbb{N}}(V_n + A_n)$. Let $k \in \mathbb{N}$. Then $U_k^o \subseteq \frac{1}{\lambda_k} A_n$ for all $n \geq k$. Choose $\mu_k \geq 1/\lambda_k$ with $U_k^o \subseteq \mu_k V_n$ for all $n < k$. Thus $U_k^o \subseteq \mu_k(V_n + A_n)$ for all n, i.e. $U_k^o \subseteq \mu_k W$, i.e. W is bornivorous and hence a 0-nbhd in E^* by $\boxed{4.15}$. We claim that $(x_0^* + W) \cap A_n = \emptyset$ for all n and hence $x_0^* \notin \bar{U}$, since otherwise $\emptyset \neq (x_0^* + W) \cap A_n \subseteq (x_0^* + V_n + A_n) \cap A_n$, i.e. $\exists v \in V_n$, $\exists a, a' \in A_n$: $a = x_0^* + v + a'$. Hence $x_0^* + v = a - a' \in 2A_n$ and thus $(x_0^* + V_n) \cap 2A_n \neq \emptyset$, a contradiction.

So the barrel $\bar{U} \subseteq V$. Since E^* is assumed to be barrelled, we are done. □

4.17 Definition. (DF)-spaces (See [**Jar81**, 12.4.1 p.257], [**MV92**, 25.6 p.279]).
An lcs E is called (DF)-SPACE, iff it has a countable base of the bounded sets and is infra-countably-barrelled (see $\boxed{4.12}$), i.e. every bornivorous subsets which is the intersection of countable many closed absolutely convex 0-neighborhoods is a 0-neighborhood.

An lcs E is called (df)-SPACE iff it has a countable base of its bornology and is infra-c_0-barrelled.

4.18 Proposition.

1. *The dual of any Fréchet space is a complete (DF) space* (See [**MV92**, 25.7 p.280], [**Jar81**, 12.4.5 p.260]).
2. *The dual of any (DF) space is a Fréchet space* (See [**MV92**, 25.9 p.280], [**Jar81**, 12.4.1 p.257]).

In [**Jar81**, 12.4.1 p.257] it is shown that: E^* is Fréchet $\Leftrightarrow$ $(E, \mu(E, E^*))$ is (df).

Proof. ($\boxed{1}$) This is $\boxed{4.15}$, since for the bornological space E the dual $E^* = E'$ is complete, and a countable basis of the bornology is given by the family U_n^o, where $\{U_n : n \in \mathbb{N}\}$ is a 0-nbhd basis of E.

($\boxed{2}$) By assumption a (DF)-space E has a countable base $\{B_n : n \in \mathbb{N}\}$ of bornology and hence $(E^*, \beta(E^*, E))$ a countable 0-nbhd basis $\{B_n^o : n \in \mathbb{N}\}$, so is metrizable. Let $(x_n^*)_n$ be Cauchy in E^*. Then x_n^* converges pointwise to some linear $x_\infty^* : E \to \mathbb{K}$. Let $V_n := \{x_n^*\}_o$ and $V_\infty := \bigcap_{n\in\mathbb{N}} V_n$. Since $(x_n^*)_n$ is Cauchy, it is bounded, thus contained in $\lambda_k B_k^o$ for some $\lambda_k > 0$. Hence $B_k \subseteq \lambda_k \{x_n^*\}_o = \lambda_k V_n$ and so $B_k \subseteq \lambda_k V_\infty$, i.e. V_∞ is bornivorous and hence a 0-nbhd since E is infra-countably-barrelled. Furthermore, $x_n^* \in (V_n)^o \subseteq (V_\infty)^o$, hence $x_\infty^* \in (V_\infty)^o \subseteq E^*$. And since (by $\boxed{4.3}$) the Cauchy-sequence x_n^* converges to x_∞^* uniformly on B_k for any $k \in \mathbb{N}$, we get that $x_n^* \to x_\infty^*$ in E^*. □

4.19 Corollary (See [**MV92**, 25.10 p.51]).
The bidual of any Fréchet space is a Fréchet space. □

Duals of Köthe sequence spaces

In this section we describe the duals of Köthe sequence spaces and characterize reflexivity (and the Montel property) of $\lambda^\infty(A)$ (and of $c_0(A)$). We also give an example of a Köthe sequence space, whose strong dual fails to be (infra-)barrelled.

4.20 Seminorms of $\lambda^p(A)^*$ (See [**MV92**, 27.13 p.314]).
Let A be countable, $\lambda := \lambda^p(A)$ for $1 \le p < \infty$ or $\lambda := c_0(A)$ for $p = \infty$. Then the Minkowski-functionals $(B_b^p)^o$ for $B_b^p := \{x : \|x/b\|_{\ell^p} \le 1\}$ (see 2.10) are given by

$$\|y\|_b := \|y \cdot b\|_{\ell^q}, \text{ for } b \in \lambda^\infty(A) \text{ and } \frac{1}{p} + \frac{1}{q} = 1,$$

and are a basis of the seminorms of

$$\lambda^* \cong \Big\{y \in \mathbb{K}^{\mathbb{N}} : \forall b \in \lambda^\infty(A) : \|y\|_b < \infty\Big\} = \Big\{y \in \mathbb{K}^{\mathbb{N}} : \exists a \in A : \|y\|_a^* < \infty\Big\},$$

where $\|\_\|_a^*$ is the Minkowski-functional of $\big\{x \in \lambda : \|x\|_a \le 1\big\}^o$ (see 1.24).

Proof. By 1.22 we have $\lambda^* \cong \lambda^1(\lambda)$ via $\big(x \mapsto \sum_{j=0}^\infty x_j\, y_j\big) \leftarrow\!\shortmid y$.

By 2.10 the sets $B_b^p := \{x : \|x/b\|_{\ell^p} \le 1\}$ (resp. $B_b^o : B_b^\infty \cap c_0(A)$) for $b \in \lambda^\infty(A)$ (w.l.o.g. $\forall j : b_j > 0$) form a basis of the bornology on $\lambda^p(A)$ (resp. $c_0(A)$). Let $y \in \lambda^1(\lambda) \cong \lambda^*$ and $\frac{1}{q} + \frac{1}{p} := 1$, then by 1.23 the Minkowski-functional $p_{(B_b^p)^o}$ is given by

$$\sup_{x \in B_b^p} |y(x)| = \sup_{x \in B_b^p} \Big|\sum_{j=0}^\infty x_j y_j\Big| = \sup\Big\{\Big|\sum_j \frac{x_j}{b_j} b_j y_j\Big| : \|x/b\|_{\ell^p} \le 1\Big\} = \|y \cdot b\|_{\ell^q} =: \|y\|_b.$$

$\lambda^* \cong \big\{y \in \mathbb{K}^{\mathbb{N}} : \forall b \in \lambda^\infty(A) : \|y\|_b < \infty\big\}$, since $y \in \mathbb{K}^{\mathbb{N}}$ acts as bounded(=continuous) linear functional $\Leftrightarrow \forall b \in \lambda^\infty(A) : \|y\|_b < \infty$.

$\lambda^* \cong \big\{y \in \mathbb{K}^{\mathbb{N}} : \exists a \in A : \|y\|_a^* < \infty\big\}$, since $\lambda^* = \bigcup_{a \in A} \lambda^*_{(U_a)^o}$, where $U_a := \{x \in \lambda : \|x\|_a < 1\}$, and $\|\_\|_a^*$ is the Minkowski-functional for $(U_a)^o$ by 1.24. □

4.21 $c_0(A)^{**} \cong \lambda^\infty(A)$ (See [**MV92**, 27.14 p.314]).

Proof. By 4.20 the family $(\|\_\|_b)_{b \in \lambda^\infty(A)}$ is a basis of seminorms for $c_0(A)^*$ and

$$c_0(A)^* \cong \Big\{y \in \mathbb{K}^{\mathbb{N}} : \|y\|_b := \sum_{j \in \mathbb{N}} |y_j b_j| < \infty\ \forall b \in \lambda^\infty(A)\Big\}. \Rightarrow$$

($\supseteq$) $\forall b \in \lambda^\infty(A)$: $y \mapsto \sum_{j \in \mathbb{N}} y_j b_j$ is in $c_0(A)^{**}$.

($\subseteq$) Let $x \in c_0(A)^{**}$: $\forall y \in c_0(A)^*$: $y = \sum_j y_j\, e_j$. $\Rightarrow$

$$x(y) = x\Big(\sum_j y_j e_j\Big) = \sum_j y_j \underbrace{x(e_j)}_{=:x_j} = \sum_j y_j x_j.$$

The family $(U_a^o)_{a \in A}$ is a basis of the bornology for $c_0(A)^*$ (see 4.18.1). $\Rightarrow$

$$\forall a \in A : \infty > \sup_{y \in U_a^o} |x(y)| \overset{1.24}{=\!=} \sup\Big\{\sum_j \Big|\frac{y_j}{a_j} a_j x_j\Big| : \sum_j \Big|\frac{y_j}{a_j}\Big| \le 1\Big\} = \sup_j |x_j a_j|,$$

i.e. $x \in \lambda^\infty(A)$.

Thus $c_0(A)^{**} = \lambda^\infty(A)$ as linear spaces and, by the closed graph theorem, also as lcs. □

4.22 Reflexivity of $\lambda^\infty(A)$ (See [MV92, 27.15 p.315]).
Let A be countable. Then

1. $c_0(A) = \lambda^\infty(A)$;

$\Leftrightarrow$ 2. $c_0(A)$ *is (M);*

$\Leftrightarrow$ 3. $c_0(A)$ *is reflexiv;*

$\Leftrightarrow$ 4. $\lambda^\infty(A)$ *is reflexiv.*

Proof. ($\boxed{1} \Rightarrow \boxed{2} \Rightarrow \boxed{3}$) by $\boxed{3.28}$ and $\boxed{3.22}$.
($\boxed{3} \Rightarrow \boxed{4}$) by $\boxed{4.21}$ $\lambda^\infty(A) = c_0(A)^{**}$ and duals of reflexive spaces are reflexiv by $\boxed{4.26}$.

($\boxed{4} \Rightarrow \boxed{1}$) $c_0(A) \subseteq \lambda^\infty(A)$ closed $\xRightarrow{\boxed{3.17}\text{, see also }\boxed{3.16}}$ $c_0(A)$ semi-reflexive $\Rightarrow c_0(A) = c_0(A)^{**} = \lambda^\infty(A)$, by $\boxed{4.21}$. $\square$

For $1 < p < \infty$ the space $\lambda^p(A)$ is reflexive by $\boxed{3.20}$ and thus distinguished by $\boxed{4.10}$. What about $\lambda^1(A)$?

4.23 Distinguishedness of $\lambda^1(A)$ (See [MV92, 27.17 p.316]).
*Let $A = \{a^{(k)} : k \in \mathbb{N}\}$ be countable and $\mathbb{R}_+ := \{t \in \mathbb{R} : t > 0\}$.
Then $\lambda^1(A)$ is distinguished* $\Leftrightarrow$

$$\Leftrightarrow \ \forall D : \mathbb{N} \to \mathbb{R}_+ \ \exists D' : \mathbb{N} \to \mathbb{R}_+ \ \forall C > 0 \ \forall n \ \exists n' \ \forall j :$$
$$\min\Big\{C a_j^{(n)}, \sup_{k\in\mathbb{N}} \frac{a_j^{(k)}}{D'_k}\Big\} \le \max\Big\{\frac{a_j^{(k)}}{D_k} : k \le n'\Big\}.$$

Proof. Since $E := \lambda^1(A)$ is Fréchet, E^* has a countable basis $\{U_n^o : n \in \mathbb{N}\}$ of its bornology. Hence a basis of the bornivorous disks is given by the absolutely convex hulls of $\bigcup_k \varepsilon_k U_k^o$ with $\varepsilon : \mathbb{N} \to \mathbb{R}_+$. Thus E^* is bornological iff

(1) $\qquad \forall \varepsilon : \mathbb{N} \to \mathbb{R}_+ \ \exists b \in \lambda^\infty(A) : (B_b^1)^o \subseteq \Big\langle \bigcup_k \varepsilon_k U_k^o \Big\rangle_{\text{abs.conv.}} \qquad$ (by $\boxed{2.10}$).

($\Rightarrow$) Let $D : \mathbb{N} \to \mathbb{R}_+$, $\varepsilon_k := 1/D_k \Rightarrow \exists b \in \lambda^\infty(A)$ as in $\boxed{1}$ $\xRightarrow{\boxed{2.9}}$ $\exists D' : \mathbb{N} \to \mathbb{R}_+$: w.l.o.g. $b : j \mapsto \inf_k D'_k/a_j^{(k)}$. Let $C > 0$, $n \in \mathbb{N}$ and $\xi : j \mapsto \min\{C a_j^{(n)}, 1/b_j\} \Rightarrow \xi \in (B_b^1)^o = \{y \in \lambda^* : \|y \cdot b\|_{\ell^\infty} \le 1\}$ by $\boxed{4.20}$ $\Rightarrow \exists n' \ \forall k \le n' \ \exists \xi^k \in \varepsilon_k U_k^o \ \exists \lambda_k \in \mathbb{R}$: $\sum_{k \le n'} |\lambda_k| \le 1$ and $\xi = \sum_{k\le n'} \lambda_k \xi^k$ by $\boxed{1}$. By $\boxed{1.24}$ $U_a^o = \{y : \|\frac{y}{a}\|_\infty = \|y\|_a^* \le 1\}$.

$$\Rightarrow \ \forall j : \min\Big\{C a_j^{(n)}, \sup_k a_j^{(k)}/D'_k\Big\} = \xi_j \le \sum_{k\le n'} |\lambda_k \xi_j^k| \le \max_{k\le n'} |\xi_j^k| \le \max_{k\le n'} a_j^{(k)}/D_k.$$

($\Leftarrow$) Let $\varepsilon : \mathbb{N} \to \mathbb{R}_+$, $D_k := 2^k/\varepsilon_k \Rightarrow \exists D' : \mathbb{N} \to \mathbb{R}_+$ as above. Let $b : j \mapsto \inf_k D'_k/a_j^{(k)}$ $\xRightarrow{\boxed{2.9}}$ $b \in \lambda^\infty(A)$ $\xRightarrow{\boxed{2.10}}$ B_b^1 bounded. Let $\xi \in (B_b^1)^o$ $\xRightarrow{\boxed{4.20}}$ $|\xi_j| \le \frac{1}{b_j} = \sup_k a_j^{(k)}/D'_k$ and $\exists a \in A : \xi \in E^*_{(U_a)^o} \Rightarrow \exists C > 0 \ \exists n$: $|\xi_j| \le \min\{C a_j^{(n)}, 1/b_j\} \le \max_{k\le n'} a_j^{(k)}/D_k$ for some n' by assumption. $\Rightarrow \forall j \ \exists k_j \le n'$: $|\xi_j| \le a_j^{(k_j)}/D_{k_j}$. Let

$$\xi^k : j \mapsto \begin{cases} \xi_j & \text{for } k = k_j \\ 0 & \text{otherwise} \end{cases}.$$

$\overset{\boxed{1.24}}{\Longrightarrow} 2^k\xi^k \in 2^k \frac{1}{D_k} U_k^o = \varepsilon_k U_k^o$ for $k \le n' \Rightarrow$

$$\xi = \sum_{k=1}^{n'} \xi^k = \sum_{k=1}^{n'} \frac{1}{2^k} 2^k \xi^k \in \Big\langle \bigcup_k \varepsilon_k U_k^o \Big\rangle_{\text{abs.conv.}}$$

Since $\xi \in (B_b^1)^o$ is arbitrary we are done. □

4.24 None-distinguishedness of $\boldsymbol{\lambda^1(A)}$ (See [**MV92**, 27.18 p.318])**.**
Let $A = \{a^{(k)} : k \in \mathbb{N}\}$, where the $a^{(k)} : \mathbb{N}^2 \to \mathbb{R}_+$ satisfy the following conditions:

1. $\forall i \ge k\ \forall j$: $a_{i,j}^{(k)} = a_{i,j}^{(0)}$.
2. $\forall m : \lim_{j\to\infty} a_{m,j}^{(m)}/a_{m,j}^{(m+1)} = 0$.

Then $\lambda^1(A)$ is not distinguished.

Proof. Suppose $\lambda^1(A)$ is distinguished $\overset{\boxed{4.23}}{\Longrightarrow} \forall k : D_k := 1,\ C := 2,\ n := 0$

$$\exists D' : \mathbb{N} \to \mathbb{R}_+\ \exists n'\ \forall i,j : \min\{2a_{i,j}^{(0)}, \sup_k a_{i,j}^{(k)}/D'_k\} \le \max_{k\le n'} a_{i,j}^{(k)}$$

For $i := n'$ we get $2a_{n',j}^{(0)} > a_{n',j}^{(0)} = \max_{k\le n'} a_{n',j}^{(k)}$ by ($\boxed{1}$) and hence

$$\forall j : a_{n',j}^{(n'+1)}/D'_{n'+1} \le \sup_k a_{n',j}^{(k)}/D'_k \le \max_{k\le n'} a_{n',j}^{(k)} = a_{n',j}^{(0)} = a_{n',j}^{(n')},$$

a contradiction to ($\boxed{2}$) for $m := n'$. □

4.25 Example (See [**MV92**, 27.19 p.318])**.**
Let $a_{i,j}^{(k)} := j^i$ for $k \le i$ und $a_{i,j}^{(k)} := j^k$ for $k > i$ and $A := \{a^{(k)} : k \in \mathbb{N}\}$.
Then $\lambda^1(A)$ is not distinguished, so $(\lambda^1(A))^*$ is (DF) but not infra-barrelled.

Semi-reflexivity and stronger conditions on dual spaces

4.26 Proposition (See [**Jar81**, 11.4.5 p.228])**.**
E reflexive $\Rightarrow$ E^* reflexive.

Proof. By assumption $\delta_E : E \to E^{**}$ is an isomorphism. Thus also $(\delta_E)^* : (E^{**})^* \to E^*$. We claim that $\mathrm{id}_{E^*} = (\delta_E)^* \circ \delta_{E^*} : E^* \to (E^*)^{**} = (E^{**})^* \xrightarrow{\cong} E^*$:

$$((\delta_E)^* \circ \delta_{E^*})(x^*)(x) = (\delta_E)^*(\delta_{E^*}(x^*))(x) = \delta_{E^*}(x^*)(\delta_E(x)) = \delta_E(x)(x^*) = x^*(x).$$

So $\delta_{E^*} = ((\delta_E)^*)^{-1} = ((\delta_E)^{-1})^*$ is an isomorphism. □

4.27 Proposition (See [**Jar81**, 11.5.4 p.230])**.**
E Montel $\Rightarrow$ E^* Montel.

Proof. Let E be Montel and $B \subseteq E^*$ bounded. Thus B is equicontinuous (since E is infra-barrelled by definition) and therefore relatively compact with respect to $\tau_{pc}(E^*, E)$ by the Alaŏglu-Bourbaki Theorem $\boxed{3.4}$. Since E is semi-Montel $\tau_{pc}(E^*, E) = \beta(E^*, E)$, so E^* is semi-Montel.

Since E is reflexive by $\boxed{3.22}$, the dual E^* is reflexive by $\boxed{4.26}$ and hence is (infra-)barrelled by $\boxed{4.10}$. Together this shows that E^* is Montel. □

4.28 Proposition. Schwartz versus quasi-normable spaces
(See [**Kri07a**, 6.5], [**Jar81**, 10.7.3 p.215]).
An lcs is Schwartz iff it is quasi-normable and every bounded set is precompact.

An lcs E is called QUASI-NORMABLE (see [**Jar81**, 10.7.1 p.214]) iff

$$\forall U\ \exists V\ \forall \varepsilon > 0\ \exists B \text{ bounded} : V \subseteq B + \varepsilon\, U.$$

Note that any normed space is quasi-normable. In fact we may take $V = B := U$.

Proof. In the proof of 3.31 we have shown that each bounded set in a Schwartz space is precompact.
By definition an lcs E is Schwartz iff

$$\forall U\ \exists V\ \forall \varepsilon > 0\ \exists M \text{ finite} : V \subseteq M + \varepsilon\, U.$$

Thus every Schwartz space is quasi-normable. And if every bounded set B is precompact, then there is a finite set $M \subseteq E$ such that $B \subseteq M + \varepsilon\, U$, and we have the converse implication. $\square$

4.29 Counter-example.
Note that $E := \mathbb{R}^X$ is Schwartz and even nuclear for all sets X by 3.73.1.
However, if X is uncountable then the dual $E^* = \mathbb{R}^{(X)}$ is not quasi-normable (hence neither Schwartz nor nuclear).
Suppose E^* were quasi-normable. Recall that the typical seminorms on $\mathbb{R}^{(X)}$ are given by $f \mapsto \sum_x c_x |f_x|$ with $c_x \geq 0$, see [**Kri07b**, 4.6.1]. Thus for the seminorm with $c_x := 1$ for all x there exist another seminorm given by some corresponding $c_x > 0$ such that for all $\varepsilon > 0$ there is some bounded set B_ε with

$$(1) \qquad \Big\{f : \sum_x c_x |f_x| \leq 1\Big\} \subseteq B_\varepsilon + \varepsilon \cdot \Big\{f : \|f\|_{\ell^1} := \sum_x |f_x| \leq 1\Big\}.$$

For some $\delta > 0$ the set $I := \{x : c_x \leq \frac{1}{\delta}\}$ has to be (uncountably) infinite. Now choose $\varepsilon = \frac{\delta}{2}$. Then B_ε is contained in a finite subsum, so there is some $x \in I$ with $\mathrm{pr}_x(B_\varepsilon) = \{0\}$. Since $\delta \cdot e_x$ is an element of the left hand side of 1, there has to exist a $b \in B_\varepsilon$ and an f with $\|f\|_{\ell^1} \leq 1$ with $\delta \cdot e_x = b + \varepsilon \cdot f$ and hence $\mathrm{pr}_x(b) \geq \delta - \frac{\delta}{2} > 0$, a contradiction.

4.30 Proposition (See [**Jar81**, 10.7.1 p.214]).
Any lcs E is quasi-normable iff $\forall U\ \exists V \subseteq U : (U^o, \beta(E^*, E)) \hookrightarrow E^*_{V^o}$ is a topological embedding.

Proof. This inclusion is continuous (and then an embedding, since $E^*_{V^o} \rightarrowtail E^*_\beta$ is continuous) iff $\forall \lambda > 0\ \exists B$ bounded closed absolutely convex with $U^o \cap B^o \subseteq \lambda V^o$.
($\Leftarrow$) $(B + U)^o \subseteq U^o \cap B^o \subseteq \lambda V^o \Rightarrow V \subseteq \lambda((B + U)^o)_o = \lambda\, \overline{B + U} \subseteq \lambda B + 2\lambda U$.
($\Rightarrow$) $V \subseteq B + \lambda U \Rightarrow 2V^o \supseteq 2(B + \lambda U)^o \supseteq B^o \cap (\lambda U)^o \Rightarrow 2\lambda V^o \supseteq U^o \cap \lambda B^o$. $\square$

4.31 Proposition (See [**Jar81**, 12.3.1 p.254]).
Let $(A_n)_{n\in\mathbb{N}}$ be an absorbent (bornivorous) sequence of subsets in E and $\mathcal{U}$ a 0-nbhd basis consisting of absolutely convex sets.
Then the absolutely convex hulls of $\bigcup_{k\geq 1} A_k \cap U_k$ with $U_k \in \mathcal{U}$ (resp. the absolutely convex sets $\bigcap_{k\geq 0}(A_k + U_k)$) form a basis for the finest locally convex topology, which coincides with the given one on each A_k.

By an ABSORBING (resp. BORNIVOROUS) sequence $(A_n)_{n\in\mathbb{N}}$ in an lcs E we understand a sequence of absolutely convex subsets $A_n \subseteq E$ with $A_0 := \{0\}$, $2A_n \subseteq A_{n+1}$,

and such that each finite (resp. bounded) subsets of E is absorbed by (and hence contained in) A_n for some n (See [**Jar81**, 12.3 p.253]).

Proof. It is easy to see, that these absolutely convex hulls form a basis for a locally convex topology $\tau^{\mathcal{A}}$, which is finer than the given one and which coincides with the given one on each A_n: In fact $\tau^{\mathcal{A}} \to E$ is continuous ($U_k := U$ and use $E = \bigcup_k A_k$) and $A_n \to \tau^{\mathcal{A}}$ is continuous ($A_n \cap \bigcup_k A_k \cap U_k \supseteq A_n \cap A_n \cap U_k$).

Let now τ be another topology with that property and V be an absolutely convex 0-nbhd for τ. Thus for each n there is a 0-nbhd U_n with $A_n \cap U_n \subseteq V$, hence the absolutely convex hull of $\bigcup_n A_n \cap U_n$ is contained in V, i.e. $\tau^{\mathcal{A}} \geq \tau$.

Remains to show that the two bases are equivalent:
($\supseteq$) Let $V := \bigcap_{k\geq 0}(A_k + U_k)$. Choose V_k with $((V_k)^o)_o \subseteq U_k$ and $U_k' \subseteq \bigcap_{i<k} V_i$. Since $A_m \subseteq A_k$ for all $k \geq m$, we get $A_m \cap U_m' \subseteq \bigcap_{k\geq m} A_k \cap \bigcap_{k<m} V_k \subseteq \bigcap_k \big(A_k + ((V_k)^o)_o\big) \subseteq V$, thus V contains the absolutely convex hull of $\bigcup_m A_m \cap U_m'$.
($\subseteq$) Let now U be the absolutely convex hull of $\bigcup_{m\geq 1} A_m \cap U_m$. Let $k_n := 2n+1$. Then $A_{n+1} \subseteq 2^{-n} A_{k_n}$ and there exists V_n with $V_n \subseteq 2^{-n} U_{k_n}$ and $2((V_{n+1})^o)_o \subseteq V_n$. We claim that $V := \bigcap_{n\geq 0}(A_n + V_{n+2}) \subseteq U$: Let $x \in V$, i.e. $x = y_n + v_n$ with $y_n \in A_n$ (thus $y_0 = 0$) and $v_n \in V_{n+2}$. Thus $x = v_n + \sum_{i=1}^n x_i$, where $x_i := y_i - y_{i-1}$. So $x_i \in A_i + A_{i-1} \subseteq A_{i+1}$ and $x_i = v_{i-1} - v_i \in V_{i+1} - V_{i+2} \subseteq V_i$. Hence $x_i \in A_{i+1} \cap V_i$ for all $1 \leq i \leq n$. By the properties of $\mathcal{A} := (A_n)_n$ we have $x \in A_n$ for some n, hence $x - y_n = v_n \in 2A_n \cap V_{n+2} \subseteq A_{n+2} \cap V_{n+1}$, thus

$$x = \sum_{i=1}^n x_i + v_n \in \sum_{i=1}^n A_{i+1} \cap V_i + A_{n+2} \cap V_{n+1} \subseteq \sum_{i=1}^{n+1} 2^{-i} \cdot (A_{k_i} \cap U_{k_i}) \subseteq U. \quad \square$$

4.32 Proposition (See [**Jar81**, 12.3.5 p.255]).
Let E be (quasi-)countably-barrelled and $(A_n)_{n\in\mathbb{N}}$ an absorbent (bornivorous) sequence of subsets in E. Let $0 < \rho_n \nearrow \infty$. Then an absolutely convex set U is a 0-nbhd in E iff $U \cap \rho_n A_n$ is a 0-nbhd in $\rho_n A_n$ for each n.

Proof. ($\Leftarrow$) Let U be absolutely convex and $U \cap \rho_n A_n$ a 0-nbhd in $\rho_n A_n$ for each n. So let U_n be absolutely convex 0-nbhds in E with $U_n \cap \rho_n A_n \subseteq U$. Thus $V := \bigcap_n \overline{U \cap \rho_n A_n + U_n}$ is an intersection of countably many closed absolutely convex 0-nbhds. Let $B \subseteq E$ be finite (resp. bounded). Thus $B \subseteq \rho A_m$ for some $\rho > 0$ and $m \in \mathbb{N}$. Since $\rho_n \nearrow \infty$ we may assume that $B \subseteq \rho_m A_m$. Choose $\sigma \geq 1$ with $B \subseteq \sigma U_k$ for all $k \leq m$. Then

$$B \subseteq \sigma(U_m \cap \rho_m A_m) \subseteq \sigma(U \cap \rho_m A_m) \subseteq \sigma(U \cap \rho_k A_k) \text{ for all } k \geq m.$$

Thus $B \subseteq \sigma((U \cap \rho_k A_k) + U_k)$ for all k, and hence $B \subseteq \sigma V$. Since E is (quasi-)countably-barrelled, V is a 0-nbhd. Thus it suffices to show $V \subseteq 3U$: Let $x \in V$. Take m with $x \in \rho_m A_m$. Then $V \subseteq \overline{(U \cap \rho_m A_m) + U_m} \subseteq (U \cap \rho_m A_m) + 2U_m$, i.e. $x = y + z$ with $y \in U \cap \rho_m A_m$ and $z \in 2U_m$. So $x - y = z \in (\rho_m A_m + U \cap \rho_m A_m) \cap 2U_m \subseteq 2(\rho_m A_m \cap U_m) \subseteq 2U$ and hence $x \in 3U$. $\square$

4.33 Corollary (See [**Jar81**, 12.3.6 p.256]).
Let E be (quasi-)countably-barrelled. Then for every absorbent (bornivorous) sequence $(A_n)_{n\in\mathbb{N}}$ of subsets in E the induced locally convex topology is the given one.

Proof. Obviously the final topology induced by the $A_n \hookrightarrow E$ coincides on nA_n with the given one. So every 0-nbhd U for this locally convex topology is a 0-nbhd for the original topology by 4.32. $\square$

4.34 Lemma (See [**Kri07a**, 3.46], [**Jar81**, 12.4.7 p.260]).
Every (DF) space is quasi-normable.

Proof. Let $\{B_n : n \in \mathbb{N}\}$ be a basis of the bornology and $U = (U^o)_o$ a 0-nbhd. Consider the equicontiuous sets $A_k := k\, U^o \cap B_k^o = (\frac{1}{k} U \cup B_k)^o$.
We claim that $A := \bigcup_{k\in\mathbb{N}} A_k$ is equicontinuous: $\forall k\ \exists n_k \geq k : B_k \subseteq n_k\, U$. Thus

$$B_k \cap \tfrac{1}{n_k} U = B_k \cap \Big(\bigcap_{n\leq n_k} \tfrac{1}{n} U\Big) \cap \Big(\bigcap_{n>n_k} B_n\Big) \subseteq B_k \cap \Big(\bigcap_{n\in\mathbb{N}} \tfrac{1}{n} U \cup B_n\Big) \subseteq$$
$$\subseteq B_k \cap \bigcap_n (A_n)_o = B_k \cap A_o \subseteq A_o,$$

So the absolutely convex hull of $\bigcup_{k\in\mathbb{N}} \frac{1}{n_k} U \cap B_k \subseteq A_o$, thus A_o is a 0-nbhd in the (DF)-space E by 4.31 and 4.33, and hence $A \subseteq E^*$ is equicontinuous.

We claim that U^o with the topology induced from $\beta(E^*, E)$ continuously embeds into $E^*_{V^o}$ for $V := A_o$: For the typical 0-nbhd $\frac{1}{k} V^o$ in $E^*_{V^o}$ we have that the $\beta(E^*, E)$-0-nbhd $U^o \cap \frac{1}{k} B_k^o$ in U^o satisfies

$$U^o \cap \frac{1}{k} B_k^o = \frac{1}{k} A_k \subseteq \frac{1}{k} A \subseteq \frac{1}{k} V^o.$$

Thus E is quasi-normable by 4.30. □

4.35 Proposition (See [**Jar81**, 11.6.1 p.231]).
A Fréchet space is Montel iff E^* is Schwartz.

Proof. ($\Rightarrow$) We use 3.33, so for every 0-nbhd $B^o \subseteq E^*$ we have to find a 0-nbhd C^o with $B^{oo} \subseteq E^{**}_{C^{oo}}$ being compact. Since E is reflexive by 3.22, this means that for closed bounded $B \subseteq E$ we have to find such a C with B in E_C compact. Since E is Montel, B is compact and hence contained in the closed absolutely convex hull of a 0-sequence (x_n) in E by 3.6. Since E is metrizable we find $\lambda_n \to \infty$ with (the closed absolutely convex hull C of) $\{\lambda_n x_n : n \in \mathbb{N}\}$ bounded by [**Kri14**, 2.1.6]. Then $x_n \to 0$ in E_C (since $p_C(x_n) \leq \frac{1}{\lambda_n}$) and thus its closed absolutely convex hull in the Banach space E_C is (pre)compact and contains B.

($\Leftarrow$) Since E is Fréchet, the dual E^* is complete, hence semi-Montel by 3.31. Thus every bounded=equicontinuous subset of E^* is relatively compact. Hence $\beta(E^*, E) = \tau_{pc}(E^*, E) = \gamma(E^*, E)$ by 3.24. Since $\beta(E^*, E) \geq \mu(E^*, E)$ always and $(E^*_\gamma)^* = \tilde{E} = E$ by [**Kri07b**, 5.5.7] we have $\beta(E^*, E) = \mu(E^*, E)$. The Fréchet space E is reflexiv, since every continuous linear functional on $(E^*, \beta(E^*, E)) = (E^*, \mu(E^*, E))$ belongs to E by definition of $\mu(E^*, E)$. By 3.18 and 4.26 E^* is (infra-)barrelled, hence Montel by 3.31 and thus also $E \cong (E^*)^*$ by 4.27. □

4.36 Lemma. Schwartzification (See [**Jar81**, 10.4.4 p.203]).
The topology τ_S of uniform convergence on $\mathcal{E}$-0-sequences is the finest Schwartz topology coarser than the given one.

A sequence $x_n^* \in E^*$ is said to be an $\mathcal{E}$-0-SEQUENCE, iff there exists some equicontinuous set U^o with $x_n^* \to 0$ in $E^*_{U^o}$, i.e. x_n^* is Mackey-convergent to 0 with respect to the bornology of equicontinuous sets ($\mathcal{E}$ stands for equicontinuous).
We will also write E_S for the SCHWARTZIFICATION (E, τ_S) of E. Note, that the topology τ_S is denoted $\mathcal{T}_{c_0}$ and E_S is denoted E_0 in [**Jar81**, 10.4.3 p.203].

Proof.
($E \geq \tau_S$) since $\mathcal{E}$-0-sequences are equicontinuous.

τ_S is Schwartz by 3.33, since for every polar A_o of an $\mathcal{E}$-0-sequence x_n^* there exist $\lambda_n \to \infty$ such that $y_n^* := \lambda_n x_n^*$ is still a 0-sequence in $E_{U^o}^*$ and hence $(A_o)^o$ (the $\sigma = \beta$ compact closure of the absolutely convex hull of $\{x_n^* : n \in \mathbb{N}\}$) is compact in $E_{(B_o)^o}^*$, where $B := \{y_n^* : n \in \mathbb{N}\}$.

Now let $\tau' \leq E$ be a Schwartz topology and U be a closed absolutely convex 0-nbhd with respect to τ'. Then by 3.33 there exists a $(\tau'$-$)$0-nbhd $V \subseteq U$ with $U^o \subseteq E_{V^o}^*$ compact and hence contained in the closed convex hull of a 0-sequence in $E_{V^o}^*$. Since V is also a 0-nbhd in E, this sequence is an $\mathcal{E}$-0-sequence and hence $U = (U^o)_o$ (since U is also E-closed) is a τ_S-0-nbhd. □

4.37 Proposition. Universal Schwartz space
(See [**Kri07a**, 6.26], [**Jar81**, 10.5.1 p.204]).
The Schwartz spaces are exactly the subspaces of products of the Schwartzification of c_0, or of its completion $\widehat{(c_0)_S} = (\ell^\infty, \mu(\ell^\infty, \ell^1))$.

The first statement is sometimes also called Schur's lemma, see [**Jar81**, p.218]. There is however no universal (FS)-space, see [**Jar81**, 10.9 p.218].

Proof. In fact by 4.36 Schwartz spaces have a basis of 0-neighborhoods given by the polars $V := \{x_n^* : m \in \mathbb{N}\}_o$ of $\mathcal{E}$-0-sequences $(x_n^*)_{n\in\mathbb{N}}$ in E^*. Since $p_V(x) = \sup\{|x_n^*(x)| : n \in \mathbb{N}\}$, the map $T : x \mapsto (x_n^*(x))_{n\in\mathbb{N}}$ defines a continuous linear map from $E \to c_0$ and factors over $\iota_V : E \twoheadrightarrow E_V$ as $T = \tilde{T} \circ \iota_V$ with an isometric mapping $\tilde{T} : E_V \hookrightarrow c_0$. Since $(\iota_V)_V : E \hookrightarrow \prod_V E_V$ is an embedding, we get an embedding $E \hookrightarrow \prod_V c_0$.

It is easy to see that $\tilde{T} : (E_V)_S \to (c_0)_S$ is continuous: Let $S \in \mathcal{L}(E, F)$ and $y_n^* \to 0$ in $(F_V)^* \cong F_{V^o}^*$. Then $S^{-1}\big(\{y_n^* : n \in \mathbb{N}\}_o\big) = \{S^*(y_n^*) : n \in \mathbb{N}\}_o$ and $S^*(y_n^*) \to 0$ in $E_{S^*(V^o)}^* \subseteq E_{(S^{-1}V)^o}^*$, i.e. $S \in \mathcal{L}(E_S, F_S)$.
It is an embedding, since $\tilde{T}^* : \ell^1 \to (E_V)^* = E_{V^o}^*$ is a quotient map between Banach spaces, hence every 0-sequence in the image is the image of a 0-sequence in the domain.

Remains to show that E embeds into the reduced projective system of the $(E_V)_S$: Obviously $E = E_S \to \varprojlim_V (E_V)_S$ is continuous.

Conversely, let $V := \{x_n^* : n \in \mathbb{N}\}_o$ with $x_n^* \to 0$ in $(E_{V'})^*$ for some V' and $A := \{x_n^* : n \in \mathbb{N}\} \cup \{0\} \subseteq (E_{V'})^*$. Since $\iota_{V'}^*(A)_o = \iota_{V'}^{-1}(A_o)$, we have $E_{V'} \supseteq A_o = \iota_{V'}(\iota_{V'}^{-1}(A_o)) = \iota_{V'}(\iota_{V'}^*(A)_o) = \iota_{V'}(V)$, a 0-nbhd in $(E_{V'})_S$ with $\iota_V^{V'}(A_o) = \iota_V^{V'}(\iota_{V'}(V)) = \iota_V(V)$, thus $\iota_V^{V'} : (E_{V'})_S \to E_V$ is continuous and hence also ι_V from $E \subseteq \varprojlim_V (E_V)_S$ into E_V.

$$\begin{tikzcd}
E \arrow[d, hook] \arrow[dd, bend right, "\iota_V"'] \arrow[ddrr, bend left, "\iota_{V'}"] & & \\
\varprojlim_V E_V \arrow[d] \arrow[r, dotted, bend left] & \varprojlim_V (E_V)_S \arrow[l, dotted, "\mathrm{id}"'] \arrow[d] \arrow[dl, dotted] & \\
E_V & (E_{V'})_S \arrow[l] & E_{V'} \arrow[l, two heads, "\mathrm{id}"'] \arrow[ll, bend left, "\iota_V^{V'}"]
\end{tikzcd}$$

Thus the identity from (the subspace E of) $\varprojlim_V (E_V)_S \to \varprojlim_V E_V$ is continuous.

That $\widetilde{(c_0)_S} = (\ell^\infty, \mu(\ell^\infty, \ell^1))$ can be found in [**Jar81**, 10.5.3 p.206]. □

4.38 Proposition. Nuclearification (See [**Jar81**, 21.9.1 p.508]).
The finest nuclear locally convex topology coarser than the given one is the topology τ_N of uniform convergence on $\mathcal{E}$-nuclear sequences.

A sequence x_n^* in E^* is called $\mathcal{E}$-NUCLEAR (cf. 3.79), iff for each $k \in \mathbb{N}$ there is a 0-nbhd U_k such that $(n^k x_n^*)_{n\in\mathbb{N}}$ is a 0-sequence (or ℓ^p for $0 < p < \infty$) in $E_{U_k^o}^*$ (See

[**Jar81**, 21.7.1 p.500]).
We will also write E_N for the NUCLEARIFICATION (E, τ_N) of E.

Proof. $(E \geq \tau_N)$ since $\mathcal{E}$-nuclear-sequences are equicontinuous (take $k := 0$).

(τ_N is nuclear)
By definition the polars of $\mathcal{E}$-0-sequences $(a_n)_{n\in\mathbb{N}}$ form a 0-nbhd basis for E_N. So let such an $U := \{a_n : n \in \mathbb{N}\}_o$ be given and put $V := \{n^2 a_n : n \in \mathbb{N}\}_o \subseteq U$.
The mapping $S : \ell^1 \to E^*_{U^o}$ defined by $x = (x_n)_{n\in\mathbb{N}} \mapsto \sum_n x_n\, a_n$ is obviously well-defined continuous linear and it is onto, since any $x^* \in E^*_{U^o}$ is in $C\, U^o$ for some $C > 0$ hence of the form $x^* = \sum_n x_n\, a_n$ with $(x_n)_{n\in\mathbb{N}} \in C\, o\ell^1$.
Similarly we have $T : \ell^1 \to E^*_{V^o}$ defined by $x \mapsto \sum_n n^2\, x_n\, a_n$.

$$\begin{tikzcd} \ell^1 \arrow[r, "D"] \arrow[d, "S"'] & \ell^1 \arrow[d, "T"] \\ E^*_{U^o} \arrow[r, tail, "\iota"] & E^*_{V^o} \end{tikzcd}$$

Let $D : \ell^1 \to \ell^1$ be the (by 3.58) nuclear diagonal mapping $(x_n)_{n\in\mathbb{N}} \mapsto (\frac{1}{n^2} x_n)$ and $\iota = (\iota_U^V)^* : E^*_{U^o} \rightarrowtail E^*_{V^o}$ the natural inclusion. Thus $\iota \circ S = T \circ D$ is nuclear and thereby $S^* \circ \iota^*$ is nuclear by 4.45 below and in particular absolutely summing by 3.62. The adjoint S^* of a quotient mapping S between Banach spaces is a topological embedding (use $(E/F)^* \cong F^o$), thus $\iota^* = (\iota_U^V)^{**}$ is absolutely summing and hence also its restriction $\widetilde{E_V} \to \widetilde{E_U}$, i.e. (E, τ_N) is nuclear by 3.70.

Now let $\tau' \leq E$ be some nuclear topology and let U be a closed absolutely convex τ'-0-nbhd, which we may assume to be the unit-ball of a Hilbert seminorm by 3.72. By 3.79 there exists a fast-falling ONB $(e_n)_{n\in\mathbb{N}}$ of $E^*_{U^o}$, i.e. $\forall k\, \exists V : \{n^k e_n : n \in \mathbb{N}\} \subseteq V^o$. Let $\rho := \|(\frac{1}{n})_{n\in\mathbb{N}}\|_{\ell^2}$ and $a_n := \rho\, n\, e_n$. Then $U^o \subseteq (\{a_n : n \in \mathbb{N}\}_o)^o$: For $x^* \in U^o$ define $(x_n)_n \in o\ell^1$ by $x_n := \frac{\langle x^*, e_n\rangle}{\rho\, n}$. Then

$$\Big\langle \sum_n x_n a_n, e_k \Big\rangle = \Big\langle \sum_n \langle x^*, e_n\rangle\, e_n, e_k \Big\rangle = \sum_n \langle x^*, e_n\rangle\, \langle e_n, e_k\rangle = \langle x^*, e_k\rangle,$$

i.e. $x^* = \sum_n x_n\, a_n \in (\{a_n : n \in \mathbb{N}\}_o)^o$.
Since the sets V from above are also 0-nbhds in E, the sequence $(a_n)_{n\in\mathbb{N}}$ is an $\mathcal{E}$-nuclear-sequence and hence $U = (U^o)_o \supseteq \{a_n : n \in \mathbb{N}\}_o$ is a τ_N-0-nbhd. □

4.39 Proposition (See [**Jar81**, 12.5.8 p.265], [**Woz13**, 4.42 p.85])**.**
An lcs is the dual of an (FM)-space iff it is (S) and a complete (DF)-space.
It is then even ultra-bornological.

Proof. ($\Rightarrow$) Let $F = E^*$ with E an (FM)-space. Then F is a complete (DF)-space by 4.18.1 and is (S) by 4.35. It is then even barrelled by 3.22 and 4.10 and hence ultra-bornological by 4.16.

($\Leftarrow$) By 4.18.2 the dual $E := F^*$ of the (DF)-space F is (F) and it is (M): Let $B \subseteq E$ be bounded. Since E is metrizable it is enough to show that every countable subset of B is relatively compact. W.l.o.g. let $B = \{b_n : n \in \mathbb{N}\}$ and consider $B_o = \bigcap_{n\in\mathbb{N}} (b_n)_o$, a countable intersection of closed 0-nbhds in F. This set B_o is bornivorous, hence a 0-nbhd in F by the infra-countably-barrelledness of the (DF) space F: In fact, let $A \subseteq F$ be bounded. Then A^o is a 0-nbhd in E and hence absorbs the bounded set B. Thus B_o absorbs A. Since F is Schwartz, there is a $\mathcal{E}$-0-sequence (y_n^*) in F^* with $B_o \supseteq \{y_n^* : n \in \mathbb{N}\}_o$ by 4.36 and hence B is contained in the (compact) closed absolutely convex hull of $\{y_n^* : n \in \mathbb{N}\}$. Thus $E = F^*$ is semi-Montel and as (F) space even Montel.

Since $E = F^*$ is a Montel space, $\beta(F^{**}, F^*) = \tau_c(F^{**}, F^*)$. Hence

$$\beta^*(F, F^*) = \beta(F^{**}, F^*)|_F = \tau_c(F^{**}, F^*)|_F = \tau_S,$$

where $\beta^*(F,F^*)$ denotes the topology of uniform convergence on bounded sets in F^*_β and τ_S is the topology on F of uniform convergence on $\mathcal{E}$-0-sequences:
To see the last inequality, observe that $\mathcal{E}$-0-sequences are relatively $\beta(F^*,F)$-compact by 3.6, hence $\tau_S \le \tau_c(F^{**},F^*)|_F$. To show the converse, note that by 3.6 for every relatively compact set A in the Fréchet space F^*, there is a 0-sequence $(a_n)_{n\in\mathbb{N}}$ in the F^* (which is also an $\mathcal{E}$-0-sequence by 2.2) such that A is contained in the closed absolutely convex hull of the a_n, and therefore $\{a_n : n \in \mathbb{N}\}^o \subseteq A^o$.

Since F is complete and Schwartz, it is semi-reflexive by 3.31 and 3.22 and, by what we have just shown, it carries the topology $\tau_S = \beta^*(F,F^*)$, which is quasi-barrelled ($A \subseteq F^*_\beta$ is bounded $\Leftrightarrow$ A_o is a bornivorous barrel). Thus F is reflexive, hence $F = F^{**} = E^*$. □

4.40 Proposition (See [**Flo71**, 5.4 p.164])**.**
Each locally complete (LF)-space is regular.

An (LF)-space is the reduced inductive limit of a sequence of Fréchet spaces and similarly an (LB)-space is the reduced inductive limit of a sequence of Banach spaces.

Proof. Let B be bounded closed and absolutely convex, thus E_B is a Banach space by local completeness. By Grothendieck's factorization theorem 2.6 $E_B \rightarrowtail E$ factors over some $\iota^n : E_n \rightarrowtail E = \varinjlim_k E_k$ to a continuous linear mapping $E_B \to E_n$, hence B is bounded in E_n. □

4.41 Raikov's completeness theorem
(See [**Rai59**], [**Flo71**, 4.1 p.162], [**Sch12**, 2.11 p.36])**.**
Let E be an lcs and $(A_n)_{n\in\mathbb{N}}$ be an absorbent sequence of subsets of E satisfying:

1. *The lcs E carries the final locally convex topology with respect to $(A_n)_{n\in\mathbb{N}}$.*
2. *Every Cauchy-net in any A_n is convergent in E.*

Then E is complete.

Proof. Let $(x_j)_{j\in J}$ be a Cauchy net in E and $\mathcal{U}$ be a 0-nbhd basis of absolutely convex sets. We claim the following:

$$\exists n_0 \in \mathbb{N}\, \forall U \in \mathcal{U}\, \forall j \in J\, \exists i \succ j : x_i \in U + A_{n_0}.$$

Otherwise, $\forall n\, \exists U_n\, \exists j_n\, \forall j' \succ j_n : x_{j'} \notin U_n + A_n$. Put $V := \bigcap_n (U_{n+1} + A_n)$. Let $x \in (U_m + A_{m-1}) \cap A_{m-1}$ and $n \ge m$, then $x = x' + u$ with $x, x' \in A_{m-1}$ and $u \in U_m$, thus $u = x - x' \in 2A_{m-1} \subseteq A_n$, i.e. $x = 0 + x \in U_{n+1} + A_n$. Therefore $V \cap A_m = \bigcap_{n<m}(U_{n+1} + A_n) \cap A_m$ is a 0-nbhd in A_m, hence a 0-nbhd in E by 1. Since $(x_j)_{j\in J}$ is Cauchy, there exists $j \in J$ such that $x_{j'} - x_{j''} \in V$ for all $j', j'' \succ j$. Since $(A_n)_{n\in\mathbb{N}}$ is absorbing there exists an n with $x_{j''} \in A_{n-1}$ and hence $x_{j'} \in x_{j''} + V \subseteq A_{n-1} + V \subseteq A_{n-1} + (U_n + A_{n-1}) \subseteq U_n + A_n$ for all $j' \succ j$, a contradiction.

Now consider the net $\tilde{x} : J \times \mathcal{U} \to A_{n_0} \subseteq E$, which assigns to each (j,U) an element $\tilde{x}_{j,U} := x_i - u \in A_{n_0}$ with $i \succ j$ and $u \in U$. This net is Cauchy and hence converges to some x_∞ in E by 2, since for $U \in \mathcal{U}$ there exist $W \in \mathcal{U}$ with $3W \subseteq U$ and $j \in J$ such that $x_{i'} - x_{i''} \in W$ for all $i', i'' \succ j$. So

$$\tilde{x}_{j',U'} - \tilde{x}_{j'',U''} = (x_{i'} - u') - (x_{i''} - u'') = (x_{i'} - x_{i''}) - u' + u'' \in 3W \subseteq U$$

for all $(j',U'), (j'',U'') \succ (j,W)$ and hence $u' \in U' \subseteq W$, $u'' \in U'' \subseteq W$, $i' \succ j' \succ j$, and $i'' \succ j'' \succ j$.
It follows that $(x_j)_{j\in J}$ converges to x_∞: For any $U \in \mathcal{U}$ there exist $W \in \mathcal{U}$ with

$3W \subseteq U$ and $j \in J$ with $x_{j'} - x_{i'} \in W$ for all $i' \succ j' \succ j$ and $\tilde{x}_{j',U'} - x_\infty \in W$ for all $(j', U') \succ (j, W)$, and thus $x_{j'} - x_\infty = x_{j'} - x_{i'} + u' + \tilde{x}_{j',U'} - x_\infty \in 3W \subseteq U$. □

4.42 Proposition
(See [**Sch12**, 2.14 p.38], [**Flo71**], [**Rai59**], cf. [**Jar81**, 12.5.2 p.263]).
An (LB)-space is complete if and only if it is quasi-complete.

Proof. (⇐) Let $(E, \tau) = \varinjlim_n (E_n, \tau_n)$ denote an (LB)-space, and let $B_n := oE_n$ be the closed unit ball of the Banach space (E_n, τ_n). We will apply 4.41 for $A_n := 2^n B_n$.

Since, for each $n \in \mathbb{N}$, we may assume B_n to be continuously injected into B_{n+1}, the sequence $(A_n)_{n\in\mathbb{N}}$ is an absorbing sequence.

To prove 1, let $V \subseteq E$ be an absolutely convex set such that $V \cap A_n$ is a 0-neighborhood of $(A_n, \tau|_{A_n})$ for each $n \in \mathbb{N}$ and thus also a 0-neighborhood of $(A_n, \tau_n|_{A_n})$. Since A_n is a closed 0-neighborhood of (E_n, τ_n), we see that $V \cap A_n$ and hence $V \cap E_n \supseteq V \cap A_n$ are also 0-neighborhoods of (E_n, τ_n). This holds for all $n \in \mathbb{N}$, which means that V has to be a 0-neighborhood of the inductive limit (E, τ).

Remains to show condition 2, i.e. that each τ-Cauchy net contained in some A_n converges in E. But this is clear by the quasi-completeness of E since the sets A_n and hence their Cauchy nets are bounded. . □

Let $E = \varinjlim_n E_n$ be a reduced inductive limit with compact connecting mappings $T_n : E_n \rightarrowtail E_{n+1}$, i.e. which map some absolutely convex 0-nbhd $U_n \subseteq E_n$ to a relative compact subsets of $T_n(U_n) \subseteq E_{n+1} \rightarrowtail E$. Let B_n be the (compact) closure of the bounded set $T_n(U_n) \subseteq E$. Thus T_n factors over the normed space E_B generated by B and this space is complete by 4.3 (for $\tau := p_B$ and $\tau' := E|_B$), since B is compact and hence complete. Thus we can rewrite E as reduced projective limit of a sequence of Banach spaces with compact connecting homomorphisms.

4.43 Proposition (See [**Flo71**, 7.5,7.6 p.170], [**Sch12**, 2.8 p.33]).
Let $E = \varinjlim_n E_n$ a reduced inductive limit of a sequence of Banach spaces with compact connecting homomorphisms $E_n \to E_{n+1}$. Then the limit is complete and regular.

Proof. In view of 4.40 it is enough to show completeness using 4.41: Let $A_n := 2^n\, oE_n$ where w.l.o.g. $oE_n \subseteq oE_{n+1}$. Obviously $(A_n)_{n\in\mathbb{N}}$ is an absorbing sequence. Ad 1: Let $U \subseteq E$ be absolutely convex with $U \cap A_n$ a 0-nbhd for each n and hence also in the (finer) topology induced from E_n on A_n. Since A_n is a 0-nbhd in E_n, also $U \cap A_n$ is one and hence also $U \cap E_n \supseteq U \cap A_n$. Thus U is 0-nbhd for the inductive topology of E.
Remains to show 2: So let $(x_j)_{j\in J}$ be a Cauchy-net in A_n. Since A_n is compact in E_{n+1}, the sequence x_j has an accumulation point x_∞ in E_{n+1} and hence also in E. But as Cauchy-net it has to converge to x_∞. □

4.44 Proposition
(See [**Jar81**, 12.5.9 p.266], [**Woz13**, 4.30 p.75], [**MV92**, 25.20 p.57]).
Let F be an lcs. Then

1. *F is the dual of an (FS)-space;*

⇔ 2. *F is a bornological (DF)-space where each bounded set is relative compact in F_A for some bounded Banach-disk A;*

$\Leftrightarrow$ 3. *F is an inductive limit of a sequence of Banach-spaces with compact connecting mappings;*

$\Leftrightarrow$ 4. *F is a complete (DF)-space, is (S), and every 0-sequence is Mackey-convergent.*

A space satisfying these equivalent conditions is also called SILVA-space.

Proof. (1$\Rightarrow$2) Let $F = E^*$ with E a (FS)-space. By 4.39 F is ultra-bornological (DF). By 3.33 $\forall U \subseteq E\ \exists V \subseteq U$: $U^o \subseteq F_{V^o}$ is compact. Since the polars of 0-nbhds are basis of the bounded sets in F (by infra-barrelledness of E) the condition on the bounded sets in 2 is satisfied.

(2$\Rightarrow$3) Let $\{B_n : n \in \mathbb{N}\}$ be a countable basis of the bornology of F. By assumption we recursively find bounded Banach-disks $A_n \supseteq A_{n-1}$ with $\bigcup_{k<n}(A_k \cup B_k)$ relative compact in F_{A_n}. Obviously $\bigcup_{n\in\mathbb{N}} F_{A_n} = F$, the identity $F_{A_{n-1}} \to F_{A_n}$ is a compact operator and the identity $\varinjlim_n F_{A_n} \to F$ is continuous. Conversely, let $B \subseteq F$ be bounded, so there exists an $k \in \mathbb{N}$ with $B \subseteq B_k$, thus B is bounded in $F_{A_{k+1}}$ and hence also in $\varinjlim_n F_{A_n}$. Therefore the identity $F \to \varinjlim_n F_{A_n}$ is bounded and since F is bornological it is continuous.

(3$\Rightarrow$4) Let $F = \varinjlim_n F_n$ with $F_n \to F_{n+1}$ being compact between Banach spaces, hence it is ultra-bornological, complete and regular by 4.43. In particular, F has a countable basis of bornology formed by the multiples of the unit-balls of the F_n and thus is (DF) and (M). Moreover, every 0-sequence is bounded, hence relatively compact in some F_n and thus Mackey-convergent. In view of 4.28 it remains to show quasi-normability as characterized in 4.30: For every bounded=compact set U^o in the (FM)-space F^* (by 4.18.1 and 4.27) there exists a (Mackey-)0-sequence $x_n^* \to 0$ such that U^o is contained in its closed absolutely convex hull. Let $\lambda_n \to \infty$ be such that $\{\lambda_n x_n^* : n \in \mathbb{N}\}$ is bounded in F^* and thus contained in some V^o since F is barrelled. Then $x_n^* \to 0$ in $F^*_{V^o}$ and hence U^o is compact in $F^*_{V^o}$ and thus homeomorphic to its image in $(F^*, \beta(F^*, F))$.

(4$\Rightarrow$1) By 4.39 $F = E^*$ for some (FM)-space E. By 3.34 the (FM) space E is (S) iff it is separable (which is automatically satisfied by 3.27) and $\sigma(E^*, E)$-convergent sequences are $\beta(E^*, E)$-convergent by 3.27, hence equicontinuously=Mackey convergent by 4. □

4.45 Lemma (See [**Jar81**, 17.3.6 p.379])**.**
Let $T : E \to F$ be nuclear between Banach spaces. Then $T^* : F^* \to E^*$ is nuclear.

Proof. By assumption $T = \sum_n x_n^* \otimes y_n$ with $\sum_n \|x_n^*\|\,\|y_n\| < \infty$. Thus

$$T^*(y^*)(x) = y^*(T(x)) = y^*\Big(\sum_n x_n^*(x)\, y_n\Big) = \sum_n x_n^*(x)\, y^*(y_n)$$

$$= \sum_n \mathrm{ev}_{y_n}(y^*)\, x_n^*(x) = \Big(\sum_n \mathrm{ev}_{y_n}(y^*)\, x_n^*\Big)(x) = \Big(\sum_n \mathrm{ev}_{y_n} \otimes x_n^*\Big)(y^*)(x),$$

i.e. $T^* = \sum_n \mathrm{ev}_{y_n} \otimes x_n^*$ with

$$\sum_n \|\mathrm{ev}_{y_n}\|\,\|x_n^*\| = \sum_n \|y_n\|\,\|x_n^*\| < \infty. \quad \square$$

4.46 Proposition (See [**Jar81**, 21.5.1 p.491])**.**
The dual E^* of an lcs E is nuclear iff $\forall B\ \exists B' : \iota_{B'}^B : \widetilde{E_B} \to \widetilde{E_{B'}}$ is nuclear

An lcs E satisfying these equivalent conditions is (sometimes) called CO-NUCLEAR, see [**Jar81**, 21.5 p.491].

Proof. A typical 0-nbhd in E^*_β is B^o for some bounded (absolutely convex and closed) $B \subseteq E$.

($\Leftarrow$) By assumption there is some bounded $B' \supseteq B$ such that $\widetilde{E_B} \to \widetilde{E_{B'}}$ is nuclear. Then its dual mapping $(E_{B'})^* \to (E_B)^*$ is nuclear by 4.45. Now note that $(E^*)_{B^o}$ is isometrically embedded into $(E_B)^*$: The inclusion $E_B \to E$ induces a morphism $E^* \to (E_B)^*$, which factors over $(E^*)_{B^o}$ via an embedding, since $\|x^*\|_{(E_B)^*} = \sup\{|x^*(x)| : p_B(x) \le 1\} = \sup\{|x^*(x)| : x \in B = (B^o)_o\} = p_{B^o}(x^*)$. So the connecting morphism from $(E^*)_{(B')^o} \to (E^*)_{B^o}$ is absolutely summing as restriction of the (by 3.62) absolutely summing map $(E_{B'})^* \to (E_B)^*$, i.e. E^*_β is nuclear by 3.70.

($\Rightarrow$) Let E^* be nuclear, so for each closed absolutely convex bounded B there is another one B', such that $(E^*)_{(B')^o} \to (E^*)_{B^o}$ is nuclear. Hence the adjoint $E^{**}_{B^{oo}} = ((E^*)_{B^o})^* \to ((E^*)_{(B')^o})^* = E^{**}_{(B')^{oo}}$ is nuclear and thus the restriction to $\widetilde{E_B} \to \widetilde{E_{B'}}$ (since $B = E \cap B^{oo}$) is absolutely summing and a composition of 6 such maps is nuclear, see the proof of (6⇒7) in 3.70 . □

4.47 Lemma (See [**Jar81**, 12.5.1,12.5.2 p.263])**.**
For (DF)-spaces E and their Schwartzification $E_S := (E, \tau_S)$ we have

$$\beta(E^*, E) = \eta(E^*, E) = \eta(E^*, E_S) = \gamma(E^*, E_S) = \tau_c(E^*, \widetilde{E_S}) = \beta(E^*, \widetilde{E_S}).$$

In particular, $\beta(E^*, E) = \beta(E^*, \tilde{E})$ provided E is (DF).

The (DF) condition can be weakend to (df) in this lemma using the same proof, but with the sharpening mentioned in 4.18 instead of Proposition 4.18.

Proof. Note, that obviously $E \to E_S \to (E, \sigma(E, E^*))$ are continuous, hence $E^* = (E_S)^*$. We always have:

$$\beta(E^*, E) \le \eta(E^*, E) \le \eta(E^*, E_S) \overset{(S)}{=\!=} \gamma(E^*, E_S) \overset{3.24}{=\!=} \tau_c(E^*, \widetilde{E_S}) \overset{(s.-M)}{=\!=} \beta(E^*, \widetilde{E_S})$$

The first $\le$ holds, since $(E, \eta(E^*, E)) := \varinjlim_U E^*_{U^o}$ and $E^*_{U^o} \rightarrowtail E^*_\beta$ is continuous.
The second one holds, since $\mathrm{id} : E \rightarrowtail\!\!\!\to E_S$ is continuous, so the injective limit $\eta(E^*, E)$ has more steps than $\eta(E^*, E_S)$.
The first equality holds since E_S is (S): In fact, $E^*_{U^o} \rightarrowtail (E^*, \sigma(E^*, E))$ is continuous, so $\mathrm{id} : \eta(E^*, E) := \varinjlim_U E^*_{U^o} \rightarrowtail\!\!\!\to \gamma(E^*, E)$ (recall 3.24) is continuous. Conversely, let E be Schwartz, i.e. for every 0-nbhd U there exists a 0-nbhd V with $U^o \subseteq E^*_{V^o}$ compact by 3.33, and hence the induced (compact) topology from $E^*_{V^o}$ on U^o coincides with the restriction of $\sigma(E^*, E)$, and the inclusion from U^o with this topology into $E^*_{V^o}$ is continuous. Thus $\gamma(E^*, E) \rightarrowtail\!\!\!\to \eta(E^*, E)$ is continuous.
The last equality holds, since $\widetilde{E_S}$ is a complete Schwarz space, hence semi-Montel by 3.31, thus the closed bounded subsets coincide with the compact ones.

($\beta(E^*, E) \ge \eta(E^*, E_S)$) Since E is (DF), E^*_β is (F), by 4.18. Let $x^*_n \to 0$ in E^*_β, then x^*_n is Mackey-convergent by 2.2, so there exists a sequence $\lambda_n \to \infty$ with $\lambda_n^2 x^*_n \to 0$ in E^*_β. Since the (DF)-space E is infra-c_0-barrelled, $\lambda_n^2 x^*_n \in U^o$ for some 0-nbhd $U \subseteq E$. Thus $\lambda_n x^*_n$ is an $\mathcal{E}$-0-sequence and hence $W := \{\lambda_n x^*_n : n \in \mathbb{N}\}_o$ is a 0-nbhd for τ_S (see 4.36). Since $x^*_n \to 0$ in $E^*_{W^o}$ and hence in $\varinjlim_W E^*_{W^o} =: \eta(E^*, E_S)$, the inclusion $\beta(E^*, E) \to \eta(E^*, E_S)$ is (sequentially-)continuous.

The particular case follows, since by the universal property $E \to E_S \hookrightarrow \widetilde{E_S}$ factors over $E \hookrightarrow \tilde{E}$. Thus $\beta(E^*, E) \le \beta(E^*, \tilde{E}) \le \beta(E^*, \widetilde{E_S})$. □

4.48 Proposition (See [**Jar81**, 21.5.3 p.491])**.**
For metrizable and for (DF)-spaces nuclearity and co-nuclearity are equivalent.

Proof. (nuclear $\stackrel{(F)}{\Rightarrow}$ co-nuclear) Let p_n be an increasing sequence of seminorms defining the topology of E such that the connecting morphisms $T_n : E_{p_{n+1}} \to E_{p_n}$ are nuclear, and hence admit representations $T_n = \sum_k \lambda_{n,k} x^*_{n,k} \otimes y_{n,k}$ with $x^*_{n,k} \in o(E_{p_{n+1}})^*$, $y_{n,k} \in o(E_{p_n})$ and $\lambda_n := \sum_k |\lambda_{n,k}| < \infty$. Now let $B \subseteq E$ be a closed bounded disk, $\sigma_n := \sup\{p_{n+1}(b) : b \in B\}$, let $\rho_n := \max\{\sigma_n, \lambda_n \sigma_n\}$, and set $C := \Big\{x \in E : q_C(x) := \sum_n \frac{p_n(x)}{2^n \rho_n} \leq 1\Big\}$. For $x \in B$ we have $p_n(x) \leq p_{n+1}(x) \leq \sigma_n$, hence $\sum_n \frac{p_n(x)}{2^n \rho_n} \leq \sum_n \frac{\sigma_n}{2^n \sigma_n} = 1$, i.e. $B \subseteq C$. Furthermore C is bounded since $p_n(C) \leq 2^n \rho_n$. The connecting morphism $E_B \to E_C$ is absolutely summable, since for arbitrary finitely many $x_i \in E_B \subseteq E$ we have

$$\begin{aligned}\sum_i p_n(x_i) &= \sum_i p_n(T_n(x_i)) \leq \sum_i \sum_k p_n\big(\lambda_{n,k}\, x^*_{n,k}(x_i)\, y_{n,k}\big)\\ &\leq \sum_k |\lambda_{n,k}| \sum_i |x^*_{n,k}(x_i)| \leq \lambda_n \sup_{x^* \in U^o_{n+1}} \sum_i |x^*(x_i)|\\ &\leq \lambda_n \sup_{x^* \in \sigma_n B^o} \sum_i |x^*(x_i)| \leq \lambda_n \sigma_n \sup_{x^* \in B^o} \sum_i |x^*(x_i)| \leq \rho_n \sup_{x^* \in B^o} \sum_i |x^*(x_i)|.\end{aligned}$$

Thus $\sum_i q_C(x_i) = \sum_{n,i} \frac{1}{2^n} \frac{p_n(x_i)}{\rho_n} \leq \sup\{\sum_i |x^*(x_i)| : x^* \in B^o\}$ and hence the identity $E_B \to E_C$ is absolutely summing by 3.62. Since $\mathcal{S}^6_1 \subseteq \mathcal{A}_1 \subseteq \mathcal{N}$ we may assume that it is even nuclear, and hence E is co-nuclear.

(nuclear $\stackrel{(DF)}{\Rightarrow}$ co-nuclear) By 4.47 $E^*_\beta = \tilde{E}^*_\beta$, so we may assume that E is a complete nuclear (DF). That $\tilde{E}$ is (DF) can be seen as follows: By 4.47 we have $\beta(E^*, \tilde{E}) = \beta(E^*, E)$ and hence is metrizable and $\tilde{E}$ has a basis of its bornology formed by closures of bounded sets in E, since for every bounded $\tilde{B} \subseteq \tilde{E}$ we find a bounded set $B \subseteq E$ such that the 0-nbhds $B^o \subseteq \tilde{B}^o$ and hence $\tilde{B} \subseteq ((\tilde{B})^o)_o \subseteq (B^o)_o = \overline{B}^{\tilde{E}}$. That $\tilde{E}$ is quasi-c_0-barrelled is obvious (recall [**Kri07b**, 4.10.3]).
Let $\{B_n : n \in \mathbb{N}\}$ be a basis of the bornology consisting of closed absolutely convex sets with $B_{n+1} \supseteq 2B_n$. Put $E_n := E_{B_n}$. Since E is complete (S) hence semi-(M) and thus semi-reflexive, $E_B \cong ((E^*)_{B^o})^*$ via $x \mapsto \delta(x)|_{(E^*)_{B^o}}$:
This mapping is onto, for let $\lambda : (E^*)_{B^o} \to \mathbb{K}$ be continuous and linear, and $x \in E$ be such that $\delta_x := \delta(x) = \lambda \circ \iota_{B^o} \in E^{**}$, i.e. $\delta_x(B^o) = \{\delta_x(x^*) = x^*(x) : x^* \in B^o\}$ is bounded by $C := \|\lambda\|$ and thus $x \in C\,(B^o)_o = C\,B \subseteq E_B$.
It is also injective, for let $x \in E_B$ be such that $\delta(x)|_{(E^*)_{B^o}} = 0$, hence $0 = \delta(x) \circ (\iota^B)^* = \delta_x : E^* \to \mathbb{K}$, hence $x = 0$.

$$\begin{tikzcd} & E^* \arrow[dl, "\iota_{B^o}"'] \arrow[d, "(\iota^B)^*"] & & E \\ (E^*)_{B^o} \arrow[r, hook, dotted] \arrow[dr, "\lambda"'] & (E_B)^* \arrow[d, "\delta_x"] & & E_B \arrow[u, "\iota^B"'] \\ & \mathbb{K} & & \end{tikzcd}$$

We claim that $\ell^1\{E\} = \text{cbs-}\varinjlim \ell^1\{E_n\}$, i.e. every bounded $S \subseteq \ell^1\{E\}$ is contained and bounded in $\ell^1\{E_n\}$ for some n (recall 3.41):
Suppose indirectly, that for each n we find $x^{(n)} \in S$ with $\pi_n(x^{(n)}) := \sum_{k=0}^\infty \|x^{(n)}_k\|_n > 2^n$. So there exists a finite set $F_n \subseteq \mathbb{N}$ with $\sum_{k \in F_n} \|x^{(n)}_k\|_n > 2^n$. Choose $a^{(n)}_k \in B^o_n$ with $\sum_{k \in F_n} |a^{(n)}_k(x^{(n)}_k)| > 2^n$. Then

$$\forall n, r \in \mathbb{N}\, \forall k \in F_{n+r} : 2^r\, p_{B^o_n}(a^{(n+r)}_k) \leq p_{B^o_{n+r}}(a^{(n+r)}_k) \leq 1.$$

Thus the sequence $(a_k)_k$ formed by all these finite subsequences $(a_k^{(n)})_{k\in F_n}$ for $n \in \mathbb{N}$ converges to 0 in E^* and hence forms an equicontinuous set $A \subseteq E^*$ by the (df)-property. Thus A_o is a 0-nbhd in E and its Minkowski functional $p_{A_o} : x \mapsto \sup\{|a(x)| : a \in A\}$ is a continuous seminorm on E. Hence $\pi_{A_o} : (x_k)_{k\in\mathbb{N}} \mapsto \sum_k p_{A_o}(x_k)$ is a continuous seminorm on $\ell^1\{E\}$. Thus $\pi_{A_o}(S)$ has to be bounded, in contradiction to

$$\pi_{A_o}(x^{(n)}) = \sum_k \sup_{a\in A} |a(x_k^{(n)})| \geq \sum_{k\in F_n} |a_k^{(n)}(x_k^{(n)})| > 2^n.$$

The canonical map $\ell^1[E_n] \to \ell^1[E]$ is continuous and $\ell^1[E] = \ell^1\{E\}$ by 3.70, so the image of $S := o\,\ell^1[E_n]$ is bounded in $\ell^1\{E\}$ and, by what we have just shown, even bounded in $\ell^1\{E_{n'}\}$ for some $n' \geq n$, i.e. the connecting mapping $E_n \to E_{n'}$ is absolutely summing, hence E is co-nuclear by 4.46.

(nuclear $\overset{(F)}{\Leftarrow}$ co-nuclear) By assumption and 4.18.1 E^* is a nuclear (DF)-space. Hence by the second part E^{**} is nuclear and so is E as a subspace by 3.73.2.

(nuclear $\overset{(DF)}{\Leftarrow}$ co-nuclear) By assumption and 4.18.2 E^* is a nuclear Fréchet space. Hence by the first part E^{**} is nuclear. In order to apply 3.73.2 it remains to show that $\delta : E \to E^{**}$ is an embedding, i.e. E is infra-barrelled: The bounded=pre-compact (since E^* is (S)) sets in E^* are contained in the bipolar of some 0-sequence in E^* by 3.6 and, since E is (df) and hence quasi-c_0-barrelled, the 0-sequences are equicontinuous, hence the topology of E (which is that of uniform convergence on equicontinuous sets) coincides with that induced from E^{**}. □

This proof works also for (df) instead of (DF), however the last argument shows, that (co-)nuclear (df) spaces are infra-barrelled and in particular (DF) spaces.

4.49 Proposition (See [**Kri07a**, 6.31], [**Jar81**, 21.5.5 p.493])**.**
Every strict inductive limit of a sequence of nuclear Fréchet spaces is co-nuclear.

Proof. Since strict inductive limits are regular this is immediate by 4.46. □

4.50 Theorem. Density of finite dimensional operators
(See [**Kri07a**, 4.44], [**Jar81**, 18.1.1 p.398])**.**
Let E be a locally convex space and $\mathcal{B}$ be a bornology on E. We consider on the function spaces $\mathcal{L}(E, \_)$ the topology of uniform convergence on all sets in $\mathcal{B}$, and hence denote them by $\mathcal{L}_\mathcal{B}$. Then

1. *$E^* \otimes F$ is dense in $\mathcal{L}_\mathcal{B}(E, F)$ for every locally convex space F;*

⇔ 2. *$E^* \otimes F$ is dense in $\mathcal{L}_\mathcal{B}(E, F)$ for every Banach space F;*

⇔ 3. *$E^* \otimes E$ is dense in $\mathcal{L}_\mathcal{B}(E, E)$;*

⇔ 4. *id_E is a limit in $\mathcal{L}_\mathcal{B}(E, E)$ of a net in $E^* \otimes E$.*

Proof. (1⇒2) is trivial.
(2⇒1) A typical 0-neighborhood in $\mathcal{L}_\mathcal{B}(E, F)$ is given by $N_{B,V} := \{T : T(B) \subseteq V\}$ with $B \in \mathcal{B}$ and V a 0-neighborhood in F. Let $\iota_V : F \twoheadrightarrow F_V$ be the canonical surjection. Since F_V is a normed space $\iota_V \circ T : E \to F \twoheadrightarrow F_V \hookrightarrow \tilde{F}_V$ can be uniformly approximated on B with respect to $p_V : F_V \to \mathbb{K}$ by finite dimensional operators $E \to \tilde{F}_V$ by 2. Since F_V is dense with respect to p_V in $\tilde{F}_V$ we may assume that the finite operators belong to $\mathcal{L}(E, F_V)$. Taking inverse images of the vector components, we may even assume that they belong to $\mathcal{L}(E, F)$.
(1⇒3) and (3⇒4) are trivial.

(4⇒1) Let T_i be a net of finite dimensional operators converging to id_E, then the net $T \circ T_i$ of finite dimensional operators converges to $T \circ \mathrm{id} = T$. □

Let E be complete and assume that the equivalent statements of 4.50 are true for some bornology $\mathcal{B}$. And w.l.o.g. let $B \in \mathcal{B}$ be absolutely convex. Since the identity on E can be approximated uniformly on B by finite dimensional operators, we conclude that the inclusion $E_B \to E$ can be approximated by finite dimensional operators $E_B \to E$ uniformly on the unit ball of E_B. Hence it has to have relatively compact image on the unit ball by the following lemma 4.51, i.e. B has to be relatively compact.

4.51 Lemma.
The set $\mathcal{K}(E, F)$ of COMPACT OPERATORS *from a normed space E into a complete space F is closed in $\mathcal{L}(E, F)$.*

Proof. To see this use that $F = \varprojlim_V \widetilde{F_V} \subseteq \prod_V \widetilde{F_V}$, hence a subset K of F is relatively compact iff $\iota_V(K)$ is relatively compact in $\widetilde{F_V}$ for all V. Now let $T_i \in \mathcal{K}(E, F)$ converge to $T \in \mathcal{L}(E, F) = L(E, F)$. Then the $\iota_V \circ T_i \in \mathcal{K}(E, \widetilde{F_V})$ converge to $\iota_V \circ T$ in $L(E, \widetilde{F_V})$. Since $\widetilde{F_V}$ is a Banach spaces it can be shown as in [**Kri07b**, 6.4.8] that $\iota_V \circ T \in \mathcal{K}(E, \widetilde{F_V})$. Hence $\iota_V(T(oE))$ is relatively compact in $\widetilde{F_V}$ and thus $T(oE)$ is relatively compact in F. □

4.52 Definition.
A complete lcs is said to satisfy the APPROXIMATION PROPERTY iff the equivalent statements in 4.50 are true for the bornology $\mathcal{B} = \mathrm{cp}$ of all relatively compact subsets of E. A non-complete space E is said to have the *approximation property*, iff its completion $\tilde{E}$ has it. Note that the finite dimensional operators may be taken in $\mathcal{L}(E, E)$ in this situation.

4.53 Remark (See [**Kri07a**, 4.63], [**Jar81**, 18.5.8 p.414])**.**
For a long time it was unclear if there are spaces without the approximation property at all. It was known that, if such a Banach space exists, then there has to be a subspace of c_0 failing this property. It was [**Enf73**] who found a subspace of c_0 without this property. In [**Sza78**] it was shown that $L(\ell^2, \ell^2) \cong L(\ell^2, (\ell^2)^*) \cong (\ell^2 \hat{\otimes}_\pi \ell^2)^*$ doesn't have the approximation property. In contrast $\ell^2 \otimes_\pi \ell^2$ has the approximation property, since by [**Jar81**, 18.2.9 p.403] every completed projective tensor product of Fréchet spaces with the approximation property has it. Note however, that for Banach spaces one can show that if E^* has the approximation property then so does E, see [**Jar81**, 18.3.5 p.407]. Due to [**H.77**] is the existence of a Fréchet-Montel space without the approximation property, see [**Jar81**, p416].

4.54 Lemma. "Kelley-fication" of the completion (See [**Kri07a**, 4.76])**.**
The bijection $(E^*_\gamma)^*_\gamma \rightarrowtail\!\!\!\!\rightarrow \tilde{E}$ given by Grothendiecks completeness theorem is continuous, both spaces have the same compact subsets and $(E^*_\gamma)^*_\gamma$ carries the final locally convex topology with respect to these subsets. If $\tilde{E}$ is compactly generated, and hence in particular if E is metrizable, then we have equality.

Proof. Recall that by Grothendiecks completeness theorem [**Kri07b**, 7.5.7] we have a bijection $\tilde{E} \rightarrowtail\!\!\!\!\rightarrow \mathcal{L}_{\mathrm{equi}}(E^*_\gamma, \mathbb{K})$ into the space of linear functionals, which are continuous on each equicontinuous set $U^o \subseteq E^*$ with its compact topology $\sigma(E^*, E)|_{U^o}$, supplied with the topology of uniform convergence on each equicontinuous set. Whereas $(E^*_\gamma)^*_\gamma$ is the same space, but with the final locally convex

topology induced by the inclusions of W^o with their compact topology $\sigma(W^o, E^*)$ for all 0-nbhds $W \subseteq E^*$ with respect to $\gamma(E^*, E) = \tau_c(E^*, \tilde{E})$ by 3.24.

In order to show that $(E^*_\gamma)^*_\gamma \to \tilde{E}$ is continuous, denote with $\tilde{\tau}$ the topology of $\tilde{E}$, let the polars be with respect to the duality $(\tilde{E}, E^*)$, and consider W^o for a 0-nbhd $W \subseteq E^*_\gamma$. Since $\gamma(E^*, E) = \tau_c(E^*, \tilde{E})$ there exists a compact set $K \subseteq \tilde{E}$ with $W \supseteq K^o$. By [**Kri07b**, 6.4.2] the closed absolutely convex hull $(K^o)_o$ of K is precompact and hence compact in $\tilde{E}$ and hence the same is true for the closed subset $W_o \subseteq (K^o)_o$. So on W_o the (compact) topology of $\tilde{\tau}$ coincides with that of $\sigma(W_o, E^*)$, and hence $(W^o, \sigma(W^o, E^*)) \rightarrowtail (\tilde{E}, \tilde{\tau})$ is continuous.

Conversely, let now $K \subseteq \tilde{E}$ be compact. Then K^o is a 0-nbhd in $(E^*, \tau_c(E^*, \tilde{E})) = E^*_\gamma$ and thus the inclusion of the (compact) equicontinuous set $((K^o)^o, \sigma((E^*_\gamma)^*, E^*)) \rightarrowtail (E^*_\gamma)^*_\gamma$ is continuous. Since the inclusion $(K, \tilde{\tau}) \rightarrowtail \sigma(\tilde{E}, E^*)$ is continuous, we get that K is compact in $(E^*_\gamma)^*_\gamma$ and $(E^*_\gamma)^*_\gamma$ carries the final locally convex topology with respect to the compact sets. $\square$

4.55 Proposition. Approximation property versus ε-product
(See [**Kri07a**, 4.68], [**Jar81**, 18.1.8 p.400]).
A complete space E has the approximation property iff $F \otimes_\varepsilon E$ is dense in the so-called ε-PRODUCT $F \,\varepsilon\, E := \mathcal{L}_{equi}(F^*_\gamma, E)$ for every locally convex space F.

Note that the topology of $F \otimes_\varepsilon E$ is by definition 3.44 initial with respect to the inclusion $F \otimes E \hookrightarrow L_{\text{equi}}(F^*, E)$ and has in fact values in $\mathcal{L}((F^*, \sigma(F^*, F)), E) \subseteq \mathcal{L}(F^*_\gamma, E)$.

Proof. Note that $F \otimes E$ is mapped into $\mathcal{L}(F^*_\gamma, E)$, since for $y \in F$ we have $\delta(y) \in (F^*_\gamma)^*$ by [**Kri14**, 5.5.7].

($\Leftarrow$) Consider the following commuting diagram: By assumption for $F := E^*_\gamma$ the inclined arrow on the left hand side has dense image. The arrow on the right hand side is an embedding, since $(E^*_\gamma)^*_\gamma \to \tilde{E} = E$ is a continuous bijection and the equi-continuous subsets in $(E^*_\gamma)^*_\gamma$ are exactly the relatively compact subsets of $\tilde{E} = E$ by 4.54.

$$\begin{tikzcd} E^*_\gamma \otimes E \arrow[rr] \arrow[dr] & & \mathcal{L}_{cp}(E,E) \arrow[dl, hook] \\ & \mathcal{L}_{equi}((E^*_\gamma)^*_\gamma, E) & \end{tikzcd}$$

($\Rightarrow$) Let $T \in \mathcal{L}(F^*_\gamma, E)$ and let a 0-neighborhood $N_{V^o,U}$ in this space be given. Since T is continuous on the compact space $(V^o, \sigma(F^*, F))$, we have that $K := T(V^o)$ is compact in E. By assumption $E^* \otimes E$ is dense in $\mathcal{L}_{cp}(E, E)$. Hence there exists a finite dimensional operator $S \in \mathcal{L}(E, E)$ with $(\mathrm{id}_E - S)(K) \subseteq U$. Then $S \circ T : F^*_\gamma \to E \to E$ is finite dimensional and since $(F^*_\gamma)^* = \tilde{F}$ by [**Kri14**, 5.5.7] it belongs to $\tilde{F} \otimes E$ and $(T - S \circ T)(V^o) = (\mathrm{id} - S)(K) \subseteq U$. Thus $T - S \circ T \in N_{V^o,U}$. Hence $\tilde{F} \otimes_\varepsilon E$ is dense in $\mathcal{L}_{\text{equi}}(F^*_\gamma, E)$ and, since $F \otimes E$ is dense in $\tilde{F} \otimes_\varepsilon E$, it is also dense in $\mathcal{L}_{\text{equi}}(F^*_\gamma, E)$. $\square$

4.56 Proposition (See [**Jar81**, J18.2.1 p.401]).
Let E be the reduced projective limit of spaces E_j with the approximation property. Then E has the approximation property.

Proof. We may assume that all E_j and E is complete (since taking completions commutes with reduced projective limits, see [**Jar81**, 3.4.6 p.63]). Let $K \subseteq E$ be compact and $U \subseteq E$ a 0-nbhd, w.l.o.g. of the form $\iota_k^{-1}(U_k)$ for some $k \in J$ and 0-nbhd $U_k \subseteq E_k$. By reducedness $F_k := \iota_k(E)$ is dense in E_k hence has the approximation property. So there are $a_i \in E^*_k$ and $x_i \in E$ such that $(\mathrm{id}_{F_k} - S)(\iota_k(K)) \subseteq U_k$

for $S := \sum_{i=1}^n a_i \otimes \iota_k(x_i)$. Thus $(\mathrm{id}_E - \tilde{S})(K) \subseteq U$ for the finite dimensional operator $\tilde{S} := \sum_{i=1}^n \iota_k^*(a_i) \otimes x_i$. □

4.57 Proposition. Consequences of nuclearity
(See [**Kri07a**, 6.19.2], [**Jar81**, 21.2.2 p.483]).
Each nuclear space has the approximation property.

Proof. Since by E is a reduced projective limit of Hilbert-spaces, it satisfies the approximation property, by 4.56 and since Hilbert spaces have the approximation property: Let $(e_i)_{i\in I}$ be an orthonormal basis. Then the net of ortho-projections $P_J : x \mapsto \sum_{i\in J}\langle x, e_i\rangle e_i$ with finite $J \subseteq I$ converges pointwise to id and is equicontinuous, since $\|p_J(x)\|_{\ell^2} = \left(\sum_{i\in J} |\langle x_i, e_i\rangle|^2\right)^{1/2} \le \|x\|_{\ell^2}$. So it converges for the topology $\tau_{pc} = \tau_c$. □

4.58 Lemma (See [**Kri07a**, 4.70]).
For complete spaces E and F we have $F \varepsilon E \cong E \varepsilon F$.

Proof. We only have to show bijectivity, since $F \varepsilon E = \mathcal{L}_{equi}(F^*_\gamma, E) \subseteq L(F^*, E)$ embeds into the space $L(F^*, E^{*\prime}) \cong L(F^*, E^*; \mathbb{K})$. To every continuous $T : F^*_\gamma \to E$ we associate the continuous $T^* : E^*_\gamma \to (F^*_\gamma)^*_\gamma$ (in fact every equi-continuous set U^o of E^* is mapped to $T^*(U^o) = \{x^* \circ T : x^* \in U^o\} \subseteq \{y^* : y^* \in (T^{-1}(U))^o\}$, the polar of a 0-neighborhood in F^*_γ). And by Grothendieck's completion result (See [**Kri14**, 5.5.7]) we are done since by the lemma 4.54 the identity $(F^*_\gamma)^*_\gamma \to \mathcal{L}_{equi}(F^*_\gamma, \mathbb{K}) = \tilde{F}$ is continuous. □

Let us consider $E^* \hat{\otimes}_\varepsilon F$ now. If F is complete and satisfies the approximation property, then $E^*_\gamma \hat{\otimes}_\varepsilon F \cong \mathcal{L}_{equi}((E^*_\gamma)^*_\gamma, F)$ by 4.55.

4.59 Proposition (See [**Kri07a**, 4.73]).
If E and F are complete, E is Montel and F (or E) satisfies the approximation property, then

$$E \hat{\otimes}_\varepsilon F \cong E \,\varepsilon\, F := \mathcal{L}_{equi}(E^*_\gamma, F) \cong \mathcal{L}_b(E^*_\beta, F),$$

In more detail, for complete spaces E and F we have under the indicated assumptions the following identities:

$$E \hat{\otimes}_\varepsilon F \overset{F\ app.prop.}{=\!=\!=} E \,\varepsilon\, F = \mathcal{L}_{equi}(E^*_\gamma, F) \overset{E\ semi\text{-}Montel}{=\!=\!=}$$
$$= \mathcal{L}_{equi}(E^*_\beta, F) \overset{E\ infra\text{-}barreled}{=\!=\!=} \mathcal{L}_b(E^*_\beta, F) \overset{E^*_\beta\ bornological}{=\!=\!=} L(E^*_\beta, F)$$

Proof. In the first statement the first isomorphism follows from the definition 3.44 of $E \otimes_\varepsilon F \subseteq \mathcal{L}_{\mathrm{equi}}(E^*_\gamma, F) \hookrightarrow L(E^*, F)$ and the approximation property (that it hold also if E instead of F satisfies the approximation property follows from 4.58). And the second one follows, since Montel spaces are barrelled by 3.22 and 3.18 and since $E^*_\gamma = \tau_c(E^*, \tilde{E}) = \beta(E^*, E)$ by 3.24 and E being semi-Montel.

Note that the strong dual of a semi-reflexive space is barreled 4.10. If E is in addition metrizable, then E^* is bornological by 4.16, and hence we have

$$\mathcal{L}_b(E^*_\beta, F) = L(E^*, F). \quad \square$$

4.60 Proposition (See [**Kri07a**, 4.74]).
For complete spaces E^*_β and F we have under the indicated assumptions the following identities:

$$E^*_\beta \hat{\otimes}_\varepsilon F \xlongequal{F\ app.prop.} E^*_\beta\, \varepsilon\, F := \mathcal{L}_{equi}((E^*_\beta)^*_\gamma, F) \xlongequal{E\ Montel} \\ = \mathcal{L}_b((E^*_\beta)^*_\beta, F) \xlongequal{E\ reflexive} \mathcal{L}_b(E,F) \xlongequal{E\ bornological} L(E,F),$$

Proof. This follows, since the strong dual E^*_β of a Montel space E is Montel by 4.27. Note that a Montel-space E is reflexive by 3.22, i.e. $(E^*_\beta)^*_\beta = E$. Furthermore $E^*_\beta = E'_\beta$ is complete, provided E is bornological. □

4.61 Theorem (See [**Kri07a**, 6.32], [**Jar81**, 21.5.9 p.496]).
Let E and F be Fréchet spaces with E nuclear.
Then we have the following isomorphisms:

1. $E\hat{\otimes}_\pi F \cong E\hat{\otimes}_\varepsilon F \cong L(E^*, F)$;
2. $E^*\hat{\otimes}_\pi F \cong E^*\hat{\otimes}_\varepsilon F \cong L(E, F)$;
3. $E^*\hat{\otimes}_\pi F^* \cong E^*\hat{\otimes}_\varepsilon F^* \cong L(E, F^*) \cong (E\hat{\otimes}_\pi F)^*$;

Proof. (1) Recall that we have shown in 4.59 that for complete spaces we have $E\hat{\otimes}_\varepsilon F \cong L(E^*_\beta, F)$ provided E satisfies the approximation property, is Montel and E^*_β is bornological. These conditions are satisfied if E is a nuclear Fréchet space by 4.57, 3.60, 3.31, and 4.39.

(2) Recall that we have shown in 4.60 that for complete spaces E^*_β and F we have $E^*_\beta\hat{\otimes}_\varepsilon F \cong L(E,F)$ provided E^*_β satisfies the approximation property and E is Montel and bornological. This is all satisfied if E is a nuclear Fréchet space, since then E^*_β is nuclear by 4.48.

(3) the same argument as in (2) applies and hence $E^*\hat{\otimes}_\varepsilon F^* \cong L(E, F^*)$. In general we have $L(E,F^*) = L(E,F') \cong L(E,F;\mathbb{K}) = \mathcal{L}(E,F;\mathbb{K}) \cong (E\hat{\otimes}_\pi F)^*$, since E and F are Fréchet. □

4.62 Proposition (See [**Jar81**, 16.4.1 p.353], [**Jar81**, 21.8.9 p.507]).
Let $\mathcal{B}$ be a bornology on $E \neq \{0\} \neq F$.
Then $\mathcal{L}_\mathcal{B}(E,F)$ is Schwartz/nuclear iff $E^*_\mathcal{B} := \mathcal{L}_\mathcal{B}(E,\mathbb{K})$ and F are Schwartz/nuclear.

Proof. ($\Rightarrow$) is obvious by 3.73.2, since F and $E^*_\mathcal{B}$ can be considered as (complemented) subspaces.

($\Leftarrow$) First one shows that a 0-neighborhood basis in $\mathcal{L}_\mathcal{B}(E,F)$ is given by the sets $N := N_{\{x_n\},\{y_n^*\}^o} := \{T : |T(x_n)(y_m^*)| \leq 1\ \forall n,m\}$, where x_n is Mackey-convergent to 0 in E with respect to $\mathcal{B}$ and y_n^* is Mackey convergent to 0 in F^* with respect to the bornology of equicontinuous sets, in fact the polars of these sequences form bases by 4.36. Without loss of generality we may replace x_n by $\lambda_n x_n$ and y_n^* by $\mu_n y_n^*$ with λ and μ in c_0. The functionals $\ell_{j,k} : \mathcal{L}_\mathcal{B}(E,F) \to \mathbb{K}$ given by $T \mapsto y_j^*(T(x_k))$ form an equicontinuous family, since N is mapped into $\{\lambda \in \mathbb{K} : |\lambda| \leq 1\}$. Thus $\lambda_k\mu_j\ell_{j,k}$ are Mackey-convergent to 0 with respect to the bornology of equicontinuous subsets. Hence its polar (which is a subset of N) is a neighborhood in the Schwartzification τ_S of $\mathcal{L}_\mathcal{B}(E,F)$.

The proof for nuclearity is analogous using that by 4.38 the nuclearification is given by the topology of uniform convergence on $\mathcal{E}$-nuclear sequences $x_n^* \in E^*$. □

4.63 Corollary (See [**Kri07a**, 6.21 p.142])**.**
The ε-tensor product of Schwartz spaces is Schwartz.

Proof. This follows from $\boxed{4.62}$ since $E \otimes_\varepsilon F \subseteq E\,\varepsilon\,F \subseteq \mathcal{L}(E^*_\gamma, F)$ and $(E^*_\gamma)^*_\gamma = \tilde{E}$ is Schwartz. □

Dual morphisms

4.64 Definition. Short exact sequences.
If $T \in \mathcal{L}(E,F)$ is an embedding then $T^* \in \mathcal{L}(F^*, E^*)$ is onto by Hahn-Banach. If $T \in \mathcal{L}(E,F)$ is onto (or at has at least dense image) then $T^* \in \mathcal{L}(F^*, E^*)$ is injective. In order ot treat both cases simultaneoulsy we can consider short sequences of continuous linear mappinngs

$$0 \to E \to F \to G \to 0.$$

A sequence $\dots \to E_{n-1} \xrightarrow{T_{n-1}} E_n \xrightarrow{T_n} E_{n+1} \to \dots$ is called (ALGEBRAICALLY) EXACT iff $\ker T_n = \operatorname{img} T_{n-1} := T_{n-1}(E_{n-1})$ for all n. It is called TOPOLOGICALLY EXACT iff T_{n-1} induces an isomorphism $E_{n-1}/\ker T_{n-1} \to \ker T_n$ of lcs for all n.
Thus a short sequence $0 \to E \xrightarrow{S} F \xrightarrow{Q} G \to 0$ is algebraically exact iff S is injective, $\operatorname{img}(S) = \ker(Q)$, and Q is onto. It is topologically exact iff in addition S is a topological embedding and Q is a quotient mapping.
Every injective mapping (embedding) $S : E \to F$ with closed image gives rise to the short (topologically) exact sequence $0 \to E \xrightarrow{S} F \twoheadrightarrow F/\operatorname{img} S \to 0$. And every surjective (quotient) mapping $Q : F \to G$ gives rise to the short (topologically) exact sequence $0 \to \ker Q \hookrightarrow F \xrightarrow{Q} G \to 0$.

4.65 Remark.
Let $E = \lim_j E_j$ be a limit. Then E can be identified with the closed subspace of $\prod_{j\in J} E_j$ formed by all $x = (x_j)_{j\in J}$ with $\mathcal{F}(f)(x_j) = x_{j'}$ for all $f : j \to j'$. We get a short exact sequence $0 \to E \hookrightarrow \prod_j E_j \twoheadrightarrow (\prod_j E_j)/E \to 0$. We can give an explicite description of the linear space $(\prod_j E_j)/E$, namely the subspace of $\prod_{f:j\to j'} E_{j'}$ formed by the image of the mapping $Q : \prod_j E_j \to \prod_{f:j\to j'} E_{j'}$ which given by $\operatorname{pr}_{f:j\to j'} \circ Q := \mathcal{F}(f) \circ \operatorname{pr}_j - \operatorname{pr}'_j$. Even for projective limits of a sequence it however not clear, whether Q is onto or is a quotient map onto its image.

4.66 Lemma.
Every short exact sequence of (F) spaces is topologically exact.

Proof. Let $T : E \to F$ be a continuous linear mapping between Fréchet spaces. By the open mapping theorem we get: If T is onto, then it is open hence a quotient mapping. If T is injective with closed image, then it is an homeomorphism onto its image, hence an embedding. □

4.67 Lemma (See [**MV92**, 26.4 p.291])**.**
Let $0 \to E \xrightarrow{S} F \xrightarrow{Q} G \to 0$ *be topologically exact.*
Then the dual sequence $0 \leftarrow E^* \xleftarrow{S^*} F^* \xleftarrow{Q^*} G^* \leftarrow 0$ *is algebraically exact.*

Proof. (S embedding $\Rightarrow$ S^* onto) by Hahn-Banach.
(Q onto $\Rightarrow$ Q^* injective) obviously.
($\ker Q = \operatorname{img} S$ and Q quotient mapping $\Rightarrow \ker S^* = \operatorname{img} Q^*$) For $y^* \in F^*$ we have: $y^* \in \ker S^* \Leftrightarrow y^* \circ S = 0 \Leftrightarrow y^*|_{\operatorname{img} S} = 0 \Leftrightarrow y^*|_{\ker Q} = 0 \Leftrightarrow \exists z^* \in G^* : y^* = z^* \circ Q = Q^*(z^*) \Leftrightarrow y^* \in \operatorname{img} Q^*$. □

Now the question arises, whether the dual of a topological short exact sequence is also topologically exact. Since the topology on the dual space is generated by the polars of bounded sets and (for infra-barrelled spaces) the bornology is generated by the polars of 0-nbhds, we need to determine how polars behave under adjoint mappings:

4.68 Lemma (See [**Jar81**, 6.8.2.a p.161])**.**
Let $T : E \to F$ be continous linear and $A \subseteq E$. Then

1. $(T^*)^{-1}(A^o) = T(A)^o$.
2. $A^o \cap \operatorname{img} T^* = T^*(T(A)^o)$.

Proof. ($\boxed{1}$) $(T^*)^{-1}(A^o) = \{y^* : \forall a \in A : |y^*(T(a))| = |T^*(y^*)(a)| \le 1\} = \{y^* : \forall b \in T(A) : |y^*(b)| \le 1\} = T(A)^o$.

($\boxed{2}$) $T^*(T(A)^o) \overset{\boxed{1}}{=\!=} T^*((T^*)^{-1}(A^o)) = A^o \cap \operatorname{img} T^*$ □

4.69 Definition. Special cbs-morphisms.
Among the various structures on the dual space E^* of an lcs E the bornology formed by the equicontinuous subsets is most closely related to (the topology of) E. It will thus be essential, to consider properties of morphisms between convex bornological spaces.

A bounded linear mapping T between separated convex bornological spaces is called a (BORNOLOGICAL) EMBEDDING (or CBS-EMBEDDING) iff $T^{-1}(B)$ is bounded for each bounded B. Any cbs-embedding is automatically injective, since its kernel is a bounded linear subspace hence 0. It is called (BORNOLOGICAL) QUOTIENT MAPPING (or CBS-QUOTIENT MAPPING) iff each bounded B has a bounded lift B', i.e. $T(B') = B$. It is enough to assume $T(B') \supseteq B$, since then we may replace B' by $B' \cap T^{-1}(B)$. Any cbs-quotient mapping is automatically onto, since each point is bounded, hence the inverse image is non-empty.

Let us denote the functors ${}^b(\_) : \underline{lcs} \to \underline{cbs}$ given by assigning the von Neuman bornology and ${}^t(\_) : \underline{cbs} \to \underline{lcs}$ given by assigning the topology formed by the bornivorous absolutely convex subsets. These functors are adjoint to each other, i.e. $\underline{lcs}({}^tE, F) \cong \underline{cbs}(E, {}^bF)$, see [**Kri07a**, 3.15]. The bornological locally convex spaces are exactly the fixpoints under ${}^t(\_) \circ {}^b(\_)$, i.e. the image of ${}^t(\_)$.

4.70 Lemma.
If $T : E \to F$ is an lcs-embedding, then $T : {}^bE \to {}^bF$ is a cbs-embedding.

Proof. Let $B \subseteq E$ be such that $T(B) \subseteq {}^bF$ is bounded. Let U be a 0-nbhd in E. By assumption there is a 0-nbhd V in F with $U = T^{-1}(V)$. Since $T(B) \subseteq \lambda V$ for some $\lambda > 0$ we have $B = T^{-1}(T(B)) \subseteq T^{-1}(\lambda V) = \lambda U$ by injectivity of T. Thus B is bounded in ${}^bE$. □

The converse is not true: Let F be a bornological lcs and E a (closed) lcs-subspace which is not bornological, e.g. $\boxed{4.81}$. Then its bornologification E_{born} has the same bounded sets as E, is cbs-embedded in F, but does not carry the lcs-subspace structure.

4.71 Lemma.
If $T : E \to F$ is a cbs-quotient mapping, then $T : {}^tE \to {}^tF$ is an lcs-quotient mapping.

Proof. We show that $T : {}^tE \to {}^tF$ is an open mapping. Let U be an absolutely convex 0-nbhd in ${}^tE$. Then $T(U)$ is absolutely convex and bornivorous, since any

bounded $B \subseteq F$ is image of some bounded $A \subseteq E$, thus $A \subseteq \lambda U$ for some $\lambda > 0$ and hence $B = T(A) \subseteq \lambda T(U)$. Hence $T(U)$ is a 0-nbhd in ${}^t F$. $\square$

The converse is not true, as the example 4.80 (based on 3.36 and 4.79) shows: A Köthe sequence space $\lambda^p(A)$ which is (FM), but has ℓ^p as quotient, hence the bounded unit-ball cannot be lifted, since otherwise it would be compact.

Definition. External duality functors.
Consider duality as functor $(\_)^* : \underline{lcs} \to \underline{cbs}$, which maps lcs E to the dual formed by the continuous linear functionals together with the bornology of equicontinuous sets, and the duality $(\_)' : \underline{cbs} \to \underline{lcs}$, which maps cbs E to the dual formed by the bounded linear functionals together with the topology of uniform convergence on the bounded sets of E.

These two dualities form a pair of adjoint functors, since

$$\underline{lcs}(E, F') \cong \underline{cbs}(F, E^*) = \underline{cbs}^{\mathrm{op}}(E^*, F),$$

see [**Kri07a**, 3.16].

By what we have already mentioned (see [**Kri07b**, 7.4.11]) the canonical mapping $E \hookrightarrow (E^*)'$ is an $\underline{\text{lcs}}$-embedding. And also $E^*_\beta \hookrightarrow ({}^b E)'$ is an embedding by definition of $\beta(E^*, E)$.

4.72 Proposition (See [**Kri07a**, 3.18])**.**

1. *The duality $(\_)' : \underline{cbs} \to \underline{lcs}$ carries $\underline{cbs}$-quotient mappings to $\underline{lcs}$-embeddings.*
2. *The duality $(\_)^* : \underline{lcs} \to \underline{cbs}$ carries $\underline{lcs}$-quotient mappings to $\underline{cbs}$-embeddings.*
3. *Let $T : E \to F$ be continuous and linear. Then T is an $\underline{lcs}$-embedding iff T^* is a $\underline{cbs}$-quotient mapping for the equicontinuous bornologies.*
4. *Furthermore, T is a dense $\underline{lcs}$-embedding iff T^* is a $\underline{cbs}$-isomorphism.*

Proof. 1 Since $\underline{\text{cbs}}$ quotient mappings $T : E \to F$ are onto, we conclude that $T^* : F' \to E'$ is injective. Since $T^*(T(B)^o) = T^*((T^*)^{-1}(B^o)) = B^o \cap T^*(F')$, by 4.68.2, and since the sets $T(B)^o$ form a 0-neighborhood basis of F', we are done.

2 Let U be an absolutely convex 0-nbhd in E. Since $T : E \to F$ is an $\underline{\text{lcs}}$-quotient mapping $V := T(U)$ is an absolutely convex 0-nbhd in F and by 4.68.1 $(T^*)^{-1}(U^o) = T(U)^o = V^o$, thus T^* is a $\underline{\text{cbs}}$-embedding.

3 ($\Rightarrow$) Let $T : E \hookrightarrow F$ be an $\underline{\text{lcs}}$-embedding and U a 0-nbhd in E. Let p_U be the Minkowski-functional of U and $\tilde{p}$ an extension to F, i.e. $\tilde{p} \circ T = p_U$, and let $V := \{y \in F : \tilde{p}(y) \le 1\}$. Remains to show that $U^o \subseteq T^*(V^o)$. So let $x^* \in U^o$, i.e. $|x^*| \le p$. By Hahn-Banach there exists an $y^* \in F^*$ with $T^*(y^*) = y^* \circ T = x^*$ and $|y^*| \le \tilde{p}$, hence $y^* \in V^o$.

3 ($\Leftarrow$) If $T^* : F^* \to E^*$ is a $\underline{\text{cbs}}$-quotient map, then $(T^*)^* : (F^*)' \to (E^*)'$ is a topological embedding by 1 and using the embedding $E \hookrightarrow (E^*)' = L((E^*, \mathcal{E}), \mathbb{K})$ of [**Kri07b**, 7.4.11] and the commutative diagram

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[d, "T"'] & L(E^*, \mathbb{K}) \arrow[r, equal] \arrow[d, "L(T^*, \mathbb{K})"] & (E^*)' \arrow[d, "(T^*)^*"] \\
F \arrow[r, hook] & L(F^*, \mathbb{K}) \arrow[r, equal] & (F^*)'
\end{tikzcd}$$

shows that T is an embedding as well.

$\boxed{4}$ If T is a dense lcs-embedding, then T^* is injective and by $\boxed{3}$ a cbs-quotient mapping, hence a cbs-isomorphism. Conversely, if T^* is a cbs-isomorphism, then T is an lcs-embedding by $\boxed{3}$ and since the continuous linear functionals separate points from closed linear subspaces, T has dense image by the injectivity of T^*. □

4.73 Remark.
Surjectivity of linear operators D, means solvability of inhomogeneous equations $D(u) = s$ for arbitrary s with respect to u.
For example, by the Malgrange-Ehrenpreis Theorem (see [**Kri07b**, 8.3.1]) every linear partial differential operator (PDO) $D := P(\frac{1}{i}\partial)$ with constant coefficients $C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is onto. This can be shown, by considering the formal adjoint operator $D^t := P^t(\frac{1}{i}\partial) : \mathcal{D} \to \mathcal{D}$ and its adjoint $\tilde{D} := (D^t)^*$ on the space of distributions $\mathcal{D}^*$ (see [**Kri07b**, 4.9]), proving the existence of a fundamental solution $\varepsilon \in \mathcal{D}^*$ (i.e. $\tilde{D}(\varepsilon) = \delta$) via Fourier transform (see [**Kri07b**, 8.3.1]), and obtaining the solution of $D(u) = s$ as $u := \varepsilon \star s$ (see [**Kri07b**, 4.7.7]). Here P is a polynomial $z \mapsto \sum_{|k|\le m} a_k\, z^k$ and P^t is the polynomial $z \mapsto \sum_{|k|\le m}(-1)^{|k|} a_k z^k$.

In [**DGC71**] it is shown that every linear partial differential operator $C^\omega(\mathbb{R}^2) \to C^\omega(\mathbb{R}^2)$ is onto, where $C^\omega(\mathbb{R}^n)$ denotes the space of real-analytic scalar valued functions on $\mathbb{R}^n$. In contrast, the PDO $(\frac{\partial}{\partial x})^2 + (\frac{\partial}{\partial y})^2 : C^\omega(\mathbb{R}^3) \to C^\omega(\mathbb{R}^3)$ is not onto.

4.74 Surjectivity criterium (See [**MV92**, 26.1 p.289])**.**
Let $T : E \to F$ be continuous linear between Fréchet spaces. Then

1. *T is onto;*

⇔ 2. *T is an lcs-quotient mapping;*

⇔ 3. *$T^* : F^* \to E^*$ is a cbs-embedding,*
i.e. B equicontinuous $\Rightarrow$ $(T^*)^{-1}(B)$ equicontinuous.

⇔ 4. *$T^* : {}^b(F^*_\beta) \to {}^b(E^*_\beta)$ is a cbs-embedding,*
i.e. w.r.t. the von Neumann bornologies.

Proof. ($\boxed{1}\Rightarrow\boxed{2}$) by the open mapping theorem.

($\boxed{2}\Rightarrow\boxed{3}$) is $\boxed{4.72}$

($\boxed{3}\Leftrightarrow\boxed{4}$) since E and F are Fréchet (hence quasi-barrelled) the β-bounded sets are exactly the equicontinuous ones.

($\boxed{3}\Rightarrow\boxed{1}$) Let U be an absolutely convex 0-nbhd $\Rightarrow U^o$ equicontinuous $\xRightarrow{\boxed{3}}$ $T(U)^o = (T^*)^{-1}(U^o)$ (by $\boxed{4.68.1}$) is equicontinuous $\Rightarrow \overline{T(U)} = (T(U)^o)_o$ 0-nbhd $\xRightarrow{F \text{ not meager}}$ $T(E)$ not meager (and hence T is onto by [**Kri14**, 4.3.6]): Suppose $T(E) \subseteq \bigcup_n A_n$ with A_n closed. Then $E = \bigcup_n T^{-1}(A_n)$ with $T^{-1}(A_n)$ closed, hence $\exists n$: $\operatorname{int}(T^{-1}(A_n)) \neq \emptyset$. Let $x \in \operatorname{int}(T^{-1}(A_n))$ and U be a 0-nbhd with $x+U \subseteq T^{-1}(A_n)$. Then $T(x) + T(U) \subseteq A_n$ and also $T(x) + \overline{T(U)} \subseteq A_n$, i.e. the interior of A_n is not empty. □

4.75 Lemma of Baernstein (See [**MV92**, 26.26 p.303])**.**
Let $T : E \to F$ continuous linear between (DF) spaces, E be (M).
Then $T : E \to F$ is an lcs-embedding iff $T : {}^bE \to {}^bF$ is an cbs-embedding.

Proof.
($\Rightarrow$) is $\boxed{4.70}$.

($\Leftarrow$) By 4.18.2 E^* and F^* are Fréchet and $T^* \in \mathcal{L}(F^*, E^*)$. By 3.22 E is reflexive. Since $T^{**} = T : E^{**} = E \to F \hookrightarrow F^{**}$ it follows that T^* is onto by 4.74. Let U be an absolutely convex closed 0-nbhd in E. Thus U^o is bounded and hence compact in the (M)-space E^*. By 3.6 this can be lifted to a compact set $K \subseteq F^*$ which has to be contained in the closed absolutely convex hull of a 0-sequence (y_n^*) in the (F)-space F^*. Let $V := \bigcap_n V_n$ with absolutely convex 0-nbhds $V_n := \{y_n^*\}_o$. The set V is bornivorous, since $y_n^* \to 0$, and hence a 0-nbhd, since as (DF)-space F is quasi-countably-barrelled. Since $K \subseteq \overline{\langle\{y_n^* : n \in \mathbb{N}\}\rangle_{\text{abs.conv.}}} \subseteq V^o$, we get $T^*(V^o) \supseteq T^*(K) = U^o$ and hence $U = (U^o)_o \supseteq (T^*(V^o))_o = T^{-1}(V)$, i.e. $T(U)$ is a 0-nbhd in the trace topology on $\operatorname{img} T$. □

4.76 Theorem of Eidelheit (See [**MV92**, 26.27 p.305]).
Let E be (F) and $(x_k^*)_{k\in\mathbb{N}}$ linearly independent in E^*. Then
$\forall y \in \mathbb{R}^{\mathbb{N}}\ \exists x \in E\ \forall k \in \mathbb{N} : x_k^*(x) = y_k \Leftrightarrow \forall U : \dim\big(E^*_{U^o} \cap \big\langle\{x_k^* : k \in \mathbb{N}\}\big\rangle_{lin.sp}\big) < \infty.$

Proof. By assumption $T := (x_k^*)_{k\in\mathbb{N}} : E \to \mathbb{K}^{\mathbb{N}}$ is continuous linear. Its adjoint $T^* : \mathbb{K}^{(\mathbb{N})} = (\mathbb{K}^{\mathbb{N}})^* \to E^*$ is given by $T^*(y) = \sum_k x_k^* \otimes y_k$, since

$$T^*(y)(x) = y(T(x)) = \sum_k y_k\, x_k^*(x) = \Big(\sum_k x_k^* \otimes y_k\Big)(x).$$

Hence T^* is bijective onto $\langle\{x_k^* : k \in \mathbb{N}\}\rangle_{\text{lin.sp.}}$ (since the x_k^* are linearly independent).
By 4.74 T is onto iff $(T^*)^{-1}(B)$ is bounded in $\mathbb{K}^{(\mathbb{N})}$ for each bounded $B \subseteq E^*$, i.e. for each 0-nbhd U the set $T(U)^o = (T^*)^{-1}(U^o) = \big\{y \in \mathbb{K}^{(\mathbb{N})} : \sum_k x_k^* \otimes y_k \in U^o\big\}$ has to be bounded and hence has to be contained in some finite subsum $\mathbb{K}^N$. Since T^* is injective, it induces a linear isomorphism

$$\begin{aligned}\big\langle T(U)^o\big\rangle_{\text{lin.sp.}} &= \bigcup_{\lambda>0} \lambda \cdot T(U)^o \cong \bigcup_{\lambda>0} \lambda \cdot T^*(T(U)^o) \overset{4.68.2}{=\!=\!=} \bigcup_{\lambda>0} \lambda \cdot U^o \cap \operatorname{img} T^* \\ &= E^*_{U^o} \cap \big\langle\{x_k^* : k \in \mathbb{N}\}\big\rangle_{\text{lin.sp.}}.\end{aligned}$$

($\Rightarrow$) Since $T(U)^o$ has to be contained in some $\mathbb{K}^N$, we have that $\dim\big(E^*_{U^o} \cap \big\langle\{x_k^* : k \in \mathbb{N}\}\big\rangle_{\text{lin.sp.}}\big) < \infty$ for each U.

($\Leftarrow$) The condition implies that the closed absolutely convex set $A := (T^*)^{-1}(U^o) = T(U)^o$ is contained in a finite dimensional linear subspace $\mathbb{K}^N$ and contains no $\mathbb{R}_+ \cdot x^*$ for $x^* \neq 0$, since otherwise $T^*(x^*)|_U = 0$ and hence $T^*(x^*) = 0$, thus $x^* = 0$. This implies that A is bounded, otherwise choose $a_n \in A \subseteq \mathbb{K}^N$ with $1 \leq \|a_n\| \to \infty$ and let $a_\infty \in A$ be an accumulation point of $\frac{1}{\|a_n\|} a_n \in A$. Then $\lambda\, a_\infty \in A$ for all $\lambda > 0$ since $A \ni \frac{\lambda}{\|a_n\|} a_n \to \lambda\, a_\infty$ for $\|a_n\| \geq \lambda$. □

4.77 Corollary. (F) spaces with $\mathbb{K}^{\mathbb{N}}$ as quotient (See [**MV92**, 26.28 p.305]).
Let E be (F) and not Banach. Then $\mathbb{K}^{\mathbb{N}}$ is a topological quotient of E.

Proof. Let (U_n) be a falling 0-nbhd basis of E. Since E is not Banach, we may assume that $\exists x_k^* \in E^*_{U_k^o} \setminus E^*_{U_{k-1}^o}$. Then $(x_k^*)_k$ is linear independent and the mapping $Q := (x_k^*)_{k\in\mathbb{N}} : E \to \mathbb{K}^{\mathbb{N}}$ satisfies the assumptions of 4.76 hence is onto. □

4.78 Borels theorem (See [**MV92**, 26.29 p.305]).
$\forall y \in \mathbb{K}^{\mathbb{N}}\ \exists f \in C^\infty([-1,1],\mathbb{K})\ \forall k \in \mathbb{N} : f^{(k)}(0) = y_k.$

Proof. Let $\|f\|_k := \max_{j\leq k} \|f^{(j)}\|_\infty$ and $U_k := \{f : \|f\|_k \leq 1\}$. Consider $x_k^* : E := C^\infty([-1,1],\mathbb{K}) \to \mathbb{K}$ given by $x_k^*(f) := f^{(k)}(0)$. Obviously $(x_k^*)_{k\in\mathbb{N}}$ is linearly

independent (on monomials). For finite sequences ξ the functional $\sum_j \xi_j x_j^* \in E^*_{(U_k)^o}$ iff $\xi_j = 0$ for all $j > k$ (Choose f with small derivatives of order $< j$ but high one of order j). Thus $(x_k^*)_{k\in\mathbb{N}}$ is onto by 4.76. □

4.79 $\lambda^p(A)$ with quotient ℓ^p (See [**MV92**, 27.22 p.320]).
Let $A = \{a^{(k)} \in \mathbb{R}_+^{\mathbb{N}\times\mathbb{N}} : k \in \mathbb{N}\}$ with $a_{i,j}^{(1)} \geq 1$ and $a_{i,k}^{(k)} = a_{1,k}^{(k)}$.
Then $\lambda^p(A)$ has ℓ^p as quotient for $1 \leq p < \infty$ and $c_0(A)$ has c_0 as quotient.

Proof. $Q : \lambda^p(A) \to \ell^p$ defined by $Q(x) := \Big(\sum_j x_{i,j}/2^j\Big)_i$ is continuous and linear, since $\sum_i \big|\sum_j x_{i,j}/2^j\big|^p \leq \sum_i \big(\|(x_{i,j})_j\|_{\ell_p} \cdot \|(\frac{1}{2^j})_j\|_{\ell^q}\big)^p \leq \|x\|_{\ell^p}^p \cdot 1 \leq \|x \cdot a^{(1)}\|_{\ell^p}^p$.

Claim: Q is onto (we will use 4.74): $Q^* : \ell^q \to \lambda^p(A)^*$, $y \in \ell^q$, $x \in \lambda^p(A)$:

$$(Q^*y)(x) = y(Qx) = \sum_i y_i \sum_j x_{i,j}/2^j = \sum_{i,j} \frac{y_i}{2^j} x_{i,j} \Rightarrow Q^*(y) = (y_i/2^j)_{i,j}.$$

For $k \in \mathbb{N}$ and $U_k := \{x : \|x\|_k \leq 1\}$ we have:

$$(Q^*)^{-1}(U_k^o) \overset{\boxed{4.68.1}}{=\!=} Q(U_k)^o \overset{\boxed{1.24}}{=\!=} \Big\{y \in \ell^q : |y(Qx)| = \Big|\sum_{i,j} x_{i,j} y_i/2^j\Big| \leq \|x\|_k\Big\}.$$

Let $y \in (Q^*)^{-1}(U_k^o) \subseteq \ell^q$, $\xi \in \ell^p$, $x : (i,j) \mapsto \xi_i \delta_{j,k}$. Then

$$\begin{aligned}|y(\xi)| &= \Big|\sum_i \xi_i y_i\Big| = 2^k \Big|\sum_i \xi_i y_i/2^k\Big| = 2^k \Big|\sum_{i,j} x_{i,j} y_i/2^j\Big| \\ &\leq 2^k \|x\|_k = 2^k \Big(\sum_i |\xi_i a_{i,k}^{(k)}|^p\Big)^{1/p} = 2^k a_{1,k}^{(k)} \|\xi\|_{\ell^p}\end{aligned}$$

$\Rightarrow \|y\|_{\ell^q} \leq 2^k a_{1,k}^{(k)}$, i.e. $(Q^*)^{-1}(U_k^o)$ is bounded $\overset{\boxed{4.74}}{\Longrightarrow}$ Q is a quotient mapping.

For $c_0(A)$ the proof is analogous. □

4.80 Counter-example for <u>cbs</u>-quotient mapping (See [**MV92**, 27.23 p.321]).
Let A be as in 3.36, $1 \leq p < \infty$, $Q : \lambda^p(A) \twoheadrightarrow \ell^p$ a quotient mapping as in 4.79.
Then Q is not a bornological quotient mapping.

Proof. The unit ball in ℓ^p is not compact and $\lambda^p(A)$ is Montel, hence a bounded lift would be compact. □

4.81 Counter-example for inheritance of reflexivity and bornologicity (See [**MV92**, 27.24 p.321]).
There is a reflexive (even (S)) ultra-bornological (DF) space with a closed not infra-barrelled and hence not reflexive subspace.

Proof. Let $\lambda^p(A)$ with $1 \leq p < \infty$ be the (FM) space of 3.36. By 3.22 it is reflexive and by 4.27 its dual $E := \lambda^p(A)^*$ is Montel (by 4.35 even (S)), hence reflexive and bornological by 4.16, and (DF) by 4.18.1. Let $Q : \lambda^p(A) \twoheadrightarrow \ell^p$ be the quotient mapping as in 4.79 and consider the closed subspace $F := \operatorname{img}(Q^*) = \ker(Q)^o$ in $\lambda^p(A)^*$, using 4.67. Let W be the unit ball in ℓ^p, then $U := Q^{-1}(W)$ is a 0-nbhd with $Q(U) = W$. By 4.68.2 we have $Q^*(W^o) = Q^*(Q(U)^o) = U^o \cap \operatorname{img} Q^* = U^o \cap F$, hence $Q^*(W^o)$ is absolutely convex and closed in F. It is a bornivorous barrel, since each bounded set B in F has bounded inverse $(Q^*)^{-1}(B)$ in ℓ^q by 4.74 and hence is absorbed by the unit-ball W^o. Infra-barrelledness of F would imply that $Q^*(W^o)$ is a 0-nbhd in F and is bounded as image of the unit-ball.

Since E is Montel it would be even relatively compact in F. Thus F would be finite dimensional, which is a contradiction to the injectivity of Q^*. $\square$

4.82 Surjectivity of dense mappings (See [**MV92**, 26.2 p.289])**.**
Let $T : E \to F$ be continuous linear with dense image between (F) spaces. Then T is onto $\Leftrightarrow \forall U : U^o \cap \operatorname{img}(T^*)$ is a Banach-disk .

Proof. By $\boxed{4.68.2}$ $A^o \cap \operatorname{img}(T^*) = T^*(T(A)^o)$.

($\Rightarrow$) U 0-nbhd $\Rightarrow$ $T(U)$ 0-nbhd $\xRightarrow{[\text{Kri14}, 5.4.12]+[\text{Kri14}, 5.4.17]}$ $T(U)^o$ Banach-disk. $T(E)$ dense $\Rightarrow$ T^* injective $\Rightarrow$ $E^*_{T^*(T(U)^o)} \cong F^*_{T(U)^o}$ Banach, i.e. $U^o \cap \operatorname{img} T^*$ is a Banach-disk.

($\Leftarrow$) U 0-nbhd $\Rightarrow$ $B := U^o \cap \operatorname{img} T^*$ Banach-disk, $(E^*)_B \to (E^*, \sigma(E^*, E))$ bd. Let $(V_n)_n$ be a 0-nbhd-basis in F $\Rightarrow$ $F^* = \bigcup_n V_n^o$ $\Rightarrow$ $(E^*)_B \subseteq \operatorname{img} T^* = \bigcup_n T^*(V_n^o)$. V_n^o is $\sigma(F^*, F)$-cp $\Rightarrow$ $T^*(V_n^o)$ is $\sigma(E^*, E)$-cp $\Rightarrow$ $T^*(V_n^o) \cap (E^*)_B$ is closed in $(E^*)_B$. $\xRightarrow{[\text{Kri14}, 4.1.11]}$ $\exists m$: $\exists x$ in the interior of $T^*(V_m^o) \cap (E^*)_B$ $\Rightarrow$ $-x$ as well $\Rightarrow$ 0 as well $\Rightarrow$ $\exists \varepsilon > 0$: $\varepsilon B \subseteq T^*(V_m^o)$ $\xRightarrow{T^* \text{ inj}}$ $(T^*)^{-1}(B) \subseteq \frac{1}{\varepsilon} V_m^o$ $\Rightarrow$ $(T^*)^{-1}(B)$ bd.
$\xRightarrow{\boxed{4.74}}$ T surjective. $\square$

4.83 Theorem on closed image (See [**MV92**, 26.3 p.290], [**Kri14**, 9.11])**.**
Let $T : E \to F$ be continuous linear between (F) spaces. Then

1. $\operatorname{img}(T)$ *is closed;*
$\Leftrightarrow$ 2. $\operatorname{img}(T) = \ker(T^*)^o$*;*
$\Leftrightarrow$ 3. $U^o \cap \operatorname{img}(T^*)$ *is a Banach-disk for each 0-nbhd U;*
$\Leftrightarrow$ 4. $U^o \cap \operatorname{img}(T^*)$ *is ($\sigma(E^*, E)$ or) $\beta(E^*, E)$ closed for each U;*
$\Leftrightarrow$ 5. $\operatorname{img}(T^*)$ *is closed;*
$\Leftrightarrow$ 6. $\operatorname{img}(T^*) = \ker(T)^o$*;*
$\Leftrightarrow$ 7. $T : E/\ker(T) \to \operatorname{img}(T)$ *is a homeomorphism.*

Proof. ($\boxed{1}\Leftrightarrow\boxed{2}$) $\overline{\operatorname{img} T} = \ker(T^*)^o$ by [**Kri14**, 5.4.3].

($\boxed{1}\Rightarrow\boxed{7}$) $T : E/\ker(T) \to \operatorname{img}(T)$ bijective continuous linear. Homeomorphism $\Leftrightarrow$ $\operatorname{img} T$ closed.

($\boxed{7}\Rightarrow\boxed{6}$) $\operatorname{img} T \cong E/\ker T$. $\operatorname{img}(T^*) \subseteq \ker(T)^o$ is obvious. Conversely, $x^* \in \ker(T)^o \Rightarrow \exists y^* \in (\operatorname{img} T)^* \cong (E/\ker T)^*$, $y^* \circ T = x^* \Rightarrow \exists z^* \in F^* : z^*|_{\operatorname{img} T} = y^*$, i.e. $T^*(z^*) = z^* \circ T = y^* \circ T = x^*$.

($\boxed{6}\Rightarrow\boxed{5}$) Obvious, since $\ker(T)^o$ is closed.

($\boxed{5}\Rightarrow\boxed{4}$) $\operatorname{img}(T^*)$ and U^o are closed $\Rightarrow$ $U^o \cap \operatorname{img}(T^*)$ is $\beta(E^*, E)$-closed.

($\boxed{4}\Rightarrow\boxed{3}$) $B := U^o \cap T^*(F^*)$ closed, $E^* = E'$ complete $\Rightarrow$ $(E^*)_B$ complete, see [**Kri07a**, 2.27].

($\boxed{3}\Rightarrow\boxed{1}$) Let $T_0 : E \to \overline{T(E)} (\overset{\iota}{\hookrightarrow} F)$ $\xRightarrow{[\text{Kri14}, 5.1.5]}$ ι^* is onto $\Rightarrow$ $\operatorname{img}(T_0^*) = \operatorname{img}(T^*)$ $\xRightarrow{\boxed{4.82}, \boxed{3}}$ T_0 surj $\Rightarrow$ $\overline{T(E)} = T_0(E) = T(E)$, i.e. $\operatorname{img} T$ is closed.

($\boxed{4}$:$\beta(E^*, E)$-closed$\Rightarrow$$\sigma(E^*, E)$-closed). By ($\boxed{4}\Rightarrow\boxed{3}\Rightarrow\boxed{1}\Rightarrow\boxed{6}$) we have $U^o \cap \operatorname{img}(T^*) = U^o \cap \ker(T)^o$, which is $\sigma(E^*, E)$-closed. $\square$

4.84 Exactness for dual sequence (See [**MV92**, 26.4 p.291])**.**
Let $E \xrightarrow{S} F \xrightarrow{Q} G$ be a short sequence in (F). Then $0 \to E \to F \to G \to 0$ is exact $\Leftrightarrow$ $0 \leftarrow E^* \leftarrow F^* \leftarrow G^* \leftarrow 0$ is algebraically exact.

Proof. ($\Rightarrow$) This is $\boxed{4.67}$.

($\Leftarrow$) $\operatorname{img}(Q^*) = \ker(S^*)$ and $\operatorname{img}(S^*) = E^* \Rightarrow \operatorname{img}(Q^*)$ and $\operatorname{img}(S^*)$ are closed $\Rightarrow$ $\operatorname{img}(Q)$ and $\operatorname{img}(S)$ are closed by $\boxed{4.83}$.
Since $\operatorname{img}(T^*)_o = \{x : y^*(Tx) = T^*(y^*)(x) = 0\ \forall y^*\} = \ker(T)$, we get

$$\begin{aligned}
\ker(S) &= \operatorname{img}(S^*)_o = (E^*)_o = 0 \\
\operatorname{img}(S) &= \overline{\operatorname{img}(S)} \xlongequal{\text{[Kri07b, 7.4.3]}} \ker(S^*)_o = \operatorname{img}(Q^*)_o = \ker(Q) \\
\operatorname{img}(Q) &= \overline{\operatorname{img}(Q)} = (\operatorname{img}(Q)^o)_o \xlongequal{\text{[Kri07b, 7.4.3]}} \ker(Q^*)_o = \{0\}_o = G. \quad \square
\end{aligned}$$

4.85 Corollary. Duals of subspaces and quotient spaces of (F) spaces (See [**MV92**, 26.5 p.292], [**Kri14**, 5.4.4]).
Let $F \hookrightarrow E$ be an embedding in (F) and let $0 \to (F/E)^* \to F^* \to E^* \to 0$ be topologically exact. Then $E^* \cong F^*/E^o$ and $(F/E)^* \cong E^o$.

Proof.

$$\begin{tikzcd}
(F/E)^* \arrow[r, hook] \arrow[d, tail] & F^* \arrow[r, two heads] \arrow[dr] & E^* \\
E^o \arrow[ur, hook] & & F^*/E^o \arrow[u, tail, two heads]
\end{tikzcd}$$

By [**Kri14**, 5.4.4] the vertical arrows in this diagram are continuous bijections. The left one is an iso, since $(F/E)^* \to F^*$ is assumed to be an embedding and the right one is an iso, since $F^* \to E^*$ is assumed to be a quotient mapping. $\square$

4.86 Definition. The canonical resolution.
Let $E_\infty = \varprojlim_n E_n$ be a reduced projective limit of a sequence of Fréchet spaces with connecting morphisms $f_k^{k+1} : E_{k+1} \to E_k$. Then the short sequence

$$0 \to E_\infty \xrightarrow{\iota} \prod_k E_k \xrightarrow{\pi} \prod_k E_k \to 0,$$
$$\text{where } \pi\big((x_k)_{k\in\mathbb{N}}\big) := \big(x_k - f_k^{k+1}(x_{k+1})\big)_{k\in\mathbb{N}},$$

is called CANONICAL RESOLUTION of the projective limit.

If E is a Fréchet space with basis of seminorms $\|\_\|_k$ for $k \in \mathbb{N}$, then E is isomorphic to the reduced projective limit formed by the Banach spaces $E_k := \widetilde{E_{U_k}}$, where $U_k := \{x \in E : \|x\|_k \le 1\}$ and the corresponding short exact sequence is called the CANONICAL RESOLUTION of the Fréchet space.

4.87 Exactness of the canonical resolutions (See [**MV92**, 26.15 p.299]).
The canonical resolution of any reduced projective limit of a sequence of (F) spaces is exact. In particular, the canonical resolution of any Fréchet space is exact.

Proof. Let $E = \varprojlim_n E_n$ with (F) spaces E_n, i.e. $E = \{x \in \prod_k E_k : x_k = f_k^{k+1}(x_{k+1})$ for all $k\}$ which is the kernel of the mapping $\pi : \prod_k E_k \to \prod_k E_k$ given by $\pi(x) = (x_k - f_k^{k+1}(x_{k+1}))_{k\in\mathbb{N}}$. Let $\iota : E \hookrightarrow \prod_{k\in\mathbb{N}} E_k$ be the inclusion. Obviously $\prod_k E_k^* \cong (\prod_k E_k)^*$, via $(x_k^*)_{k\in\mathbb{N}} \mapsto ((x_k)_{k\in\mathbb{N}} \mapsto \sum_k x_k^*(x_k))$. Using this

identification the adjoint mappings are given by:

$$\iota^*(x^*)(x) := x^*(\iota(x)) = \sum_k x_k^*(\iota_k(x)) = \Big(\sum_k x_k^* \circ \iota_k\Big)(x) \;\Rightarrow$$
$$\iota^* : (x_k^*)_{k\in\mathbb{N}} \mapsto \sum_k x_k^* \circ \iota_k$$
$$\pi^*(y^*)\big((x_k)_{k\in\mathbb{N}}\big) := y^*\Big(\pi\big((x_k)_{k\in\mathbb{N}}\big)\Big) = \sum_k y_k^*\big(x_k - f_k^{k+1}(x_{k+1})\big)$$
$$= \sum_k \Big(y_k^* - y_{k-1}^* \circ f_{k-1}^k\Big)\big((x_k)_{k\in\mathbb{N}}\big) \;\Rightarrow$$
$$\pi^* : (y_k^*)_{k\in\mathbb{N}} \mapsto (y_k^* - y_{k-1}^* \circ f_{k-1}^k)_{k\in\mathbb{N}}, \text{ where } y_{-1}^* := 0.$$

Since ι is an embedding, ι^* is onto by Hahn-Banach. The adjoint π^* is injective, since $y_k^* - y_{k-1}^* f_{k-1}^k = 0$ for all k recursively gives $y_k^* = 0$ for all k.

Obviously $0 = 0^* = (\pi \circ \iota)^* = \iota^* \circ \pi^*$. So let $x^* \in \coprod_k E_k^*$ with $\iota^*(x^*) = 0$. Remains to find $y^* \in \coprod_k E_k^*$ with $x^* = \pi^*(y^*)$, i.e.

$$x_k^* = y_k^* - y_{k-1}^* \circ f_{k-1}^k \text{ for all } k \in \mathbb{N}.$$

Recursively we get $y_{-1}^* := 0$ and $y_k^* := \sum_{j\le k} x_j^* \circ f_j^k$:

$$y_k^* = x_k^* + y_{k-1}^* \circ f_{k-1}^k = x_k^* + \Big(\sum_{j\le k-1} x_j^* \circ f_j^{k-1}\Big) \circ f_{k-1}^k = \sum_{j\le k} x_j^* \circ f_j^k.$$

Since $x^* \in \coprod_k E_k^*$ there exists an $n \in \mathbb{N}$ with $x_j^* = 0$ for all $j \ge n$. For $m \ge n$ we thus have

$$0 = \iota^*(x^*) = \sum_k x_k^* \circ \iota_k = \sum_{k<n} x_k^* \circ (f_k^m \circ \iota_m) = \Big(\sum_{k\le m} x_k^* \circ f_k^m\Big) \circ \iota_m = y_m^* \circ \iota_m.$$

Since ι_m has dense image we get $y_m^* = 0$ for all $m \ge n$, i.e. $y^* \in \coprod_k E_k^*$.

Thus the dual sequence is exact and by 4.84 the canonical resolution itself is exact. □

4.88 Proposition (See [**Bon91**]+[**MV92**, 26.14 p.298])**.**
Let E be a Fréchet space with increasing sequence of seminorms $\|\text{-}\|_k$. And let $\|\text{-}\|_k^*$ be the Minkowski functional of the polar of $U_k := \{x : \|x\|_k \le 1\}$, i.e. $\|x^*\|_k^* = \sup\{|x^*(x)| : x \in U_k\}$. Then

1. *E is quasi-normable, i.e. $\forall U\ \exists U'\ \forall \varepsilon > 0\ \exists B$ bd $: U' \subseteq B + \varepsilon\, U$.*

⇔ 2. $\forall p\ \exists p' > p\ \forall q > p'\ \forall \varepsilon > 0\ \exists \varepsilon' > 0 : \|\text{-}\|_{p'}^* \le \varepsilon'\, \|\text{-}\|_q^* + \varepsilon\, \|\text{-}\|_p^*$;

⇔ 3. $\forall p\ \exists p' > p\ \forall q > p'\ \forall \varepsilon > 0\ \exists \varepsilon' > 0 : \varepsilon'\, U_q^o \cap U_p^o \subseteq \varepsilon\, U_{p'}^o$.

⇔ 4. $\forall p\ \exists p' > p\ \forall q > p'\ \forall \varepsilon > 0\ \exists \varepsilon' > 0 : U_{p'} \subseteq \varepsilon'\, U_q + \varepsilon\, U_p$;

⇔ 5. *Every 0-sequence in E_β^* is Mackey-convergent.*

⇔ 6. *$F_{born}^* = \varinjlim_n E_{U_n^o}^*$ is sequentially retractive.*

We need and prove only the equivalence of the first 4 conditions.

Proof.
(1⇒2) By 1: $\forall p\ \exists p' > p\ \forall \varepsilon > 0\ \exists B$ bd $: U_{p'} \subseteq B + \varepsilon\, U_p$. By the boundedness of B: $\forall q > p'\ \exists \varepsilon' > 0 : B \subseteq \varepsilon'\, U_q$. Thus

$$\|y\|_{p'}^* = \sup\big\{|y(x)| : x \in U_{p'}\big\} \le \sup\big\{|y(x_1 + x_2)| : x_1 \in \varepsilon' U_q, x_2 \in \varepsilon\, U_p\big\}$$
$$\le \sup\big\{|y(x_1)| : x_1 \in \varepsilon' U_q\big\} + \sup\big\{|y(x_2)| : x_2 \in \varepsilon\, U_p\big\} = \varepsilon'\, \|y\|_q^* + \varepsilon\, \|y\|_p^*.$$

($\boxed{2} \Rightarrow \boxed{3}$) For given $\alpha > 0$ let $\varepsilon := \frac{\alpha}{2}$ and we have an $\varepsilon' > 0$ with

$$\|y\|_{p'}^* \le \varepsilon' \|y\|_q^* + \varepsilon \|y\|_p^*.$$

Put $\alpha' := \frac{\alpha}{2\varepsilon'}$ and let $y \in \alpha' U_q^o \cap U_p^o$. Then $\|y\|_q^* \le \alpha'$ and $\|y\|_p^* \le 1$, hence

$$\|y\|_{p'}^* \le \varepsilon' \cdot \alpha' + \varepsilon \cdot 1 = \alpha, \text{ i.e. } y \in \alpha U_{p'}^o.$$

($\boxed{3} \Rightarrow \boxed{4}$) Let $\varepsilon' U_q^o \cap U_p^o \subseteq \varepsilon U_{q'}^o$. Since $(\frac{1}{\varepsilon'} U_q + U_p)^o \subseteq \varepsilon' U_q^o \cap U_p^o$ we get by polarization and bipolar theorem

$$U_{q'} \subseteq (U_{q'}^o)_o \subseteq \varepsilon\Big(\big(\tfrac{1}{\varepsilon'} U_q + U_p\big)^o\Big)_o = \varepsilon\, \overline{\tfrac{1}{\varepsilon'} U_q + U_p} \subseteq \varepsilon\Big(U_q + \tfrac{1}{\varepsilon'} U_q + U_p\Big) = \varepsilon'' U_q + \varepsilon U_p$$

with $\varepsilon'' := \varepsilon(1 + \frac{1}{\varepsilon'})$.

($\boxed{4} \Rightarrow \boxed{1}$) [Bon91] W.l.o.g. let $U := U_0$ and by assumption $\boxed{4}$ $(p + 1 = p')$

$$\forall p\, \forall q\, \forall \varepsilon > 0\; \exists \varepsilon' > 0 : U_{p+1} \subseteq \varepsilon' U_q + \varepsilon U_p.$$

$\Rightarrow \exists\, \varepsilon_1' > 0$: $U_1 \subseteq \varepsilon_1' U_2 + \frac{\varepsilon}{2} U_0$
$\Rightarrow \exists\, \bar\varepsilon_2' > 0$: $U_2 \subseteq \bar\varepsilon_2' U_3 + \frac{\varepsilon}{2^2 \varepsilon_1'} U_1$, i.e. $\varepsilon_1' U_2 \subseteq \varepsilon_2' U_3 + \frac{\varepsilon}{2^2} U_1$ with $\varepsilon_2' := \varepsilon_1' \bar\varepsilon_2'$. $\Rightarrow \dots$
$\Rightarrow \exists\, \varepsilon_k'$: $\varepsilon_{k-1}' U_k \subseteq \varepsilon_k' U_{k+1} + \frac{\varepsilon}{2^k} U_{k-1}$.
Let $z \in U_1$. Then $z = \varepsilon_1' u_2 + \frac{\varepsilon}{2} v_1$ with $u_2 \in U_2$ and $v_1 \in U_0$ and $\varepsilon_{k-1}' u_k = \varepsilon_k' u_{k+1} + \frac{\varepsilon}{2^k} v_k$ with $u_{k+1} \in U_{k+1}$ and $v_k \in U_{k-1} \subseteq U_0$. $\Rightarrow \exists x := \sum_{k=1}^\infty \frac{\varepsilon}{2^k} v_k \in \varepsilon U_0$, since F is Fréchet. The set $B := \bigcap_k (\varepsilon_k' + \varepsilon)\, U_k$ is bounded and $z - x \in B$, since

$$\begin{aligned} z - x &= \Big(z - \sum_{j=1}^{k} \frac{\varepsilon}{2^j} v_j\Big) - \sum_{j=k+1}^{\infty} \frac{\varepsilon}{2^j} v_j = \varepsilon_k' u_{k+1} - \sum_{j=k+1}^{\infty} \frac{\varepsilon}{2^j} v_j \\ &\in \varepsilon_k' U_{k+1} + \frac{\varepsilon}{2^k} U_k \subseteq (\varepsilon_k' + \varepsilon) U_k. \quad \square \end{aligned}$$

4.89 Canonical resolution and quasi-normability (See [MV92, 26.16 p.299]).
Let E be (F) and π the quotient mapping of the canonical resolution of E.
If π^* is an embedding, then E is quasi-normable.

Proof. Indirectly assume E is not quasi-normable. $(E_k)^* \cong E^*_{U_k^o}$ and the dual norm $\|\text{-}\|_k$ on $(E_k)^*$ is just $\|\text{-}\|_k^*$. $\xRightarrow{\boxed{4.88}}$

(1) $\quad \exists m\, \forall k > m\, \exists k' > k\, \exists \varepsilon_k > 0\, \forall S > 0\, \exists y \in E_m^* : \|y\|_k > S \|y\|_{k'} + \varepsilon_k \|y\|_m$.

By assumption $(\pi^*)^{-1} : \operatorname{img}(\pi^*) \to \coprod_k E_k^*$ is continuous and $p : \eta \mapsto \sum_k \frac{k}{\varepsilon_k} \|\eta_k\|_k$ a continuous seminorm on $\coprod_k E_k^*$.
$\Rightarrow \exists \tilde p$ SN of $\coprod_k E_k^*$: $p((\pi^*)^{-1}(\eta)) \le \tilde p(\eta)$ for all $\eta \in \operatorname{img}(\pi^*) = \ker(\iota^*)$.
$\Rightarrow \exists D_k \ge 0$: $\tilde p(\eta) \le \sum_k D_k \|\eta_k\|_k$.
$\boxed{1} \Rightarrow \forall k > \max\{m, D_m\}\, \exists k' > k\, \exists \varepsilon_k > 0$ for $S := D_{k'} \varepsilon_k / k\; \exists y \in E_m^*$ with

$$\|y\|_k > S \|y\|_{k'} + \varepsilon_k \|y\|_m$$

Let $\eta \in \ker(\iota^*)$ be given by $\eta_m := y$, $\eta_{k'} := -y \in E_m^* \subseteq E_{k'}^*$ and $\eta_j = 0$ otherwise.
$\xRightarrow{\text{Proof of } \boxed{4.87}}$ $(\pi^*)^{-1}(\eta) = (\sum_{j \le k} \eta_j)_k$. Since $m < k < k'$ we have:

$$\begin{aligned} &\frac{k}{\varepsilon_k} \|y\|_k \le p\Big(\Big(\sum_{j \le k} \eta_j\Big)_k\Big) \le p\big((\pi^*)^{-1}(\eta)\big) \le \tilde p(\eta) \le D_m \|y\|_m + D_{k'} \|y\|_{k'} \\ &\Rightarrow\; \|y\|_k \le D_m \frac{\varepsilon_k}{k} \|y\|_m + D_{k'} \frac{\varepsilon_k}{k} \|y\|_{k'} \le \varepsilon_k \|y\|_m + S \|y\|_{k'}, \end{aligned}$$

a contradiction. $\square$

4.90 Sequences of bounded subsets in (F) spaces (See [**MV92**, 26.6 p.292]).
Let E be metrizable. Then

1. *$B_n \subseteq E$ bounded $\Rightarrow \exists \delta_n > 0$: $\bigcup_{n\in\mathbb{N}} \delta_n B_n$ bounded.*
2. *$\forall$ B bounded $\exists$ $C \supseteq B$ bounded: $E_C \supseteq B \to E$ is an embedding.*

Proof. Let (U_n) be a 0-nbhd basis of E.
($\boxed{1}$) $\forall j, n$ $\exists \lambda_{j,n} > 0$: $B_j \subseteq \lambda_{j,n} U_n$. Let $\lambda_n := \max\{\lambda_{j,n} : j \le n\} \Rightarrow \forall j \le n$: $\lambda_{j,n} \le \lambda_n \Rightarrow \forall j$ $\exists \alpha_j \ge 1$ $\forall$n: $\lambda_{j,n} \le \alpha_j \lambda_n \Rightarrow \forall j, n$: $B_j \subseteq \lambda_{j,n} U_n \subseteq \alpha_j \lambda_n U_n \Rightarrow \forall j$: $B_j \subseteq \alpha_j \bigcap_n \lambda_n U_n$ and $B := \bigcap_n \lambda_n U_n$ is bounded. Thus $\bigcup_j \frac{1}{\alpha_j} B_j \subseteq B$ is bounded.

($\boxed{2}$) W.l.o.g. B absolute convex. $\exists \lambda_n > 0 : B \subseteq \lambda_n U_n \Rightarrow B \subseteq C := \bigcap_n n\lambda_n U_n$ and C is bounded and $E_C \supseteq B \to E$ continuous. In order to show the converse, let $x \in B$ and $\varepsilon > 0$. $\exists m$: $\varepsilon > \frac{2}{m}$, $\exists k$: $U_k \subseteq \bigcap_{n<m} \varepsilon n \lambda_n U_n \Rightarrow$

$$\begin{aligned}
2B \cap U_k \subseteq \Big(\bigcap_{n\ge m} \varepsilon n\lambda_n U_n\Big) \cap \bigcap_{n<m} \varepsilon n \lambda_n U_n = \varepsilon \bigcap_n n\lambda_n U_n = \varepsilon\, C \\
\Rightarrow B \cap (x + U_k) = x + (B - x) \cap U_k \subseteq x + 2B \cap U_k \subseteq x + \varepsilon\, C \\
\Rightarrow B \cap (x + U_k) \subseteq B \cap (x + \varepsilon\, C). \quad \square
\end{aligned}$$

4.91 Theorem. Dual of surjective mappings (See [**MV92**, 26.7 p.293]).
Let $Q : F \twoheadrightarrow G$ be continuous linear between (F) and onto. Then
Q is a $\underline{cbs}$-quotient mapping $\Leftrightarrow$ Q^* is an $\underline{lcs}$-embedding.

Proof. ($\Rightarrow$) This is $\boxed{4.72.1}$
($\Leftarrow$) $B \subseteq G$ bd $\Rightarrow B^o$ 0-nbhd $\Rightarrow Q^*(B^o)$ 0-nbhd in $\mathrm{img}(Q^*) \subseteq F^* \Rightarrow \exists M \subseteq F$ bd. $M^o \cap \mathrm{img}(Q^*) \subseteq Q^*(B^o) \xRightarrow{Q^* \text{ inj}}$

$$\begin{aligned}
Q(M)^o \overset{\boxed{4.68.1}}{=\!=} (Q^*)^{-1}(M^o) = (Q^*)^{-1}(M^o \cap \mathrm{img}(Q^*)) \subseteq B^o \\
\Rightarrow B \subseteq (B^o)_o \subseteq (Q(M)^o)_o = \overline{Q(M)} \\
\xRightarrow{\boxed{4.90.2} \text{ for } \overline{Q(M)}} \exists B' \subseteq G \text{ bd.} : \overline{Q(M)} = \overline{Q(M)}^{G_{B'}}
\end{aligned}$$

$\Rightarrow \forall B \subseteq G$ bd. $\exists M \subseteq F$ bd. $\exists B' \subseteq G$ bd. $\forall \varepsilon > 0 : B \subseteq Q(M) + \varepsilon B'$. $\xRightarrow{\text{recursion}}$ $\forall B_0 \subseteq G$ bd. $\forall n$ $\exists M_n \subseteq F$ bd. $\exists B_n \subseteq G$ bd. $\forall \varepsilon > 0$: $B_n \subseteq Q(M_n) + \varepsilon B_{n+1}$. $\xRightarrow{\boxed{4.90.1}}$ $\exists M, B$ bd. $\exists \lambda_n > 0$: $M_n \subseteq \lambda_n M$, $B_n \subseteq \lambda_n B$. W.l.o.g. M closed and $\lambda_0 = \frac{1}{2}$. $\Rightarrow$ Take $\varepsilon_0 := 1$ and $\varepsilon_n \le 1$ such that $\varepsilon_n \lambda_n \le 1/2^{n+1}$.
$\forall b_0 \in B_0$ $\forall n$ $\exists m_n \in M_n$ $\exists b_{n+1} \in B_{n+1}$: $b_n = Q(m_n) + \varepsilon_{n+1} b_{n+1}$. $\Rightarrow$

$$b_0 = Q\Big(\sum_{j\le k} \varepsilon_0 \cdots \varepsilon_j m_j\Big) + \varepsilon_0 \cdots \varepsilon_{k+1} b_{k+1}$$

$$\varepsilon_0 \cdots \varepsilon_j m_j \in e_0 \cdots \varepsilon_j M_j \subseteq \varepsilon_j \lambda_j M \subseteq \frac{1}{2^{j+1}} M$$

$$\varepsilon_0 \cdots \varepsilon_{k+1} b_{k+1} \in e_0 \cdots \varepsilon_{k+1} B_{k+1} \subseteq \varepsilon_{k+1} \lambda_{k+1} B \subseteq \frac{1}{2^{k+2}} B$$

$$\Rightarrow \; \exists m := \sum_{j=0}^{\infty} \varepsilon_0 \cdots \varepsilon_j m_j \in M \text{ and } Q(m) = \sum_{j=0}^{\infty} \varepsilon_0 \cdots \varepsilon_j (b_j - \varepsilon_{j+1} b_{j+1}) = b_0,$$

i.e. $B_0 \subseteq Q(M)$. $\square$

4.92 Proposition. (See [**MV92**, 26.11 p.295]).
Let $0 \to E \xrightarrow{S} F \xrightarrow{Q} G \to 0$ *be exact between (F).*
Q *a* *cbs-quotient mapping* $\Rightarrow$ S^* *is an* *lcs-quotient mapping.*

Proof. Let $(U_n)_n$ and $(V_n)_n$ 0-nbhd-bases of F and G.

Claim. $\forall n\ \exists m\ \forall B \subseteq V_m$ bd $\exists M \subseteq U_n$ bd: $Q(M) = B$:
Indirect: $\exists n\ \forall m\ \exists B_m \subseteq V_m$ bd $\forall M \subseteq F$ bd $Q(M) = B \Rightarrow M \not\subseteq U_n$.
$\Rightarrow B := \bigcup 2^m B_m \subseteq G$ bd (in fact: $2V_{n+1} \subseteq V_n$, $\Rightarrow \bigcup_{m \geq n} 2^m B_m \subseteq 2^n V_n$) $\Rightarrow$ $\exists M \subseteq F$ bd: $Q(M) = B \Rightarrow \exists \varepsilon > 0$: $\varepsilon M \subseteq U_n$, $\exists m$: $\varepsilon 2^m \geq 1 \Rightarrow Q(\varepsilon M) = \varepsilon B \supseteq \varepsilon 2^m B_m \supseteq B_m$. $\Rightarrow M_0 := \varepsilon M \cap Q^{-1}(B_m) \subseteq U_n$, $Q(M_0) = B_m$, a contradiction.

Claim. $\forall B \subseteq G$ bd $\exists M \subseteq F$ bd $\forall n\ \exists k$: $Q(M \cap U_n) \supseteq B \cap V_k$:
$\forall n\ \exists m_n\ \forall B \subseteq G$ bd $\exists M_n \subseteq U_n$: $Q(M_n) = B_n := B \cap V_{m_n}$.
Put $M := \langle \bigcup_n M_n \rangle_{\text{abs.conv}}$. $\Rightarrow M \subseteq F$ bd, since $\bigcup_{k \geq n} M_k \subseteq \bigcup_{k \geq n} U_k = U_n$. $Q(M \cap U_n) \supseteq Q(M_n) = B_n = B \cap V_{m_n}$

Claim. $\forall L \subseteq F$ bd $\exists D \subseteq E$ bd $\forall n\ \exists m$: $(L + U_m) \cap E \subseteq D + U_n$
(See [**MV92**, 26.10 p.295]):
Let $L \subseteq F$ bd, $B := Q(L) \Rightarrow \exists M \subseteq F$ bd $\forall n\ \exists k$: $Q(M \cap U_n) \supseteq B \cap V_k$.
Put $D := (L + M) \cap E \subseteq E$ bd. $\forall n\ \exists \bar{n} \geq n$: $2U_{\bar{n}} \subseteq U_n$ $\exists k$: $Q(M \cap U_{\bar{n}}) \supseteq B \cap V_k$.
Q continuous $\Rightarrow \exists m \geq \bar{n}$: $Q(U_m) \subseteq V_k$.
Let $x \in (L + U_m) \cap E$. $\Rightarrow \exists l \in L\ \exists u \in U_m$: $x = l - u$, $Q(l) - Q(u) = Q(x) = 0$.
$Q(l) = Q(u) \in B \cap Q(U_m) \subseteq B \cap V_k \subseteq Q(M \cap U_{\bar{n}}) \Rightarrow$
$\exists \xi \in M \cap U_{\bar{n}}$: $Q(\xi) = Q(l) = Q(u) \Rightarrow$

$$x = (l - \xi) + (\xi - u) \in (L + M) \cap E + U_{\bar{n}} + U_m \subseteq (L + M) \cap E + 2U_{\bar{n}} \subseteq D + U_n$$

Claim. S^* is a quotient mapping (See [**MV92**, 26.9 p.295]):
$\forall L \subseteq F$ bd $\exists D \subseteq E$ bd $\forall n\ \exists m$: $(L + U_m) \cap E \subseteq D + U_n$.
Let $y \in D^o$. $\Rightarrow \exists n$: $y \in U_n^o \Rightarrow y \in D^o \cap U_n^o \subseteq 2(D + U_n)^o \Rightarrow y \in 2((L + U_m) \cap E)^o$
$\xRightarrow{\text{Hahn Banach}} \exists \tilde{y} \in 2(L + U_m)^o \subseteq 2L^o$: $\tilde{y} \circ S = y$, i.e. $S^*(L^o) \supseteq \frac{1}{2} D^o$. □

4.93 Theorem. Exactness of dualized sequence (See [**MV92**, 26.12 p.296]).
Let $0 \to E \to F \xrightarrow{Q} G \to 0$ *be exact between (F) spaces. Then*
$0 \leftarrow E^* \leftarrow F^* \leftarrow G^* \leftarrow 0$ *is topologically exact* $\Leftrightarrow$ Q *is bornological quotient mapping.*

Proof. $\boxed{4.84}$, $\boxed{4.91}$, $\boxed{4.92}$ $\Rightarrow$ □

4.94 Lifting compact sets along quotient mappings in (F)
(See [**MV92**, 26.21 p.302]).
Let $Q : E \to F$ *be continuous linear surjective between (F) spaces.*
Then Q *is a* *cbs-quotient mapping for the bornologies of relatively compact subsets.*

Proof. $K \subseteq F$ compact $\xRightarrow{\boxed{3.6}}$ $\exists x_n \to 0$, $K \subseteq \langle \{x_n : n \in \mathbb{N}\} \rangle \overset{(!)}{\Rightarrow} \exists z_n \to 0$ in E with $Q(z_n) = x_n \xRightarrow{\boxed{3.6}} L := \langle \{z_n : n \in \mathbb{N}\} \rangle$ cp, $Q(L) \subseteq K$. □

4.95 Theorem. Dual of sequences in (F) with (M) quotient (See [**MV92**, 26.22 p.303]).
Let $0 \to E \to F \to G$ *be a sequence in (F) and let* G *be (M). Then*
$0 \to E \to F \to G \to 0$ *is exact* $\Leftrightarrow$ $0 \leftarrow E^* \leftarrow F^* \leftarrow G^* \leftarrow 0$ *is topologically exact.*

Proof. G (M) $\Rightarrow$ bounded sets are relatively compact, hence have bounded lifts along Q by $\boxed{4.94}$. Thus Q is a cbs-quotient mapping, hence the dual sequence is topologically exact by $\boxed{4.93}$. The converse follows from $\boxed{4.84}$. $\square$

4.96 Exactness for quasi-normable spaces (See [**MV92**, 26.13 p.296]).
Let $0 \to E \to F \xrightarrow{Q} G \to 0$ be exact in (F) and let E quasi-normable. Then Q is a cbs-quotient mapping.

Proof. $(W_n)_n$ 0-nbhd-basis of F. $\xRightarrow{\boxed{4.88.4}}$ $\exists n_k$, $U_k := W_{n_k}$, $V_k := U_k \cap E$:

(1) $$\forall k\, \forall \varepsilon > 0\, \exists \varepsilon' > 0 : V_k \subseteq \varepsilon' V_{k+1} + \varepsilon V_{k-1}.$$

Let $B \subseteq G$ bd. $\xRightarrow{Q \text{ open}}$ $\forall k\, \exists C_k$: $B \subseteq C_k\, Q(U_k)$. Put $C'_k := C_k + C_{k+1}$. We use recursion to construct

(2) $$\forall k \geq 2\, \exists \varepsilon_k > 0 : V_k \subseteq \varepsilon_k V_{k+1} + \frac{1}{2^k D_k} V_{k-1} \text{ with } D_k := C'_k + D_{k-1}\varepsilon_{k-1}.$$

In fact, put $D_1 := 0$, and in the induction step let $\varepsilon := 1/(2^k D_k)$ and take $\varepsilon_k := \varepsilon'$ as in $\boxed{1}$.
For $\|\_\|_k := p_{U_k}$ let $M := \{x \in F : \forall k \geq 2 : \|x\|_k \leq C_k + D_k\varepsilon_k + C'_k + 1\}$. Then M is bounded.

Claim: $Q(M) \supseteq B$.
Let $\xi \in B \subseteq C_k\, Q(U_k) = Q(C_k\, U_k) \Rightarrow \exists x_k \in C_k\, U_k$, $Q(x_k) = \xi$. Put $y_k := x_k - x_{k+1}$. $\Rightarrow Q(y_k) = Q(x_k - x_{k+1}) = \xi - \xi = 0$, i.e. $y_k \in E = \ker(Q)$ and

$$y_k = x_k - x_{k+1} \in C_k\, U_k + C_{k+1}\, U_{k+1} \subseteq (C_k + C_{k+1})\, U_k = C'_k\, U_k$$

$\Rightarrow \forall k : y_k \in C'_k\, U_k \cap E = C'_k\, V_k$.
We use induction to construct $v_k \in D_k\varepsilon_k V_{k+1}$ and $u_k \in 2^{-k} V_{k-1}$ such that

$$y_k + v_{k-1} = v_k + u_k.$$

Let $v_0 := 0$ and v_{k-1} already be given. Then $y_k + v_{k-1} \in (C'_k + \varepsilon_{k-1} D_{k-1}) V_k = D_k V_k \xRightarrow{\boxed{2}} \exists v_k \in D_k\varepsilon_k V_{k+1}$, $u_k \in 2^{-k} V_{k-1}$: $y_k + v_{k-1} = v_k + u_k$.
$\Rightarrow \exists b_k := v_{k-1} - \sum_{j \geq k} u_j \in E$, since $\sum_{j>k} \|u_j\|_k \leq \sum_{j>k} \|u_j\|_{j-1} \leq \sum_{j>k} \frac{1}{2^j} = \frac{1}{2^k}$.

$$\|b_k\|_k = \Big\|\underbrace{v_{k-1} - u_k}_{= v_k - y_k} - \sum_{j>k} u_j\Big\|_k \leq \|v_k\|_k + \|y_k\|_k + \frac{1}{2^k} \leq D_k\varepsilon_k + C'_k + 1.$$

Let $x := x_2 + b_2$. Since $b_{k+1} - b_k = v_k - v_{k-1} + u_k = y_k = x_k - x_{k+1}$, we have $\|x\|_k = \|x_k + b_k\|_k \leq \|x_k\|_k + \|b_k\|_k \leq C_k + (D_k\varepsilon_k + C'_k + 1) \Rightarrow x \in M$ and $Q(x) = Q(x_2) + Q(b_2) = \xi + 0$. $\square$

4.97 Theorem. Dual sequences and quasi-normability (See [**MV92**, 26.17 p.300]).
Let E be (F). Then

1. *E is q-normable*

$\Leftrightarrow$ 2. *If $Q : F \twoheadrightarrow G$ is an lcs-quotient mapping in (F) with kernel E, then Q is a cbs-quotient mapping.*

$\Leftrightarrow$ 3. *If $0 \to E \to F \to G \to 0$ is exact in (F), then $0 \leftarrow E^* \leftarrow F^* \leftarrow G^* \leftarrow 0$ is topologically exact.*

Proof. (1⇒2) by 4.96.
(2⇒3) by 4.93.
(3⇒1) 4.89 for the canonical resolution of 4.87. □

4.98 Corollary. Quasi-normable (F) are distinguished
(See [MV92, 26.18 p.301]).
q-normable (F) ⇒ distinguished.

Proof. $\xRightarrow{4.97}$ dual of canonical resolution of 4.87 is topological exact. ⇒ E^* is quotient of countable coproduct of Banach spaces, hence bornological by 2.5. □

4.99 Dual sequences for Schwartz spaces (See [MV92, 26.24 p.303]).
Let $0 \to E \to F \to G \to 0$ be exact in (F) and let one of these 3 spaces be in (S). Then the dual sequence is topologically exact.

Proof. Closed subspaces and quotients of (FS) are (FS) by 3.73, hence (M) and quasi-normable. Thus 4.97 and 4.95 yield the result. □

4.100 Corollary (See [MV92, 26.25 p.303]).
Let F be (FS) and $E \subseteq F$ be closed. Then $E^* \cong F^*/E^o$ and $(F/E)^* \cong E^o$.

Proof. 4.99, 4.85 ⇒ □

4.101 Example (See [MV92, 27.19 p.318]).
Let $A = \{a^{(k)} : k \in \mathbb{N}\}$ as in 4.25. Then the Fréchet space $\lambda^1(A)$ is not distinguished, not quasi-normable and $(\lambda^1(A))^*$ is (DF) but not infra-barrelled.

Proof. In 4.25 we have shown that $\lambda^1(A)$ is not distinguished and hence $(\lambda^1(A))^*$ is (DF) but not infra-barrelled. By 4.98 $\lambda^1(A)$ is not quasi-normable. □

4.102 Quasi-normability of $\lambda^p(A)$ (See [MV92, 27.20 p.318]).
For $A = \{a^{(n)} : n \in \mathbb{N}\}$ let $E := \lambda^p(A)$ with $1 \le p < \infty$ oder $E := c_0(A)$. Then

1. *E is q-normable*

⇔ 2. $\forall p\, \exists p'\, \forall \varepsilon > 0\, \forall q\, \exists \varepsilon' > 0\, \forall j : 1/a_j^{(p')} \le \varepsilon'/a_j^{(q)} + \varepsilon/a_j^{(p)}$.

⇔ 3. $\forall p\, \exists p'\, \forall J \subseteq \mathbb{N}$*:* $\inf_{j\in J} a_j^{(p)}/a_j^{(p')} > 0 \Rightarrow \forall q \ge p'$*:* $\inf_{j\in J} : a_j^{(p)}/a_j^{(q)} > 0$.

Proof. $E = \lambda^p(A)$ (For $E = c_0(A)$ analogously)

(1⇒2) $U_p := \{x \in \lambda^p(A) : \|x\|_p \le 1\}$. $\xRightarrow{4.88}$

$$\forall p\, \exists p' > p\, \forall \varepsilon > 0\, \forall q \in \mathbb{N}\, \exists \varepsilon' > 0\, \forall y \in E^* : \|y\|_{U_{p'}^o} \le \varepsilon' \|y\|_{U_q^o} + \varepsilon \|y\|_{U_p^o}$$

(2) is obvious for j with $a_j^{(p)} = 0$. Otherwise $a_j^{(p)} \ne 0 \xRightarrow{1.24} e_j \in \langle U_p^o \rangle$ and

$$1/a_j^{(p')} = \|e_j\|_{U_{p'}^o} \le \varepsilon' \|e_j\|_{U_q^o} + \varepsilon \|e_j\|_{U_p^o} = \varepsilon'/a_j^{(q)} + \varepsilon/a_j^{(p)}.$$

(2⇒3) $\forall p\, \exists p'$ satisfying (2). $I \subseteq \mathbb{N}$ with $\inf_{j\in I} a_j^{(p)}/a_j^{(p')} =: \eta > 0$. Put $\varepsilon := \eta/2$. For $q \ge p'$ $\exists \varepsilon' > 0$ satisfying $a_j^{(p)}/a_j^{(p')} \le \varepsilon' a_j^{(p)}/a_j^{(q)} + \varepsilon$ for all $j \Rightarrow a_j^{(p)}/a_j^{(q)} \ge (a_j^{(p)}/a_j^{(p')} - \varepsilon)/\varepsilon' \ge \frac{\eta}{2\varepsilon'}$.

($\boxed{3}\Rightarrow\boxed{1}$) $\forall p\ \exists p' > p$ satisfying ($\boxed{3}$). Let $\varepsilon > 0$, $q \in \mathbb{N}$. Put $I := \{j : a_j^{(p)}/a_j^{(p')} \geq \varepsilon\}$ $\xRightarrow{(\boxed{3})}$ $\forall q \geq p'\ \exists \varepsilon' > 0$: $\inf_{j \in I} a_j^{(p)}/a_j^{(q)} \geq \frac{1}{\varepsilon'}$.

Claim: $U_{p'} \subseteq \varepsilon' U_q + \varepsilon U_p$ ($\xRightarrow{\boxed{4.88.4}}$ $\lambda^p(A)$ q-normable):
Let $x \in U_{p'}$ and $z := x - y$ with $y_j := x_j$ for $j \in I$ and 0 otherwise. $\forall j \in I$: $a_j^{(p')} \geq a_j^{(p)} \geq a_j^{(q)}/\varepsilon' \Rightarrow$

$$\|y\|_q = \left\| y \cdot a^{(p')} \cdot \frac{1}{a^{(p')}} \cdot a^{(q)} \right\|_{\ell^p} \leq \|x\|_{p'}\, \varepsilon' \leq \varepsilon'$$

$\forall j \notin I$: $a_j^{(p)} < \varepsilon a_j^{(p')} \Rightarrow$

$$\|z\|_p = \left\| z \cdot a^{(p')} \cdot \frac{1}{a^{(p')}} \cdot a^{(p)} \right\|_{\ell^p} \leq \|x\|_{p'}\, \varepsilon \leq \varepsilon$$

$\Rightarrow x = y + z \in \varepsilon' U_q + \varepsilon U_p$. $\square$

Splitting sequences

In this section we describe situations, where short exact sequences split and reformulate this in terms of the derived functor Ext of the Hom-functor. For sequences with a power series space as kernel the characterizing property on the quotient is (DN). And for sequences with a power series space of infinite (resp. finite) type as quotient the characterizing property on the kernel is (Ω) (resp. $(\overline{\overline{\Omega}})$). We show that the spaces in (DN) are exactly the subspaces of generalized power series spaces $\lambda^\infty_\infty(a)$ of infinite type. And the spaces in (Ω) are exactly the quotients of generalized power series spaces $\lambda^1_\infty(a)$ of infinity type. We give some applications to extensions of non-linear mappings and introduce universal linearizer for that.

4.103 Continuously solving PDE's.
In the early fifties of the 20^{th} century L. Schwartz posed the problem of determining when a linear partial differential operator $P(\partial)$ has a (continuous linear) right inverse. Grothendieck has shown that for $n \geq 2$ no elliptic operator has such an inverse on $C^\infty(U)$ for $U \subseteq \mathbb{R}^n$. By [**Vog84**, Theorem 3.3 p.365] the same holds more generally for hypoelliptic operators. This is in contrast to that fact, that hyperbolic PDO's have continuous linear right inverses on $C^\infty(\mathbb{R}^n)$.

A partial differential operator D defined on an open subset $U \subseteq \mathbb{R}^n$ is called HYPOELLIPTIC if for every distribution u defined on an open subset $V \subseteq U$ such that Du is C^∞ is itself C^∞. The Laplacian Δ is elliptic and thus hypoelliptic. The heat equation operator $D(u) := \frac{\partial}{dt}u - \Delta(u)$ is hypoelliptic (even parabolic) but not elliptic. The wave equation operator $D(u) := (\frac{\partial}{dt})^2 u - \Delta(u)$ is not hypoelliptic, it is hyperbolic; i.e. the Cauchy problem is uniquely solvable in a neighborhood of each point p for any initial data given on a non-characteristic hypersurface passing through p.

4.106 Continuously extending functions or jets.
The exact sequence

$$\{f \in C^\infty([-1,1]) : f \text{ is } \infty\text{-flat at } 0\} \hookrightarrow C^\infty([-1,1]) \twoheadrightarrow \mathbb{R}^{\mathbb{N}}$$

of Borel's theorem 4.78 is not splitting. Otherwise, there would be an embedding $\mathbb{R}^{\mathbb{N}} \hookrightarrow C^\infty([-1,1])$ and the ∞-norm on $C^\infty([-1,1])$ would induce a seminorm on $\mathbb{R}^{\mathbb{N}}$ with kernel $\{0\}$. But each seminorm $x \mapsto \max_{i\leq n}|x_i|$ of the usual basis of seminorms for the product $\mathbb{R}^{\mathbb{N}}$ has an infinite dimensional kernel $\{x \in \mathbb{R}^{\mathbb{N}} : x_i = 0 \text{ for all } i \leq n\}$.

For subsets $\iota : A \subseteq \mathbb{R}^n$ let us consider the property, that the restriction operator $\iota^* : C^\infty(A,\mathbb{R}) \twoheadrightarrow C^\infty(\mathbb{R}^n,\mathbb{R})$ has a continuous linear right inverse.

By [**See64**] $\mathbb{R}_{t\geq 0} \subseteq \mathbb{R}$ has this property and by [**Tid79**, Satz 4.5 p.308] for each $r > 1$ the set $A := \{x \in \mathbb{R}^n : 0 \leq x_1 \leq 1, x_2^2 + \dots + x_n^2 \leq x_1^{2r}\} \subseteq \mathbb{R}^n$ has it. Whereas, by [**Tid79**, Beispiel 2 p.301] the set $A := \{(x,y) : x \geq 0, |y| \leq \varphi(x)\}$ does not have it, when $\varphi \in C^\infty(\mathbb{R},\mathbb{R})$ is ∞-flat at 0.

4.107 Proposition.
The functor $\mathcal{L}(\_,F) : \underline{lcs}^{op} \to \underline{lcs}$ is left exact,
i.e. if $0 \leftarrow E^+ \xleftarrow{Q} E \xleftarrow{S} E^-$ is topologically exact, then

$$0 \to \mathcal{L}(E^+,F) \xrightarrow{Q^*} \mathcal{L}(E,F) \xrightarrow{S^*} \mathcal{L}(E^-,F)$$

is also exact, i.e. $\mathcal{L}(\_,F)$ is a LEFT EXACT FUNCTOR.

Proof. (Q^* is injective) Let $0 = Q^*(\varphi) = \varphi \circ Q$. Then $\varphi = 0$, since Q is onto.

$(\ker(S^*) = \operatorname{img}(Q^*))$ Let $0 = S^*(\varphi) = \varphi \circ S$, i.e. φ vanishes on $\operatorname{img}(S) = \ker(Q)$ and hence factors to a $\tilde\varphi : E^+ \to F$ with $\varphi = \tilde\varphi \circ Q = Q^*(\tilde\varphi)$. The converse inclusion is obvious by $Q \circ S = 0$. □

The functor $\mathcal{L}(\_, F)$ is not exact (i.e. maps short exact sequences to such sequences) in general, since exactness at $\mathcal{L}(E^-, F)$ would mean that for closed embeddings $S : E^- \hookrightarrow E$ the adjoint $S^* : \mathcal{L}(E, F) \to \mathcal{L}(E^-, F)$ is onto, i.e. every morphism $\varphi : E^- \to F$ must have an extension to E.

4.108 Definition. Injective spaces and extension of maps.
An (F) space E is called INJECTIVE FRÉCHET SPACE iff for every embedding $S : H \hookrightarrow G$ of (F) spaces every $T \in \mathcal{L}(H, E)$ has an extension $\tilde T \in \mathcal{H}(G, E)$, i.e. $\tilde T \circ S = T$.

$$\begin{tikzcd} H \arrow[r, hook, "S"] \arrow[dr, "T"'] & G \arrow[d, dotted, "\tilde T"] \\ & E \end{tikzcd}$$

Thus the Fréchet spaces F for which $\mathcal{L}(\_, F)$ preserves exactness of all short exact sequences in (F) are exactly the injective ones.
By Hahn-Banach $\mathbb{K}$ is an injective (Fréchet) space.
More generally, for every set X the Banach space $\ell^\infty(X)$ of bounded functions on X is an injective Fréchet space: Let $S : H \hookrightarrow G$ be an embedding and $T : H \to \ell^\infty(X)$ be continuous, i.e. $p := \|\_\|_{\ell^\infty} \circ T$ is a continuous seminorm on H and hence has an extension to a continuous seminorm $\tilde p$ on G. Now $T_x := \operatorname{ev}_x \circ T \in H^*$ for each $x \in X$ and $|T_x(y)| \le \|T(y)\|_\infty = p(y)$ for all $y \in H$. By Hahn Banach we find $\tilde T_x \in G^*$ with $|\tilde T_x(z)| \le \tilde p(z)$ for all $z \in G$. Thus $\tilde T : G \to \ell^\infty(X)$ defined by $\tilde T(z)(x) := \tilde T_x(z)$ for each $z \in G$ and $x \in X$ is a continuous linear extension of T, since $\|\tilde T(z)\|_\infty = \sup\{|\tilde T_x(z)| : x \in X\} \le \tilde p(z) < \infty$ for all $z \in G$.

Every Fréchet space F is subspace of an injective Fréchet space: We can embed F into a countable product of Banach spaces and every Banach space G can be embedded into the space of bounded linear functionals on G^* and thus into $\ell^\infty(oG^*)$. Since a countable product of injective Fréchet spaces is obviously an injective Fréchet space we are done.

4.109 Proposition.
The functor $\mathcal{L}(E, \_) : \underline{lcs} \to \underline{lcs}$ is also left exact.

Proof. Let $0 \to F^- \xrightarrow{S} F \xrightarrow{Q} F^+$ be topologically exact and consider

$$0 \to \mathcal{L}(E, F^-) \xrightarrow{S_*} \mathcal{L}(E, F) \xrightarrow{Q_*} \mathcal{L}(E, F^+).$$

It is exact at $\mathcal{L}(E, F^-)$, since S_* is obviously injective.

It is exact at $\mathcal{L}(E, F)$, since $\varphi \in \mathcal{L}(E, F)$ is in $\ker(Q_*) \Leftrightarrow 0 = Q_*(\varphi) = Q \circ \varphi \Leftrightarrow \operatorname{img}(\varphi) \subseteq \ker(Q) = \operatorname{img}(S) \Leftrightarrow \varphi$ factors to a morphism $\tilde\varphi : E \to F^-$ over the embedding $S : F^- \hookrightarrow F \Leftrightarrow \varphi \in \operatorname{img}(S_*)$. □

The functor $\mathcal{L}(E, \_)$ is not exact in general, since exactness at $\mathcal{L}(E, F^+)$ would mean that for quotient mappins $p : F \twoheadrightarrow F^+$ the adjoint $p_* : \mathcal{L}(E, F) \to \mathcal{L}(E, F^+)$ is onto, i.e. every morphism $\varphi : E \to F^+$ can be lifted along $p : F \twoheadrightarrow F^+$ to a morphism $\tilde\varphi : E \to F$.

4.110 Remark. Projective spaces and lifting of maps.

An (F) space E is called PROJECTIVE FRÉCHET SPACE iff for every quotient mapping $Q : G \twoheadrightarrow H$ of (F) spaces every $T \in \mathcal{L}(E, H)$ has a lift $\tilde T \in \mathcal{L}(E, G)$, i.e. $Q \circ \tilde T = T$.

$$\begin{tikzcd} H & G \arrow[l, two heads, "Q"'] \\ & E \arrow[ul, "T"] \arrow[u, dotted, "\tilde T"'] \end{tikzcd}$$

Obviously every finite dimensional space is projective, since all linear mappings on it are continuous. It was shown in [**Gej78**] that there are no other projective Fréchet spaces.

4.111 Theorem. Splitting sequences (See [**Vog87**, 1.8 p.171]).
Let E and F be (F). Then

1. *Let $0 \to F \to G \to H \to 0$ be exact in (F).*
   Then $0 \to \mathcal{L}(E,F) \to \mathcal{L}(E,G) \to \mathcal{L}(E,H) \to 0$ is exact.

⇔ 2. *Let $0 \to F \to G \to H \to 0$ be exact in (F).*
   Then any $T \in \mathcal{L}(E,H)$ lifts along $G \twoheadrightarrow H$.

⇔ 3. *Let $0 \to F \to G \to E \to 0$ be exact in (F).*
   Then $G \to E$ has a right inverse.

⇔ 4. *Let $0 \to F \to G \to E \to 0$ be exact in (F). Then it splits,*
   i.e. is isomorphic to the sequence $0 \to F \xrightarrow{\mathrm{inj}_1} F \oplus E \xrightarrow{\mathrm{pr}_2} E \to 0$

⇔ 5. *Let $0 \to F \to G \to E \to 0$ be exact in (F).*
   Then $F \to G$ has a left inverse.,

⇔ 6. *Let $0 \to H \to G \to E \to 0$ be exact in (F).*
   Then any $T \in \mathcal{L}(H,F)$ extends along $H \rightarrowtail G$.

⇔ 7. *Let $0 \to H \to G \to E \to 0$ be exact in (F).*
   Then $0 \to \mathcal{L}(E,F) \to \mathcal{L}(G,F) \to \mathcal{L}(H,F) \to 0$ is exact.

Proof. (1⇔2) Since $\mathcal{L}(E,\_)$ is left exact by 4.109, 1 holds iff $\mathcal{L}(E,G) \to \mathcal{L}(E,H)$ is onto, i.e. any $T \in \mathcal{L}(E,H)$ has a lift $\tilde{T} \in \mathcal{L}(E,G)$.

(2⇒3) By 2 the identity $\mathrm{id}_E : E \to E$ has a lift $\widetilde{\mathrm{id}_E} : E \to G$ along $G \to E$.

(3⇒4) Let $\mathrm{id} = Q \circ S : E \to G \to E$. The isomorphism $F \oplus E \to G$ is given by $(y,x) \mapsto (y, S(x))$ with inverse $G \to F \oplus E$, $z \mapsto (z - S(Q(z)), Q(z))$.

(4⇒2) Consider the pull-back

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r, hook] \arrow[d, equal] & G \arrow[r, two heads] & H \arrow[r] & 0 \\
0 \arrow[r] & F \arrow[r, hook, dotted, "\mathrm{inj}_1"] \arrow[d, equal] & G \times_H E \arrow[r, two heads, "\mathrm{pr}_2"] \arrow[u, "\mathrm{pr}_1"'] & E \arrow[r] \arrow[u, "T"'] \arrow[d, equal] & 0 \\
0 \arrow[r] & F \arrow[r, hook, "\mathrm{inj}_1"] & F \oplus E \arrow[r, two heads, "\mathrm{pr}_2"] \arrow[u, dotted, "\cong"', "\Phi"] & E \arrow[r] \arrow[l, bend left, dotted, "\mathrm{inj}_2"] & 0
\end{tikzcd}$$

The bottom row is again an exact sequence, hence splits by 4, and thus gives a lifting

$$\tilde{T} := \mathrm{pr}_1 \circ \Phi \circ \mathrm{inj}_2,$$

where Φ is the isomorphism.

(7⇔6⇔5⇔4) is obtained by dualizing the arguments of (1⇔2⇔3⇔4). □

4.112 Remark. Exactness of tensor-functors.
The algebraic tensor product functor $\_ \otimes F$ is exact, since short exact sequences of linear spaces are splitting and applying the functor $\_ \otimes F$ to the splitting gives splitting exact sequences.

The injective tensor product functor $\_ \otimes_\varepsilon F$ preserves embeddings S, since we have by definition natural embeddings $E \otimes_\varepsilon F \hookrightarrow L(E^*, F)$ and $L(E^*, \_)$ obviously preserves embeddings $S : F_1 \hookrightarrow F_2$, in fact

$$(S_*)^{-1}(N_{B,V}) = \{T : (S \circ T)(B) \subseteq V\} = \{T : T(B) \subseteq S^{-1}(V)\} =: N_{B,S^{-1}(V)}.$$

It does not preserve quotient mappings, see [**Kri07a**, 4.29].

The projective tensor product functor $\_\otimes_\pi F$ preserves quotient mappings $Q : E_1 \to E_2$, since the image of $U\otimes V \subseteq E_1\otimes_\pi V$ under the linear map $Q\otimes F$ is the absolutely convex hull of the image $(\otimes \circ (Q \times F))(U_1 \times V) = \otimes(Q(U_1) \times V)$ and hence is a 0-neighborhood in $E_2 \otimes_\pi F$.

The completion functor $(\_)^\sim$ preserves short topologically exact sequences $0 \to E \hookrightarrow F \twoheadrightarrow G \to 0$, since $\tilde{E}$ can be obtained as closure of E in $\tilde{F}$. Thus $E = \tilde{E} \cap F$, since $y \in \tilde{E}\cap F = \overline{E}^{\tilde{F}} \cap F \Rightarrow \exists x_i \in E : x_i \to y \in F \Rightarrow 0 = Q(x_i) = \tilde{Q}(x_i) \to \tilde{Q}(y) = Q(y)$, i.e. $y \in \ker Q = E$. Therefore, the mapping $G \cong F/E = F/(\tilde{E}\cap F) \to \tilde{F}/\tilde{E}$ induced from $F \hookrightarrow \tilde{F}$ is injective. It is an embeddings, since for every continuous seminorm q in G, we can extend $p := q \circ Q$ to a continuous seminorm $\tilde{p}$ in $\tilde{F}$ which has to vanish on the closure $\tilde{E}$ of E in $\tilde{F}$ hence factors to a continuous seminorm $\tilde{q}$ on $\tilde{F}/\tilde{E}$, which induces q on F/E. Since $F \hookrightarrow \tilde{F}$ has dense image and $\tilde{F} \twoheadrightarrow \tilde{F}/\tilde{E}$ is onto the embedding $F/E \hookrightarrow \tilde{F}/\tilde{E}$ has dense image, and since $\tilde{F}/\tilde{E}$ is a Fréchet space, it is the completion $\tilde{G}$ of G.

In order to describe the obstruction to exactness of the functor $\mathcal{L}$ we need injective resolutions:

4.113 Proposition. Injective Resolution.
For every Fréchet space F there exists an injective resolution, i.e. a long exact sequence $0 \to F \to I_0 \to I_1 \to I_2 \to \cdots$, where I_k is an injective Fréchet space for each $k \in \mathbb{N}$.

Proof. Let I_0 be an injective Fréchet space into which F embeds by 4.108. Recursively, we may embed $I_k/\operatorname{img}(I_{k-1})$ (where $I_{-1} := F$) into an injective Fréchet space I_{k+1} and take as connecting map the composite of the quotient map $I_k \twoheadrightarrow I_k/\operatorname{img}(I_{k-1})$ and this embedding $I_k/\operatorname{img}(I_{k-1}) \hookrightarrow I_{k+1}$. □

Using injective resolutions we can construct the derived functors using homological algebra:

4.114 Theorem. Derived functors.
There are functors $\operatorname{Ext}^k : \underline{lcs}^{op} \times \underline{lcs} \to \underline{vs}$ for $k \in \mathbb{Z}$ (called the RIGHT-DERIVED FUNCTORS *of $\mathcal{L}$) and natural transformations δ such that:*

1. $\operatorname{Ext}^k(E, F) = 0$ *for* $k < 0$.
2. $\operatorname{Ext}^0 \cong \mathcal{L}$.
3. $\operatorname{Ext}^k(E, F) = 0$ *for all* $k > 0$ *if* F *is injective.*
4. *For every short exact sequence* $0 \to E^- \to E \to E^+ \to 0$ *there is a long exact sequence*

$$\cdots \to \operatorname{Ext}^k(E^+, F) \to \operatorname{Ext}^k(E, F) \to \operatorname{Ext}^k(E^-, F) \xrightarrow{\delta} \operatorname{Ext}^{k+1}(E^+, F) \to \cdots .$$

For every short exact sequence $0 \to F^- \to F \to F^+ \to 0$ *there is a long exact sequence*

$$\cdots \to \operatorname{Ext}^k(E, F^-) \to \operatorname{Ext}^k(E, F) \to \operatorname{Ext}^k(E, F^+) \xrightarrow{\delta} \operatorname{Ext}^{k+1}(E, F^-) \to \cdots .$$

For fixed F the functor $\operatorname{Ext}^*_R(\_, F)$ together with the natural transformation δ is up to isomorphisms uniquely determined by 1-4. And similarly for each fixed E.

Proof.
(1) By 4.113 there is an injective resolution I of F:

$$0 \to F \to I_0 \to I_1 \to I_2 \to \cdots$$

Applying $\mathcal{L}(E, \_)$ to I (only!) gives a cochain complex

$$0 \to \mathcal{L}(E, I_0) \to \mathcal{L}(E, I_1) \to \mathcal{L}(E, I_2) \to \cdots$$

and we define $\operatorname{Ext}^k(E, F) := H^k(\mathcal{L}(E, I))$.

By [**KriSS**, 9.19] and [**KriSS**, 8.23] the linear spaces $\operatorname{Ext}^k(E, F)$ are independent on the injective resolution of F.

($\boxed{2}$) By definition $\operatorname{Ext}^0_R(E, F)$ is just the kernel of $\mathcal{L}(E, I_0) \to \mathcal{L}(E, I_1)$ and by left exactness in $\boxed{4.109}$ the sequence $0 \to \mathcal{L}(E, F) \to \mathcal{L}(E, I_0) \to \mathcal{L}(E, I_1) \to \cdots$ is exact, hence this kernel is isomorphic to $\mathcal{L}(E, F)$.

($\boxed{3}$) If F is injective then $I_0 := F$ and $I_k := 0$ for $k > 0$ gives an injective resolution. Hence $\mathcal{L}(E, I_k) = 0$ and thus also $\operatorname{Ext}^k(E, F) = H^k(\mathcal{L}(E, I)) = 0$ for $k > 0$.

($\boxed{4}$) Let $0 \leftarrow E^+ \leftarrow E \leftarrow E^- \leftarrow 0$ be short exact and I be an injective resolution of F. Since I_k is injective we have short exact sequences

$$0 \to \mathcal{L}(E^+, I_k) \to \mathcal{L}(E, I_k) \to \mathcal{L}(E^-, I_k) \to 0$$

and this gives a short exact sequence of cochain complexes since $\mathcal{L}$ is a bifunctor:

$$0 \to \mathcal{L}(E^+, I) \to \mathcal{L}(E, I) \to \mathcal{L}(E^-, I) \to 0$$

By [**KriSS**, 7.30] we get a long exact sequence in (co)homology:

$$\cdots \to \operatorname{Ext}^k(E^+, F) \to \operatorname{Ext}^k(E, F) \to \operatorname{Ext}^k(E^-, F) \xrightarrow{\delta} \operatorname{Ext}^{k+1}(E^+, F) \to \cdots.$$

Let $0 \to F^- \to F \to F^+ \to 0$ be short exact and I^- and I^+ be corresponding injective resolutions of F^- and F^+. We construct an injective resolution I of E and an exact sequence of resolutions with

$$\begin{tikzcd}
0 \arrow[r] & E^- \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, dotted] & E^+ \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^- \arrow[r, dotted] & I \arrow[r, dotted] & I^+ \arrow[r] & 0
\end{tikzcd}$$

Take $I_k := I^-_k \oplus I^+_k$ and put

$$d_k := (\tilde{d}^-_k, d^+_k \circ \mathrm{pr}_2) : I_k \to I^-_{k+1} \oplus I^+_{k+1},$$

where $\tilde{d}^-_k$ is an extension of $d^-_k : I^-_k \to I^-_{k+1}$ along $I^-_k \hookrightarrow I_k$.
This makes I into a chain complex and $\mathrm{inj}_1 : I^- \to I$ and $\mathrm{pr}_2 : I \to I^+$ into chain mappings.

$$\begin{tikzcd}
I^-_{k-1} \arrow[r, tail, "\mathrm{inj}_1"'] \arrow[d, "d'_{k-1}"'] & I_{k-1} \arrow[r, two heads, "\mathrm{pr}_2"'] \arrow[d, dotted, "d_{k-1}"] \arrow[dl, dotted] & I^+_{k-1} \arrow[d, "d''_{k-1}"] \\
I^-_k \arrow[r, tail] \arrow[d, "d^-_k"'] & I_k \arrow[r, two heads, "\mathrm{pr}_2"'] \arrow[d, dotted, "d_k"] \arrow[dl, dotted, "\tilde{d}^-_k"'] & I^+_k \arrow[d, "d^+_k"] \\
I^-_{k+1} \arrow[r, tail, "\mathrm{inj}_1"'] & I_{k+1} \arrow[r, two heads, "\mathrm{pr}_2"'] \arrow[l, dotted, bend right, "\mathrm{pr}_1"'] & I^+_{k+1}
\end{tikzcd}$$

Since I'_k is injective, the sequences $0 \to I^-_k \to I_k \to I^+_k \to 0$ split and hence also $0 \to \mathcal{L}(E, I'_k) \to \mathcal{L}(E, I_k) \to \mathcal{L}(E, I''_k) \to 0$ splits and, in particular, is exact. By [**KriSS**, 7.30] we get a long exact sequence in (co)homology:

$$\cdots \to \operatorname{Ext}^k(E, F^-) \to \operatorname{Ext}^k(E, F) \to \operatorname{Ext}^k(E, F^+) \xrightarrow{\delta} \operatorname{Ext}^{k+1}(E, F^-) \to \cdots.$$

(Uniqueness) We proceed by induction on k. For $k \le 0$ we have uniqueness by $\boxed{1}$ and $\boxed{2}$. So we assume that we have two sequences of functors Ext^*, which are naturally isomorphic till order k, and we have natural connecting morphisms. Then a diagram chase starting at a short exact sequence $0 \to F^- \to F \to F^+ \to 0$ with

injective F shows that they are also isomorphic in order $k+1$ on (E,F^-):

$$\begin{tikzcd} \dots \arrow[r] & 0 \arrow[r] & \mathrm{Ext}^k(E,F^+) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathrm{Ext}^{k+1}(E,F^-) \arrow[r] \arrow[d, dotted, "\cong"] & 0 \arrow[r] & \dots \\ \dots \arrow[r] & 0 \arrow[r] & \overline{\mathrm{Ext}}^k_R(E,F^+) \arrow[r, "\cong"] & \overline{\mathrm{Ext}}^{k+1}_R(E,F^-) \arrow[r] & 0 \arrow[r] & \dots \end{tikzcd}$$

□

4.115 Proposition.
The statement $\mathrm{Ext}^1(E,F)=0$ *is equivalent to the equivalent conditions of* 4.111.

Proof. (⇒4.111.1) By 4.114.4 we have the exact sequence

$$0\to\mathcal{L}(E,F)\to\mathcal{L}(E,G)\to\mathcal{L}(E,H)\to\underbrace{\mathrm{Ext}^1(E,F)}_{=0}\to\cdots$$

(⇐4.111.1) Choose an injective I into which F embeds by 4.108 and consider the short exact sequence $0\to F\hookrightarrow I\twoheadrightarrow I/F\to 0$. Hence $0\to\mathcal{L}(E,F)\to\mathcal{L}(E,I)\to\mathcal{L}(E,I/F)\to 0$ is exact and in particular $\mathcal{L}(E,I)\to\mathcal{L}(E,I/F)$ is onto. Investigating the long exact sequence

$$0\to\mathcal{L}(E,F)\to\mathcal{L}(E,I)\twoheadrightarrow\mathcal{L}(E,I/F)\xrightarrow{0}\mathrm{Ext}^1(E,F)\to\underbrace{\mathrm{Ext}^1(E,I)}_{=0}\to\cdots$$

using 4.114.3, gives $\mathrm{Ext}^1(E,F)=0$. □

For an additive description of (DN) and later of (Ω) similar to 4.88.2 for quasi-normed spaces, we need the following:

4.119. Lemma.
Let $a,b>0$ *and* $\alpha,\beta\geq 0$. *Then* $\inf\{r^a\alpha+\frac{1}{r^b}\beta : r>0\}=\frac{a+b}{a}\left(\frac{a}{b}\right)^{\frac{b}{a+b}}\alpha^{\frac{b}{a+b}}\beta^{\frac{a}{a+b}}$

Proof. Let $f(r):=r^a\alpha+\frac{1}{r^b}\beta=\left(r^{a+b}\alpha+\beta\right)r^{-b}$. Then $f'(r)=a\,\alpha\,r^{a-1}-b\,\frac{1}{r^{b+1}}\,\beta$ and hence $f'(r)=0\Leftrightarrow r^{a+b}\alpha=\frac{b}{a}\,\beta$. Thus

$$\begin{aligned} f(r)&\geq\left(\frac{b}{a}\,\beta+\beta\right)\left(\frac{b\beta}{a\alpha}\right)^{-\frac{b}{a+b}}=\alpha^{\frac{b}{a+b}}\beta^{1-\frac{b}{a+b}}\left(1+\frac{b}{a}\right)\left(\frac{a}{b}\right)^{-\frac{b}{a+b}}\\ &=\alpha^{\frac{b}{a+b}}\beta^{\frac{a}{a+b}}\,\frac{a+b}{a}\,\left(\frac{a}{b}\right)^{\frac{b}{a+b}}. \end{aligned}$$

Note that $f(r)\to+\infty$ for $r\searrow 0$ if $\alpha>0$ and for $r\nearrow+\infty$ if $\beta>0$, hence the infimum is attained if $\alpha,\beta>0$. Otherwise $f(r)\to 0$ for $r\to 0$ or $r\to+\infty$, hence the statement is valid in this case as well. □

4.123 Theorem. Characterization of the property (DN).
Let E *be a Fréchet space with an increasing basis of semi-norms* $\|\text{-}\|_k$ *corresponding closed unit balls* U_k *and polars* $B_k:=(U_k)^o$.

1. *E is (DN) (see* 3.13*), i.e.*
   $\exists q\;\forall p\;\exists p'\;\exists C>0:\|\text{-}\|_p^2\leq C\,\|\text{-}\|_q\cdot\|\text{-}\|_{p'}$;

⇔ 2. $\exists q\;\forall p\;\exists p'\;\exists C>0\;\forall r>0:\|\text{-}\|_p\leq r\|\text{-}\|_q+\frac{C}{r}\|\text{-}\|_{p'}$;

⇔ 3. $\exists q\;\forall p\;\exists p'\;\exists C>0\;\forall r>0:U_p^o\subseteq r\,U_q^o+\frac{C}{r}U_{p'}^o$;

⇔ 4. $\exists q\;\forall\, 0<\delta<1\;\forall p\;\exists p'\;\exists C>0:\|\text{-}\|_p\leq C\|\text{-}\|_q^{1-\delta}\cdot\|\text{-}\|_{p'}^{\delta}$;

⇔ 5. $\exists q\;\exists d>0\;\forall p\;\exists p'\;\exists C>0:\|\text{-}\|_p^{1+d}\leq C\|\text{-}\|_q^{d}\cdot\|\text{-}\|_{p'}$;

$\Leftrightarrow$ 6. *There exists a log-convex basis of semi-norms* $|||\text{-}|||_p$ *on* E, *i.e.*
$\forall p : |||\text{-}|||_p^2 \le |||\text{-}|||_{p-1} \cdot |||\text{-}|||_{p+1}$.

Note that in all these conditions we may assume w.l.o.g. that $q < p < p'$, since for $p'' \ge p'$ we have $\|\text{-}\|_{p''} \ge \|\text{-}\|_{p'}$ and for $p \le q$ we may take $p' = p$ and $C = 1$.
Note furthermore, that for $\|\text{-}\|_{p'} \ge \|\text{-}\|_q$ only δ near 0 (and hence $d = \frac{1-\delta}{\delta}$ near ∞) are relevant, since for $\delta < \delta'$ (and $\|y\|_q \ne 0$) we get

$$\|y\|_q^{1-\delta}\, \|y\|_{p'}^{\delta} = \|y\|_q \underbrace{\left(\frac{\|y\|_{p'}}{\|y\|_q}\right)}_{\ge 1}^{\delta} \le \|y\|_q \left(\frac{\|y\|_{p'}}{\|y\|_q}\right)^{\delta'} = \|y\|_q^{1-\delta'}\, \|y\|_{p'}^{\delta'}$$

Proof.
($\boxed{1} \Leftrightarrow \boxed{2}$) the minimum of $r \mapsto r\|x\|_q + \frac{C}{r}\|x\|_{p'}$ is $2\sqrt{C\|x\|_q\|x\|_{p'}}$ by $\boxed{4.119}$. Hence the inequality in $\boxed{6}$ for all $r > 0$ is equivalent to $\|\text{-}\|_p^2 \le 4C\|\text{-}\|_q\|\text{-}\|_{p'}$.

($\boxed{2} \Leftrightarrow \boxed{3}$) From $\|\text{-}\|_p \le r\|\text{-}\|_q + \frac{C}{r}\|\text{-}\|_{p'}$ we get

$$\tfrac{1}{r}U_q \cap \tfrac{r}{C}U_{p'} \subseteq 2\,U_p \text{ and hence } U_p^o \subseteq 2\Big(r\,U_q^o + \tfrac{C}{r}U_{p'}^o\Big).$$

Conversely,

$$U_p^o \subseteq r\,U_q^o + \tfrac{C}{r}U_{p'}^o$$

implies that any $u \in U_p^o$ can be written as $u = r\,v + \frac{C}{r}u'$ with $v \in U_q^o$, $u' \in U_{p'}^o$, i.e.

$$|u(x)| \le r|v(x)| + \tfrac{C}{r}|u'(x)| \le r\|x\|_q + \tfrac{C}{r}\|x\|_{p'}$$

for all $x \in E$. Hence $\boxed{2}$ holds, since $\|x\|_q = \sup_{u \in U_q}|u(x)|$.

($\boxed{1} \Rightarrow \boxed{6}$) Define a new basis of semi norms $|||\text{-}|||_k$ recursively by: $|||\text{-}|||_0 := \|\text{-}\|_q$;
$\exists p_0'\, \exists C_0 \ge 1$: $|||\text{-}|||_0^2 \le |||\text{-}|||_0 \cdot C_0\, \|\text{-}\|_{p_0'} = |||\text{-}|||_0 \cdot |||\text{-}|||_1$, where $|||\text{-}|||_1 := C_0\|\text{-}\|_{p_0'}$;
$\exists p_k'\, \exists C_k \ge 1$: $|||\text{-}|||_k^2 \le |||\text{-}|||_0 \cdot C_k\, \|\text{-}\|_{p_k'} \le |||\text{-}|||_{k-1} \cdot |||\text{-}|||_{k+1}$, with $|||\text{-}|||_{k+1} := C_k\, \|\text{-}\|_{p_k'}$.

($\boxed{6} \Rightarrow \boxed{1}$) From $|||\text{-}|||_k^2 \le |||\text{-}|||_{k-1}\, |||\text{-}|||_{k+1}$ we obtain that all $|||\text{-}|||_k$ are norms and using $|||x|||_k/|||x|||_{k-1} \le |||x|||_{k+1}/|||x|||_k$ for all $x \ne 0$, we get for all $k \in \mathbb{N}$ the inequality

$$\frac{|||x|||_k}{|||x|||_0} = \prod_{j=1}^{k}\frac{|||x|||_j}{|||x|||_{j-1}} \le \prod_{j=k+1}^{2k}\frac{|||x|||_j}{|||x|||_{j-1}} = \frac{|||x|||_{2k}}{|||x|||_k}, \text{i.e. } |||x|||_k^2 \le |||x|||_0 |||x|||_{2k}.$$

($\boxed{1} \Rightarrow \boxed{4}$) Put $p_0 := p$ and apply $\boxed{1}$ iteratively to get $p_{\nu+1} \ge p_\nu$ and a C_ν with

$$\|\text{-}\|_{p_\nu}^2 \le C_\nu\, \|\text{-}\|_q\, \|\text{-}\|_{p_{\nu+1}}.$$

Let $0 < \delta < 1$ and $m \in \mathbb{N}$ with $\frac{1}{m+1} < \delta$. Since $\|\text{-}\|_q$ is a norm, we have

$$\left(\frac{\|\text{-}\|_p}{\|\text{-}\|_q}\right)^m \le \prod_{\nu=0}^{m-1}\frac{\|\text{-}\|_{p_\nu}}{\|\text{-}\|_q} \le \prod_{\nu=0}^{m-1} C_\nu \frac{\|\text{-}\|_{p_{\nu+1}}}{\|\text{-}\|_{p_\nu}} \le \left(\prod_{\nu=0}^{m-1} C_\nu\right)\frac{\|\text{-}\|_{p_m}}{\|\text{-}\|_p}.$$

If we put $C := \Big(\prod_{\nu=0}^{m-1} C_\nu\Big)^{\frac{1}{m+1}}$, we get

$$\|\text{-}\|_p \le C\|\text{-}\|_q^{1-\frac{1}{m+1}}\, \|\text{-}\|_{p_m}^{\frac{1}{m+1}} = C\, \|\text{-}\|_q \cdot \left(\frac{\|\text{-}\|_{p_m}}{\|\text{-}\|_q}\right)^{1/(m+1)} \le C\, \|\text{-}\|_q \cdot \left(\frac{\|\text{-}\|_{p_m}}{\|\text{-}\|_q}\right)^{\delta}$$

($\boxed{4} \Rightarrow \boxed{5}$) This follows directly with $d := \frac{1-\delta}{\delta}$.

($\boxed{5} \Rightarrow \boxed{1}$) We have $\forall p\ \exists p', C_p\ :\ \|x\|_p^{1+d} \leq C_p\, \|x\|_q^d\, \|x\|_{p'}$ and $\forall p'\ \exists p'', C_{p'}\ :$ $\|x\|_{p'}^{1+d} \leq C_{p'}\, \|x\|_q^d\, \|x\|_{p''}$. Thus for $d' = 2d + d^2 > 2d$ we get

$$\|x\|_p^{1+d'} = \|x\|_p^{(1+d)^2} \leq C_p^{1+d}\, \|x\|_q^{d(1+d)}\, C_{p'}\, \|x\|_q^d\, \|x\|_{p''} = C_{p'}\, C_p^{1+d}\, \|x\|_q^{d'}\, \|x\|_{p''}$$

So we get $\boxed{5}$ for some $d \geq 1$ by induction and thus also for $d = 1$. $\square$

4.124 Definition. Generalized power series spaces (See [**Vog85**, p.256])**.**
For a set J and $a : J \to \{t \in \mathbb{R} : t \geq 1\}$ we consider the following GENERALIZED POWER SERIES SPACES OF INFINITE TYPE:

$$\lambda_\infty^\infty(a) := \Big\{ f \in \mathbb{K}^J : \|f\|_k := \sup_{t \in J} |f(t)|\, a(t)^k < \infty \text{ for all } k \in \mathbb{N} \Big\},$$
$$\lambda_\infty^1(a) := \Big\{ f \in \mathbb{K}^J : \|f\|_k := \sum_{t \in J} |f(t)|\, a(t)^k < \infty \text{ for all } \in \mathbb{N} \Big\}.$$

These Fréchet spaces $\lambda_\infty^p(a)$ are usally denoted $\Lambda^p(J,a)$ for $p \in \{1, \infty\}$ and for $p = 1$ the index is often dropped.

4.125 Lemma (cf. [**Vog77a**, 2.4. Corollary p.115] **).**
For each $a \in \mathbb{R}^J_{\geq 1}$ the space $\lambda_\infty^\infty(a)$ has property (DN).

Proof. By definition $\|f\|_k := \|f \cdot a^k\|_{\ell^\infty}$. Thus

$$\|f\|_p^2 = \|f \cdot a^p\|_{\ell^\infty}^2 = \|f^2 \cdot a^{2p}\|_{\ell^\infty} \leq \|f\|_{\ell^\infty} \cdot \|f \cdot a^{2p}\|_{\ell^\infty} = \|f\|_0 \cdot \|f\|_{2p}$$

Thus $\lambda_\infty^\infty(a)$ satisfies condition $\boxed{4.123.1}$ for $q := 0$, $p' := 2p$, and $C := 1$. $\square$

4.126 Lemma (See [**Vog85**, Example (3) p.256])**.**
Let I be a set and $a : \mathbb{N} \times I \to \mathbb{N}$ be given by $a(n,i) := n+1$.
Then $\ell^\infty(I) \hat{\otimes} s \cong \lambda_\infty^\infty(a)$ and $\ell^1(I) \hat{\otimes} s \cong \lambda_\infty^1(a)$.

Proof. By $\boxed{3.48}$ $E' \otimes_\varepsilon F \hookrightarrow L(E,F)$. Since $c_0(I)$ is Banach with dual $c_0(I)^* = \ell^1(I)$ and $\ell^1(I)^* = \ell^\infty(I)$ we get embeddings

$$\ell^1(I) \otimes_\varepsilon s \hookrightarrow \mathcal{L}(c_0(I), s) \text{ and } \ell^\infty(I) \otimes_\varepsilon s \hookrightarrow \mathcal{L}(\ell^1(I), s).$$

Let $P_k : s \to s$ be defined by $P_k(x)_n := x_n$ for $n < k$ and 0 otherwise. Then $P_k \to \mathrm{id}$ and for $T \in \mathcal{L}(\ell^1(I), s)$ we have $P_k \circ T \to T$ with $P_k \circ T \in \ell^1(I)^* \otimes s$. So $\ell^\infty(I) \hat{\otimes} s \cong \mathcal{L}(\ell^1(I), s)$ and analogously $\ell^1(I) \hat{\otimes} s \cong \mathcal{L}(c_0(I), s)$.

Furthermore, $\mathcal{L}(\ell^1(I), E) \cong \ell^\infty(I, E)$ and $\mathcal{L}(c_0(I), E) \cong \ell^1(I, E)$: In fact, $T \in \mathcal{L}(\ell^1(I), E)$ is uniquely determined by $x^i := T(e_i) \in E$ for $i \in I$. Since T is bounded, $\{x^i : i \in I\} \subseteq E$ has to be bounded. And conversely a bounded family $\{x^i : i \in I\} \subseteq E$ defines a continuous linear operator $T : \ell^1(I) \to E$, $(y_i)_{i \in I} \mapsto \sum_i y_i x^i \in E$. And the same arguments work also for the second isomorphism.

Finally note, that $\ell^\infty(I, s) \cong \lambda_\infty^\infty(a)$ and $\ell^1(I, s) \cong \lambda_\infty^1(a)$: In fact, the seminorm $\|\text{\_}\|_k : x \mapsto \sup\{|(n+1)^k x_n| : n \in \mathbb{N}\}$ of s induces the seminorm of $\ell^\infty(I, s)$ by taking the ℓ^∞-Norm of $(\|x^i\|_k)_{i \in I}$ and corresponds to the seminorm $\|\text{\_}\|_k$ of $\lambda_\infty^\infty(a)$. Replacing the supremum by the 1-Norm, gives the second isomorphism. $\square$

4.127 Proposition (See [**Vog82**, 1.1 p.540], [**Vog85**, Lemma 1.3 p.258], [**Vog87**, 4.3 p.185], and [**Vog77a**, Satz 1.5 p.111])**.**
Let F and G be Fréchet spaces and assume that G has property (DN).
Then any exact sequence

$$0 \to \lambda_\infty^\infty(a) \to F \xrightarrow{Q} G \to 0$$

with $a \in \mathbb{R}^J_{\geq 1}$ splits.

Proof. W.l.o.g. let $E := \lambda^\infty_\infty(a) \hookrightarrow F$ be the inclusion of a subspace. We have to prove that it is complemented, i.e. there exists a left inverse $\phi : F \to E$ to it.

Let $\mathrm{ev}_j \in E^*$ be given by $\mathrm{ev}_j(x) := x(j)$ for $x \in E$. Since $|a(j)^k \mathrm{ev}_j(x)| = |a(j)^k x(j)| \le \|x\|_k$ the set $\{a(j)^k \mathrm{ev}_j : j \in J\}$ is equicontinuous for each $k \in \mathbb{N}$. By Hahn-Banach we can extend ev_j to $\widetilde{\mathrm{ev}}^k_j \in F^*$ for each $k \in \mathbb{N}$ such that $\left\{a(j)^k \widetilde{\mathrm{ev}}^k_j : j \in J\right\}$ is equicontinuous, thus contained in U^o_k for a suitable neighbourhood U_k of $0 \in F$. We can assume that $U_{k+1} \subseteq U_k$ for all $k \in \mathbb{N}$.

Thus

$$g^k_j := \widetilde{\mathrm{ev}}^{k+1}_j - \widetilde{\mathrm{ev}}^k_j \in \tfrac{1}{a(j)^k}\left(\tfrac{1}{a(j)}\, U^o_{k+1} + U^o_k\right) \cap E^o \subseteq \tfrac{2}{a(j)^k}\, U^o_{k+1} \cap E^o =: \tfrac{1}{a(j)^k} B_k \subseteq E^o.$$

Since $Q^* : G^* \cong (F/E)^* \cong E^o \subseteq F^*$ as cbs for the equicontinuous subsets by [**Kri07b**, 7.4.4] and 4.72.2 and since G has property (DN), there exists a bounded set $B \subseteq E^o$, which satisfies the conditions of 4.123.3 for a fixed fundamental system of bounded sets B_k in E^o. W.l.o.g. (by enlargeing B_{k+1}) we can assume that

$$\forall k \in \mathbb{N}\, \forall r > 0 : B_k \subseteq rB + \frac{2^{-k-2}}{r} B_{k+1}$$

In particular, for $r := a(j)2^{-k-1}$ we get by multiplication with $2a(j)^{-k}$

$$2a(j)^{-k} B_k \subseteq a(j)^{-k+1} 2^{-k} B + a(j)^{-k-1} B_{k+1}. \tag{1}$$

We now choose for fixed j recursively b^k_j with $b^k_j \in a(j)^{-k} B_k \subseteq E^o$:
Put $b^0_j := 0$. If $b^k_j \in a(j)^{-k} B_k$ is already chosen, we have $g^k_j + b^k_j \in 2a(j)^{-k} B_k$. Hence by (1) there exists a $b^{k+1}_j \in a(j)^{-k-1} B_{k+1}$ such that

$$g^k_j + b^k_j - b^{k+1}_j \in 2^{-k} a(j)^{-k+1} B.$$

If we put

$$\phi^k_j := \widetilde{\mathrm{ev}}^k_j - b^k_j \in F^*,$$

we get for $k \ge 1$:

$$\phi^{k+1}_j - \phi^k_j = g^k_j - b^{k+1}_j + b^k_j \in 2^{-k} a(j)^{-k+1} B \subseteq 2^{-k} B.$$

Hence

$$\exists \phi_j := \lim_{k\to\infty} \phi^k_j \in F^*.$$

Since $\phi^{n+1}_j = \widetilde{\mathrm{ev}}^{n+1}_j - b^{n+1}_j \in 2a(j)^{-n} U^o_{n+1}$ we have for $k > n$:

$$a(j)^n \phi^k_j = a(j)^n \phi^{n+1}_j + a(j)^n \sum_{\nu=n+1}^{k-1} (\phi^{\nu+1}_j - \phi^\nu_j) \in 2U^o_{n+1} + 2^{-n} B.$$

Thus $a(j)^n \phi_j \in 2U^o_{n+1} + 2^{-n} B$, i.e. $\{a(j)^n \phi_j : j \in J\}$ is equicontinuous in F^*. Therefore $x \to (\phi_j(x))_{j\in J}$ defines a continuous linear left inverse to $E \hookrightarrow F$, since for $x \in E$ we have

$$\phi_j(x) = \lim_{k\to\infty} \phi^k_j(x) = \lim_{k\to\infty} \widetilde{\mathrm{ev}}^k_j(x) - b^k_j(x) = \mathrm{ev}_j(x) - 0 = x(j). \quad \square$$

In fact, it can be shown that the condition (DN) yields even a characterization:

4.128 Theorem [**Vog87**, 4.3 p.185]**.**
Let $\sup_n \frac{\alpha_{n+1}}{\alpha_n} < \infty$, $r \le +\infty$, and E a Fréchet space. Then

1. *E is (DN);*

$\Leftrightarrow$ 2. *$\mathrm{Ext}^1(E, \lambda^\infty_r(\alpha)) = 0$, i.e. any ses $0 \to \lambda^\infty_r(\alpha) \to G \to E \to 0$ splits;*

$\Leftrightarrow$ 3. *If $Q : G \twoheadrightarrow H$ is a quotient mapping with kernel $\lambda^\infty_r(\alpha)$ then $Q_* : \mathcal{L}(E, G) \to \mathcal{L}(E, H)$ is onto;*

$\Leftrightarrow$ 4. *If $S : H \hookrightarrow G$ is a closed embedding with quotient E then $S^* : \mathcal{L}(G, \lambda_r^\infty) \to \mathcal{L}(H, \lambda_r^\infty)$ is onto.*

Proof. ($\boxed{1} \Rightarrow \boxed{2}$) for $r = +\infty$ is $\boxed{4.127}$.

($\boxed{1} \Leftarrow \boxed{2}$) is shown in [**Vog87**, 4.3 p.185].

($\boxed{2} \Leftrightarrow \boxed{3} \Leftrightarrow \boxed{4}$) is $\boxed{4.111}$ and $\boxed{4.115}$. □

4.129 Corollary [**See64**].
The restriction $\mathrm{incl}^* : C^\infty(\mathbb{R}, \mathbb{R}) \twoheadrightarrow C^\infty(\mathbb{R}_{\geq 0}, \mathbb{R})$ *has a continuous linear right inverse.*

Proof. We show first that the restriction map $C^\infty_{[-2,2]}(\mathbb{R}) \twoheadrightarrow C^\infty([-1.1])$ has a continuous linear right inverse: By $\boxed{1.16.4}$ $C^\infty([-1,1]) \cong s$ and by $\boxed{1.16.3}$ $C^\infty_{[-2,2]}(\mathbb{R}) \cong s$. Moreover the kernel of the restriction map is the subspace

$$\Big\{f \in C^\infty(\mathbb{R}) : f(t) = 0\ \forall |t| \geq 2 \text{ and } f(t) = 0\ \forall |t| \leq 1\Big\} =$$
$$= C^\infty_{[-2,-1]}(\mathbb{R}) \oplus C^\infty_{[1,2]}(\mathbb{R}) \cong s \oplus s \cong s :$$

In fact $s \cong s \times s$ via $(x_k)_{k\in\mathbb{N}} \mapsto ((x_{2k})_{k\in\mathbb{N}}, (x_{2k+1})_{k\in\mathbb{N}})$: This mapping is obviously linear and injective. It is continuous, since $|(k+1)^q x_{2k}| \leq |(2k+1)^q x_{2k}|$ and $|(k+1)^q x_{2k+1}| \leq |(2k+2)^q x_{2k+1}|$. It is onto $s \times s$, since given $y, z \in s$ the inverse image is given by $x_{2k} := y_k$ and $x_{2k+1} := z_k$ with

$$|(n+1)^q x_n| = \begin{cases} |(2k+1)^q y_k| \leq |2^q (k+1)^q y_k| & \text{for } n = 2k, \\ |(2k+2)^q z_k| \leq |2^q (k+1)^q z_k| & \text{for } n = 2k+1. \end{cases}$$

Thus we have a short exact sequence $s \hookrightarrow s \twoheadrightarrow s$, which splits by $\boxed{4.127}$ since s is a power series space of infinite type by $\boxed{1.15.4}$ and hence has property (DN) by $\boxed{3.14.3}$.

Using translation it suffices to consider the restriction map $C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}_{\geq -1})$. We choose a function $\varphi \in C^\infty(\mathbb{R}, [0,1])$ with $\varphi(t) = 0$ for all $t \geq 0$ and $\varphi(t) = 1$ for all $t \leq -\frac{1}{2}$ and decompose $f \in C^\infty(\mathbb{R}_{\geq -1})$ as $f = (1-\varphi)\cdot f + \varphi \cdot f$. Since $(1-\varphi)\cdot f$ is 0 on $[-1, -\frac{1}{2}]$ we can extend it by 0 to $\tilde{f}^0 \in C^\infty(\mathbb{R})$. By what we have shown before the restriction of $\varphi \cdot f$ to $[-1,1]$ has an extension $\tilde{f}^1 \in C^\infty_{[-2,2]}(\mathbb{R}) \subseteq C^\infty(\mathbb{R})$. Then $\tilde{f}^2 : t \mapsto \varphi(t - \frac{1}{2}) \cdot \tilde{f}^1(t)$ is an extension of $\varphi \cdot f$ restricted to $[-1, +\infty)$, since $\varphi(t - \frac{1}{2}) = 1$ for all t with $\varphi(t) \neq 0$ and $\varphi(t - \frac{1}{2}) = 0 = \varphi(t)$ for all $t \geq \frac{1}{2}$. Thus $\tilde{f} := \tilde{f}^0 + \tilde{f}^2$ is the desired extension of f, and it depends continuously and linearly on f, since all intermediate steps do so. □

More generally, it is shown in [**Tid79**, Folgerung 2.4 p.296] for compact $K \subseteq \mathbb{R}^n$: $C^\infty(\mathbb{R}^n) \twoheadrightarrow \mathcal{E}(K)$ has a continuous linear right inverse $\Leftrightarrow \mathcal{E}(K)$ (DN) $\Leftrightarrow \mathcal{E}(K) \cong s$. Here $\mathcal{E}(K) \cong C^\infty(\mathbb{R}^n)/\{f \in C^\infty(\mathbb{R}^n) : f|_K = 0\}$ denotes the Fréchet space of Whitney jets on K.

Another application is:

4.130 Proposition. [**Vog87**, 7.1 p.193].
Let $D := P(\partial)$ be an elliptic linear PDO with constant coefficients on $\mathbb{R}^n$ with $n \geq 2$ and $U \subseteq \mathbb{R}^n$ open and E a Fréchet space. Then
$D_* : \mathcal{L}(E, C^\infty(U)) \to \mathcal{L}(E, C^\infty(U))$ *is onto* $\Leftrightarrow \mathrm{Ext}^1(E, \ker D) = 0 \Leftrightarrow E$ *is (DN).*

4.131 Corollary (See [**Vog83**, 6.1. Satz p.197], [**Vog85**, 2.6 p.260]).
A Fréchet space F is (DN) $\Leftrightarrow \exists J\ \exists a \in \mathbb{R}^J_{\geq 1}$: $F \hookrightarrow \lambda^\infty_\infty(a)$.

Proof.
($\Rightarrow$) Let $J := \bigcup_k B_k$ for some basis of equicontinuous sets $B_k \subseteq F^*$. Then F can be embedded into $(\ell^\infty(J))^{\mathbb{N}}$ in a natural way.

By Borel's theorem $0 \to C^\infty_{[-1,0]}(\mathbb{R}) \times C^\infty_{[0,1]}(\mathbb{R}) \hookrightarrow C^\infty_{[-1,1]} \twoheadrightarrow \mathbb{R}^{\mathbb{N}} \to 0$ is exact and $C^\infty_{[a,b]}(\mathbb{R}) \cong s$ by 1.16.3. Moreover $s \cong s \times s$ via $(x_k)_{k\in\mathbb{N}} \mapsto ((x_{2k})_{k\in\mathbb{N}}, (x_{2k+1})_{k\in\mathbb{N}})$ by what we have shown in 4.129. By tensoring this exact sequence of nuclear (F) spaces with $\ell^\infty(J)$ (i.e. applying $\mathcal{L}((\_)^*, \ell^\infty(J))$ with the injective (F) space $\ell^\infty(J)$) we get the (using 4.99) exact sequence of (F) spaces:

$$0 \to s\hat{\otimes}\ell^\infty(J) \to s\hat{\otimes}\ell^\infty(J) \to \mathbb{R}^{\mathbb{N}}\hat{\otimes}\ell^\infty(J) \to 0.$$

Since $s\hat{\otimes}\ell^\infty(J) \cong \lambda^\infty_\infty(a)$ by 4.126, where $a : \mathbb{N} \times J \to \mathbb{N}$ is given by $(n,j) \mapsto n+1$, and $\mathbb{R}^{\mathbb{N}}\hat{\otimes}\ell^\infty(J) \cong (\mathbb{R}^{(\mathbb{N})})^*\hat{\otimes}\ell^\infty(J) \cong \mathcal{L}(\mathbb{R}^{(\mathbb{N})}, \ell^\infty(J)) \cong \ell^\infty(J)^{\mathbb{N}}$, this sequence is

$$0 \to \lambda^\infty_\infty(a) \to \lambda^\infty_\infty(a) \xrightarrow{Q} (\ell^\infty(J))^{\mathbb{N}} \to 0.$$

Since F embeds into $(\ell^\infty(J))^{\mathbb{N}}$ we may consider the pullback(=preimage) $Q^{-1}(F)$ of F under Q, and get the short exact sequence

$$0 \to \lambda^\infty_\infty(a) \to Q^{-1}(F) \to F \to 0.$$

By 4.127 the sequence splits if F has property (DN). We therefore get the embedding $F \hookrightarrow Q^{-1}(F) \subseteq \lambda^\infty_\infty(a)$.

$$\begin{tikzcd}
0 \arrow[r] & s\hat{\otimes}\ell^\infty(J) \arrow[r, hook] \arrow[d, equal] & s\hat{\otimes}\ell^\infty(J) \arrow[r, two heads, "Q"] \arrow[d, equal] & \mathbb{R}^{\mathbb{N}}\hat{\otimes}\ell^\infty(J) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \lambda^\infty_\infty(a) \arrow[r, hook] & \lambda^\infty_\infty(a) \arrow[r, two heads, "Q"] & (\ell^\infty(J))^{\mathbb{N}} \arrow[r] & 0 \\
0 \arrow[r] & \lambda^\infty_\infty(a) \arrow[r, dotted, "(2)"] \arrow[u, equal] & Q^{-1}(F) \arrow[r, two heads] \arrow[u, hook] & F \arrow[r] \arrow[u, hook, dotted, "(1)"'] \arrow[l, dotted, bend right, "(3)"'] & 0
\end{tikzcd}$$

($\Leftarrow$) Since $\lambda^\infty_\infty(a)$ has property (DN) by 4.125, the converse follows from 3.14.2.

Now we consider the dual situation.

4.132 Lemma. Characterization of the property (Ω).
Let $\|\_\|_k$ be an increasing basis of seminorms of a Fréchet space E, denote with $U_k := \{x \in E : \|x\|_k \le 1\}$ the corresponding unit-balls and $\|\_\|_{-k}$ the Minkowski functionals of U_k^o, i.e. $\|y\|_{-k} := \|y\|_k^* := \sup\{|y(x)| : x \in U_k\} = \sup\{\frac{|y(x)|}{\|x\|_k} : x \in E\}$ for $y \in E^*$ (cf. property (DN) in 4.123).

1. $\forall p\ \exists p'\ \forall k\ \exists C > 0\ \exists\, 0 < \delta < 1 : \|\_\|_{-p'} \le C\,(\|\_\|_{-p})^{1-\delta} \cdot (\|\_\|_{-k})^\delta$;

$\Leftrightarrow$ 2. $\forall p\ \exists p'\ \forall k\ \exists C > 0\ \exists d > 0 : \|\_\|_{-p'}^{1+d} \le C\,\|\_\|_{-k} \cdot \|\_\|_{-p}^d$
[**Vog83**, p.194]. [**VW80**, Korollar 2.2 p.232]. [**Vog85**, p.255].

$\Leftrightarrow$ 3. $\forall p\ \exists p'\ \forall k\ \exists k'\ \exists C > 0\ \forall r > 0 : \|\_\|_{-p'} \le C\, r^{k'} \|\_\|_{-k} + \frac{1}{r}\|\_\|_{-p}$
[**VW80**, Korollar 2.1 p.232];

$\Leftrightarrow$ 4. $\forall p\ \exists p'\ \forall k\ \exists k'\ \exists C > 0\ \forall r > 0 : U_{p'} \subseteq C\, r^{k'} U_k + \frac{1}{r} U_p$
[**VW80**, Definition 1.1 p.225];

A Fréchet space E is said to be (Ω) iff these equivalent conditions are satisfied.

Note that we may assume that $p' > p$ and it suffices that $k > p'$ and $d \in \mathbb{N}$, since $q \geq p' \Leftrightarrow \|\text{-}\|_q \geq \|\text{-}\|'_p \Leftrightarrow U_{p'} \supseteq U_q \Leftrightarrow \|\text{-}\|_{-p'} \geq \|\text{-}\|_{-q}$ thus $\boxed{1}$ holds for each $p'' > p'$ as well and $\boxed{2}$ holds for each $d' > d$ as well.

Proof.
($\boxed{1}\Leftrightarrow\boxed{2}$) $\delta = \frac{d}{d+1}$.

($\boxed{2}\Leftrightarrow\boxed{3}$) since the infimum of $r \mapsto \alpha r^{k'} + \beta \frac{1}{r}$ is $C_{k'} \sqrt[k'+1]{\alpha \beta^{k'}}$ by $\boxed{4.119}$.

($\boxed{3}\Rightarrow\boxed{4}$) Let $\|\text{-}\|_{-p'} \leq C r^{k'} \|\text{-}\|_{-k} + \frac{1}{r}\|\text{-}\|_{-p}$. Then

$$\frac{1}{2Cr^{k'}} U_k^o \cap \frac{r}{2} U_p^o \subseteq U_{p'}^o$$

and by taking polars

$$U_{p'} \subseteq ((U_{p'})^o)_o \subseteq \Big(\frac{1}{2Cr^{k'}} U_k^o \cap \frac{r}{2} U_p^o\Big)_o \subseteq 3Cr^{k'} U_k + \frac{2}{r} U_p.$$

($\boxed{3}\Leftarrow\boxed{4}$) Let $U_{p'} \subseteq C\, r^{k'} U_k + \frac{1}{r} U_p$. Then every $x \in U_{p'}$ can be written as as $x = C r^n a + \frac{1}{r} b$ with $a \in U_k$ and $b \in U_p$. Thus for $x^* \in E^*$ we get

$$|x^*(x)| \leq C r^{k'} \|x^*\|_{-k} + \frac{1}{r}\|x^*\|_{-p}$$

and taking the sup over $x \in U_{p'}$ gives $\boxed{3}$. □

4.133 Inheritance properties of (Ω)
(See [**VW80**, Satz 2.5 p.236], [**MV92**, 29.11 p.347]).

1. *(Ω) is a topological invariant.*
2. *(Ω) is inherited by quotients.*
3. *$\lambda^q_r(\alpha)$ has (Ω) for all $r \leq \infty$ and $1 \leq q < \infty$.*
4. *$\lambda^1_\infty(a)$ has (Ω) for all $a \in \mathbb{R}^J_{\geq 1}$.*

Proof. ($\boxed{1}$) is obvious in view of $\boxed{4.132.4}$.

($\boxed{2}$) Let $F \hookrightarrow E$ be a closed subspace, $\pi : E \to E/F$ the canonical quotient mapping, p a seminorm on E, and $\tilde{p}$ the corresponding norm on the quotient. Then $\tilde{p}_{<1} = \pi(p_{<1})$ thus applying π to $\boxed{4.132.4}$ for the open unit balls of E gives the same for E/F.

($\boxed{3}$) For $\lambda^q_r(\alpha)$ let $\frac{1}{q} + \frac{1}{q'} = 1$ and $a(j) := e^{\alpha_j}$. Then

$$\begin{aligned} \|y\|_{-k} \cdot \|y\|^d_{-p} &= \Big\|\frac{y}{a^k}\Big\|_{\ell^{q'}} \cdot \Big\|\frac{y}{a^p}\Big\|^d_{\ell^{q'}} = \Big\|\frac{y}{a^k}\Big\|_{\ell^{q'}} \cdot \Big\|\Big(\frac{y}{a^p}\Big)^d\Big\|_{\ell^{q'/d}} \\ &\geq \Big\|\frac{y}{a^k} \cdot \Big(\frac{y}{a^p}\Big)^d\Big\|_{\ell^{1/(1/q'+d/q')}} = \Big\|\frac{y^{1+d}}{a^{k+p\,d}}\Big\|_{\ell^{q'/(1+d)}} \\ &= \Big\|\frac{y}{a^{(k+p\,d)/(1+d)}}\Big\|^{1+d}_{\ell^{q'}} = \|y\|^{1+d}_{-p'}, \end{aligned}$$

where $k > p' > p$ and d is the solution of $p' = \frac{k+p\,d}{1+d}$, i.e. $d = \frac{k-p'}{p'-p}$.

($\boxed{4}$) follows by the same arguments as in $\boxed{3}$ but for uncountable index sets J. □

4.134 Theorem
(See [**Vog77b**, Theorem 2.3], [**Vog85**, Lemma 1.3 p.258], and [**Vog87**, 4.1 p.183]).

Let E and F be Fréchet spaces and assume that E has property (Ω). Then any exact sequence

$$0 \to E \to F \xrightarrow{Q} \lambda^1_\infty(a) \to 0$$

with $a \in \mathbb{R}^J_{\geq 1}$ splits.

Proof. We assume that $E = \ker Q$ is a subspace of F. Using 4.132.4 we find an decreasing 0-nbhd basis in E of absolutely convex sets U_k and ν_k such that

(1) $$U_{k-1} \subseteq r^{\nu_{k-1}} U_k + \frac{1}{r} U_{k-2} \text{ for all } r \geq 2 \text{ and } k \geq 2.$$

Let $(W_k)_{k\in\mathbb{N}}$ be a corresponding decreasing 0-nbhd basis in F with $W_k \cap E = U_k$ and let e_j denote the j-th unit vector in $\lambda^1_\infty(a)$. For the canonical norms

$$\|x\|_k := \sum_j a(j)^k |x_j|$$

we have $\|e_j\|_k = a(j)^k$. By the open mapping theorem, $Q(W_k) \subseteq \lambda^1_\infty(a)$ is open. Hence, for every k there exists an $n_k \in \mathbb{N}$ and a $C_k \geq 1$ with

$$\frac{e_j}{a(j)^{n_k}} \in \big\{x : \|x\|_{n_k} \leq 1\big\} \subseteq C_k\, Q(W_k).$$

Thus there are $d^k_j \in C_k a(j)^{n_k} W_k \cap Q^{-1}(e_k) \subseteq F$. We may assume that

$$n_{k+1} \geq (1+\nu_{k-1}) n_k \geq n_k \text{ and } C_{k+1} \geq 2^{k\nu_{k-1}} (3C_k)^{1+\nu_{k-1}} \geq C_k$$

for all $k \in \mathbb{N}$. Thus

$$d^k_j - d^{k-1}_j \in \Big(C_k a(j)^{n_k} W_k + C_{k-1} a(j)^{n_{k-1}} W_{k-1}\Big) \cap \ker Q \subseteq 2C_k a(j)^{n_k} U_{k-1}$$

We claim that there are $a^k_j \in C_{k+1} a(j)^{n_{k+1}}\, U_k$ with

$$R^k_j := d^k_j - a^k_j \in C_k a(j)^{n_k}\, W_k + C_{k+1} a(j)^{n_{k+1}}\, U_k \subseteq 2C_{k+1} a(j)^{n_{k+1}}\, W_k.$$

Let $a^0_j := 0$ and assume a^{k-1}_j is already constructed. Then

$$d^k_j - d^{k-1}_j + a^{k-1}_j \in 2C_k a(j)^{n_k} U_{k-1} + C_k a(j)^{n_k} U_{k-1} \subseteq \underbrace{3C_k a(j)^{n_k}}_{=:\rho\geq 1}\, U_{k-1}.$$

Multiplying 1 for $r := \rho\, 2^k$ with ρ gives the existence of

$$a^k_j \in \rho\, r^{\nu_{k-1}}\, U_k = (3C_k a(j)^{n_k})^{1+\nu_{k-1}}\, 2^{k\,\nu_{k-1}}\, U_k \subseteq C_{k+1} a(j)^{n_{k+1}}\, U_k$$

with

$$R^k_j - R^{k-1}_j = (d^k_j - d^{k-1}_j + a^{k-1}_j) - a^k_j \in \frac{\rho}{r} U_{k-2} = 2^{-k} U_{k-2}$$

Thus

$$\exists\, R_j := \lim_{l\to\infty} R^l_j = R^k_j + \sum_{l>k} (R^l_j - R^{l-1}_j) \in 2C_{k+1} a(j)^{n_{k+1}}\, W_k + \sum_{l>k} 2^{-l}\, U_{l-2} \subseteq$$

$$\subseteq \Big(2C_{k+1} a(j)^{n_{k+1}} + 2^{-k}\Big) W_{k-1} \subseteq \big(1 + 2C_{k+1}\big)\, a(j)^{n_{k+1}}\, W_{k-1} \subseteq F.$$

So we can define

$$R(x) := \sum_j x_j R_j \in F \text{ for all } x = (x_j)_{j\in\mathbb{N}} \in \lambda^1_\infty(a),$$

since

$$R(x) = \sum_j a(j)^{n_{k+1}}\, x_j\, \frac{R_j}{a(j)^{n_{k+1}}} \in \|x\|_{n_{k+1}} (1 + 2C_{k+1})\, W_{k-1}.$$

Thus, for each $k > 0$,

$$p_{W_{k-1}}(R(x)) \leq (1 + 2C_{k+1})\, \|x\|_{n_{k+1}},$$

i.e. $R \in \mathcal{L}(\lambda^1_\infty(a), F)$ and, since

$$Q(R_j) = \lim_{k\to\infty} Q(R_j^k) = \lim_{k\to\infty} Q(d_j^k - a_j^k) = e_j - 0,$$

we get $Q \circ R = \mathrm{id}$.

In fact, it has be shown that the condition (Ω) gives even a characterization:

4.135 Proposition [Vog87, 4.1 p.183]**.**
Let $\sup_n \frac{\alpha_{n+1}}{\alpha_n} < \infty$ and F be a Fréchet space. Then

1. *F is (Ω);*

$\Leftrightarrow$ 2. *$\mathrm{Ext}^1(\lambda^1_\infty(\alpha), F) = 0$, i.e. any ses $0 \to F \to G \to \lambda^1_\infty(\alpha) \to 0$ splits.*

$\Leftrightarrow$ 3. *If $Q : G \twoheadrightarrow H$ is a quotient mapping with kernel F then $Q_* : \mathcal{L}(\lambda^1_\infty(\alpha), G) \to \mathcal{L}(\lambda^1_\infty(\alpha), H)$ is onto;*

$\Leftrightarrow$ 4. *If $S : H \hookrightarrow G$ is a closed embedding with quotient $\lambda^1_\infty(\alpha)$ then $S^* : \mathcal{L}(G, F) \to \mathcal{L}(H, F)$ is onto.*

Proof. (1$\Rightarrow$2) is 4.134.

(1$\Leftarrow$2) is shown in [Vog87, 4.1 p.183].

(2$\Leftrightarrow$3$\Leftrightarrow$4) is 4.111 and 4.115. □

And similar to 4.131 one obtains:

4.136 Corollary [Vog85, 3.2 p.263]**.**
A Fréchet space F is (Ω) $\Leftrightarrow$ F is a quotient of $\lambda^1_\infty(a)$ for some $a \in \mathbb{R}^J_{\geq 1}$.

Proof.
($\Rightarrow$) We have the canonical resolution

$$0 \to E \to \prod_k E_k \to \prod_k E_k \to 0.$$

Let $F := \{x = (x_k)_k \in \prod_k E_k : \|x\| := \sum \|x_k\|_k < \infty\}$, a Banach space which contains each E_k as direct summand (and let F_k be a complement of E_k in F). Let $\{x_i : i \in I\}$ be a (w.l.o.g. infinite) dense subset in F and $0 \to K \hookrightarrow \ell^1(I) \twoheadrightarrow F \to 0$ be the resulting exact sequence. Taking the tensor product with the ses $0 \to s \to s \to \mathbb{K}^\mathbb{N} \to 0$ gives by 4.112 a diagram with exact rows and columns (since all factors are Fréchet and always one of them is nuclear). This gives a right exact diagonal sequence

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & F\hat{\otimes}s \arrow[r] \arrow[u] & F\hat{\otimes}s \arrow[r] \arrow[u] & F^\mathbb{N} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \ell^1(I)\hat{\otimes}s \arrow[r] \arrow[u] & \ell^1(I)\hat{\otimes}s \arrow[r] \arrow[u] & \ell^1(I)^\mathbb{N} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & K\hat{\otimes}s \arrow[r] \arrow[u] & K\hat{\otimes}s \arrow[r] \arrow[u] & K^\mathbb{N} \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

$(\ell^1(I)\hat{\otimes}s)\oplus(K\hat{\otimes}s) \to \ell^1(I)\hat{\otimes}s \xrightarrow{Q} F^\mathbb{N} \to 0$

and let N denote the kernel of Q.

Taking the direct sum of the canonical resolution of E with $0 \to 0 \to \prod_k F_k \to \prod_k F_k \to 0$ gives the exact sequence:

$$0 \to E \to F^{\mathbb{N}} \to F^{\mathbb{N}} \to 0$$

and by 4.133.2 also every quotient of $\lambda^1_\infty(a)$. Now take the pullback H to obtain the diagram on the right side.

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & E \arrow[r] & F^{\mathbb{N}} \arrow[r] \arrow[u] & F^{\mathbb{N}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & E \arrow[r] \arrow[u, equal] & H \arrow[r] \arrow[u] & \ell^1(I)\hat{\otimes}s \arrow[r] \arrow[u] & 0 \\
 & & N \arrow[r, equal] \arrow[u] & N \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Its second row splits by 4.134 (i.e. $H \cong E \oplus (\ell^1(I)\hat{\otimes}s)$) and taking the pullback G of its two columns gives:

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & N \arrow[r] & E \oplus (\ell^1(I)\hat{\otimes}s) \arrow[r] \arrow[u] & F^{\mathbb{N}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[u, equal] & G \arrow[r] \arrow[u] & \ell^1(I)\hat{\otimes}s \arrow[r] \arrow[u] & 0 \\
 & & N \arrow[r, equal] \arrow[u] & N \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Since N is the quotient of $(\ell^1(I)\hat{\otimes}s) \oplus (K\hat{\otimes}s) \cong (\ell^1(I) \oplus K)\hat{\otimes}s$ and hence of $\ell^1(I \sqcup K)\hat{\otimes}s \cong \lambda^1_\infty(a)$ by 4.126 we have that N has property (Ω). Therefore the second row splits and the first column shows that E is a quotient of $G \cong N \oplus (\ell^1(I)\hat{\otimes}s)$. Thus E is also a quotient of $(\ell^1(I) \oplus K \oplus \ell^1(I))\hat{\otimes}s$. Since K also contains a dense subset of cardinality $\leq |I|$ it is a quotient of $\ell^1(I)$ and since $\ell^1(I)^3 \cong \ell^1(I \sqcup I \sqcup I) \cong \ell^1(I)$ we conclude that E is a quotient of $\ell^1(I)\hat{\otimes}s$. $\square$

For power series spaces $\lambda^1_0(\alpha)$ of finite type one needs the stronger condition $(\overline{\overline{\Omega}})$:

4.137 Proposition. [**Vog87**, 4.2 p.184]**.**
Let $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$ *and E be a Fréchet space. Then*

1. *E has $(\overline{\overline{\Omega}})$, i.e.* $\forall p\ \exists p'\ \forall k\quad \forall d > 0\ \exists C > 0 : \|\text{-}\|^{1+d}_{-p'} \leq C\,\|\text{-}\|_{-k} \cdot \|\text{-}\|^d_{-p}$ *(cf. 4.132.2);*

$\Leftrightarrow$ 2. $\mathrm{Ext}^1(\lambda^1_0(\alpha), E) = 0$, *i.e. any ses* $0 \to E \to G \to \lambda^1_0(\alpha) \to 0$ *splits.*

If all involved Fréchet spaces have a basis of Hilbert seminorms then 4.127 and 4.134 can be generalized to

4.138 Splitting theorem (See [**MV92**, 30.1 p.357], [**Vog87**, 5.1 p.186])**.**
Let $0 \to E \to G \to F \to 0$ be a short exact sequence of (F) spaces having a basis of Hilbert seminorms.
If E is (Ω) and F is (DN), then the sequence splits.

4.139 Universal linearizer.
These results can also be used for lifting problems of non-linear functions:
Let $\mathcal{F}(U, E)$ be a class of functions from some set U (e.g. an open subset of some $\mathbb{K}^n$) into lcs E from a certain class.
The corresponding free space (or universal linearizer) $\lambda(U)$ should be an lcs in this class with the following universal property:
There exists a $\delta \in \mathcal{F}(U, \lambda(U))$, such for every $f \in \mathcal{F}(U, E)$ there exists a unique $\tilde{f} \in \mathcal{L}(\lambda(U), E)$ with $\tilde{f} \circ \delta = f$.

$$\begin{tikzcd}
U \arrow[r, "\delta"] \arrow[dr, "\forall f\in\mathcal{F}"'] & \lambda(U) \arrow[d, dotted, "\exists!\ \tilde{f}\in\mathcal{L}"] \\
 & \forall\, E
\end{tikzcd}$$

Let us try to find $\lambda(U)$: For $E := \mathbb{K}$ we need a bijection $\delta^* : \lambda(U)^* \to \mathcal{F}(U) := \mathcal{F}(U, \mathbb{K})$. Thus, if we have some reflexive lc-topology on $\mathcal{F}(U)$ then $\lambda(U) = \mathcal{F}(U)^*$ and δ^* should be the inverse of $\delta_{\mathcal{F}(U)} : \mathcal{F}(U) \to \mathcal{F}(U)^{**}$, i.e. $f(t) = \delta^*(\delta_{\mathcal{F}(U)}(f))(t) = \delta_{\mathcal{F}(U)}(f)(\delta(t)) = \delta(t)(f)$ for all $f \in \mathcal{F}(U)$ and all $t \in U$. So $\delta : U \to \lambda(U) := \mathcal{F}(U)^*$ is the usual evaluation map.

We need that $\delta : U \to \mathcal{F}(U)^*$, $x \mapsto (f \mapsto f(x))$ belongs to $\mathcal{F}$. Often it is the case, that switching variables gives a bijection $\mathcal{F}(U, E') \cong L(E, \mathcal{F}(U))$. For $E := \mathcal{F}(U)$, the map $\delta : U \to E^* \hookrightarrow E'$ corresponds to $\mathrm{id} \in \mathcal{L}(E, E)$, hence belongs to $\mathcal{F}(U, E')$ and, since it has values in E^*, it usually belongs even to $\mathcal{F}(U, E^*)$.

Let now E be arbitrary. In order that $\delta^* : \mathcal{L}(\lambda(U), E) \to \mathcal{F}(U, E)$, $T \mapsto T \circ \delta$, makes sense, we need that $f \in \mathcal{F}$, $T \in \mathcal{L} \Rightarrow T \circ f \in \mathcal{F}$, which is not a big limitation.

Is δ^* injective? So let $T \in \mathcal{L}(\lambda(U), E)$ be such that $f := T \circ \delta = 0$, hence $0 = x^* \circ T \circ \delta = \delta^*(x^* \circ T) : U \to \mathbb{K}$ for all $x^* \in E^*$. Since $\delta^* : \lambda(E)^* \to \mathcal{F}(U)$ is the inverse of $\delta : \mathcal{F}(U) \to \mathcal{F}(U)^{**}$ it follows that $x^* \circ T = 0$, and consequently $T = 0$.

Note, that we can deduce that the image of $\delta : U \to \lambda(U)$ generates a dense linear subspace, since every continuous linear functional T on $\lambda(U)$ which vanishes on the image of δ, i.e. $\delta^*(T) = 0$, has to be 0.

Is δ^* onto? So let $f \in \mathcal{F}(U, E)$ and consider $f^* : E^* \to \mathcal{F}(U)$, $x^* \mapsto x^* \circ f$. This is well-defined by the assumption above. In order to show that it is bounded, we consider the associated mapping $\widetilde{f^*} : U \to (E^*)'$, which is just $f : U \to E \hookrightarrow (E^*)'$ and belongs to $\mathcal{F}$. So f^* is bounded and hence $f^{**} : \lambda(U) = \mathcal{F}(U)' \to (E^*)'$ is continuous. Since $f^{**} \circ \delta = \delta \circ f : U \to E \to (E^*)'$ its values on the image of δ lie in E and, since this image generates a dense subspace, f^{**} is the required inverse image for complete E.

Thus we have shown:

Proposition.
Let $\mathcal{F}(U, E)$ be function spaces with the following properties:

1. $f \in \mathcal{F}$, $T \in \mathcal{L} \Rightarrow T \circ f \in \mathcal{F}$.
2. *If $\iota : G \hookrightarrow E$ is a closed embedding, then $f \in \mathcal{F}(U, G) \Leftrightarrow \iota \circ f \in \mathcal{F}(U, E)$.*
3. $\mathcal{F}(U, E') \cong L(E, \mathcal{F}(U))$ *by switch of variables.*
4. $\mathcal{F}(U)$ *carries a reflexive lc-topology.*

Let $\lambda(U) := \mathcal{F}(U)^*$ then $\delta^* : \mathcal{L}(\lambda(U), E) \cong \mathcal{F}(U, E)$ is a linear bijection for each complete lcs E with complete dual E^*. □

Examples.
($\boxed{1}$) $T \in \mathcal{L}, f \in \mathcal{F} \Rightarrow T \circ f \in \mathcal{F}$:
For ℓ^∞, C^∞ (See [**KM97**, 2.11 p.24]), H ([**KN85**, 2.6 p.283]), and C^ω ([**KM90**, 1.9 p.10]) this is easily checked.

($\boxed{2}$) $\iota \circ f \in \mathcal{F} \Rightarrow f \in \mathcal{F}$:
For ℓ^∞, C^∞, H, and C^ω this is obvious since these mappings can be tested by the continuous linear functionals.

($\boxed{3}$) $\mathcal{F}(U, E') \cong L(E, \mathcal{F}(U))$:
For C^∞ see [**FK88**, 4.4.5], for H see [**KN85**, 2.14 p.288], for C^ω see [**KM90**, 6.3.3 p.37], and for ℓ^∞ see [**Kri07a**, 4.7.4].

($\boxed{4}$) $\mathcal{F}(U)$ reflexive:
$C^\infty(U)$ is nuclear (F) and has (Ω), but not (DN) ($\cong s^{\mathbb{N}}$).

$H(U)$ is nuclear (F) and a power series space, it has always (Ω) and only for $U \cong \mathbb{C}$ (DN).
$C^\omega(U)$ is complete ultrabornological (N) and its dual is complete nuclear (LF).
ℓ^∞: For bornological spaces X one has $\ell^\infty(X) = (\ell^1(X))^*$ by [**FK88**, 5.1.25] and $\ell^1(X) = (c_0(X))^*$ by [**FK88**, 5.1.19], where

$$\ell^1(X) := \{f \in \mathbb{R}^X : \mathrm{carr}(f) \text{ is bounded and } \|f\|_1 < \infty\} \text{ and}$$

$$c_0(X) := \{f \in \mathbb{R}^X : \mathrm{carr}(f) \text{ countable and } \forall B\, \forall \varepsilon > 0 : \{x : |f(x)| > \varepsilon\} \text{ finite.}\}.$$

However, $\lambda(X) = \ell^1(X)$ for $\mathcal{F} := \ell^\infty$ by [**Kri07a**, 4.7.4].

In many situations one can show better density conditions for the image of δ (like Mackey-denseness) and hence gets the universal property for spaces E being less complete (like Mackey-complete).

For $U \subseteq \mathbb{R}^n$ is open, it has been show in [**FK88**, 5.1.8] that $\lambda(U) = C^\infty(U,\mathbb{R})^*$ is universal for C^∞-mappings into Mackey-complete spaces. For open $U \subseteq \mathbb{C}^n$, it has been shown in [**Sie95**] that $\lambda(U) = H(U)^*$ is universal for H-mappings into Mackey-complete spaces. The free convenient vector space for real-analytic mappings has been considered in [**KM90**] and for sequentially complete spaces in [**BD01**]. In [**FK88**, 5.1.24] it is shown that $\lambda(X) = \ell^1(X)$ is universal for ℓ^∞-mappings into Mackey-complete spaces.

4.140 Parameter dependance of PDO solutions.
Particular cases for surjective PDO's $D := P(\partial) : \mathcal{G}(W) \twoheadrightarrow \mathcal{G}(W)$ have been considered and ($\mathcal{F}$-)parameter dependence of the solutions discussed: Let $E_i := \mathcal{G}(W_i)$ and $D : E_1 \twoheadrightarrow E_2$ be onto. Is $D_* : \mathcal{F}(U, E_1) \to \mathcal{F}(U, E_2)$ onto? Using the universal linearizer $\lambda(U)$ for the function space $\mathcal{F}(U, \_)$, this question is reduced to the surjectivity of $D_* : \mathcal{L}(\lambda(U), E_1) \to \mathcal{L}(\lambda(U), E_2)$. Using the suggested isomorphism one obtains under appropriate conditions the following descriptions for the extension of $D : \mathcal{G}(W_1) \to \mathcal{G}(W_2)$:

$$\begin{tikzcd}
\mathcal{F}(U,\mathcal{G}(W_1)) \arrow[r, phantom, "\cong"] \arrow[d, "D_*"] & \mathcal{L}(\mathcal{F}(U)^*,\mathcal{G}(W_1)) \arrow[r, phantom, "\cong"] \arrow[d, "D_*"] & \mathcal{F}(U)\hat{\otimes}\mathcal{G}(W_1) \arrow[r, phantom, "\cong"] \arrow[d, "\mathcal{F}(U)\otimes D"] & \mathcal{L}(\mathcal{G}(W_1)^*,\mathcal{F}(U)) \arrow[r, phantom, "\cong"] \arrow[d, "D^{**}"] & \mathcal{G}(W_1,\mathcal{F}(U)) \arrow[d, "\tilde{D}"] \\
\mathcal{F}(U,\mathcal{G}(W_2)) \arrow[r, phantom, "\cong"] & \mathcal{L}(\mathcal{F}(U)^*,\mathcal{G}(W_2)) \arrow[r, phantom, "\cong"] & \mathcal{F}(U)\hat{\otimes}\mathcal{G}(W_2) \arrow[r, phantom, "\cong"] & \mathcal{L}(\mathcal{G}(W_2)^*,\mathcal{F}(U)) \arrow[r, phantom, "\cong"] & \mathcal{G}(W_2,\mathcal{F}(U))
\end{tikzcd}$$

[**BD98**, Corollary 39 p.34] If $D : C^\omega(\mathbb{R}) \twoheadrightarrow C^\omega(\mathbb{R})$ onto then one can find solutions depending holomorphically on a parameter in $\mathbb{C}$. By [**BD01**, Proposition 9 p.501] for every elliptic surjective linear PDO $D := P(\partial) : C^\omega(U) \to C^\omega(U)$ with constant coefficients and open $U \subseteq \mathbb{R}^n$ the extension $D \otimes E : C^\omega(U, E) \to C^\omega(U, E)$ is surjective if E is (F) or the strong dual of a (F)-space with (DN).

In contrast, by [**BD01**, Theorem 8 p.501] for every elliptic surjective linear PDO $D := P(\partial) : C^\omega(\mathbb{R}^2) \to C^\omega(\mathbb{R}^2)$ with constant coefficients the extension $D \otimes H(\bar{\mathbb{D}}) : C^\omega(\mathbb{R}^2, H(\bar{\mathbb{D}})) \to C^\omega(\mathbb{R}^2, H(\bar{\mathbb{D}}))$ is not surjective.

[**BD01**, Theorem 6 p.499] and [**BD98**, Theorem 38 p.33]: For open sets $U_i \subseteq \mathbb{R}^{n_i}$ let $T : C^\omega(U_1) \to C^\omega(U_2)$ be a continuous linear surjective mapping. Then $T \otimes E : C^\omega(U_1, E) \to C^\omega(U_2, E)$ is onto provided E is (F)+(DN) or (E is complete+(LB) and E^* is $(\overline{\overline{\Omega}})$) or E is a (F)-quojection, i.e. every quotient with a continuous norm is a Banach space.

For (sequentially) complete lcs E and open $U \subseteq \mathbb{R}^n$ one has a linear bijection $C^\omega(U, E) \cong C^\omega(U)\varepsilon E = \mathcal{L}(C^\omega(U)^*_\beta, E) \cong \mathcal{L}((E^*, \tau_c), C^\omega(U))$ by [**BD01**, Theorem 2 p.496]

Locally bounded linear mappings

In this section, we describe situations where continuous linear mappings are even locally bounded. If the domain space is a power series space of finite type, then the characterizing property for the range space is (DN). And if the range space is such a power series space, then the characterizing property of the domain space is $(\overline{\overline{\Omega}})$. For power series spaces of infinite type, the characterizing properties for the other involved space are (LB_∞) and (LB^∞). We give applications to vector valued real-analytic mappings and mention applications to holomorphic functions on Fréchet spaces.

4.141 Definition and Remark. Locally bounded operators.
A linear map $T : E \to F$ between lcs is called LOCALLY BOUNDED if there exists a 0-nbhd U with $T(U)$ bounded. We will denote by $\mathcal{LB}(E, F)$ the space of all locally bounded linear maps from E to F.

We have $\mathcal{LB}(E, F) \subseteq \mathcal{L}(E, F)$: Let $U \subseteq E$ be a 0-nbhd with $T(U)$ bounded and $V \subseteq F$ be an arbitrary 0-nbhd. Then $\exists C > 0$: $T(U) \subseteq C\,V$ and hence $\frac{1}{C}U \subseteq T^{-1}(V)$, i.e. T is continuous.

We are interested in pairs (E, F) for which $\mathcal{LB} = \mathcal{L}$.

If E or F is a normed space, then $\mathcal{LB}(E, F) = \mathcal{L}(E, F)$: Let $T \in \mathcal{L}(E, F)$. If E is normed, than $T(U)$ is bounded for the unit ball $U := oE$. If F is normed, than $U := T^{-1}(oF)$ is a 0-nbhd with $T(U) \subseteq oF$ bounded.

Note that $\mathrm{id}_E \in \mathcal{LB}(E, E) \Leftrightarrow E$ is normable, since $U = \mathrm{id}(U)$ is a bounded 0-nbhd.

If $Q : E \twoheadrightarrow E_1$ is a quotient mapping and $S : F_1 \hookrightarrow F$ and is an embedding then $\mathcal{LB}(E, F) = \mathcal{L}(E, F) \Rightarrow \mathcal{LB}(E_1, F_1) = \mathcal{L}(E_1, F_1)$: For $T \in \mathcal{L}(E_1, F_1)$ we have that $S \circ T \circ Q \in \mathcal{L}(E, F) = \mathcal{LB}(E, F)$, hence there exists a 0-nbhd $U \subseteq E$ with $S(T(Q(U))) \subseteq F$ bounded. Since Q is open, the set $U_1 := Q(U) \subseteq E_1$ is a 0-nbhd and since S is an embedding $T(U_1) = T(Q(U))$ is bounded.

Let $\mathcal{LB}(E, F) = \mathcal{L}(E, F)$. If there is an embedding $E \hookrightarrow F$, then E is normed, since then $\mathcal{LB}(E, E) = \mathcal{L}(E, E)$. And if there is a quotient mapping $E \twoheadrightarrow F$, then F is normed, since then $\mathcal{LB}(F, F) = \mathcal{L}(F, F)$.

If E is a Fréchet space and $\mathcal{LB}(E, \mathbb{K}^\mathbb{N}) = \mathcal{L}(E, \mathbb{K}^\mathbb{N})$, then E is normable: If E is not normable, then there exists a quotient mapping $Q \in \mathcal{L}(E, \mathbb{R}^\mathbb{N})$ by 4.77, hence $\mathbb{K}^\mathbb{N}$ would have to normable but is not.

4.142 Proposition [**BD98**, Theorem 16 p.22]**,** [**BD01**, Theorem 2 p.496]**.**
The bijection $\delta^* : \mathcal{L}(C^\omega(U)^*, E) \to C^\omega(U, E)$ from 4.139 for open $U \subseteq \mathbb{R}^n$ and Fréchet spaces E maps $\mathcal{LB}(C^\omega(U)^*, E)$ onto $C^\omega_t(U, E)$, the space of topologically real-analytic mappings, i.e. mappings which are locally representable by a convergent power series.

$$\begin{tikzcd}
\mathcal{L}(C^\omega(U)^*, E) \arrow[r, tail, two heads, "\delta^*"] & C^\omega(U, E) \\
\mathcal{LB}(C^\omega(U)^*, E) \arrow[u, hook] \arrow[r, tail, two heads] & C^\omega_t(U, E) \arrow[u, hook]
\end{tikzcd}$$

Sketch of proof. It is easy to see that $f \in C^\omega_t(\mathbb{R}, E)$ is locally C^ω into some E_B and by 4.90.1 even globally, hence corresponds to an element in $\mathcal{L}(C^\omega(\mathbb{R})^*, E_B) = \mathcal{LB}(C^\omega(\mathbb{R})^*, E_B) \subseteq \mathcal{LB}(C^\omega(\mathbb{R})^*, E)$.

Conversely, $T \in \mathcal{LB}(C^\omega(\mathbb{R})^*, E) \Rightarrow \exists B : T \in \mathcal{LB}(C^\omega(\mathbb{R})^*, E_B)$. Thus $\delta^*(T) \in C^\omega(\mathbb{R}, E_B) \subseteq C^\omega_t(\mathbb{R}, E)$, by a Baire argument, see [**KM90**, 1.6 p.8]. □

Thus, in order to get $C^\omega(\mathbb{R}, E) = C^\omega_t(\mathbb{R}, E)$ we have determine whether $\mathcal{L} = \mathcal{LB}$?

4.143 Definition.
Let E and F be (F) with increasing bases of seminorms $(\|\_\|_k)_{k\in\mathbb{N}}$ and $(\|\_\|_n)_{n\in\mathbb{N}}$. For linear $T : E \to F$ consider

$$\|T\|_{k,n} := \sup_{\|x\|_k \le 1} \|Tx\|_n \in [0, +\infty].$$

Note that $\|T\|_{k+1,n} \le \|T\|_{k,n} \le \|T\|_{k,n+1}$ and

(1) $$T \in \mathcal{L}(E, F) \Leftrightarrow \forall n \in \mathbb{N}\ \exists k_n \in \mathbb{N} : \|T\|_{k_n,n} < \infty$$

(2) $$T \in \mathcal{LB}(E, F) \Leftrightarrow \exists k' \in \mathbb{N}\ \forall n \in \mathbb{N} : \|T\|_{k',n} < \infty$$

Proof.
(1) $T \in \mathcal{L}(E, F) \Leftrightarrow \forall n \in \mathbb{N}\ \exists k_n \in \mathbb{N}\ \exists C > 0 : \|T(x)\|_n \le C\|x\|_{k_n}$.
(2) $T \in \mathcal{LB}(E, F) \Leftrightarrow \exists k' \in \mathbb{N}\ \forall n \in \mathbb{N}\ \exists C > 0\ \forall x : \|x\|_{k'} \le 1 \Rightarrow \|T(x)\|_n \le C$. □

4.144 Lemma. Characterizing $\mathcal{L} = \mathcal{LB}$ (See [**Vog83**, 1.1 p.183])**.**
Let E and F be (F) with increasing bases of seminorms $(\|\_\|_k)_{k\in\mathbb{N}}$ and $(\|\_\|_n)_{n\in\mathbb{N}}$. Then

1. $\mathcal{L}(E, F) = \mathcal{LB}(E, F)$;

$\Leftrightarrow$ 2. $\forall k \in \mathbb{N}^\mathbb{N}\ \exists k'\ \forall n\ \exists n'\ \exists C > 0\ \forall T \in \mathcal{L}(E, F) : \|T\|_{k',n} \le C \max_{m\le n'} \|T\|_{k_m,m}$.

W.l.o.g. $k \nearrow \infty$, since validity of 2 for k implies it for any $\underline{k} \le k$.

Proof. For $k \in \mathbb{N}^\mathbb{N}$ consider

$$G_k := \Big\{T \in \mathcal{L}(E, F) : \|T\|_{k_n,n} < \infty \text{ for all } n \in \mathbb{N}\Big\},$$

a Fréchet space with respect to the seminorms $\|\_\|_{k_n,n}$ for $n \in \mathbb{N}$.
For each $k' \in \mathbb{N}$ let

$$H_{k'} := \Big\{T \in \mathcal{L}(E, F) : \|T\|_{k',n} < \infty \ \text{ for all } n \in \mathbb{N}\Big\},$$

a Fréchet spaces with respect to the seminorms $\|\_\|_{k',n}$ for $n \in \mathbb{N}$.
Since $\{x : \|x\|_{k'+1} \le 1\} \subseteq \{x : \|x\|_{k'} \le 1\}$ we have $\|T\|_{k'+1,n} \le \|T\|_{k',n}$ and thus continuous inclusions $H_{k'} \subseteq H_{k'+1}$. By 4.143.1 $\mathcal{L}(E, F) = \bigcup_k G_k$ and by 4.143.2 $\mathcal{LB}(E, F) = \bigcup_{k'} H_{k'}$.

(1⇒2) By 1 we have $G_k \subseteq \mathcal{L}(E, F) = \mathcal{LB}(E, F) = \bigcup_{k'} H_{k'}$ Since the inclusions $G_k \subseteq \mathcal{L}(E, F)$ and $H_{k'} \subseteq \mathcal{L}(E, F)$ are continuous (for $B \subseteq C\, U_{k'}$ we have $\sup\{\|T(x)\|_n : x \in B\} \le C\,\|T\|_{k',n}$) we can apply Grothendieck's Factorization Theorem 2.6 to obtain a $k' \in \mathbb{N}$ such that $G_k \subseteq H_{k'}$ and the inclusion is continuous, i.e.

$$\forall n \in \mathbb{N}\ \exists n' \in \mathbb{N}\ \exists C > 0 : \|T\|_{k',n} \le C \max_{m\le n'} \|T\|_{k_m,m}.$$

1⇐2) Let $T \in \mathcal{L}(E, F)$. By 4.143.1 $\exists k \in \mathbb{N}^\mathbb{N} : T \in G_k$ and by 2:

$$\exists k'\ \forall n\ \exists n'\ \exists C > 0 : \|T\|_{k',n} \le C \max_{m\le n'} \|T\|_{k_m,m}.$$

Hence

$$\|T\|_{k',n} < \infty \text{ for all } n,$$

i.e. $T \in H_{k'} \subseteq \mathcal{LB}(E, F)$.

4.145 Lemma (See [Vog83, 1.3 p.184]).
Let $B = \{b^{(k)} : k \in \mathbb{N}\}$ be a Köthe matrix and F a Fréchet space with increasing basis of seminorms $\|\_\|_k$. Then

1. $\mathcal{L}(\lambda^1(B), F) = \mathcal{LB}(\lambda^1(B), F)$;

$\Leftrightarrow$ 2. $\forall k \in \mathbb{N}^{\mathbb{N}}\ \exists k'\ \forall n\ \exists n'\ \exists C > 0\ \forall j\ \forall y \in F : \frac{\|y\|_n}{b_j^{(k')}} \le C \max_{m \le n'} \frac{\|y\|_m}{b_j^{(k_m)}}$.

W.l.o.g. $k \nearrow \infty$.

Proof.
(1⇒2) follows from 4.144 for $T := \mathrm{pr}_j \otimes y$ with $y \in F$ and $\mathrm{pr}_j(x) := x_j$ for $x \in \lambda^1(B) =: E$, since

$$\|T\|_{k,n} = \sup\{\|Tx\|_n : \|x\|_k \le 1\} = \sup\Big\{|x_j|\,\|y\|_n : \|x \cdot b^{(k)}\|_{\ell^1} \le 1\Big\} = \frac{\|y\|_n}{b_j^{(k)}}.$$

(1⇐2) Since e_j is an (absolute) Schauder-basis of $E := \lambda^1(B)$ by 1.21 every $T \in \mathcal{L}(E, F)$ is of the form

$$T(x) = T\Big(\sum_j \mathrm{pr}_j(x)\, e_j\Big) = \sum_j \mathrm{pr}_j(x)\, y_j, \text{ where } y_j := T(e_j),$$

$$\|T(x)\|_n \le \sum_{j\in\mathbb{N}} b_j^{(k')}\, |\mathrm{pr}_j(x)| \cdot \sup_{j\in\mathbb{N}} \frac{\|y_j\|_n}{b_j^{(k')}} = \|x\|_{k'} \cdot \sup_{j\in\mathbb{N}} \frac{\|y_j\|_n}{b_j^{(k')}},$$

$$\text{and}\quad \forall m\ \exists k_m\ \exists C_m > 0 : \|y_j\|_m = \|T(e_j)\|_m \le C_m \|e_j\|_{k_m} = C_m\, b_j^{(k_m)}.$$

By 2 we have $\exists k'\ \forall n\ \exists n'\ \exists C > 0$:

$$\|T\|_{k',n} \le \sup_{j\in\mathbb{N}} \frac{\|y_j\|_n}{b_j^{(k')}} \le \sup_{j\in\mathbb{N}} \Big(C \max_{m \le n'} \frac{\|y_j\|_m}{b_j^{(k_m)}}\Big) \le C \max_{m \le n'} C_m < \infty,$$

i.e. $T \in \mathcal{LB}(E, F)$ by 4.143.2. □

4.146 Theorem (See [Vog83, 2.1 p.186]).
Let $\beta := (\beta_j)_{j\in\mathbb{N}}$ be a SHIFT-STABLE SEQUENCE, *i.e.* $\sup_n \frac{\beta_{n+1}}{\beta_n} < \infty$, *and F a Fréchet space with increasing basis of seminorms $\|\_\|_k$. Then*

1. $\mathcal{L}(\lambda_0^1(\beta), F) = \mathcal{LB}(\lambda_0^1(\beta), F)$;

$\Leftrightarrow$ 2. *F has property (DN) (See 4.123).*

(1⇐2) is valid without the assumption on β.

The shift-stability is equivalent to $\lambda_r^p(\beta) \cong \mathbb{K} \oplus \lambda_r^p(\beta)$, via $\Phi : x \mapsto (x_0, S(x))$, where $S(x)_j := x_{j+1}$.

Proof. By 1.26.1 we may replace $\lambda_0^1(\beta)$ by the isomorphic space $\lambda_1^1(\beta)$. Let $0 < \rho_k \nearrow 1$ for $k \to \infty$, i.e. $b_j^{(k)} := e^{\rho_k \beta_j}$ describes the Köthe-matrix B for $\lambda_1^1(\beta) := \lambda(B)$.

(1⇐2) Let $k \in \mathbb{N}^{\mathbb{N}}$. By 4.123.5 the property (DN) means:

$$\exists q\ \exists d > 0\ \forall p\ \exists p' \ge q\ \exists C \ge 1 : \|\_\|_p^{1+d} \le C\, \|\_\|_q^d\, \|\_\|_{p'}.$$

(Note, that because of 4.131, it would be enough to consider $F = \lambda_\infty^\infty(a)$ and hence $q = 0$, $d = 1$, $p' = 2p$, and $C = 1$ by 4.125.)
Now choose a $k' > k_q$ such that

$$d > \frac{1 - \rho_{k'}}{\rho_{k'} - \rho_{k_q}} \ \big(\searrow 0 \text{ for } k' \to \infty\big).$$

Let $y \in F$ and $j \in \mathbb{N}$ be fixed.
If

$$\|y\|_q < e^{(\rho_{k_q}-\rho_{k'})\beta_j}\|y\|_p,$$

then

$$\|y\|_p^{1+d} \le C\,\|y\|_q^d\,\|y\|_{p'} \le C\,e^{d(\rho_{k_q}-\rho_{k'})\beta_j}\,\|y\|_p^d\,\|y\|_{p'}.$$

By hypothesis $d(\rho_{k_q}-\rho_{k'}) \le \rho_{k'}-1 \le \rho_{k'}-\rho_{k_{p'}}$, so we get

$$\|y\|_p \le C\,e^{(\rho_{k'}-\rho_{k_{p'}})\beta_j}\,\|y\|_{p'}.$$

Otherwise,

$$\|y\|_p \le e^{(\rho_{k'}-\rho_{k_q})\beta_j}\,\|y\|_q.$$

In any case

$$\frac{\|y\|_p}{e^{\rho_{k'}\beta_j}} \le \max\Big\{\frac{\|y\|_q}{e^{\rho_{k_q}\beta_j}}, C\frac{\|y\|_{p'}}{e^{\rho_{k_{p'}}\beta_j}}\Big\} \le C\max_{m\le p'}\frac{\|y\|_m}{e^{\rho_{k_m}\beta_j}}$$

and 4.145.2 gives 2.

(1 ⇒ 2) By 4.145.2 for the sequence $k := \mathrm{id} \in \mathbb{N}^{\mathbb{N}}$ we have

$$\exists k'\ \forall n\ \exists n'\ \exists C>0\ \forall y\in F\ \forall j: \|y\|_n\,e^{-\rho_{k'}\beta_j} \le C\max_{m\le n'}\|y\|_m\,e^{-\rho_m\beta_j}.$$

W.l.o.g. $n' \ge \max\{n, k'+1\}$ and thus

$$\|y\|_n\,e^{-\rho_{k'}\beta_j} \le C\max_{m\le n'}\|y\|_m\,e^{-\rho_m\beta_j} \le C\max\left\{\|y\|_{k'}, \|y\|_{n'}\,e^{-\rho_{k'+1}\beta_j}\right\},\ \text{since}$$

$$\|y\|_m\,e^{-\rho_m\beta_j} \le \begin{cases}\|y\|_{k'}\cdot 1 & \text{for } m\le k',\\ \|y\|_{n'}\,e^{-\rho_{k'+1}\beta_j} & \text{for } k'<m\le n'.\end{cases}$$

Let $b := \sup_n \frac{\beta_{n+1}}{\beta_n} < \infty$ and take $y \in F$. If there exists a $j \in \mathbb{N}$ such that

$$\|y\|_n\,e^{-\rho_{k'}\beta_{j+1}} \le C\|y\|_{k'} < \|y\|_n\,e^{-\rho_{k'}\beta_j}\ (\searrow 0 \text{ for } j\to\infty),$$

then

$$\begin{aligned}\|y\|_n &\le e^{\rho_{k'}\beta_j}\,C\,\max\left\{\|y\|_{k'}, \|y\|_{n'}\,e^{-\rho_{k'+1}\beta_j}\right\} = C\,\|y\|_{n'}\,e^{(\rho_{k'}-\rho_{k'+1})\beta_j}\\ &\le C\|y\|_{n'}\,e^{-\rho_{k'}\beta_{j+1}\frac{\rho_{k'+1}-\rho_{k'}}{b\,\rho_{k'}}} \le C\|y\|_{n'}\Big(C\frac{\|y\|_{k'}}{\|y\|_n}\Big)^d,\end{aligned}$$

where $d := \frac{\rho_{k'+1}-\rho_{k'}}{b\,\rho_{k'}}$, i.e. $\|y\|_n^{1+d} \le C^{1+d}\,\|y\|_{n'}\,\|y\|_{k'}^d$.
If no such j exists, then $e^{-\rho_{k'}\beta_0}\,\|y\|_n \le C\|y\|_{k'}$ and we get

$$\|y\|_n^{1+d} \le \|y\|_{n'}\,\|y\|_n^d \le \|y\|_{n'}\,(C\,e^{\rho_{k'}\beta_0})^d\,\|y\|_{k'}^d.$$

Hence in both cases

$$\|y\|_n^{1+d} \le C'\,\|y\|_{n'}\,\|y\|_{k'}^d \text{ with } C' := C^d\max\{C, e^{\rho_{k'}\beta_0 d}\},$$

which is equivalent to (DN) by 4.123.5 with $q := k'$, $p := n$, and $p' := n'$. □

4.147 Proposition (See [**Vog83**, 1.4 p.185])**.**
Let $A = \{a^{(k)} \in \mathbb{R}_+^J : k \in \mathbb{N}\}$ be a Köthe matrix, E a Fréchet space with decreasing 0-nbhd basis $\{U_k : k \in \mathbb{N}\}$ and Minkowski-functionals $\|\text{-}\|_{-k}$ of the polars U_k^o. Then

1. $\mathcal{L}(E, \lambda^\infty(A)) = \mathcal{LB}(E, \lambda^\infty(A))$;

⇔ 2. $\forall k \in \mathbb{N}^{\mathbb{N}}\ \exists k'\ \forall n\ \exists n'\ \exists C>0\ \forall j\ \forall x^*: a_j^{(n)}\|x^*\|_{-k'} \le C\max_{m\le n'} a_j^{(m)}\|x^*\|_{-k_m}$.

Proof. This proof is similar to the proof of 4.145.

(1⇒2) follows from 4.144 for $T := x^* \otimes e_j$ with $j \in \mathbb{N}$ and $x^* \in E^*$.

(1⇐2) Let $T \in \mathcal{L}(E, \lambda^\infty(A))$ and put $x_j^* := \mathrm{pr}_j \circ T \in E^*$. Then

$$\begin{aligned}\|T(x)\|_n &\le \sup_j |a_j^{(n)}\, x_j^*(x)| \le \|x\|_{k'} \sup_{j\in\mathbb{N}} a_j^{(n)} \|x_j^*\|_{-k'} \\ &\le \|x\|_{k'}\, C \max_{m\le n'} \underbrace{\big(\sup_{j\in\mathbb{N}} a_j^{(m)} \|x_j^*\|_{-k_m}\big)}_{=:\|T\|_{k_m,m}},\end{aligned}$$

by 2. This implies

$$\|T\|_{k',n} \le C \max_{m\le n'} \|T\|_{k_m,m},$$

i.e. $T \in \mathcal{LB}(E, \lambda^\infty(A))$ by 4.144. □

4.148 Theorem. [Vog83, 4.2 Satz p.190].
Let $(\alpha_j)_{j\in\mathbb{N}}$ be a shift-stable sequence and E a Fréchet space. Then

1. *$\mathcal{L}(E, \lambda_0^\infty(\alpha)) = \mathcal{LB}(E, \lambda_0^\infty(\alpha))$;*

⇔ 2. *E has property $(\overline{\overline{\Omega}})$ (see 4.137).*

(1⇐2) is valid without the assumption on α.

Proof. By 1.26.1 we may replace $\lambda_0^\infty(\alpha)$ by the isomorphic space $\lambda_1^\infty(\alpha)$. Let $0 < \rho_k \nearrow 1$ for $k \to \infty$, i.e. $a_j^{(k)} := e^{\rho_k \alpha_j}$ describes the Köthe-matrix A for $\lambda_1^\infty(\alpha) := \lambda^\infty(A)$. W.l.o.g. we may assume that $\lim_{k\to\infty} \frac{1-\rho_k}{\rho_k-\rho_{k-1}} = 0$, e.g. take $\rho_k := 1 - \frac{1}{k!}$. Let $\{U_k : k \in \mathbb{N}\}$ be an increasing 0-nbhd basis of E and $\|\text{-}\|_{-k}$ the Minkowski-functional of $U_k^o \subseteq E^*$.

(1⇐2) Let $k \in \mathbb{N}^\mathbb{N}$ be given. For $p := k_0$ choose p' according to $(\overline{\overline{\Omega}})$, i.e.

$$\forall n\ \ \forall d > 0\ \exists C \ge 1 : \|\text{-}\|_{-p'}^{1+d} \le C\, \|\text{-}\|_{-n} \cdot \|\text{-}\|_{-p}^d$$

For every $n \in \mathbb{N}$ let $n' \ge p$ with $\rho_{n'} > \rho_n$ and $d > 0$ with $d(\rho_n - \rho_0) \le \rho_{n'} - \rho_n$. Thus there exists a $C \ge 1$ such that

$$\|\text{-}\|_{-p'}^{1+d} \le C\, \|\text{-}\|_{-k_{n'}} \cdot \|\text{-}\|_{-k_0}^d.$$

For $x^* \in E^*$ and $j \in \mathbb{N}$ either $e^{\rho_n\alpha_j}\|x^*\|_{-p'} < e^{\rho_0\alpha_j}\|x^*\|_{-k_0}$ or

$$\|x^*\|_{-p'}^{1+d} \le C\, \|x^*\|_{-k_{n'}} \cdot \|x^*\|_{-k_0}^d \le C\, \|x^*\|_{-k_{n'}}\, e^{d(\rho_n-\rho_0)\alpha_j}\, \|x^*\|_{-p'}^d,$$

i.e. $\|x^*\|_{-p'} \le C\, e^{(\rho_{n'}-\rho_n)\alpha_j}\, \|x^*\|_{-k_{n'}}$.
In both cases we have 4.147.2

$$e^{\rho_n\alpha_j}\|x^*\|_{-p'} \le \max\big\{e^{\rho_0\alpha_j}\|x^*\|_{-k_0}, C\, e^{\rho_{n'}\alpha_j}\|x^*\|_{-k_{n'}}\big\} \le C \max_{m\le n'} e^{\rho_m\alpha_j}\|x^*\|_{-k_m},$$

with $k' := p'$, hence $\mathcal{L} = \mathcal{LB}$ by 4.147.

(1⇒2) Let $p \in \mathbb{N}$ and consider the sequence $k : n \mapsto p + n$. By 4.147.2

$\exists p'\ \forall n\ \exists n'\ \exists C_n \ge 1\ \forall x^* \in E^*\ \forall j :$

$$\begin{aligned}e^{\rho_n\alpha_j}\, \|x^*\|_{-p'} &\le C_n \max_{m\le n'} e^{\rho_m\alpha_j}\|x^*\|_{-k_m} \\ &\le C_n \max\Big\{e^{\rho_{n-1}\alpha_j}\|x^*\|_{-p}, e^{\alpha_j}\|x^*\|_{-k_n}\Big\},\end{aligned}$$

since

$$e^{\rho_m \alpha_j} \|x^*\|_{-k_m} \le \begin{cases} e^{\rho_{n-1}\alpha_j} \|x^*\|_{-k_0} & \text{for } m < n, \\ e^{\alpha_j} \|x^*\|_{-k_n} & \text{for } n \le m \le n'. \end{cases}$$

Let $x^* \in E^*$. If there exists a $j \in \mathbb{N}$ such that

$$e^{(\rho_n - \rho_{n-1})\alpha_{j-1}} \|x^*\|_{-p'} \le C_n \|x^*\|_{-p} < e^{(\rho_n - \rho_{n-1})\alpha_j} \|x^*\|_{-p'} \ (\nearrow 0 \text{ for } j \to \infty),$$

then, since $d_n := \sup_j \frac{(1-\rho_n)\alpha_j}{(\rho_n - \rho_{n-1})\alpha_{j-1}}$,

$$\begin{aligned} \|x^*\|_{-p'} &\le e^{-\rho_n \alpha_j} C_n \max\Big\{ e^{\rho_{n-1}\alpha_j} \|x^*\|_{-p}, e^{\alpha_j} \|x^*\|_{-k_n} \Big\} = C_n\, e^{(1-\rho_n)\alpha_j} \|x^*\|_{-k_n} \\ &\le C_n\, e^{(\rho_n - \rho_{n-1})\alpha_{j-1} d_n} \|x^*\|_{-k_n} \le C_n \left(C_n \frac{\|x^*\|_{-p}}{\|x^*\|_{-p'}} \right)^{d_n} \|x^*\|_{-k_n}, \end{aligned}$$

i.e. $\|x^*\|_{-p'}^{1+d_n} \le C_n^{1+d_n} \|x^*\|_{-k_n} \|x^*\|_{-p}^{d_n}$.

If no such j exists, then $C_n \|x^*\|_{-p} < e^{(\rho_n - \rho_{n-1})\alpha_0} \|x^*\|_{-p'}$. Hence

$$\|x^*\|_{-p'} \le e^{-\rho_n \alpha_j} C_n \max\Big\{ e^{\rho_{n-1}\alpha_j} \|x^*\|_{-p}, e^{\alpha_j} \|x^*\|_{-k_n} \Big\} = C_n\, e^{(1-\rho_n)\alpha_0} \|x^*\|_{-k_n}$$

and thus we obtain in both cases

$$\|x^*\|_{-p'}^{1+d_n} \le C_n' \|x^*\|_{-k_n} \cdot \|x^*\|_{-p}^{d_n}, \text{ where } C_n' := C_n \max\{C_n^{d_n}, e^{(1-\rho_n)\alpha_0}\}.$$

Since $k_n \to +\infty$ and $d_n \to 0$, condition

$(\overline{\overline{\Omega}})$ $\qquad \forall p\, \exists p'\, \forall k\ \forall d > 0\, \exists C \ge 1 : \|\text{-}\|_{-p'}^{1+d} \le C\, \|\text{-}\|_{-k} \cdot \|\text{-}\|_{-p}^{d}$

follows. □

For power series spaces $\lambda_\infty^p(\alpha)$ of infinite type one needs new (smaller) classes:

4.149 Theorem. [Vog83, 3.2 Satz p.188].
Let $(\beta_j)_{j\in\mathbb{N}}$ be a shift-stable sequence and F be a Fréchet space with increasing basis of seminorms $\|\text{-}\|_k$. Then

1. $\mathcal{L}(\lambda_\infty^1(\beta), F) = \mathcal{LB}(\lambda_\infty^1(\beta), F)$*;*

$\Leftrightarrow$ 2. *F has property (LB_∞), i.e.*

$$\forall \rho \in \mathbb{R}_+^{\mathbb{N}}\, \exists k'\, \forall n\, \exists n'\, \exists C > 0\, \forall y\, \exists m \in [n, n'] : \|y\|_n^{1+\rho_m} \le C\, \|y\|_{k'}^{\rho_m} \|y\|_m.$$

($\boxed{1} \Leftarrow \boxed{2}$) is valid without the assumption on β.

Similary as in $\boxed{4.123}$ we may assume $k' \le n \le n'$ and $\rho \nearrow \infty$:

$$\frac{\|y\|_n}{\|y\|_m} \le C \left(\frac{\|y\|_{k'}}{\|y\|_n} \right)^{\rho_m} \le C \left(\frac{\|y\|_{k'}}{\|y\|_n} \right)^{\rho'_m} \text{ for each } \rho' \le \rho.$$

Obviously (use $\boxed{4.123.5}$ and $\rho := \text{const}_d$) one has: $(\text{LB}_\infty) \Rightarrow (\text{DN})$.
In fact, recall:

(DN) $\qquad \exists k'\, \exists d > 0\, \forall n\, \exists n'\, \exists C > 0 : \|\text{-}\|_n^{1+d} \le C\, \|\text{-}\|_{k'}^{d}\, \|\text{-}\|_{n'};$

Proof.
($\boxed{1} \Leftarrow \boxed{2}$) Let $k \in \mathbb{N}^{\mathbb{N}}$ with $k \nearrow +\infty$ be arbitrary. By $\boxed{2}$ we have for $\rho := k$

$$\exists k'\, \forall n\, \exists n' \ge k'\, \exists C > 0\, \forall y\, \exists m \in [n, n'] : \|y\|_n^{1+k_m} \le C\, \|y\|_{k'}^{k_m} \|y\|_m.$$

Put $k'' := k_{k'} + 1$. For given j either

$$\|y\|_n\, e^{-k''\beta_j} \le \|y\|_{k'}\, e^{-k_{k'}\beta_j}$$

or

$$\|y\|_n^{1+k_m} \le C\, \|y\|_{k'}^{k_m} \|y\|_m \le C\, e^{(k_{k'} - k'')\beta_j k_m} \|y\|_n^{k_m} \|y\|_m$$

and hence

$$\|y\|_n \underbrace{e^{-k''\beta_j}}_{\le 1} \le \|y\|_n \le C\,\|y\|_m\, e^{(k_{k'}-k'')\beta_j k_m} = C\,\|y\|_m\, e^{-k_m\beta_j}.$$

In any case we have

$$\|y\|_n\, e^{-k''\beta_j} \le C \max_{m\le n'} \|y\|_m\, e^{-k_m\beta_j},$$

i.e. condition 4.145.2 is satisfied.

(1⇒2) Let $\rho \in \mathbb{R}_+^{\mathbb{N}}$ with $\rho \nearrow +\infty$ and let

$$\|\xi\|_k := \sum_{j\in\mathbb{N}} |\xi_j| e^{\sigma_k\beta_j} \text{ with } \sigma_k := \rho_k^2$$

the basis of seminorms on $\lambda^1_\infty(\beta)$. By 4.145.2 for $k :=$ id we have

$$\exists k'\ \forall n\ \exists n'\ \exists C\ge 1\ \forall y\in F\ \forall j : \|y\|_n e^{-\sigma_{k'}\beta_j} \le C \max_{m\le n'} \|y\|_m e^{-\sigma_m\beta_j}.$$

We pick a j_0, such that for $j \ge j_0$ we have

$$1 > C e^{(\sigma_{k'}-\sigma_{k'+1})\beta_j}\ (\searrow 0 \text{ for } j\to\infty).$$

Thus

$$\|y\|_n\, e^{-\sigma_{k'}\beta_j} \le C \max\{\|y\|_m\, e^{-\sigma_m\beta_j} : m\in[0,k']\cup[n+1,n']\},$$

since for $k' < m \le n$ we have

$$C\,\|y\|_m\, e^{-\sigma_m\beta_j} \le C\,\|y\|_n\, e^{-\sigma_{k'+1}\beta_j} < \|y\|_n\, e^{-\sigma_{k'}\beta_j}.$$

Let $y\in F$. Then either

$$\|y\|_n \le C e^{\sigma_{k'}\beta_{j_0}} \|y\|_{k'}$$

or there exists a $j \ge j_0$ with

$$\|y\|_n e^{-\sigma_{k'}\beta_{j+1}} \le C\|y\|_{k'} < \|y\|_n e^{-\sigma_{k'}\beta_j}\ (\searrow 0 \text{ for } j\to\infty)$$

and, since then for $m \le k'$

$$C\,\|y\|_m\, e^{-\sigma_m\beta_j} \le C\,\|y\|_{k'} < \|y\|_n\, e^{-\sigma_{k'}\beta_j},$$

the maximum is attained for some m with $k' < n < m \le n'$, i.e.

$$\|y\|_n \le C\,\|y\|_m e^{(\sigma_{k'}-\sigma_m)\beta_j} \le C\,\|y\|_m\, e^{-\sigma_{k'}\beta_{j+1}\frac{\sigma_m-\sigma_{k'}}{b\,\sigma_{k'}}} \le C\,\|y\|_m \Big(C\frac{\|y\|_{k'}}{\|y\|_n}\Big)^{\frac{\sigma_m-\sigma_{k'}}{b\,\sigma_{k'}}},$$

where $b := \sup_j \frac{\beta_{j+1}}{\beta_j} < \infty$. Thus

$$\|y\|_n^{1+d_m} \le C'_{n'}\, \|y\|_{k'}^{d_m}\, \|y\|_m \text{ with } d_m := \frac{\sigma_m-\sigma_{k'}}{b\,\sigma_{k'}} \text{ and } C'_{n'} := C^{1+\frac{\sigma_{n'}-\sigma_{k'}}{b\,\sigma_{k'}}}.$$

Hence in both cases

$$\|y\|_n^{1+d_m} \le C''_{n'}\, \|y\|_{k'}^{d_m}\, \|y\|_m, \text{ where } C''_{n'} := \max\Big\{(C\, e^{\sigma_{k'}\beta_{j_0}})^{\frac{\sigma_{n'}-\sigma_{k'}}{b\,\sigma_{k'}}}, C'_{n'}\Big\}.$$

For $n\in\mathbb{N}$ choose $\bar n \ge n$ such that $d_m := \frac{\rho_m^2-\sigma_{k'}}{b\,\sigma_{k'}} \ge \rho_m$ for all $m \ge \bar n$. By what we have just shown

$$\exists \bar n'\ \exists C''_{\bar n'} > 0\ \forall y\ \exists m\in[\bar n,\bar n'] : \|y\|_n^{1+\rho_m} \le \|y\|_{\bar n}^{1+\rho_m} \le C''_{\bar n'}\, \|y\|_{k'}^{\rho_m}\, \|y\|_m,$$

i.e. 2 is satisfied (with $n' := \bar n'$ and $C' := C''_{\bar n'}$). □

4.150 Theorem (See [**Vog83**, Satz 5.2 p.193])**.**
Let $\alpha = (\alpha_j)_{j\in\mathbb{N}}$ be a shift-stable sequence and E a Fréchet space with decreasing 0-nbhd basis $\{U_k : k\in\mathbb{N}\}$ and Minkowski-functionals $\|\_\|_{-k}$ of the polars U_k^o. Then

1. *$\mathcal{L}(E,\lambda^\infty_\infty(\alpha)) = \mathcal{LB}(E,\lambda^\infty_\infty(\alpha))$;*

$\Leftrightarrow$ 2. *E has the property (LB^∞), i.e.*

$\forall \rho \in \mathbb{R}_+^{\mathbb{N}},\ \rho \nearrow \infty\ \forall p\ \exists p'\ \forall n\ \exists n'\ \exists C\ \forall x^*\ \exists m \in [n,n'] : \|x^*\|_{-p'}^{1+\rho_m} \le C\,\|x^*\|_{-p}^{\rho_m}\,\|x^*\|_{-m}.$

($\boxed{1}\Leftarrow\boxed{2}$) is valid without the assumption on α and more generally for $\lambda_\infty^\infty(\alpha)$ replaced by $\lambda_\infty^\infty(a)$ with arbitrary $a \in \mathbb{R}_{\ge 1}^J$.

Similary as in $\boxed{4.132}$ we may assume that $p \le p' \le n$.
Obviously (use $\boxed{4.137.1}$ and $\boxed{4.132.2}$) one has: $(\overline{\overline{\Omega}}) \Rightarrow (\mathrm{LB}^\infty) \Rightarrow (\Omega)$.
In fact, recall:

$(\overline{\overline{\Omega}})$ $\qquad \forall p\ \exists p'\ \forall n\ \forall d>0\ \exists C>0 : \|\text{-}\|_{-p'}^{1+d} \le C\,\|\text{-}\|_{-p}^{d}\,\|\text{-}\|_{-n}.$

(Ω) $\qquad \forall p\ \exists p'\ \forall n\ \exists C>0\ \exists d>0 : \|\text{-}\|_{-p'}^{1+d} \le C\,\|\text{-}\|_{-p}^{d}\,\|\text{-}\|_{-n}.$

Proof.
($\boxed{1}\Leftarrow\boxed{2}$) We will verify condition $\boxed{4.147.2}$:

$$\forall k \in \mathbb{N}^{\mathbb{N}}\ \exists k'\ \forall n\ \exists n'\ \exists C>0\ \forall j\ \forall x^* : a_j^{(n)}\|x^*\|_{-k'} \le C \max_{m\le n'} a_j^{(m)}\|x^*\|_{-k_m},$$

where $a_j^{(n)} := e^{n\alpha_j}$. So let w.l.o.g. $k \nearrow \infty$ be given. The property (LB^∞) does not depend on the specific basis of seminorms of E so we may assume that it holds for the seminorms $|||\text{-}|||_n := \|\text{-}\|_{k_n}$, i.e.

$\forall \rho \in \mathbb{R}_+^{\mathbb{N}}\ \forall p\ \exists p'\ \forall \bar n\ \exists \bar n'\ \exists C \ge 1\ \forall x^*\ \exists m \in [\bar n, \bar n'] : |||x^*|||_{-p'}^{1+\rho_m} \le C\,|||x^*|||_{-p}^{\rho_m}\,|||x^*|||_{-m}.$

Now we choose $\rho \nearrow \infty$ such that $\lim_{m\to\infty} \frac{\rho_m}{m} = 0$ and take $p := 0$ and obtain a corresponding p'. To given $n \in \mathbb{N}$ we next choose $\bar n > n$ such that $n\,\rho_m \le m-n$ for all $m \ge \bar n$. For each $x^* \in E^*$ and j either

$$e^{n\alpha_j}|||x^*|||_{-p'} \le |||x^*|||_{-0} \le C\,e^{0\,\alpha_j}|||x^*|||_{-0}$$

or

$$\begin{aligned}|||x^*|||_{-p'}^{1+\rho_m} &\le C\,|||x^*|||_{-0}^{\rho_m}\,|||x^*|||_{-m} < C\left(e^{n\alpha_j}|||x^*|||_{-p'}\right)^{\rho_m}|||x^*|||_{-m}\\ &= C\,e^{n\alpha_j\rho_m}\,|||x^*|||_{-p'}^{\rho_m}\,|||x^*|||_{-m} \le C\,e^{(m-n)\alpha_j}\,|||x^*|||_{-p'}^{\rho_m}\,|||x^*|||_{-m}\end{aligned}$$

holds. Let $k' := k_{p'}$, $n' := \bar n'$ then we have in both cases $\boxed{4.147.2}$:

$$e^{n\alpha_j}\|x^*\|_{-k'} = e^{n\alpha_j}|||x^*|||_{-p'} \le C \max_{m\le \bar n'} e^{m\alpha_j}|||x^*|||_{-m} = C \max_{m\le n'} e^{m\alpha_j}\|x^*\|_{-k_m}.$$

($\boxed{1}\Rightarrow\boxed{2}$) Let $\rho \nearrow \infty$ and $p \in \mathbb{N}$.
By $\boxed{4.147.2}$ for $k : m \mapsto p+m$ and $a_j^{(n)} := e^{\rho_n\,\alpha_j}$ we get:

$$\exists p'>p\ \forall n\ \exists n'\ \exists C \ge 1\ \forall x^*\ \forall j : e^{\rho_n\alpha_j}\|x^*\|_{-p'} \le C \max_{m\le n'} e^{\rho_m\alpha_j}\|x^*\|_{-k_m}.$$

For fixed p' and $n > p'-p$ choose j_0 such that for all $j \ge j_0$

$$C < e^{(\rho_n-\rho_{n-1})\alpha_j}\ \big(\nearrow \infty \text{ for } j \to \infty\big)$$

holds. Then

$$e^{\rho_n\alpha_j}\|x^*\|_{-p'} \le C\,\max\Big\{e^{\rho_m\alpha_j}\,\|x^*\|_{-k_m} : m \in [0,p'-p] \cup [n,n']\Big\},$$

since for $p'-p < m \le n-1$

$$e^{\rho_n\alpha_j}\,\|x^*\|_{-p'} > C\,e^{\rho_{n-1}\alpha_j}\,\|x^*\|_{-p'} \ge C\,e^{\rho_m\alpha_j}\,\|x^*\|_{-(p+m)} = C\,e^{\rho_m\alpha_j}\,\|x^*\|_{-k_m}.$$

For $x^* \in E^*$, either

$$C\,\|x^*\|_{-k_0} = C\,\|x^*\|_{-p} \le e^{(\rho_n-\rho_{p'-p})\alpha_{j_0}}\|x^*\|_{-p'},$$

and, since then for $m \le p' - p$

$$C\, e^{\rho_m \alpha_{j_0}}\, \|x^*\|_{-k_m} \le C\, e^{\rho_{p'-p}\, \alpha_{j_0}}\, \|x^*\|_{-k_0} < e^{\rho_n\, \alpha_{j_0}}\, \|x^*\|_{-p'},$$

we get

$$\exists m \in [n, n'] :\ e^{\rho_n \alpha_{j_0}} \|x^*\|_{-p'} \le C\, e^{\rho_m \alpha_{j_0}}\, \|x^*\|_{-k_m},$$

and hence for any $d > 0$

$$\|x^*\|_{-p'}^{1+d} \le \left(C\, e^{(\rho_m - \rho_n)\alpha_{j_0}}\, \|x^*\|_{-k_m}\right) \|x^*\|_{-p'}^{d} \le C\, e^{(\rho_{n'} - \rho_n)\alpha_{j_0}}\, \|x^*\|_{-k_m}\, \|x^*\|_{-p}^{d}.$$

Otherwise, there exists a $j \ge j_0$ with

$$e^{(\rho_n - \rho_{p'-p})\alpha_{j-1}} \|x^*\|_{-p'} < C \|x^*\|_{-p} \le e^{(\rho_n - \rho_{p'-p})\alpha_j} \|x^*\|_{-p'}\ \left(\nearrow \infty \text{ for } j \to \infty\right)$$

and, since for $m \le p' = p$

$$C\, e^{\rho_m \alpha_j}\, \|x^*\|_{-k_m} \le C\, e^{\rho_{p'-p}\, \alpha_j}\, \|x^*\|_{-k_0} < e^{\rho_n\, \alpha_j}\, \|x^*\|_{-p'},$$

we also get

$$\begin{aligned}\exists m \in [n, n'] :\ \|x^*\|_{-p'} &\le C\, e^{(\rho_m - \rho_n)\alpha_j}\, \|x^*\|_{-k_m} \\ &\le C\, e^{(\rho_n - \rho_{p'-p})\alpha_{j-1}\, b\frac{\rho_m - \rho_n}{\rho_n - \rho_{p'-p}}}\, \|x^*\|_{-k_m} \\ &< C \left(C \frac{\|x^*\|_{-p}}{\|x^*\|_{-p'}}\right)^{d} \|x^*\|_{-k_m},\end{aligned}$$

where $d := b\frac{\rho_m - \rho_n}{\rho_n - \rho_{p'-p}}$ with $b := \sup_j \frac{\alpha_j}{\alpha_{j-1}}$, i.e.

$$\|x^*\|_{-p'}^{1+d} \le C^{1+d}\, \|x^*\|_{-k_m}\, \|x^*\|_{-p}^{d},$$

For given $\underline{n}$ we may now choose the n from above such that $n > \max\{p' - p, \underline{n}\}$ and $\frac{b}{\rho_n - \rho_{p'-p}} \le 1$ and thus $d \le \left(\frac{b}{\rho_n - \rho_{p'-p}}\right)\rho_m \le \rho_m$. Hence, in both cases we have for $C' := C\, \max\{e^{(\rho_{n'} - \rho_n)\alpha_{j_0}}, C^d\}$ and $\underline{n}' := n' + p$ the condition 2:

$$\exists p'\ \forall \underline{n}\ \exists \underline{n}'\ \exists C > 0\ \exists\, m \in [\underline{n}, \underline{n}'] : \|\text{-}\|_{-p'}^{1+\rho_m} \le C\, \|\text{-}\|_{-m}\, \|\text{-}\|_{-p}^{\rho_m}. \quad \square$$

4.151 Corollary (See [**Vog83**, 6.2. Satz p.198])**.**
Let E and F be Fréchet spaces. If E has property (LB^∞) and F property (DN), then

$$\mathcal{L}(E, F) = \mathcal{LB}(E, F).$$

Proof. By 4.131 there exists an $a : M \to \mathbb{R}_{\ge 1}$ such that F embeds as closed subspace into $\lambda^\infty_\infty(a)$. By 4.150 for $\lambda^\infty_\infty(a)$ we have

$$\mathcal{L}(E, \lambda^\infty_\infty(a)) = \mathcal{LB}(E, \lambda^\infty_\infty(a)).$$

Thus, by 4.141,

$$\mathcal{L}(E, F) = \mathcal{LB}(E, F)$$

as well. $\square$

An application of these results is:

Proposition [**BD98**, Thm. 18 p.23], [**BD01**, Theorem 3 p.497]**.**
Let F be (F). Then

1. *F is (DN);*

$\Leftrightarrow$ 2. *$C^\omega(U, F) = C^\omega_t(U, F)\ \forall(\exists \emptyset \ne) U \subseteq \mathbb{R}^n$ open.*

Proof. For sake of simplicity we consider only the case $U = \mathbb{R}$ treated in [**BD98**, Thm. 18 p.23]. By 4.142: 2 $\Leftrightarrow \mathcal{L}(C^\omega(\mathbb{R})^*, F) = \mathcal{LB}(C^\omega(\mathbb{R})^*, F)$.

($\Leftarrow$) By [**BD98**, Proposition 5 p.17] there exists a quotient map $q : C^\omega(\mathbb{R})^* \twoheadrightarrow H(\mathbb{D})$ (since $C^\omega_{\text{per}}(\mathbb{R}) \cong H(\overline{\mathbb{D}})$) thus $\mathcal{L}(H(\mathbb{D}), F) = \mathcal{LB}(H(\mathbb{D}), F)$.

($\Rightarrow$) Let $T \in \mathcal{L}(C^\omega(\mathbb{R})^*, F)$. Since $C^\omega(\mathbb{R})^* = \varinjlim_n E_n$ with $E_n \cong H(\mathbb{D})$ by [**BD98**, Proposition 3 p.16] and $H(\mathbb{D}) = \lambda_0(\text{id})$ by 1.15.6, there exists for every $n \in \mathbb{N}$ a 0-nbhd $U_n \subseteq E_n$ with $T(U_n)$ bounded by 4.146. By 4.90.1 there are $\delta_n > 0$ such that $\bigcup_n \delta_n T(U_n)$ is bounded. Thus T is bounded on the absolutely convex hull U_∞ (which is a 0-nbhd in $\varinjlim$) of $\bigcup_n \delta_n U_n$. □

Similarly, the following can be shown:

4.152 Proposition [**BD01**, Theorem 5 p.498] (See [**BD98**, Theorem 21 p.24]).
Let F be a complete (LB). Then

1. *F^* is $(\overline{\overline{\Omega}})$;*

$\Leftrightarrow$ 2. *$C^\omega(U, F) = C^\omega_t(U, F)$ $\forall(\exists \emptyset \neq)U \subseteq \mathbb{R}^n$ open.*

These results have been generalized to

4.153 Proposition [**HH03**, Theorem B p.286].
Let F be a Fréchet space having property (LB_∞) then $C^\omega(U, F) = C^\omega_t(U, F)$ for every open set U in a Fréchet space E.

4.154 Proposition [**HH03**, Theorem A p.286].
Let F be a Fréchet space.

1. *F is (DN);*

$\Leftrightarrow$ 2. *$C^\omega(U, F) = C^\omega_t(U, F)$ $\forall U \subseteq E$ open, where E is (F)+(N)+$(\tilde{\Omega})$;*

$\Leftrightarrow$ 3. *$C^\omega(U, F) = C^\omega_t(U, F)$ $\forall U \subseteq E$ open, where E is (F)+(S)+$(\tilde{\Omega})$ and has an absolute basis.*

A Fréchet space E is said to have property $(\tilde{\Omega})$ iff

$$\forall p\ \exists p'\ \exists d > 0\ \forall k\ \exists C > 0 : \|\text{-}\|^{1+d}_{-p'} \leq C \|\text{-}\|_{-k} \|\text{-}\|^d_{-p}.$$

This property has been used in [**DMV84**, Theorem 9 p.54] to characterize (NF) spaces in which not every bounded set is uniformly polar. One has the implications: $(\overline{\overline{\Omega}}) \Rightarrow (\tilde{\Omega}) \Rightarrow (\text{LB}_\infty) \Rightarrow (\Omega)$.

Another application is:

4.155 Proposition [**MV86**, 2.3 p.150] **and** [**MV86**, 3.4 p.157].
Let E be a Fréchet space. If every entire function $f : E \to \mathbb{C}$ is of uniformly bounded type (i.e. there is some 0-nbhd, where the function is bounded on each multiple) then E satisfies (LB^∞).

A nuclear Fréchet space E has $(\overline{\overline{\Omega}})$ iff every holomorphic functions on polycylindrical $U \subseteq E$ (i.e. finite intersection of sets of the form $\{x : |x^*(x)| < 1\}$ for $x^* \in E^*$) is of uniformly bounded type (i.e. is bounded on each q-bounded subset, which has positive q-distance to the complement, for some seminorm q for which U is open)

The subspaces and the quotients of s

Note: Quotient and subspaces of s via (N) and $\operatorname{Ext}^1 = 0$ ([Vog84, 2.4 p.362] and [Vog84, 2.3 p.361]) [Vog84, 2.5 p.363] Quotient and subspaces of s [MV92, 31 p.369],
nuclear-(DN) are the subspaces of s [MV92, 31.5 p.372],
nuclear-(Ω) are the quotients of s [MV92, 31.6 p.373],
nuclear-($DN \cap \Omega$) are the direct summand of s [MV92, 31.7 p.375]

4.156 Definition. Vector-valued sequence space s.
Let E be an lcs. Then

$$s(\mathbb{N}, F) := \Big\{x \in F^{\mathbb{N}} : \{(1+n)^k x_k : n \in \mathbb{N}\} \text{ is bounded in } F \text{ for each } k\Big\}.$$

Supplied with the norms $p_k(x) := \sup\{(1+n)^k p(x_n) : n \in \mathbb{N}\}$ for $k \in \mathbb{N}$ and seminorms p of F it is an lcs and Fréchet if F is Fréchet.

4.157 Proposition. Universal linearizer for s.
Let F be an lcs. Then $\mathcal{L}(s^*, F) \cong s(\mathbb{N}, F)$ via $T \mapsto (T(\mathrm{pr}_n))_{n\in\mathbb{N}}$.

Proof. Let $T \in \mathcal{L}(s^*, F)$ and $x_n := T(\mathrm{pr}_n)$. For $k \in \mathbb{N}$ and seminorms p of F we have

$$\begin{aligned} p_k\big((x_n)_{n\in\mathbb{N}}\big) &:= \sup\Big\{(1+n)^k p(x_n) : n \in \mathbb{N}\Big\} = \sup\Big\{p\Big(T\big((1+n)^k \,\mathrm{pr}_n\big)\Big) : n \in \mathbb{N}\Big\} \\ &\le \sup\Big\{p(T(x^*)) : x^* \in U_k^o\Big\}, \end{aligned}$$

since for the standard seminorms (given by $\|x\|_k := \sup_n (1+n)^k |x_n|$, see 1.15.4) on $s = c_0(A)$ the polar of the corresponding 0-nbhd U_k is by 1.24

$$U_k^o := \Big\{x^* \in s^* : \|x^*\|_{U_k^o} \le 1\Big\} = \Big\{y \in \mathbb{K}^{\mathbb{N}} : \sum_n |y_n|(1+n)^{-k} \le 1\Big\} \ni (1+n)^k \,\mathrm{pr}_n\,.$$

Thus $\mathcal{L}(s^*, F) \to s(\mathbb{N}, F)$, $T \mapsto (T(\mathrm{pr}_n))_{n\in\mathbb{N}}$, is welldefined, linear, and continuous.

It is bijective, since for $x = (x_n)_{n\in\mathbb{N}} \in s(\mathbb{N}, F)$ the only possible inverse image $T \in \mathcal{L}(s^*, F)$ is given by

$$\begin{aligned} &x^*\Big(: x \mapsto x^*(x) = x^*\Big(\sum_n \mathrm{pr}_n(x)\, e_n\Big) = \sum_n x^*(e_n)\ \mathrm{pr}_n(x)\Big) \mapsto \\ &\mapsto T(x^*) = T\Big(\sum_n x^*(e_n)\ \mathrm{pr}_n\Big) := \sum_n x^*(e_n)\, T(\mathrm{pr}_n) = \sum_n x^*(e_n)\, x_n. \end{aligned}$$

This definition for T makes sense, since any $x^* \in S^*$ is contained in some U_k^o, i.e. $\sum_n |x^*(e_n)|(1+n)^{-k} \le 1$ and $\{(1+n)^k x_n : n \in \mathbb{N}\}$ is bounded.
Moreover, the so defined T is continuous, since

$$\begin{aligned} (p \circ T)(x^*) = p\Big(\sum_n x^*(e_n)\, x_n\Big) &\le \sum_n \frac{|x^*(e_n)|}{(1+n)^k}\,(1+n)^k p(x_n) \\ &\le \sum_n \frac{|x^*(e_n)|}{(1+n)^k} \cdot \sup_n (1+n)^k p(x_n) \le \|x^*\|_{U_k^o}\, p_k(x). \end{aligned}$$

This shows at the same time, that the inverse $s(\mathbb{N}, F) \to \mathcal{L}(s^*, F)$, $(x_n)_{n\in\mathbb{N}} \mapsto T$, is continuous as well. □

4.158 $s(\mathbb{N}, s) \cong s$ (See [MV92, 31.1 p.369]).

Proof. Any $x = (x_n)_{n\in\mathbb{N}} \in s^{\mathbb{N}}$ is in $s(\mathbb{N}, s)$ iff $\forall k : \{(1+i)^k x_i : i \in \mathbb{N}\}$ is bounded in s, i.e. $\forall k\, \forall l : \{(1+j)^l(1+i)^k x_{i,j} : i, j \in \mathbb{N}\}$ is bounded in $\mathbb{K}$. Take the bijection $\mathbb{N} \cong \mathbb{N}\times\mathbb{N}$, $n \leftrightarrow (i, j)$ given by the usual diagonal procedure. Then n is smaller than the number $\frac{(m+1)(m+2)}{2}$ of lattice points in the triangle with vertices $(0, 0)$, $(m, 0)$, and $(0, m)$, where $m := i + j$. And on the other hand $i, j \le n$. Thus

$$(1+j)^l(1+i)^k \le (1+n)^{k+l}$$

$$\text{and } (1+n)^k \le \Big(\tfrac{(1+m)(2+m)}{2}\Big)^k \le (1+i+j)^{2k} \le (1+i)^{2k}(1+j)^{2k}.$$

So the seminorms of $s(\mathbb{N}, s)$ and s can be dominated by each other under this bijection. □

4.159 $s \to s \to s^{\mathbb{N}}$ (See [**MV92**, 31.3 p.370])**.**
There is a short exact sequence

$$0 \to s \hookrightarrow s \twoheadrightarrow s^{\mathbb{N}} \to 0.$$

Proof. By $\boxed{4.78}$ (see the proof of $\boxed{4.131}$) we have the short exact sequence $0 \to s \hookrightarrow s \overset{Q}{\twoheadrightarrow} \mathbb{K}^{\mathbb{N}} \to 0$. By $\boxed{4.99}$ the dual sequence $0 \to \mathbb{K}^{(\mathbb{N})} \overset{Q^*}{\to} s^* \to s^* \to 0$ is topologically exact and by $\boxed{4.107}$ the functor $\mathcal{L}(\_, s)$ is left exact. So we obtain

$$\begin{tikzcd}
0 \arrow[r] & s\hat{\otimes}s \arrow[r, tail] \arrow[d, equal, "\boxed{4.61}"] & s\hat{\otimes}s \arrow[r, two heads, "Q\otimes s"] \arrow[d, equal] & \mathbb{K}^{\mathbb{N}}\hat{\otimes}s \arrow[r] \arrow[d, equal, "\boxed{4.61}"] & 0 \\
0 \arrow[r] & \mathcal{L}(s^*, s) \arrow[r, tail] \arrow[d, equal, "\boxed{4.157}"] & \mathcal{L}(s^*, s) \arrow[r, "Q^{**}"] \arrow[d, equal] & \mathcal{L}(\mathbb{K}^{(\mathbb{N})}, s) \arrow[d, equal] & \\
0 \arrow[r] & s(\mathbb{N}, s) \arrow[r, tail] \arrow[d, equal, "\boxed{4.158}"] & s(\mathbb{N}, s) \arrow[r] \arrow[d, equal] & \mathcal{L}(\mathbb{K}, s)^{\mathbb{N}} \arrow[d, equal] & \\
0 \arrow[r] & s \arrow[r, tail, dotted] & s \arrow[r, two heads, dotted] & s^{\mathbb{N}} \arrow[r] & 0
\end{tikzcd}$$

In order to see that these isomorphic sequences are short exact we use that any $z \in \mathbb{K}^{\mathbb{N}}\hat{\otimes}_\pi s$ can be represented by $\boxed{3.40}$ as $z = \sum_n \lambda_n x_n \otimes y_n$ with $\lambda \in \ell^1$, $\{x_n : n \in \mathbb{N}\}$ bounded in $\mathbb{K}^{\mathbb{N}}$ and $\{y_n : n \in \mathbb{N}\}$ bounded in s. Since $\mathbb{K}^{\mathbb{N}}$ is (FM) we find a set $\{\tilde{x}_n : n \in \mathbb{N}\}$ bounded in s with $Q(\tilde{x}_n) = x_n$. Then $\tilde{z} := \sum_n \lambda_n\, \tilde{x}_n \otimes y_n \in s\hat{\otimes}_\pi s$ with $(Q\hat{\otimes}s)(\tilde{z}) = z$. Since all these tensor products are Fréchet, the top row is a topologically exact sequence and hence also the bottom row. □

4.160 Characterizing the subspaces of s (See [**MV92**, 31.5 p.372])**.**
$\exists\iota : E \hookrightarrow s \Leftrightarrow E$ *is (N)+(F)+(DN).*

Proof. ($\Rightarrow$) By $\boxed{1.15.4}$ $s \cong \lambda_\infty(\alpha)$ with $\alpha(n) := \ln(n+1)$, by $\boxed{3.78.1}$ and $\boxed{4.125}$ $\lambda_\infty(\alpha)$ is (N) and (DN), and by $\boxed{3.73.2}$ and $\boxed{3.14}$ E is (N) and (DN).

($\Leftarrow$) By $\boxed{4.159}$ there is an exact sequence $0 \to s \to s \to s^{\mathbb{N}} \to 0$ and by $\boxed{3.81}$ there is an embedding $E \hookrightarrow s^{\mathbb{N}}$. So the pullback gives another short exact sequence (where $\alpha(n) := \ln(n+1)$)

$$\begin{tikzcd}
s \arrow[r, hook] \arrow[d, equal, "\boxed{1.15.4}"'] & s \arrow[r, two heads] & s^{\mathbb{N}} \\
\lambda^\infty_\infty(\alpha) \arrow[r, tail, dotted] & s \times_{s^{\mathbb{N}}} E \arrow[r, two heads] \arrow[u, hook] & E \arrow[u, hook] \arrow[l, dotted, bend right]
\end{tikzcd}$$

which splits by $\boxed{4.127}$. Thus $E \hookrightarrow s \times_{s^{\mathbb{N}}} E \hookrightarrow s$. □

4.161 Characterizing the quotients of s (See [**MV92**, 31.6 p.373])**.**
$\exists \pi : s \twoheadrightarrow E \Leftrightarrow E$ *is (N)+(F)+(Ω).*

Proof.
($\Rightarrow$) $s \cong \lambda_\infty(\alpha)$ has (N) and (Ω) by 3.78.1 and 4.133.4. Thus E has (N) and (Ω) by 3.73.4 and 4.133.2.

($\Leftarrow$) By 3.81 there is an embedding $E \hookrightarrow s^{\mathbb{N}}$. Then $Q := s^{\mathbb{N}}/E$ is (NF), and thus there exists a short exact sequence $0 \to s \xrightarrow{j_2} \tilde{Q} \xrightarrow{p_2} Q \to 0$ as in the proof of 4.160 with $\tilde{Q} \hookrightarrow s$ and hence $\tilde{Q}$ has (DN) by 3.14.2.
Let $H := \{(x, y) \in s^{\mathbb{N}} \times \tilde{Q} : p_1(x) = p_2(x)\}$ be the pullback. Then the diagram on the right side has exact rows and columns and by 4.138 $H \cong E \times \tilde{Q}$ since E is (Ω).

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & E \arrow[r, hook] & s^{\mathbb{N}} \arrow[r, two heads] \arrow[u] & Q \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & E \arrow[r] \arrow[u, equal] & H \arrow[r] \arrow[u] & \tilde{Q} \arrow[r] \arrow[u] & 0 \\
 & & s \arrow[r, equal] \arrow[u] & s \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Take the left column as top row and proceed analogously with the sequence from 4.159 as right column to obtain another diagram with exact rows and columns. Again by 4.138 (or by 4.127) $G \cong s \times s$.
Thus we have quotient mappings

$$s \cong s \times s \cong G \twoheadrightarrow H \cong E \times \tilde{Q} \twoheadrightarrow E.$$

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & s \arrow[r] & H \arrow[r] \arrow[u] & s^{\mathbb{N}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & s \arrow[r] \arrow[u, equal] & G \arrow[r] \arrow[u] & s \arrow[r] \arrow[u] & 0 \\
 & & s \arrow[r, equal] \arrow[u] & s \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$
$\square$

4.162 Characterizing the complemented subspaces of s (See [**MV92**, 31.7 p.375])**.**
$\exists \iota : E \overset{\oplus}{\hookrightarrow} s \Leftrightarrow E$ *(N)+(F)+(DN)+(Ω).*

Here $\overset{\oplus}{\hookrightarrow}$ denotes an embedding as direct summand (i.e. having a left inverse).

Proof.
($\Rightarrow$) follows from 4.160 and 4.161.

($\Leftarrow$) Proceed as in the proof of 4.161, where $H \cong E \times \tilde{Q} \hookrightarrow s \times s \cong s$, hence is (DN). By 4.138 not only the bottom row but also the left column in the second diagram split, i.e. $s \cong s \times s \cong G \cong H \times s \cong E \times \tilde{Q} \times s$. Hence $E \overset{\oplus}{\hookrightarrow} s$. $\square$

4.163 $s \overset{\oplus}{\hookrightarrow} E \overset{\oplus}{\hookrightarrow} s \Rightarrow E \cong s$ (See [**MV92**, 31.2 p.370])**.**

Proof.

$$\exists E_0 : E \cong E_0 \times s \text{ and } \exists E_1 : s \cong E \times E_1 \Rightarrow$$
$$\Rightarrow s \cong E \times E_1 \cong E_0 \times s \times E_1 \cong E_0 \times E_2 \text{ with } E_2 := s \times E_1$$
$$\xRightarrow{4.158} s \cong s(\mathbb{N}, s) \cong s(\mathbb{N}, E_0) \times s(\mathbb{N}, E_2)$$
$$s(\mathbb{N}, E_0) \cong E_0 \times s(\mathbb{N}, E_0) \Rightarrow$$
$$\Rightarrow s \cong s(\mathbb{N}, E_0) \times s(\mathbb{N}, E_2) \cong E_0 \times s(\mathbb{N}, E_0) \times s(\mathbb{N}, E_2) \cong E_0 \times s \cong E \quad \square$$

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