Exercises for Analysis on Manifolds

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1. Surfaces of any genus.
Let $f : \mathbb{R} \to \mathbb{R}$ be C^∞. Under which conditions on $\varepsilon > 0$ describes the equation

$$(f(x) + y^2)^2 - \varepsilon\,(f(x) + y^2) + z^2 = 0$$

a manifold? Furthermore show: If f is a polynomial with $2g$ simple zeros and positive highest coefficient, then, for appropriately chosen ε, this manifold is an oriented surface of genus g.
Hint: Consider the intersection curves with the planes parallel to the y-z plane for $f(x) < 0$ and for $f(x) > 0$.

2. Quadrics.
Let $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be bilinear, symmetric and $a \in L(\mathbb{R}^n, \mathbb{R})$. Find sufficient conditions under which the quadric $M := \{x \in \mathbb{R}^n : b(x, x) + a(x) = 1\}$ is a manifold of dimension $n - 1$. Identify paraboloid, hyperboloid and ellipsoid as special casees.

3. Conformal linear mappings.
Show that a bijective linear map $f : E \to E$ on an Euclidean space E is conformal (i.e. angle preserving) if and only if a $\lambda > 0$ exists, such that $\langle f(x)|f(y)\rangle = \lambda\langle x|y\rangle$ holds for all $x, y \in E$, i.e. $\frac{1}{\sqrt{\lambda}} f$ is an isometry.
Hint: ($\Rightarrow$) For $v \in E$, define $\lambda(v) > 0$ by $\|f(v)\|^2 = \lambda(v)\,\|v\|^2$. Let $(e_1, \dots, e_n)$ be an orthonormal basis. Then $e_i + e_j \perp e_i - e_j$ and thus also for their images under f. Deduce that $\lambda(e_i) = \lambda(e_j)$ and furthermore that λ is constant. Finally use the polarization equation to obtain the desired identity.

4. Conformity of the stereographic projection.
Show that the stereographic projection $S^n \to \mathbb{R}^n$ angle preserving, i.e. its derivative at each point is conformal.
Hint: Show that its inverse mapping $h : \mathbb{R}^n \to S^n$ is conformal.

5. The image of hyperspheres under the stereographic projection.
Show that the stereographic projection $S^n \to \mathbb{R}^n$ maps all $(n-1)$-spheres to $(n-1)$-spheres or hyperplanes of the $\mathbb{R}^n$.
Hint: The equation of an $(n-1)$ sphere or hyperplane is

$$\alpha(x_1^2 + \dots + x_n^2) + \beta_1 x_1 + \dots + \beta_n x_n + \gamma = 0$$

with $4\alpha\gamma < \beta_1^2 + \dots + \beta_n^2$ and the stereographic projection maps $(y_1, \dots, y_{n+1})$ to $(x_1, \dots, x_n)$ with $x_i = \frac{y_i}{1-y_{i+1}}$.

6. Quaternions.
Show: The set $\left\{\left(\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right) : a, b \in \mathbb{C}\right\}$ is a subring of the the ring of complex 2×2-matrices and even a skew field (i.e. division ring). If one identifies $\mathbb{C}^2$ with these ring, by virtue of the linear mapping $(a, b) \mapsto \left(\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right)$, then $\mathbb{C}^2$ becomes also a skew field $\mathbb{H}$, whose elements are called quaternions. The square of the norm of (a, b) is the determinant of $\left(\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right)$. Thus $|(a_1, b_1) \cdot (a_2, b_2)| = |(a_1, b_1)| \cdot |(a_2, b_2)|$ holds and the set $S^3 \subseteq \mathbb{H}$ of the unitary quaternions is a subgroup of $\mathbb{H}$. If one identifies $\mathbb{C}^2$ with $\mathbb{R} \times \mathbb{R}^3$, then multiplication takes the following form: $(t, x) \cdot (s, y) = (ts - \langle x, y\rangle, ty + sx - x \times y)$ for $(t, x), (s, y) \in \mathbb{R} \times \mathbb{R}^3$.

Show furthermore that: $(\forall x \in \{0\} \times \mathbb{R}^3 : xy = yx) \Rightarrow y \in \mathbb{R} \times \{0\}$ and $(\forall x \in \mathbb{H} : xy = zx) \Rightarrow y = z \in \mathbb{R} \times \{0\}$. By differentiating the equation $xx^{-1} = 1$, calculate the derivative of the map $\mathrm{inv} : x \mapsto x^{-1}$.

7. Smoothness of the mapping "taking the image".
Show that $T \mapsto \mathrm{im}(T)$, $L_r(m.n) \to G(r,n)$ is C^∞.
Hint: Describe this mapping locally as composition

$$L_r(m,n) \to L_r(r,n) \to V(r,n) \to G(r,n)$$

where the first (local!) mapping is given by restricting to a suitably chosen r-dimensional subspace, the second one is Gram-Schmidt orthonormalization of the columns of the matrix, and the last one is "taking the image" treated in the lectures.

8. Smoothness of the mapping "taking the kernel".
Show that $T \mapsto \ker(T)$, $L_r(m,n) \to G(m-r,m)$ is C^∞.
Hint: $\operatorname{Ker} T = (\operatorname{im} T^t)^\perp$.

9. Möbius strip, part 1.
Let $M := [-1,1] \times (-1,1)/\sim$, where $\sim$ is the equivalence relation generated by $\forall s : (-1,-s) \sim (1,s)$, and $q : [-1,1] \times (-1,1) \to M$, the quotient map $(t,s) \mapsto [(t,s)]$.
Furthermore, let $\bar\varphi_0, \bar\varphi_1 : (-1,1) \times (-1,1) \to \mathbb{R}^2$ be given by

$$\bar\varphi_0(t,s) := (t,s) \text{ and } \bar\varphi_1(t,s) := \begin{cases} (t+1,s) & \text{for } t < 0 \\ (t-1,-s) & \text{for } t \geq 0 \end{cases}$$

and $\varphi_i := q \circ \bar\varphi_i$. Then $\bar\varphi_1$ exchanges the left rectangle $(1,0) \times (-1,1)$ with the right one $[0,1) \times (-1,1)$ and mirrors them vertically. Show that $\{\varphi_0, \varphi_1\}$ is a C^∞-atlas for M.

10. Möbius strip, part 2.
Show that the map

$$f : \mathbb{R}^2 \to \mathbb{R}^3, \quad (t,s) \mapsto \Big(\big(1 + s\cos(\tfrac{\pi}{2}t)\big)\cos(\pi t), \big(1 + s\cos(\tfrac{\pi}{2}t)\big)\sin(\pi t), s\sin(\tfrac{\pi}{2}t)\Big)$$

induces a diffeomorphism $\tilde f : [(t,s)] \mapsto f(t,s)$ of M from example 9 with the submanifold Möb:$= f(\mathbb{R} \times (-1,1)) \subseteq \mathbb{R}^3$.

Hint: Use f is a local parameterization of Möb and that $f(x) = f(y) \Leftrightarrow x \sim y$ for all $x, y \in [-1,1] \times (-1,1)$.

11. Tangent space of the space of mappings of rank k.
Determine the tangent space of the manifold $L_k(n,m)$ at

$$f : (x,y) \mapsto (x,0), \quad \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k \times \mathbb{R}^{m-k}.$$

12. Tangent space of the Grassmann manifold.
Determine the tangent space of the Grassmann manifold $G(k,n)$ at the point $P : \mathbb{R}^k \hookrightarrow \mathbb{R}^n$, $x \mapsto (x,0)$.

13. Tangent space of the Stiefel manifold.
Determine the tangent space of the Stiefel manifold $V(k,n)$ at the point $A : \mathbb{R}^k \hookrightarrow \mathbb{R}^n$, $x \mapsto (x,0)$.

14. Normal space of a surface.
Show that the normal space $(T_pM)^\perp$ for each 2-dimensional manifold $M \subseteq \mathbb{R}^3$ is generated by the gradient $\mathrm{grad}_p f$ of a regular equation and is also generated by the cross product $\partial_1\varphi(0) \times \partial_2\varphi(0)$ for a parameterization φ with $\varphi(0) = p$.

15. Chain Rule.
Prove lemma [1, 10.4] for abstract manifolds.

Hint: In order to determine T_pf, evaluate this expression at $\partial \in \mathrm{Der}_p(C^\infty(M,\mathbb{R}),\mathbb{R})$ and the result on $h \in C^\infty(M,\mathbb{R})$, i.e. consider $(T_pf)(\partial)(h)$. For the product rule use the isomorphism $\mathrm{Der}_p(C^\infty(\mathbb{R},\mathbb{R}),\mathbb{R}) \cong \mathbb{R}$, $\partial \mapsto \partial(\mathrm{id})$.

16. Embedding of the projective space.
Show that the space $\mathbb{P}^n$ of the straight line in the $\mathbb{R}^{n+1}$ can be embedded into $\mathbb{R}^{2n}$.

Hint: Let $h : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{2n+1}$ be given by

$$(x_0,\dots,x_n;y_0,\dots,y_n) \mapsto \Big(\sum_{i+j=0;i,j\le n}^{k} x_i y_j\Big)_{k=0}^{2n} =$$
$$= \Big(x_0y_0, x_0y_1 + x_1y_0, \dots, \sum_{i=0}^{n} x_i y_{n-i}, \dots, x_{n-1}y_n + x_n y_{n-1}, x_n y_n\Big)$$

and let $g : S^n \to S^{2n}$ be given by $g(x) = \frac{h(x,x)}{|h(x,x)|}$. Then $g(x_1) = g(x_2) \Leftrightarrow x_1 = \pm x_2$ holds (if $h(x,x) = \lambda^2 h(y,y)$, then $h(x+\lambda y, x-\lambda y) = 0$ and therefore $x + \lambda y = 0$ or $x - \lambda y = 0$) and thus provides an injective mapping $\mathbb{P}^n \to \{(z_0,\dots,z_{2n}) \in S^{2n} : z_0 \ge 0\}$.

17. Universal vector bundle.
Show that $E(k,n) := \{(\varepsilon, v) \in G(k,n)\times\mathbb{R}^n : v \in \varepsilon\} \to G(k,n)$, $(\varepsilon,v)\mapsto\varepsilon$ is a (the so-called universal) vector bundle over the Grassmann manifold. Its fiber over a point in $G(k,n)$, i.e. over a k-plane ε in $\mathbb{R}^n$, is just this plane.

Hint: In order to recognize $E(k,n)$ as a vector subbundle of $G(k,n)\times\mathbb{R}^n$ (and thus especially as a manifold), consider the locally defined mapping $\varphi : G(k,n) \to GL(n)$,

$$\varphi : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & 0 \\ C & 1 \end{pmatrix}.$$

Show that $\varphi(\varepsilon)(\mathbb{R}^k \times \{0\}) = \varepsilon$ and thus $(\varepsilon, v) \mapsto (\varepsilon, \varphi(\varepsilon)\cdot v)$ is a local diffeomorphism of $G(k,n)\times\mathbb{R}^n$, which locally maps the subspace $G(k,n)\times\mathbb{R}^k\times\{0\}$ to $E(k,n)$.

18. Universality of $E(k,s) \to G(k,s)$.
It is $p : E \to M$ a k-plane bundle and $f : E \to M \times \mathbb{R}^s$ a VB monomorphism over id_M. Show that E is isomorphic to the pullback bundle $g^*(E(k,s))$, where g is the classifying map described in [1, 27.23].
Hint: Using [1, 27.11], show that the natural map $E \to M \times_{G(k,s)} E(k,s)$ is a VB isomorphism

19. Smooth normality.
Let M be a paracompact Hausdorff manifold and let $A_0, A_1 \subseteq M$ be closed and disjoint. Show the existence of a smooth function $f : M \to \mathbb{R}$ with $f|_{A_i} = i$ for $i \in \{0,1\}$.
Hint: Consider the partition of 1, which is subordinated to the covering $\{M \setminus A_0, M \setminus A_1\}$.

20. Denseness of smooth functions.
Let M be a paracompact Hausdorff manifold, $g : M \to \mathbb{R}$ and $\varepsilon : M \to (0,+\infty)$ continuous. Show the existence of a smooth function $h : M \to \mathbb{R}$ with $|h(x) - g(x)| < \varepsilon(x)$ for all $x \in M$.
Hint: Use a partition $\mathcal{F}$ of unity, which is subordinated to the covering with the sets $U_x := \{y : |g(y) - g(x)| < \varepsilon(y)\}$ for $x \in M$ and put $h(x) := \sum_{f\in\mathcal{F}} f(x)\, g(x_f)$, where $\mathrm{Trg}(f) \subseteq U_{x_f}$.

21. Special indexing of partitions of unity.
Show that the partition $\mathcal{F}$ of unity subordinated to a covering $\mathcal{U}$ can be chosen in such a way, that $\mathcal{F} = \{f_U : U \in \mathcal{U}\}$ with $\mathrm{Trg}(f_U) \subseteq U$ for all $U \in \mathcal{U}$.

Hint: Let $\mathcal{F}$ be any subordinated partition of unity, i.e. for each $f \in \mathcal{F}$ exists a $U \in \mathcal{U}$ with $\operatorname{Trg}(f) \subseteq U$. Choose such a U_f for each $f \in \mathcal{F}$ and define a new partition of unity by $f_U := \sum_{f \in \mathcal{F}: U_f = U} f$.

22. A non integrables subbundle.
Show directly that the subvector bundle of $T\mathbb{R}^3$ defined in [1, 18.3.3] is not integrable.
Hint: Determine the Lie-backet of the two generating vector fields.

In the following Exercises 23-28 we consider $S^2 \subseteq \mathbb{R}^3$ as a Riemann manifold with the metric inherited from $\mathbb{R}^3$. We use as charts off the poles

- the spherical coordinates $(\theta, \varphi) \mapsto (\cos\theta \cos\varphi, \cos\theta \sin\varphi, \sin\theta)$ from [1, 3.4]
- and the stereographic coordinates $(t, s) \mapsto \frac{1}{t^2+s^2+1}(2t, 2s, t^2 + s^2 - 1)$ from [1, 3.5].

23. Restricting a vector field to S^2.
Consider the velocity field $(x, y, z) \mapsto (-y, x, 0)$ on $\mathbb{R}^3$, which corresponds to the rotation around the z axis. Express the restriction ξ of this vector field to S^2 in the two coordinates mentioned above.

Furthermore, do the analoguous calculation for the vector field $\eta : (x, y, z) \mapsto (xz, yz, -x^2 - y^2)$.

24. Riemann metrics on S^2.
Describe the Riemann metric of S^2 as a 2-fold contravariant tensor field in the coordinates from above.

25. Associated 1-form on S^2.
Describe the 1-forms associated via $\sharp$ to the vector fields from Exercise 23 in the coordinates from above.

26. Volume form on S^2.
Describe the volume form of S^2 in the coordinates from above.

27. $\wedge$ product on S^2.
Find the $\wedge$ product of the 1-forms from Exercise 25 and compare it to the volume form from Exercise 26.

28. Pullback of 1-forms along a curve in S^2.
Determine the pullback of the 1-forms from Exercise 25 along the mapping $f : \mathbb{R} \to S^2$, $t \mapsto (\sin t, \frac{3}{5}\cos t, \frac{4}{5}\cos t)$.

29. 1-forms of related vector fields.
Let $f : M \to \tilde{M}$ be a smooth mapping between manifolds using Riemann metrics g and $\tilde{g}$. Continue to be $\xi \in \mathfrak{X}(M)$ and $\tilde{\xi} \in \mathfrak{X}(\tilde{M})$. Show: If ξ is f-related to $\tilde{\xi}$ and $g = f^*\tilde{g}$, so is $\sharp\xi = f^*(\sharp\tilde{\xi})$.

30. Hodge star operator.
Let E be an oriented m-dimensional Euclidean vector space. Then $\dim\left(\bigwedge^k E\right) = \binom{m}{k}$ and thus $\bigwedge^k E \cong \bigwedge^{m-k} E$. We want to describe an isomorphism $* : \bigwedge^k E \to \bigwedge^{m-k} E$, which does not depend on the choice of a basis. This is called the Hodge star operator and is given by the following implicit equation:

$$\eta \wedge *\omega = \langle \eta, \omega \rangle \cdot \det \text{ for } \eta, \omega \in \bigwedge^k E.$$

Where the inner product on $\bigwedge^k$ is defined by $e^{i_1} \wedge \cdots \wedge e^{i_k}$ to be an orthonormal base if (e^i) is one of E. Show that this really uniquely determines a linear operator.

To do so, calculate the coefficients of $*(e^{i_1} \wedge \cdots \wedge e^{i_k})$ relative to the associated base of $\bigwedge^{n-k} E$.

31. Inverse of Hodge star operator.
Show that the Hodge star operator is an isometry that satisfies $*\circ* = (-1)^{k(m-k)} : \bigwedge^k E \to \bigwedge^{m-k} E \to \bigwedge^k E$.

32. Hodge star operator for Riemann manifolds.
For an oriented Riemann manifold (M, g) of dimension m, we define the Hodge star operator $* : \Omega^k(M) \to \Omega^{m-k}(M)$ by $(*\omega)(x) := *(\omega(x))$. Show that $* : C^\infty(M, \mathbb{R}) = \Omega^0(M) \to \Omega^m(M)$ is given by $f \mapsto f \cdot \mathrm{vol}$ and $\mathfrak{X}(M) \cong \Omega^1(M) \to \Omega^{m-1}(M)$ by $\xi \mapsto i_\xi \mathrm{vol}$.

33. Inner product on the dual space.
Let E be a finite dimensional Euclidean vector space and E^* its dual space. Define the canonical inner product on E^* by $\langle v, w\rangle := \langle \flat v, \flat w\rangle$ for all $v, w \in E^*$. Show that the dual base of each orthonormal base of E is an orthonormal basis and $(\langle g^i, g^j\rangle)_{i,j}$ is the inverse matrix to $(\langle g_i, g_j\rangle)_{i,j}$ for each base $(g_i)_i$ of E with dual base g^i.

34. Divergence of vector fields.
The divergence of a vector field $\xi \in \mathfrak{X}(M)$ is defined by

$$\operatorname{div} \xi := *\Big(d\big(\iota_\xi \mathrm{vol}_M\big)\Big) \overset{32}{=\!=} (*\circ d \circ * \circ \sharp)(\xi) \in C^\infty(M, \mathbb{R}).$$

Show that $\operatorname{div} \xi \cdot \mathrm{vol}_M = \mathcal{L}_\xi \mathrm{vol}_M$ holds and determine the local formula for $\operatorname{div} \xi$.

Hint: For the latter, use the local formula from Exercise 32:

$$\iota_\xi \mathrm{vol}_M = \sqrt{G} \sum_j (-1)^{j-1} \xi^j du^1 \wedge \ldots \wedge \widehat{du^j} \wedge \ldots \wedge du^m$$

35. Inverse images and intersections of submanifolds.
Prove [1, 27.9] using [1, 27.8].

Hint: $g : X \to Y$ is for $SS1$.

36. Poincaré Lemma.
Let ω be a closed k-form on an open and (with respect to 0) star-shaped set $U \subseteq \mathbb{R}^n$. Determine an explicit formula for a solution η of $d\eta = \omega$.

Hint: After proof of the homotopyi axiom is $\eta = I_0^1(i_\xi(H^*(\omega))$ where $H : U \times \mathbb{R} \to U$ is the homotopy $(x, t) \mapsto tx$ and $\xi = \frac{\partial}{\partial t}$.

37. Volume element of S^n.
Use [1, 28.10] to identify the volume element of S^n as

$$\mathrm{vol}_{S^n} = \iota^*\Big(\sum_k (-1)^k x^k\, dx^0 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^n\Big)$$

. Express this term for $n = 2$ in spherical coordinates and determine the surface area $\int_{S^2} \mathrm{vol}_{S^2}$.

38. Decomposition of volume forms on a product.
Let M and N be two oriented manifolds of dimension m and n. For $\omega \in \Omega_c^m(M)$ and $\eta \in \Omega_c^n(N)$ put $\omega \wedge \eta := \mathrm{pr}_1^*(\omega) \wedge \mathrm{pr}_2^*(\eta) \in \Omega_c^{m+n}(M \times N)$. For $f \in C^\infty(M \times N, \mathbb{R})$ define $g \in C^\infty(M, \mathbb{R})$ by $g(x) := \int_N f(x, \_)\, \eta$. Show that:

$$\int_{M\times N} f \cdot \omega \wedge \eta = \int_M g \cdot \omega$$

Each $m+n$-form on $M\times N$ can be written as $f\cdot\omega\wedge\eta$ with appropriate $f\in C^\infty(M\times N,\mathbb{R})$, $\omega\in\Omega^m(M)$, and $\eta\in\Omega^n(N)$.

39. Cohomology with compact carriers of cylinders.
Determine $H^k_c(S^j\times\mathbb{R}^n)$ by induction to j using the Mayer-Vietoris sequence for compact carriers.

40. Five-Lemma.
Show that in the proof of [1, 29.22] all squares (needed for the application of the Five-Lemma) commute with the exception of

$$\begin{array}{ccc} H^{k-1}(U\cap V) & \longrightarrow & H^k(U\cup V) \\ \downarrow & & \downarrow \\ H^{l+1}_c(U\cap V)^* & \longrightarrow & H^l_c(U\cup V)^* \end{array}$$

which only commutes up to a sign.
Hint: In the proof of [1, 26.3.4], $\varphi_U := h_V\varphi$ and $\varphi_V := -h_U\varphi$ is the correct definition.

Literatur

[1] A. Kriegl. *Analysis on Manifolds.* Univ. Wien, SS 2018.