Riemannian Geometry

Andreas Kriegl
email:andreas.kriegl@univie.ac.at

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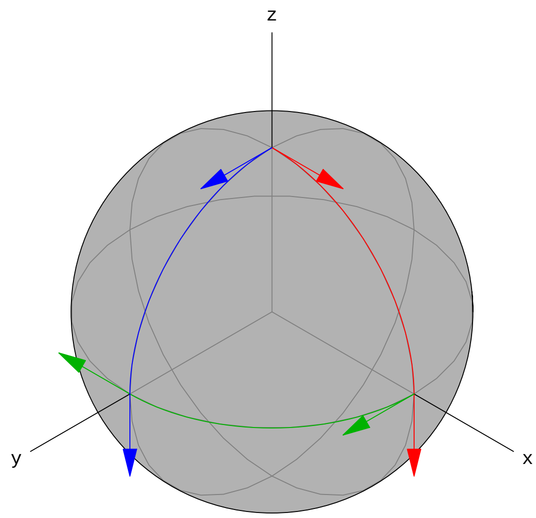


This is preliminary english version of the script for my homonymous lecture course in the Winter Semester 2018. It was translated from the german original using a pre and post processor (written by myself) for google translate. Due to the limitations of google translate – see the following article by Douglas Hofstadter www.theatlantic.com/.../551570 – heavy corrections by hand had to be done afterwards. However, it is still a rather rough translation which I will try to improve during the semester.

It consists of selected parts of the much more comprehensive differential geometry script (in german), which is also available as a PDF file under
http://www.mat.univie.ac.at/∼kriegl/Skripten/diffgeom.pdf.

In choosing the content, I followed the curricula, so the following topics should be considered:

- Levi-Civita connection
- Geodesics
- Completeness
- Hopf-Rhinov theoroem
- Selected further topics from Riemannian geometry.

Prerequisite is the lecture course 'Analysis on Manifolds'.

The structure of the script is thus the following:

Chapter I deals with isometries and conformal mappings as well as Riemann surfaces - i.e. 2-dimensional Riemannian manifolds - and their relation to complex analysis.

In Chapter II we look again at differential forms in the context of Riemannian manifolds, in particular the gradient, divergence, the Hodge star operator, and most importantly, the Laplace Beltrami operator. As a possible first application, a section on classical mechanics is included.

In Chapter III we first develop the concept of curvature for plane curves and space curves, then for hypersurfaces, and finally for Riemannian manifolds. Of course, we will also treat geodesics, parallel transport and the covariant derivative.

During the semester, I will post a detailed list of the treated sections in
http://www.mat.univie.ac.at/∼kriegl/LVA-2018-WS.html.

Of course, the attentive reader will be able to find (typing) errors. I kindly ask to let me know about them (consider the german saying: shared suffering is half the suffering). Future generations of students might appreciate it.

Andreas Kriegl, Vienna in July 2018

Contents

I. Conformal structures and Riemannian surfaces

1. Conformal mappings

A RIEMANN METRIC on a smooth manifold M is a 2-fold covariant tensor field

$$g \in \mathfrak{T}^2_0(M) = C^\infty(M \leftarrow T^*M \otimes T^*M) \cong L^2_{C^\infty(M,\mathbb{R})}(\mathfrak{X}(M), \mathfrak{X}(M); C^\infty(M,\mathbb{R})),$$

which is pointwise a positive-definite symmetric bilinear form, see [**95**, 24.1] and [**95**, 20.1]. It has therefore a representation with respect to local coordinates (u^i) of the following form:

$$g = \sum_{i,j} g_{i,j}\, du^i \otimes du^j.$$

A RIEMANNIAN MANIFOLD (M,g) is a smooth manifold M which is provided with an distinguished Riemann metric g, see [**95**, 18.11]. On Riemannian manifolds (M,g) we can define the LENGTH OF TANGENTIAL VECTORS $\xi_x \in T_xM$ as $|\xi_x| := \sqrt{g_x(\xi_x,\xi_x)}$ and, in analogy to [**82**, 6.5.12], the LENGTH OF SMOOTH CURVES $c : [0,1] \to M$ as

$$L(c) := \int_0^1 \sqrt{g_{c(t)}(c'(t), c'(t))}\, dt.$$

1.1 Definition (Parameterization according to arc length).

A parameterization c of a curve is called PARAMETERIZATION BY ARC LENGTH if $|c'(t)| = 1$ for all t. For the length with respect to such parameterizations we thus have $L_a^b(c) = b - a$ by [**87**, 2.7].

1.2 Proposition (Parameterization by arc length).

Each curve has a parameterization by arc length. Each two such parameterizations of the same curve are equivalent via a parameter change of the form $t \longmapsto \pm t + a$.

Proof. Existence: Let $c : I \to (M,g)$ be a curve (which we always assume to be regular, i.e. $c'(t) \neq 0$ for all t), a a point in the interval I and $s(t) := L_a^t(c) = \int_a^t |c'(t)|\, dt$ the length function $s : \mathbb{R} \supset I \to \mathbb{R}$, with derivative $s'(t) = |c'(t)| > 0$. In particular, $s(I)$ is connected and thus again an interval (see [**81**, 3.4.3]). The inverse function $\varphi : s(I) \to I$, $s \longmapsto t(s)$ is smooth by the Inverse Function Theorem (see [**81**, 4.1.10] and [**82**, 6.3.5]). The parameterization $\bar{c} := c \circ \varphi$ is the required parameterization by arc length, since

$$\frac{d\bar{c}}{ds} = \frac{dc}{dt} \cdot \frac{dt}{ds} = \frac{dc}{dt} \cdot \frac{1}{\frac{ds}{dt}} = \frac{dc}{dt} \cdot \frac{1}{|c'(t)|} = \frac{dc}{dt} \cdot \frac{1}{|\frac{dc}{dt}|}.$$

Uniqueness: If c and $c \circ \varphi$ are two parameterizations by arc length, then:

$$1 = |(c \circ \varphi)'(t)| = |c'(\varphi(t))| \cdot |\varphi'(t)| = |\varphi'(t)| \text{ for all } t,$$

since $|(c \circ \varphi)'(t)| = 1 = |c'(\varphi(t))|$. Hence $\varphi' = \pm 1$ and thus $\varphi(t) = \varphi(a) + \int_a^t \varphi'(r)dr = \varphi(a) \pm (t - a) = \pm t + (\varphi(a) \mp a)$. □

On connected Riemannian manifolds (M, g), we obtain a metric $d_g : M \times M \to \mathbb{R}^+$ in sense of topology

$$d_g(p, q) := \inf\Big\{L(c) : c \in C^\infty(\mathbb{R}, M);\ c(0) = p,\ c(1) = q\Big\}.$$

We have shown in [**95**, 18.12] that this metric d_g generates the topology of M.

1.3 Definition (Isometry).

If (M, g) and (N, h) are two Riemannian manifolds and $f : M \mapsto N$ is smooth, then f is called ISOMETRY if and only if

$$T_x f : (T_x M, g_x) \to (T_{f(x)} N, h_{f(x)})$$

is a linear isometry for all x (see [**87**, 1.2]). Note that f is an isometry if and only if $f^* h = g$ is.

Remark.

1. If f is an isometry and $c : \mathbb{R} \to M$ is smooth, then
$$L_h(f \circ c) = L_{f^*h}(c) = L_g(c) :$$
Thus we obtain $d_h(f(x), f(y)) \le d_{f^*h}(x, y) = d_g(x, y)$ for the distance, that is, the isometry can not increase the distance. If f is a diffeomorphism and an isometry then $d(x, y) = d(f(x), f(y))$.

2. If the set of fixed points of an isometry can be parameterized as a smooth curve c, then this curve is locally the shortest connection of each of two points: We will see in 10.8 that locally the shortest connections exist and are unique. But since the isometric image of such a curve has the same length, it must agree with it, that is, must be contained in the fixed point set.

1.4 Theorem of Nash.

Each abstract and connected m-dimensional Riemannian manifold (M, g) can be isometrically embedded in $\mathbb{R}^{(2m+1)(6m+14)}$.

Without proof, see [**125**].

1.5 Proposition (Existence of Riemannian metrics).

Each paracompact smooth manifold admits a complete Riemannian metric, that is, a Riemannian metric g, whose associated metric d_g on M is complete.

Proof. We only need to embed (the connected components of) M into an $\mathbb{R}^n$ and then take the metric induced by the standard metric to obtain a Riemannian metric on M.

Or we can use charts to find Riemannian metrics locally and glue them to a global Riemannian metric by using a partition of unity. This works, since "being a Riemann metric" is a convex condition.

The existence of complete Riemannian metrics will be shown in 13.14. □

1.6 Proposition (Lie group of isometries).

Let (M, g) be a connected m-dimensional Riemannian manifold, then

$$\mathrm{Isom}(M) := \{f \in \mathrm{Diff}(M) : f \text{ is an isometry}\}$$

can be made into a Lie group of dimension at most $\frac{1}{2}m(m+1)$.

The group $\text{Isom}(M)$ is thus finite-dimensional in contrast to the group $\text{Diff}(M)$ of all diffeomorphisms. For example, both $\text{Isom}(\mathbb{R}^m) = O(m) \ltimes \mathbb{R}^m$ and $\text{Isom}(S^m) = O(m+1)$ have dimension $\frac{m(m-1)}{2} + m = \frac{(m+1)(m+1-1)}{2}$.

Without proof. See [78, 2.1.2].

Since one can define angles between vectors by

$$\cos \sphericalangle(x,y) := \frac{\langle x,y\rangle}{\sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle}}$$

by means of an inner product $\langle \_, \_ \rangle$, we can measure angles α between tangent vectors on each Riemannian manifold (M,g) and thus between curves c_1 and c_2 in their intersection points (say $c_1(0) = c_2(0)$) in the following way:

$$\cos\alpha := \frac{g(c_1'(0), c_2'(0))}{\sqrt{g(c_1'(0), c_1'(0))}\sqrt{g(c_2'(0), c_2'(0))}} .$$

1.7 Definition (Conformal mappings).

A smooth mapping $f : (M,g) \to (N,h)$ is called ANGLE PRESERVING (or CONFORMAL) if $T_x f : T_x M \to T_{f(x)} N$ is angle preserving for all $x \in M$.

1.8 Theorem (Lie group of conformal diffeomorphisms).

The set of conformal diffeomorphism of an m-dimensional paracompact connected Riemannian manifold forms a Lie group of dimension at most $\frac{1}{2}(m+1)(m+2)$.

For example, for $M = \mathbb{R}^m$ this group is the group of similarity maps of dimension $\dim(O(m)) + \dim(\mathbb{R}^m) + 1 = \frac{m(m-1)}{2} + m + 1 = \frac{m^2+m+2}{2}$ by the Proposition of Liouville [87, 52.11] (or 1.11), and for $M = S^2$ by 1.11 its connected connected component is $SL_{\mathbb{C}}(2)/\mathbb{Z}_2$ (the group of Moebius transformations) of dimension $6 = \frac{1}{2} \cdot 3 \cdot 4$.

Without proof. See [78, 4.6.1].

1.9 Lemma (Linear conformal mappings).

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be linear and injective, then the following statements are equivalent:

1. *f is angle preserving,*
2. *$\exists\, \lambda > 0 : \langle f(x), f(y)\rangle = \lambda\langle x,y\rangle$ for all $x,y \in \mathbb{R}^n$;*
3. *$\exists\, \mu > 0$: μf is an isometry.*

Proof.

(2⇔3) is obvious using $\lambda\,\mu^2 = 1$.

(1⇐2) Let α be the angle between the vectors x and y, and α' the angle between the vectors $f(x)$ and $f(y)$. Then

$$\cos\alpha' = \frac{\langle f(x), f(y)\rangle}{|f(x)|\cdot|f(y)|} = \frac{\lambda\langle x,y\rangle}{\sqrt{\lambda}|x|\sqrt{\lambda}|y|} = \cos\alpha.$$

So $\alpha = \alpha'$, and f is angle preserving.

(1⇒2) We implicitly define $\lambda(v) \ge 0$ by $\langle f(v), f(v)\rangle =: \lambda(v)\langle v,v\rangle$.

For vectors v and w, we have $v + w \perp v - w \Leftrightarrow 0 = \langle v+w, v-w\rangle = |v|^2 - |w|^2 \Leftrightarrow |v| = |w|$. Since f is conformal, the following holds for vectors with $|v| = 1 = |w|$:

$$0 = \langle f(v+w), f(v-w)\rangle = \langle f(v), f(v)\rangle - \langle f(w), f(w)\rangle = \lambda(v) - \lambda(w).$$

So λ is constant on the unit sphere and thus also on $\mathbb{R}^n \setminus \{0\}$, because for $w = |w|\, v$ with $v := \frac{1}{|w|} w \in S^{n-1}$ we have

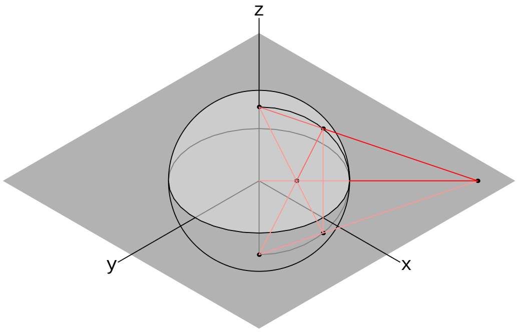
$$\begin{aligned}\lambda(w)\langle w, w\rangle &= \langle f(w), f(w)\rangle = \langle f(|w|\, v), f(|w|\, v)\rangle = \langle |w|\, f(v), |w|\, f(v)\rangle \\ &= |w|^2 \langle f(v), f(v)\rangle = \langle w, w\rangle\, \lambda(v)\, 1.\end{aligned}$$

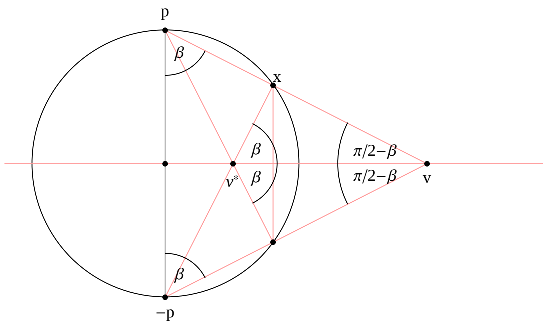
Thus, for any two vectors, v and w:

$$\begin{aligned}\langle f(v), f(w)\rangle &= \frac{1}{4}\Big(|f(v)+f(w)|^2 - |f(v)-f(w)|^2\Big) = \frac{1}{4}\Big(|f(v+w)|^2 - |f(v-w)|^2\Big) \\ &= \frac{1}{4}\,\lambda\,\Big(|v+w|^2 - |v-w|^2\Big) = \lambda\,\langle v, w\rangle. \quad \square\end{aligned}$$

1.10 Examples of conformal mappings.

1. The reflection $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$, $z \mapsto \frac{z}{|z|^2}$ along the unit sphere. It is the chart changeing mapping for the stereographic projections with respect to antipodal points:





The mapping f is conformal since $f'(z)(v) = \frac{v\langle z,z\rangle - 2z\langle z,v\rangle}{\langle z,z\rangle^2}$ and thus

$$\begin{aligned}\langle f'(z)(v), f'(z)(w)\rangle &= \left\langle \frac{v\langle z,z\rangle - 2z\langle z,v\rangle}{\langle z,z\rangle^2}, \frac{w\langle z,z\rangle - 2z\langle z,w\rangle}{\langle z,z\rangle^2} \right\rangle = \\ &= \frac{1}{\langle z,z\rangle^4}\Big(\langle v,w\rangle\,\langle z,z\rangle^2 - 4\langle z,z\rangle\langle z,v\rangle\langle z,w\rangle + 4\langle z,z\rangle\langle z,v\rangle\langle z,w\rangle\Big) = \frac{\langle v,w\rangle}{\langle z,z\rangle^2}.\end{aligned}$$

2. The stereographic projection $S^n \to \mathbb{R}^n$ (see [**91**, 2.20]).

1.11 Proposition.

Let f be a smooth (not necessarily regular) mapping between 2-dimensional Riemannian manifolds. We call it CONFORMAL *if $T_z f$ is a multiple of an isometry for each $z \in U$ (so $T_z f$ might be zero). Then:*

1. *$f : \mathbb{C} \supseteq U \to \mathbb{C}$ is conformal $\Leftrightarrow$ f or $\bar f$ is holomorphic.*
2. *$f : S^2 \to \mathbb{C}$ is conformal $\Leftrightarrow$ f is constant.*
3. *$f \neq \infty : \mathbb{C} \supseteq U \to S^2$ is conformal $\Leftrightarrow$ the chart representation of f or $\bar f$ with respect to stereographic projection $\mathbb{C} \subseteq S^2$ is meromorphic, i.e. is holomorphic up to poles.*
4. *$f : S^2 \to S^2$ is conformal $\Leftrightarrow$ the chart representation of f or $\bar f$ with respect to the stereographic projection $\mathbb{C} \subseteq S^2$ is rational, i.e. is the quotient of two polynomials.*
5. *$f : S^2 \to S^2$ is a conformal diffeomorphism $\Leftrightarrow$ the chart representation of f or $\bar f$ with respect to the stereographic projection $\mathbb{C} \subseteq S^2$ is a Möbius transformation, i.e. is a quotient of form $z \mapsto (a\,z+b)/(c\,z+d)$ with $ad - bc \neq 0$.*

6. *$f : \mathbb{R}^2 \to \mathbb{R}^2$ is a conformal diffeomorphism $\Leftrightarrow$ f is a similarity map, that is, a motion composed with a uniform scaling.*

Here S^2 is considered as complex manifold, see [91, 2.18] or 2.5.1, i.e. it is the 1-point compactification $\mathbb{C} \cup \{\infty\}$, where the chart at ∞ is given by the inversion $z \mapsto \frac{1}{z}$,

By analogy with the definition of holomorphy in [87, 30.9] (see [91, 2.3,2.5]), a function $f : \mathbb{C} \supseteq U \to \mathbb{C}$ is called ANTIHOLOMORPHIC if $f : \mathbb{R}^2 \supseteq U \to \mathbb{R}^2$ is smooth and $f'(z)$ is conjugate complex-linear, that is $f'(z)(iv) = -i f'(z)(v)$ for all v, z. This is exactly the case if $\bar{f}$ is holomorphic.

Proof. The implications ($\Leftarrow$) are easy to verify, see [91, 2.10]. In (1) this follows, since the derivative of a holomorphic mapping is given at each point by multiplication with a complex number, hence is conformal. In (5) this works as follows. Let $f : z \mapsto \frac{az+b}{cz+d}$ be a Möbius transformation. Then $f : \mathbb{C} \setminus \{-d/c\} \to \mathbb{C} \setminus \{a/c\}$ is a conformal diffeomorphism, with inverse $w \mapsto \frac{dw-b}{-cw+a}$, because

$$f(z) = w \Leftrightarrow az + b = (cz + d)w \Leftrightarrow z = \frac{dw - b}{-cw + a}.$$

If $c \neq 0$, then we extend it by $f(-d/c) := \infty$ and $f(\infty) := a/c$ to a bijection $S^2 \to S^2$. This extension is holomorphic at $-d/c$ because $z \mapsto 1/f(z) = (cz+d)/(az+b)$ is holomorphic on a neighborhood of $z = -d/c$ (note that $a(-d/c)+b = -(ad-bc)/c \neq 0$). It is holomorphic at ∞ as well (see [91, 2.18]), because

$$z \mapsto f(1/z) = (a/z + b)/(c/z + d) = (bz + a)/(dz + c),$$

is holomorphic on a neighborhood of 0 (note that $d\,0 + c = c \neq 0$).
If $c = 0$, then we extend f by $f(\infty) := \infty$. This extension is holomorphic at ∞, because

$$1/f(1/z) = (c/z + d)/(a/z + b) = (dz + c)/(bz + a)$$

(note that $a \neq 0$ since $ad = ad - bc \neq 0$). So every Möbius transformation f defines a conformal diffeomorphism $S^2 \to S^2$ (see also [91, 2.18])

For the reverse implications ($\Rightarrow$) we proceed as follows:

(1) Each linear isometry $\mathbb{R}^2 \to \mathbb{R}^2$ is a rotation (possibly composed with a reflection), see [87, 1.2]. So $f'(z)$ or $\overline{f'(z)} = \overline{f}'(z)$ is multiplication by a complex number by 1.9 and the Cauchy-Riemann differential equations $\frac{\partial u}{\partial x} = \pm\frac{\partial v}{\partial y}$ and $-\frac{\partial u}{\partial y} = \pm\frac{\partial v}{\partial x}$ are satisfied for $f =: u + i\,v$. It remains to show that these signs $\pm$ (of the determinant of the Jacobi matrix of f) are independent on z: We obtain

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \pm\frac{\partial^2 v}{\partial x \partial y} \mp \frac{\partial^2 v}{\partial y \partial x} = 0,$$

i.e. $u = \mathfrak{Re} f$ is a harmonic mapping. We are looking for some w, s.t. $u + i\,w$ is holomorphic, i.e. satisfies the Cauchy-Riemann differential equations. So $dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy = -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy$ should hold, which we achieve using

$$w(z) := \int_{z_0}^{z} \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy,$$

because of the integrability condition

$$d\Big(-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy\Big) = -\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big)\,dx \wedge dy = 0$$

the integrand is a closed form. Since $u + i\,w$ is holomorphic, the points z with $(u + iw)'(z) = 0$ ($\Leftrightarrow du = 0 \Leftrightarrow f'(z) = 0$) are isolated, hence the determinant of the Jacobi matrix of f has constant sign apart from these points, and thus f or $\bar f$ is holomorphic.

(2) Let $f : S^2 \to \mathbb{C}$ be conformal. Then the composition $\mathbb{C} \hookrightarrow S^2 \to \mathbb{C}$ with the stereographic parameterization is also conformal, i.e. holomorphic or antiholomorphic by (1). Since $f(S^2)$ is compact, this composition is bounded and hence constant by the Theorem of Liouville (see [**91**, 3.42] or [**132**, S.116]).

(3) Let $f : \mathbb{C} \supseteq U \to S^2$ be conformal and $z_0 \in U$. If $f(z_0) \in \mathbb{C} \subseteq S^2$, then $f : \mathbb{C} \to \mathbb{C}$ is locally conformal and hence (anti-)holomorphic by (1). Otherwise, $f(z_0) = \infty$ and thus $z \mapsto \frac{1}{f(z)}$ is locally conformal by 1.10.1 and hence is (anti-)holomorphic and has z_0 as an isolated zero. Thus f has z_0 as isolated singularity and locally around z_0 its values are near ∞, hence not dense in $\mathbb{C}$ and consequently z_0 is not an essential singularity by the theorem of Casorati-Weierstrass (see [**132**, S. 166]), but instead z_0 is a pole. So f or $\bar f$ is meromorphic.

(4) By (3), $f|_{\mathbb{C}} : S^2 \supseteq \mathbb{C} \to S^2$ (or $\bar f$) is meromorphic and has only finitely many poles z_j, since these are isolated on S^2. The Laurent development at the pole z_j is of the form $f(z) = \sum_{k=-n_j}^{\infty}(z-z_j)^k f_k^j$ with some $n_j \in \mathbb{N}$. So $z \mapsto f(z) - \sum_{k=1}^{n_j}(z-z_j)^{-k} f_{-k}^j$ is holomorphic around z_j. If also ∞ is a pole then the Laurent development there is $f(\frac{1}{z}) = \sum_{k=-n_\infty}^{\infty} z^k f_k^\infty$, so $f(z) - \sum_{k=1}^{n_\infty} z^k f_{-k}^\infty$ is holomorphic at ∞. So

$$z \mapsto f(z) - \sum_j \sum_{k=1}^{n_j}(z-z_j)^{-k} f_{-k}^j - \sum_{k=1}^{n_\infty} z^k f_{-k}^\infty$$

is holomorphic $S^2 \to \mathbb{C}$ and constant by (2), i.e. f is rational.

(5) By (4), $f = \frac{p}{q}$ with relatively prime polynomials p and q. Suppose the degree of p or q is greater than 1, then $h(z) := p(z) - c\,q(z)$ (for suitable c) has degree greater than 1. Since f is injective, only one solution $z = z_0$ of $h(z) = 0$ may exist, that is $h(z) = k(z-z_0)^n$ for some $n \geq 2$ and $0 \neq k \in \mathbb{C}$. Then $p(z_0) = c\,q(z_0)$ and also $0 = h'(z_0) = p'(z_0) - c\,q'(z_0)$ and thus $f'(z_0) = \frac{q\,p' - p\,q'}{q^2}(z_0) = 0$ yields a contradiction to the fact that f is a diffeomorphism.

(6) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a conformal diffeomorphism. W.l.o.g. (replace f by $\bar f$ if necessary), f is holomorphic by (1) and satisfies $f(0) = 0$ (replace f with $f - f(0)$). Let $\iota : z \mapsto \frac{1}{z}$. Then $\tilde f := \iota \circ f \circ \iota : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is a holomorphic diffeomorphism. Since f is a diffeomorphism at 0 we have that f^{-1} is locally bounded and hence for each $\varepsilon > 0$ a $\delta > 0$ exists with $|f^{-1}(w)| \leq \frac{1}{\delta}$ for all $|w| \leq \frac{1}{\varepsilon}$. Thus, $|z| < \delta \Rightarrow |\iota(z)| > \frac{1}{\delta} \Rightarrow |f(\iota(z))| > \frac{1}{\varepsilon} \Rightarrow |\tilde f(z)| = |\iota(f(\iota(z)))| < \varepsilon$, that is, $\tilde f$ is continuously extendable to a holomorphic function on $\mathbb{C}$ with $\tilde f(0) = 0$ (see [**91**, 3.31] or [**132**, S.115]). The same argument holds for the inverse function $\tilde f^{-1}$, i.e. f can be extended to a conformal diffeomorphism $S^2 \to S^2$. Thus, by (5), f is a Möbius transformation $z \mapsto \frac{a\,z+b}{c\,z+d}$ with $\infty \mapsto \infty$, i.e. $c = 0$, and hence f is a similarity map. □

2. Riemann surfaces

2.1 Definition (Riemann surface).

A Riemann surface is a 2-dimensional Riemannian manifold.

2.2 Theorem of Korn-Lichtenstein.

On each Riemann surface there are conformal local coordinates (also called ISOTHERMAL COORDINATES*).*

For a sketch of proof see 5.1.

2.3 Definition (Complex manifold).

A COMPLEX MANIFOLD is a smooth manifold with an atlas whose chart changes are complex differentiable (i.e. holomorphic).

An ORIENTED MANIFOLD is a smooth manifold with an atlas whose chart changes are orientation-preserving. For a more detailed study of orientability see Section [**95**, 27].

2.4 Corollary.

Each oriented Riemann surface is a complex manifold.

Proof. Choose an atlas according to 2.2, whose chart changes are conformal and orientation-preserving, i.e. holomorphic by 1.11.1. □

2.5 Examples of conformal diffeomorphisms.

(1) The S^2 has as an atlas the stereographic projection from the North and South Poles. The chart change is the inversion on the unit circle, so it is conformal but reverses the orientations. We change the orientation of one chart and get a holomorphic atlas. This is also called the RIEMANN SPHERE, see also [**91**, 2.22]. We now consider the AUTOMORPHISM GROUP of S^2. This is the set of all biholomorphic maps $f : S^2 \to S^2$, where the BIHOLOMORPHIC MAPS are exactly the conformal, orientation-preserving diffeomorphisms. By 1.11.5, via the stereographic projection of $S^2 \to \mathbb{C}$, the following description holds:

$$\mathrm{Aut}(S^2) = \left\{ z \mapsto \frac{az+b}{cz+d} : ad - bc = 1 \right\}.$$

This group of MÖBIUS TRANSFORMATIONS can also be identified with the following matrix group, up to multiplication by ± 1:

$$\mathrm{SL}_{\mathbb{C}}(2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\}.$$

Thus, the group $\mathrm{Aut}(S^2)$ is isomorphic to $\mathrm{SL}_{\mathbb{C}}(2)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is the discrete subgroup given by $\mathbb{Z}_2 := \{\mathrm{id}, -\mathrm{id}\}$. Hence $\mathrm{Aut}(S^2)$ is a Lie group of dimension $4 \cdot 2 - 2 = 6$.

(2) The automorphism group of $\mathbb{C}$ consists of those Möbius transformations of $\mathrm{Aut}(S^2)$ that leave $\mathbb{C} \subset S^2$ or, equivalently, the North Pole $\triangleq \infty \in \mathbb{C}$ invariant (see 1.11.6): In fact, if f is an automorphism of $\mathbb{C}$, then $f_\infty : z \mapsto 1/f(1/z)$ is holomorphic on the pointed plane. Since f is a diffeomorphism, f_∞ is continuously extendable by $f_\infty(0) = 0$, so ∞ is a removable singularity and f can be extended by $f(\infty) := \infty$ to a holomorphic diffeomorphism $S^2 \to S^2$, i.e. a Möbius transformation $z \mapsto \frac{az+b}{cz+d}$. Because of

$$\frac{az+b}{cz+d} = \frac{a + \frac{b}{z}}{c + \frac{d}{z}} \xrightarrow{z\to\infty} \frac{a}{c},$$

the Möbius transformation $z \mapsto \frac{az+b}{cz+d}$ maps ∞ to a/c, and thus ∞ is invariant if and only if $c = 0$ and $a \neq 0$. The Möbius transformation then has the form

$$\frac{az+b}{d} = \frac{a}{d}z + \frac{b}{d}.$$

Hence

$$\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az + b : a \neq 0,\ a, b \in \mathbb{C}\} \cong \left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \neq 0\ a, b \in \mathbb{C}\right\}.$$

This is also called the "$az + b$-group", see [**87**, 14.2]. It is complex 2-dimensional.

(3) For the open unit disk $\mathbb{D}$, the automorphism group consists of those Möbius transformations of S^2 that leave $\mathbb{D}$ invariant, i.e.

$$\mathrm{Aut}(\mathbb{D}) = \left\{z \mapsto \frac{az+b}{\bar{b}z + \bar{a}} : a\bar{a} - b\bar{b} = 1\right\} \cong SU(2,1)/\mathbb{Z}_2.$$

It is easy to see that any such Möbius transformation leaves $\mathbb{D}$ invariant. For the converse we need

Schwarz's Lemma.

Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all z. More precisely, one of the following two cases occurs:

- *$|f'(0)| < 1$ and $|f(z)| < |z|$ for all $z \neq 0$;*
- *$f(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$ and all z.*

For a proof, see [**91**, 3.43].

Let f be an automorphism of $\mathbb{D}$ with $f(0) =: c$. The map $z \mapsto \frac{z-c}{1-\bar{c}\,z}$ is a Möbius transformation of the given form. If we compose f with it, 0 is left invariant, so w.l.o.g. $f(0) = 0$. According to Schwarz's Lemma, $|f'(0)| \leq 1$ and since f is a diffeomorphism, $f'(0) \neq 0$. The same holds for the inverse mapping f^{-1}. Because of $f^{-1} \circ f = \mathrm{id}$ we have $(f^{-1})'(0) \circ f'(0) = 1$ and thus $|f'(0)| = 1$, i.e. $f(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$ by Schwarz's Lemma. Which is also a Möbius transformation of the desired form.

The group $\mathrm{Aut}(\mathbb{D})$ is 3-dimensional: Let $a = a_1 + ia_2$ and $b = b_1 + ib_2$. Then ${a_1}^2 + {a_2}^2 - {b_1}^2 - {b_2}^2 = 1$ and by

(1) $$r_{1,1} := a_1 + b_1 \qquad r_{1,2} := a_2 + b_2$$

(2) $$r_{2,1} := a_2 - b_2 \qquad r_{2,2} := a_1 - b_1$$

an element $\begin{pmatrix} r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2} \end{pmatrix} \in \mathrm{SL}(2)/\mathbb{Z}_2$ is defined. Thus, we obtain an isomorphism $\mathrm{SU}(2,1)/\mathbb{Z}_2 \cong \mathrm{Aut}(\mathbb{D}) \cong \mathrm{SL}(2)/\mathbb{Z}_2$, see Exercise [**87**, 72.62].

2.6 The hyperbolic disk.

We define another Riemannian metric on $\mathbb{D}$ by

$$g_z(v, w) := \frac{1}{(1 - |z|^2)^2}\langle v, w\rangle.$$

This is a conformal equivalent metric, i.e. $\mathrm{id} : (\mathbb{D}, \langle \cdot, \cdot \rangle) \to (\mathbb{D}, g)$ is a conformal diffeomorphism. Thus

$$\mathrm{Aut}(\mathbb{D}, g) = \mathrm{Aut}(\mathbb{D}, \langle \cdot, \cdot \rangle).$$

For $f(z) := \frac{az+b}{\bar{b}z+\bar{a}}$, i.e. $f \in \mathrm{Aut}(\mathbb{D})$, we have

$$g_z(v, v) = \frac{|v|^2}{(1 - |z|^2)^2} = \frac{|f'(z)(v)|^2}{(1 - |f(z)|^2)^2} = g_{f(z)}(f'(z)v, f'(z)v),$$

because

$$(1 - |z|^2)|f'(z)| = 1 - |f(z)|^2.$$

Therefore $\mathrm{Aut}(\mathbb{D}, g) = \mathrm{Isom}(\mathbb{D}, g)$. This Riemann surface $(\mathbb{D}, g)$ is called the HYPERBOLIC DISK. Each of its angle-preserving diffeomorphism is thus actually length-preserving.

3. Riemann mapping theorem and uniformization theorem

3.1 Riemann's Mapping Theorem.

Each complex 1*-dimensional, simply connected manifold is biholomorphic to* $\mathbb{D}$*,* $\mathbb{C}$*, or* S^2*.*

This is a generalization of [**87**, 5.3].

Without proof. See [**6**, S.158].

The universal covering $\tilde{M}$ of a complex manifold M constructed in [**87**, 24.31] (see also [**92**, 6.29]) is itself a complex manifold and the covering mapping is locally biholomorphic, which is obvious since the canonical chart change mappings of $\tilde{M}$ are identical to such of M (see also [**92**, 6.34]).

Because of 3.1, the universal covering map of any connected 2-dimensional, complex manifold is $\mathbb{D}$, $\mathbb{C}$, or S^2.

3.2 The universal covering of the punctured plane.

The mapping $p : \mathbb{R}^+ \times \mathbb{R} \twoheadrightarrow \mathbb{R}^+ \times S^1 \cong \mathbb{C} \setminus \{0\}$, given by $(r, \varphi) \mapsto (r, e^{i\varphi}) \mapsto re^{i\varphi}$, is obviously a covering map. Since $\mathbb{R}^+ \times \mathbb{R}$ is simply connected, p is also a universal covering. So p is an isometry with respect to the pull-back Riemann metric on $\mathbb{R}^+ \times \mathbb{R}$, which is given by

$$|(s, \psi)|^2_{(r,\varphi)} := |p'(r, \varphi)(s, \psi)|^2 = s^2 + r^2\psi^2$$

since

$$p'(r, \varphi)(s, \psi) = s\,\tfrac{\partial p}{\partial r} + \psi\,\tfrac{\partial p}{\partial \varphi} = s\,e^{i\varphi} + r\psi\,ie^{i\varphi}.$$

The mapping $h : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C}$, given by

$$h : (r, \varphi) \mapsto \ln(r) + i\varphi = (\ln(r), \varphi),$$

is a conformal diffeomorphism: That h is a diffeomorphism is obvious because of $h^{-1} : (x, y) \mapsto (e^x, y)$. We have

$$\begin{aligned} h'(r, \varphi)(s, \psi) &= (\tfrac{1}{r}s, \psi) \\ |h'(r, \varphi)(s, \psi)|^2 &= \tfrac{s^2}{r^2} + \psi^2 = \tfrac{1}{r^2}(s^2 + r^2\psi^2) = \tfrac{1}{r^2}|(s, \psi)|^2_{(r,\varphi)}, \end{aligned}$$

so h is also conformal. Thus

$$\begin{aligned} p \circ h^{-1} &: \mathbb{C} \to \mathbb{R}^+ \times \mathbb{R} \to \mathbb{C} \setminus \{0\} \\ z &= x + iy \mapsto (e^x, y) \mapsto e^x \cdot e^{iy} = e^{x+iy} = e^z. \end{aligned}$$

is the universal covering as Riemann surfaces.

We now want to describe Riemann surfaces M by means of their universal covering $\tilde{M}$. For this we will use [**87**, 24.18]: $M \cong \tilde{M}/G$, where G is the group of deck transformations of the universal covering $\tilde{M} \to M$.

3.3 The deck transformations of $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$.

We want to determine the deck transformations of the universal covering map $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$, $z \mapsto e^z$. We already know by 2.5.2 that

$$\mathrm{Aut}(\mathbb{C}) = \{f : \mathbb{C} \to \mathbb{C} : f \text{ is biholomorph } \} = \{z \mapsto az + b : a, b \in \mathbb{C},\ a \neq 0\}.$$

Now we define for $z_1, z_2 \in \mathbb{C}$:

$$z_1 \sim z_2 :\Leftrightarrow \exp(z_1) = \exp(z_2) \Leftrightarrow e^{x_1} e^{iy_1} = e^{x_2} e^{iy_2} \Leftrightarrow (x_1 = x_2) \wedge (y_1 - y_2 \in 2\pi\mathbb{Z}).$$

Each deck transformation $g \in \{h \in \mathrm{Aut}(\mathbb{C}) : h(z) \sim z \ \forall\ z\}$ can be written as $z \mapsto az + b$. So let $az + b \sim z$ for all z. For $z := 0$ we conclude $b \sim 0$ and thus $b \in 2i\pi\mathbb{Z}$. Furthermore, for $z := 1$, it follows that $a + 0 \sim a + b \sim 1$, i.e. $\mathfrak{Re}(a) = 1$, and $\mathfrak{Im}(a) \in 2\pi\mathbb{Z}$. If $z := i$, we conclude analogously that $ai = -\mathfrak{Im}(a) + i\mathfrak{Re}(a) \sim i$, i.e. $\mathfrak{Im}(a) = 0 \Rightarrow a = \mathfrak{Re}(a) = 1$. Thus we have determined the group G of the deck transformations for this universal covering map:

$$G = \{z \mapsto z + 2i\pi k : k \in \mathbb{Z}\}.$$

3.4 Uniformization Theorem.

Let M be a 2-dimensional, connected, oriented Riemannian manifold. Then M is conformal diffeomorphic to $\tilde{M}/G$, with $\tilde{M} \in \{S^2, \mathbb{C}, \mathbb{D}\}$ and G being a group of Möbius transformations in $\mathrm{Aut}(\tilde{M})$. Conversely, let G be a group of Möbius transformations on $M_1 \in \{S^2, \mathbb{C}, \mathbb{D}\}$, which acts STRICTLY DISCONTINUOUS, *that is $\forall\ x\ \exists\, U(x)$ a neighborhood of x with $U(x) \cap g(U(x)) = \emptyset$ for all $g \neq \mathrm{id}$, then*

1. *M_1/G is a manifold,*
2. *The quotient mapping $M_1 \to M_1/G$ is a covering map, and*
3. *G is the group of deck transformations of it.*

Proof. The universal covering $\tilde{M}$ (existing by [**87**, 24.31], see also [**92**, 6.29]) is one of the three spaces S^2, $\mathbb{C}$, $\mathbb{D}$ by the Riemann Mapping Theorem 3.1, and M is isomorphic to $\tilde{M}/G$, where G is the set of deck transformations and hence a group of Möbius transformations which acts strictly discontinuous on $\tilde{M}$ by [**87**, 24.18] (see also [**92**, 6.27]). Conversely, each such group G provides a covering $\tilde{M} \to \tilde{M}/G =: M$ by [**87**, 24.19] (see also [**92**, 6.2]). □

II. Differential forms on Riemannian manifolds

4. Volume form and Hodge-Star operator

4.1 Recap: Musical isomorphisms

In [95, 24.2] we introduced the “musical” isomorphisms $\sharp : T_xM \xrightarrow{\cong} (T_xM)^*$ and its inverse $\flat$ for Riemannian manifolds M. The basis elements $\frac{\partial}{\partial u^i}$ of T_xM are mapped to $\sharp(\frac{\partial}{\partial u^i}) = \sum_j g_{j,i}\, du^j$ with $g_{j,i} := g(\frac{\partial}{\partial u^j}, \frac{\partial}{\partial u^i})$. It follows that $TM \cong T^*M$ in a canonical manner, and thus the space of vector fields $\mathfrak{X}(M)$ is canonically isomorphic to the space of 1-forms $\Omega^1(M)$. In particular, for functions $f \in C^\infty(M,\mathbb{R})$, the gradient $\operatorname{grad}(f) \in \mathfrak{X}(M)$ is defined by $\sharp(\operatorname{grad} f) = df \in \Omega^1(M)$. More generally, for tensor fields (see [95, 23.1]) we have the natural isomorphisms

$$\mathcal{T}^q_p(M) \cong \mathcal{T}^0_{p+q}(M) \cong \mathcal{T}^{p+q}_0(M)$$

4.2 Recap: Volume form

The determinant function det for oriented Euclidean vector spaces gave us the volume form $\mathrm{vol}_M \in \Omega^m(M)$ on oriented Riemannian manifolds (M,g) in 4.2

$$\mathrm{vol}_M(x) := \det \in L^m_{\mathrm{alt}}(T_xM;\mathbb{R}).$$

Its value on the basis $(g_i := \frac{\partial}{\partial u^i})$ of T_xM is

$$\mathrm{vol}(\tfrac{\partial}{\partial u^1},\dots,\tfrac{\partial}{\partial u^m}) = \det(g_1,\dots,g_m) = \sqrt{G} \text{ with } G := \det(g_{i,j})_{i,j}.$$

And we get the following isomorphism:

$$C^\infty(M,\mathbb{R}) \xrightarrow{\cong} \Omega^{\dim M}(M), \quad f \mapsto f\cdot \mathrm{vol}_M\,.$$

In [95, 28.10] we have considered oriented codimension 1 submanifolds N of $(n+1)$-dimensional oriented Riemannian manifolds M. If ν_x for $x \in N$ designates the uniquely determined vector in T_xM for which $(\nu_x, e_1,\dots,e_n)$ is a positive-oriented orthonormal basis in T_xM for an oriented orthonormal basis $(e_1,\dots,e_n)$ of T_xN and is extended to a vector field ν on all of M, then

$$\mathrm{vol}_N = \mathrm{inkl}^*(\iota_\nu(\mathrm{vol}_M)) \text{ on } N.$$

This applies in particular to the canonically oriented boundary $N = \partial M$ of an oriented Riemannian manifold M with boundary. In this case, the vector ν is the outward-pointing unit normal vector, see [95, 28.9].

4.3 Recap: Extension of the inner product

In Exercise [99, 33], we defined an inner product on the dual space E^* of an oriented Euclidean vector space E, by requiring that the dual basis (e^i) in E^* of a positive-oriented orthonormal basis (e_i) of E is again an orthonormal basis. On $\bigotimes^k E$ we define a scalar product by requiring that the basis $(e_{i_1}\otimes\ldots\otimes e_{i_k})_{i_1,\dots,i_k}$ is an orthonormal basis and similarly for $\bigwedge^k E$ and the basis $(e_{i_1} \wedge \ldots \wedge e_{i_k})_{i_1 < \cdots < i_k}$. This definition is independent of the bases, because the scalar product on E^* is given by the following formula:

$$\langle x^*, y^* \rangle_{E^*} = \langle \flat x^*, \flat y^* \rangle_E, \text{ because } \langle e^i, e^j \rangle_{E^*} = \langle e_i, e_j \rangle_E = \delta_{i,j}.$$

Thus $\sharp$ and $\flat$ are isometries by definition.

On $\bigotimes^k E$, the scalar product is analogously given by:

$$\big\langle x_1 \otimes \ldots \otimes x_k, y_1 \otimes \ldots \otimes y_k \big\rangle_{\otimes^k E} = \langle x_1, y_1 \rangle_E \cdot \ldots \cdot \langle x_k, y_k \rangle_E$$

On $\bigwedge^k E$, the scalar product is analogously given by:

$$\begin{aligned} \big\langle x_1 \wedge \ldots \wedge x_k, y_1 \wedge \ldots \wedge y_k \big\rangle_{\bigwedge^k E} &= \det\Big((\langle x_i, y_j \rangle_E)_{i,j} \Big) \\ &= \frac{1}{k!} \big\langle x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k \big\rangle_{\otimes^k E}. \end{aligned}$$

Caution: The restriction of the scalar product of $\bigotimes^k E$ to the subspace $\bigwedge^k E$ has an additional factor $k!$, because

$$x_1 \wedge \cdots \wedge x_k = k! \operatorname{alt}(x_1 \otimes \cdots \otimes x_k) = \sum_\sigma \operatorname{sign}(\sigma)\, x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}$$

and thus

$$\begin{aligned} &\big\langle x_1 \wedge \cdots \wedge x_k, x_1 \wedge \cdots \wedge x_k \big\rangle_{\otimes^k E} = \\ &\quad = \sum_{\sigma,\pi} \operatorname{sign}(\sigma) \operatorname{sign}(\pi) \big\langle x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}, x_{\pi(1)} \otimes \cdots \otimes x_{\pi(k)} \big\rangle_{\otimes^k E} \\ &\quad = \sum_{\sigma,\pi} \operatorname{sign}(\sigma) \operatorname{sign}(\pi) \big\langle x_{\sigma(1)}, x_{\pi(1)} \big\rangle \ldots \big\langle x_{\sigma(k)}, x_{\pi(k)} \big\rangle \\ &\quad = \sum_{\sigma,\pi} \operatorname{sign}(\sigma) \operatorname{sign}(\pi \circ \sigma) \big\langle x_{\sigma(1)}, x_{\pi(\sigma(1))} \big\rangle \ldots \big\langle x_{\sigma(k)}, x_{\pi(\sigma(k))} \big\rangle \\ &\quad = k! \sum_\pi \operatorname{sign}(\pi) \big\langle x_1, x_{\pi(1)} \big\rangle \ldots \big\langle x_k, x_{\pi(k)} \big\rangle \\ &\quad = k! \big\langle x_1 \wedge \ldots \wedge x_k, x_1 \wedge \ldots \wedge x_k \big\rangle_{\bigwedge^k E} \end{aligned}$$

4.4 Recap: Hodge star operator

In Exercise [**99**, 30] we have defined the Hodge star operator $* : \bigwedge^k E \to \bigwedge^{m-k} E$ for oriented m-dimensional Euclidean vector spaces E by the following implicit equation:

$$\eta \wedge (*\omega) = \langle \eta, \omega \rangle \cdot \det \text{ for } \eta, \omega \in \bigwedge^k E.$$

In Exercise [**99**, 31] we checked that $*$ is an isometry and satisfies

$$* \circ * = (-1)^{k(m-k)} : \bigwedge^k E \to \bigwedge^{m-k} E \to \bigwedge^k E.$$

And in Exercise [**99**, 32] we defined the Hodge star operator $* : \Omega^k(M) \to \Omega^{m-k}(M)$ for oriented Riemannian manifolds (M, g) of dimension m by $(*\omega)(x) := *(\omega(x))$ and showed that

$$* : C^\infty(M, \mathbb{R}) = \Omega^0(M) \to \Omega^m(M) \text{ is given by } f \mapsto *f = f \cdot \text{vol and}$$
$$* : \mathfrak{X}(M) \cong \Omega^1(M) \to \Omega^{m-1}(M) \text{ is given by } \xi \mapsto *\sharp\,\xi = i_\xi \text{vol}.$$

4.5 Recap: Divergence

In Exercise [99, 34] we defined the divergence of a vector field $\xi \in \mathfrak{X}(M)$

$$\operatorname{div} \xi := *\big(d(\iota_\xi \operatorname{vol}_M)\big) \xlongequal{\text{Exercise [99, 32]}} (* \circ d \circ * \circ \sharp)(\xi) \in C^\infty(M, \mathbb{R})$$

and showed that $\operatorname{div} \xi \cdot \operatorname{vol}_M = \mathcal{L}_\xi \operatorname{vol}_M$. Moreover we obtained the local formula

$$\operatorname{div} \xi = \frac{1}{\sqrt{G}} \sum_i \frac{\partial(\sqrt{G}\xi^i)}{\partial u^i}.$$

4.6 Remark

For vector fields ξ on Riemannian manifolds M with boundary one has:

$$\operatorname{incl}^*(\iota_\xi \operatorname{vol}_M) = \langle \xi, \nu_{\partial M} \rangle \cdot \operatorname{vol}_{\partial M},$$

since for an orthonormal basis $(e_i)_{i=1}^m$ of $T_x(\partial M)$ we get

$$\begin{aligned}
(\iota_\xi \operatorname{vol}_M)(e_1, \dots, e_m) &= \operatorname{vol}_M(\xi, e_1, \dots, e_m) = \\
&= \operatorname{vol}_M\Big(\underbrace{\langle \xi, \nu\rangle \nu}_{\in (T(\partial M))^\perp} + \underbrace{\xi - \langle \xi, \nu \rangle \nu}_{\in T(\partial M)}, e_1, \dots, e_m\Big) \\
&= \langle \xi, \nu \rangle \cdot \operatorname{vol}_M(\nu, e_1, \dots, e_m) + 0 \\
&\xlongequal{\boxed{4.2}} \langle \xi, \nu \rangle \cdot \operatorname{vol}_{\partial M}(e_1, \dots, e_m).
\end{aligned}$$

4.7 Green's Theorem.

Let M be an oriented Riemannian manifold with boundary and let $\xi \in \mathfrak{X}(M)$ be of compact support. Then

$$\int_M \operatorname{div} \xi \cdot \operatorname{vol}_M = \int_{\partial M} \langle \xi, \nu_{\partial M} \rangle \cdot \operatorname{vol}_{\partial M}.$$

This formula justifies the term SOURCE DENSITY (german: Quelldichte) for div.

Proof. The following holds:

$$\begin{aligned}
\int_M \operatorname{div} \xi \cdot \operatorname{vol}_M &\xlongequal{\boxed{4.5}} \int_M \mathcal{L}_\xi \operatorname{vol}_M \xlongequal{[95,\ 25.9]} \int_M (d \circ \iota_\xi + \iota_\xi \circ d) \operatorname{vol}_M \\
&= \int_M d(\iota_\xi \operatorname{vol}_M) + 0 \xlongequal{\text{Stokes [95, 28.11]}} \int_{\partial M} \operatorname{incl}^*(\iota_\xi \operatorname{vol}_M) \\
&\xlongequal{\boxed{4.6}} \int_{\partial M} \langle \xi, \nu_{\partial M} \rangle \cdot \operatorname{vol}_{\partial M}. \qquad \square
\end{aligned}$$

5. The Laplace Beltrami operator

5.1 Laplace operator.

We now generalize the Laplace operator to oriented Riemannian manifolds. For this, the CODIFFERENTIAL OPERATOR d^* is defined by the commuting diagram:

$$\begin{array}{ccc} \Omega^p & \xrightarrow{(-1)^p\, d^*} & \Omega^{p-1} \\ {\scriptstyle *}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle *} \\ \Omega^{m-p} & \xrightarrow[d]{} & \Omega^{m-p+1} \end{array} \qquad \text{or} \qquad \begin{array}{ccc} \Omega^p & \xrightarrow{d^*} & \Omega^{p-1} \\ {\scriptstyle *}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\uparrow{\scriptstyle *} \\ \Omega^{m-p} & \xrightarrow[(-1)^{pm+m+1}\, d]{} & \Omega^{m-p+1} \end{array}$$

It should be noted that this is not a graded derivation. The sign is chosen so that d^* becomes formally adjoint to d, as we will show in 5.5. To show the equivalence of the two diagrams one calculates as follows:

$$d \circ * = * \circ (-1)^p d^* \Leftrightarrow * \circ (-1)^{pm+m+1} d \circ * = * \circ (-1)^{pm+m+1} * \circ (-1)^p d^* \overset{4.4}{=\!=} \\ = (-1)^{(pm+m+1)+p+(p-1)(m-p+1)} d^* = d^*$$

In particular, for 3-dimensional oriented Riemannian manifolds (and in particular for M open in $\mathbb{R}^3$) we have (compare with [95, 25.11] and 4.5):

$$\begin{array}{ccccccc} \Omega^3 & \xrightarrow{-d^*} & \Omega^2 & \xrightarrow{d^*} & \Omega^1 & \xrightarrow{-d^*} & \Omega^0 \\ {\scriptstyle *}\updownarrow{\scriptstyle \cong} & & {\scriptstyle *}\updownarrow{\scriptstyle \cong} & & {\scriptstyle *}\updownarrow{\scriptstyle \cong} & & {\scriptstyle *}\updownarrow{\scriptstyle \cong} \\ \Omega^0 & \xrightarrow[d]{} & \Omega^1 & \xrightarrow[d]{} & \Omega^2 & \xrightarrow[d]{} & \Omega^3 \\ \| & & {\scriptstyle \sharp}\uparrow{\scriptstyle \cong} & & {\scriptstyle *\circ\sharp}\uparrow{\scriptstyle \cong} & & {\scriptstyle \cdot\,\mathrm{vol}}\uparrow{\scriptstyle \cong} \\ C^\infty & \xrightarrow[\mathrm{grad}]{} & \mathfrak{X} & \xrightarrow[\mathrm{rot}]{} & \mathfrak{X} & \xrightarrow[\mathrm{div}]{} & C^\infty \end{array}$$

The mapping $\Delta := d\,d^* + d^* d : \Omega^p \to \Omega^p$ is called the LAPLACE BELTRAMI OPERATOR.

In general, for functions $f \in C^\infty(M, \mathbb{R})$ the formula $\Delta f = -\operatorname{div} \operatorname{grad} f$ holds, because

$$\begin{aligned} \Delta f &= d^* d\, f + 0 = (-1)^{1m+m+1} * d * d\, f \\ &= - * d * \sharp \operatorname{grad} f \overset{4.4}{=\!=} - * d\, \iota_{\operatorname{grad} f} \operatorname{vol}_M \overset{4.5}{=\!=} -\operatorname{div}(\operatorname{grad} f). \end{aligned}$$

Thus the Laplace operator defined here has perhaps an unfamiliar sign, which serves to make it a positive operator, see 5.5.

Sketch of the proof for Theorem 2.2 of Korn-Lichtenstein.
Using a standard result on PDEs (see [15, p228, §5.4]) there exist locally around $p \in M$ smooth solution $u : M \to \mathbb{R}$ of $\Delta u = 0$ with prescribed values $u(p)$ and $du(p)$ on each Riemannian manifold M. Thus we find local harmonic (i.e. $\Delta u^i = 0$ for all i) coordinates $(u^1, \dots, u^m)$ by using linear independent initial values $du^j(p)$.

For local harmonic coordinates (u^1, u^2) on Riemannian surfaces, we have:
(u^1, u^2) is conformal $\Leftrightarrow du^2 = \pm * du^1$.

($\Leftarrow$) Since the Hodge-star operator is an isometry, we have that $|du^2| = |\pm * du^1| = |du^1| =: c$ and $du^2 \perp du^1$ (with respect to the inner product on $(T_x M)^*$) because

$$\langle du^1, du^2 \rangle \operatorname{vol}_M = \langle du^1, \pm * du^1 \rangle \operatorname{vol}_M \overset{4.4}{=\!=} du^1 \wedge (\pm * * du^1) = \mp du^1 \wedge du^1 = 0.$$

Hence pointwise $g^{i,j} := \langle du^i, du^j\rangle = c^2\,\delta^{i,j}$. Since the matrix with entries $g_{i,j} := \langle \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\rangle$ is inverse to $(g^{i,j})_{i,j=1\dots m}$ by [95, 24.2], we have that $g_{i,j} = \frac{1}{c^2}\,\delta_{i,j}$, i.e. $(u^1, u^2) : \mathbb{R}^2 \to M$ is conformal.

($\Rightarrow$) Conversely, if (u^1, u^2) is conformal, then $du^1 \perp du^2$ and have the same length. By the previous argument $*\,du^1$ is also orthogonal to du^1 and has the same length, hence $du^2 = \pm * du^1$.

So let u^1 be a local harmonic function with $du^1(p) \neq 0$. Since $0 = \Delta u^1 = (dd^* + d^*d)u^1 = d^*du^1 = (-1)^1 {*d*d}\, u^1$, we have $d{*du^1} = 0$ and hence by Poincare's lemma $\exists\, u^2$ with $du^2 = *du^1$. Moreover u^2 is also harmonic, since $\Delta u^2 = -*\,d*d\,u^2 = -*\,d**\,d\,u^1 = *\,d^2\,u^1 = 0$. And, by what we have show just before, (u^1, u^2) are conformal (i.e. isothermal) local coordinates on M. □

5.2 Product rules.

For $f, g \in C^\infty(M, \mathbb{R})$ and $\xi \in \mathfrak{X}(M)$:

$$\begin{aligned} \operatorname{grad}(f\cdot g) &= g\cdot \operatorname{grad}(f) + f\cdot\operatorname{grad}(g)\\ \operatorname{div}(f\cdot\xi) &= f\cdot\operatorname{div}(\xi) + df\cdot\xi = f\cdot\operatorname{div}(\xi) + \langle\operatorname{grad}(f),\xi\rangle\\ \varDelta(f\cdot g) &= f\cdot\varDelta(g) + \varDelta(f)\cdot g - 2\langle\operatorname{grad}(f),\operatorname{grad}(g)\rangle, \end{aligned}$$

see Exercise [87, 72.69].

5.3 Green's formulas.

Let M be a compact oriented Riemannian manifold with boundary and let f and h be in $C^\infty(M, \mathbb{R})$. Then:

(1) $$\int_M \Big(\langle\operatorname{grad} f, \operatorname{grad} h\rangle - f\cdot\varDelta h\Big)\cdot\operatorname{vol} = \int_{\partial M} f\cdot\langle\operatorname{grad} h, \nu\rangle\cdot\operatorname{vol}$$

(2) $$\int_M (f\cdot\varDelta h - h\cdot\varDelta f)\cdot\operatorname{vol} = -\int_{\partial M}(f\,dh - h\,df)(\nu)\cdot\operatorname{vol}$$

Proof. 1 We have

$$\operatorname{div}(f\cdot\operatorname{grad} h) \overset{5.2}{=\!=} f\cdot\operatorname{div}(\operatorname{grad} h) + \langle\operatorname{grad} f,\operatorname{grad} h\rangle = -f\cdot\varDelta h + \langle\operatorname{grad} f,\operatorname{grad} h\rangle$$

and thus for $\xi := f\cdot\operatorname{grad} h$ we obtain

$$\begin{aligned} \int_M \Big(\langle\operatorname{grad} f,\operatorname{grad} h\rangle - f\cdot\varDelta h\Big)\cdot\operatorname{vol}_M &= \int_M \operatorname{div}(f\cdot\operatorname{grad} h)\cdot\operatorname{vol}_M = \\ &= \int_M \operatorname{div}\xi\cdot\operatorname{vol}_M \overset{4.7}{=\!=} \int_{\partial M}\langle\xi,\nu_{\partial M}\rangle\cdot\operatorname{vol}_{\partial M}\\ &= \int_{\partial M}\langle f\cdot\operatorname{grad} h,\nu_{\partial M}\rangle\cdot\operatorname{vol}_{\partial M} = \int_{\partial M} f\cdot\langle\operatorname{grad} h,\nu_{\partial M}\rangle\cdot\operatorname{vol}_{\partial M}\\ &= \int_{\partial M} f\cdot dh(\nu_{\partial M})\cdot\operatorname{vol}_{\partial M}. \end{aligned}$$

2 If one exchanges f and h in 1 and subtracts the result of 1, one obtains the second Green's formula. □

5.4 Corollary (Subharmonic functions are constant).

Let M be a compact, oriented Riemannian manifold without boundary. Then each SUBHARMONIC FUNCTION *$f \in C^\infty(M, \mathbb{R})$ - i.e. $\varDelta f \le 0$ - is constant. This holds in particular for harmonic functions, i.e. the stationary points f of the heat conduction equation $\Delta f = 0$.*

Proof. If we choose the function h constant to 1 in the 2nd Green's formula 5.3.2, we get $\int_M -\Delta f \cdot \mathrm{vol}_M = \int_\emptyset df(\nu)\cdot \mathrm{vol} = 0$. Because of $\Delta f \le 0$ we have $\Delta f = 0$, i.e. f is harmonic. By the 1st Green's formula 5.3.1 for $h = f$ we get analogously

$$0 \overset{5.3.1}{=\!=} \int_M \Big(|\operatorname{grad} f|^2 - f\cdot \underbrace{\Delta f}_{=0}\Big)\cdot \mathrm{vol}_M = \int_M |\operatorname{grad} f|^2 \cdot \mathrm{vol}_M,$$

so $\operatorname{grad} f = 0$ and thus f is constant. □

5.5 The Laplace Beltrami operator is symmetric.

What can be said in general about the Laplace Beltrami operator $\Delta := dd^* + d^*d : \Omega(M) \to \Omega(M)$ of a compact oriented Riemannian manifold M? On each homogeneous part $\Omega^k(M)$ we have an inner product by 4.3:

$$\langle \alpha, \beta\rangle_{\Omega^k(M)} := \int_M \langle \alpha(.), \beta(.)\rangle_{\bigwedge^k T^*M}\ \mathrm{vol}_M \in \mathbb{R}.$$

The operators d and d^* are formally adjoint with respect to this inner product, because

$$\alpha \wedge *\beta \overset{4.4}{=\!=} \langle \alpha, \beta\rangle \cdot \mathrm{vol} \text{ for } \alpha, \beta \in \Omega^k(M)$$

and for $\alpha \in \Omega^{k-1}$ and $\beta \in \Omega^k$ we calculate as follows:

$$\begin{aligned}
\Big(\langle d\alpha, \beta\rangle - \langle \alpha, d^*\beta\rangle\Big)\ \mathrm{vol} &\overset{5.1}{=\!=} \langle d\alpha, \beta\rangle\ \mathrm{vol} - \Big\langle \alpha, (-1)^{km+m+1} * d * \beta\Big\rangle\ \mathrm{vol} \\
&\overset{4.4}{=\!=} d\alpha \wedge *\beta + (-1)^{km+m} \alpha \wedge * * d * \beta \\
&\overset{4.4}{=\!=} d\alpha \wedge *\beta + (-1)^{km+m} \alpha \wedge (-1)^{(m-k+1)(k-1)} d * \beta \\
&= d\alpha \wedge *\beta + (-1)^{k-1} \alpha \wedge d * \beta \\
&= d(\alpha \wedge *\beta)
\end{aligned}$$

$$\Rightarrow \int_M \langle d\alpha, \beta\rangle\ \mathrm{vol} = \int_M \langle \alpha, d^*\beta\rangle\ \mathrm{vol} + \underbrace{\int_M d(\alpha \wedge *\beta)}_{=0}.$$

Thus, the Laplace Beltrami operator $\Delta = dd^* + d^*d$ is symmetric, i.e.

$$\langle \Delta\alpha, \beta\rangle = \langle \alpha, \Delta\beta\rangle$$

It is also positive, because

$$\langle \Delta\alpha, \alpha\rangle = \big\langle (dd^* + d^*d)\alpha, \alpha\big\rangle = \langle d^*\alpha, d^*\alpha\rangle + \langle d\alpha, d\alpha\rangle \ge 0.$$

This implies

$$\Delta\alpha = 0 \Leftrightarrow d\alpha = 0 = d^*\alpha, \text{ i.e. } \ker(\Delta) = \ker(d) \cap \ker(d^*).$$

The forms in the kernel of Δ are also called HARMONIC FORMS.

The operator Δ is a linear differential operator of degree 2. It can be shown to be elliptic, see [**147**, 6.35 S.258], and the following lemmas apply:

5.6 Lemma.

A sequence of k-forms $\alpha_n \in \Omega^k(M)$, for which both $\{\|\alpha_n\|^2 := \langle \alpha_n, \alpha_n\rangle : n \in \mathbb{N}\}$ and $\{\|\Delta(\alpha_n)\|^2 : n \in \mathbb{N}\}$ are bounded, has a Cauchy subsequence in the normed (incomplete) space $\Omega^k(M)$.

Without proof, see [**147**, 6.6 S.231] and [**147**, 6.33 S.258].

Each $\alpha \in \Omega^k(M)$ defines a continuous linear functional $\tilde{\alpha} \in L(\Omega^k(M), \mathbb{R})$ by $\tilde{\alpha}(\varphi) := \langle \alpha, \varphi \rangle$; but not vice versa! However:

5.7 Lemma.

Any WEAK SOLUTION *α of $\Delta\alpha = \gamma$ with $\gamma \in \Omega^k(M)$ is a real solution, that is from $\tilde{\alpha} \in L(\Omega^k(M), \mathbb{R})$ with $\langle \gamma, \varphi \rangle = \tilde{\alpha}(\Delta(\varphi))$ for all $\varphi \in \Omega^k(M)$ follows that an $\alpha \in \Omega^k(M)$ exists with $\tilde{\alpha}(\varphi) = \langle \alpha, \varphi \rangle$ for all $\varphi \in \Omega^k(M)$.*

Without proof, see [**147**, 6.5 S.231] and [**147**, 6.32 S.253].

Note: $\Delta\alpha = \gamma \Leftrightarrow \forall\, \varphi : \langle \gamma, \varphi \rangle = \langle \Delta\alpha, \varphi \rangle = \langle \alpha, \Delta\varphi \rangle = \tilde{\alpha}(\Delta\varphi)$.

5.8 Theorem of Hodge.

Let M be a compact oriented Riemannian manifold, then the following holds:

1. $\dim(\ker \Delta) < \infty$.
2. $\Delta : (\ker \Delta)^\perp \to \operatorname{im} \Delta$ *is an open mapping.*
3. $\operatorname{im} \Delta = (\ker \Delta)^\perp$.

Proof.

1 Suppose $\ker \Delta$ were infinite-dimensional, then there exists an orthonormal sequence $\alpha_n \in \ker \Delta$. This has by 5.6 a Cauchy subsequence, which is a contradiction to $\|\alpha_n - \alpha_m\|^2 = \|\alpha_n\|^2 + \|\alpha_m\|^2 = 2$.

2 Of course, $\Delta : (\ker \Delta)^\perp \to \operatorname{im} \Delta$ is bijective.

Claim: $\exists\, c\ \forall\, \alpha \in (\ker \Delta)^\perp : \|\alpha\| \leq c\|\Delta\alpha\|$ (so $\Delta^{-1} : \operatorname{im} \Delta \to (\ker \Delta)^\perp$ is continuous with respect to the norm).

Suppose indirectly: $\exists\, \alpha_n \in (\ker \Delta)^\perp$ with $\|\alpha_n\| = 1$ and $\|\Delta\alpha_n\| \to 0$. According to Lemma 5.6, we may assume that α_n is a Cauchy sequence with respect to the (incomplete) norm. So there exists

$$\tilde{\alpha}(\varphi) := \lim_{n\to\infty} \langle \alpha_n, \varphi \rangle \text{ for each } \varphi \in \Omega^k.$$

The linear functional $\tilde{\alpha} : \Omega^k \to \mathbb{R}$ is bounded, because $|\tilde{\alpha}(\varphi)| \leq \sup_n |\langle \alpha_n, \varphi \rangle| \leq 1 \cdot \|\varphi\|$ and $\tilde{\alpha}|_{\ker \Delta} = 0$, because

$$\varphi \in \ker \Delta \Rightarrow \tilde{\alpha}(\varphi) = \lim_n \langle \alpha_n, \varphi \rangle \overset{\alpha_n \in (\ker \Delta)^\perp}{=\!=\!=\!=} \lim_n 0 = 0$$

and furthermore $\tilde{\alpha}|_{\operatorname{im} \Delta} = 0$ (i.e. $\tilde{\alpha}$ is a weak solution of $\Delta\tilde{\alpha} = 0$), because $\tilde{\alpha}(\Delta\varphi) = \lim_{n\to\infty} \langle \alpha_n, \Delta\varphi \rangle = \lim_{n\to\infty} \langle \Delta\alpha_n, \varphi \rangle = 0$. By Lemma 5.7 it is a real solution, i.e. $\exists\, \alpha \in \Omega^k : \tilde{\alpha}(\varphi) = \langle \alpha, \varphi \rangle$ for all $\varphi \in \Omega^k$. Thus $\alpha \in (\ker \Delta)^\perp$, because $\langle \alpha, \varphi \rangle = \tilde{\alpha}(\varphi) = 0$ for all $\varphi \in \ker \Delta$, and $\alpha \neq 0$, in fact even $\|\alpha\| = \lim_n \|\alpha_n\| = 1$. But $0 = \tilde{\alpha}(\Delta\varphi) = \langle \alpha, \Delta\varphi \rangle = \langle \Delta\alpha, \varphi \rangle$ for all $\varphi \in \Omega^k$ and thus $\Delta\alpha = 0$. This is a contradiction to $0 \neq \alpha \in (\ker \Delta)^\perp$.

3 The idea behind the proof of $\operatorname{im} \Delta = (\ker \Delta)^\perp$ is the equation $(\ker T)^\perp = \operatorname{im}(T^*)$ for linear mappings T between finite-dimensional vector spaces. In infinite-dimensions, this is no longer true, but by means of ellipticity we can show it for $T := \Delta$ now.

($\subseteq$) $\operatorname{im} \Delta \subseteq (\ker \Delta)^\perp$ holds, because $\langle \Delta\alpha, \varphi \rangle = \langle \alpha, \Delta\varphi \rangle = 0$ for $\varphi \in \ker \Delta$ since Δ is symmetric.

($\supseteq$) Let $\gamma \in (\ker \Delta)^\perp$. We define $\tilde{\alpha}(\Delta\varphi) := \langle \gamma, \varphi \rangle$ for all $\varphi \in \Omega^k$. Then $\tilde{\alpha} : \operatorname{im} \Delta \to \mathbb{R}$ is well-defined, because $\Delta\varphi_1 = \Delta\varphi_2$ implies $\varphi_1 - \varphi_2 \in \ker \Delta$ and thus

$\langle \gamma, \varphi_1 - \varphi_2 \rangle = 0$. And $\tilde{\alpha} : \operatorname{im} \Delta \to \mathbb{R}$ is bounded because for the part ψ of φ, which is orthogonal to $\ker \Delta$, we have $\Delta\varphi = \Delta\psi$ and thus by 2:

$$|\tilde{\alpha}(\Delta\varphi)| = |\tilde{\alpha}(\Delta\psi)| = |\langle \gamma, \psi \rangle| \le \|\gamma\| \cdot \|\psi\| \le c \cdot \|\gamma\| \cdot \|\Delta\psi\| = c \cdot \|\gamma\| \cdot \|\Delta\varphi\|$$

So $\tilde{\alpha}$ is extendable to a $\|\_\|$-bounded linear functional on Ω^k by the Hahn-Banach theorem (see [**85**, 7.2]). This extension however is a weak solution of $\Delta\tilde{\alpha} = \gamma$, and thus there exists an $\alpha \in \Omega^k$ by Lemma 5.7 with $\langle \alpha, \varphi \rangle = \tilde{\alpha}(\varphi)$ for all φ, so $\langle \Delta\alpha, \varphi \rangle = \langle \alpha, \Delta\varphi \rangle = \tilde{\alpha}(\Delta\varphi) = \langle \gamma, \varphi \rangle$. Hence $\gamma = \Delta\alpha \in \operatorname{im} \Delta$. □

5.9 Corollary (Orthogonal decomposition of forms).

For compact orientable Riemannian manifolds M we have the following orthogonal decompositions:

$$\Omega = \ker \Delta \oplus \operatorname{im} \Delta \quad \text{and} \quad \operatorname{im} \Delta = \operatorname{im} d \oplus \operatorname{im} d^*$$

Proof. The first direct sum decomposition was shown in 5.8. Now for the second:

($\supseteq$) The linear subspaces $\operatorname{im} d$ and $\operatorname{im} d^*$ are included in $\operatorname{im} \Delta = (\ker \Delta)^\perp$ because $\langle d\alpha, \beta \rangle = \langle \alpha, d^*\beta \rangle = \langle \alpha, 0 \rangle = 0$ and $\langle d^*\alpha, \beta \rangle = \langle \alpha, d\beta \rangle = \langle \alpha, 0 \rangle = 0$ for all $\beta \in \ker \Delta = \ker(d) \cap \ker(d^*)$ by 5.5.

($\subseteq$) This is obvious because of $\Delta = d\,d^* + d^* d$.

($\oplus$) The sum is orthogonal, because $\operatorname{im} d$ is normal to $\operatorname{im} d^*$ since $\langle d\alpha, d^*\beta \rangle = \langle d^2\alpha, \beta \rangle = \langle 0, \beta \rangle = 0$. □

5.10 Definition (Green operator).

Because of 5.8.2 and 5.8.3, $\Delta : (\ker \Delta)^\perp \to \operatorname{im} \Delta = (\ker \Delta)^\perp$ is an open bijection and, if we denote the orthonormal projection with $H : \Omega \to \ker \Delta$, the Green operator G defined by $G := (\Delta|_{\operatorname{im} \Delta})^{-1} \circ H^\perp : \Omega \to (\ker \Delta)^\perp \to (\ker \Delta)^\perp$ with $H^\perp := \operatorname{id}_\Omega - H$ is the uniquely determined solution operator of $\Delta(G(\alpha)) = H^\perp(\alpha)$ for all $\alpha \in \Omega$.

Consequently, G is a bounded operator and - as an inverse to the symmetric elliptic differential operator Δ - it is symmetric and compact.

5.11 Corollary.

If $T : \Omega \to \Omega$ is a linear operator that commutes with Δ, i.e. $T \circ \Delta = \Delta \circ T$, then it also commutes with G. This holds in particular to d, d^* and Δ.

Proof. From $T \circ \Delta = \Delta \circ T$ it follows that $\ker \Delta = \operatorname{im} H$ and $\operatorname{im} \Delta = (\ker \Delta)^\perp = \operatorname{im} H^\perp$ are both T-invariant. Thus T commutes with H and $H^\perp$, hence with $G = \Delta^{-1} \circ H^\perp$: In fact, $T(H(x)) \in \ker \Delta$, $T(H^\perp(x)) \in \operatorname{im} \Delta$, and $T(H(x)) + T(H^\perp(x)) = T(x) = H(T(x)) + H^\perp(T(x))$, thus $T(H(x)) = H(T(x))$ and $T(H^\perp(x)) = H^\perp(T(x))$). □

5.12 Corollary (Harmonic representatives).

The cohomology $H(M)$ of M is isomorphic to the space $\ker \Delta$ of the harmonic forms. More precisely, in every cohomology class there is exactly one harmonic representative.

Proof. By 5.9 we have $\Omega = \ker \Delta \oplus \operatorname{im} d \oplus \operatorname{im} d^*$.
We claim that $\ker d = \ker \Delta \oplus \operatorname{im} d$:
($\supseteq$) By 5.5 we have $\ker \Delta = \ker d \cap \ker d^* \subseteq \ker d$ and $\operatorname{im} d \subseteq \ker d$ because of $d^2 = 0$.

(⊆) Let $\omega \in \ker d$. By 5.9 we have $\omega = \omega_1 + \omega_2 + \omega_3$ with $\omega_1 \in \ker \Delta$, $\omega_2 \in \operatorname{im} d$ and $\omega_3 \in \operatorname{im} d^*$ and thus $0 = d\omega = d\omega_1 + d\omega_2 + d\omega_3$ with $d\omega_1 = 0 = d\omega_2$ because of (⊇), and hence also $d\omega_3 = 0$. Since $\omega_3 \in \operatorname{im} d^*$, there exists an α with $d^*\alpha = \omega_3$ and thus $\|\omega_3\|^2 = \|d^*\alpha\|^2 = \langle d^*\alpha, d^*\alpha\rangle = \langle dd^*\alpha, \alpha\rangle = \langle d\omega_3, \alpha\rangle = \langle 0, \alpha\rangle = 0$. So $\omega = \omega_1 + \omega_2 \in \ker \Delta \oplus \operatorname{im} d$. □

5.13 Corollary (Finite-dimensional cohomology).

The cohomology of any compact, orientable manifold is finite-dimensional, that is, all Betti numbers are finite.

Proof. We choose a Riemann metric on M, then $H(M) \cong \ker \Delta$ by 5.12 and thus is finite-dimensional by 5.8. □

5.14 Definition (Poincaré duality).

For each compact oriented m-dimensional Riemannian manifold (M, g), the mapping

$$\Omega^{m-k}(M) \times \Omega^k(M) \to \mathbb{R} \text{ given by } (\alpha, \beta) \mapsto \int_M \alpha \wedge \beta$$

induces a bilinear mapping $H^{m-k}(M) \times H^k(M) \to \mathbb{R}$, the so-called Poincaré duality.

This definition makes sense, because $\alpha_2 - \alpha_1 = d\alpha$ implies $\alpha_2 \wedge \beta - \alpha_1 \wedge \beta = d\alpha \wedge \beta = d(\alpha \wedge \beta) \pm \alpha \wedge d\beta$, where $d\beta = 0$ since $[\beta] \in H^k(M) = \ker d / \operatorname{im} d$. Thus, according to the Theorem [95, 28.11] of Stokes $\int_M \alpha_2 \wedge \beta = \int_M \alpha_1 \wedge \beta$.

5.15 Lemma.

The Poincaré duality induces an isomorphism $H^{m-k} \cong (H^k)^*$, i.e. the Betti numbers satisfy $\beta_k = \beta_{m-k}$.

In [95, 29.22] we have generalized this to an isomorphism $H^k(M) \to H_c^{m-k}(M)^*$ for connected oriented (triangulated) manifolds M.

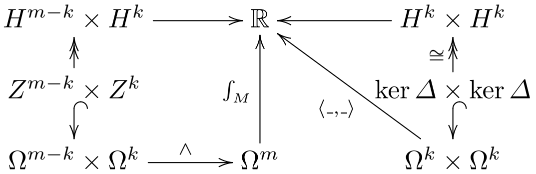
Proof. We first show that the Poincaré duality is not degenerate.

Let $0 \neq [\alpha] \in H^{m-k}$. Because of 5.12 we may assume that α is harmonic and thus also $d^*\alpha = 0$. If we put $\beta := *\alpha$, then $d\beta = d * \alpha = \pm * d^*\alpha = 0$ and $\int_M \alpha \wedge \beta = \int_M \alpha \wedge *\alpha = \int_M \langle \alpha, \alpha \rangle \operatorname{vol} > 0$, since $\alpha \neq 0$.

Each bilinear non-degenerate map $b : E \times F \to \mathbb{R}$ induces an isomorphism $b^\vee : E \to F^*$ on finite-dimensional vector spaces:
The induced mapping $E \ni v \mapsto b(v, \cdot) \in F^*$ is injective, because $b(v, w) = 0$ for all $w \in F$ implies $v = 0$. So $\dim E \le \dim(F^*) = \dim F$, and for reasons of symmetry $\dim E = \dim F$. Thus, the induced mapping is an isomorphism. □

Remark.

Since H^k is finite-dimensional by 5.13, each inner product on H^k provides an isomorphism $\sharp : H^k \to (H^k)^*$, and thus an isomorphism $H^{m-k} \to (H^k)^* \leftarrow H^k$ by 5.15. Using in particular the isomorphism $H(M) \cong \ker \Delta \subseteq \Omega(M)$ and the inner product of 5.5 induced by $\Omega(M)$, the above isomorphism $H^{m-k} \cong H^k$ can be described as follows:



$$H^{m-k} \xrightarrow[\cong]{} (H^k)^* \xleftarrow[\cong]{\sharp} H^k$$

$$[\alpha] \leftrightarrow \Big([\beta] \mapsto \int_M \alpha\wedge\beta\Big) \leftrightarrow \gamma \in \ker \Delta,$$

$$\text{with}\quad \int_M \alpha\wedge\beta = \sharp(\gamma)(\beta) := \langle\gamma,\beta\rangle_{\Omega^k(M)} := \int_M \langle\gamma,\beta\rangle_{\Lambda^k(M)}\,\mathrm{vol}_M$$
$$= \int_M \beta\wedge *\gamma = (-1)^{k(m-k)}\int_M *\gamma\wedge\beta$$

for all $[\beta]\in H^k(M)$, thus $[\alpha] = (-1)^{k(m-k)}[*\gamma]$ or $[\gamma] = (-1)^{k(m-k)}[**\gamma] = [*\alpha]$, i.e. the isomorphism is given on representatives by the Hodge-Star operator. Note that

$$\begin{aligned}
\Delta(*\gamma) &= (dd^* + d^*d)*\gamma\\
&= (-1)^{1+m+m(m-k)} d*d**\gamma + (-1)^{1+m+m(m-k+1)} *d*d*\gamma\\
&= (-1)^{1+m+m(m-k)+k(m-k)} d*d\gamma + (-1)^{1+m+m(m-k+1)} *d*d*\gamma\\
&= *\Big((-1)^{m(m-1-2k)}(-1)^{1+m+m(k+1)} *d*d+\\
&\qquad + (-1)^{m(m-1-2k)}(-1)^{2m}(-1)^{1+m+mk} d*d*\Big)\gamma\\
&= (-1)^{m(m-1-2k)} *(d^*d+dd^*)\gamma = *\Delta\gamma = *0 = 0, \qquad \text{for } \gamma\in\ker\Delta,
\end{aligned}$$

i.e. $*$ maps harmonic forms to such.

5.16 Corollary.

If M is a compact connected orientable m-dimensional manifold then $H^m(M)\cong\mathbb{R}$, i.e. $\beta_m = 1$. The isomorphism is given by integrating the representatives.

Compare this with [**95**, 29.5].

Proof. The Poincaré duality provides the isomorphism $H^m \cong (H^0)^*$, and $H^0\cong\mathbb{R}$, by [**95**, 26.5.2] because M is connected. The composition of the isomorphisms $H^m\cong (H^0)^*\cong H^0\cong\mathbb{R}$ is $[\omega]\mapsto\int_M\omega\wedge 1 = \int_M\omega$. □

5.17 Corollary.

If M is compact, orientable and of odd dimension, then the Euler characteristic $\chi = \sum_k (-1)^k\beta_k$ vanishes.

Proof. Let $\dim M = 2n+1 = m$, then

$$\begin{aligned}
\chi &= \sum_{k=0}^{m}(-1)^k\beta_k = \sum_{k=0}^{n}(-1)^k\beta_k + \sum_{k=n+1}^{m}(-1)^k\beta_k\\
&= \sum_{k=0}^{n}(-1)^k\beta_k + \sum_{k=0}^{n}(-1)^{m-k}\beta_{m-k} = \sum_{k=0}^{n}(-1)^k\underbrace{(\beta_k-\beta_{m-k})}_{=0,\text{ by } 5.15} = 0.
\end{aligned}$$ □

5.18 Can one hear the shape of a drum? [72].

To get that very vivid problem into a mathematical formulation, let's imagine a drum as a bounded surface in $\mathbb{R}^2$. If we let it vibrate with the edge held tight, it has certain natural frequencies that we could hear - at least with absolute pitch. Now the question arises whether the surface is already completely determined up to isometries by this spectrum of natural frequencies.

More generally, we can also pose this problem for arbitrary-dimensional, abstract oriented Riemannian manifolds. Since we only want to bring them a little bit out of the rest position, it does not matter in which surrounding space the manifold is isometrically embedded, most easily in $M \times \mathbb{R}$. Now let $u(x,t)$ be the distance of point $x \in M$ at time t from its rest position. Then, as in the usual equation of the vibrating string (see, for example, [83, 9.3.1]), u satisfies the 2nd order partial differential equation

$$\frac{\partial^2 u}{\partial t^2} + \Delta u = 0 \text{ with } u|_{\partial M} = 0,$$

where Δ is the Laplace Beltrami operator of the Riemannian manifold.

The usual solution method uses the separate variable approach (see [83, 9.3.2]), that is $u(x,t) := \varphi(x)\cdot\psi(t)$. The equation then translates into $\frac{\Delta\varphi}{\varphi}(x) = -\frac{\psi''}{\psi}(t)$ and thus both sides must be constant, e.g. equal to λ. Thus we are looking for eigenvalues $\lambda \in \mathbb{R}$ and eigenfunctions $\varphi \in C^\infty(M,\mathbb{R})$ of the operator $\Delta : C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$.

If M is compact, all eigenvalues are real by 5.5 and the eigenfunctions for different eigenvalues are orthogonal (since Δ is symmetric). The eigenvalues are all not negative (since Δ is positive) and can be ordered into a monotonically increasing sequence (λ_k), which accumulates only at infinity, because otherwise an associated orthonormal sequence of eigenfunctions by 5.6 would have a Cauchy subsequence. Using an orthonormal sequence of associated eigenfunctions $\varphi_k \in C^\infty(M,\mathbb{R})$, the wave equation can be solved by means of Fourier series

$$u(x,t) = \sum_{k=0}^{\infty}\Big(a_k \cos(\sqrt{\lambda_k}t) + b_k \sin(\sqrt{\lambda_k}t)\Big)\cdot\varphi_k(x),$$

the constants a_k and b_k being determined by the initial conditions. The sound wave of the manifold is then a suitable mean:

$$s(t) = \sum_{k=0}^{\infty}\Big(\alpha_k \cos(\sqrt{\lambda_k}t) + \beta_k \sin(\sqrt{\lambda_k}t)\Big).$$

And so we can (in some sense) hear the λ_k.

This sequence (λ_k) is called the SPECTRUM OF THE RIEMANNIAN MANIFOLD. For example, it can be shown that the spectrum of S^n is the sequence $(k(k+n-1))_{k=0}^{\infty}$, where each $k > 0$ occurs with multiplicity $\frac{(n+2k-1)!(n+k-2)!}{(n-1)!k!}$.

It was also shown that the following things can be heard, i.e. are are uniquely determined by the spectrum alone: The dimension, the volume, and the Euler characteristic, and thus the genus (of a 2-dimensional manifold without boundary) and the total scalar curvature (see 14.13).

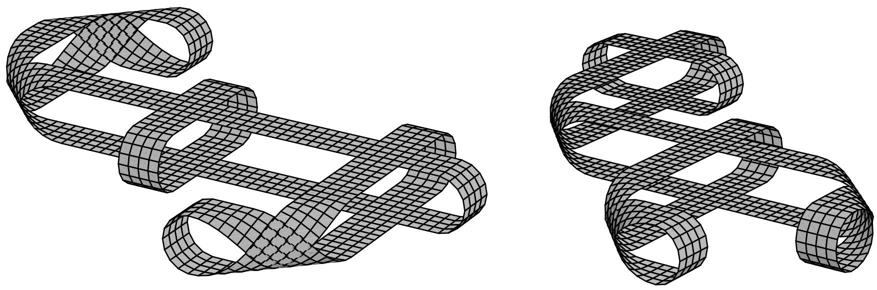
It was furthermore shown that the following Riemannian manifolds with their canonical metric can be recognized by listening: the spheres S^n, the real projective spaces $\mathbb{P}^{2n-1}$ for $n \le 3$, the flat torus $S^1 \times S^1$, as well as all compact 3-dimensional manifolds with constant curvature $K > 0$.

However, there are ISOSPECTRAL RIEMANNIAN MANIFOLDS that are not isometric. The first example was found by Milnor in [118] and was two 16-dimensional tori.

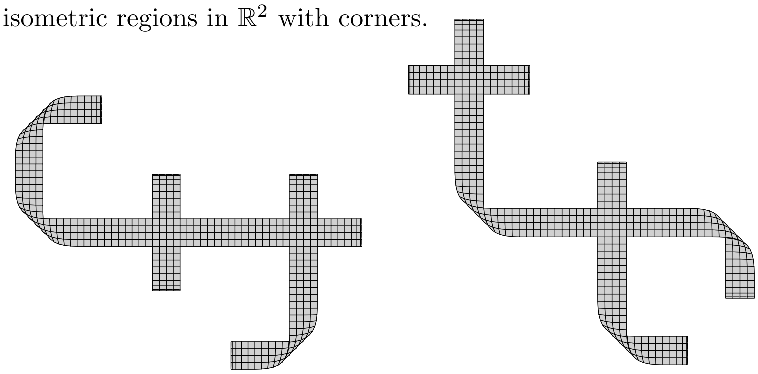
Marie-France Vignéras [**145**, Théoreme 8, S.31] constructed 2-dimensional examples obtained as quotients of the hyperbolic half-plane modulo discrete groups of isometries. That there are even isospectral deformations of Riemannian manifolds has been shown by Gordon and Wilson [**51**]. Sunada adapted a method of number theory in [**139**, Theorem 1, S.170]: Let $M \to M_0$ be a normal (see [**92**, 6.25]) Riemann covering map with finite deck transformation group G. If all conjugate classes of G meet two subgroups G_1 and G_2 in the same number of elements, the total spaces of the associated coverings $M_1 \to M_0$ and $M_2 \to M_0$ are isospectral. Building on this, Gordon, Webb and Wolpert finally constructed in [**50**] a surface M with boundary, composed of $168 = 7 \cdot 24$ crosses, and on which the elements of group $SL_{\mathbb{Z}_2}(3)$ of order 168 act as fixed-point-free isometries. The respective subgroups

$$G_1 := \left\{ \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\} \quad \text{und} \quad G_2 := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \right\}$$

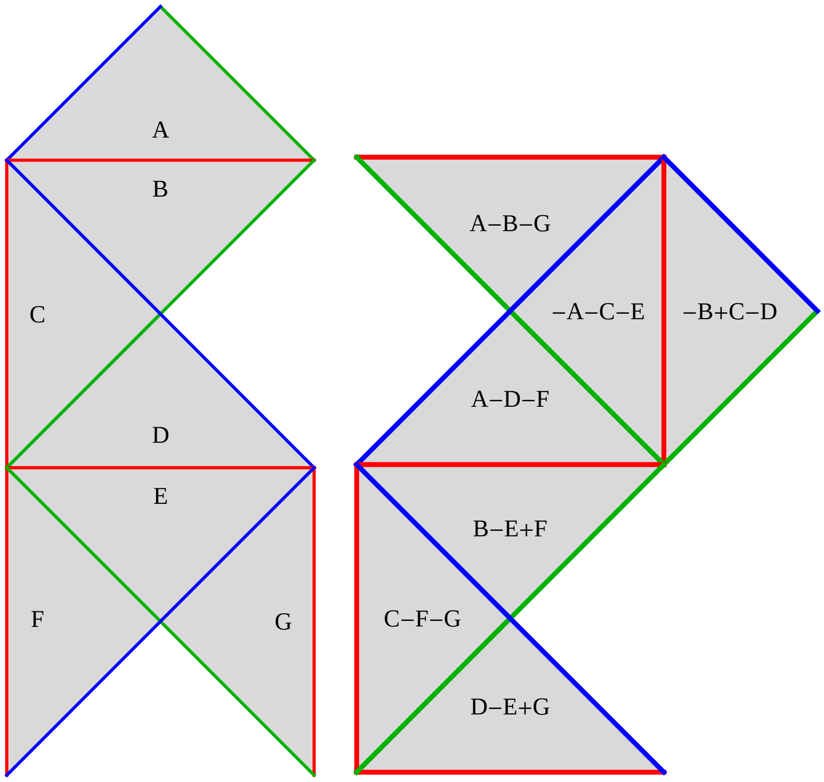
with $24 = 2{\cdot}2{\cdot}6$ elements then provide two 24-fold covering maps $M \to M/G_i =: M_i$ with M_1 and M_2 isospectral but not isometric.



Factorizing the obvious isometric involution $\tau_i : M_i \to M_i$ results in two isospectral but not isometric regions in $\mathbb{R}^2$ with corners.

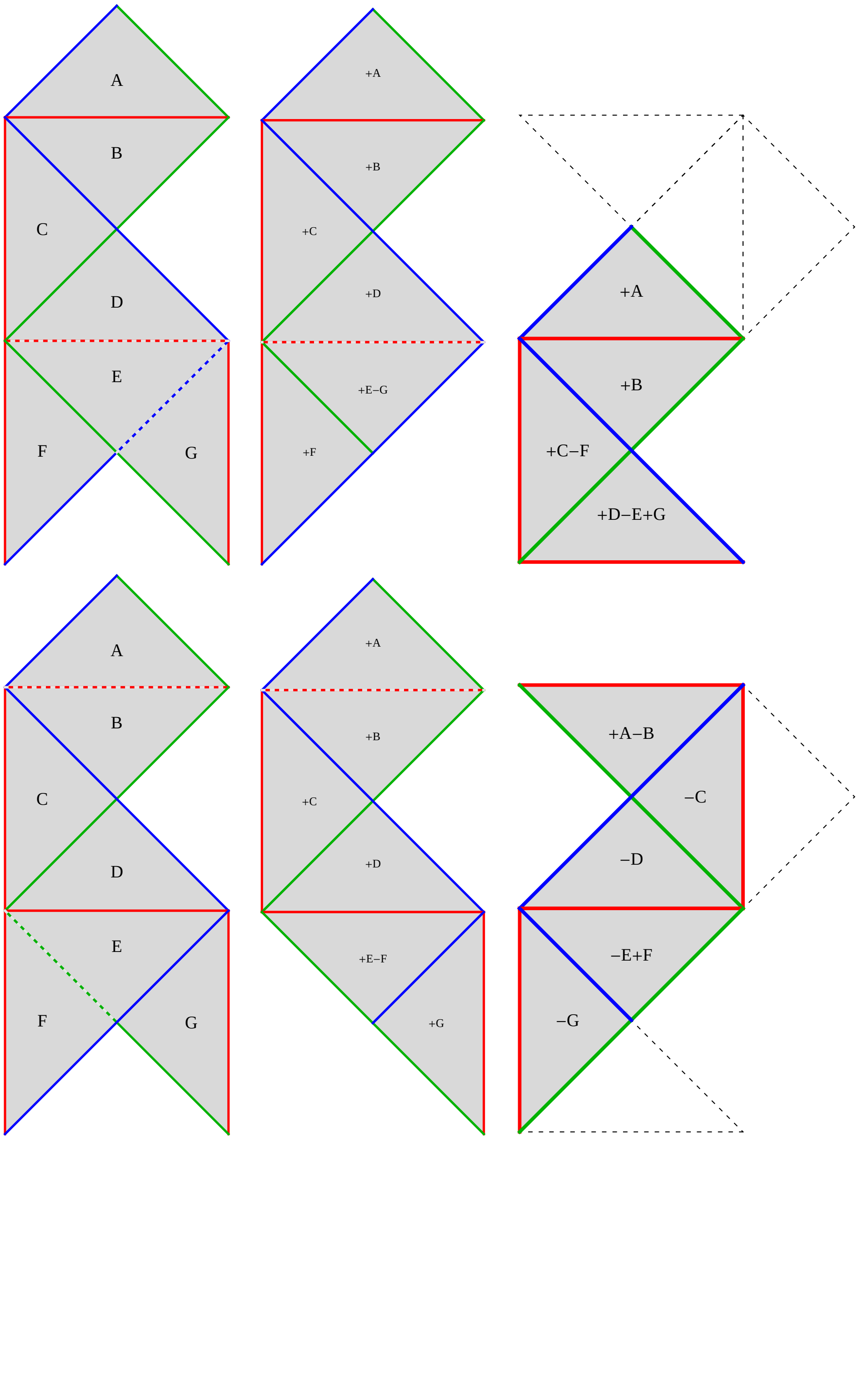


An elementary geometric proof of Sunada's theorem was provided by Buser in [**22**] by constructing an isometry $L^2(M_1) \to L^2(M_2)$ which identifies the eigenspaces to the same eigenvalue: Consider the following two domains in $\mathbb{R}^2$ consisting of 7 identical triangles each. The restrictions of some eigenfunction to the triangles on the left are denoted $A, \dots, G$.



Now construct an eigenfunction for the domain on the right hand side by taking a sum of these parts composed with the motion mapping the corresponding triangles to one another. A minus indicates that one has to use a reflection as well. It is not hard to see that the obtained function vanishes on the boundary and is at least C^1 (by the property that Δ commutes with motions and reflections), hence a weak solution of the eigenvalue equation, and by 5.7 a true solution. This mapping is easily seen to be injective, and similiarly we get a mapping in the opposite direction. Hence the eigenspaces of the two domains are isomorphic.
Note, that the necessary combinations are easily determined: Lets start by putting A on the top triangle. In order that the new function vanishes on the red hypothenuses we have to subtract B. In order that it prolongs C^1 to the next triangle, we have to use $-A-C$ there. And when we prolong along the red edge to the third triangle, we need $-B+C$ there. This vanishes on the blue vertex, but not on the green one, so we have to add $-D$ on the third, $-E$ on the second, and $-G$ on the first triangle. Then the function obtained so far will be C^1 on all 3 triangles and vanish on the outer boundary edges. So we extend to the next triangle, and so on.

An even more geometric argument is given by folding the domain on the left along the dotted lines, to get some subset of the domain on the right side. The corresponding function will be continuous and vanish on the boundary, but will not be C^1. So we do this in 3 different ways and finally sum up the 3 partial functions obtained. The resulting combination is exactly the function described above, which is C^1 also on the interior edges as seen before.


A
B
C
D
E
F
G
+A
+B
+C
+D
+E−G
+F
+A
+B
+C−F
+D−E+G
A
B
C
D
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+D
+E−F
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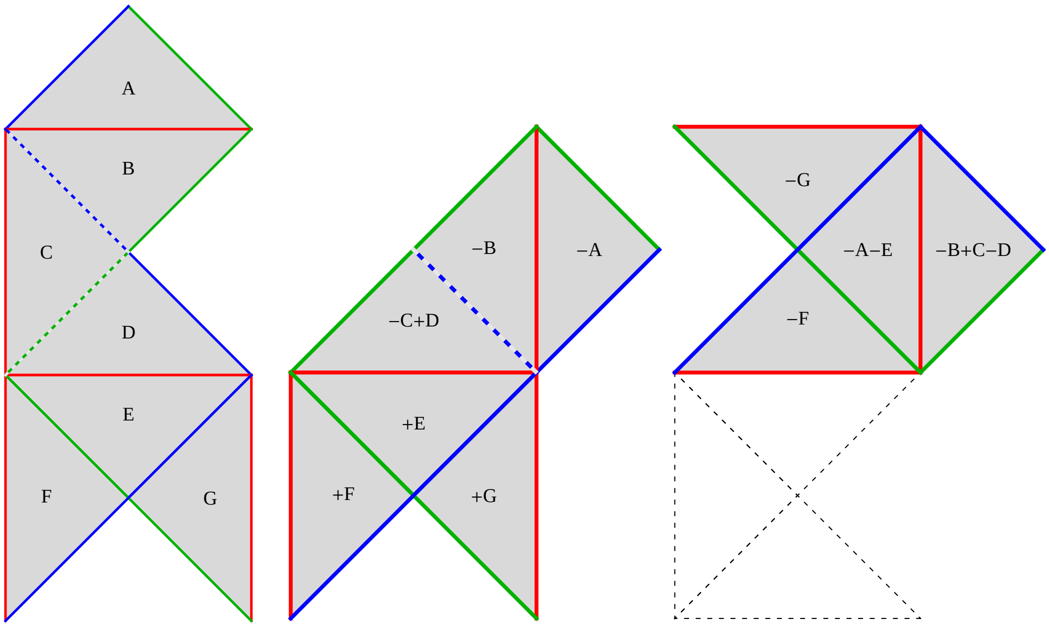
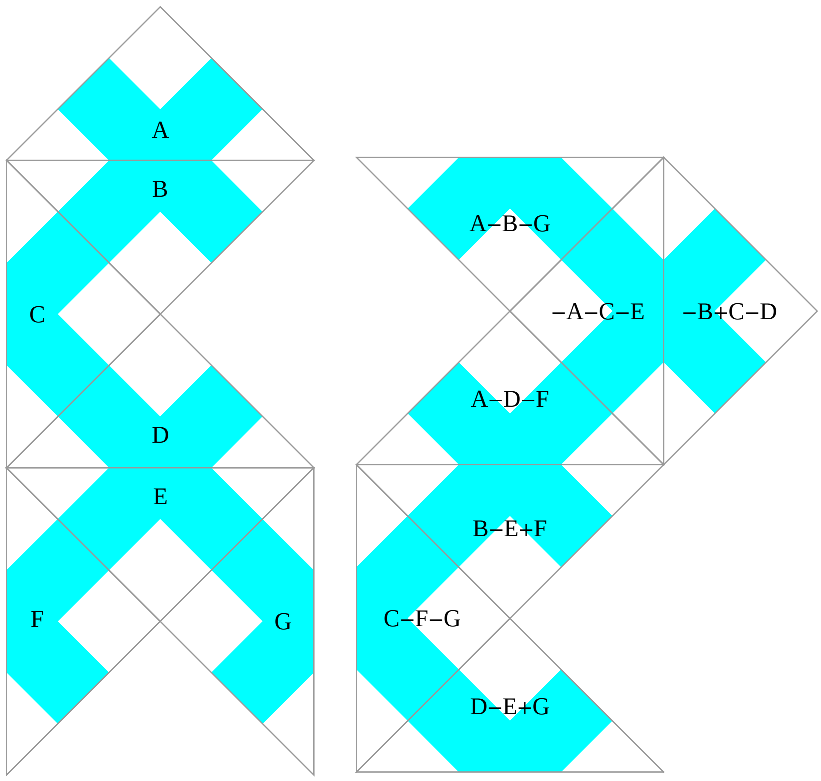
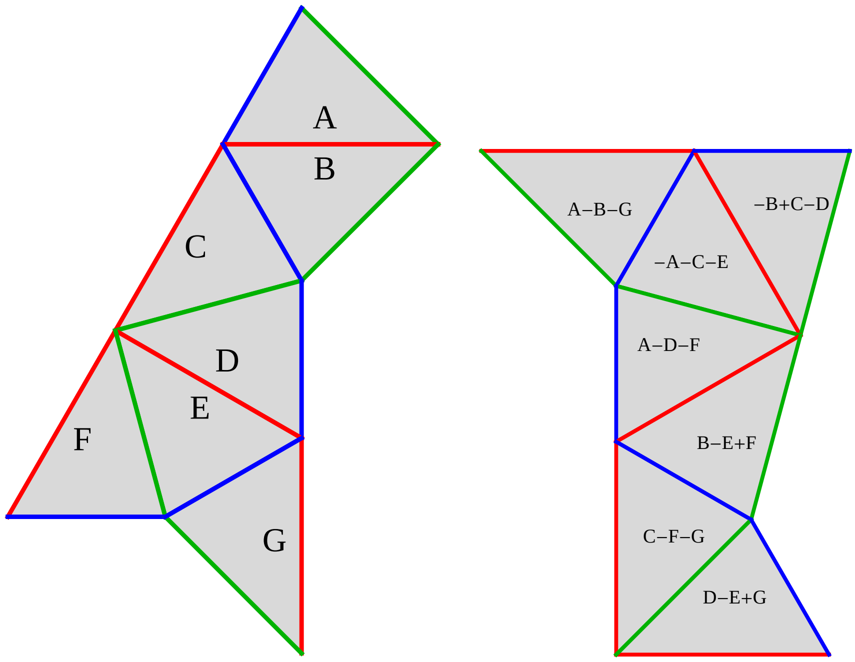
+A−B
−C
−D
−E+F
−G



This was also used by Berard in [8] to obtain, among others, the example of [50] by using instead of the entire triangle an appropriate subset:



For an even simpler example, see [23, S.3], where one uses instead of a rectangular triangle one with appropriate;y choosen angles:



6. Classical mechanics

6.1 Force Law of Newton.

For the force F, the mass m, and the acceleration $\ddot{x}$, the following formula holds:

$$F(x) = m \cdot \ddot{x},$$

here and in the following we restrict ourselves to time-independent forces for the sake of simplicity. We also set $m = 1$, because a general m can be absorbed in F.

For now, let the space $Q \subseteq \mathbb{R}^n$ of positions x be open. The function $F : Q \to \mathbb{R}^n$ can then be interpreted as a vector field. Particularly important is the case when F is a gradient field, that is, a potential $U : Q \to \mathbb{R}$ exists with $F = -\operatorname{grad} U$. This is a local (integrability) condition $dF = 0$ and a global (cohomologic) condition $H^1(Q) = 0$ at Q, see [**95**, 26.5.6] and [**95**, 26.5.7].

Newton's equation is an ordinary differential equation of second order. Thus can be rewritten as a (system of) ordinary differential equation(s) of 1st order on $TQ = Q \times \mathbb{R}^n$ by using the velocity vector $v = \dot{x}$ as an additional variable:

$$\dot{x} =: v$$
$$\dot{v} = F(x).$$

The simplest invariant of this DG is the Energy

$$E(x, v) := \tfrac{|v|^2}{2} + U(x),$$

because

$$\frac{d}{dt} E(x, \dot{x}) = \langle \dot{x}, \ddot{x} \rangle + U'(x) \cdot \dot{x} = \langle \dot{x}, -\operatorname{grad}(U)(x) \rangle + U'(x) \cdot \dot{x} = 0.$$

$m \frac{|\dot{x}|^2}{2}$ is the kinetic and $U(x)$ the potential energy.

6.2 Newton's law on manifolds.

As we noted in [95, 14.1], a 1st order ordinary differential equation on a manifold Q is described by a vector field $\xi : Q \to TQ$, because the first derivative of a curve $x : \mathbb{R} \to Q$ is a curve $\dot{x} : \mathbb{R} \to TQ$ with values in the tangent bundle TQ. If $(x^1, \dots, x^n)$ are local coordinates on Q, then the derivations $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n})$ form a basis of the tangent space T_xQ. If $(v^1, \dots, v^n)$ are the coordinates with respect to this basis, then $(x^1, \dots, x^n; v^1, \dots, v^n)$ are local coordinates of the tangent bundle TQ, the foot point map $\pi_Q : TQ \to Q$ is given in local coordinates by the assignment

$$(x^1, \dots, x^n; v^1, \dots, v^n) \mapsto (x^1, \dots, x^n),$$

and the derivative of the curve $t \mapsto x(t) \in Q$ is given by

$$t \mapsto (x^1(t), \dots, x^n(t); \dot{x}^1(t), \dots, \dot{x}^n(t)).$$

What corresponds to an ordinary differential equation of second order, as it is represented by the law of force? The second derivative of a curve $x : \mathbb{R} \to Q$ is a curve $\ddot{x} : \mathbb{R} \to T(TQ) =: T^2Q$ with values in the second tangent bundle of Q. If $(x^1, \dots; v^1, \dots)$ are local coordinates on TQ as above, then the derivatives $(\frac{\partial}{\partial x^1}, \dots; \frac{\partial}{\partial v^1}, \dots)$ form a basis of the tangent space $T_{(x,v)}(TQ)$ to the manifold TQ in the point $(x, v) \in TQ$. If $(y^1, \dots, y^n; w^1, \dots, w^n)$ are the coordinates with respect to this basis, $(x^1, \dots; v^1, \dots; y^1, \dots; w^1, \dots)$ are local coordinates of the second tangent bundle T^2Q, and the second derivative of the curve $t \mapsto x(t) \in Q$ is as follows:

$$\ddot{x} = (x^1, \dots, x^n; \dot{x}^1, \dots, \dot{x}^n; \dot{x}^1, \dots, \dot{x}^n; \ddot{x}^1, \dots, \ddot{x}^n).$$

With respect to these coordinates on T^2Q, the foot point map $\pi_{TQ} : T^2Q \to TQ$ is given by $(x^1, \dots; v^1, \dots; y^1, \dots; w^1, \dots) \mapsto (x^1, \dots; v^1, \dots)$, whereas the derivativew $T\pi_Q : T^2Q \to TQ$ of it is given by $(x^1, \dots; v^1, \dots; y^1, \dots; w^1, \dots) \mapsto (x^1, \dots; y^1, \dots)$. An ordinary differential equation of the second order $\ddot{x} = X(x, \dot{x})$ is therefore given by a map $X : TQ \to T(TQ)$ which has the following appearance in coordinates:

$$X(x, v) = (x^1, \dots, x^n; v^1, \dots, v^n; v^1, \dots, v^n; X^1(x, v), \dots, X^n(x, v)).$$

Or using the basis vector fields we have

$$X(x, v) = \sum_{i=1}^{n} v^i \, \tfrac{\partial}{\partial x^i} + \sum_{i=1}^{n} X^i(x, v) \, \tfrac{\partial}{\partial v^i}.$$

The mapping $X : TQ \to T(TQ)$ is thus a vector field on TQ, which additionally has the property that $T\pi_Q \circ X = \mathrm{id}$, i.e. the second and third components are the same. One can also formulate this additional condition by $\kappa_Q \circ X = X$, where $\kappa_Q : T^2Q \to T^2Q$ denotes the canonical flip which swaps the two middle components (this is globally defined!). A vector field X on TQ with this additional property is called a spray. So these describe ordinary differential equations of 2nd order on Q. For the solution curves $c : \mathbb{R} \to TQ$ of the corresponding differential equation of 1st order on TQ we have therefore

$$\tfrac{d}{dt}(\pi_Q \circ c) = T\pi_Q \circ X \circ c = \mathrm{id} \circ c = c.$$

6.3 Variation problem.

With the philosophy that nature proceeds in a minimalistic way, one will try to find a functional in the space of the curves whose critical points are precisely the solution curves of the differential equation. Let's look at the case that $Q \subseteq \mathbb{R}^n$ is open. The critical points of a function I of the form

$$I(x) := \int_a^b L(x(t), \dot{x}(t)) \, dt$$

are just the solutions of the Euler-Lagrange equation

$$\frac{\partial}{\partial x^i} L(x,\dot x) = \frac{d}{dt}\frac{\partial}{\partial \dot x^i} L(x,\dot x) \text{ for } i = 1,\dots,n$$

of an implicit differential equation of second order.

To see this, note that the functional

$$x \mapsto I(x) := \int_a^b L(x(t),\dot x(t))\,dt$$

has exactly x as a critical point when the direction derivative $\frac{d}{ds}\big|_{s=0} I(x+s\,v)$ vanishes for all v (with $v(a) = 0 = v(b)$). We calculate them now:

$$\begin{aligned}
\frac{d}{ds}\Big|_{s=0} I(x+s\,v) &= \int_a^b \frac{\partial L}{\partial x}(x(t),\dot x(t))\cdot v(t) + \frac{\partial L}{\partial v}(x(t),\dot x(t))\cdot \dot v(t)\;dt \\
&= \int_a^b \Big(\frac{\partial L}{\partial x}(x(t),\dot x(t)) - \frac{d}{dt}\Big(\frac{\partial L}{\partial v}(x(t),\dot x(t))\Big)\Big)\cdot v(t)\;dt \\
&\quad + \int_a^b \frac{d}{dt}\Big(\frac{\partial L}{\partial v}(x(t),\dot x(t))\cdot v(t)\Big)\;dt \\
&= \int_a^b \Big(\frac{\partial L}{\partial x}(x(t),\dot x(t)) - \frac{d}{dt}\Big(\frac{\partial L}{\partial v}(x(t),\dot x(t))\Big)\Big)\cdot v(t)\;dt + 0.
\end{aligned}$$

Since v was arbitrary, all components must therefore be

$$\frac{\partial L}{\partial x^i}(x(t),\dot x(t)) - \frac{d}{dt}\Big(\frac{\partial L}{\partial v^i}(x(t),\dot x(t))\Big) = 0.$$

For simplicity, suppose that the variables x and $\dot x$ are separated in L, that is, $L(x,\dot x) = f(x) + g(\dot x)$, then the Euler-Lagrange equation is:

$$\frac{\partial}{\partial x^i} f(x) = \frac{d}{dt}\frac{\partial}{\partial \dot x^i} g(\dot x) = \sum_{j=1}^n \frac{\partial^2}{\partial \dot x^j\,\partial \dot x^i} g(\dot x)\cdot \ddot x^j \text{ for } i = 1,\dots,n.$$

By comparison with the Newton equation $-\operatorname{grad} U(x) = F(x) = \ddot x$, we obtain as simplest solution for L the terms

$$g(\dot x) := \tfrac{|\dot x|^2}{2} \quad\text{ und }\quad f(x) := -U(x)$$

and thus the so-called Lagrange function

$$L(x,v) = f(x) + g(v) = \tfrac{|v|^2}{2} - U(x).$$

The time evolution is thus determined by a real-valued function $L : TQ \to \mathbb{R}$ instead of the more complicated object of a spray $X : TQ \to T(TQ)$. However, the beautiful explicite second order differential equation (Newton's law of force) has to be replaced by an in ccordinates implicite second order differential equation (the Euler-Lagrange equation), for which we have not developed a theory on manifolds.

6.4 Lagrangian formalism.

Conversely, let us try to obtain the vector field $X : TQ \to T^2Q$ and the energy $E : TQ \to \mathbb{R}$ from a general Lagrangian function $L : TQ \to \mathbb{R}$ on a manifold Q. The Euler-Lagrange equation looks in coordinates again as follows:

$$\tfrac{\partial}{\partial x^i} L = \tfrac{d}{dt}\tfrac{\partial}{\partial \dot x^i} L = \sum_j \dot x^j \tfrac{\partial^2}{\partial x^j\,\partial \dot x^i} L + \sum_j \ddot x^j \tfrac{\partial^2}{\partial \dot x^j\,\partial \dot x^i} L$$

Let's also write the desired vector field X_L in the local coordinates as:

$$X_L(x,v) = (x^1,\dots;v^1,\dots;v^1,\dots;X^1(x,v),\dots,X^n(x,v)).$$

we obtain the implicit equation

$$\tfrac{\partial}{\partial x^i}\,L = \tfrac{d}{dt}\,\tfrac{\partial}{\partial \dot{x}^i}\,L = \sum_j \dot{x}^j\,\tfrac{\partial^2}{\partial x^j\,\partial \dot{x}^i}\,L + \sum_j X^j\,\tfrac{\partial^2}{\partial \dot{x}^j\,\partial \dot{x}^i}\,L$$

for the coefficients X^i by substituting $\ddot{x}^i = X^i(x,\dot{x})$. If the matrix $(\frac{\partial^2 L}{\partial \dot{x}^i\,\partial \dot{x}^j})_{i,j=1}^n$ is invertible, we can calculate the X^i by multiplying it by the inverse matrix $L^{k,i}$:

$$\begin{aligned}\sum_i L^{k,i}\Big(\tfrac{\partial}{\partial x^i}\,L - \sum_j \dot{x}^j\,\tfrac{\partial^2}{\partial x^j\,\partial \dot{x}^i}\,L\Big) &= \sum_i\sum_j L^{k,i} X^j\,\tfrac{\partial^2}{\partial \dot{x}^j\,\partial \dot{x}^i}\,L\\ &= \sum_j X^j\,\delta^k_j = X^k\end{aligned}$$

The vector field

$$X_L = \sum_i v^i\,\tfrac{\partial}{\partial x^i} + \sum_i\sum_k L^{i,k}\Big(\tfrac{\partial}{\partial x^k}\,L - \sum_j \dot{x}^j\,\tfrac{\partial^2}{\partial x^j\,\partial \dot{x}^k}\,L\Big)\tfrac{\partial}{\partial v^i}$$

defined thereby is then called Lagrange vector field to L. We still have to check if this definition really defines something independent of coordinates. We will show that later.

Since we can solve the implicit equation for the Lagrangian vector field only under additional conditions, we will try to determine the simplest motion invariant, the energy E, directly from L.

In the special case where $Q \subseteq \mathbb{R}^n$ is open and $L(x,v) = \frac{|v|^2}{2} - U(x)$, we try to obtain the kinetic energy $|v|^2/2$ from L. In coordinates we can do that via

$$|v|^2 = \tfrac{d}{dt}\big|_{t=1}\,L(x,t\,v) = \tfrac{d}{dt}\big|_{t=0}\,L(x,v+tv) = \sum_{i=1}^n v^i\,\tfrac{\partial}{\partial v^i}L(x,v).$$

Thus, for a general vector bundle $V \to Q$ and a function $L : V \to \mathbb{R}$, we define the so-called fiber derivative $d_f L : V \to V^*$ of L by

$$d_f L(\xi)(\eta) := \tfrac{d}{dt}\big|_{t=0}\,L(\xi + t\eta).$$

If $(x^1,\dots;v^1,\dots)$ are local vector bundle coordinates of V with basis point coordinates $(x^1,\dots,x^n)$, then

$$(d_f L)(x,v)(x,w) = \sum_i \frac{\partial L}{\partial v^i}(x,v)\cdot w^i.$$

For a Lagrangian function $L : TQ \to \mathbb{R}$ of a general manifold Q we define the action $A : TQ \to \mathbb{R}$

$$A(\xi) := d_f L(\xi)\cdot\xi \text{ that is, } A(x,v) = \sum_{i=1}^n v^i\,\tfrac{\partial}{\partial v^i}L(x,v)$$

and the Energy $E : TQ \to \mathbb{R}$ as

$$E := A - L \text{ that is } E(x,v) = \sum_{i=1}^n v^i\,\tfrac{\partial}{\partial v^i}L(x,v) - L(x,v).$$

We can easily calculate that the energy is indeed a motion invariant, since

$$\begin{aligned}\frac{d}{dt}E(x(t),v(t)) &= \frac{d}{dt}\left(\sum_i v^i(t)\,\tfrac{\partial}{\partial v^i}L(x(t),v(t)) - L(x(t),v(t))\right)\\ &= \sum_i \dot{v^i}\,\frac{\partial L}{\partial v^i} + \sum_i v^i\,\frac{d}{dt}\frac{\partial L}{\partial v^i} - \sum_i\left(\frac{\partial L}{\partial x^i}\,v^i + \frac{\partial L}{\partial v^i}\,\dot{v^i}\right)\\ &= 0,\end{aligned}$$

because of the Euler-Lagrange equation.

6.5 Mechanics on Riemannian manifolds.

On a (pseudo) Riemannian manifold (Q, g), the Lagrangian function with respect to a potential $U : Q \to \mathbb{R}$ is defined in analogy by

$$L(\xi) = \tfrac{1}{2} g(\xi, \xi) - U(\pi(\xi)),$$

that is, in local coordinates

$$L(x, v) = \tfrac{1}{2} \sum_{i,j} g_{i,j}(x)\, v^i v^j - U(x).$$

The fiber derivative is obviously

$$d_f L(\xi) \cdot \eta = g(\xi, \eta)$$

and thus the action is $A(\xi) = g(\xi, \xi)$ and the energy is

$$E(\xi) = \tfrac{1}{2} g(\xi, \xi) + U(\pi(\xi)).$$

In the case of $U = 0$, the vector field X_L is called geodetic spray. There $\frac{\partial}{\partial x^i} L = \frac{1}{2} \sum_{j,k} \frac{\partial g_{j,k}}{\partial x^i} v^j v^k$ and $\frac{\partial}{\partial v^i} L = \sum_j g_{i,j}(x) v^j$. And thus, the matrix $(\frac{\partial^2 L}{\partial v^i \, \partial v^j})$ is just the coefficient matrix $(g_{i,j})$ of the metric and its inverse $(L^{i,j})$ is usually denoted $(g^{i,j})$, see [**95**, 24.2]. Furthermore, we have

$$\frac{\partial^2}{\partial x^k \, \partial v^i} L = \frac{\partial}{\partial x^k} \sum_j g_{i,j}(x) v^j = \sum_j \frac{\partial g_{i,j}}{\partial x^k} v^j$$

So the explicit Euler-Lagrange equation (see 6.4) is

$$\begin{aligned}
\ddot{x}^k &= \sum_i g^{k,i} \left(\frac{1}{2} \sum_{j,r} \frac{\partial g_{j,r}}{\partial x^i} v^j v^r - \sum_j \dot{x}^j \sum_r \frac{\partial g_{i,r}}{\partial x^j} v^r \right) \\
&= \sum_{i,j,r} g^{k,i} \, \dot{x}^j \, \dot{x}^r \left(\frac{1}{2} \frac{\partial g_{j,r}}{\partial x^i} - \frac{\partial g_{i,r}}{\partial x^j} \right) \\
&= - \sum_i g^{k,i} \sum_{j,r} \dot{x}^j \, \dot{x}^r \, \frac{1}{2} \left(- \frac{\partial g_{j,r}}{\partial x^i} + \frac{\partial g_{i,r}}{\partial x^j} + \frac{\partial g_{i,j}}{\partial x^r} \right) \\
&= - \sum_{j,r} \dot{x}^j \, \dot{x}^r \sum_i g^{k,i} \Gamma_{j,r,i} \\
&= - \sum_{j,r} \dot{x}^j \, \dot{x}^r \, \Gamma^k_{j,r}.
\end{aligned}$$

This is the differential equation of the geodesics, where the $\Gamma_{j,r,i}$ are the Christoffel symbols of 1st type and $\Gamma^k_{j,r}$ are that of 2nd type, see 10.5. So the integral curves in Q of the geodetic spray's X_L are just the geodesics, which we have also recognized as critical points of arc length.

For a general U and an $\varepsilon > U(x)$ for all $x \in Q$, one can define a new (the so-called Jacobi metric) $g_\varepsilon := (\varepsilon - U) \cdot g$. It can then be shown that the integral curves c of X_L with energy $\varepsilon = g(\dot{c}(t), \dot{c}(t))$ for all t, are up to reparametrization exactly the geodesics of the Jacobi metric g_ε with energy 1.

6.6 Relationship between Lagrange vector field X_L and energy E.

It would be nice if there were a similar relationship between X_L and E, as it exists between gradient and potential. In addition, let us recall that the gradient of a potential $U : Q \to \mathbb{R}$ with respect to a Riemannian metric g is given on Q by:

$$g_x(\operatorname{grad} U(x), \eta) = dU(x) \cdot \eta \text{ for all } \eta \in T_x Q,$$

where dU denotes the total differential of U. So we are looking for a bilinear form ω, which fullfills

$$\omega_{x,v}(X_L, Y) = dE(x,v)\cdot Y \qquad \text{for all } Y \in T_{(x,v)}(TQ).$$

Let X_L be the vector field on TQ describing the Euler-Lagrange equation for a sufficiently regular Lagrange function L. In coordinates, each vector field X_L, which describes an ordinary differential equation of second order, has the following form by what we have shown in 6.2:

$$X_L(x,v) = \sum_{i=1}^n v^i \tfrac{\partial}{\partial x^i} + \sum_{i=1}^n X^i(x,v)\, \tfrac{\partial}{\partial v^i}.$$

For the energy we have the formula

$$E(x,v) = \sum_{i=1}^n v^i \tfrac{\partial}{\partial v^i} L(x,v) - L(x,v)$$

according to 6.4 and for its differential

$$\begin{aligned} dE(x,v) &= \sum_j \tfrac{\partial}{\partial x^j} E(x,v)\, dx^j + \sum_j \tfrac{\partial}{\partial v^j} E(x,v)\, dv^j \\ &= \sum_j \left(\sum_{i=1}^n v^i \tfrac{\partial}{\partial x^j} \tfrac{\partial}{\partial v^i} L - \tfrac{\partial}{\partial x^j} L \right) dx^j \\ &\quad + \sum_j \left(\tfrac{\partial}{\partial v^j} L + \sum_{i=1}^n v^i \tfrac{\partial}{\partial v^j} \tfrac{\partial}{\partial v^i} L - \tfrac{\partial}{\partial v^j} L \right) dv^j \\ &= \sum_j \left(\sum_{i=1}^n v^i \tfrac{\partial}{\partial x^j} \tfrac{\partial}{\partial v^i} L - \tfrac{\partial}{\partial x^j} L \right) dx^j + \sum_j \left(\sum_{i=1}^n v^i \tfrac{\partial}{\partial v^j} \tfrac{\partial}{\partial v^i} L \right) dv^j. \end{aligned}$$

A general bilinear form ω on TQ, i.e. a 2-fold covariant tensor field on TQ, is given with respect to the local coordinates $(x^1, \dots; v^1, \dots)$ by:

$$\begin{aligned} \omega = &\sum_{i,j} \omega(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j})\, dx^i \otimes dx^j + \sum_{i,j} \omega(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial v^j})\, dx^i \otimes dv^j \\ &+ \sum_{i,j} \omega(\tfrac{\partial}{\partial v^i}, \tfrac{\partial}{\partial x^j})\, dv^i \otimes dx^j + \sum_{i,j} \omega(\tfrac{\partial}{\partial v^i}, \tfrac{\partial}{\partial v^j})\, dv^i \otimes dv^j. \end{aligned}$$

Consequently,

$$\begin{aligned} \sum_{i=1}^n v^i\, \omega(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^j}) + \sum_{i=1}^n X^i\, \omega(\tfrac{\partial}{\partial v^i}, \tfrac{\partial}{\partial x^j}) = \omega(X_L, \tfrac{\partial}{\partial x^j}) = dE \cdot \tfrac{\partial}{\partial x^j} &= \\ = \sum_{i=1}^n v^i\, \tfrac{\partial}{\partial x^j} \tfrac{\partial}{\partial v^i} L - \tfrac{\partial}{\partial x^j} L & \\ \sum_{i=1}^n v^i\, \omega(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial v^j}) + \sum_{i=1}^n X^i\, \omega(\tfrac{\partial}{\partial v^i}, \tfrac{\partial}{\partial v^j}) = \omega(X_L, \tfrac{\partial}{\partial v^j}) = dE \cdot \tfrac{\partial}{\partial v^j} &= \\ = \sum_{i=1}^n v^i\, \tfrac{\partial}{\partial v^j} \tfrac{\partial}{\partial v^i} L. & \end{aligned}$$

From the second equation we obtain by coefficient comparison

$$\begin{aligned} \omega(\tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial v^j}) &= \tfrac{\partial}{\partial v^j} \tfrac{\partial}{\partial v^i} L \\ \omega(\tfrac{\partial}{\partial v^i}, \tfrac{\partial}{\partial v^j}) &= 0. \end{aligned}$$

If we insert the implicit Euler-Lagrange equation

$$\tfrac{\partial}{\partial x^i} L = \tfrac{d}{dt}\tfrac{\partial}{\partial v^i} L = \sum_j v^j \tfrac{\partial^2}{\partial x^j\,\partial v^i} L + \sum_j X^j \tfrac{\partial^2}{\partial v^j\,\partial v^i} L$$

into the first equation, we get

$$\sum_{i=1}^n v^i\,\omega(\tfrac{\partial}{\partial x^i},\tfrac{\partial}{\partial x^j}) + \sum_{i=1}^n X^i\,\omega(\tfrac{\partial}{\partial v^i},\tfrac{\partial}{\partial x^j}) =$$
$$= \sum_{i=1}^n v^i \tfrac{\partial}{\partial x^j}\tfrac{\partial}{\partial v^i} L - \sum_{i=1}^n v^i \tfrac{\partial^2}{\partial x^i\,\partial v^j} L - \sum_{i=1}^n X^i \tfrac{\partial^2}{\partial v^i\,\partial v^j} L$$

and we are forced to put $\omega(\tfrac{\partial}{\partial v^i},\tfrac{\partial}{\partial x^j}) := -\tfrac{\partial^2}{\partial v^i\,\partial v^j} L$ and

$$\omega(\tfrac{\partial}{\partial x^i},\tfrac{\partial}{\partial x^j}) := \tfrac{\partial^2}{\partial x^j\,\partial v^i} L - \tfrac{\partial^2}{\partial x^i\,\partial v^j} L..$$

In the local coordinates $(x^1,\dots;v^1,\dots)$, ω is given by:

$$\omega = \sum_{i,j}\Big(\tfrac{\partial^2 L}{\partial x^j\,\partial v^i} - \tfrac{\partial^2 L}{\partial x^i\,\partial v^j}\Big)\,dx^i\otimes dx^j$$
$$+ \sum_{i,j}\tfrac{\partial^2 L}{\partial v^j\,\partial v^i}\,dx^i\otimes dv^j - \sum_{i,j}\tfrac{\partial^2 L}{\partial v^i\,\partial v^j}\,dv^i\otimes dx^j$$

It can thus be seen that the bilinear form ω is skew-symmetric, and thus represents a 2-form $\omega\in\Omega^2(TQ)$. Using $du^i\wedge du^j := du^i\otimes du^j - du^j\otimes du^i$, it is given in local coordinates by

$$\omega = \sum_{i,j}\tfrac{\partial^2 L}{\partial x^j\,\partial v^i}\,dx^i\wedge dx^j + \sum_{i,j}\tfrac{\partial^2 L}{\partial v^j\,\partial v^i}\,dx^i\wedge dv^j.$$

It is even exact, because the 1-form

$$\vartheta := \sum_{i=1}^n \tfrac{\partial L}{\partial v^i}\,dx^i \in \Omega^1(TQ)$$

has $d\vartheta = -\omega$ as outer derivative.

However, we have not yet checked if ω and ϑ are really coordinate independent. We will also do that in 6.7.

For each spray $X: TQ\to T^2Q$ we have

$$\vartheta X = \Big(\sum_i \frac{\partial L}{\partial v^i}\,dx^i\Big)\Big(\sum_j v^j\frac{\partial}{\partial x^j} + X^j(x,v)\frac{\partial}{\partial v^j}\Big) = \sum_i \frac{\partial L}{\partial v^i}\,v^i = A.$$

Globally, the defining implicit equation for the Lagrange vector field X can be written as

$$\iota_X\,\omega = dE,$$

where i is the insertion operator $(\iota_X\,\omega)(Y) := \omega(X,Y)$.

Unfortunately, these differential forms depend on L, so we better write $\omega_L := \omega$ and $\vartheta_L := \vartheta$.

6.7 Hamilton formalism.

To get rid of this dependence of the differential forms ω_L and ϑ_L on the Lagrange function L, we want to introduce new coordinates. Of course, the partial derivatives $p_i := \frac{\partial L}{\partial v^i}$ suggest themselves. We simply rename the coordinates x^i in the basis manifold to $q^i := x^i$.

The form ϑ_L is then given in the new coordinates by

$$\vartheta_0 := \sum_{i=1}^n p_i\,dq^i$$

and its outer derivative $-\omega_0 := d\vartheta_0$ by

$$\omega_0 := \sum_{i=1}^n dq^i \wedge dp_i.$$

But we have the problem, whether the q^i and p_i really are coordinates of a manifold (TQ?). For this we need on the one hand, that the $\frac{\partial p_i}{\partial v^j} := \frac{\partial^2 L}{\partial v^j \partial v^i}$ form an invertible matrix and on the other hand we have to determine the change of coordinates. If $(\bar{x}^1, \dots, \bar{x}^n)$ are other coordinates on Q, then for the basis of TQ we have

$$\tfrac{\partial}{\partial \bar{x}^i} = \sum_j \tfrac{\partial x^j}{\partial \bar{x}^i}\,\tfrac{\partial}{\partial x^j}$$

and for the components v^j with respect to these bases

$$v^j = \sum_i \tfrac{\partial x^j}{\partial \bar{x}^i}\bar{v}^i.$$

Furthermore,

$$\frac{\partial v^j}{\partial \bar{v}^i} = \frac{\partial}{\partial \bar{v}^i}\Big(\sum_k \frac{\partial x^j}{\partial \bar{x}^k}\,\bar{v}^k\Big) = \sum_k \frac{\partial x^j}{\partial \bar{x}^k}\,\frac{\partial \bar{v}^k}{\partial \bar{v}^i} = \sum_k \frac{\partial x^j}{\partial \bar{x}^k}\,\delta_i^k = \frac{\partial x^j}{\partial \bar{x}^i}.$$

Thus, for the new coordinates:

$$\bar{p}_i = \tfrac{\partial}{\partial \bar{v}^i}\,L = \sum_k \tfrac{\partial v^k}{\partial \bar{v}^i}\,\tfrac{\partial}{\partial v^k}\,L = \sum_k \tfrac{\partial x^k}{\partial \bar{x}^i}\,p_k$$

This is not the right transformation behavior for points in the tangent space. But comparison with the coordinate change in the cotangent bundle T^*Q of the components $(\eta_1, \dots, \eta_n)$ with respect to the basis $(dx^1, \dots, dx^n)$ (see [95, 19.5]):

$$dx^i = \sum_{j=1}^n \tfrac{\partial x^i}{\partial \bar{x}^j}\,d\bar{x}^j \qquad \text{und} \qquad \bar{\eta}_j = \sum_i \tfrac{\partial x^i}{\partial \bar{x}^j}\,\eta_i,$$

shows that $(q^1, \dots; p_1, \dots)$ are just the usual coordinates of a point in the cotangent bundle T^*Q. The transition from the coordinates $(x^1, \dots; v^1, \dots)$ of the tangent bundle TQ to the coordinates $(q^1, \dots; p_1, \dots)$ of the cotangent bundle T^*Q is given by the fiber derivative $d_f L : TQ \to T^*Q$, which has the representation $(d_f L)_i = \frac{\partial L}{\partial v^i}$ with respect to the basis $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n})$ and $(dx^1, \dots, dx^n)$ by 6.4, i.e.:

$$[d_f L] : (x^1, \dots; v^1, \dots) \mapsto (q^1, \dots; p_1, \dots) = \Big(x^1, \dots; \frac{\partial L}{\partial v^1}, \dots\Big).$$

We now show that the canonical 1-form ϑ_0 is really coordinate-independent and thus also $\omega_0 = -d\vartheta_0$:
The two mappings $T\pi_Q^* : TT^*Q \to TQ$ and $\pi_{T^*Q} : TT^*Q \to T^*Q$ given locally by $(q, p, v, \rho) \mapsto (q, v)$ and $(q, p, v, \rho) \mapsto (q, p)$. Define a mapping $(\pi_{T^*Q}, T\pi_Q^*) : TT^*Q \to T^*Q \times_Q TQ := \{(\alpha, \xi) : \pi^*(\alpha) = \pi(\xi)\}$. Combined with the evaluation map $\mathrm{ev} : T^*Q \times_Q TQ \to \mathbb{R}$, $(q, p; q, v) \mapsto \sum_i p_i\,v^i$ this is exactly the canonical 1-form ϑ_0, because

$$\sum_i p_i\,v^i = \sum_i p_i\,dq^i(q, p, v, \rho) = \vartheta_0(q, p, v, \rho).$$

We see immediately that

$$\vartheta_L = \sum_i \frac{\partial L}{\partial v^i}\, dx^i = (d_f L)^*\Big(\sum_i p_i\, dq^i\Big) = (d_f L)^* \vartheta_0$$

and thus also

$$\omega_L = -d\vartheta_L = -d(d_f L)^*\vartheta_0 = -(d_f L)^* d\vartheta_0 = (d_f L)^*\omega_0.$$

So these forms are also coordinate-independent.

The energy $E = \sum_i v^i \frac{\partial L}{\partial v^i} - L : TQ \to \mathbb{R}$ then corresponds to a so-called Hamilton function $H : T^*Q \to \mathbb{R}$, defined by

$$H \circ d_f L := E$$

and the vector field X_L now corresponds to the so-called Hamiltonian vector field X_H, which is $d_f L$-related to X_L defined by

$$X_H \circ d_f L := T(d_f L) \circ X_L.$$

The equation $\iota_{X_E}\omega_L = dE$ turns into

$$\iota_{X_H}\omega_0 = dH \text{ that is, } \omega_0(X_H, Y) = dH \cdot Y \text{ for all } Y.$$

This can be seen as follows:

$$\begin{aligned}
\iota_{X_H}\omega_0(T_x(d_f L)\xi) &= \omega_0\Big((X_H)_{(d_f L)(x)}, T_x(d_f L)\xi\Big) \\
&= \omega_0\Big(T_x(d_f L)X_L, T_x(d_f L)\xi\Big) = \omega_L(X_L, \xi) \\
&= (\iota_{X_L}\omega_L)(\xi) = dE(\xi) = d(H \circ d_f L)(\xi) \\
&= dH_{(d_f L)(x)} \cdot T_x(d_f L) \cdot \xi,
\end{aligned}$$

where $T_x(d_f L)\xi$ runs through all tangent vectors in $T^*_{(d_f L)(x)}Q$.

In terms of the coordinates (q^i, p_i) we have $X_H = \sum_i \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \sum_i \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$: We have $\omega_0 = \sum_i dq^i \wedge dp_i$ and let $X_H = \sum_i a^i \frac{\partial}{\partial q^i} + \sum_i b^i \frac{\partial}{\partial p_i}$. Then

$$\begin{aligned}
\sum_i \frac{\partial H}{\partial p_i} dp_i + \sum_i \frac{\partial H}{\partial q^i} dq^i &= dH = \iota_{X_H}\omega_0 \\
&= \sum_{i,j} a^i\, \iota_{\frac{\partial}{\partial q^i}}(dq^j \wedge dp_j) + \sum_{i,j} b^i\, \iota_{\frac{\partial}{\partial p_i}}(dq^j \wedge dp_j) \\
&= \sum_i a^i dp_i - \sum_i b^i dq^i
\end{aligned}$$

and a coefficient comparison yields $a^i = \frac{\partial H}{\partial p_i}$ and $b^i = -\frac{\partial H}{\partial q^i}$.

The integral curves $t \mapsto (q(t), p(t))$ of X_H are the solutions of

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \quad \text{und} \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}.$$

Since the integral curves of X_H are the derivatives of curves in the basis Q and the basis coordinates correspond to each other, the base curves for X_H and X_L are the same.

We have $A = \vartheta_0(X_H) \circ d_f L$:

$$\begin{aligned}\vartheta_0(X_H) \circ d_f L &= \operatorname{ev} \circ(\pi_{T^*Q}, T\pi_Q^*) \circ X_H \circ d_f L = \operatorname{ev} \circ(\operatorname{id} \circ d_f L, T\pi_Q^* \circ T(d_f L) \circ X_L)\\ &= \operatorname{ev} \circ(d_f L, T(\pi_Q^* \circ d_f L) \circ X_L) = \operatorname{ev} \circ(d_f L, T\pi_Q \circ X_L)\\ &= \operatorname{ev} \circ(d_f L, \operatorname{id}_{TQ}) = A\end{aligned}$$

or for $\xi \in TQ$

$$\begin{aligned}(\vartheta_0(X_H) \circ d_f L)(\xi) &= \vartheta_0|_{d_f L\cdot\xi}(X_H|_{d_f L\cdot\xi}) = \vartheta_0|_{d_f L\cdot\xi}(T(d_f L)(X_L|_\xi))\\ &= (d_f L)^*(\vartheta_0)(X_L|_\xi) = \vartheta_L(X_L|_\xi) = A(\xi)\end{aligned}$$

We have as advantages of Hamilton mechanics that the object describing the time evolution is a real-valued function $H : T^*Q \to \mathbb{R}$ which at the same time also represents a motion invariant and the associated (Hamiltonian) vector field X_H can be easily calculated from $\iota_{X_H}\omega = dH$.

Hamiltonian mechanics on Riemannian manifolds.

Let $\langle \_, \_ \rangle$ be a Riemannian metric on Q and $L : TQ \to \mathbb{R}$ the Lagrangian function given by a potential $U : Q \to \mathbb{R}$ via $L(v) := \frac{1}{2}\langle v, v\rangle - U(\pi(v))$. Then $d_f L : TQ \to T^*Q$ is the mapping $\sharp : w \mapsto \langle w, \_\rangle$ and $A(v) = \langle v, v\rangle$. So $E = A - L = \frac{1}{2}\langle v, v\rangle + U(\pi(v))$. In particular, if $U = 0$, then $A = 2E = 2L$ and the Hamilton function $H : T^*Q \to \mathbb{R}$ is then given by $H(\eta) = \frac{1}{2}\langle \eta^\flat, \eta^\flat\rangle$, i.e. in coordinates by $H(\sum_i \eta_i dx^i) = \frac{1}{2}\sum_{i,j} g^{i,j}\eta_i\eta_j$.

6.8 Legendre Transformation.

To translate the Hamiltonian formalism on T^*Q back into the Lagrange formalism on TQ, we need a description of the inverse of the Legendre transformation $d_f L : TQ \to T^*Q$ exclusively in terms of the Hamiltonian H function.

The canonical isomorphism is $d_f H \circ d_f L = \delta : TQ \to T^{**}Q$:
To see this let $\xi \in T_xQ$ and $\eta \in (T_xQ)^*$. Let $\xi_t \in T_xQ$ be such that $d_f L \cdot \xi_t = d_f L \cdot \xi + t\,\eta \in (T_xQ)^*$, thus $\xi_0 = \xi$ and $\eta = \frac{d}{dt}\big|_{t=0}(d_f L \cdot \xi + t\,\eta) = \frac{d}{dt}\big|_{t=0} d_f L \cdot \xi_t$. Then on the one hand

$$\begin{aligned}(d_f H \circ d_f L)(\xi)(\eta) &= d_f H(d_f L \cdot \xi)(\eta) = \frac{d}{dt}\Big|_{t=0} H(d_f L \cdot \xi + t\,\eta)\\ &= \frac{d}{dt}\Big|_{t=0} H(d_f L \cdot \xi_t) = \frac{d}{dt}\Big|_{t=0} E(\xi_t)\\ &= \frac{d}{dt}\Big|_{t=0} (A - L)(\xi_t) = \frac{d}{dt}\Big|_{t=0} d_f L(\xi_t)\cdot\xi_t - L(\xi_t)\\ &= \Big(T(d_f L)(\xi_0)\cdot\dot\xi_0\cdot\xi_0 + d_f L(\xi_0)\cdot\dot\xi_0\Big) - d_f L(\xi_0)\cdot\dot\xi_0\\ &= T(d_f L)(\xi_0)\cdot\dot\xi_0\cdot\xi_0\end{aligned}$$

and on the other hand

$$\delta(\xi)(\eta) = \eta(\xi_0) = \frac{d}{dt}\Big|_{t=0}(d_f L \cdot \xi_t)(\xi_0) = T(d_f L)(\xi_0)\cdot\dot\xi_0\cdot\xi_0$$

Assuming the invertibility of $d_f H$ we can thus recover

$$\begin{aligned}
E &:= (d_f H^{-1})^*(H),\\
A &:= (d_f H^{-1})^*(G) \text{ where } G := \vartheta_0(X_H),\\
L &:= A - E \text{ and}\\
X_L &:= (d_f H^{-1})^*(X_H)
\end{aligned}$$

.

6.9 Symplectic Mechanics.

More generally, instead of T^*Q, one considers symplectic manifolds M, that is, manifolds together with a so-called symplectic form ω, i.e. a non-degenerate closed-2 form $\omega \in \Omega^2(M)$. We have shown in [95, 4.6] that such a manifold must be even-dimensional, and we will further show in [87, 50.39] that one can always locally choose coordinates $(q^1, \dots, p_1, \dots)$ so that $\omega = \sum_{i=1}^n dq^i \wedge dp_i$. The n-fold wedge product of ω defines a volume form

$$\mathrm{vol}_\omega := \omega \wedge \dots \wedge \omega$$

on M. In particular, M is oriented. If $H : M \to \mathbb{R}$ is a smooth function (a so-called Hamilton function), then X_H denotes the so-called Hamiltonian vector field, which is given by the implicit equation

$$\iota_{X_H}\omega = dH.$$

In local coordinates X_H is given by

$$X_H = \sum_i \tfrac{\partial H}{\partial p_i} \tfrac{\partial}{\partial q^i} - \sum_i \tfrac{\partial H}{\partial q^i} \tfrac{\partial}{\partial p_i}.$$

The Hamilton function H is constant along the integral curves of the Hamilton vector field X_H:
Let $t \mapsto x(t)$ be an integral curve, that is $x : \mathbb{R} \to T^*Q$ with $\dot{x}(t) = X_H(x(t))$. Then

$$\begin{aligned}
(H \circ x)\dot{}(t) &= dH_{x(t)}(\dot{x}(t)) = \iota_{X_H}\omega_{x(t)}(\dot{x}(t))\\
&= \omega(X_H(x(t)), \dot{x}(t)) = \omega(X_H(x(t)), X_H(x(t))) = 0,
\end{aligned}$$

is $H \circ x$ constant.

The flow Fl_t of the vector field X_H is a symplectomorphism, i.e. leaves the symplectic form ω invariant, and consequently also the symplectic volume form $\omega^m = \mathrm{vol}_\omega$, i.e.

$$\left(Fl_t^{X_H}\right)^* \mathrm{vol}_\omega = \mathrm{vol}_\omega .$$

Because

$$\begin{aligned}
\frac{d}{dt}(\mathrm{Fl}_t)^*\omega &= (\mathrm{Fl}_t^*) \left.\frac{d}{ds}\right|_{s=0} (\mathrm{Fl}_s)^*\omega = (\mathrm{Fl}_t^*)\mathcal{L}_{X_H}\omega\\
&= (\mathrm{Fl}_t^*)(\iota_{X_H} d\omega + d\iota_{X_H}\omega) = (\mathrm{Fl}_t^*)(\iota_{X_H} 0 + ddH) = 0,
\end{aligned}$$

we have that $(\mathrm{Fl}_t)^*\omega$ is constantly the same as $(\mathrm{Fl}_0)^*\omega = \omega$.

Not every vector field X on M is a Hamiltonian vector field. To be one, a local (integrability) condition $\mathcal{L}_X\omega = 0$ must be satisfied for X and a global (cohomological) condition $H^1(M) = 0$ for M, see [87, 50.40].

III. Curvature und geodesics

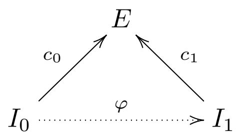
7. Curvature of curves in the plane

In this section the central concept of curvature for plane curves is studied.

7.1 Definition (curves).

A PARAMETERIZED CURVE in an Euclidean space E is a map $c : I \to E$, where I is a (usually) open interval in $\mathbb{R}$ and c is sufficiently often differentiable. For the sake of simplicity, we always assume infinitely differentiable (in short smooth) and sufficiently regular, i.e. at least $c'(t) \neq 0$ for all $t \in I$.

Since we are essentially not interested in the parameterization of the curve, but more in its geometric form, we give the following definition:
A GEOMETRIC CURVE Γ is an equivalence class of parameterized curves, where $c_0 : I_0 \to E$ and $c_1 : I_1 \to E$ are called EQUIVALENT, if a DIFFEOMORPHISM $\varphi : I_0 \to I_1$ (i.e., φ is bijective and both φ and φ^{-1} are smooth) exists with $c_1 \circ \varphi = c_0$.



An ORIENTED GEOMETRIC CURVE is an equivalence class of parameterized curves, where c_1 and c_2 are called EQUIVALENT, if a φ exists as above, which additionally satisfies $\varphi'(t) > 0$ for all t (i.e., is strictly monotonic increasing).

So, an (oriented) geometric curve is determined by specifying a PARAMETERIZATION, i.e. a parameterized curve in its class. Consequently, we can confine ourselves to developing concepts for parameterized curves, but have always to make sure that these concepts are truly geometric in nature, i.e. do not depend on the choosen representatives(=parameterizations) and are also invariant under motions.

The IMAGE OF A GEOMETRIC CURVE is the image of one (all) of its parameterizations.

7.2 The tangent.

The TANGENT TO A PARAMETERIZED CURVE c at point t is the affine line $c(t) + \mathbb{R} \cdot c'(t)$.

Lemma.

The tangent is a GEOMETRIC CONCEPT, *i.e. reparametrization invariant and also invariant under motions.*

Proof. First, we show the invariance under reparametrizations. Let (c, t) and $(\bar{c}, \bar{t})$ be two representatives of the same point of a geometric curve, and φ an associated parameter change, i. $\bar{c} = c \circ \varphi$ and $t = \varphi(\bar{t})$. The tangent of $\bar{c}$ in $\bar{t}$ is that of c in t, because

$$\bar{c}(\bar{t}) + \mathbb{R} \cdot \bar{c}'(\bar{t}) = (c \circ \varphi)(\bar{t}) + \mathbb{R} \cdot (c \circ \varphi)'(\bar{t}) = \\ = c(t) + \mathbb{R} \cdot \varphi'(\bar{t}) \cdot c'(\varphi(\bar{t})) = c(t) + \mathbb{R} \cdot c'(t)$$

(useing the chain rule and $\mathbb{R} \cdot \varphi'(\bar{t}) = \mathbb{R}$).

Now the motion invariance: Let $x \mapsto A(x) + b$ be a motion and (c, t) a point of a curve. The moved curve is then $\bar{c} : t \mapsto A(c(t)) + b$. The moved tangent of c in t is the tangent of the moved curve $\bar{c}$, because

$$A(c(t) + \mathbb{R} \cdot c'(t)) + b = A(c(t)) + b + \mathbb{R} \cdot A(c'(t)) = \\ = (A(c(t)) + b) + \mathbb{R} \cdot (A \circ c)'(t) = \bar{c}(t) + \mathbb{R} \cdot \bar{c}'(t)$$

(using the chain rule (see [**82**, 5.5.2] or [**82**, 6.1.9]) and $A'(x)(v) = A(v)$ by [**87**, 2.1], since A is linear). □

7.3 Definition (Unit tangential vector).

The UNIT TANGENTIAL VECTOR $\tau(t)$ at a point t of a parameterized curve is defined by $\tau(t) := \frac{c'(t)}{|c'(t)|}$. For well-definedness we use the regularity of the curve, i.e. $c'(t) \neq 0$. Note that this too represents a geometric concept for oriented geometric curves, but it should be noted that $c(t)$ belongs to the (affine) Euclidean space and $c'(t)$ belongs to the corresponding vector space.

7.4 Definition (Standard normal vector).

The UNIT NORMAL VECTOR ν to a parameterized curve c in the point with parameter t is $\nu(t) := \tau(t)^\perp$, where for each vector $x \neq 0$ in $\mathbb{R}^2$ we denote with $x^\perp$ the uniquely determined vector, which is normal to x, has the same length as x and is to the left of x (i.e. $(x, x^\perp)$ is positively oriented), see [**87**, 1.3]. So it is obtained from x by a rotation with angle $\frac{\pi}{2}$:

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}^\perp = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} -x^2 \\ x^1 \end{pmatrix}.$$

Like the tangent vector, the unit normal vector is a geometric concept for oriented curves. The pair (τ, ν) is called the MOVING FRAME of the curve. For each parameter value t, $(\tau(t), \nu(t))$ is a well-adapted basis for the plane.

7.5 Definition (Curvature).

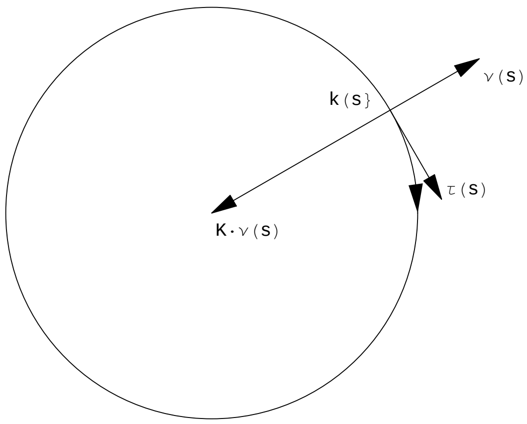
Since a circle is more curved the smaller the radius r is, we want to use the reciprocal $\frac{1}{r}$ as a measure of the CURVATURE K of the circle. We provide it with a sign, which is positive if the circle is positively oriented i.e. described a left curve, and otherwise negative. A straight line can be regarded as a limit case of a circle for $r \to \infty$, and the corresponding definition for its curvature as $K := \frac{1}{\infty} = 0$ also agrees with the notion of non-curved.

We can calculate the midpoint M of a circle from the first few derivatives at point s of an arc length parameterization $k(s) = r(\cos \frac{\pm s}{r}, \sin \frac{\pm s}{r}) + M$:

$$\tau(s) = k'(s) = \pm(-\sin\tfrac{\pm s}{r}, \cos\tfrac{\pm s}{r}), \quad \nu(s) = \tau(s)^\perp = \mp(\cos\tfrac{\pm s}{r}, \sin\tfrac{\pm s}{r}) \quad \Rightarrow$$

$$k''(s) = -\tfrac{1}{r}(\cos\tfrac{\pm s}{r}, \sin\tfrac{\pm s}{r}) = \pm\frac{1}{r}\cdot\nu(s) = K\cdot\nu(s) \quad \Rightarrow \quad |k''(s)| = \tfrac{1}{r} = \pm K \quad \text{and}$$

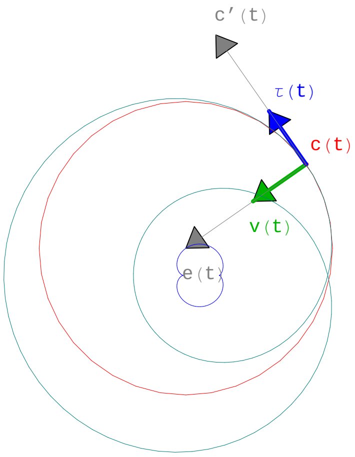
$$M = k(s) - r(\cos\tfrac{\pm s}{r}, \sin\tfrac{\pm s}{r}) = k(s) + r^2\cdot k''(s) = k(s) + \frac{k''(s)}{|k''(s)|^2}$$
$$= k(s) + \frac{1}{K}\cdot\nu(s).$$



More generally, let c be a curve parameterized by arc length with $c''(s) \neq 0$. The OSCULATING CIRCLE at point s is understood to be the circle k which touches c of order 2 at s, i.e. $c(s) = k(s)$, $c'(s) = k'(s)$ and $c''(s) = k''(s)$. The CURVATURE $K(s)$ of the curve c at s is the signed curvature of the osculating circle. The center of the osculating circle is given by the formula from above as

$$M = k(s) + \frac{k''(s)}{|k''(s)|^2} = c(s) + \frac{c''(s)}{|c''(s)|^2} = c(s) + \frac{1}{K(s)}\cdot\nu(s).$$

Its radius is $r = \frac{1}{|c''(s)|}$. This Ansatz proves also its existence. The curve formed by the centers of the osculating circles (which are well-defined, where $|c''| \neq 0$) is called EVOLUTE.



For the osculating circle k we have: $c''(s) = k''(s) = K(s) \cdot \nu(s)$ where $\nu(s)$ is the unit normal to k or c and $K(s)$ is the curvature. According to Newton's law "force = mass × acceleration", $K(s)$ measures the (scalar magnitude of the) force needed to keep the point (with unit mass) moving at scalar velocity $|c'(s)| = 1$ along the curve.

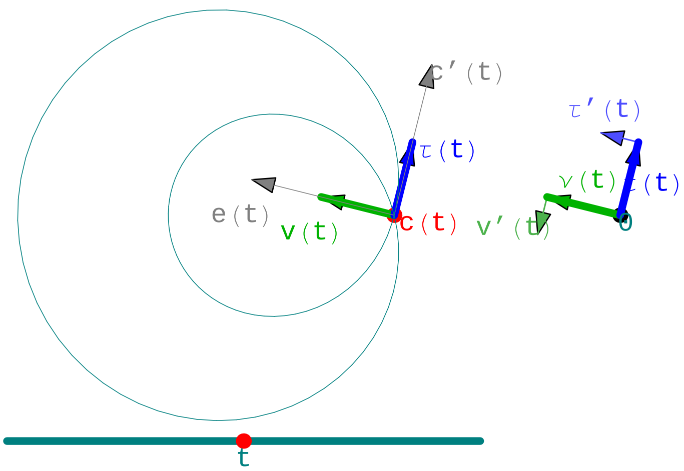
We can therefore interpret K as the coefficient of $\tau' = c''$ with respect to the second vector ν of the moving frame (τ, ν). If we apply a rotation R by $\pi/2$ to this equation $\tau' = K \cdot \nu$, then we obtain

$$\nu' = (R \circ \tau)' = R \circ \tau' = R \circ (K \cdot \nu) = K \cdot (R \circ \nu) = K \cdot (R^2 \circ \tau) = -K \cdot \tau.$$

Together these two equations are the so-called FRENET FORMULAS:

$$\begin{aligned} \tau' &= K \cdot \nu \\ \nu' &= -K \cdot \tau, \end{aligned}$$

expressing the derivative of the moving frame in the basis given by the moving frame.



With $c(s) =: (x(s), y(s))$ the following explicit formula for the curvature holds

$$\begin{aligned} K(s) &= \langle K(s)\cdot\nu(s) \,|\, \nu(s)\rangle = \langle \tau'(s) \,|\, \nu(s)\rangle = \langle \tau'(s) \,|\, \tau(s)^\perp \rangle \\ &= \det(\tau(s), \tau'(s)) = \det(c'(s), c''(s)), \end{aligned}$$

because

$$\det(x,y) = \det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = x_1 y_2 - x_2 y_1 = \left\langle \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \,\middle|\, \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \right\rangle = \langle y \,|\, x^\perp \rangle .$$

If, more generally, c is not parameterized by arc length, $t \mapsto s(t)$ is the arc length function and $\bar c = c \circ s^{-1}$ is the reparametrization by arc length, then we obtain for the curvature:

$$\begin{aligned} K_{\bar c}(s) &= \det(\bar c'(s), \bar c''(s)) = \det\Big(c'(t)\tfrac{1}{s'}, (c''(t) - c'(t)s'')\,\tfrac{1}{(s')^2}\Big) \\ &= \tfrac{1}{(s')^3}\det(c'(t), c''(t)) + 0 \Rightarrow \\ \Rightarrow K_c(t) = K_{\bar c}(s(t)) &= \frac{\det(c'(t), c''(t))}{|c'(t)|^3} = \frac{\langle c'(t)^\perp \,|\, c''(t)\rangle}{|c'(t)|^3}, \end{aligned}$$

where we used $c = \bar c \circ s$, $c' = (\bar c' \circ s)s'$, $c'' = (\bar c'' \circ s)(s')^2 + (\bar c' \circ s)s''$.

We now want to show that the osculating circle is invariant under motion. For this it suffices to show the invariance of its center

$$M(s) = c(s) + \frac{c''(s)}{|c''(s)|^2} = c(s) + \frac{K(s)\nu(s)}{|K(s)\cdot\nu(s)|^2} = c(s) + \frac{\nu(s)}{K(s)}$$

Let c be parameterized by arc length and let $\bar c(t) = R(c(t)) + a$ be the curve moved by $x \mapsto Rx + a$. The curvature of $\bar c$ is then

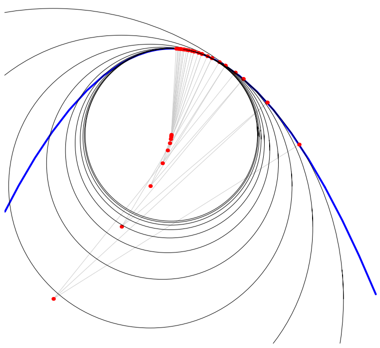
$$\begin{aligned} K_{\bar c}(t) &= \det(\bar c'(t), \bar c''(t)) = \det\big(R(c'(t)), R(c''(t))\big) \\ &= \det\big(R(c'(t), c''(t))\big) = \det R \cdot \det(c'(t), c''(t)) = K_c(t), \end{aligned}$$

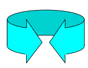
since $\det R = +1$. Thus, the curvature is invariant and therefore also the center.

7.6 Lemma (Osculating circle as limit).

Let c be a curve parameterized by arc length with $c''(s) \neq 0$. For every three different points s_1, s_2, s_3, let $M(s_1, s_2, s_3)$ be the center of the circle through the points $c(s_1), c(s_2), c(s_3)$ and $M(s)$ the center of the osculating circle of c.

Then $M(s_1, s_2, s_3) \to M(s)$ for $s_1, s_2, s_3 \to s$. The same holds to the radii.





Proof of 7.6. The bisector of the line segment between $c(t_1)$ and $c(t_2)$ is given in normal vector form by

$$\left\{ z : \left\langle c(t_2) - c(t_1) \,\middle|\, z - \frac{c(t_1) + c(t_2)}{2} \right\rangle = 0 \right\}$$

and that between $c(t_2)$ and $c(t_3)$ is in parameter form given by

$$\left\{\frac{c(t_2)+c(t_3)}{2}+\lambda\cdot\Big(c(t_3)-c(t_2)\Big)^\perp:\lambda\in\mathbb{R}\right\}.$$

The center $M(t_1,t_2,t_3)$ of the circle through the 3 points $c(t_1)$, $c(t_2)$ and $c(t_3)$ is thus at the intersection of the two bisectors, i.e. is given by

$$M(t_1,t_2,t_3):=\frac{c(t_2)+c(t_3)}{2}+\lambda\cdot\Big(c(t_3)-c(t_2)\Big)^\perp,$$

where λ is the solution of the equation

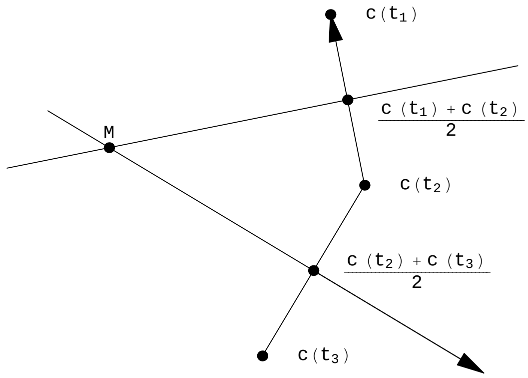
$$0=\left\langle c(t_2)-c(t_1)\;\middle|\;\frac{c(t_2)+c(t_3)}{2}+\lambda\cdot\Big(c(t_3)-c(t_2)\Big)^\perp-\frac{c(t_1)+c(t_2)}{2}\right\rangle,$$

that is

$$\lambda=\frac{\Big\langle c(t_2)-c(t_1)\;\Big|\;c(t_1)-c(t_3)\Big\rangle}{2\det\Big(c(t_3)-c(t_2),c(t_2)-c(t_1)\Big)},$$

i.e.

$$M(t_1,t_2,t_3)=\frac{c(t_2)+c(t_3)}{2}+\frac{\Big\langle c(t_2)-c(t_1)\;\Big|\;c(t_1)-c(t_3)\Big\rangle}{2\det\Big(c(t_3)-c(t_2),c(t_2)-c(t_1)\Big)}\cdot\Big(c(t_3)-c(t_2)\Big)^\perp.$$



Because of

$$\frac{c(t_2)-c(t_1)}{t_2-t_1}=\frac{1}{t_2-t_1}\int_{t_1}^{t_2}c'(t)\,dt=\int_0^1 c'\Big(t_1+s(t_2-t_1)\Big)\,ds$$

$$\frac{\frac{c(t_1)-c(t_2)}{t_1-t_2}-\frac{c(t_3)-c(t_2)}{t_3-t_2}}{t_1-t_3}=\int_0^1\int_0^1 c''\Big(t_2+s_1(t_3-t_2)+s_2s_1(t_1-t_3)\Big)\,s_1\,ds_1\,ds_2$$

we have

$$c(t_1),c(t_2),c(t_3)\to c(t)$$
$$\frac{c(t_2)-c(t_1)}{t_2-t_1},\frac{c(t_1)-c(t_3)}{t_1-t_3},\frac{c(t_3)-c(t_2)}{t_3-t_2}\to c'(t)$$
$$\left(\frac{c(t_1)-c(t_2)}{t_1-t_2}-\frac{c(t_3)-c(t_2)}{t_3-t_2}\right)/(t_1-t_3)\to c''(t)/2$$

for $t_1, t_2, t_3 \to t$, so

$$
\begin{aligned}
M(t_1,t_2,t_3) &= \frac{c(t_2)+c(t_3)}{2} + \frac{\Big\langle c(t_2)-c(t_1) \Big| c(t_1)-c(t_3) \Big\rangle}{2\det\Big(c(t_3)-c(t_2), c(t_2)-c(t_1)\Big)} \cdot \Big(c(t_3)-c(t_2)\Big)^\perp \\
&= \frac{c(t_2)+c(t_3)}{2} + \\
&\quad + \frac{\Big\langle \frac{c(t_2)-c(t_1)}{t_2-t_1} \Big| \frac{c(t_1)-c(t_3)}{t_1-t_3} \Big\rangle}{2\det\Big(\frac{c(t_3)-c(t_2)}{t_3-t_2}, \frac{c(t_2)-c(t_1)}{(t_2-t_1)\cdot(t_1-t_3)} - \frac{c(t_3)-c(t_2)}{(t_3-t_2)\cdot(t_1-t_3)}\Big)} \cdot \Big(\frac{c(t_3)-c(t_2)}{t_3-t_2}\Big)^\perp \\
&\to c(t) + \frac{\langle c'(t) \,|\, c'(t) \rangle}{\det\Big(c'(t), c''(t)\Big)} \cdot \Big(c'(t)\Big)^\perp = c(t) + \frac{1}{K(t)}\nu(t). \qquad \square
\end{aligned}
$$

7.7 Lemma (Curvature as change of direction).

Let $c : I \to \mathbb{C}$ be a curve parameterized by arc length with $c' : I \to S^1 := \{z \in \mathbb{C} : |z| = 1\}$. Let $s_0 \in I$ and $\theta : I \to \mathbb{R}$ for s near s_0 be a differentiable solution of $e^{i\theta(s)} = c'(s)$. Then $K(s) = \theta'(s)$, i.e. the curvature K measures the infinitesimal change in the angle of the tangent.

Proof. By differentiating the equation $c'(s) = e^{i\theta(s)}$, one obtains $c''(s) = i\theta'(s)e^{i\theta(s)} = \theta'(s)ic'(s) = \theta'(s)\nu(s)$, but this is the implicit equation for the curvature, hence $K(s) = \theta'(s)$. $\square$

Summary.

The curvature of a curve parameterized by arc length can be understood as:

1. The reciprocal $\frac{1}{r} = |c''(s)| = |K(s)|$ of the radius r of the osculating circle, i.e. of the circle, which best approximates c, supplied with the sign, which results from whether the osculating circle is positively or negatively oriented, see 7.5.
2. The scalar value of the acceleration $c''(s) = K(s)\nu(s)$, see 7.5.
3. The infinitesimal change of the angle of the tangent $K(s) = \theta'(s)$, according to Lemma 7.7.

7.8 Theorem (Curvature characterizes the curve).

If $K : I \to \mathbb{R}$ is a smooth mapping, there is, up to motions, exactly one curve which has an arc length parameterization c for which $K_c(s) = K(s)$ holds.

Proof. Let c be an arc length parameterized curve with curvature K, i.e. $|c'(s)| = 1$ for all s and $K(s) = \theta'(s)$ by 7.7, where θ is a lift of c' (for its existence see [**87**, 24.5] or [**92**, 6.11]), i.e. $e^{i\theta(s)} = c'(s)$ holds. Thus

$$
\begin{aligned}
\theta(s) &= \theta(0) + \int_0^s \theta'(\tau)\,d\tau = \theta(0) + \int_0^s K(\tau)d\tau \text{ and} \\
c(s) &= c(0) + \int_0^s c'(\tau)d\tau = c(0) + \int_0^s e^{i\theta(\tau)}d\tau,
\end{aligned}
$$

where $c(0)$ is the arbitrary starting point of the curve, and $\theta(0)$ is the freely chooseable angle of the initial direction. Each two such initial datas provide a motion which maps the associated curves into each other. Let c be as defined above. Then $c'(s) = e^{i\theta(s)}$, that is $|c'(s)| = 1$, i.e. c is parameterized by arc length. Since θ is the lift, $K_c(s) = \theta'(s) = K(s)$ holds. $\square$

8. Curvatures of curves in higher dimensions

8.1 Definition (Curvature and moving frame).

As a measure of the curvature of a curve parameterized by arc length c in the point t we take $|c''(t)|$, but have now no reasonable way to provide this with a sign. If $c''(t) \neq 0$, then $\nu(t) := \frac{1}{|c''(t)|} c''(t)$ is called the main normal vector of c in t.

In $\mathbb{R}^3$, we can add the two vectors τ (see 7.3) and ν to a positive oriented orthonormal basis $\{\tau, \nu, \beta\}$ by defining the binormal vector β as $\beta := \tau \times \nu$. This basis is called the moving frame of the curve.

In $\mathbb{R}^{n+1}$ we proceed as follows: Let $c'(t), c''(t), \dots, c^{(n)}(t)$ be linearly independent. By the Gram-Schmidt orthogonalization procedure we can construct an orthonormal family $\nu_0, \nu_1, \dots, \nu_{n-1}$ from them. (Induction is used to show that for k linearly independent vectors $a_1, a_2, \dots a_k$, a unique vector v exists in the linear span of $a_1, a_2, \dots a_k$, which is normal to $a_1, a_2, \dots, a_{k-1}$ and has an angle $|\alpha| < \pi/2$ to a_k. We now augment $\nu_0, \nu_1, \dots, \nu_{n-1}$ to a positively oriented orthonormal base $\nu_0, \nu_1, \dots, \nu_{n-1}, \nu_n$ of $\mathbb{R}^{n+1}$.

This orthonormal basis is called the moving frame of the curve.

8.2 Definition (Curvatures).

We now want to describe the analogue of Frenet's formulas. For each vector ν_i' we have $\nu_i' = \sum_{j=0}^{n} \langle \nu_i' \,|\, \nu_j \rangle \, \nu_j$. So we have to determine the coeffcients $\langle \nu_i' \,|\, \nu_j \rangle$ of ν_i' with respect to the basis vectors ν_j. We differentiate the equation $\langle \nu_i(s) \,|\, \nu_j(s) \rangle = \delta_{ij}$ by s and obtain

$$\langle \nu_i' \,|\, \nu_j \rangle + \langle \nu_i \,|\, \nu_j' \rangle = 0 \Rightarrow \langle \nu_i' \,|\, \nu_j \rangle = -\langle \nu_i \,|\, \nu_j' \rangle .$$

So the matrix $(\langle \nu_i' \,|\, \nu_j \rangle)_{i,j}$ is skew-symmetric. Since $\nu_i \in \langle \{c', \dots, c^{(i+1)}\} \rangle$, we may represent ν_i in the following way:

$$\begin{aligned} \nu_i &= \sum_{j=1}^{i+1} a_j \cdot c^{(j)} \\ \Rightarrow \nu_i' &= \sum_{j=1}^{i+1} \left(a_j' \cdot c^{(j)} + a_j \cdot c^{(j+1)} \right) \in \left\langle \{c', c'', \dots, c^{(i+2)}\} \right\rangle \end{aligned}$$

Because ν_j is normal to $\langle \{\nu_i : 0 \le i < j\} \rangle = \langle \{c^{(i)} : 1 \le i \le j\} \rangle$, we have that $\langle \nu_i' \,|\, \nu_j \rangle = 0$ for $i + 2 \le j$. In addition, obviously, $\langle \nu_i' \,|\, \nu_i \rangle = 0$ for all i and hence only immediately above and below the diagonal of the skew-symmetric matrix $(\langle \nu_i' \,|\, \nu_j \rangle)_{i,j}$ may be non-zero entries.

For a curve c, the term $\langle \nu_i' \,|\, \nu_{i+1} \rangle =: K_{i+1}$ is called $(i+1)$-st curvature.

$$\langle \nu_i' \,|\, \nu_j \rangle = \begin{pmatrix} 0 & K_1 & 0 & \dots & 0 \\ -K_1 & 0 & K_2 & \ddots & \vdots \\ 0 & -K_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & K_n \\ 0 & \dots & 0 & -K_n & 0 \end{pmatrix}$$

For a curve c parameterized by arc length in $\mathbb{R}^2$, the following holds:

$$K_1 = \langle \nu_0' \,|\, \nu_1 \rangle = \left\langle (c')' \,\middle|\, \frac{c''}{|c''|} \right\rangle = |c''| = |K|.$$

8.3 Frenet-Serre Formulas.

For the moving frame $(\nu_i)_{i=0}^n$ *of a curve* c *in* $\mathbb{R}^{n+1}$ *we have:*

$$\nu_i' = -K_i \cdot \nu_{i-1} + K_{i+1} \cdot \nu_{i+1},$$

where $K_0 := 0$, $K_{n+1} := 0$, $\nu_{-1} := 0$ *and* $\nu_{n+1} := 0$.

Proof.

$$\nu_i' = \sum_{j=0}^n \underbrace{\langle \nu_i' \,|\, \nu_j \rangle}_{\substack{=0 \\ \text{for } |i-j|\neq 1}} \cdot \nu_j = \underbrace{\langle \nu_i' \,|\, \nu_{i+1} \rangle}_{=K_{i+1}} \cdot \nu_{i+1} + \underbrace{\langle \nu_i' \,|\, \nu_{i-1} \rangle}_{=\langle \nu_{i-1}' \,|\, \nu_i \rangle = K_i} \cdot \nu_{i-1} \qquad \square$$

8.4 Lemma.

The moving frame and the curvatures are geometric objects.

Proof. Left to the reader! □

8.5 Remark.

Conversely, the derivatives of a curve parameterized by arc length can be written as linear combinations of the moving frame as follows:

$$\begin{aligned} c' &= \nu_0 \\ c'' &= \nu_0' = K_1\nu_1 - 0 \\ c''' &= (K_1\nu_1)' = K_1'\nu_1 + K_1\nu_1' = K_1'\nu_1 + K_1(K_2\nu_2 - K_1\nu_0) \\ &= K_1K_2\nu_2 + K_1'\nu_1 - {K_1}^2\nu_0. \end{aligned}$$

According to Taylor's theorem, a curve c can now be written as follows:

$$\begin{aligned} c(t) &= c(0) + \frac{c'(0)}{1!}t + \frac{c''(0)}{2!}t^2 + \frac{c'''(0)}{3!}t^3 + O(t^4) \\ &= c(0) + \nu_0(0)t + \frac{K_1(0)\nu_1(0)}{2}t^2 \\ &\quad + \frac{K_1(0)K_2(0)\nu_2(0) + K_1'(0)\nu_1(0) - (K_1(0))^2\nu_0(0)}{6}t^3 + O(t^4) \\ &= c(0) + \left(t - \frac{(K_1(0))^2}{6}t^3\right)\nu_0(0) \\ &\quad + \left(\frac{K_1(0)}{2}t^2 + \frac{K_1'(0)}{6}t^3\right)\nu_1(0) + \left(\frac{K_1(0)K_2(0)}{6}t^3\right)\nu_2(0) + O(t^4) \end{aligned}$$

8.6 Definition (Torsion).

If $c : \mathbb{R} \to \mathbb{R}^3$ is a space curve with $\tau := \nu_0$, $\nu := \nu_1$, $\beta := \tau \times \nu = \nu_2$, then $K := K_1$ is called the CURVATURE and $T := K_2$ is the TORSION of the curve. The Frenet-Serre formulas are then:

$$\begin{array}{rclccc} \tau' &=& & +K\nu & \\ \nu' &=& -K\tau & & +T\beta \\ \beta' &=& & -T\nu & \end{array}$$

The affine plane though $c(0)$ spanned by the main normal vector ν and the binormal vector β by is called the NORMAL PLANE, that spanned by the unit tangential vector τ and the main normal vector ν is called OSCULATING PLANE FOR A SPACE CURVE and the plane spanned by the unit tangential vector τ and the binormal vector β is called the RECTIFYING PLANE.

If x, y, z are the coordinates of c with respect to τ, ν, β at the point $c(0)$, then according to 8.5:

$$\begin{aligned} x(t) &= t & &- \frac{K^2(0)}{6} t^3 + O(t^4) \\ y(t) &= & \frac{K(0)}{2} t^2 &+ \frac{K'(0)}{6} t^3 + O(t^4) \\ z(t) &= & &\frac{K(0)T(0)}{6} t^3 + O(t^4) \end{aligned}$$

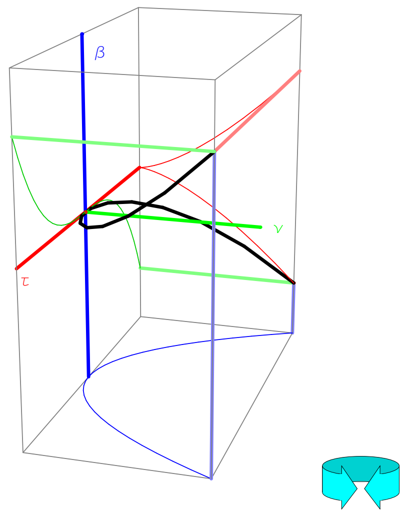
We consider the projection of the curve to the planes spenned by the moving frame:

First the projection to the osculating plane $c(0) + \beta^\perp$: We get $y = \frac{K}{2} t^2 + O(t^3)$, $x = t + O(t^3)$ and after neglecting the higher-order terms $y \approx x^2 \frac{K}{2}$.

For the projection to the rectifying plane $c(0) + \nu^\perp$ we get $x = t + O(t^3)$, $z = \frac{KT}{6} t^3 + O(t^4)$ and thus $z \approx x^3 \frac{KT}{6}$.

For the projection to the normal plane $c(0) + \tau^\perp$ we get $y = \frac{K}{2} t^2 + O(t^3)$, $z = \frac{KT}{6} t^3 + O(t^4)$ and thus $y^3 \approx (\frac{K}{2})^3 t^6 = \frac{9K}{2T^2} \frac{(KT)^2}{6^2} t^6 \approx z^2 \frac{9K}{2T^2}$.

These projections are drawn in the following 3-dimensional image:



8.7 Lemma.

The curvatures K_i $(i = 1, \dots, n)$ and the moving frame $(\nu_0, \dots, \nu_n)$ of a curve parameterized by arc length are uniquely determined by the following conditions:

$$c^{(j+1)} = \nu_0^{(j)} \equiv K_1 \cdot \ldots \cdot K_j \cdot \nu_j \mod \langle \{\nu_0, \dots, \nu_{j-1}\} \rangle,$$
$$K_i > 0 \textit{ for } i < n$$
and $(\nu_1, \dots, \nu_n)$ is a positive oriented orthonormal basis.

Proof.
($\Rightarrow$) We already know (because of 8.5) that this equation holds for $j \in \{1, 2, 3\}$. Suppose it is satisfied for j, i.e.

$$\nu_0^{(j)} = K_1 \cdot \ldots \cdot K_j \cdot \nu_j + \sum_{i=0}^{j-1} a_i \nu_i \text{ with } a_i \in \mathbb{R},$$

so we have to show the claim for $j+1$: For this purpose, we differentiate this equation and, because of the Frénet-Serre formulas, we obtain 8.3

$$\begin{aligned}\nu_0^{(j+1)} &= (K_1 \dots K_j)\cdot \nu_j' + (K_1\dots K_j)'\cdot \nu_j + \sum_{i=0}^{j-1}(a_i'\cdot \nu_i + a_i \cdot \nu_i')\\ &\equiv (K_1\dots K_j)(K_{j+1}\cdot \nu_{j+1} - K_j\cdot \nu_{j-1}) \mod \langle\{\nu_0,\dots,\nu_j\}\rangle\\ &\equiv K_1\dots K_j K_{j+1}\cdot \nu_{j+1} \mod \langle\{\nu_0,\dots,\nu_j\}\rangle,\end{aligned}$$

because ν_i' for $i<j$ and also $K_j\nu_{j-1}$ are in the linear span of $\nu_0,\dots,\nu_j$.

By construction the angle between ν_j and $c^{(j+1)}$ is less than $\pi/2$ for $j<n$. Thus

$$0 < \langle\, \nu_j \,|\, c^{(j+1)}\,\rangle = \Big\langle\, \nu_j \,\Big|\, K_1\cdot \ldots \cdot K_j\nu_j + \sum_{i<j} a_i\nu_i \Big\rangle = K_1\cdot\ldots\cdot K_j \underbrace{\langle\, \nu_j\,|\,\nu_j\,\rangle}_{1}$$

and hence $K_j>0$ is for each $j<n$.

By construction the ν_j form a positively oriented orthonormal basis.

($\Leftarrow$) $\langle\{\nu_0,\dots,\nu_{j-1}\}\rangle = \langle\{c',\dots,c^{(j)}\}\rangle$ for $1\le j<n$ and $0<\langle\,\nu_j\,|\,c^{(j+1)}\,\rangle$ follows recursively from

$$c^{(j+1)} \equiv K_1\cdot\ldots\cdot K_j\nu_j \mod \langle\{\nu_0,\dots,\nu_{j-1}\}\rangle$$

and $K_j>0$ for $j<n$. Because of the orthogonality ν_j is the moving frame. Furthermore, $K_1\cdot\dots\cdot K_j$ is the uniquely determined coefficient of ν_j, in the development of $\nu_0^{(j)}$ with respect to the basis $(\nu_0,\dots,\nu_n)$ and thus K_i the corresponding curvature. □

8.8 Corollary.

We have $K_1^nK_2^{n-1}\cdot\dots\cdot K_{n-1}^2K_n = \det(c',\dots,c^{(n+1)})$ *and, in particular,* K_n *has the same sign as* $\det(c',\dots,c^{(n+1)})$. *For space curves* $c: I\to\mathbb{R}^3$, *the torsion is thus given by* $T=\det(c',c'',c''')/K^2$.

Proof. According to the rules for calculating determinants we obtain

$$\begin{aligned}\det(c',\dots,c^{(n+1)}) &= \det\Big(\nu_0, K_1\nu_1,\dots,K_1\cdot\ldots\cdot K_n\nu_n + \sum_{i=0}^{n-1}a_i\nu_i\Big)\\ &= \underbrace{{K_1}^n{K_2}^{n-1}\dots{K_{n-1}}^2}_{>0}{K_n}^1 \quad \underbrace{\det(\nu_0,\dots,\nu_n)}_{=1 \text{ because pos. orientiert}}\end{aligned}$$

□

8.9 Theorem (Curvatures characterize the curve).

Let $K_i: I\to\mathbb{R}$ *be smooth functions for* $1\le i\le n$ *with* $K_i(t)>0$ *for* $i<n$ *and all* t. *Then there is a curve in* $\mathbb{R}^{n+1}$, *which is uniquely determined up to motions and which parameterized by arc length has exactly the* K_i *as curvatures.*

The motto is:

"Tell me how you bend and twist, and I'll tell you who you are"!

Proof. W.l.o.g. let $0\in I$. First we claim: There is exactly one arc-parameterized curve with the given curvatures and the initial conditions: $c(0)=0$ and the standard basis $e_0,\dots,e_n$ as moving frame at 0.

According to Frenet's equation 8.3, $\nu_j' = K_{j+1}\cdot\nu_{j+1} - K_j\cdot\nu_{j-1}$ must hold for $j=0,\dots,n$ and $\nu_j(0)=e_j$. This is a system of $(n+1)^2$ linear homogeneous one-dimensional differential equations and corresponding initial conditions. For such a system there exists a unique solution $(\nu_0,\dots,\nu_n)$ which we will show to be the moving frame of a curve. We assert that the ν_i are orthonormal for all times: We define $g_{ij} := \langle \nu_i \,|\, \nu_j \rangle : I \to \mathbb{R}$. Then $g_{ij}(0) = \delta_{ij}$. With

$$g'_{ij} = \langle \nu'_i \,|\, \nu_j \rangle + \langle \nu_i \,|\, \nu'_j \rangle = K_{i+1}g_{i+1,j} - K_i g_{i-1,j} + K_{j+1}g_{i,j+1} - K_j g_{i,j-1}$$

we obtain again a system of $(n+1)^2$ linear homogeneous one-dimensional differential equations and corresponding initial conditions $g_{ij}(0) = \delta_{ij}$. Again, there must be a unique solution g_{ij}. On the other hand, we see that δ_{ij} is a solution:

$$K_{i+1}\delta_{i+1,j} - K_i\delta_{i-1,j} + K_{j+1}\delta_{i,j+1} - K_j\delta_{i,j-1} = \\ = \left\{\begin{matrix} K_{i+1} - K_j & \text{for } j = i+1 \\ -K_i + K_{j+1} & \text{for } i = j+1 \\ 0 & \text{for } |i-j| \neq 1 \end{matrix}\right\} = 0 = \delta'_{i,j}$$

So $g_{ij} = \delta_{ij}$, i.e. the ν_i are orthonormal. They are also positively oriented, because $\det(\nu_0, \dots, \nu_n)(0) = 1$ and $\det(\nu_0, \dots, \nu_n)(t) = \pm 1$ for all t; So due to the intermediate value theorem it follows that $\det(\nu_0, \dots, \nu_n)(t) = 1$ for all t.

There is at most one curve c, which has the ν_i as moving frame and fulfills $c(0) = 0$, namely: $c(t) := \int_0^t \nu_0$, because c' must be equal to ν_0. Then $|c'| = |\nu_0| = 1$, so c is parameterized by arc length. By differentiating, we obtain $c^{(j+1)}(t) = \nu_0^{(j)}$, and because of the differential equation for ν_j, we have

$$\nu_0^{(j)} \equiv K_1 \cdot \ldots \cdot K_j \, \nu_j \mod \langle \{\nu_0, \dots, \nu_{j-1}\} \rangle$$

for all $j < n$ as shown in the proof of 8.7 using the Frénet-Serre formulas only. Thus

$$c^{(j+1)}(t) \equiv K_1 \cdot \ldots \cdot K_j \, \nu_j \mod \langle \{\nu_0, \dots, \nu_{j-1}\} \rangle$$

and hence the K_j are the curvatures of c and the ν_i from the moving frame by 8.7.

Finally, any other curve with these curvatures can be transformed by a unique motion into a curve with (the same curvatures and) the given initial conditions. The latter is uniquely determined by what has been said so far, so the same holds for the former. □

9. Curvatures of hypersurfaces

9.1 Definition (Hypersurface).

A HYPERSURFACE M in $\mathbb{R}^n$ is a submanifold of codimension 1, i.e. dimension $m := n - 1$. It can locally be given for example by an equation $f : \mathbb{R}^n \to \mathbb{R}$ or a parameterization $\varphi : \mathbb{R}^m \to \mathbb{R}^n$.

Examples.

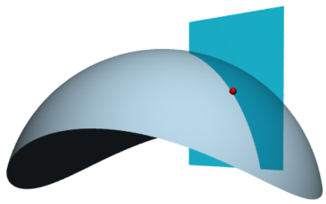
Surfaces in $\mathbb{R}^3$, spheres $S^m \subset \mathbb{R}^n$ and $SL(n) \subset L(n,n)$.

9.2 The Gauss map.

In each point $p \in M$ we have exactly two normalized normal vectors to T_pM in $\mathbb{R}^n$. If M is oriented, we can distinguish one of these normal vectors, such that $(\nu_p, e_1, \dots, e_m)$ is a positive oriented orthonormal basis of $\mathbb{R}^n$, for one (each) positive oriented orthonormal basis $(e_1, \dots, e_m)$ of T_pM, cf. [**95**, 28.9]. If M is given locally by an equation $f : \mathbb{R}^n \to \mathbb{R}$, the gradient $\operatorname{grad} f$ is a normal vector, which we only have to normalize, cf. [**95**, 27.41]. Thus, there is a local (and for oriented hypersurfaces even global) smooth mapping $M \ni p \mapsto \nu_p \in S^m \subset \mathbb{R}^n$ with $\nu_p \perp T_pM$, i.e. $T_pM = \nu_p^\perp$. A function ν choosen in that way is called GAUSS MAP.

9.3 Normal curvature.

We now want to define the curvature for hypersurfaces. Let $\nu_p \perp T_pM$ be a fixed unit normal vector and $\xi \in T_pM$ a unit tangent vector. We consider the intersection of M with the affine plane $(t,s) \mapsto p + t\nu + s\xi$ through p and directional vectors ξ and ν.



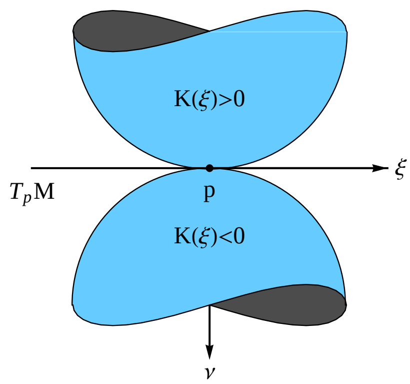
Let f be a local regular equation of M at p. Without loss of generality, $\operatorname{grad}_p f$ is normalized and has the same orientation as ν_p, i.e. $\nu_p = \operatorname{grad}_p f$. The intersection of the plane with M is then given by the equation $f(p + t\nu_p + s\xi) = 0$ in (t,s). In this implicit equation, we want to solve t in terms of s using the implicit function theorem [**95**, 2.2]. This is possible because of

$$\frac{\partial}{\partial t}\Big|_{t=0} f(p + t\nu_p + s\xi)|_{s=0} = f'(p)(\nu_p) = \langle \operatorname{grad}_p f | \nu_p \rangle = |\nu_p|^2 = 1 \neq 0.$$

So, as intersection, we get a local curve $c : s \mapsto p + t(s)\nu_p + s\xi$ in M with $c(0) = p$ and $c'(0) = \xi + t'(s)\nu_p = \xi$, since $c'(0) \in T_pM$. We may assume that c is parametrized proportional to arc length. The signed curvature of the plane curve c defined in 7.5, where we choose (ν_p, ξ) as the positively oriented basis, is called the NORMAL CURVATURE $K(\xi) := K_M(\xi) := K_c(0)$ of M at point p and direction ξ. Note that $(\xi, -\nu_p)$ is the moving frame of c in the point $p = c(0)$! A formula of $K(\xi)$ is obtained as follows: Due to $c(t) \in M$, $c'(t) \in T_{c(t)}M = {\nu_{c(t)}}^\perp$ holds, i.e. $\langle c'(t), \nu_{c(t)} \rangle = 0$. By differentiating at 0, we obtain: $\langle c''(0), \nu_p \rangle + \langle \xi, T_p\nu \cdot \xi \rangle = 0$. Consequently

$$K(\xi) = K_c(0) = \langle c''(0), -\nu_p \rangle = \langle \xi, T_p\nu \cdot \xi \rangle$$

This formula can also be used as a definition of $K(\xi)$ for $|\xi| \neq 1$.



9.4 The Weingarten map.

The tangential mapping

$$L_p := T_p\nu : T_pM \to T_{\nu_p}S^m = {\nu_p}^\perp = T_pM$$

of the Gauss mapping $\nu : M \to S^m$ is called the WEINGARTEN MAP, after Julius Weingarten, 1836–1910. Thus, the vector $L_p(\xi)$ measures the infinitesimal change of the surface normal when moving on M from p in direction $\xi \in T_pM$. By what has just been shown

$$K(\xi) = \langle \xi, L_p \cdot \xi \rangle.$$

9.5 Lemma.

The Weingarten map $L_p : T_pM \to T_pM$ is symmetrical.

First proof. Let ξ_1 and ξ_2 be two vector fields on M. We extend both ξ_i and ν locally around p to vector fields on $\mathbb{R}^n$. Because of $\langle \xi_1, \nu\rangle|_M = 0$, we have

$$\begin{aligned} 0 &= \langle \xi_1, \nu\rangle'(p)(\xi_2(p)) = \big\langle {\xi_1}'(p)(\xi_2(p)), \nu(p)\big\rangle + \big\langle \xi_1(p), \nu'(p)(\xi_2(p))\big\rangle \\ &= \big\langle {\xi_1}'(p)(\xi_2(p)), \nu_p\big\rangle + \big\langle \xi_1(p), L_p(\xi_2(p))\big\rangle. \end{aligned}$$

Thus we get

$$\begin{aligned} &\big\langle \xi_1(p), L_p(\xi_2(p))\big\rangle - \big\langle \xi_2(p), L_p(\xi_1(p))\big\rangle = \\ &\qquad = \big\langle {\xi_2}'(p)(\xi_1(p)) - {\xi_1}'(p)(\xi_2(p)), \nu_p\big\rangle = \big\langle \underbrace{[\xi_1,\xi_2](p)}_{\in T_pM}, \nu_p\big\rangle = 0. \quad \square \end{aligned}$$

Second proof. Let $\varphi : \mathbb{R}^m \to M \subset \mathbb{R}^n$ be a local parameterization centered at p. With φ_i we denote the i-th partial derivative of φ. The $\varphi_i(0)$ form a basis of T_pM for $i = 1, \dots, m$

$$\begin{aligned} \big\langle \varphi_i(0), L_p \cdot \varphi_j(0)\big\rangle &= \big\langle \tfrac{d}{dt}|_{t=0}\varphi(t\,e_i), T_p\nu \cdot \tfrac{d}{ds}|_{s=0}\varphi(s\,e_j)\big\rangle \\ &= \big\langle \tfrac{d}{dt}|_{t=0}\varphi(t\,e_i), \tfrac{d}{ds}|_{s=0}\nu(\varphi(s\,e_j))\big\rangle \\ &= \tfrac{d}{ds}|_{s=0}\big\langle \tfrac{d}{dt}|_{t=0}\varphi(t\,e_i + s\,e_j), \nu(\varphi(s\,e_j))\big\rangle \\ &\quad - \big\langle \tfrac{d}{ds}|_{s=0}\tfrac{d}{dt}|_{t=0}\varphi(t\,e_i + s\,e_j), \nu(\varphi(0\,e_j))\big\rangle \\ &= 0 - \langle \varphi_{i,j}(0), \nu_p\rangle, \end{aligned}$$

and thus is obviously symmetric in (i, j), because φ's mixed 2-nd partial derivatives $\varphi_{i,j}$ are symmetric. Since we may assume w.l.o.g. that the $\varphi_i(0)$ are orthonormal (compose φ from the right with the inverse of the Gram-Schmidt orthonormalization), symmetry for L follows. $\square$

9.6 The fundamental forms of a surface.

The symmetric bilinear form $\mathrm{II}_p(\xi_1, \xi_2) := \langle \xi_1, L_p(\xi_2)\rangle$ on T_pM is called the 2nd fundamental form of M. The 1st fundamental form is the Riemann metric, i.e. $\mathrm{I}(\xi, \eta) := \langle \xi, \eta\rangle$. We showed in 9.3 that $K(\xi) = \mathrm{II}(\xi, \xi)$ holds.

9.7 Spektrum of the Weingarten mapping.

We now want to determine the extremal values of the normal curvature. Because of the homogeneity of L, this task only makes sense if we restrict the mapping K to the unit sphere $S^{m-1} \subset T_pM$. Critical point $\xi \in S^{m-1}$ are those for which the tangent map $T_\xi K : T_\xi S^{m-1} \to \mathbb{R}$ is 0, i.e. $K'(\xi)(v) = 0$ for all $v \in T_\xi S^{m-1} = \xi^\perp$. The following holds:

$$K'(\xi)(v) = \tfrac{d}{dt}|_{t=0}\,\mathrm{II}(\xi + tv, \xi + tv) \overset{9.5}{=\!=} 2\,\mathrm{II}(\xi, v) = 2\langle L\xi, v\rangle.$$

So $L\xi \in T_xM$ has to be normal to all those v, which are normal to ξ, i.e. $L\xi$ must be proportional to ξ. This shows the following:

Theorem of Rodriguez.

The critical points ξ of the normal curvature are exactly the eigenvectors of the symmetric linear map L, and the eigenvalue λ belonging to ξ is given by

$$\lambda = \lambda\langle \xi, \xi\rangle = \langle \xi, \lambda\xi\rangle = \langle \xi, L\xi\rangle = \mathrm{II}(\xi, \xi) = K(\xi),$$

i.e. is the normal curvature of M in direction ξ. In the case of $m = 2$, the critical points are also extremal, namely the minimum and maximum of $K(\xi)$ for $|\xi| = 1$. For $m > 2$, the critical points are not necessary extremal. □

9.8 Main and Gaussian Curvature.

The eigenvalues of L are called the MAIN CURVATURES and the associated eigenvectors are MAIN CURVATURE DIRECTIONS. Since L is symmetric, there are only real eigenvalues and an orthonormal basis of T_pM of eigenvectors can be choosen (Use $0 = \langle Av, w\rangle - \langle v, Aw\rangle = (\lambda - \mu)\langle v, w\rangle$). Let K_i be the main curvatures and let ξ_i be an orthonormal basis of the associated main curvature directions. Then according to Euler we have for the normal curvature

$$K(\xi) = \mathrm{II}(\xi, \xi) = \mathrm{II}\Big(\sum_i \langle \xi, \xi_i\rangle \xi_i, \sum_j \langle \xi, \xi_j\rangle \xi_j\Big) =$$
$$= \sum_{i=j} \langle \xi, \xi_i\rangle\langle \xi, \xi_j\rangle\, \mathrm{II}(\xi_i, \xi_j) = \sum_i \langle \xi, \xi_i\rangle^2 K_i.$$

The GAUSS CURVATURE $K \in \mathbb{R}$ at point p is the product of all main curvatures, that is, the determinant of L.
The MEAN CURVATURE $H \in \mathbb{R}$ is the arithmetic mean of the main curvatures, that is $\frac{1}{m}$ of the trace of L.
A curve c in M is called a CURVATURE LINE if its derivative at each point is a main curvature direction.
A vector $\xi \neq 0$ is called ASYMPTOTIC DIRECTION if $\mathrm{II}(\xi, \xi) = K(\xi) = 0$. A curve c in M is called the ASYMPTOTIC LINE if its derivative at each point is an asymptotic direction.
Finally, two vectors $\xi_1 \neq 0$ and $\xi_2 \neq 0$ are CONJUGATED if $\mathrm{II}(\xi_1, \xi_2) = 0$.
A point p is called UMBILIC POINT (or NAVEL POINT) if all main curvatures are equal, i.e. L is a multiple of the identity. Then the normal curvature is constantly equal to the mean curvature.
If all main curvatures are 0, this is called a FLAT POINT.

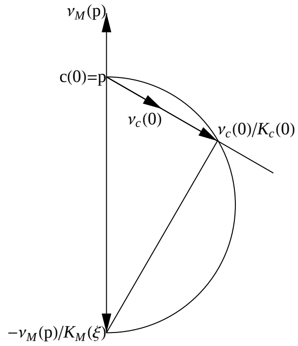
9.9 Examples.

1. Hyperplane: $\mathbb{R}^m := e_0^\perp \subset \mathbb{R}^n$. As a normal vector we use e_0. The Gaussian mapping is thus constant e_0 and the Weingarten mapping $L = 0$. So the curvatures defined above are all equal to 0. All points are flat points and all directions are main curvature directions and asymptotic directions.
2. Sphere: $S^m = \{x : |x| = R\} \subset \mathbb{R}^n$. Here we can take $\nu_x = \frac{1}{R}x$ as normal in the point $x \in S^m$. Then the Gaussian mapping is the linear mapping $\frac{1}{R}\,\mathrm{id}$ and thus this is also the Weingarten mapping. So all points are umbilic points and all directions are main curvature vectors with main curvature $\frac{1}{R}$. There are no asymptotic directions. The Gaussian curvature is thus $\frac{1}{R^m}$ and the mean curvature is $\frac{1}{R}$.
3. Cylinder: $M := \{(x, t) \in \mathbb{R}^m \times \mathbb{R} : |x| = 1\} \subset \mathbb{R}^n$. As normal in $(x, t) \in M$ we can use $\nu_{x,t} = (x, 0) \in \mathbb{R}^n \times \mathbb{R}$. The tangent space of M at this point is thus $T_{(x,t)}M := \nu_{x,t}^\perp = \{(y, s) \in \mathbb{R}^n \times \mathbb{R} : y \perp x\}$ and the Gaussian mapping is the restriction of the linear mapping $\mathrm{id} \oplus 0$ to $T_xS^{m-1} \times \mathbb{R}$. The Weingarten map looks exactly the same. One main curvature is thus 0 with curvature direction $(0, 1)$ and all other main curvatures are 1. The generators $\{x\} \times \mathbb{R}$ are the asymptotic lines. A curve $c : s \mapsto (x(s), t(s))$ is a curvature line if and only if $s \mapsto t(s)$ or $s \mapsto x(s)$ is constant.

9.10 Lemma [112].

If c is a curve parameterized by arc length on M with $c(0) = p \in M$ and $c'(0) = \xi \in T_pM$, $|\xi| = 1$, the following holds:

1. $(\nu \circ c)'(0) = L_p \cdot \xi$,
   that is, $L_p \cdot \xi$ measures the infinitesimal change of ν along c.
2. $-\langle c''(0), \nu_p\rangle = \mathrm{II}(\xi, \xi) = \langle L_p \cdot \xi, \xi\rangle = K(\xi)$,
   that is, the normal component of the acceleration depends only on the velocity vector, and is the normal curvature in its direction.
3. *We have $-K_M(\xi) = K_c(0)\langle \nu_M(p), \nu_c(0)\rangle = K_c(0)\cos\theta$, where θ is the angle between the surface normal $\nu_M(p)$ and the main normal vector $\nu_c(0)$ (or equivalent of the osculating plane) of c in p, and $K_c(0) \geq 0$ is the curvature of the space curve c.*
4. *The osculating circle to c in p has its center on the sphere around $p - \frac{1}{2K_M(\xi)}\nu_M(p)$ through p.*



Proof. 1 is just the definition of the Weingarten map.

2 To show this, we differentiate $0 = \langle c'(t), \nu_M(c(t))\rangle$ as in 9.3 and get

$$-\Big\langle c''(0), \nu_M(c(0))\Big\rangle = \Big\langle c'(0), (\nu \circ c)'(0)\Big\rangle \overset{\boxed{1}}{=\!=} \langle \xi, L \cdot \xi\rangle = \mathrm{II}(\xi, \xi) = K(\xi).$$

3 The result follows from 2 because of

$$-K_M(\xi) \overset{\boxed{2}}{=\!=} \Big\langle c''(0), \nu_M(p)\Big\rangle \overset{\boxed{8.1}}{=\!=} \Big\langle K_c(0)\nu_c(0), \nu_M(p)\Big\rangle = K_c(0)\Big\langle \nu_c(0), \nu_M(p)\Big\rangle.$$

4 According to 7.5, the center of the osculating circle (the higher-dimensional variant of the osculating circle) is given by $c(0) + \frac{1}{K_c(0)}\nu_c(0)$. Now we consider the triangle with vertices p, $p - \frac{1}{K(\xi)}\nu_M(p)$ and $p + \frac{1}{K_c(0)}\nu_c(0)$. This has a right angle at $p + \frac{1}{K_c(0)}\nu_c(0)$, because by 3 we have

$$\Big\langle \nu_M(p), \nu_c(0)\Big\rangle = -\frac{\mathrm{II}(\xi,\xi)}{K_c(0)} = -\frac{K_M(\xi)}{K_c(0)}$$

and thus

$$\Big\langle \frac{1}{K_c(0)}\nu_c(0), \frac{1}{K_M(\xi)}\nu_M(p) + \frac{1}{K_c(0)}\nu_c(0)\Big\rangle = -\frac{1}{K_c(0)K_M(\xi)}\frac{K_M(\xi)}{K_c(0)} + \frac{1}{K_c(0)^2} = 0.$$

So, the center of osculating circle lies on the circle (or in reality on the sphere) of Thales with distance from p to $p - \frac{1}{K_M(\xi)}\nu_M(p)$ as diameter. □

9.11 Formulas for parameterized surfaces.

Let $\varphi : \mathbb{R}^m \to M \subset \mathbb{R}^n$ be a local parameterization of a hypersurface M. For a point $p = \varphi(u) \in M$, a basis of the tangent space $T_pM = \operatorname{im}\varphi'(u)$ is given by $(\partial_1\varphi(u), \ldots, \partial_m\varphi(u))$, where $\partial_i\varphi(u) = \frac{\partial}{\partial u^i}\varphi(u)$ is the ith partial derivative of φ at

u, we will briefly write $\varphi_i(u)$ for this. Similarly, $\varphi_{i,j}$ will denote the second partial derivative $\frac{\partial^2}{\partial u^i \partial u^j}\varphi(u)$. Since $\langle(\nu\circ\varphi)(u), \varphi_i(u)\rangle = 0$ for all i, we get for the j-th partial derivative $0 = \langle L\cdot\varphi_j, \varphi_i\rangle + \langle\nu, \varphi_{i,j}\rangle$, i.e. $\langle L\cdot\varphi_j, \varphi_i\rangle = -\langle\nu, \varphi_{i,j}\rangle$. If the φ_i were orthonormal, then these are the matrix coefficients of L. In the general case, L is also determined by all these inner products. To see sthis we need the following lemma from linear algebra.

9.12 Lemma.

Let $(g_1, \ldots, g_m)$ be a basis of the Euclidean vector space V and $T : V \to V$ a linear map. Let $g_{i,j} := \langle g_i, g_j\rangle$ and $G = (g_{i,j})$ be the associated symmetric positive definite matrix, $[T] := (T^i_j)$ the matrix of T with respect to the basis $(g_i, \ldots, g_j)$, i.e. $Tg_j = \sum_i T^i_j g_i$, and finally A the matrix with entries $A^i_j := \langle g_i, Tg_j\rangle$. Then $[T] = G^{-1}\cdot A$ holds.

Proof. By defintion, $Tg_j = \sum_i T^i_j g_i$, where j counts the columns and i counts the rows, and thus

$$A^k_j := \langle g_k, Tg_j\rangle = \Big\langle g_k, \sum_i T^i_j g_i\Big\rangle = \sum_i \langle g_k, g_i\rangle T^i_j = \sum_i g_{k,i} T^i_j$$

So $A = G\cdot[T]$ and thus $[T] = G^{-1}\cdot A$. □

9.13 Corollary (Matrix representation of the Weingarten map).

The Weingarten map has the following matrix representation

$$[L] = -(\langle\varphi_i, \varphi_j\rangle)^{-1}\cdot(\langle\nu, \varphi_{i,j}\rangle). \quad \square$$

with respect to the basis $(\varphi_1, \ldots, \varphi_m)$

9.14 Formulas for 2-surfaces.

In particular, let now $m = 2$ (i.e. $n = 3$) and $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$, $(t, s) \mapsto \varphi(t, s)$ be a local parameterization. Then put

$$\begin{gathered} E := g_{11} = \langle\varphi_t, \varphi_t\rangle, \quad F := g_{12} = \langle\varphi_t, \varphi_s\rangle, \quad G := g_{22} = \langle\varphi_s, \varphi_s\rangle \\ \nu := \frac{\varphi_t\times\varphi_s}{|\varphi_t\times\varphi_s|}, \qquad |\varphi_t\times\varphi_s| = \sqrt{|\varphi_t|^2\cdot|\varphi_s|^2 - \langle\varphi_t, \varphi_s\rangle^2} = \sqrt{EG - F^2} \\ e := -\langle\nu, \varphi_{t,t}\rangle, \quad f := -\langle\nu, \varphi_{t,s}\rangle, \quad g := -\langle\nu, \varphi_{s,s}\rangle. \end{gathered}$$

Here we used that the length of a vector of the form $v\times w$, i.e. the surface area of the parallelogram spanned by v and w, is given as follows:

$$\begin{aligned} |v\times w| &= |v|\cdot|w|\cdot\sin\measuredangle(v, w) = |v|\cdot|w|\cdot\sqrt{1 - \cos^2\measuredangle(v, w)} \\ &= |v|\cdot|w|\cdot\sqrt{1 - \frac{\langle v, w\rangle^2}{|v|^2\cdot|w|^2}} = \sqrt{|v|^2\cdot|w|^2 - \langle v, w\rangle^2}. \end{aligned}$$

With respect to the basis (φ_t, φ_s), the fundamental forms look like this:

$$[\mathrm{I}] = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \text{ and } [\mathrm{II}] = \begin{pmatrix} e & f \\ f & g \end{pmatrix}.$$

The Weingarten map is:

$$\begin{aligned}[L] &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \cdot \begin{pmatrix} e & f \\ f & g \end{pmatrix} \\ &= \frac{1}{EG-F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \cdot \begin{pmatrix} e & f \\ f & g \end{pmatrix} \\ &= \frac{1}{EG-F^2} \begin{pmatrix} Ge - Ff & Gf - Fg \\ Ef - Fe & Eg - Ff \end{pmatrix}\end{aligned}$$

The Gaussian curvature is thus

$$K = \det L = \frac{(eg - f^2)(EG-F^2)}{(EG-F^2)^2} = \frac{eg - f^2}{EG - F^2},$$

as seen from $K = \det L = \det(\mathrm{I}^{-1} \cdot \mathrm{II}) = \det \mathrm{II} \,/\, \det \mathrm{I}$, and the mean curvature is

$$2H = \operatorname{trace} L = \frac{1}{EG-F^2} \begin{pmatrix} Ge - Ff & * \\ * & Eg - Ff \end{pmatrix} = \frac{Ge - 2Ff + Eg}{EG-F^2}.$$

The main curvatures $K_{1,2}$ are obtained as solutions of the characteristic equation

$$K^2 - \operatorname{trace} L \cdot K + \det L = 0, \text{ i.e. } K_{1,2} = H \pm \sqrt{H^2 - K}.$$

The main curvature directions are the corresponding eigen-vectors $\xi = a_t\varphi_t + a_s\varphi_s$, i.e. $L(\xi) = K_i\xi$. Thus they are determined by

$$\begin{aligned}0 = 0 \cdot (EG-F^2) &= \det\left(\begin{pmatrix} a_t \\ a_s \end{pmatrix} \quad [L]\begin{pmatrix} a_t \\ a_s \end{pmatrix} \right) \cdot (EG - F^2) \\ &= \det \begin{pmatrix} a_t & (Ge - Ff)a_t + (Gf - Fg)a_s \\ a_s & (Ef - Fe)a_t + (Eg - Ff)a_s \end{pmatrix} \\ &= {a_t}^2(Ef - Fe) + a_t a_s (Eg - Ge) + {a_s}^2 (Fg - Gf) \\ &= \det \begin{pmatrix} {a_t}^2 & -a_t a_s & {a_s}^2 \\ g & f & e \\ G & F & E \end{pmatrix}\end{aligned}$$

9.15 Determinant formulas for the curvature.

We now want to determine which quantities are INTRINSIC, that is, do not change when we pass to an isometric surface. So these are the quantities that can be recognized by a being living in the surface, without being aware of the surrounding space. Of course, they can measure lengths and angles, and hence the 1st fundamental form is intrinsic. But not the 2nd fundamental form, since it is defined by the derivative of the normal vector. So we know from the outset, of none of the curvatures whether they are intrinsic. If we compare the cylinder and the plane, we see that both the main curvatures and the mean curvature are not intrinsic. Now let us show that the Gaussian curvature is nevertheless intrinsic. For this we first need formulas for e, f and g in which ν does not appear:

$$\begin{aligned}e = -\langle \nu, \varphi_{t,t} \rangle &= -\left\langle \frac{\varphi_t \times \varphi_s}{|\varphi_t \times \varphi_s|}, \varphi_{t,t} \right\rangle \\ &= -\frac{1}{\sqrt{EG-F^2}} \langle \varphi_t \times \varphi_s, \varphi_{t,t} \rangle = -\frac{1}{\sqrt{EG-F^2}} \det(\varphi_t, \varphi_s, \varphi_{t,t}) \\ \text{and } f = -\frac{1}{\sqrt{EG-F^2}} &\det(\varphi_t, \varphi_s, \varphi_{t,s}) \text{ and } g = -\frac{1}{\sqrt{EG-F^2}} \det(\varphi_t, \varphi_s, \varphi_{s,s}).\end{aligned}$$

Here we have used $|\varphi_t \times \varphi_s| = \sqrt{EG - F^2}$ from 9.14.

We now try to represent the Gaussian curvature solely by the coefficients of the first fundamental form, as well as their partial derivatives. Let $D := \sqrt{EG - F^2}$. The following holds:

$$
\begin{aligned}
KD^4 &\overset{9.14}{=\!=\!=} (eg - f^2)D^2 = (-eD)(-gD) - (-fD)^2 \\
&= \det(\varphi_t, \varphi_s, \varphi_{t,t}) \cdot \det(\varphi_t, \varphi_s, \varphi_{s,s}) - \det(\varphi_t, \varphi_s, \varphi_{t,s})^2 \\
&= \det((\varphi_t, \varphi_s, \varphi_{t,t})^* \cdot (\varphi_t, \varphi_s, \varphi_{s,s})) - \det((\varphi_t, \varphi_s, \varphi_{t,s})^* \cdot (\varphi_t, \varphi_s, \varphi_{t,s})) \\
&= \det\begin{pmatrix} \varphi_t^*\varphi_t & \varphi_t^*\varphi_s & \varphi_t^*\varphi_{s,s} \\ \varphi_s^*\varphi_t & \varphi_s^*\varphi_s & \varphi_s^*\varphi_{s,s} \\ \varphi_{t,t}^*\varphi_t & \varphi_{t,t}^*\varphi_s & \varphi_{t,t}^*\varphi_{s,s} \end{pmatrix} - \det\begin{pmatrix} \varphi_t^*\varphi_t & \varphi_t^*\varphi_s & \varphi_t^*\varphi_{t,s} \\ \varphi_s^*\varphi_t & \varphi_s^*\varphi_s & \varphi_s^*\varphi_{t,s} \\ \varphi_{t,s}^*\varphi_t & \varphi_{t,s}^*\varphi_s & \varphi_{t,s}^*\varphi_{t,s} \end{pmatrix} \\
&= \det\begin{pmatrix} E & F & \langle\varphi_t, \varphi_{s,s}\rangle \\ F & G & \langle\varphi_s, \varphi_{s,s}\rangle \\ \langle\varphi_{t,t}, \varphi_t\rangle & \langle\varphi_{t,t}, \varphi_s\rangle & \langle\varphi_{t,t}, \varphi_{s,s}\rangle \end{pmatrix} \\
&\quad - \det\begin{pmatrix} E & F & \langle\varphi_t, \varphi_{t,s}\rangle \\ F & G & \langle\varphi_s, \varphi_{t,s}\rangle \\ \langle\varphi_{t,s}, \varphi_t\rangle & \langle\varphi_{t,s}, \varphi_s\rangle & \langle\varphi_{t,s}, \varphi_{t,s}\rangle \end{pmatrix} \\
&= \det\begin{pmatrix} E & F & \langle\varphi_t, \varphi_{s,s}\rangle \\ F & G & \langle\varphi_s, \varphi_{s,s}\rangle \\ \langle\varphi_{t,t}, \varphi_t\rangle & \langle\varphi_{t,t}, \varphi_s\rangle & \langle\varphi_{t,t}, \varphi_{s,s}\rangle - \langle\varphi_{t,s}, \varphi_{t,s}\rangle \end{pmatrix} \\
&\quad - \det\begin{pmatrix} E & F & \langle\varphi_t, \varphi_{t,s}\rangle \\ F & G & \langle\varphi_s, \varphi_{t,s}\rangle \\ \langle\varphi_{t,s}, \varphi_t\rangle & \langle\varphi_{t,s}, \varphi_s\rangle & 0 \end{pmatrix} \\
&= \det\begin{pmatrix} E & F & F_s - \frac12 G_t \\ F & G & \frac12 G_s \\ \frac12 E_t & F_t - \frac12 E_s & F_{t,s} - \frac12(E_{s,s} + G_{t,t}) \end{pmatrix} - \det\begin{pmatrix} E & F & \frac12 E_s \\ F & G & \frac12 G_t \\ \frac12 E_s & \frac12 G_t & 0 \end{pmatrix},
\end{aligned}
$$

because of the development formula for determinants and since

$$
\begin{gathered}
E = \langle\varphi_t, \varphi_t\rangle, \quad G = \langle\varphi_s, \varphi_s\rangle, \quad F = \langle\varphi_t, \varphi_s\rangle \\
E_t = 2\langle\varphi_{t,t}, \varphi_t\rangle, \quad G_t = 2\langle\varphi_{s,t}, \varphi_s\rangle, \quad F_t = \langle\varphi_{t,t}, \varphi_s\rangle + \langle\varphi_t, \varphi_{s,t}\rangle \\
E_s = 2\langle\varphi_{t,s}, \varphi_t\rangle, \quad G_s = 2\langle\varphi_{s,s}, \varphi_s\rangle, \quad F_s = \langle\varphi_{t,s}, \varphi_s\rangle + \langle\varphi_t, \varphi_{s,s}\rangle, \\
F_s - \frac{1}{2}G_t = \langle\varphi_t, \varphi_{s,s}\rangle, \quad F_t - \frac{1}{2}E_s = \langle\varphi_{t,t}, \varphi_s\rangle, \\
\tfrac12 E_{s,s} = \langle\varphi_{t,s,s}, \varphi_t\rangle + \langle\varphi_{t,s}, \varphi_{t,s}\rangle, \\
F_{t,s} = \langle\varphi_{t,t,s}, \varphi_s\rangle + \langle\varphi_{t,t}, \varphi_{s,s}\rangle + \langle\varphi_{t,s}, \varphi_{s,t}\rangle + \langle\varphi_t, \varphi_{s,t,s}\rangle, \\
F_{t,s} - \tfrac12 E_{s,s} = \langle\varphi_{t,t,s}, \varphi_s\rangle + \langle\varphi_{t,t}, \varphi_{s,s}\rangle, \\
\tfrac12 G_{t,t} = \langle\varphi_{s,t,t}, \varphi_s\rangle + \langle\varphi_{s,t}, \varphi_{s,t}\rangle, \\
F_{t,s} - \tfrac12(E_{s,s} + G_{t,t}) = \langle\varphi_{t,t}, \varphi_{s,s}\rangle - \langle\varphi_{s,t}, \varphi_{s,t}\rangle.
\end{gathered}
$$

By expanding the determinants of the above formula for K we obtain:

$$
\begin{aligned}
4(EG - F^2)^2 K = {} & E(E_s G_s - 2F_t G_s + {G_t}^2) \\
& + F(E_t G_s - E_s G_t - 2E_s F_s + 4F_t F_s - 2F_t G_t) \\
& + G(E_t G_t - 2E_t F_s + {E_s}^2) \\
& - 2(EG - F^2)(E_{s,s} - 2F_{t,s} + G_{t,t}).
\end{aligned}
$$

A more symmetric formula for K is the following:

$$K = -\frac{1}{4D^4}\det\begin{pmatrix} E & E_t & E_s \\ F & F_t & F_s \\ G & G_t & G_s \end{pmatrix} - \frac{1}{2D}\left(\partial_s \frac{E_s - F_t}{D} + \partial_t \frac{G_t - F_s}{D}\right)$$

where again $D := \sqrt{EG - F^2}$. This can easily be verified by expanding the determinant and differentiating it.

9.16 Theorema Egregium [45]**.**

If two surfaces can be developed into one another, i.e. they are (locally) isometric, then they have the same Gaussian curvature in corresponding points. The Gaussian curvature K is thus an intrinsic concept, i.e. it depends only on the metric of the surface and not on the surrounding space.

For a partial reversal, see 11.11.

Proof. Because of the formula in 9.15, the Gaussian curvature depends only on the coefficients of the Riemann metric and its 1st and 2nd partial derivatives. However, these are the same for two isometric surfaces, because for a local isometry f and a parameterization φ of the range manifold, $\psi = f \circ \varphi$ is a parameterization of the domain manifold, and thus the basis vector fields are Tf related and hence the coefficients of the Riemannian metric in the corresponding coordinates are the same. □

9.17 Lemma (Jacobi equation).

If (t, s) are geodesic coordinates on M (i.e. $E = 1$ and $F = 0$ for the associated parameterization), the Gaussian curvature satisfies the Jacobi equation*:*

$$K = -\frac{1}{\sqrt{G}}\left(\frac{\partial}{\partial t}\right)^2 \sqrt{G}$$

We will show in 10.9 and 10.10 that such coordinates always exist. Note that condition $E = 1$ tells us that the parameter lines $t \mapsto \varphi(t, s)$ are parameterized by arc length and $F = 0$ tells us that the other parameter lines $s \mapsto \varphi(t, s)$ are orthogonal thereto.

Proof. The above determinant formula for K yields in this case:

$$K \cdot G^2 = \det\begin{pmatrix} 1 & 0 & -\frac{1}{2}G_t \\ 0 & G & \frac{1}{2}G_s \\ 0 & 0 & -\frac{1}{2}G_{t,t} \end{pmatrix} - \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & G & \frac{1}{2}G_t \\ 0 & \frac{1}{2}G_t & 0 \end{pmatrix} = -\frac{1}{2}GG_{t,t} + \frac{1}{4}{G_t}^2$$

$$\Rightarrow \quad K = -\frac{1}{\sqrt{G}}\left(\frac{\partial}{\partial t}\right)^2 \sqrt{G} \qquad \square$$

Since we have shown that the Gauß-curvature is intrinsic for surfaces, it suggests that we may use the determinant formula from 9.15 resp. the Jacobi identity from 9.17 to define a related curvature also for general Riemannian surfaces. For this we will have to prove its invariance under coordinate changes, and we will do so in 14.8.

9.18 Definition (Surface of revolution).

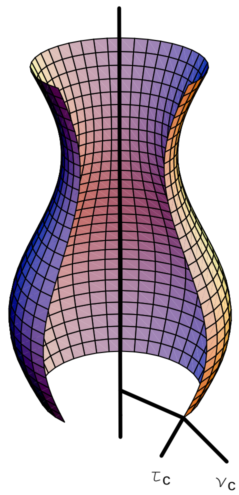
A SURFACE OF REVOLUTION is the (hyper-)surface which is created when a curve in the (x, z) plane is rotated around the z axis. So let $c : s \mapsto (r(s), z(s))$ be this curve, which we may assume to be parameterized by arc length. Then, the surface of revolution M generated this way is given by

$$M := \{(r(s)x, z(s)) \in \mathbb{R}^m \times \mathbb{R} : x \in S^{m-1}\}.$$

So in particular, if $m = 2$, we can parameterize S^1 by $\theta \mapsto (\cos\theta, \sin\theta)$ and get a **parametrization**

$$\varphi : (s, \theta) \mapsto (r(s)\cos\theta, r(s)\sin\theta, z(s))$$

of M.



We want to calculate the **Gaussian curvature**. The partial derivatives of φ are

$$\left.\begin{aligned}\varphi_s(s,\theta) &= (r'(s)\cos\theta, r'(s)\sin\theta, z'(s))\\ \varphi_\theta(s,\theta) &= (-r(s)\sin\theta, r(s)\cos\theta, 0)\end{aligned}\right\} \Rightarrow \left\{\begin{aligned}&E = r'(s)^2 + z'(s)^2 = 1\\ &F = 0\\ &G = r(s)^2\end{aligned}\right\}$$

$$\overset{9.17}{\Longrightarrow} \quad K = -\frac{1}{\sqrt{G}}\left(\frac{\partial}{\partial s}\right)^2\sqrt{G} = -\frac{1}{r(s)}\left(\frac{\partial}{\partial s}\right)^2(r(s)) = -\frac{r''(s)}{r(s)}.$$

One can use Dupin's theorem [87, 52.6] to determine the **curvature lines**. Let

$$\Psi(u^1, u^2) := c(u^1) + u^2\,\nu(u^1) = (r(u^1) - u^2\,z'(u^1), z(u^1) + u^2\,r'(u^1)),$$

be where ν denotes the unit normal to c. Then

$$\partial_1\Psi(u^1, u^2) = c'(u^1) + u^2\,\nu'(u^1) = (1 - u^2\,K(u^1))\,\tau(u^1) \perp \nu(u^1) = \partial_2\Psi(u^1, u^2).$$

Therefore, $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$, given by

$$\begin{aligned}&\Phi(u^1, u^2, u^3) :=\\ &= ((r(u^1) - u^2\,z'(u^1))\,\cos(u^3), (r(u^1) - u^2\,z'(u^1))\,\sin(u^3), z(u^1) + u^2\,r'(u^1)),\end{aligned}$$

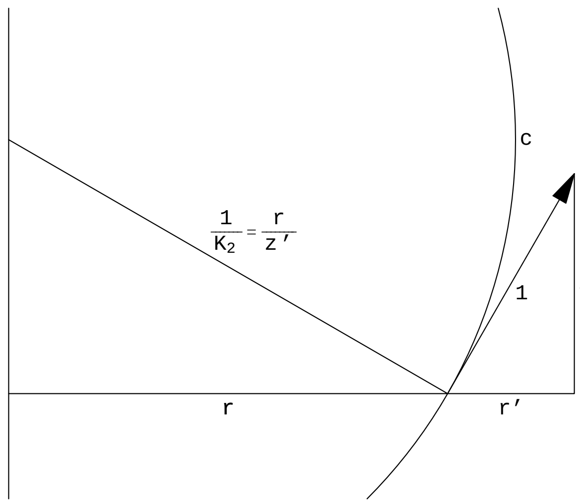
meets the requirements of Dupin's Theorem. Thus, both the MERIDIANS $s \mapsto \Phi(s, 0, \theta) = \varphi(s, \theta)$ and the LATITUDINAL CIRCLES $\theta \mapsto \Phi(s, 0, \theta) = \varphi(s, \theta)$ are curvature lines.

Main curvature in the direction of the meridians: A meridian is the intersection of M with a plane through the z axis. The normal curvature in the direction ξ of the meridian is thus just the curvature of the meridian or the generating curve c by 9.3, provided we use $-\nu_c = (z', -r')$ for ν_M (see also 9.10.3).
That this is a main curvature can be seen also directly:
The normal vector ν_M to the surface in each point of the meridian lies in this plane. Hence, if we differentiate it in the direction of ξ of the meridian, the result $L\xi$ is again in the plane, so it must be proportional to ξ. Since c is parameterized by arc length, according to 7.5 we have: $(r'', z'') = \tau' = K_c \cdot \nu = K_c(-z', r')$ and thus $K_1 := K(\xi) = K_c = -\frac{r''}{z'} = \frac{z''}{r'}$.

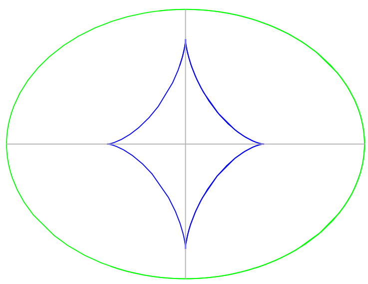
Main curvature in the direction of the latitudinal circles: The fact that the circles of fixed latitudes are also curvature lines, follows directly from 9.8, because they are normal to the meridians. Using the Gaussian curvature we obtain for the second main curvature $K_2 = \frac{K}{K_1} = \frac{z'}{r} = -\frac{r''\,r'}{z''\,r}$. Conversely, Meusnier's Theorem 9.10.3 provides a geometric method to calculate the second main curvature directly and thus the **Gaussian curvature**:
The unit normal ν_M to the surface is just $(z', 0, -r')$, up to a rotation around the z axis by the angle θ. The main normal to the circle of fixed latitude is the equally rotated vector $(-1, 0, 0)$. The curvature of the circle of fixed latitude is $\frac{1}{r}$ and the normal curvature in the direction of its tangent is therefore

$$K_2 \overset{9.10.3}{=\!=\!=} -\frac{1}{r}\Big\langle (z', 0, -r'), (-1, 0, 0) \Big\rangle = \frac{z'}{r}.$$

Umbilical points: These are given by the equation $K_1 = K_2$, that is, by $-\frac{r''}{z'} = \frac{z'}{r}$, or equivalent to $-r''\,r = (z')^2 = 1 - (r')^2$. The center of the osculating circle to the intersection curve with the plane generated by ν_M and φ_θ through the point $\varphi(s, 0)$ lies on the normal $\nu_c(s)$ at a distance $1/K_2 = \frac{r}{z'}$. This center is (because of similar triangles) the point of intersection of the normal with the axis of rotation. Therefore $\varphi(s, 0)$ (and thus $\varphi(s, \theta)$ for all θ) is an umbilical point if and only if it is also the center of osculating circle to the meridian c at the parameter s.



For example, the only umbilical points of an ellipsoid of revolution (which is not a sphere) are the poles.



Evolute of the ellipse

Asymptotic directions: $\xi = \xi^1\varphi_s + \xi^2\varphi_\theta$ is such a direction if and only if

$$\begin{aligned} 0 = K(\xi) = \langle \xi, L\xi\rangle &= \langle \xi^1\varphi_s + \xi^2\varphi_\theta, \xi^1 K_1\varphi_s + \xi^2 K_2\varphi_\theta\rangle \\ &= (\xi^1)^2 K_1 E + (\xi^2)^2 K_2 G + \xi^1\xi^2(K_1+K_2)F \\ &= K_1\,(\xi^1)^2 + G\,K_2\,(\xi^2)^2. \end{aligned}$$

Together with

$$\begin{aligned} 1 = |\xi|^2 &= \langle \xi^1\varphi_s + \xi^2\varphi_\theta, \xi^1\varphi_s + \xi^2\varphi_\theta\rangle \\ &= E(\xi^1)^2 + G(\xi^2)^2 + 2F\xi^1\xi^2 \\ &= (\xi^1)^2 + G\,(\xi^2)^2, \end{aligned}$$

this system of linear equations in $(\xi^1)^2$ and $(\xi^2)^2$ has a unique solution if and only if

$$0 \neq \det\begin{pmatrix} K_1 & G\,K_2 \\ 1 & G \end{pmatrix} = G\cdot(K_1 - K_2),$$

i.e. when $K_1 \neq K_2$, given by $(\xi^1)^2 := \frac{K_2}{K_2-K_1}$ and $(\xi^2)^2 := -\frac{1}{G}\frac{K_1}{K_2-K_1}$. Only for $K = K_1\cdot K_2 \le 0$ exist real solutions (ξ^1,ξ^2).

9.19 Example.

We consider the torus with radius A of the central core and radius $a < A$ of the meridians. It is generated by rotation of the arc length parameterized curve

$$c(s) := (r(s), z(s)) := (A, 0) + a\left(\cos\left(\tfrac{s}{a}\right), \sin\left(\tfrac{s}{a}\right)\right).$$

Consequently, $K_1 := \frac{1}{a} > 0$ is the main curvature in the direction of the meridians, the Gauss curve is

$$K = -\frac{r''(s)}{r(s)} = \frac{\cos(s/a)/a}{A + a\cos(s/a)},$$

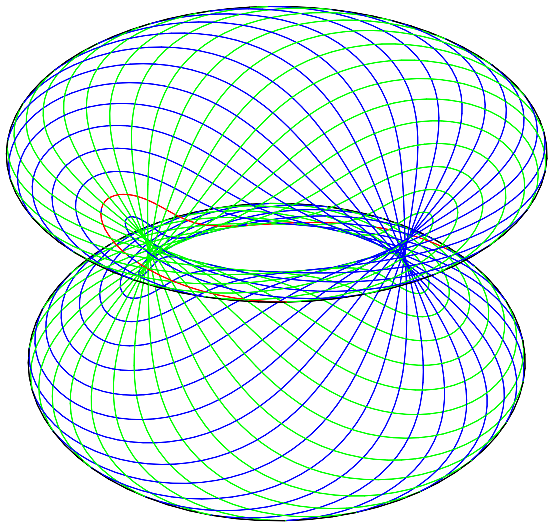
and finally the main curvature in the direction of circles of fixed latitude

$$K_2 := \frac{K}{K_1} = \frac{\cos(s/a)}{A + a\cos(s/a)} = \frac{1}{a + A/\cos(s/a)}.$$

Thus, the Gaussian curvature vanishes at the north and south pole-circles ($s/a = \pm\pi/2$). It is positive on the OUTER HEMI-TORUS (given by $|s/a| < \pi/2$) and negative on the inside. There are no asymptomatic directions on the outer hemi-torus. The POLE-CIRCLES are asymptotic lines. There are exactly two asymptotic directions $\xi = \xi^1\varphi_s + \xi^2\varphi_\theta$ with

$$\begin{aligned} (\xi^1)^2 &= \frac{K_2}{K_2 - K_1} = \frac{\frac{1}{a + A/\cos(s/a)}}{\frac{1}{a + A/\cos(s/a)} - \frac{1}{a}} = -\frac{a\cos(s/a)}{A} \\ (\xi^2)^2 &= \frac{1}{G}\,\frac{K_1}{K_1 - K_2} = \frac{1}{A(A + a\cos(s/a))} \end{aligned}$$

in each point of the inner hemi-torus



Parameterization of the inner hemi-torus by means of asymptotic lines

9.20 Surfaces of revolution with constant Gaussian curvature.

In order to find surfaces of revolution that have constant Gaussian curvature $K = -\frac{r''}{r}(s)$, we need to solve the system of differential equations

$$\begin{aligned} r''(s) + K r(s) &= 0 \\ r'(s)^2 + z'(s)^2 &= 1. \end{aligned}$$

The first equation has, as a second-order linear differential equation, a 2-dimensional linear solution space.

The **case $\boldsymbol{K = 0}$** is not very interesting, because then $r'' = 0$, that is $r(s) = as + b$ and thus $z(s) = \sqrt{1-a^2}\, s$. So the solution curve is a straight line and the surface a cone for $0 < a < 1$ and in the degenerate cases for $a = 0$ a cylinder and for $a = 1$ a plane.

For $K \neq 0$, one obtains generators for the solution space via the Ansatz $r(s) := e^{\lambda\, s}$, resulting in $\lambda^2 = -K$.

First, consider the **case $\boldsymbol{K > 0}$**. There $s \mapsto e^{\pm i\sqrt{K}s}$ is a complex generating system of the solutions. The general real solution is $r : s \mapsto a\cos(\sqrt{K}s) + b\sin(\sqrt{K}s)$. If we represent (a, b) in polar coordinates, i.e. put

$$(a, b) =: r_0\Big(\cos(-\sqrt{K}s_0), \sin(-\sqrt{K}s_0)\Big)$$

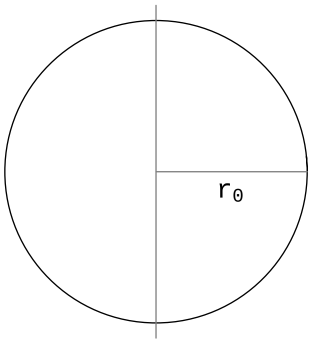
with $r_0 \geq 0$, then

$$r(s) = r_0 \cos(\sqrt{K}(s - s_0)).$$

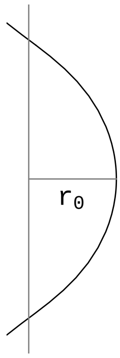
W.l.o.g., $s_0 = 0$ (after a time shift) and $r_0 > 0$ (for $r_0 = 0$ the map φ does not parametrize a surface). Thus,

$$z(s) = \int_0^s \sqrt{1 - r'(\sigma)^2}\, d\sigma = \int_0^s \sqrt{1 - r_0^2 K \sin^2(\sqrt{K}\,\sigma)}\, d\sigma,$$

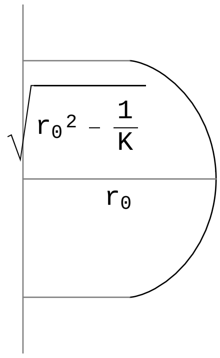
which is a Legendre integral. For $r_0^2 K = 1$ this yields a sphere, for $r_0^2 K < 1$ a so-called SPINDLE SURFACE and for $r_0^2 K > 1$ a BULDGE SURFACE (german: Wulstfläche). These are all locally isometric (see 11.11), but can not be transformed into one another by a motion.



Sphere



Spindle surface



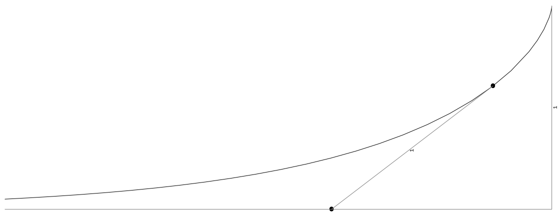
Buldge surface

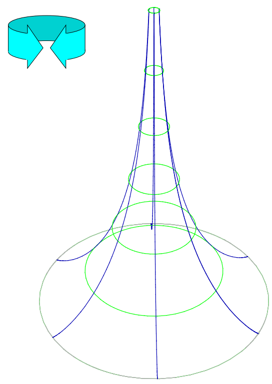
Now consider the **case $K < 0$**, then $r(s) = ae^{\sqrt{-K}s} + be^{-\sqrt{-K}s}$ with any a and b is the general solution.

If $ab = 0$, then by mirroring the time parameter s we may assume $a = 0$. By a time shift of $-\ln(b)/\sqrt{-K}$ we obtain $b = 1/\sqrt{-K}$ and by a stretch with factor $\sqrt{-K}$ and simultaneous reparameterization with factor $1/\sqrt{-K}$ the solution is then

$$r(s) = e^{-s}, \text{ and thus } z(s) = \int_0^s \sqrt{1 - e^{-2\sigma}}\, d\sigma \text{ for } s \geq 0.$$

This is the arc length parameterization of the traktrix. The associated surface of revolution is called PSEUDOSPHERE.





We can determine $z(s)$ explicitely:

$$\begin{aligned}
\int \sqrt{1 - e^{-2s}}\, ds &= \int -\sqrt{1 - r^2}\, \frac{dr}{r} = \int -\sqrt{1 - \left(\tfrac{2u}{1+u^2}\right)^2}\, \frac{1 + u^2}{2u}\, \frac{2(1 - u^2)}{(1 + u^2)^2}\, du \\
&= \int \frac{(1 - u^2)^2}{u(1 + u^2)^2}\, du = \int \frac{1}{u} - \frac{4u}{(1 + u^2)^2}\, du \\
&= c + \ln u + \frac{2}{1 + u^2} = c + \ln\left(\frac{1 + \sqrt{1 - r^2}}{r}\right) + 1 - \sqrt{1 - r^2}
\end{aligned}$$

$$= c + 1 + \operatorname{Arcosh}\left(\frac{1}{r}\right) - \sqrt{1-r^2},$$

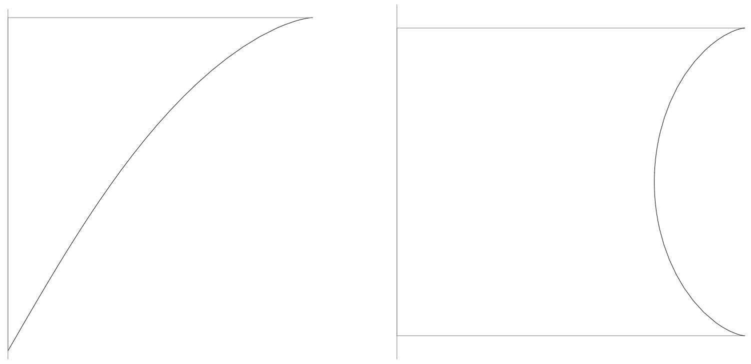
$$\text{hence} \quad z(s) = \operatorname{Arcosh}\left(\frac{1}{r(s)}\right) - \sqrt{1-r(s)^2}$$

In case $ab \neq 0$ we may assume $a = -b$ or $a = b$: In fact, if we replace s with $s - c$, then $r(s-c) = a\,e^{-\sqrt{-K}c}\,e^{\sqrt{-K}s} + b\,e^{\sqrt{-K}c}\,e^{-\sqrt{-K}s}$, and $|a\,e^{-\sqrt{-K}c}| = |b\,e^{\sqrt{-K}c}|$ with $e^{2\sqrt{-K}c} := |\frac{a}{b}|$.
The resulting surfaces

$$r(s) := a\,\sinh(\sqrt{-K}s), \text{ and thus } z(s) := \int_0^s \sqrt{1 + a^2 K \cosh^2(\sqrt{-K}\sigma)}\,d\sigma;$$

$$r(s) := a\,\cosh(\sqrt{-K}s), \text{ and thus } z(s) := \int_0^s \sqrt{1 + a^2 K \sinh^2(\sqrt{-K}\sigma)}\,d\sigma$$

are called of CONE TYPE or of THROAT TYPE.

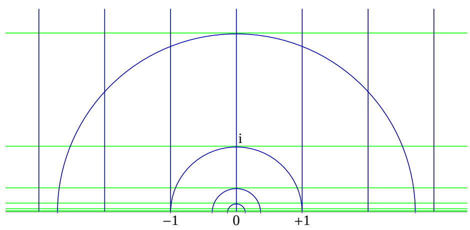


cone type throat type

9.21 Geodesic coordinates of the Poincaré half-plane.

The POINCARÉ HALF-PLANE M is the upper half-plane $\{(x,y) \in \mathbb{R}^2 : y > 0\}$ provided with the Riemann metric $g : (ds)^2 = \frac{1}{y^2}\big((dx)^2 + (dy)^2\big)$.

We want to find geodesic coordinates for it. In 10.10 we will learn a method of constructing them using geodesics. It turns out that the geodesics are the circles centered on the x axis, i.e. those circles that intersect the x axis at right angles.



The circles through ∞ are the straight lines parallel to the y axis. We want to parametrize the latter according to arc length. So be $c(t) := (x,t)$. Then the arc length is

$$s(t) = \int |c'(t)|_{c(t)}\,dt = \int \frac{dt}{t} = \ln(t)$$

with inverse function $t(s) = e^s$.
As parameterization of M we now use

$$\varphi : (s,x) \mapsto (x, e^s).$$

The derivative of φ is thus

$$\varphi'(s,x) = \begin{pmatrix} 0 & 1 \\ e^s & 0 \end{pmatrix}$$

and for the coefficients of the metric we obtain:

$$E = g(\partial_s\varphi, \partial_s\varphi) = \frac{e^{2s}}{e^{2s}} = 1, \quad F = g(\partial_s\varphi, \partial_x\varphi) = 0, \quad G = g(\partial_x\varphi, \partial_x\varphi) = e^{-2s}.$$

Therefore by 9.17 we have

$$K = -\frac{1}{\sqrt{G}}\left(\frac{\partial}{\partial s}\right)^2\sqrt{G} = -1, \qquad \sqrt{G}(0,x) = 1, \qquad \frac{\partial}{\partial s}\sqrt{G}(0,x) = -1.$$

The coefficients of the first fundamental form with respect to these coordinates coincide with ($G(s,\theta) = r(s)^2 = e^{-2s}$) constructed in 9.18 for the speudo sphere from 9.20, so we obtain an isometry of the subset $\{(x,y) : y \geq 1\}$ of the Poincaré half-plane to the pseudosphere as follows:

Poincare's half-plane ⟶ Pseudosphäre

$$(x,y) \longmapsto \left(\tfrac{\cos x}{y}, \tfrac{\sin x}{y}, z(\ln(y))\right)$$

$$(x,y) = (x,e^s), \qquad \left(\tfrac{\cos x}{y}, \tfrac{\sin x}{y}, z(\ln(y))\right) = (e^{-s}\cos\theta, e^{-s}\sin\theta, z(s))$$

$$(x,e^s) \longmapsto (e^{-s}\cos\theta, e^{-s}\sin\theta, z(s))$$

$$(s,x) = (s,\theta) \longmapsto (x,e^s), \qquad (s,x) = (s,\theta) \longmapsto (e^{-s}\cos\theta, e^{-s}\sin\theta, z(s))$$

$$\mathbb{R}^+ \times \mathbb{R}$$

This can also be verified by a direct calculation.

9.22 Geodesic coordinates of the hyperbolic disk.

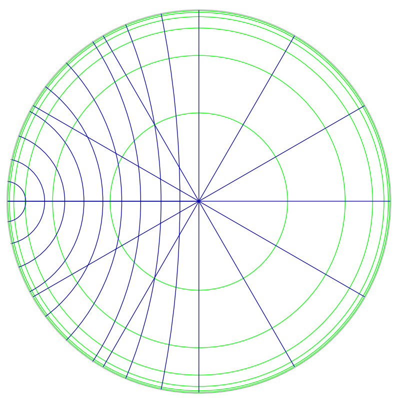
The hyperbolic disk is, according to 2.6, the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ provided with the Riemann metric (up to the constant factor 4)

$$g : (ds)^2 = \frac{4}{\big(1-(x^2+y^2)\big)^2}\big((dx)^2 + (dy)^2\big).$$

Again we want to use the method from 10.10 to find geodesic coordinates.
The geodesics are those circles (and lines) which intersect the unit circle orthogonally. We want to parametrize the geodesics through 0 by arc length. So let $c(t) := (t,0)$. Then the arc length is

$$s(t) = \int |c'(t)|_{c(t)}\,dt = \int \frac{2\,dt}{1-t^2} = \ln\left(\frac{1+t}{1-t}\right)$$

with inverse function $t(s) = \frac{e^s-1}{e^s+1} = \tanh(\frac{s}{2})$.



As parameterization of $\mathbb{D}$ we use

$$\varphi : (s,\theta) \mapsto \big(\theta, \tanh(\frac{s}{2})\big) \mapsto \Big(\tanh(\frac{s}{2})\cos\theta, \tanh(\frac{s}{2})\sin\theta\Big).$$

Its derivative is

$$\varphi'(s,\theta) = \begin{pmatrix} \cosh^{-2}(\frac{s}{2})\cdot\frac{\cos(\theta)}{2} & -\tanh(\frac{s}{2})\cdot\sin\theta \\ \cosh^{-2}(\frac{s}{2})\cdot\frac{\sin(\theta)}{2} & \tanh(\frac{s}{2})\cdot\cos\theta \end{pmatrix}$$

and for the coefficients of the metric we get:

$$E = g(\partial_s\varphi, \partial_s\varphi) = \frac{4\cosh(\frac{s}{2})^2}{4\cosh(\frac{s}{2})^2} = 1, \quad F = 0, \quad \text{and } G = g(\partial_\theta\varphi, \partial_\theta\varphi) = \frac{\sinh^2(s)}{2}.$$

These are just the coefficients of the first fundamental form of a throat-type surface for $a := 1/\sqrt{2} =: -b$ and $K := -1$ with coordinates

$$r(s) := \frac{1}{2\sqrt{2}}\left(e^s - e^{-s}\right) = \frac{1}{\sqrt{2}}\sinh(s) \text{ and } z(s) := \int_0^s \sqrt{1 - r'(\sigma)^2}\, d\sigma$$

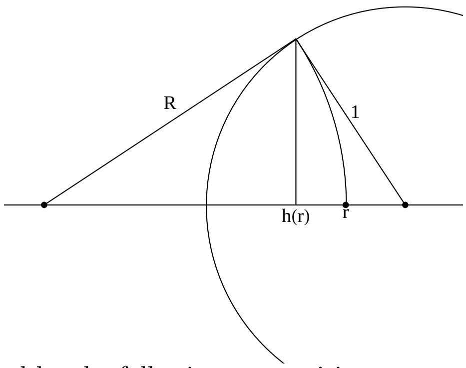
from 9.20. Thus, analogous to 9.21 and [87, 54.5], we obtain a local isometry of the hyperbolic disk with an surface of throat type and thus (as we will show in 11.11) also with the pseudosphere.

9.23 Another description of the hyperbolic disk.

We now want to distort the hyperbolic disk by a diffeomorphism so that the geodesics become exactly the straight lines.

This diffeomorphism should leave the disk invariant and fix its center and its boundary pointwise.

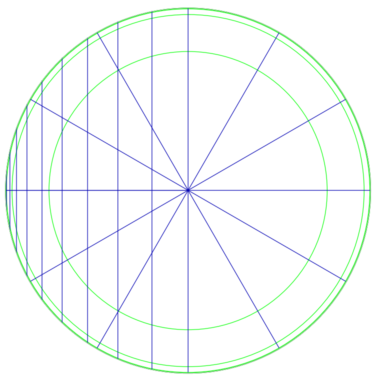
In between, we must therefore deform the circles orthogonal to the boundary so that they become straight lines through the same points of intersection with the edges. The elementary geometric consideration $h(r) : 1 = 1 : (R + r)$ and $R^2 + 1 = (R + r)^2$ and thus $R = \frac{1-r^2}{2r}$ shows that this is achieved in polar coordinates by $h \times \mathrm{id} : (r, \theta) \mapsto (\frac{2r}{1+r^2}, \theta)$.



Our desired diffeomorphism is thus obtained by the following composition:

$$\begin{array}{ccc}
(r,\theta) & \xrightarrow[h\times\mathrm{id}]{} & (\frac{2r}{1+r^2}, \theta) \\
\| & & \downarrow \text{kart.Koord.} \\
(\sqrt{x^2+y^2}, \arctan\frac{y}{x}) & & (\frac{2r}{1+r^2}\cos\theta, \frac{2r}{1+r^2}\sin\theta) \\
\uparrow \text{Pol.Koord.} & & \| \\
(x,y) & & (\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2})
\end{array}$$

Note that the derivative of $r \mapsto \frac{2r}{1+r^2}$ is given by $r \mapsto \frac{2(1-r^2)}{(1+r^2)^2}$ and the inverse function by $\frac{1-\sqrt{1-r^2}}{r} \leftarrow r$ (the choice of the solution of the quadratic equation results from $r < \frac{2r}{1+r^2}$). Note, however, that in this way we do not obtain geodesic coordinates. To achieve this we would have to parametrize the radial geodesics as in 9.22.



9.24 Isomorphism of the Poincaré half-plane and hyperbolic disk.

Consider the Möbius transformation (see [**91**, 2.16,2.30]) $\mu : z \mapsto \frac{az+b}{cz+d}$, which has the following particular values:

$$0 \mapsto -i, \quad i \mapsto 0, \quad \pm 1 \mapsto \pm 1, \quad \infty \mapsto i.$$

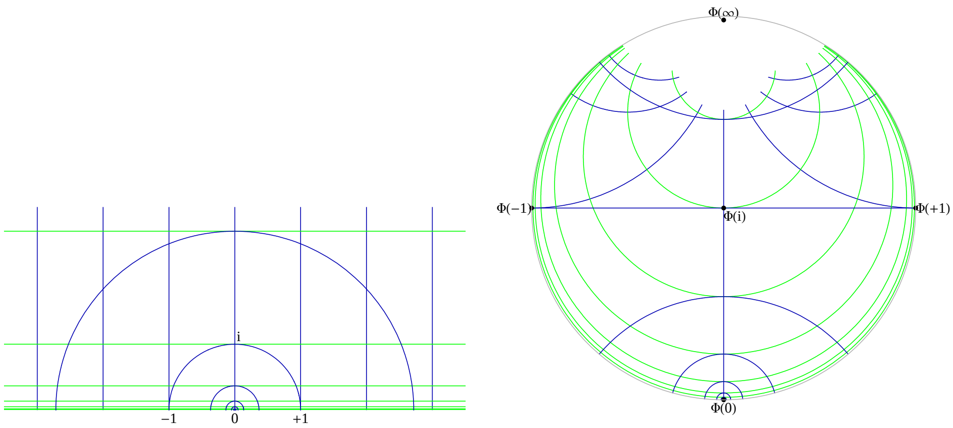
From these equations we get: $d = ib$, $b = -ia$, $c = -ia$ and finally $a = d = 1/\sqrt{2}$ and $b = c = -i/\sqrt{2}$ because of $1 = ad - bc = 2a^2$, that is (when we expand with $i\sqrt{2}$)

$$\mu : z \mapsto \frac{iz+1}{z+i}.$$

It maps the upper half-plane onto the unit disk. Their inverse function is given by $w \mapsto (w+i)/(iw+1)$. If we pull back the hyperbolic metric from 9.22 on $\mathbb{D}$ to the upper half-plane using μ, we obtain the metric:

$$|v|_z := |\mu'(z)v|_{\mu(z)} = \frac{2|\mu'(z)|}{1-|\mu(z)|^2}\,|v| = \frac{2|v|}{i(\bar{z}-z)} = \frac{|v|}{\mathfrak{Im}(z)}.$$

This is the metric of the Poincaré half plane from 9.21. Thus, the Poincaré half-plane is isometrically diffeomorphic to the hyperbolic disk, and with 9.21 and 9.22 also the pseudosphere is locally isometric with an surface of throat type.



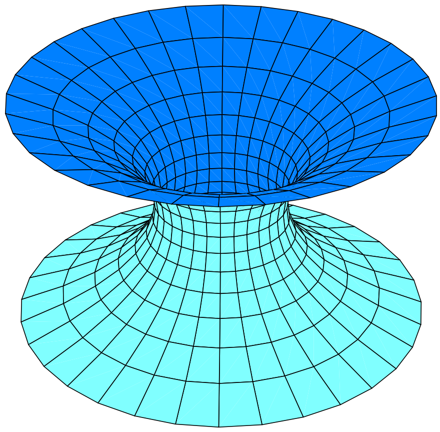
9.25 Minimal surfaces of revolution.

We want to determine those surfaces of revolution which fulfill $H = 0$, i.e. have locally minimal surface area (see 9.27). Since the two main curvatures are $-\frac{r''}{z'}$ and $\frac{z'}{r}$, we have to solve the system of differential equations $r''r = (z')^2 = 1-(r')^2$. This is equivalent to

$$\left(\frac{r^2}{2}\right)'' = rr'' + (r')^2 = 1 \text{ with the general solution } r(s)^2 = (s+a)^2 \pm b^2.$$

After a time shift we may assume $a = 0$ and thus $2r(s)r'(s) = 2s$. In case $b = 0$ we have (after mirroring) the solution $r(s) = s$ and $z(s) = 0$, a plane. The case $-b^2 < 0$ can not occur because then $r'(s) = \frac{s}{r(s)} = \frac{s}{\sqrt{s^2-b^2}} > 1$ gives a contradiction.

For $r(s)^2 = s^2 + b^2$ with $b > 0$ we get $z(s) = \int_0^s \sqrt{1 - r'(\sigma)^2}\, d\sigma = \int_0^s \sqrt{1 - \frac{\sigma^2}{\sigma^2+b^2}}\, d\sigma = \int_0^s \frac{b}{\sqrt{\sigma^2+b^2}}\, d\sigma = b\,\mathrm{Arsinh}(s/b)$, that is $s = b\sinh(z/b)$ and $r = b\sqrt{1+\sinh^2(z/b)} = b\cosh(z/b)$. So this is just the arc length parameterization of the catenary $r/b = \cosh(z/b)$.



9.26 Definition (Minimal surfaces).

A surface is called a MINIMAL SURFACE if it is (locally, i.e. if we vary it only locally or on a compact part) a critical point for the surface area. So we do not need to consider the whole surface (which can be infinite), but only the part that changes.

By [83, 8.1.5], the area of a surface given by a parameterization $\varphi : \mathbb{R}^2 \supseteq K \to M \subseteq \mathbb{R}^3$ with compact J-measurable K is

$$\mathrm{vol}(M) := \int_K \|\partial_1\varphi \times \partial_2\varphi\|,$$

i.e.

$$\begin{aligned}\mathrm{vol}(M) &:= \int_K \sqrt{\|\partial_1\varphi\|^2\|\partial_2\varphi\|^2 - \langle\partial_1\varphi|\partial_2\varphi\rangle^2} = \int_K \sqrt{E\,G - F^2}\\ &= \int_K \sqrt{\det((g_{i,j})_{i,j\in\{1,2\}})},\end{aligned}$$

where $g_{i,j} := \langle g_i|g_j\rangle$ are the coefficients of the first fundamental form with $g_i := \partial_i\varphi$. More generally, the $n-1$-dimensional volume of parameterized hypersurfaces $M \subseteq \mathbb{R}^n$ is given by

$$\mathrm{vol}(M) = \int_K \sqrt{\det((g_{i,j})_{i,j})}.$$

9.27 Proposition, [112].

A surface is a minimal surface if and only if $H = 0$.

Proof. This variation problem amounts to determining the local minima of the function $M \mapsto \mathrm{vol}(M) := \int_M \mathrm{vol}_M \in \mathbb{R}$. Let the surface M be a critical point of this functional. Each surface near M can (by definition) be represented as $\{x + f(x)\nu(x) : x \in M\}$ with a real-valued smooth function $f : M \to \mathbb{R}$ with compact support. Thus, $\frac{d}{dt}\big|_0 \mathrm{vol}(M^t) = 0$ hast to hold, where M^t is the surface $\{x + t\,f(x)\nu(x)\}$. For this we have to determine $\frac{d}{dt}\big|_0 \mathrm{vol}_{M^t}$. We choose a local parameterization $\varphi : U \to M$ of M with associated local coordinates $(u^1, \dots, u^m)$. A local parameterization of M^t is then $\varphi^t(u) = \varphi(u) + t\,f(\varphi(u))\nu(\varphi(u))$. Locally

$$\mathrm{vol}_{M^t} = \sqrt{\det(g^t_{i,j})}\, du^1 \wedge \cdots \wedge du^m,$$

where

$$g^t_{i,j} := \langle g^t_i, g^t_j\rangle \text{ with } g^t_i := \partial_i\varphi^t = \partial_i\varphi + t\Big(\partial_i(f\circ\varphi)\cdot(\nu\circ\varphi) + (f\circ\varphi)\cdot\partial_i(\nu\circ\varphi)\Big).$$

So

$$\begin{aligned}
\tfrac{d}{dt}\big|_0\, g^t_i &= \Big(\partial_i(f\circ\varphi)\cdot(\nu\circ\varphi) + (f\circ\varphi)\cdot L(\partial_i\varphi)\Big) = \tfrac{\partial f}{\partial u^i}\cdot\nu + f\cdot L(g_i)\\
\tfrac{d}{dt}\big|_0\, g^t_{i,j} &= \Big\langle g_i, \tfrac{d}{dt}\big|_0\, g^t_j\Big\rangle + \Big\langle \tfrac{d}{dt}\big|_0\, g^t_i, g_j\Big\rangle\\
&= f\cdot\Big(\langle\partial_i\varphi, L(\partial_j\varphi)\rangle + \langle L(\partial_i\varphi), \partial_j\varphi\rangle\Big) =: 2f\, h_{i,j},
\end{aligned}$$

and

$$\begin{aligned}
\tfrac{d}{dt}\big|_0 \sqrt{\det(g^t_{i,j})} &= \frac{1}{2}\frac{1}{\sqrt{\det(g_{i,j})}}\det(g_{i,j})\,\mathrm{trace}\Big((g_{i,j})^{-1}(\tfrac{d}{dt}\big|_0\, g^t_{i,j})\Big)\\
&= \sqrt{\det(g_{i,j})}\cdot(f\circ\varphi)\cdot\mathrm{trace}\Big((g_{i,j})^{-1}(h_{i,j})\Big)\\
&= \sqrt{\det(g_{i,j})}\cdot(f\circ\varphi)\cdot\mathrm{trace}\, L\\
&= m\sqrt{\det(g_{i,j})}\cdot(f\circ\varphi)\cdot H.
\end{aligned}$$

We used that $\det'(A)(B) = \det A\cdot\mathrm{trace}(A^{-1}B)$. Thus

$$\tfrac{d}{dt}\big|_0\, \mathrm{vol}_{M^t} = mf\,H\;\mathrm{vol}_M$$

and finally

$$\tfrac{d}{dt}\big|_{t=0}\mathrm{vol}(M^t) = \tfrac{d}{dt}\big|_{t=0}\int_M \mathrm{vol}_{M^t} = \int_M \tfrac{d}{dt}\big|_{t=0}\mathrm{vol}_{M^t} = m\int_M f\,H\,vol_M.$$

If this vanishes for all variations of the surfaces M, i.e. for all $f : M\to\mathbb{R}$, then $H = 0$ (choose $f = H$); and vice versa. □

10. Geodesics

We now want to solve the problem of the shortest connection paths on general hypersurfaces.

10.1 Definition (Geodesic).

A GEODESIC is a curve in M, which is a critical point for arc length.

10.2 Proposition (Characterization of geodesics).

A curve c in a hypersurface M is a geodesic if and only if for some parameterization of c the following holds: $c''(t) \in T_{c(t)}M^\perp$ for all t, i.e. the acceleration only serves to keep the curve on the manifold. This parameterization is then automatically proportional to arc length.

Proof. Let $c : [a,b] \to M$ be a curve which is w.l.o.g. parametrized by arc length. It is a critical point for arc length, if for all 1-PARAMETER FAMILIES OF CURVES (c^s) with $s\in\mathbb{R}$, the derivative $\frac{d}{ds}\big|_{s=0} L(c^s)$ equals 0, where 1-parameter families of curves are smooth mappings $\mathbb{R}^2\to M$, $(t,s)\mapsto c^s(t)$, which satisfy $c^s(a) = c(a)$,

$c^s(b) = c(b)$, $\forall\ s$ and $c^0 = c$. Let's calculate this derivative where we put $c(t,s) := c^s(t)$:

$$\begin{aligned}
\tfrac{d}{ds}\big|_{s=0} L(c^s) &= \tfrac{d}{ds}\big|_{s=0} \int_a^b \left|\tfrac{\partial}{\partial t} c(t,s)\right| dt = \int_a^b \tfrac{\partial}{\partial s}\big|_{s=0} \left|\tfrac{\partial}{\partial t} c(t,s)\right| dt \\
&= \int_a^b \frac{1}{2} \frac{\tfrac{\partial}{\partial s}\big|_{s=0} \langle \tfrac{\partial}{\partial t} c(t,s), \tfrac{\partial}{\partial t} c(t,s)\rangle}{\underbrace{|\tfrac{\partial}{\partial t} c(t,0)|}_{=1}} dt \\
&= \int_a^b \Big\langle \tfrac{\partial}{\partial s}\big|_{s=0} \tfrac{\partial}{\partial t} c(t,s), \tfrac{\partial}{\partial t} c(t,0) \Big\rangle dt \\
&= \int_a^b \Big\langle \tfrac{\partial}{\partial t}\ \tfrac{\partial}{\partial s}\big|_{s=0} c(t,s), \tfrac{\partial}{\partial t} c(t,0) \Big\rangle dt \qquad \text{(part.integr.)} \\
&= \Big[\Big\langle \tfrac{\partial}{\partial s}\big|_{s=0} c(t,s), \tfrac{\partial}{\partial t} c(t,0) \Big\rangle\Big]_{t=a}^b \\
&\qquad - \int_a^b \Big\langle \tfrac{\partial}{\partial s}\big|_{s=0} c(t,s), \big(\tfrac{\partial}{\partial t}\big)^2 c(t,0) \Big\rangle dt \\
&= 0 - \int_a^b \langle \eta(t), c''(t)\rangle dt = \int_a^b h \cdot \underbrace{\big(\langle c'', \nu \circ c\rangle^2 - \langle c'', c''\rangle\big)}_{\le 0 \ \text{(Cauchy-Schwarz)}}
\end{aligned}$$

Where we have chosen $\eta(t) := \tfrac{\partial}{\partial s}\big|_{s=0} c(t,s)$ for the last equality sign such that

$$\eta(t) = h(t)\Big(c''(t) - \langle c''(t), \nu(c(t))\rangle\, \nu(c(t))\Big) \in T_{c(t)}M$$

with some smooth function $h : [a,b] \to \mathbb{R}_+$ with $h(a) = 0 = h(b)$. This is possible because η is a vector field on M along c, which only needs to satisfy $\eta(a) = 0 = \eta(b)$.

The derivative thus vanishes for all such η if and only if equality holds in the Cauchy-Schwarz inequality: $\langle c'', c''\rangle = \langle c'', \nu \circ c\rangle^2$, i.e. $c''(t) \parallel \nu(c(t))$. In other words, if $c''(t) \in T_{c(t)}M^\perp$ for all t.

Conversely, let c be a parameterization which satisfies $c''(t) \in T_{c(t)}M^\perp$. Then, in particular, $c''(t) \perp c'(t)$ and thus $\langle c'(t), c'(t)\rangle$ is constant, i.e. c is parameterized proportional to arc length. Thus the calculation for $\tfrac{d}{ds}\big|_{s=0} L(c^s)$ from above applies to c (with the exception of the last equality sign) and yields 0. □

The above question is of course a variation problem and the standart method for solving it is that of Euler-Lagrange, see [**83**, 9.4.16] - [**83**, 9.4.18].

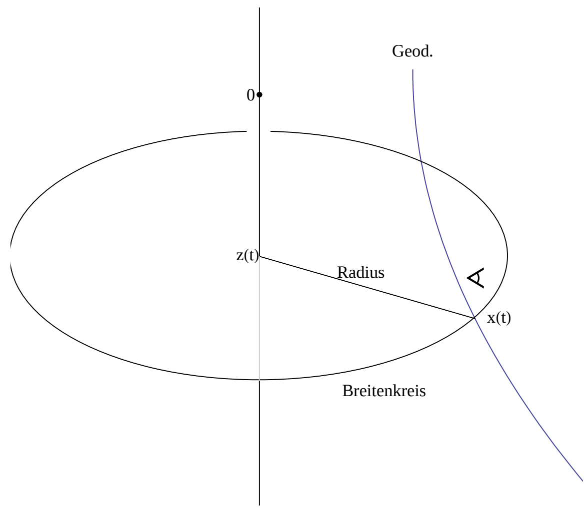
10.3 Examples.

1. In a hyperplane, the straight lines are obviously the geodesics.
2. Each great circle on the sphere S^m, i.e. the intersection of a plane through 0 with S^m, is a geodesic, because the second derivative of the circle points to the center, i.e. exactly in the direction of the normal vector to the sphere.
3. More generally, on surfaces of revolution, the meridians are geodesics and also those latitudinal circles which are critical points for the radius (so-called EQUATORS).
4. On a (hyper-)cylinder, we can also easily specify geodesics in further directions, namely: $t \mapsto (c(t\cos(\varphi)), t\sin(\varphi))$, where c is an arc length parameterized geodesic on the equatorial sphere (circle) and $\varphi \in \mathbb{R}$.

10.4 Proposition of Clairaut.

On each surface of revolution, the product of the distance from the axis of rotation with the cosine of the angle between a geodesic and the latitudinal circle is constant along the geodesic:

$$\textit{Radius}\cdot \cos\sphericalangle(\textit{geodesic, latitudinal circle}) = \text{const}\,.$$



Proof. Let $c : t \mapsto (x(t); z(t)) \in \mathbb{R}^2 \times \mathbb{R}$ be a geodesic on M parameterized by arc-length, i.e. $c''(t) \perp T_{c(t)}M$. Then $(x(t)^\perp, 0)$ is tangential to the parallels (i.e. circles of latitude) through $c(t)$ and $|c'(t)| = 1$, therefore

$$\begin{aligned}\Big(\text{ radius}\cdot\cos\sphericalangle(\text{ geodesic, parallel})\Big)'(t) &= \\ &= \frac{d}{dt}\Big(|x(t)|\cdot\cos\sphericalangle\Big(c'(t),\begin{pmatrix}x(t)^\perp\\0\end{pmatrix}\Big)\Big)\\ &= \frac{d}{dt}\Big(|x(t)|\cdot\frac{\Big\langle\binom{x'(t)}{z'(t)},\binom{x(t)^\perp}{0}\Big\rangle}{|c'(t)|\,|x(t)^\perp|}\Big) = \frac{d}{dt}\Big\langle\begin{pmatrix}x'(t)\\z'(t)\end{pmatrix},\begin{pmatrix}x(t)^\perp\\0\end{pmatrix}\Big\rangle\\ &= \Big\langle\begin{pmatrix}x''(t)\\z''(t)\end{pmatrix},\begin{pmatrix}x(t)^\perp\\0\end{pmatrix}\Big\rangle + \Big\langle\begin{pmatrix}x'(t)\\z'(t)\end{pmatrix},\begin{pmatrix}x'(t)^\perp\\0\end{pmatrix}\Big\rangle\\ &= \Big\langle c''(t),\begin{pmatrix}x(t)^\perp\\0\end{pmatrix}\Big\rangle + \big\langle x'(t), x'(t)^\perp\big\rangle = 0+0,\end{aligned}$$

because $c''(t) \in (T_{c(t)}M)^\perp$ and thus $c''(t) \perp (x(t)^\perp, 0)$. Consequently, the claimed expression is constant. □

We have shown:
A curve c is a geodesic $\Leftrightarrow$ $\forall\ t : c''(t) \in T_{c(t)}M^\perp$, i.e. the following differential equation is fulfilled:

$$c''(t) = \langle c''(t), \nu_{c(t)}\rangle\nu_{c(t)} = -\langle c'(t), L\,c'(t)\rangle\nu_{c(t)}.$$

We want to describe it in local coordinates. For this we develop the second partial derivatives $\varphi_{i,j}$ of the parameterization φ in the basis $(\varphi_1,\dots,\varphi_m;\nu)$ of $\mathbb{R}^n$ formed by the partial derivatives $\varphi_i := \frac{\partial}{\partial u^i}\varphi(u)$ and the standard normal vector ν:

10.5 Analogon of the Frenét equations for surfaces.

The second partial derivatives $\varphi_{i,j} : u \mapsto \frac{\partial^2}{\partial u^i \partial u^j}\varphi(u)$ *of a local parameterization* φ *have the following development in the basis* $(\varphi_1, \dots, \varphi_m, \nu)$*:*

$$\varphi_{i,j}(u) = \sum_{k=1}^{m} \Gamma^k_{i,j}(u)\varphi_k(u) - h_{i,j}(u)\nu_{\varphi(u)},$$

where $h_{i,j} := -\langle \varphi_{i,j}, \nu\rangle = \langle \varphi_i, L\,\varphi_j\rangle$ *and* $\Gamma^k_{i,j}$ *are the correspondingly chosen coefficients, these are also called* Christoffel symbols of the 2nd kind.

The Christoffelsymbols $\Gamma^k_{i,j}$ *of the 2nd kind can be calculated from the* Christoffelsymbols of the 1st kind

$$\Gamma_{i,j,k} := \langle \varphi_{i,j}, \varphi_k\rangle = \tfrac{1}{2}\big(\partial_j g_{i,k} + \partial_i g_{k,j} - \partial_k g_{i,j}\big)$$

as follows:

$$\Gamma^k_{i,j} = \sum_{l=1}^{m} \Gamma_{i,j,l}\, g^{l,k} \qquad \text{with } (g^{l,k}) := (g_{l,k})^{-1} \text{ and } g_{l,k} := \langle \varphi_l, \varphi_k\rangle.$$

Proof. In order to calculate the coefficients of $\varphi_{i,j}$, we first form the inner product with ν and obtain $\langle \varphi_{i,j}, \nu\rangle = 0 - h_{i,j}\cdot 1$ for the coefficient of ν. By calculating the inner product with φ_l we get:

$$\Gamma_{i,j,l} := \langle \varphi_{i,j}, \varphi_l\rangle = \left\langle \sum_{k=1}^{m} \Gamma^k_{i,j}(u)\varphi_k(u), \varphi_l(u) \right\rangle + 0 = \sum_{k=1}^{m} \Gamma^k_{i,j} g_{k,l}.$$

Multiplying with the inverse matrix $(g^{l,p})$ yields:

$$\sum_{l=1}^{m} \Gamma_{i,j,l}\, g^{l,p} = \sum_{l=1}^{m}\sum_{k=1}^{m} \Gamma^k_{i,j} g_{k,l} g^{l,p} = \sum_{k=1}^{m} \Gamma^k_{i,j} \sum_{l=1}^{m} g_{k,l} g^{l,p} = \sum_{k=1}^{m} \Gamma^k_{i,j}\delta^p_k = \Gamma^p_{i,j}.$$

The following holds:

$$\partial_k g_{i,j} = \partial_k \langle \varphi_i, \varphi_j\rangle = \langle \varphi_{i,k}, \varphi_j\rangle + \langle \varphi_i, \varphi_{j,k}\rangle.$$

By cyclical permutation we obtain:

$$\begin{aligned} \partial_i g_{j,k} &= \langle \varphi_{j,i}, \varphi_k\rangle + \langle \varphi_j, \varphi_{k,i}\rangle \\ \partial_j g_{k,i} &= \langle \varphi_{k,j}, \varphi_i\rangle + \langle \varphi_k, \varphi_{i,j}\rangle. \end{aligned}$$

The alternating sum of these 3 equations is

$$2\Gamma_{k,j,i} = 2\langle \varphi_{k,j}, \varphi_i\rangle = \partial_j g_{k,i} - \partial_i g_{j,k} + \partial_k g_{i,j}. \qquad \square$$

10.6 Remark.

Let M be a surface in $\mathbb{R}^3$ (or even an abstract Riemann surface) and E, F, G the coefficients of the 1st fundamental form, that is

$$\begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \quad \text{und} \quad \begin{pmatrix} g^{1,1} & g^{1,2} \\ g^{2,1} & g^{2,2} \end{pmatrix} = \frac{1}{D^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix},$$

where $D := \sqrt{E\,G - F^2}$. The Christoffel symbols of the 1st order have the following form:

$$
\begin{aligned}
\Gamma_{i,j,k} &:= \frac{1}{2}\,\Big(\partial_i(g_{j,k}) + \partial_j(g_{i,k}) - \partial_k(g_{i,j})\Big) \quad \Rightarrow\\
\Gamma_{1,1,1} &:= \frac{1}{2}\,E_1\\
\Gamma_{1,2,1} &:= \frac{1}{2}\,E_2\\
\Gamma_{2,2,1} &:= \frac{1}{2}\,(2F_2 - G_1)\\
\Gamma_{1,1,2} &:= \frac{1}{2}\,(2F_1 - E_2)\\
\Gamma_{1,2,2} &:= \frac{1}{2}\,G_1\\
\Gamma_{2,2,2} &:= \frac{1}{2}\,G_2
\end{aligned}
$$

And for those of second order we get:

$$
\begin{aligned}
\Gamma^k_{i,j} &:= \Gamma_{i,j,1}\,g^{1,k} + \Gamma_{i,j,2}\,g^{2,k} \quad \Rightarrow\\
\Gamma^1_{1,1} &:= \Gamma_{1,1,1}\,g^{1,1} + \Gamma_{1,1,2}\,g^{2,1}\\
&= \frac{E_1}{2}\,\frac{G}{D^2} + (F_1 - \frac{E_2}{2})\,\frac{-F}{D^2} = \frac{G\,E_1 - 2\,F\,F_1 + F\,E_2}{2\,D^2}\\
\Gamma^1_{1,2} &:= \Gamma_{1,2,1}\,g^{1,1} + \Gamma_{1,2,2}\,g^{2,1}\\
&= \frac{E_2}{2}\,\frac{G}{D^2} + \frac{G_1}{2}\,\frac{-F}{D^2} = \frac{G\,E_2 - F\,G_1}{2\,D^2}\\
\Gamma^1_{2,2} &:= \Gamma_{2,2,1}\,g^{1,1} + \Gamma_{2,2,2}\,g^{2,1}\\
&= (F_2 - \frac{G_1}{2})\,\frac{G}{D^2} + \frac{G_2}{2}\,\frac{-F}{D^2} = \frac{2\,G\,F_2 - G\,G_1 - F\,G_2}{2\,D^2}\\
\Gamma^2_{1,1} &:= \Gamma_{1,1,1}\,g^{1,2} + \Gamma_{1,1,2}\,g^{2,2}\\
&= \frac{E_1}{2}\,\frac{-F}{D^2} + (F_1 - \frac{E_2}{2})\,\frac{E}{D^2} = \frac{-F\,E_1 + 2\,E\,F_1 - E\,E_2}{2\,D^2}\\
\Gamma^2_{1,2} &:= \Gamma_{1,2,1}\,g^{1,2} + \Gamma_{1,2,2}\,g^{2,2}\\
&= \frac{E_2}{2}\,\frac{-F}{D^2} + \frac{G_1}{2}\,\frac{E}{D^2} = \frac{-F\,E_2 + E\,G_1}{2\,D^2}\\
\Gamma^2_{2,2} &:= \Gamma_{2,2,1}\,g^{1,2} + \Gamma_{2,2,2}\,g^{2,2}\\
&= (F_2 - \frac{G_1}{2})\,\frac{-F}{D^2} + \frac{G_2}{2}\,\frac{E}{D^2} = \frac{-2\,F\,F_2 + F\,G_1 + E\,G_2}{2\,D^2}
\end{aligned}
$$

10.7 Local Geodesic Equation The differential equation for geodesics $c := \varphi \circ u$ with local representation $u(t) = (u^1(t), \dots, u^m(t))$ looks in local coordinates as follows:

$$c(t) = (\varphi \circ u)(t) \quad \Rightarrow \quad c'(t) = \sum_i \frac{\partial \varphi}{\partial u^i} \cdot \frac{du^i}{dt}$$

$$\Rightarrow \quad c''(t) = \sum_i \left(\sum_j \frac{\partial^2 \varphi}{\partial u^i \partial u^j} \cdot \frac{du^i}{dt} \cdot \frac{du^j}{dt} + \frac{\partial \varphi}{\partial u^i} \cdot \frac{d^2 u^i}{dt^2} \right) \in$$

$$\in \sum_k \left(\sum_{i,j} \Gamma^k_{i,j} \frac{du^i}{dt} \frac{du^j}{dt} + \frac{d^2 u^k}{dt^2} \right) \varphi_k + \mathbb{R} \cdot \nu, \qquad \text{by } 10.5.$$

Hence c is a geodesic, i.e. $c''(t) \in T_{c(t)}M^\perp$, if and only if

$$\frac{d^2 u^k}{dt^2}(t) + \sum_{i,j=1}^m \Gamma^k_{i,j}(u(t)) \cdot \frac{du^i}{dt}(t) \cdot \frac{du^j}{dt}(t) = 0 \qquad \text{for } k = 1, \dots, m.$$

or for short:

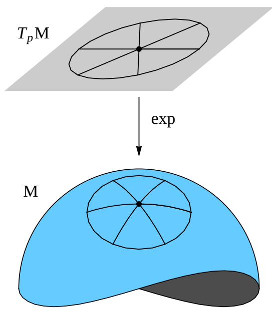
$$\ddot{u}^k + \sum_{i,j} \Gamma^k_{i,j} \dot{u}^i \dot{u}^j = 0,$$

where $\dot{u}^k$ denotes the derivative $t \mapsto \frac{du^k}{dt}(t)$ with respect to time t. This system of ordinary differential equations of 2nd order has locally a unique solution for given initial data $u^k(0)$ and $\frac{du^k}{dt}(0)$.

10.8 Lemma. The exponential mapping.

For each $x \in M$ and $\xi \in T_xM$ there exists a unique geodesic $c_\xi : I \to M$ with maximal interval $I \subseteq \mathbb{R}$ of definition, and with constant scalar velocity and initial conditions $c_\xi(0) = x$ and $c'_\xi(0) = \xi$.

If one now associates to $\xi \in TM$ the value $c_\xi(1)$ of the geodesic c_ξ with initial condition ξ, then the result is called $\exp(\xi)$. The exponential function $\exp$ is defined on an open neighborhood of the zero section M in TM. It is smooth there, has values in M, and $\exp_x := \exp|_{T_xM} : T_xM \to M$ satisfies $\exp_x(0_x) = x$ and $T_{0_x}(\exp_x) = \mathrm{id}_{T_xM}$. The geodesic c_ξ with initial value ξ is then given by $c_\xi(t) = \exp(t\,\xi)$.



The reason for the notation "exp" is that for $M := S^1 \subset \mathbb{C}$ with $TM = \{(x, t\,x^\perp) : |x| = 1,\, t \in \mathbb{R}\} \cong S^1 \times \mathbb{R}$, the exponential map is given by $\exp(x, t\,x^\perp) = x\,e^{i\,t}$.

Proof. The local formula in 10.7 for the geodesic equation shows the existence and uniqueness of a maximally defined geodesics c_ξ, as well as its smooth dependence from the initial value ξ, i.e. there is an open neighborhood V of 0 in T_xM and a $\delta > 0$, s.t. $c_\xi(t)$ exists for all $\xi \in V$ and $|t| \le \delta$, and $(t, \xi) \mapsto c_\xi(t)$ is smooth.

Another way to see this for hypersurfaces without using local coordinates goes as follows:

If c is a geodesic then $c''(t) = \lambda(t) \cdot \nu_{c(t)}$ holds, where

$$\lambda(t) = \langle c''(t), \nu(c(t))\rangle = \langle c'(t), (\nu \circ c)'(t)\rangle = -\langle c', L \circ c'\rangle(t) = -K(c'(t)),$$
$$\text{i.e. } c \text{ is a geodesic } \Leftrightarrow c'' = -\langle c', (\nu \circ c)'\rangle(\nu \circ c).$$

If we choose a local equation f for M, then $\nu = \frac{1}{|\operatorname{grad} f|} \operatorname{grad} f$ makes sense not only on M but also locally in the surrounding $\mathbb{R}^n$. Thus, the above geodesic equation is a 2nd order ordinary differential equation on $\mathbb{R}^n$ and hence has, for a given initial condition for $c(0)$ and $c'(0)$, a unique solution $c : I \to \mathbb{R}^n$ which smoothly depends on the initial data. In particular, there is an open neighborhood V of 0_x in TM and a $\delta > 0$, s.t. $c_\xi(t)$ exists for all $\xi \in V$ and $|t| \le \delta$, and is smooth in (ξ, t).
It still has to be shown that the curve c stays in M. Since $\langle c', \nu \circ c\rangle' = \langle c'', \nu \circ c\rangle + \langle c', (\nu \circ c)'\rangle = 0$ is valid for a solution, $\langle c', \nu \circ c\rangle$ is constant equal to $\langle c'(0), \nu_{c(0)}\rangle = \langle \xi, \nu_x\rangle = 0$. Thus

$$(f \circ c)'(t) = \langle \operatorname{grad}_{c(t)} f, c'(t)\rangle = |\operatorname{grad}_{c(t)} f| \cdot \langle \nu_{c(t)}, c'(t)\rangle = 0,$$

So $f \circ c$ is constant equal to $f(c(0)) = f(x) = 0$, i.e. $c(t) \in f^{-1}(0) = M$.

If c_ξ denotes the geodesic with initial value $c'(0) = \xi$, then for $t \in \mathbb{R}$ the curve $s \mapsto c_\xi(t\,s)$ is the geodesic with initial value $\frac{d}{ds}\big|_{s=0} c_\xi(t\,s) = t\,c'_\xi(0) = t\,\xi$, i.e. the following homogeneity relation holds:

$$c_\xi(t\,s) = c_{t\,\xi}(s).$$

Let now $\xi = \delta\eta \in U := \delta V$, then $c_\eta(\delta) = c_{\delta\eta}(1) = c_\xi(1) =: \exp(\xi)$ exists and is smooth with respect to ξ. Thus $t \mapsto \exp(t\,\xi) = c_{t\,\xi}(1) = c_\xi(t)$ is the geodesic with initial value ξ, and furthermore, $\exp(0_x) = c_{0_x}(1) = x$ and $T_{0_x} \exp_x \cdot \xi = \frac{d}{dt}\big|_0 \exp_x(t\,\xi) = \frac{d}{dt}\big|_0 c_\xi(t) = \xi$. □

10.9 Geodesic polar coordinates.

We can now show the existence of local coordinates φ with $E = 1$ and $F = 0$ on each Riemann surface.

Because of $T_{0_x} exp_x = \mathrm{id}_{T_x M}$, the mapping $\exp_x$ is a local diffeomorphism from $T_x M$ to M and so we have a distinguished chart $\exp_x$ centered at x. In order to describe it in coordinates, we choose a unit vector $v \in T_x M$ and one (of the two) normal vector(s) $v^\perp \in T_x M$ and consider polar coordinates

$$(r, \vartheta) \mapsto r \cdot \underbrace{\left(\cos(\vartheta) \cdot v + \sin(\vartheta) \cdot v^\perp\right)}_{=:v(\vartheta)}$$

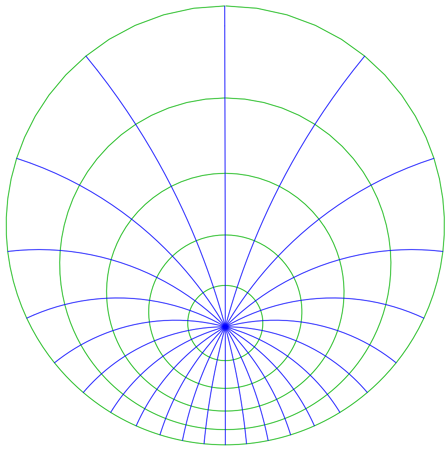
to get a parameterization (for $r \ne 0$)

$$\varphi : (r, \vartheta) \mapsto \exp_x(r\,v(\vartheta)) \text{ with } \varphi(0, \vartheta) = x.$$

Thus, $t \mapsto \varphi(t, \vartheta) = \exp_x(t\,v(\vartheta))$ is the (according to $|v(\vartheta)| = 1$) arc length parameterized geodesic with initial value $v(\vartheta) \in T_x M$, a so-called RADIAL GEODESIC and we have:

$$\left.\begin{aligned} E = |\varphi_r|^2 = |v_\vartheta|^2 = 1 \quad &\Rightarrow \quad \langle \varphi_r, \varphi_{r,\vartheta}\rangle = 0 \\ \varphi_{r,r} \perp TM \quad &\Rightarrow \quad \langle \varphi_{r,r}, \varphi_\vartheta\rangle = 0 \end{aligned}\right\} \Rightarrow$$
$$\Rightarrow \quad F_r = \frac{\partial}{\partial r}\langle \varphi_r, \varphi_\vartheta\rangle = \langle \varphi_{r,r}, \varphi_\vartheta\rangle + \langle \varphi_r, \varphi_{\vartheta,r}\rangle = 0.$$

In addition, $\varphi(0, \vartheta) = x$ and thus $\varphi_\vartheta(0, \vartheta) = 0$, hence $F = \langle \varphi_r(r, \vartheta), \varphi_\vartheta(r, \vartheta)\rangle = \langle \varphi_r(0, \vartheta), \varphi_\vartheta(0, \vartheta)\rangle = 0$. Finally, $G = \langle \varphi_\vartheta, \varphi_\vartheta\rangle \ge 0$ and locally vanishes only for $r = 0$.



Geodesic polar coordinates of Poincaré's half-plane

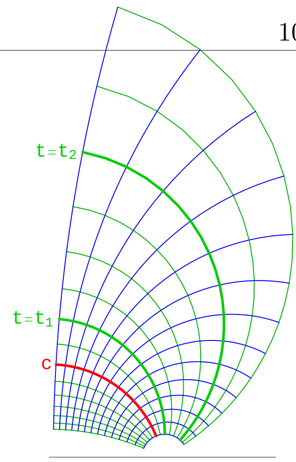
The closed curves $\vartheta \mapsto \varphi(r, \vartheta)$ are called GEODESIC CIRCLES with radius r. These are of course in general not geodesics!

Examples of geodesic polar coordinates.

- Sphere
- Torus
- Paraboloid
- Hyperbolic paraboloid
- One-sheet hyperboloid
- Two-sheet hyperboloid
- Helicoid
- Catenoid
- Ennepers surface
- pseudosphere
- Möbius strip
- Plueckers cone
- Sherks surface

10.10 Proposition. Geodesic parallel coordinates.

The coefficients of a Riemann metric have locally the form $E = 1$, $F = 0$, and $G > 0$ if and only if $t \mapsto \varphi(t, s)$ are arc length parameterized geodesics which intersect curves $s \mapsto \varphi(t, s)$ orthogonally. In particular, the length of the segments of these geodesics from $t = t_1$ to $t = t_2$ is $t_2 - t_1$, and thus is independent on s.
For each regular curve $c : \mathbb{R} \to M$, unique local coordinates φ exist having the above properties and satisfy $\varphi(0, s) = c(s)$.



For a geodesic circle c these are just the geodesic polar coordinates from 10.9.

Proof. Existence: For this, we choose a unit vector field ξ along c, which is normal to c' and define a map $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ by $\varphi(t, s) := \exp_{c(s)}(t\xi(s))$. Then $\varphi(0, s) = c(s)$ and $t \mapsto \varphi(t, s)$ is arc length parameterized geodesic with initial vector $\xi(s)$. So $\varphi_s(0, s) = c'(s) \neq 0$ as well as $\varphi_t(0, s) = \xi(s)$ and thus φ is a local diffeomorphism.

Since $t \mapsto \varphi(t, s)$ are geodesics, as shown in the proof of 10.9, $E = g_{1,1} = \langle\varphi_t, \varphi_t\rangle = 1$ and $F_1 = 0$. Because of $F(0, s) = g_{1,2}(0, s) = \langle\xi(s), c'(s)\rangle = 0$ we obtain $F = 0$.

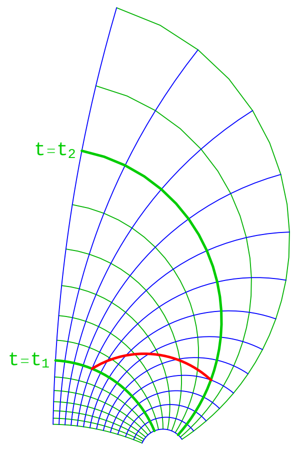
Conversely, if $E = 1$ and $F = 0$, then according to 10.6 (see also 11.2)

$$\Gamma^1_{1,1} = \frac{G\,E_1 - 2\,F\,F_1 + F\,E_2}{2\,D^2} = 0 \quad \text{und} \quad \Gamma^2_{1,1} = \frac{-F\,E_1 + 2\,E\,F_1 - E\,E_2}{2\,D^2} = 0.$$

Thus, curves $u^1(t) := t$ and $u^2 = \text{const}$ satisfy the geodesic equation 10.7 (see also 11.2), are parameterized by arc length because of $E = 1$, and due to $F = 0$ intersect the curves with constant u^1 orthogonally. □

10.11 Lemma.
If $\varphi : \mathbb{R}^2 \supseteq U \to M$ is a parameterization by geodesic coordinates, then each curve of the form $\varphi \circ u$ with a curve u in U which connects (t_1, s_1) with (t_2, s_2) is at least as long as each geodesic $t \mapsto \varphi(t, s)$ for $t \in [t_1, t_2]$ and fixed s.

See 13.11 for the generalization of this to Riemannian manifolds.
This result shows that certain geodesics are minimal among all sufficiently nearby curves. Globally this is not true, as an arc of a great circle on the sphere of length greater than π shows.



Proof. Let s be fixed, $c_0(t) := \varphi(t, s)$ and $c_1(\tau) := \varphi(u(\tau))$ with $u^1(\tau_i) = t_i$ for $i \in \{1, 2\}$, then

$$\begin{aligned} L(c_1) &= \int_{\tau_1}^{\tau_2} \sqrt{(\tfrac{du^1}{d\tau})^2 + G(\tfrac{du^2}{d\tau})^2}\, d\tau \geq \\ &\geq \int_{\tau_1}^{\tau_2} \left|\tfrac{du^1}{d\tau}\right| d\tau \geq \left|\int_{\tau_1}^{\tau_2} \tfrac{du^1}{d\tau} d\tau\right| = |u^1(\tau_2) - u^1(\tau_1)| = |t_2 - t_1| = L(c_0) \end{aligned}$$

and equality holds if and only if u^1 is monotone and $\frac{du^2}{d\tau} = 0$, i.e. u^2 is constant. □

11. Integral theorem of Gauß - Bonnet

The following geometric formulas for the Gaussian curvature hold:

11.1 Theorem (Gaussian curvature as deviation of the measures of circles).

Let geodesic polar coordinates φ be chosen at $x \in M$. With $L(r)$ we denote the length and with $A(r)$ the area of the interior of the geodesic circle $\vartheta \mapsto \varphi(r, \vartheta)$. Then

1. $K_x = \frac{3}{\pi} \lim_{r \searrow 0} \frac{2r\pi - L(r)}{r^3}$ *Bertrand & Puiseaux 1848*
2. $K_x = \frac{12}{\pi} \lim_{r \searrow 0} \frac{r^2\pi - A(r)}{r^4}$ *Diquet 1848*

The Gaussian curvature thus measures infinitesimally by how much the circumference and the area of a geodesic circle, is too small compared to an Euclidean circle with same radius.

Proof. Geodesic polar coordinates φ are given by 10.9 by $\varphi(r, \vartheta) = \exp_x(r\, v(\vartheta))$ with $v(\vartheta) = \cos(\vartheta)\, v + \sin(\vartheta)\, v^\perp$. We already know the following about $\sqrt{G} := |\varphi_\vartheta|$: The function $G = |\varphi_\vartheta|^2$ is smooth and vanishes only for $r = 0$. So also $\sqrt{G}$ is smooth for $r \neq 0$ but not necessarily for $r = 0$. However, we have to study the behavior at 0. For this we use the Jacobi equation $K\sqrt{G} + \left(\frac{\partial}{\partial r}\right)^2 \sqrt{G} = 0$ from 9.17. The Taylor formula of order 1 with integral remainder (see [82, 6.3.11])

$$f(x) = f(0) + f'(0)(x) + \int_0^1 (1-t) f''(tx)(x, x)\, dt$$

gives

$$\begin{aligned}
\varphi(r, \vartheta) &= \exp_x(r\, v(\vartheta)) \\
&= \underbrace{\exp_x(0)}_{=x} + \underbrace{\exp_x'(0)}_{=\mathrm{id}}(r\, v(\vartheta)) + \int_0^1 (1-t) \exp_x''(t\, r\, v(\vartheta))(r\, v(\vartheta), r\, v(\vartheta))\, dt \\
&= x + r\, v(\vartheta) + r^2 \underbrace{\int_0^1 (1-t) \exp_x''(t\, r\, v(\vartheta))(v(\vartheta), v(\vartheta))\, dt,}_{=: g(r,\vartheta)}
\end{aligned}$$

where g is a smooth $\mathbb{R}^n$-valued function. By partial differentiation with respect ϑ we obtain:

$$\varphi_\vartheta(r, \vartheta) = r \left(v'(\vartheta) + r \frac{\partial}{\partial \vartheta} g(r, \vartheta) \right)$$

and thus for $r \geq 0$

$$\sqrt{G(r,\vartheta)} = |\varphi_\vartheta(r, \vartheta)| = r \sqrt{|v'(\vartheta)|^2 + 2r \langle v'(\vartheta) | \tfrac{\partial}{\partial \vartheta} g(r,\vartheta)\rangle + r^2 \langle \tfrac{\partial}{\partial \vartheta} g(r,\vartheta) | \tfrac{\partial}{\partial \vartheta} g(r,\vartheta)\rangle}$$

is smooth (because of $|v'(\vartheta)| = 1$) and in particular for the right-sided derivative at 0 we have

$$\frac{\partial}{\partial r}\Big|_{r=0} \sqrt{G(r,\vartheta)} = 0 + 1.$$

By taking the limit for $r \searrow 0$ in the Jacobi equation 9.17 we obtain

$$\left(\tfrac{\partial}{\partial r}\right)^2|_{r=0}\sqrt{G(r,\vartheta)} = -K(x)\sqrt{G(0,\vartheta)} = 0 \quad \text{and by differentiating}$$
$$\frac{\partial^3\sqrt{G}}{\partial r^3}(r,\vartheta) = -\frac{\partial\sqrt{G}}{\partial r}K - \sqrt{G}\,\frac{\partial K}{\partial r}, \quad \text{and furthermore}$$
$$\frac{\partial^3\sqrt{G}}{\partial r^3}(0,\vartheta) = -1\,K(x) + 0$$

The Taylor formula of order 2 with integral remainder (see [82, 6.3.11]) thus gives

$$\sqrt{G}(r,\vartheta) = 0 + r + 0 + \int_0^1 \frac{(1-t)^2}{2!}\frac{\partial^3\sqrt{G}}{\partial r^3}(t\,r,\vartheta)\,r^3\,dt$$

$$\lim_{r\to 0+}\frac{\sqrt{G}(r,\vartheta)-r}{r^3} = \lim_{r\to 0+}\int_0^1 \frac{(1-t)^2}{2!}\frac{\partial^3\sqrt{G}}{\partial r^3}(t\,r,\vartheta)\,dt$$
$$= \frac{\partial^3\sqrt{G}}{\partial r^3}(0,\vartheta)\int_0^1\frac{(1-t)^2}{2!}\,dt = -K(x)\,\frac{1}{3!}.$$

Thus, because of $L(r) = \int_0^{2\pi}|\varphi_\vartheta(r,\vartheta)|\,d\vartheta = \int_0^{2\pi}\sqrt{G(r,\vartheta)}\,d\vartheta$

$$K(x) = \frac{3}{\pi}\int_0^{2\pi}\frac{K(x)}{6}\,d\vartheta = \lim_{r\to 0+}\frac{3}{\pi}\int_0^{2\pi}\frac{r-\sqrt{G(r,\vartheta)}}{r^3}\,d\vartheta$$
$$= \frac{3}{\pi}\lim_{r\to 0+}\frac{2r\pi - L(r)}{r^3}$$

For the area we obtain, according to the rule of De L'Hospital (see [81, 4.1.18]),

$$A(r) \overset{9.26}{=\!=} \int_0^r\int_0^{2\pi}\sqrt{G(\rho,\vartheta)}\,d\vartheta\,d\rho$$
$$\Rightarrow A'(r) = \int_0^{2\pi}\sqrt{G(r,\vartheta)}\,d\vartheta = L(r)$$
$$\Rightarrow \lim_{r\to 0+}\frac{r^2\pi - A(r)}{r^4} = \lim_{r\to 0+}\frac{2r\pi - L(r)}{4r^3} = \frac{1}{4}\frac{K(x)\pi}{3} = K(x)\frac{\pi}{12}. \qquad \square$$

11.2 Christoffel symbols in geodesic coordinates.

We choose a geodesic parameterization φ on M, i.e. $E = 1$ and $F = 0$, with associated local coordinates $(r,\vartheta) = (u^1,u^2)$. Then, for the coefficients of the Riemann metric (see also 10.6) we have:

$$g_{1,1} = E = 1 \qquad g^{1,1} = 1$$
$$g_{1,2} = g_{2,1} = F = 0 \qquad g^{1,2} = g^{2,1} = 0$$
$$g_{2,2} = G > 0 \qquad g^{2,2} = \tfrac{1}{G}.$$

For the Christoffel symbols of the first kind 10.5 the result is (see 10.6):

$$\Gamma_{2,2,2} = \tfrac{1}{2}\tfrac{\partial G}{\partial\vartheta}$$
$$\Gamma_{1,2,2} = \Gamma_{2,1,2} = \tfrac{1}{2}\tfrac{\partial G}{\partial r}$$
$$\Gamma_{2,2,1} = -\tfrac{1}{2}\tfrac{\partial G}{\partial r}$$
$$\Gamma_{i,j,k} = 0 \qquad \text{for all other } i,\ j,\ k.$$

For those of the second kind:

$$\begin{aligned}\Gamma^1_{2,2} &= -\tfrac{1}{2}\tfrac{\partial G}{\partial r}\\ \Gamma^2_{1,2} = \Gamma^2_{2,1} &= \tfrac{1}{2G}\tfrac{\partial G}{\partial r}\\ \Gamma^2_{2,2} &= \tfrac{1}{2G}\tfrac{\partial G}{\partial \vartheta}\\ \Gamma^k_{i,j} &= 0 \qquad \text{for all other } i,\ j,\ k.\end{aligned}$$

A geodesic must therefore satisfy the following equations (see 10.7):

$$\begin{aligned}\tfrac{d^2u^1}{dt^2} + \Gamma^1_{2,2}\tfrac{du^2}{dt}\tfrac{du^2}{dt} &= 0\\ \tfrac{d^2u^2}{dt^2} + 2\Gamma^2_{1,2}\tfrac{du^1}{dt}\tfrac{du^2}{dt} + \Gamma^2_{2,2}\tfrac{du^2}{dt}\tfrac{du^2}{dt} &= 0.\end{aligned}$$

By inserting we get:

$$\begin{aligned}\tfrac{d^2u^1}{dt^2} - \tfrac{1}{2}\tfrac{\partial G}{\partial r}\left(\tfrac{du^2}{dt}\right)^2 &= 0\\ \tfrac{d^2u^2}{dt^2} + \tfrac{1}{G}\tfrac{\partial G}{\partial r}\tfrac{du^1}{dt}\tfrac{du^2}{dt} + \tfrac{1}{2G}\tfrac{\partial G}{\partial \vartheta}\left(\tfrac{du^2}{dt}\right)^2 &= 0.\end{aligned}$$

11.3 Geodesic curvature, a special case.

Let $(u^1, u^2) = (r, \vartheta)$ be local geodesic coordinates as in 11.2. For an arc length parameterized geodesic $t \mapsto u(t) = (u^1(t), u^2(t)) = (r(t), \vartheta(t))$, let the angle between it and the radial geodesics u^2=constant be denoted by $\Theta(t)$, i.e.

$$\cos\Theta(t) = \langle u'(t)|\tfrac{\partial}{\partial r}\rangle = \left\langle \tfrac{dr(t)}{dt}\tfrac{\partial}{\partial r} + \tfrac{d\vartheta(t)}{dt}\tfrac{\partial}{\partial\vartheta}\middle|\tfrac{\partial}{\partial r}\right\rangle = \tfrac{dr(t)}{dt} = \tfrac{d}{dt}u^1.$$

Thus we get

$$\tfrac{1}{2}\tfrac{\partial G}{\partial r}\left(\tfrac{du^2}{dt}\right)^2 \overset{11.2}{=\!=\!=} \tfrac{d^2u^1}{dt^2} = \tfrac{d}{dt}\cos\Theta(t) = -\sin\Theta(t)\tfrac{d\Theta(t)}{dt}.$$

On the other hand, due to $|u'| = 1 = |\frac{\partial}{\partial r}|$, $|\frac{\partial}{\partial\vartheta}| = \sqrt{G}$ and $\frac{\partial}{\partial\vartheta} \perp \frac{\partial}{\partial r}$,

$$\begin{aligned}\sin\Theta(t) &= \mathrm{vol}(\tfrac{\partial}{\partial r}, u'(t)) = \mathrm{vol}(\tfrac{\partial}{\partial r}, \tfrac{dr(t)}{dt}\tfrac{\partial}{\partial r} + \tfrac{d\vartheta(t)}{dt}\tfrac{\partial}{\partial\vartheta})\\ &= \sqrt{G}\tfrac{d\vartheta}{dt} = \sqrt{G}\tfrac{du^2}{dt}.\end{aligned}$$

Finally we get

$$\begin{aligned}\tfrac{1}{2}\tfrac{\partial G}{\partial r}\left(\tfrac{d\vartheta}{dt}\right)^2 &= -\sin\Theta(t)\,\tfrac{d\Theta}{dt} = -\sqrt{G}\tfrac{d\vartheta}{dt}\tfrac{d\Theta}{dt},\\ \text{i.e.}\quad \tfrac{d\Theta}{dt} &= -\tfrac{\partial\sqrt{G}}{\partial r}\tfrac{d\vartheta}{dt},\end{aligned}$$

or if we may reparametrize the geodesic by ϑ:

$$\tfrac{d\Theta}{d\vartheta} = -\tfrac{\partial\sqrt{G}}{\partial r}.$$

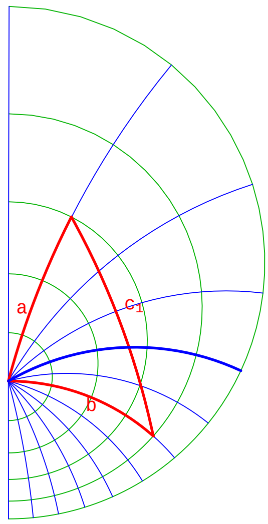
11.4 Theorema elegantissimum of Gauss.

Let Δ be a geodesic triangle in M - i.e. its sides are geodesics - with interior angles α, β, and γ. Then

$$\int_\Delta K\ \mathrm{vol}_M = \alpha + \beta + \gamma - \pi.$$

In particular, for the plane, this gives the proposition that the sum of angles in any triangle is 180 degrees (i.e. π).

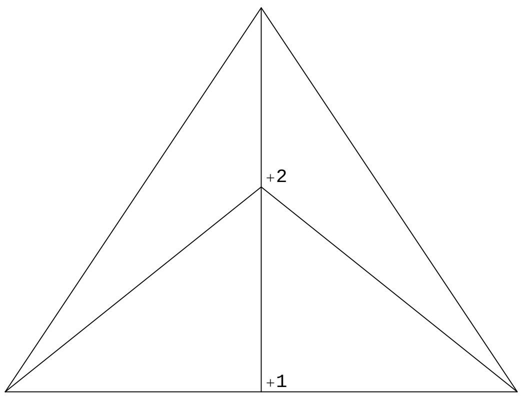
Proof.



For the time being we assume that the geodesic triangle is contained entirely in the image of geodesic polar coordinates φ around the vertex C. The edges a and b correspond in polar coordinates to two straight lines through 0. And we can choose the 0 direction to be the tangent to b. The edge c can then be described in polar coordinates by an equation of the form $r = r(\vartheta)$ with $\vartheta \in [0, \gamma]$. Note that c can nowhere be tangent to a radial geodesic, otherwise it would be one. Let $\Theta(\vartheta)$ be the angle between $\frac{\partial}{\partial r}$ and c. Clearly $\Theta(0) = \pi - \alpha$ and $\Theta(\gamma) = \beta$. Thus

$$\begin{aligned}
\int_\Delta K \operatorname{vol}_M &= \int_{\varphi^{-1}(\Delta)} \varphi^*(K \operatorname{vol}_M) = && \text{by } \boxed{9.17} \\
&= \int_{\varphi^{-1}(\Delta)} -\tfrac{1}{\sqrt{G}} \tfrac{\partial^2 \sqrt{G}}{\partial r^2} \sqrt{G}\, dr \wedge d\vartheta = \\
&= \int_0^\gamma \int_0^{r(\vartheta)} -\tfrac{\partial^2 \sqrt{G}}{\partial r^2}(r, \vartheta)\, dr\, d\vartheta = && \text{by the proof of } \boxed{11.1}, \text{ cf. } \boxed{11.11} \\
&= \int_0^\gamma 1 - \tfrac{\partial \sqrt{G}}{\partial r}(r(\vartheta), \vartheta)\, d\vartheta = && \text{by } \boxed{11.3} \\
&= \int_0^\gamma 1 + \tfrac{d\Theta}{d\vartheta}(\vartheta)\, d\vartheta = \gamma + \Theta(\gamma) - \Theta(0) = \gamma + \beta - (\pi - \alpha).
\end{aligned}$$

For general geodesic triangles, the result follows, by dividing them into smaller geodesic triangles: If we add up the results for the sub-triangles, we get $\int_\Delta K \operatorname{vol}_M$ on the left side and the sum of all interior angles on the right - i.e. the sum of the angles at the original vertices plus π times the number of remaining vertices on the boundary plus 2π times the number inner vertices - reduced by π times the number of sub-triangles.



Since every division of an inner edge divides the two bounding triangles into 4 triangles, and every division of a a boundary edge divides the bounding triangle into two, the following holds: The sum of the vertices on the boundary (without the original 3 vertices) plus twice the sum of the interior vertices is the number of triangles minus 1. Thus, this combination of π's precisely gives $-\pi$ and the formula also holds in the general case. □

If a, b and c are the lengths of the edges of a geodesic triangle Δ and $\bar\alpha$, $\bar\beta$ and $\bar\gamma$ are the angles of the Euclidean triangle with the same lengths, then

$$\alpha - \bar\alpha = \frac{\operatorname{vol}(\Delta)}{3} K + o\left(a^2 + b^2 + c^2\right)$$

and analogously for the other angles, see [**12**, 10.5.5.6 S.387].

11.5 Corollary. Global version of Gauss-Bonnet.

Let M be a compact oriented Riemann surface. Then

$$\frac{1}{2\pi}\int_M K\,\mathrm{vol}_M = \chi(M) = 2-2g.$$

Proof. We decompose the surface into small geodesic triangles. Then, for the Euler characteristic according to [**95**, 26.5.10] we have:

$$\chi(M) = \#\text{vertices} - \#\text{edges} + \#\text{triangles}.$$

Since each triangle is bounded by exactly 3 edges and each edge belongs to exactly two triangles, we have

$$3\cdot\#\text{triangles} = 2\cdot\#\text{edges}$$

and thus

$$\chi(M) = \#\text{vertices} - \frac{1}{2}\cdot\text{triangles}.$$

On the other hand

$$\begin{aligned}\int_M K\,\mathrm{vol}_M &= \sum_\Delta \int_\Delta K\,\mathrm{vol}_M\\ &= \text{Sum of all interiour angles} - \pi\cdot\#\text{triangles}\\ &= 2\,\pi\cdot\#\text{vertices} - \pi\cdot\#\text{triangles}\\ &= 2\,\pi\,\chi(M).\end{aligned}$$

The second equality $\chi(M) = 2\,(1-g)$ was shown in [**95**, 26.5.9], where g denotes the genus of M. □

If K is constant, then $\mathrm{vol}(M) = 4\pi\frac{1}{K}(1-g)$ follows.

11.6 Corollary.

Let M be a compact connected Riemann surface, then:

1. *If $K \geq 0$ is not constant 0, then $\chi(M) = 2$, i.e. M is diffeomorphic to the sphere S^2, or $\chi(M) = 1$, i.e. M is diffeomorphic to the projective plane $\mathbb{P}^2$.*
2. *If $K = 0$, then $\chi(M) = 0$, i.e. M is diffeomorphic to the torus or the Klein bottle.*
3. *If $K \leq 0$ is not constant 0, then $\chi(M) < 0$, i.e. M is diffeomorphic to a sphere with at least 2 handles or at least 3 Möbius strips.*

Proof. If M is not oriented, then we pass to the orientation covering M^{or} with $\chi(M^{\mathrm{or}}) = 2\chi(M)$ (see [**95**, 29.5]). We can then read off the Euler characteristic from 11.5 and, in particular, there are points $p \in M$ with $\mathrm{sgn}(K(p)) = \mathrm{sgn}(\chi(M))$. The statements now follow from the classification theorem [**95**, 1.2] for compact orientable surfaces (i.e. spheres with $g \geq 0$ handles, where $\chi(M) = 2\,(1-g)$ by [**95**, 26.5.9]), and that for non-orientable surfaces [**95**, 1.4] (i.e. spheres to which $g > 0$ Möbius strips are glued, where $\chi(M) = 2-g$ by [**95**, 26.5.9]). □

We now want to generalize the integral formula of Gauss-Bonnet to polygons with non-geodesic edges.

11.7 Remark.

We choose geodesic coordinates φ on M, i.e. $E = 1$ and $F = 0$. Thus, $e_1 := \varphi_1$, $e_2 := \frac{1}{\sqrt{G}}\varphi_2$ is an orthonormal basis. Let $t \mapsto u(t)$ be the local representation of an arc length parameterized curve c. Let τ be the unit tangent vector and ξ a tangent vector of M which is normal to τ. Finally, let $K_g(t) := \langle c''(t), \xi(t)\rangle$ be the geodesic curvature, cf. [87, 55.3]. Note that $K_g = 0$ holds if and only if $c''(t) \perp T_{c(t)}M$, i.e. c is a geodesic.

Lemma.

There is a function $\Theta : \mathbb{R} \to \mathbb{R}$ *with*

$$\tau(t) = \cos(\Theta(t))\, e_1(u(t)) + \sin(\Theta(t))\, e_2(u(t)),$$

unique up to $2\pi\mathbb{Z}$, *and the following holds:*

$$K_g(t) = \Theta'(t) + \tfrac{\partial\sqrt{G}}{\partial u^1}\tfrac{du^2}{dt}.$$

This is a generalization of the corresponding formula $K(s) = \Theta'(s)$ in 7.7 for planar curves and also in 11.3 for geodesics c.

Proof. As in [87, 3.9], the existence and uniqueness of the function Θ holds. Then

$$\xi(t) = -\sin(\Theta(t))\, e_1(u(t)) + \cos(\Theta(t))\, e_2(u(t)) \quad \text{and}$$
$$\tau'(t) = \Theta'(t)\,\xi(t) + \cos\Theta(t)\,\tfrac{d}{dt}e_1(u(t)) + \sin\Theta(t)\,\tfrac{d}{dt}e_2(u(t))$$

From $\langle e_i, e_j\rangle = \delta_{i,j}$ it follows that $\langle \frac{d}{dt}e_i, e_j\rangle + \langle e_i, \frac{d}{dt}e_j\rangle = 0$. If we now insert the representations for $\tau' = c''$ and for ξ in the formula for the geodesic curvature, we obtain:

$$\begin{aligned} K_g := \langle\tau',\xi\rangle &= \Theta'\,\langle\xi,\xi\rangle + \cos^2\Theta\langle\tfrac{d}{dt}e_1, e_2\rangle - \sin^2\Theta\langle\tfrac{d}{dt}e_2, e_1\rangle + \\ &\qquad + \sin\Theta\cos\Theta\Big(\underbrace{\langle\tfrac{d}{dt}e_2, e_2\rangle - \langle\tfrac{d}{dt}e_1, e_1\rangle}_{=0}\Big) \\ &= \Theta' + \Big\langle\tfrac{d}{dt}e_1, e_2\Big\rangle \end{aligned}$$

and furthermore

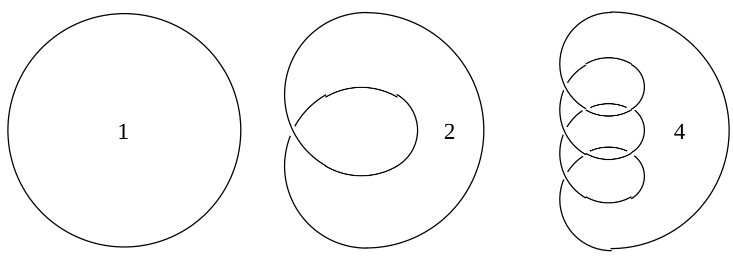
$$\begin{aligned} \big\langle\tfrac{d}{dt}e_1, e_2\big\rangle &= \Big\langle\varphi_{1,1}\tfrac{du^1}{dt} + \varphi_{1,2}\tfrac{du^2}{dt}, \tfrac{1}{\sqrt{G}}\varphi_2\Big\rangle \\ &= \tfrac{1}{\sqrt{G}}\Big(\Gamma_{1,1,2}\tfrac{du^1}{dt} + \Gamma_{1,2,2}\tfrac{du^2}{dt}\Big) \overset{11.2}{=\!=} \tfrac{1}{2\sqrt{G}}\tfrac{\partial G}{\partial r}\tfrac{du^2}{dt} = \tfrac{\partial\sqrt{G}}{\partial r}\tfrac{du^2}{dt} \end{aligned}$$

and thus

$$K_g(t) = \Theta'(t) + \tfrac{\partial\sqrt{G}}{\partial u^1}\tfrac{du^2}{dt}. \qquad \square$$

11.8 Definition (Turning number)

Let $c : [0, 2\pi] \to \mathbb{R}^2$ be a closed regular curve. Then we call $U(c) := W_0(c') = \frac{1}{2\pi i}\int_c \frac{dz}{z}$ (see [87, 4.2]) the TURNING NUMBER of the curve.



11.9 Umlaufsatz of Hopf, 1939.

If c *is a simply closed curve, then* $U(c) = \pm 1$.

Proof. Let $c : [0, L] \to \mathbb{R}^2$ be parameterized by arc length.

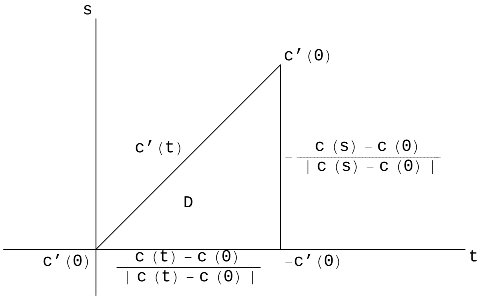
We consider

$$\Gamma(t, s) := \frac{c(t) - c(s)}{|c(t) - c(s)|}$$

with domain an open triangle

$$D := \{(t, s) : 0 < s < t < L\}.$$

This Γ can be continued to the closure $\overline{D}$: This is obvious on the catheti $\{(t, 0) : t \in]0, L[\}$ and $\{(L, s) : s \in]0, L[\}$.



On the hypothenuse $\{(t, t) : t \in [0, L]\}$, this is done by the following formula:

$$\Gamma(r, r) := \lim_{s,t\to r} \Gamma(t, s) = c'(r),$$

because $\gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ defined by

$$\gamma(t, s) := \int_0^1 c'(s + (t - s)\rho)d\rho = \frac{c(t) - c(s)}{t - s}$$

is continuous and $\lim_{s,t\to r} \gamma(t, s) = \int_0^1 c'(r + \rho(r - r))d\rho = c'(r)$. Furthermore,

$$\begin{aligned}\Gamma(t, s) &= \frac{\gamma(t, s) \cdot (t - s)}{|\gamma(t, s) \cdot (t - s)|} = \frac{\gamma(t, s)}{|\gamma(t, s)|} \operatorname{sgn}(t - s)\\ \lim_{\substack{t,s\to r\\ t>s}} \Gamma(t, s) &= \lim_{\substack{t,s\to r\\ t>s}} \frac{\gamma(t, s)}{|\gamma(t, s)|} = \frac{c'(r)}{|c'(r)|} = c'(r).\end{aligned}$$

For the vertex $(L, 0)$:

$$\lim_{\substack{t\nearrow L\\ s\searrow 0}} \Gamma(t, s) \overset{L-t'=t}{=\!=\!=} \lim_{\substack{t'\searrow 0\\ s\searrow 0}} \Gamma(L - t', s) \overset{c \text{ ist } L\text{-periodisch}}{=\!=\!=\!=\!=}$$

$$= \lim_{t',s\searrow 0} \Gamma(-t', s) = \lim_{t',s\searrow 0} \frac{\gamma(-t', s)}{|\gamma(-t', s)|} \operatorname{sgn}(-t' - s) = -c'(0)$$

So we put $\Gamma(L, 0) := -c'(0)$ and obtain a continuous continuation of Γ to $\overline{D}$.

We consider a straight line parallel to the x-axis, which touches c, such that c lies in the upper half-plane. Let 0 be a parameter for which c touches this line. W.l.o.g. let $c'(0) = \binom{1}{0}$. Since Γ restricted to the hypothenuse is the curve c', which by virtue of Γ is homotopic to Γ restricted to the catheti, it follows that

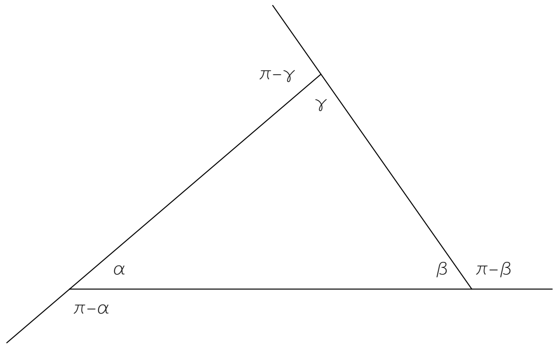
$$\begin{aligned}U(c) = W_0(c') = W_0(\Gamma|_{\text{hypothenuse}}) &\overset{[87,\ 4.6]}{=\!=\!=} W_0(\Gamma|_{\text{catheti}})\\ &= \frac{1}{2\pi}\Big((\pi - 0) + ((\pi + \pi) - (0 + \pi))\Big) = \frac{2\pi}{2\pi} = 1,\end{aligned}$$

since $c(t) - c(0)$ stays in the upper halfplane. □

The Hopf Umlaufsatz can also be applied to curves with corners, i.e. piecewise C^∞ curves, which have no cusps in the vertices, i.e. the left-hand tangent vector and the right-hand one are not oppositely oriented: Let c be smooth on the sub-intervals $[t_i, t_{i+1}]$, where t_i are the vertices of c, then we define the turning number of c by

$$U(c) = \tfrac{1}{2\pi}\Big(\sum_i (\gamma_i(t_{i+1}) - \gamma_i(t_i)) + \sum_i \varphi_i\Big)$$

where γ_i is a lift from $\frac{c'}{|c'|}\Big|_{[t_i,t_{i+1}]}$ and $\varphi_i \in\,]-\pi, +\pi[$ is the angle between left and right tangents at $c(t_i)$.



As an example, consider the turning number of a triangle $\triangle$:

$$U(\triangle) = \frac{1}{2\pi}\Big((\pi-\alpha)+(\pi-\beta)+(\pi-\gamma)\Big) = \frac{1}{2\pi}\Big(3\pi-(\alpha+\beta+\gamma)\Big) = 1.$$

11.10 Proposition. Gauss-Bonnet for polygons.

Let $\varphi : U \to M$ be a chart of M and let P be a polygon in U and α_i the outside angles (i.e. π minus interior angle) at the vertices of $\varphi(P)$. Then

$$\int_{\varphi(P)} K\ \mathrm{vol}_M + \int_{\varphi(\partial P)} K_g + \sum_i \alpha_i = 2\pi.$$

Proof. We first assume that all of $\varphi(P)$ can be parameterized by geodesic coordinates. Then

$$K \overset{\boxed{9.17}}{=\!=} -\tfrac{1}{\sqrt{G}}\tfrac{\partial^2\sqrt{G}}{\partial r^2} = \tfrac{1}{\sqrt{G}}\left(\tfrac{\partial}{\partial r}\left(\sqrt{G}\cdot(-\tfrac{1}{\sqrt{G}}\tfrac{\partial\sqrt{G}}{\partial r})\right)+\tfrac{\partial}{\partial\vartheta}0\right) \overset{\boxed{4.5}}{=\!=} \operatorname{div}\xi,$$

where $\xi := \xi^r\frac{\partial}{\partial r}+\xi^\vartheta\frac{\partial}{\partial\vartheta} := (-\frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial r})\frac{\partial}{\partial r}+0\frac{\partial}{\partial\vartheta}$. So according to Green's theorem:

$$\begin{aligned}\int_{\varphi(P)} K\ \mathrm{vol}_M &= \int_{\varphi(P)} \operatorname{div}\xi\ \mathrm{vol}_M \overset{\boxed{4.7}}{=\!=} \int_{\varphi(\partial P)} \langle\xi,\nu_{\varphi(\partial P)}\rangle\ \mathrm{vol}_{\varphi(\partial P)}\\ &\overset{\boxed{4.6}}{=\!=} \int_{\varphi(\partial P)} \mathrm{incl}^*(\iota_\xi\,\mathrm{vol}_M)\\ &\overset{\boxed{4.2}}{=\!=} \int_{\varphi(\partial P)} \mathrm{incl}^*\Big(\iota_{\xi^r\frac{\partial}{\partial r}+\xi^\vartheta\frac{\partial}{\partial\vartheta}}\sqrt{G}\,dr\wedge d\vartheta\Big)\\ &= \int_{\varphi(\partial P)} \sqrt{G}\,\xi^r\,d\vartheta-\sqrt{G}\,\xi^\vartheta\,dr = -\int_{\varphi(\partial P)}\frac{\partial\sqrt{G}}{\partial r}\,d\vartheta.\end{aligned}$$

Using the formula from the lemma in 11.7 for the geodesic curvature of the boundary curve, we get for each edge I of the polygon:

$$-\int_{\varphi(I)}\frac{\partial\sqrt{G}}{\partial r}d\vartheta = \int_I \Theta'(t)dt - \int_I K_g(t)dt.$$

Because of the Umlaufsatz 11.9 in the plane, we have in the case of the Euclidean metric with corresponding angle Θ:

$$\sum_i\Big(\int_{I_i}\Theta'(t)dt+\alpha_i\Big) = 2\pi.$$

A general Riemann metric can be connected to the Euclidean metric (i.e. $G = 1$) by $s \mapsto G^s := s + (1-s)G$ affinely, and similarly we get the function Θ^s and the angles α_i^s, and these depend continuously on s. Because of

$$\sum_i \Big(\int_{I_i} (\Theta^s)' + \alpha_i^s \Big) \equiv \sum_i \Big(\big(\Theta^s(\max I_i) - \Theta^s(\min I_i)\big) + \big(\Theta^s(\min I_i) - \Theta^s(\max I_{i-1})\big) \Big)$$
$$= 0 \bmod 2\pi,$$

this expression has to be constant in s, and thus coincides everywhere with its value 2π at $s = 1$.

If the polygon is not completely inside the chart, we subdivide it finely enough and apply the result for each of the parts. The sum of the integrals over inner edges vanishes, since they are traversed exactly twice and with opposite orientation. With E^o, E^∂, and $E = E^o \cup E^\partial$ we denote the sets of vertices inside, on the boundary, and in total; with K^o, K^∂, and $K = K^o \cup K^\partial$ the sets of the corresponding edges and with F the set of the subpolygons. For each polygon $\Delta \in F$, let E_Δ and K_Δ be the set of vertices and edges of Δ. With β_j^Δ we denote the interior angle of Δ in the vertex $j \in E_\Delta$. This gives us on the other side

$$\begin{aligned}
\sum_{\Delta \in F} \Big(2\pi - \sum_{j \in E_\Delta} (\pi - \beta_j^\Delta) \Big) &= 2\pi\,|F| - \pi \sum_{\Delta \in F} |E_\Delta| + \sum_{\Delta \in F} \sum_{j \in E_\Delta} \beta_j^\Delta \\
&= 2\pi\,|F| - \pi(2\,|K| - |K^\partial|) + \pi\,(2\,|E| - |E^\partial|) - \sum_i \alpha_i \\
&= 2\,\pi\,(|E| - |K| + |F|) - \sum_i \alpha_i \overset{[95,\ 29.24]}{=\!=\!=} 2\,\pi - \sum_i \alpha_i
\end{aligned}$$

and thus the general formula, because $|E_\Delta| = |K_\Delta|$, $|K^\partial| = |E^\partial|$,

$$\sum_{\Delta \in F} |E_\Delta| = \sum_{\Delta \in F} |K_\Delta| = 2\,|K^o| + |K^\partial| = 2\,|K| - |K^\partial|)$$

and

$$\sum_{\Delta \in F} \sum_{j \in E_\Delta} \beta_j^\Delta = \pi\,(2\,|E^o| + |E^\partial|) - \sum_i \alpha_i = \pi\,(2\,|E| - |E^\partial|) - \sum_i \alpha_i. \quad \square$$

11.11 Theorem of Minding.

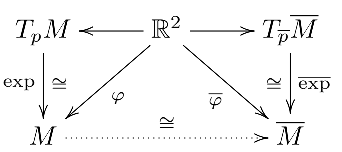
Riemann surfaces of constant Gaussian curvature are locally isometric.

Proof. We choose geodesic polar coordinates in points $p \in M$ and $\overline{p} \in \overline{M}$ using 10.9. Then (see the proof of 11.1):

$$\begin{aligned}
G &= \langle \varphi_\vartheta, \varphi_\vartheta \rangle, & \overline{G} &= \langle \bar\varphi_\vartheta, \bar\varphi_\vartheta \rangle, \\
G(0,\vartheta) &= 0, & \overline{G}(0,\vartheta) &= 0, \\
\tfrac{\partial}{\partial r}\big|_{r=0} \sqrt{G}(r,\vartheta) &= 1, & \tfrac{\partial}{\partial r}\big|_{r=0} \sqrt{\overline{G}}(r,\vartheta) &= 1.
\end{aligned}$$

Both $\sqrt{G}(.,\vartheta)$ and $\sqrt{\overline{G}}(.,\vartheta)$ are solutions γ of the Jacobi equation 9.17

$$\tfrac{\partial^2}{\partial r^2}\gamma(r) = -K \cdot \gamma(r).$$



The solution is uniquely determined by specifying $\gamma(0) = 0$ and $\gamma'(0) = 1$. So the Riemann metrics in geodesic polar coordinates have the same coefficients, and so the surfaces are locally isometric. $\square$

We are now trying to give a global version of the above proposition. But in general this can not be true. For example, the plane and the cylinder are locally but not globally isometric.

11.12 Definition (Geodesic completeness).

A Riemannian manifold M is called GEODESICALLY COMPLETE if all geodesics are infinitely long. Since for the length of a geodesic $c : [a, b] \to M$ parameterized by arc length the following holds $L(c) = \int_a^b |c'(t)|dt = \int_a^b 1dt = b - a$. So the geodesic is infinitely long if and only if its parameter interval is all of $\mathbb{R}$.

11.13 Proposition.

Each two abstract simply connected geodesically complete Riemann surfaces with the same constant Gaussian curvature are isometrically isomorphic.

Proof. Because of the simple connectedness this can be shown using the local result 11.11 together with 13.12.4, see [**12**, 11.2.1 S.407]. □

11.14 Corollary.

Each simply connected geodesically complete abstract Riemann surface with $K = 1$ is up to isometries the sphere by 11.13; Each one with $K = -1$ is up to isometries the Poincaré half-plane (or hyperbolic disk) and each one with $K = 0$ is the plane. Compare this with the Riemann Mapping Theorem 3.1.

By passing to the universal covering, it follows that each Riemann surface is the orbital space of a group of conformal mappings acting discretely on a simply connected Riemann surface. Compare this with the Uniformisation Theorem 3.4.

The only non-trivial discrete group on the sphere is that generated by the antipodal mapping. So there are only two such surfaces with constant Gaussian curvature $K > 0$, namely the sphere and the projective plane. The geometry of the projective plane is also called elliptic, in it all geodesics are closed.

There are only the following discrete groups acting on the plane:

1. *those generated by a translation,*
2. *those generated by a translation composed with a reflection,*
3. *those generated by two translations,*
4. *those generated by a translation together with a translation composed with a reflection.*

Thus, in the case of $K = 0$, there is a metric with vanishing Gaussian curvature only on the cylinder, the Möbius strip, the torus, and the Klein bottle.

On the other hand, every compact surface of the genus $g \geq 2$ has a metric with constant negative curvature, see [**12**, 11.2.5 S.409].

By [**32**] *and* [**62**] *(see* [**75**, 6.2.8 S.105]*), two isometric surfaces in $\mathbb{R}^3$ of strictly positive Gaussian curvature are identical up to a motion.*

11.15 Theorem (Surfaces in $\mathbb{R}^3$ of constant curvature).

Let M be a closed connected surface in $\mathbb{R}^3$ with constant Gaussian curvature K.

1. *If $K > 0$, then M is a sphere* [**110**].
2. *If $K = 0$, then M is a generalized cylinder* [**111**], [**60**].

3. If $K < 0$, then M does not exist [63].
This has been generalized by [38] to: There is no closed surface with a Gaussian curvature bounded above by a constant $k < 0$.

Without proof.

11.16 Lemma.

If M is a compact surface in $\mathbb{R}^3$, then there is a point where the Gaussian curvature is positive.

Proof. This is obvious since the surface has positive curvature at each point of contact with a sphere with minimal radius containing the surface. □

11.17 Corollary.

There is no compact minimal surface. □

12. Parallel transport

Next, we try, while walking along some curve c in a surface, to keep a tangent vector as parallel as possible, i.e. changing its direction as little as possible.

12.1 Definition (Parallel vector field).

A vector field along a curve c is called PARALLEL if its scalar velocity is pointwise minimal.

12.2 Lemma (Characterization of parallel vector fields).

A vector field w along a curve c on M is parallel if and only if $w'(t) \in T_{c(t)}M^\perp$ for all t.

Proof. From $w(t) \in T_{c(t)}M$ it follows that $\langle w(t), v(t)\rangle = 0$, with $v(t) := \nu(c(t))$. If we differentiate this equation in t, we get

$$\langle w'(t), v(t)\rangle + \langle w(t), v'(t)\rangle = 0.$$

In order for

$$|w'(t)|^2 = |w'(t) - \langle w'(t), v(t)\rangle v(t)|^2 + |\langle w'(t), v(t)\rangle|^2$$

to be minimal with given normal part $\langle w'(t), v(t)\rangle = -\langle w(t), v'(t)\rangle$, the tangential part $w'(t) - \langle w'(t), v(t)\rangle\, v(t)$ must be as small as possible, preferably 0. This is exactly the case, when $w'(t) = \langle w'(t), v(t)\rangle v(t)$, i.e. $w'(t) \in T_{c(t)}M^\perp$. □

In particular we have:

12.3 Corollary.

A curve c parameterized proportional to arc length is a geodesic if and only if the vector field c' is parallel along c.

12.4 Parallel vector fields in local coordinates.

Let φ be a local parameterization of M and $t \mapsto u(t)$ the local representation of a curve $c = \varphi \circ u$. The differential equation for parallel vector field $t \mapsto w(t)$ along c is determined as follows (compare with 10.7):

$$\begin{aligned} w(t) &= \sum_{i=1}^{m} w^i(t) \cdot (\partial_i \varphi)(u(t)) \Rightarrow \\ \Rightarrow \quad w' &= \sum_i \tfrac{dw^i}{dt} \cdot \varphi_i + \sum_{i,j} w^i \cdot \varphi_{i,j} \tfrac{du^j}{dt} \\ &\overset{10.5}{=\!=} \sum_i \tfrac{dw^i}{dt} \cdot \varphi_i + \sum_{i,j,k} \tfrac{du^j}{dt} w^i \cdot \Big(\Gamma^k_{i,j} \cdot \varphi_k + \langle \varphi_{i,j} | \nu \rangle \, \nu \Big) \in \\ &\in \sum_k \Big(\tfrac{dw^k}{dt} + \sum_{i,j} w^i \tfrac{du^j}{dt} \Gamma^k_{i,j} \Big) \varphi_k + \mathbb{R} \cdot \nu. \end{aligned}$$

Hence w is parallel along c if and only if

$$\tfrac{dw^k}{dt}(t) + \sum_{i,j=1}^{m} \Gamma^k_{i,j}(u(t)) \cdot w^i(t) \cdot \tfrac{du^j}{dt}(t) = 0 \text{ for } k = 1, \dots, m.$$

or for short

$$\dot{w}^k + \sum_{i,j} \Gamma^k_{i,j} w^i \dot{u}^j = 0 \text{ for } k = 1, \dots, m.$$

This system of ordinary linear differential equations has a unique global solution for given initial data $w^k(0)$.

12.5 Lemma (Existence of parallel vector fields).

For each smooth curve $c : \mathbb{R} \to M$ and initial vector $w_0 \in T_{c(0)}M$, there exists a uniquely determined parallel curve $w : \mathbb{R} \to TM$ with $w(t) \in T_{c(t)}M$ for all t and $w(0) = w_0$.

With $\operatorname{ptp}(c,t)(v_0)$ we denote the parallel curve v along c with initial value v_0 at time t. This is also called PARALLEL TRANSPORT *along c. It satisfies:*

1. $\operatorname{ptp}(c,t) : T_{c(0)}M \to T_{c(t)}M$ *is a linear isometry*
2. $\operatorname{ptp}(c,t)^{-1} = \operatorname{ptp}(c(\,.+t), -t)$
3. $\operatorname{ptp}(c, g(t)) = \operatorname{ptp}(c \circ g, t) \circ \operatorname{ptp}(c, g(0))$ *for any smooth $g : \mathbb{R} \to \mathbb{R}$.*

Proof.

1 Clearly, the solution $\operatorname{ptp}(c, .)(v_0)$ of a linear differential equation is linearly dependent on the initial value v_0. Moreover,

$$\langle \xi, \eta \rangle' = \langle \xi', \eta \rangle + \langle \xi, \eta' \rangle = 0 + 0,$$

holds if ξ and η are parallel vector fields along c, i.e. $\xi', \eta' \in TM^\perp$. So $\langle \xi, \eta \rangle$ is constant and $\operatorname{ptp}(c,t)$ is an isometry.

2 and 3 easily follow from the uniqueness of the solutions of linear differential equations. □

12.6 Example.

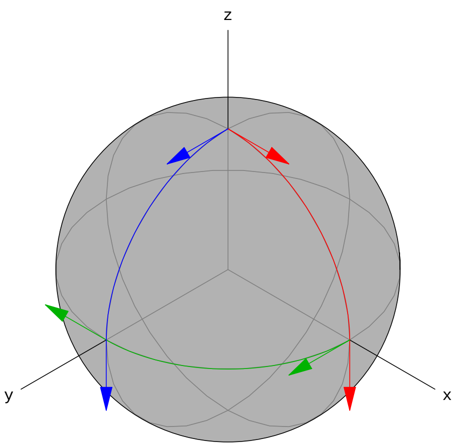
1. In each hyperplane, exactly the constant vector fields are the parallel ones, since the derivative of a vector field tangent to the plane is again in the plane.

2. As we will show in 13.7, being a parallel vector field is an intrinsic property, and therefore, on surfaces with everywhere vanishing Gaussian curvature, they are just the constant vector fields in an unwinding to the plane by 11.11.

   A RULED SURFACE is a surface which can be parametrized locally by $\varphi : (s, \vartheta) \mapsto c(\vartheta) + s\ w(\vartheta)$, i.e. it is spanned by straight lines (called GENERATORS) with (varying) direction vector w along the curve c. For φ to be regular one obviously has to assume that $\varphi_s(s, \vartheta) = w(\vartheta)$ and $\varphi_\vartheta(s, \vartheta) = c'(\vartheta) + s\ w'(\vartheta)$ are linearly independent. If the normal vector along each generator is constant, one speaks of a DEVELOPABLE SURFACE or TORSE.

   By [87, 55.4] the developable surfaces are precisely those ruled surfaces, for which $K = 0$, i.e. those which are locally isometric to the plane by 11.11.

3. Parallel vector fields along closed curves may well have different start and end values: Start on the north pole of the sphere and transport a vector towards a meridian to the equator. Then transport this vector, which is normal to the equator, along the equator to another meridian, and finally transport it along that other meridian back to the north pole. There, the angle between the transported vector and the starting vector is that of the two meridians.



A general possibility for constructing parallel vector fields is the following:

12.7 Proposition (Parallel fields via Schmiegtorse).

If $c : \mathbb{R} \to M$ is a curve on M that admits a Schmiegtorse, then a vector field is parallel along c in M if and only if it is parallel along c in the Schmiegtorse.

Proof. The Schmiegtorse is given locally $\varphi(t, s) = c(s) + t\xi(s)$ by [87, 55.6], where ξ is a vector field along c which is pointwise linearly independent of c'. Clearly, the tangent space of M in $c(t)$ is identical to that of the Schmiegtorse, and thus a vector field η along c is tangential to M if and only if it is tangential to the Schmiegtorse. So, the condition "being parallel" looks the same for both surfaces. □

12.8 Definition (Holonomy).

The subgroup

$$\Big\{ \mathrm{ptp}(c, 2\pi) : c \text{ is closed curve through } x \Big\}$$

of the group $O(T_xM)$ is called the HOLONOMY GROUP of M (at x). It is a Lie group and for different x in a connected M these subgroups are conjugated. For example, the holonomy group of S^2 is just the $S0(2) \cong S^1$ by 12.6.3 and 12.5.1.

We will characterize in 14.12 when this group is trivial.

13. Covariant derivative

Unfortunately, the above descriptions for geodesics and for parallel vector fields make use of the surface normal and thus the surrounding vector space. But these notions should also make sense for abstract Riemannian manifolds.

Instead of saying that a vector like w' is normal to the surface, we can also say that its tangential component, that is, its projection to the tangent space, vanishes. We try to describe this condition intrinsically.

13.1 Definition (Covariant derivative).

Let w be a vector field along a curve c in a hypersurface M. Then let's call the normal projection of the derivative of the vector field to the tangent space the COVARIANT DERIVATIVE ∇ (pronounced "nabla" or "del") and let's denote it by

$$\nabla w : t \mapsto w'(t) - \langle w'(t), \nu_{c(t)} \rangle \nu_{c(t)} \in T_{c(t)}M.$$

Note, that this does not depend on the choice of ν. This measures the infinitesimal change of w as seen in M and ignores the component normal on M.
The formula for the covariant derivative ∇w of a vector field w along a curve $c = \varphi \circ u$ looks in local coordinates by 12.4 as follows:

$$\nabla w = \sum_{k=1}^{m} \Big(\tfrac{dw^k}{dt} + \sum_{i,j}^{m} \Gamma^k_{i,j} w^i \tfrac{du^j}{dt} \Big) \tfrac{\partial}{\partial u^k}, \text{ where } w = \sum_k w^k \tfrac{\partial}{\partial u^k}.$$

Note that the geodesics are exactly the solutions of equation $\nabla c' = 0$ (where c' is to be understood as a vector field along c) and the vector fields w that are parallel along a curve c are exactly the solutions of equation $\nabla w = 0$.
The following formulas hold for ∇:

$$\begin{aligned} \nabla(\xi + \eta) &= \nabla\xi + \nabla\eta \\ \nabla(f \cdot \xi) &= f \cdot \nabla\xi + f' \cdot \xi \text{ for } f \in C^\infty(\mathbb{R}, \mathbb{R}) \\ \langle \xi, \eta \rangle' &= \langle \nabla\xi, \eta \rangle + \langle \xi, \nabla\eta \rangle, \end{aligned}$$

because

$$\begin{aligned} \nabla(f\,\xi) &= (f'\,\xi + f\,\xi') - \langle f'\,\xi + f\,\xi', \nu \rangle\, \nu = f'\,\xi + f\,\nabla\xi - 0, \\ \langle \xi, \eta \rangle' &= \langle \xi', \eta \rangle + \langle \xi, \eta' \rangle = \langle \nabla\xi + \langle \xi', \nu \rangle \nu, \eta \rangle + \langle \xi, \nabla\eta + \langle \eta', \nu \rangle\, \nu \rangle \\ &= \langle \nabla\xi, \eta \rangle + \langle \xi, \nabla\eta \rangle, \quad \text{since } \langle \nu, \eta \rangle = 0 = \langle \xi, \nu \rangle. \end{aligned}$$

13.2 Gauss equation.

For the covariant derivative, the following holds:

$$\nabla w = w' + \langle w, L\,c' \rangle\, \nu \circ c.$$

Proof. The claim immediately follows from $\langle w, \nu \circ c \rangle = 0$ by differentiating. □

13.3 Definition.

Given two vector fields ξ and η on a hypersurface M, we can define $\nabla_\eta \xi \in \mathfrak{X}(M)$ as $(\nabla_\eta \xi)(x) = \nabla(\xi \circ c)(0)$, where c is an integral curve of vector field η with initial condition $c(0) = x$. This can also be written as follows using the notation $\xi'(x) := \mathrm{pr}_2 \circ T_x\xi : T_xM \to \mathbb{R}^n$:

$$(\nabla_\eta \xi)_x = \xi'(x) \cdot \eta_x - \langle \xi'(x) \cdot \eta_x, \nu_x \rangle \nu_x = \xi'(x) \cdot \eta_x + \langle \xi(x), L_x \cdot \eta_x \rangle \nu_x.$$

13.4 Lemma (Properties of the covariant derivative).

The operator ∇ maps from $\mathfrak{X}(M) \times \mathfrak{X}(M)$ to $\mathfrak{X}(M)$ for hypersurfaces M and has the following properties:

1. *∇ is bilinear.*
2. *$\nabla_\eta \xi$ is $C^\infty(M,\mathbb{R})$-linear in η.*
3. *$\nabla_\eta(f\xi) = f\,\nabla_\eta \xi + \eta(f)\,\xi$ for $f \in C^\infty(M,\mathbb{R})$.*
4. *$\nabla_\eta \xi - \nabla_\xi \eta = [\eta, \xi]$.*
5. *$\eta\,\langle \xi_1, \xi_2 \rangle = \langle \nabla_\eta \xi_1, \xi_2 \rangle + \langle \xi_1, \nabla_\eta \xi_2 \rangle$.*

Proof. The operator has values in $\mathfrak{X}(M)$ because $T\xi$ and evaluation are smooth.

1 and 2 are clear.

3:

$$\begin{aligned}(\nabla_\eta(f\xi))(x) &= (f\xi)'(x) \cdot \eta_x - \langle (f\xi)'(x) \cdot \eta_x | \nu_x \rangle \nu_x \\ &= f'(x)(\eta_x) \cdot \xi_x + f(x) \cdot \xi'(x) \cdot \eta_x \\ &\qquad - \Big\langle f'(x)(\eta_x) \cdot \xi_x + f(x) \cdot \xi'(x) \cdot \eta_x, \nu_x \Big\rangle \nu_x \\ &= \eta(f)(x) \cdot \xi_x + f(x) \cdot \xi'(x) \cdot \eta_x - 0 - f(x) \cdot \langle \xi'(x) \cdot \eta_x | \nu_x \rangle \nu_x \\ &= (\eta(f) \cdot \xi)(x) + f(x) \cdot (\nabla_\eta \xi)(x) \\ &= \Big(\eta(f) \cdot \xi + f \cdot \nabla_\eta \xi\Big)(x)\end{aligned}$$

4: Because of the Gauss equation and the symmetry of L we have:

$$\begin{aligned}(\nabla_\eta \xi - \nabla_\xi \eta)(x) &\overset{13.2}{=\!=} \Big(\xi'(x) \cdot \eta_x + \langle \xi_x, L_x \eta_x \rangle \nu_x\Big) - \Big(\eta'(x) \cdot \xi_x + \langle \eta_x, L_x \xi_x \rangle \nu_x\Big) \\ &\overset{[95,\ 17.2]}{=\!=} [\eta, \xi](x) + 0.\end{aligned}$$

5:

$$\begin{aligned}\Big(\langle \nabla_\eta \xi_1, \xi_2 \rangle + \langle \nabla_\eta \xi_2, \xi_2 \rangle\Big)(x) &= \Big\langle \xi_1'(x)(\eta_x) - \langle \xi_1'(x)(\eta_x), \nu_x \rangle \nu_x, \xi_2(x) \Big\rangle \\ &\quad + \Big\langle \xi_2'(x)(\eta_x) - \langle \xi_2'(x)(\eta_x), \nu_x \rangle \nu_x, \xi_1(x) \Big\rangle \\ &= \langle \xi_1'(x)(\eta_x), \xi_2(x) \rangle + \langle \xi_2'(x)(\eta_x), \xi_1(x) \rangle - 0 \\ &= \langle \xi_1, \xi_2 \rangle'(x) \cdot \eta_x = \eta(\langle \xi_1, \xi_2 \rangle)(x). \quad \square\end{aligned}$$

We now want to show that there is such a differential operator ∇ also on abstract Riemannian manifolds, and it is uniquely determined by the above properties.

13.5 Proposition (Levi-Civita derivative).

Let M be an (abstract) Riemannian manifold. Then there is exactly one map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, which satisfies the properties (1) - (5) from 13.4, where *the inner product has been replaced by the Riemann metric in (5). This mapping is called* COVARIANT DERIVATIVE*, or* LEVI-CIVITA CONNECTION*, see* [**95**, 27.19].

(Coordinate-free) Proof. Existence: Because of (5) we have:

$$\begin{aligned} \xi_1\, g(\xi_2,\xi_3) &= g(\nabla_{\xi_1}\xi_2,\xi_3) + g(\xi_2,\nabla_{\xi_1}\xi_3) && (+)\\ \xi_2\, g(\xi_3,\xi_1) &= g(\nabla_{\xi_2}\xi_3,\xi_1) + g(\xi_3,\nabla_{\xi_2}\xi_1) && (+)\\ \xi_3\, g(\xi_1,\xi_2) &= g(\nabla_{\xi_3}\xi_1,\xi_2) + g(\xi_1,\nabla_{\xi_3}\xi_2) && (-). \end{aligned}$$

Hence, by adding the first two and subtracting the third equation using (4) yields:

$$\begin{aligned} &\xi_1\, g(\xi_2,\xi_3) + \xi_2\, g(\xi_3,\xi_1) - \xi_3\, g(\xi_1,\xi_2) =\\ &\quad = g(\nabla_{\xi_1}\xi_2 + \nabla_{\xi_2}\xi_1,\xi_3) + g(\nabla_{\xi_1}\xi_3 - \nabla_{\xi_3}\xi_1,\xi_2) + g(\nabla_{\xi_2}\xi_3 - \nabla_{\xi_3}\xi_2,\xi_1)\\ &\quad \overset{(4)}{=\!=} g\Big(2\nabla_{\xi_1}\xi_2 - [\xi_1,\xi_2],\xi_3\Big) - g([\xi_3,\xi_1],\xi_2) + g([\xi_2,\xi_3],\xi_1). \end{aligned}$$

And thus

$$\begin{aligned} 2\, g(\nabla_{\xi_1}\xi_2,\xi_3) = \xi_1\, g(\xi_2,\xi_3) &+ \xi_2\, g(\xi_3,\xi_1) - \xi_3\, g(\xi_1,\xi_2)\\ &+ g([\xi_1,\xi_2],\xi_3) - g([\xi_2,\xi_3],\xi_1) + g([\xi_3,\xi_1],\xi_2). \end{aligned}$$

Since the right side is linear in ξ_3, $\nabla_{\xi_1}\xi_2$ is well defined by this implicit equation, and since it is also bilinear in (ξ_1,ξ_2), (1) holds.

Now property (2):

$$\begin{aligned} 2\, g(\nabla_{f\xi_1}\xi_2,\xi_3) &= f\xi_1\, g(\xi_2,\xi_3) + \xi_2\, g(\xi_3,f\xi_1) - \xi_3\, g(f\xi_1,\xi_2)\\ &\quad + g([f\xi_1,\xi_2],\xi_3) - g([\xi_2,\xi_3],f\xi_1) + g([\xi_3,f\xi_1],\xi_2)\\ &\overset{[95,\ 17.2.3]}{=\!=\!=} f\xi_1\, g(\xi_2,\xi_3) + f\xi_2\, g(\xi_3,\xi_1) + \xi_2(f) g(\xi_3,\xi_1)\\ &\quad - f\xi_3\, g(\xi_1,\xi_2) - \xi_3(f) g(\xi_1,\xi_2) + g\Big(f[\xi_1,\xi_2] - \xi_2(f)\xi_1,\xi_3\Big)\\ &\quad - f\, g([\xi_2,\xi_3],\xi_1) + g\Big(f[\xi_3,\xi_1] + \xi_3(f)\xi_1,\xi_2\Big)\\ &= f\xi_1\, g(\xi_2,\xi_3) + f\xi_2\, g(\xi_3,\xi_1) + \xi_2(f) g(\xi_3,\xi_1)\\ &\quad - f\xi_3\, g(\xi_1,\xi_2) - \xi_3(f) g(\xi_1,\xi_2)\\ &\quad + f\, g([\xi_1,\xi_2],\xi_3) - \xi_2(f) g(\xi_1,\xi_3)\\ &\quad - f\, g([\xi_2,\xi_3],\xi_1) + f\, g([\xi_3,\xi_1],\xi_2) + \xi_3(f) g(\xi_1,\xi_2)\\ &= f\,\Big(\xi_1\, g(\xi_2,\xi_3) + \xi_2\, g(\xi_3,\xi_1) - \xi_3\, g(\xi_1,\xi_2)\\ &\quad + g([\xi_1,\xi_2],\xi_3) - g([\xi_2,\xi_3],\xi_1) + g([\xi_3,\xi_1],\xi_2)\Big)\\ &= 2\, f\, g(\nabla_{\xi_1}\xi_2,\xi_3). \end{aligned}$$

A very similar calculation shows the property (3).

Next property (4):

$$\begin{aligned} &2\, g(\nabla_{\xi_1}\xi_2 - \nabla_{\xi_2}\xi_1,\xi_3) =\\ &\quad = \xi_1\, g(\xi_2,\xi_3) + \xi_2\, g(\xi_3,\xi_1) - \xi_3\, g(\xi_1,\xi_2)\\ &\qquad + g([\xi_1,\xi_2],\xi_3) - g([\xi_2,\xi_3],\xi_1) + g([\xi_3,\xi_1],\xi_2)\\ &\quad - \xi_2\, g(\xi_1,\xi_3) - \xi_1\, g(\xi_3,\xi_2) + \xi_3\, g(\xi_2,\xi_1)\\ &\qquad - g([\xi_2,\xi_1],\xi_3) + g([\xi_1,\xi_3],\xi_2) - g([\xi_3,\xi_2],\xi_1)\\ &\quad = 2\, g([\xi_1,\xi_2],\xi_3). \end{aligned}$$

Finally property (5):

$$\begin{aligned}
2g(\nabla_{\xi_1}\xi_2,\xi_3)+2g(\xi_2,\nabla_{\xi_1}\xi_3)=\\
&=\xi_1\, g(\xi_2,\xi_3)+\xi_2\, g(\xi_3,\xi_1)-\xi_3\, g(\xi_1,\xi_2)\\
&\quad+g([\xi_1,\xi_2],\xi_3)-g([\xi_2,\xi_3],\xi_1)+g([\xi_3,\xi_1],\xi_2)\\
&\quad+\xi_1\, g(\xi_3,\xi_2)+\xi_3\, g(\xi_2,\xi_1)-\xi_2\, g(\xi_1,\xi_3)\\
&\quad+g([\xi_1,\xi_3],\xi_2)-g([\xi_3,\xi_2],\xi_1)+g([\xi_2,\xi_1],\xi_3)\\
&=2\,\xi_1\, g(\xi_2,\xi_3). \qquad \square
\end{aligned}$$

Coordinate proof. Above all, it has to be shown that the local expression for ∇ from 13.1 is independent on the chosen coordinates and for this we first determine the transformation behavior of the Christoffel symbols:

$$\begin{aligned}
\frac{\partial}{\partial u^i} &:= \varphi_i\\
\frac{\partial}{\partial \bar u^{\bar i}} &= \sum_i \frac{\partial u^i}{\partial \bar u^{\bar i}}\,\frac{\partial}{\partial u^i}\\
g_{i,j} &:= \Big\langle \frac{\partial}{\partial u^i},\frac{\partial}{\partial u^j}\Big\rangle := g\Big(\frac{\partial}{\partial u^i},\frac{\partial}{\partial u^j}\Big)\\
\bar g_{\bar i,\bar j} &= \Big\langle \sum_i \frac{\partial u^i}{\partial \bar u^{\bar i}}\,\frac{\partial}{\partial u^i},\sum_j \frac{\partial u^j}{\partial \bar u^{\bar j}}\,\frac{\partial}{\partial u^j}\Big\rangle = \sum_{i,j}\frac{\partial u^i}{\partial \bar u^{\bar i}}\,\frac{\partial u^j}{\partial \bar u^{\bar j}}\,g_{i,j}\\
\Gamma_{i,j,k} &:= \frac12\Big(\frac{\partial}{\partial u^i}(g_{j,k})+\frac{\partial}{\partial u^j}(g_{i,k})-\frac{\partial}{\partial u^k}(g_{i,j})\Big)\\
\frac{\partial \bar g_{\bar j,\bar k}}{\partial \bar u^{\bar i}} &= \frac{\partial}{\partial \bar u^{\bar i}}\,\Big(\sum_{j,k}\frac{\partial u^j}{\partial \bar u^{\bar j}}\,\frac{\partial u^k}{\partial \bar u^{\bar k}}\,g_{j,k}\Big)\\
&= \sum_{j,k}\Big(\frac{\partial}{\partial \bar u^{\bar i}}\Big(\frac{\partial u^j}{\partial \bar u^{\bar j}}\Big)\,\frac{\partial u^k}{\partial \bar u^{\bar k}}\,g_{j,k}+\frac{\partial u^j}{\partial \bar u^{\bar j}}\,\frac{\partial}{\partial \bar u^{\bar i}}\Big(\frac{\partial u^k}{\partial \bar u^{\bar k}}\Big)\,g_{j,k}+\frac{\partial u^j}{\partial \bar u^{\bar j}}\,\frac{\partial u^k}{\partial \bar u^{\bar k}}\,\frac{\partial}{\partial \bar u^{\bar i}}(g_{j,k})\Big)\\
&= \sum_{j,k}\Big(\frac{\partial^2 u^j}{\partial \bar u^{\bar i}\partial \bar u^{\bar j}}\,\frac{\partial u^k}{\partial \bar u^{\bar k}}\,g_{j,k}+\frac{\partial u^j}{\partial \bar u^{\bar j}}\,\frac{\partial^2 u^k}{\partial \bar u^{\bar i}\partial \bar u^{\bar k}}\,g_{j,k}+\frac{\partial u^j}{\partial \bar u^{\bar j}}\,\frac{\partial u^k}{\partial \bar u^{\bar k}}\,\sum_i\frac{\partial u^i}{\partial \bar u^{\bar i}}\,\frac{\partial g_{j,k}}{\partial u^i}\Big)
\end{aligned}$$

$$\begin{aligned}
\bar\Gamma_{\bar i,\bar j,\bar k} &= \frac12\Big(\frac{\partial}{\partial\bar u^{\bar i}}(g_{\bar j,\bar k}) + \frac{\partial}{\partial \bar u^{\bar j}}(g_{\bar i,\bar k}) - \frac{\partial}{\partial\bar u^{\bar k}}(g_{\bar i,\bar j})\Big)\\
&= \frac12\Big(\sum_{j,k}\Big(\underbrace{\frac{\partial^2 u^j}{\partial\bar u^{\bar i}\partial\bar u^{\bar j}}\,\frac{\partial u^k}{\partial \bar u^{\bar k}}\, g_{j,k}}_{(1)} + \underbrace{\frac{\partial u^j}{\partial\bar u^{\bar j}}\,\frac{\partial^2 u^k}{\partial\bar u^{\bar i}\partial\bar u^{\bar k}}\, g_{j,k}}_{(2)} + \underbrace{\frac{\partial u^j}{\partial\bar u^{\bar j}}\,\frac{\partial u^k}{\partial\bar u^{\bar k}}\,\sum_i \frac{\partial u^i}{\partial\bar u^{\bar i}}\,\frac{\partial g_{j,k}}{\partial u^i}\Big)}_{(3)}\\
&\quad + \sum_{i,k}\Big(\underbrace{\frac{\partial^2 u^i}{\partial\bar u^{\bar j}\partial\bar u^{\bar i}}\,\frac{\partial u^k}{\partial\bar u^{\bar k}}\, g_{i,k}}_{(1)} + \underbrace{\frac{\partial u^i}{\partial \bar u^{\bar i}}\,\frac{\partial^2 u^k}{\partial\bar u^{\bar j}\partial\bar u^{\bar k}}\, g_{i,k}}_{(4)} + \underbrace{\frac{\partial u^i}{\partial\bar u^{\bar i}}\,\frac{\partial u^k}{\partial\bar u^{\bar k}}\,\sum_j\frac{\partial u^j}{\partial\bar u^{\bar j}}\,\frac{\partial g_{i,k}}{\partial u^j}\Big)}_{(3)}\\
&\quad - \sum_{j,i}\Big(\underbrace{\frac{\partial^2 u^j}{\partial\bar u^{\bar k}\partial\bar u^{\bar j}}\,\frac{\partial u^i}{\partial\bar u^{\bar i}}\, g_{j,i}}_{(4)} + \underbrace{\frac{\partial u^j}{\partial\bar u^{\bar j}}\,\frac{\partial^2 u^i}{\partial\bar u^{\bar k}\partial\bar u^{\bar i}}\, g_{j,i}}_{(2)} + \underbrace{\frac{\partial u^j}{\partial\bar u^{\bar j}}\,\frac{\partial u^i}{\partial\bar u^{\bar i}}\,\sum_k \frac{\partial u^k}{\partial\bar u^{\bar k}}\,\frac{\partial g_{j,i}}{\partial u^k}\Big)\Big)}_{(3)}\\
&= \sum_{j,k}\overbrace{\frac{\partial^2 u^j}{\partial\bar u^{\bar i}\partial\bar u^{\bar j}}\,\frac{\partial u^k}{\partial\bar u^{\bar k}}\, g_{j,k}}^{(1)} + \sum_{i,j,k}\overbrace{\frac{\partial u^i}{\partial\bar u^{\bar i}}\,\frac{\partial u^j}{\partial\bar u^{\bar j}}\,\frac{\partial u^k}{\partial\bar u^{\bar k}}\,\frac12\Big(\frac{\partial g_{j,k}}{\partial u^i}+\frac{\partial g_{i,k}}{\partial u^j}-\frac{\partial g_{j,i}}{\partial u^k}\Big)}^{(3)}\\
&= \sum_{j,k}\frac{\partial^2 u^j}{\partial\bar u^{\bar i}\partial\bar u^{\bar j}}\,\frac{\partial u^k}{\partial\bar u^{\bar k}}\, g_{j,k} + \sum_{i,j,k}\frac{\partial u^i}{\partial\bar u^{\bar i}}\,\frac{\partial u^j}{\partial\bar u^{\bar j}}\,\frac{\partial u^k}{\partial\bar u^{\bar k}}\,\Gamma_{i,j,k}
\end{aligned}$$

$$\delta^j_i = \sum_{\bar i}\frac{\partial u^j}{\partial\bar u^{\bar i}}\,\frac{\partial\bar u^{\bar i}}{\partial u^i}$$

$$\delta^i_k =: \sum_j g^{i,j}\, g_{j,k}$$

$$\bar g^{\bar i,\bar j} = \sum_{i,j}\frac{\partial\bar u^{\bar i}}{\partial u^i}\,\frac{\partial\bar u^{\bar j}}{\partial u^j}\;\; g^{i,j}, \text{ denn}$$

$$\begin{aligned}
\sum_{\bar j}\bar g^{\bar i,\bar j}\,\bar g_{\bar j,\bar k} &= \sum_{\bar j}\sum_{i,j}\frac{\partial\bar u^{\bar i}}{\partial u^i}\,\frac{\partial\bar u^{\bar j}}{\partial u^j}\,g^{i,j}\,\sum_{l,k}\frac{\partial u^l}{\partial\bar u^{\bar j}}\,\frac{\partial u^k}{\partial\bar u^{\bar k}}\,g_{l,k}\\
&= \underbrace{\sum_{i,k}\underbrace{\sum_{j,l}\underbrace{\sum_{\bar j}\frac{\partial\bar u^{\bar j}}{\partial u^j}\,\frac{\partial u^l}{\partial\bar u^{\bar j}}}_{\delta^l_j}\,g^{i,j}\,g_{l,k}}_{\delta^i_k}\,\frac{\partial\bar u^{\bar i}}{\partial u^i}\,\frac{\partial u^k}{\partial\bar u^{\bar k}}}_{\delta^{\bar i}_{\bar k}}
\end{aligned}$$

$$
\begin{aligned}
\Gamma^k_{i,j} &:= \sum_l \Gamma_{i,j,l}\, g^{l,k} \\
\bar\Gamma^{\bar l}_{\bar i,\bar j} &= \sum_{\bar k} \bar\Gamma_{\bar i,\bar j,\bar k}\, \bar g^{\bar k,\bar l} \\
&= \sum_{\bar k}\Big(\sum_{j,k} \frac{\partial^2 u^j}{\partial \bar u^{\bar i}\partial \bar u^{\bar j}}\, \frac{\partial u^k}{\partial \bar u^{\bar k}}\, g_{j,k} + \sum_{i,j,k} \frac{\partial u^i}{\partial \bar u^{\bar i}}\, \frac{\partial u^j}{\partial \bar u^{\bar j}}\, \frac{\partial u^k}{\partial \bar u^{\bar k}}\, \Gamma_{i,j,k}\Big) \sum_{p,q} \frac{\partial \bar u^{\bar k}}{\partial u^p}\, \frac{\partial \bar u^{\bar l}}{\partial u^q}\, g^{p,q} \\
&= \sum_{j,q} \frac{\partial^2 u^j}{\partial \bar u^{\bar i}\partial \bar u^{\bar j}}\, \frac{\partial \bar u^{\bar l}}{\partial u^q}\, \underbrace{\sum_{k,p} \underbrace{\sum_{\bar k} \frac{\partial u^k}{\partial \bar u^{\bar k}}\, \frac{\partial \bar u^{\bar k}}{\partial u^p}}_{=\delta^k_p}\, g_{j,k} g^{p,q}}_{=\delta^q_j} \\
&\quad + \sum_{i,j,q} \frac{\partial u^i}{\partial \bar u^{\bar i}}\, \frac{\partial u^j}{\partial \bar u^{\bar j}}\, \frac{\partial \bar u^{\bar l}}{\partial u^q}\, \underbrace{\sum_{p,k} \underbrace{\sum_{\bar k} \frac{\partial u^k}{\partial \bar u^{\bar k}}\, \frac{\partial \bar u^{\bar k}}{\partial u^p}}_{=\delta^k_p}\, \Gamma_{i,j,k} g^{p,q}}_{=\Gamma^q_{i,j}} \\
&= \sum_{i,j,l} \frac{\partial u^i}{\partial \bar u^{\bar i}}\, \frac{\partial u^j}{\partial \bar u^{\bar j}}\, \frac{\partial \bar u^{\bar l}}{\partial u^l}\, \Gamma^l_{i,j} + \sum_l \frac{\partial^2 u^l}{\partial \bar u^{\bar i}\partial \bar u^{\bar j}}\, \frac{\partial \bar u^{\bar l}}{\partial u^l} \\
\delta^{\bar l}_{\bar i} &= \frac{\partial \bar u^{\bar l}}{\partial \bar u^{\bar i}} = \sum_l \frac{\partial \bar u^{\bar l}}{\partial u^l}\, \frac{\partial u^l}{\partial \bar u^{\bar i}} \qquad \Rightarrow \\
0 &= \frac{\partial}{\partial \bar u^{\bar j}} \delta^{\bar l}_{\bar i} \\
&= \sum_l \Big(\frac{\partial}{\partial \bar u^{\bar j}} \Big(\frac{\partial \bar u^{\bar l}}{\partial u^l}\Big)\, \frac{\partial u^l}{\partial \bar u^{\bar i}} + \frac{\partial \bar u^{\bar l}}{\partial u^l}\, \frac{\partial^2 u^l}{\partial \bar u^{\bar i}\partial \bar u^{\bar j}} \Big) \\
&= \sum_{l,j} \frac{\partial u^j}{\partial \bar u^{\bar j}}\, \frac{\partial}{\partial u^j}\Big(\frac{\partial \bar u^{\bar l}}{\partial u^l}\Big)\, \frac{\partial u^l}{\partial \bar u^{\bar i}} + \sum_l \frac{\partial \bar u^{\bar l}}{\partial u^l}\, \frac{\partial^2 u^l}{\partial \bar u^{\bar i}\partial \bar u^{\bar j}} \\
&= \sum_{i,j} \frac{\partial^2 \bar u^{\bar l}}{\partial u^i \partial u^j}\, \frac{\partial u^j}{\partial \bar u^{\bar j}}\, \frac{\partial u^i}{\partial \bar u^{\bar i}} + \sum_l \frac{\partial^2 u^l}{\partial \bar u^{\bar i}\partial \bar u^{\bar j}}\, \frac{\partial \bar u^{\bar l}}{\partial u^l} \qquad \Rightarrow \\
\bar\Gamma^{\bar l}_{\bar i,\bar j} &= \sum_{i,j,l} \frac{\partial u^i}{\partial \bar u^{\bar i}}\, \frac{\partial u^j}{\partial \bar u^{\bar j}}\, \frac{\partial \bar u^{\bar l}}{\partial u^l}\, \Gamma^l_{i,j} + \sum_l \frac{\partial^2 u^l}{\partial \bar u^{\bar i}\partial \bar u^{\bar j}}\, \frac{\partial \bar u^{\bar l}}{\partial u^l} \\
&= \sum_{i,j,l} \frac{\partial u^i}{\partial \bar u^{\bar i}}\, \frac{\partial u^j}{\partial \bar u^{\bar j}}\, \frac{\partial \bar u^{\bar l}}{\partial u^l}\, \Gamma^l_{i,j} - \sum_{i,j} \frac{\partial^2 \bar u^{\bar l}}{\partial u^i \partial u^j}\, \frac{\partial u^i}{\partial \bar u^{\bar i}}\, \frac{\partial u^j}{\partial \bar u^{\bar j}}
\end{aligned}
$$

Now we can give a coordinate definition of ∇w for vector fields $w : \mathbb{R} \to TM$ along curves $c : \mathbb{R} \to M$: In local coordinates, let

$$
\begin{aligned}
c(t) &= \varphi(u^1(t), \dots, u^m(t)) = \bar\varphi(\bar u^1(t), \dots, \bar u^m(t)) \\
w(t) &= \sum_i w^i(t)\, \frac{\partial}{\partial u^i} = \sum_{\bar i} \bar w^{\bar i}(t)\, \frac{\partial}{\partial \bar u^{\bar i}} = \sum_{\bar i, i} \bar w^{\bar i}(t)\, \frac{\partial u^i}{\partial \bar u^{\bar i}}\, \frac{\partial}{\partial u^i}
\end{aligned}
$$

and coefficient comparison yields

$$
w^i(t) = \sum_{\bar i} \bar w^{\bar i}(t)\, \frac{\partial u^i}{\partial \bar u^{\bar i}}.
$$

For hypersurfaces M the normal projection of the derivative of w to the tangent space is given in the local coordinates $(u^1,\dots,u^m)$ according to 13.1 by

$$\sum_i\left(\frac{dw^i(t)}{dt}+\sum_{j,k}\Gamma^i_{j,k}(u(t))w^j(t)\frac{du^k(t)}{dt}\right)\frac{\partial}{\partial u^i}$$

or in the coordinates $(\bar u^1,\dots,\bar u^m)$ by

$$\begin{aligned}
&\sum_{\bar i}\left(\frac{d\bar w^{\bar i}(t)}{dt}+\sum_{\bar j,\bar k}\Gamma^{\bar i}_{\bar j,\bar k}(u(t))\bar w^{\bar j}(t)\frac{d\bar u^{\bar k}(t)}{dt}\right)\frac{\partial}{\partial \bar u^{\bar i}}=\\
&=\sum_{\bar i}\left(\frac{d}{dt}\sum_i\frac{\partial\bar u^{\bar i}}{\partial u^i}(u(t))\,w^i(t)+\right.\\
&\qquad+\sum_{\bar j,\bar k}\Big(\sum_{j,k,i}\Gamma^i_{j,k}(u(t))\frac{\partial u^j}{\partial\bar u^{\bar j}}\,\frac{\partial u^k}{\partial\bar u^{\bar k}}\,\frac{\partial\bar u^{\bar i}}{\partial u^i}-\sum_{j,k}\frac{\partial^2\bar u^{\bar i}}{\partial u^j\partial u^k}\,\frac{\partial u^j}{\partial\bar u^{\bar j}}\,\frac{\partial u^k}{\partial\bar u^{\bar k}}\Big)\cdot\\
&\qquad\qquad\left.\cdot\sum_l\frac{\partial\bar u^{\bar j}}{\partial u^l}\,w^l(t)\frac{d\bar u^{\bar k}(t)}{dt}\right)\frac{\partial}{\partial\bar u^{\bar i}}\\
&=\sum_{\bar i}\left(\sum_{i,j}\frac{\partial^2\bar u^{\bar i}}{\partial u^i\partial u^j}(u(t))\,\frac{du^j(t)}{dt}\,w^i(t)+\sum_i\frac{\partial\bar u^{\bar i}}{\partial u^i}(u(t))\,\frac{dw^i(t)}{dt}+\right.\\
&\qquad+\sum_{\bar j,\bar k}\Big(\sum_{j,k,i}\Gamma^i_{j,k}(u(t))\frac{\partial u^j}{\partial\bar u^{\bar j}}\,\frac{\partial u^k}{\partial\bar u^{\bar k}}\,\frac{\partial\bar u^{\bar i}}{\partial u^i}-\sum_{j,k}\frac{\partial^2\bar u^{\bar i}}{\partial u^j\partial u^k}\,\frac{\partial u^j}{\partial\bar u^{\bar j}}\,\frac{\partial u^k}{\partial\bar u^{\bar k}}\Big)\cdot\\
&\qquad\qquad\left.\cdot\sum_l\frac{\partial\bar u^{\bar j}}{\partial u^l}\,w^l(t)\frac{d\bar u^{\bar k}(t)}{dt}\right)\frac{\partial}{\partial\bar u^{\bar i}}\\
&=\sum_{\bar i}\left(\sum_{i,j}\frac{\partial^2\bar u^{\bar i}}{\partial u^i\partial u^j}\,\frac{du^j}{dt}\,w^i+\sum_i\frac{\partial\bar u^{\bar i}}{\partial u^i}(u(t))\,\frac{dw^i(t)}{dt}\right.\\
&\qquad+\sum_{k,i}\sum_{l,j}\Gamma^i_{j,k}\underbrace{\sum_{\bar j}\frac{\partial u^j}{\partial\bar u^{\bar j}}\,\frac{\partial\bar u^{\bar j}}{\partial u^l}}_{\delta^j_l}\,w^l\,\underbrace{\sum_{\bar k}\frac{\partial u^k}{\partial\bar u^{\bar k}}\,\frac{d\bar u^{\bar k}}{dt}}_{\frac{du^k}{dt}}\,\frac{\partial\bar u^{\bar i}}{\partial u^i}\\
&\qquad\left.-\sum_k\underbrace{\sum_{\bar k}\frac{\partial u^k}{\partial\bar u^{\bar k}}\,\frac{d\bar u^{\bar k}}{dt}}_{\frac{du^k}{dt}}\,\sum_{j,l}\frac{\partial^2\bar u^{\bar i}}{\partial u^j\partial u^k}\,\underbrace{\sum_{\bar j}\frac{\partial u^j}{\partial\bar u^{\bar j}}\,\frac{\partial\bar u^{\bar j}}{\partial u^l}}_{\delta^j_l}\,w^l\right)\frac{\partial}{\partial\bar u^{\bar i}}\\
&=\sum_i\left(\frac{dw^i(t)}{dt}+\sum_{j,k}\Gamma^i_{j,k}(u(t))w^j(t)\frac{du^k(t)}{dt}\right)\frac{\partial}{\partial u^i}
\end{aligned}$$

so this expression is also well-defined for an abstract Riemannian manifold. We call this expression the covariant derivative ∇w of a vector field w along a curve c, i.e.

$$\nabla w(t):=\sum_i\left(\frac{dw^i(t)}{dt}+\sum_{j,k}\Gamma^i_{j,k}(u(t))w^j(t)\frac{du^k(t)}{dt}\right)\frac{\partial}{\partial u^i}\Big|_{u(t)}.$$

For vector fields X and Y on Riemannian manifolds M, we can now define a vector field $\nabla_X Y$ by

$$(\nabla_X Y)_x:=\nabla(Y\circ c)$$

as in 13.3, where c is the integral curve of X through x. Thus, $\nabla : \mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M)$ is a well-defined bilinear mapping given in coordinates by

$$\nabla_X Y = \sum_i \Bigl(\sum_k \frac{\partial Y^i}{\partial u^k} X^k + \sum_{j,k} \Gamma^i_{j,k} Y^j X^k \Bigr) \frac{\partial}{\partial u^i}. \quad \square$$

13.6 Local Formulas for ∇.

If we choose basis vector fields $g_i := \frac{\partial}{\partial u^i}$, $g_j := \frac{\partial}{\partial u^j}$ and $g_k := \frac{\partial}{\partial u^k}$ for ξ_1, ξ_2 and ξ_3, we obtain a local formula for the covariant derivative ∇:

$$2g(\nabla_{g_i} g_j, g_k) \overset{13.5}{=\!=} \tfrac{\partial}{\partial u^i} g_{j,k} + \tfrac{\partial}{\partial u^j} g_{k,i} - \tfrac{\partial}{\partial u^k} g_{i,j} + 0 =: 2\Gamma_{i,j,k}$$

For hypersurfaces, this is the formula in 10.5 for the CHRISTOFFEL SYMBOLS OF THE 1ST KIND.
Let us denote the coefficients of $\nabla_{g_i} g_j$ with respect to the basis (g_l) with $\Gamma^l_{i,j}$, thus

$$\nabla_{g_i} g_j =: \sum_{k=1}^m \Gamma^k_{i,j} g_k,$$

so we obtain:

$$\Gamma_{i,j,k} := g(\nabla_{g_i} g_j, g_k) = g\left(\sum_l \Gamma^l_{i,j} g_l, g_k\right) = \sum_l \Gamma^l_{i,j} g_{l,k},$$

i.e. the $\Gamma^l_{i,j}$ are (for hypersurfaces) the CHRISTOFFEL SYMBOLS OF THE 2ND KIND. Note that from the symmetry of $g_{i,j}$ the following inverse formula for the partial derivatives of the coefficients of the Riemann metric follows:

$$\tfrac{\partial}{\partial u^i} g_{j,k} = \Gamma_{i,j,k} + \Gamma_{i,k,j}.$$

Because of property 13.4.2, $\nabla_X Y$ is tensorial in X, that is, $(\nabla_X Y)(p)$ depends only on X_p and Y: Namely, if $X = 0$ is local to p and $f \in C^\infty(M, \mathbb{R})$ with $f(p) = 1$ and $f \cdot X = 0$, then $0 = (\nabla_{fX} Y)(p) \overset{(2)}{=\!=} f(p) \cdot (\nabla_X Y)(p) = (\nabla_X Y)(p)$; and if only $X_p = 0$ is assumed, then

$$(\nabla_X Y)(p) = (\nabla_{\sum_i X(u^i) \frac{\partial}{\partial u^i}} Y)(p) \overset{(2)}{=\!=} \sum_i X_p(u^i) \cdot (\nabla_{\frac{\partial}{\partial u^i}} Y)(p) = 0.$$

Similarly, $\nabla_X Y$ is local in Y, since $\nabla_X(fY) = f\,\nabla_X Y + X(f)\,Y$. Let $c : \mathbb{R} \to M$ be a curve with $c'(0) = X_p$. Then $(\nabla_{c'(0)} Y)(p)$ is well defined and is given in local coordinates by

$$\begin{aligned}
(\nabla_{c'(0)} Y)(p) &= \nabla_{\sum_j \frac{d(u^j\circ c)}{dt}(0)\cdot \frac{\partial}{\partial u^j}(p)} \Bigl(\sum_i Y^i \cdot \frac{\partial}{\partial u^i}\Bigr)(p) \\
&\overset{(2),(3)}{=\!=} \sum_{j,i} \frac{d(u^j \circ c)}{dt}(0) \cdot \Bigl(\frac{\partial}{\partial u^j}\Big|_p Y^i \cdot \frac{\partial}{\partial u^i}\Big|_p + Y^i(p) \cdot \nabla_{\frac{\partial}{\partial u^j}(p)} \frac{\partial}{\partial u^i}(p) \Bigr) \\
&= \sum_i \sum_j \frac{\partial}{\partial u^j}\Big|_p Y^i \cdot \frac{d(u^j \circ c)}{dt}(0) \cdot \frac{\partial}{\partial u^i}\Big|_p \\
&\qquad + \sum_{i,j,k} \frac{d(u^j \circ c)}{dt}(0)\; Y^i(p)\; \Gamma^k_{j,i}|_p \; \frac{\partial}{\partial u^k}\Big|_p
\end{aligned}$$

However, the right side even makes sense if Y is just a vector field along c, that is, $Y(t) \in T_{c(t)}M$ for all $t \in \mathbb{R}$, and $p = c(t)$, because by the chain rule we have

$$\sum_j \frac{\partial}{\partial u^j}\Big|_{c(t)} Y^i \cdot \frac{d(u^j \circ c)}{dt}(t) = \frac{d(Y^i \circ c)}{dt}(t),$$

and thus the covariant derivative also exists for a vector field Y along a curve c:

$$(\nabla_{c'(t)}Y)(t) = \sum_i \Big(\frac{dY^i}{dt}(t) + \sum_{j,k} \frac{d(u^j \circ c)}{dt}(t)\, Y^k(t)\, \Gamma^i_{j,k}(c(t)) \Big) \frac{\partial}{\partial u^i}\Big|_{c(t)}.$$

13.7 Remark.

If we express the corresponding differential equations (see 13.1)

- $0 = \nabla_{c'}Y$, for parallel vector fields Y along curves c, and
- $0 = \nabla_{c'}c'$, for geodesics c,

in local coordinates using 13.6, then they are (see 12.4 and 10.7)

$$0 = \frac{dY^i}{dt}(t) + \sum_{j,k} \frac{d(u^j \circ c)}{dt}(t)\, Y^k(t)\, \Gamma^i_{j,k}(c(t)) \quad \forall\, i$$

$$0 = \frac{d^2(u^i \circ c)}{dt^2}(t) + \sum_{j,k} \frac{d(u^j \circ c)}{dt}(t)\, \frac{d(u^k \circ c)}{dt}(t)\, \Gamma^i_{j,k}(c(t)) \quad \forall\, i.$$

Their respective solutions - the parallel transport $\mathrm{ptp} : C^\infty(\mathbb{R}, M) \times_M TM \to C^\infty(\mathbb{R}, TM)$ and the exponential map $\exp : TM \to M$ - thus also exist for abstract Riemannian manifolds and have the corresponding properties (see 12.5 and 10.8). That geodesics, even on abstract Riemannian manifolds, are exactly the solutions of the corresponding variational problems, will be shown in 15.15.1 and 15.16.

13.8 Lemma.

The mapping $(\pi, \exp) : TM \to M \times M$ is a diffeomorphism of a neighborhood U of the zero-section $M \subseteq TM$ onto a neighborhood of the diagonal $\{(x, x) : x \in M\} \subseteq M \times M$.

Proof. Note first, that the tangent space of TM at a point 0_x of the zero section is just $T_xM \oplus T_xM$ (see [**95**, 27.18]): The first factor is given by the tangential vectors to curves in $M \subseteq TM$ and the second by velocity vectors of curves in the fibre $T_xM \subseteq TM$. These two subspaces have a trivial intersection (because $\pi \circ c$ is constant for the latter curves c), and together give the correct dimension $\dim(TM) = 2\dim(M)$.

Now we calculate the partial derivatives of $(\pi, \exp)$. On the zero section, $(\pi, \exp) : TM \supseteq M \to M \times M$ is just the diagonal mapping $x \mapsto 0_x \mapsto (x, x)$, and on the fiber T_xM of π we have that $(\pi, \exp) : TM \supseteq T_xM \to M \times M$ is the mapping $(\mathrm{konst}_x, \exp_x)$. So the tangential mapping of $(\pi, \exp)$ to 0_x looks like this:

$$T_{0_x}(\pi, \exp) = \begin{pmatrix} \mathrm{id} & 0 \\ \mathrm{id} & T_0 \exp_x \end{pmatrix} : T_xM \oplus T_xM \to T_xM \oplus T_xM.$$

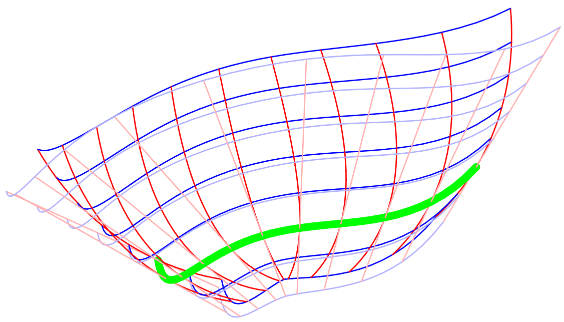
Because of $T_0 \exp_x = \mathrm{id}_{T_xM}$, the mapping $(\pi, \exp)$ is a local diffeomorphism for points close to zero cross section.

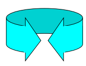
For each $x \in M$, we choose an open neighborhood U_x of 0_x in TM such that $(\pi, \exp) : U_x \to (\pi, \exp)(U_x)$ is a diffeomorphism and the fibers $U_x \cap T_yM$ are balls around 0_y. The union $U := \bigcup_{x \in M} U_x$ is then an open neighborhood of the zero section in TM and $(\pi, \exp) : U \to V := (\pi, \exp)(U)$ is a local diffeomorphism and hence its image is open.

Remains to show injectivity: If two tangent vectors have different base points, then they are separated by the first component $\pi : TM \to M$, and if they have the same base point $x \in M$, they are separated by the second component of $\exp_x$ since they are contained in balls $U_y \cap T_xM \subseteq T_xM$ (on which $\exp_y$ is injective), hence are contained in the larger of the two. □

13.9 Tubular neighborhood.

Let $M \subseteq N$ be a submanifold of the Riemannian manifold N. With $TM^\perp$ we denote the normal bundle of M in N, i.e. the vector bundle over M, which has as fiber over $x \in M$ the orthogonal complement $(T_xM)^\perp$ of T_xM in T_xN with respect to the Riemann metric of N. Then $\exp_N$ is a diffeomorphism from an open neighborhood of zero section $M \subseteq TM^\perp$ to an open neighborhood of M in N. The images of sections of constant length intersect the radial geodesics orthogonally.





Proof.

Similar to the proof of Theorem 13.8, we may decompose the tangent space of $TM^\perp \subseteq TN|_M$ at a point 0_x of the zero section as $T_xM \oplus T_xM^\perp = T_xN$ (see [**95**, 27.18]). And the tangential mapping of $\exp_N : TN \supseteq TN|_M \supseteq TM^\perp \to N$ at 0_x is

$$T_{0_x}(\exp_N|_{TM^\perp}) = \mathrm{id}_{T_xM} \oplus T_0 \exp_x|_{(T_xM)^\perp} = \mathrm{id}_{T_xN} : T_xM \oplus T_xM^\perp \to T_xN$$

So $\exp_N|_{TM^\perp} : TM^\perp \to N$ is a local diffeomorphism near the zero section $M \subseteq TM^\perp$, and the global injectivity can also be shown as in the proof of Theorem 13.8.

The statement about orthogonality of the sections of constant length follows from the following Gaussian lemma 13.10 and generalizes 10.10. □

If one chooses $M := \{x\} \subseteq N$ and in T_xN an orthonormal basis, these are the so-called Riemannian normal coordinates, see 14.9. Another special case, which is sometimes considered, is that of a double-point free curve $c : \mathbb{R} \to M$ parameterized by arc length, cf. with 10.10.

In 11.12, we have denoted a Riemannian manifold (M, g) as geodesically complete, if each geodesic has infinite length, or equivalently, is defined on all of $\mathbb{R}$. Proposition 13.12 will now provide the connection with the completeness in the sense of the metric, as we used it in Proposition 1.5. We need some preparation for that.

13.10 Gaussian lemma of Riemannian geometry.

Let M be a Riemannian manifold and $p \in M$. Let $v \in T_pM$ and $q := \exp_p(v)$. Then

$$g_q(T_v \exp_p \cdot v, T_v \exp_p \cdot w) = g_p(v, w) \;\forall\, w \in T_pM.$$

This is a generalization of 10.9 (where $v = w$ is). It tells us, that the exponential mapping is a "radial isometry": Note that only for $w_1 = v$ we have $g_q(T_v \exp_p \cdot w_1, T_v \exp_p \cdot w_2) = g_p(w_1, w_2)$.

Proof. We split w into the part $w^\perp$ which is normal to v and the part $w^\top$ which is parallel to v.

For the latter, $w^\top = r\,v$ for some $r \in \mathbb{R}$, and thus

$$\begin{aligned} T_v \exp_p \cdot w^\top &= \frac{d}{dt}\Big|_{t=0} \exp_p(v + t\,w^\top) = \frac{d}{dt}\Big|_{t=0} \exp_p((1 + t\,r)\,v) \\ &= r\,\frac{d}{ds}\Big|_{s=1} \exp_p(s\,v) = r\,T_v \exp_p \cdot v. \end{aligned}$$

Hence

$$\begin{aligned} g_q(T_v \exp_p \cdot v, T_v \exp_p \cdot w^\top) &= g_q(T_v \exp_p \cdot v, r\,T_v \exp_p \cdot v) \\ &= r\,g_q(T_v \exp_p \cdot v, T_v \exp_p \cdot v) \\ &\overset{10.9}{=\!=} r\,g_p(v, v) = g_p(v, r\,v) = g_p(v, w^\top). \end{aligned}$$

For $w^\perp$ we consider the functions $f(t, s) := t\,v + t\,s\,w^\perp$ and $\varphi := \exp_p \circ f$. Thus

$$\begin{aligned} T_v \exp_p \cdot v &= \frac{\partial}{\partial t}\Big|_{t=1} (\exp_p \circ f)(t, 0) = \partial_1\varphi(1, 0) \quad \text{and} \\ T_v \exp_p \cdot w^\perp &= \frac{\partial}{\partial s}\Big|_{s=0} (\exp_p \circ f)(1, s) = \partial_2\varphi(1, 0) \end{aligned}$$

and hence

$$g_q\big(T_v \exp_p \cdot v, T_v \exp_p \cdot w^\perp\big) = g_p\Big(\frac{\partial}{\partial t}\varphi(t, s), \frac{\partial}{\partial s}\varphi(t, s)\Big)\Big|_{t=1,s=0}.$$

For $t = 0$ we get $\frac{\partial}{\partial s}\big|_{s=0}\varphi(0, s) = \frac{\partial}{\partial s}\big|_{s=0} \exp_p(0) = 0$. Since $t \mapsto \exp_p(t\,(v + sw)) = \varphi(t, s)$ are geodesics parameterized proportional to arc length, the following holds:

$$\begin{aligned} \frac{\partial}{\partial t} g\left(\frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial s}\right)\Big|_{s=0} &\overset{13.4.5}{=\!=} g\Big(\underbrace{\nabla_{\frac{\partial\varphi}{\partial t}} \frac{\partial\varphi}{\partial t}\Big|_{s=0}}_{=0}, \frac{\partial\varphi}{\partial s}\Big|_{s=0}\Big) + g\Big(\frac{\partial\varphi}{\partial t}, \nabla_{\frac{\partial\varphi}{\partial t}} \frac{\partial\varphi}{\partial s}\Big)\Big|_{s=0} \\ &\overset{\text{see below}}{=\!=} g\Big(\frac{\partial\varphi}{\partial t}, \nabla_{\frac{\partial\varphi}{\partial s}} \frac{\partial\varphi}{\partial t}\Big)\Big|_{s=0} \overset{13.4.5}{=\!=} \frac{1}{2}\frac{\partial}{\partial s}\Big|_{s=0} \underbrace{g\Big(\frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial t}\Big)}_{=g(v+sw,v+sw)} = 0, \end{aligned}$$

because

$$\frac{\partial\varphi}{\partial s} = \sum_j \frac{\partial\varphi^j}{\partial s}\frac{\partial}{\partial u^j} \quad\text{and}\quad \frac{\partial\varphi}{\partial t} = \sum_k \frac{\partial\varphi^k}{\partial t}\frac{\partial}{\partial u^k}$$

$$\nabla_{\frac{\partial\varphi}{\partial t}}\frac{\partial\varphi}{\partial s} = \nabla_{\frac{\partial\varphi}{\partial t}}\Big(\sum_j \frac{\partial\varphi^j}{\partial s}\frac{\partial}{\partial u^j}\Big) \overset{\boxed{13.4.3}}{=\!=} \sum_j\Big(\frac{\partial^2\varphi^j}{\partial t\partial s}\frac{\partial}{\partial u^j} + \frac{\partial\varphi^j}{\partial s}\underbrace{\nabla_{\frac{\partial\varphi}{\partial t}}\frac{\partial}{\partial u^j}}_{\sum_k \frac{\partial\varphi^k}{\partial t}\nabla_{\frac{\partial}{\partial u^k}}\frac{\partial}{\partial u^j}}\Big)$$

$$\nabla_{\frac{\partial\varphi}{\partial t}}\frac{\partial\varphi}{\partial s} - \nabla_{\frac{\partial\varphi}{\partial s}}\frac{\partial\varphi}{\partial t} = \sum_{j,k}\frac{\partial\varphi^j}{\partial s}\frac{\partial\varphi^k}{\partial t}\underbrace{\Big(\nabla_{\frac{\partial}{\partial u^k}}\frac{\partial}{\partial u^j} - \nabla_{\frac{\partial}{\partial u^j}}\frac{\partial}{\partial u^k}\Big)}_{\overset{\boxed{13.4.4}}{=\!=}\big[\frac{\partial}{\partial u^k},\frac{\partial}{\partial u^j}\big]=0} = 0. \quad\square$$

Lemma.

Let $\varphi : X \to Y$ be continuous with $\varphi(X)$ closed and $U \subseteq X$ with $\varphi(U)$ open. Then each continuous curve c in Y, which leaves $\varphi(X)$, meets $\varphi(X \setminus U)$.

Proof. Suppose, this were not the case, i.e. there is a curve $c : [0,1] \to Y$ with $c(0) \in \varphi(X)$, $c(1) \notin \varphi(X)$, but $c(t) \notin \varphi(X \setminus U)$ for all t.
Consider the inverse images with respect to c of the open disjoint sets $\varphi(U)$ and $Y \setminus \varphi(X)$. They form a partition of $[0,1]$ in non-empty open sets, a contradiction to the connectedness of $[0,1]$: In fact, $0 \in c^{-1}(\varphi(X)) = c^{-1}(\varphi(U) \cup \varphi(X \setminus U)) = c^{-1}(\varphi(U)) \cup \emptyset$, $1 \in c^{-1}(Y \setminus \varphi(X))$, and if $t \notin c^{-1}(\varphi(U))$, i.e. $c(t) \notin \varphi(U)$, then $c(t) \notin \varphi(U) \cup \varphi(X \setminus U) = \varphi(X)$. $\square$

We will apply this lemma in the situation where φ is the exponential mapping $\exp_p$ of a Riemannian manifold Y for sme $p \in Y$, and $U \subseteq T_pY$ is a sufficiently small open ball and X its closure.

13.11 Corollary.

Let M be a Riemannian manifold, $p \in M$ and $\varepsilon > 0$ so small that the mapping $\exp_p : V := \{v \in T_pM : \|v\| < \varepsilon\} \to M$ is a diffeomorphism onto its image.
Then for each curve $c : [a,b] \to M$ with $c(a) = \exp_p(v_a)$ and $c(b) = \exp_p(v_b)$ for $v_a, v_b \in V$, we have $L(c) \geq \big|\|v_b\| - \|v_a\|\big|$, where equality holds if and only if c is a parameterization of a radial geodesic, i.e. $c(t) = \exp_t(r(t)\,v)$ for all t with monotonous r and $\|v\| = 1$.
Hence $d(p, \exp_p(v)) = \|v\|$ for all $v \in V$ and $\exp_p(V) = \{q \in M : d(p,q) < \varepsilon\}$.

This generalizes 10.11.

Proof. By construction, $t \mapsto \exp_p(t\,v)$ is a geodesic with scalar speed $\|v\|$, hence $\exp_p(V) \subseteq \{q : d(p,q) < \varepsilon\}$.

For the time being, let $c : [a,b] \to M$ be a curve in $\exp_p(V) \setminus \{p\}$, so $\exp_p^{-1}(c(t)) =: r(t)\,v(t)$ with $0 < r(t) < \varepsilon$ and $\|v(t)\| = 1$, hence $v(t) \perp v'(t)$.
With $\gamma(r,t) := \exp_x(r\,v(t))$ we have

$$c(t) = \gamma(r(t),t) \quad\Rightarrow\quad c'(t) = \partial_1\gamma(r(t),t)\cdot r'(t) + \partial_2\gamma(r(t),t),$$
$$\text{where } \partial_1\gamma = T_{r\,v(t)}\exp\cdot v(t) \text{ and } \partial_2\gamma = T_{r\,v(t)}\exp\cdot r v'(t), \quad\Rightarrow$$

$$\|c'\|^2 = g\big(\partial_1\gamma\cdot r' + \partial_2\gamma, \partial_1\gamma\cdot r' + \partial_2\gamma\big) \overset{\boxed{13.10}}{=\!=} |r'|^2\|\partial_1\gamma\|^2 + \|\partial_2\gamma\|^2$$
$$= |r'|^2 + \|\partial_2\gamma\|^2 \geq |r'|^2$$

and equality holds if and only if $\partial_2\gamma = 0$, i.e. v is constant. Thus

$$L(c) = \int_a^b \|c'\| \geq \int_a^b |r'| \geq \Big|\int_a^b r'\Big| = |r(b) - r(a)| = \big|\|v_b\| - \|v_a\|\big|,$$

where equality only holds if r is monotone and v is constant, i.e. c is a radial geodesic.

If, on the other hand, c leaves the set $\exp(V)\setminus\{p\}$, then its inverse image under $\exp_p$ leaves the set $V \setminus \{0\}$ and thus has as length at least that of the part of the inverse image which lies in the annulus $\{v : \min\{\|v_a\|, \|v_b\|\} \leq \|v\| \leq \max\{\|v_a\|, \|v_b\|\}\}$, i.e. has length $\geq \big|\|v_b\| - \|v_a\|\big|$: W.l.o.g. $0 < \|v_a\| < \|v_b\|$. Let $t_1 := \inf\{t : c(t) \in \exp_p(\{v : \|v\| = \|v_b\|\})$, i.e. $c(t) = \exp_p(v(t))$ with $\|v(t)\| < \|v_b\|$ for all $t < t_1$. and let $t_0 := \max\{t \leq t_1 : c(t) \in \exp_p(\{v : \|v\| = \|v_a\|\})\}$. Now consider $c|_{[t_0,t_1]}$.

Suppose there is a $q \in M \setminus \exp_p(V)$ with $d(p, q) < \varepsilon$. Let c be a curve in M with $c(0) = p$, $c(1) = q$ and $L(c) < \varepsilon$. For $t_1 := \inf\{t : c(t) \in \exp_p(\{v : \|v\| = L(c)\})\}$ we have $c(t_1) = \exp_p(v_b)$ for some v_b with $\|v_b\| = L(c)$. Because of $t_1 < 1$, we have $L(c) > L(c|_{[0,t_1]}) \geq \|v_b\| = L(c)$, a contradiction. □

13.12 Theorem of Hopf-Rinow.

For a Riemannian manifold the following three statements are equivalent:

1. *M is geodesically complete.*
2. *M is complete as a metric space, i.e. each Cauchy sequence converges.*
3. *Each closed set, which is bounded in the metric, is compact.*

Furthermore, it follows from these equivalent statements that:

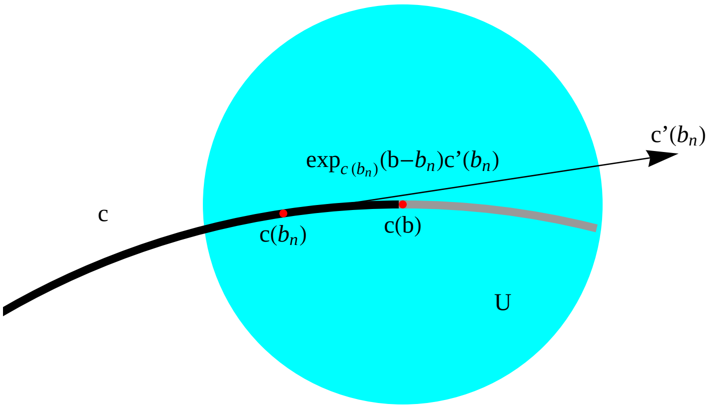
4. *Two points in the same connected component can be connected by a geodesic of minimal length.*

Proof. (3 ⇒ 2) This is a general proposition from topology, because according to Cantor's theorem (see [**80**, 3.1.4]), it suffices to prove the principle of nested intervals: So let $A_n \neq \emptyset$ be closed and monotonically falling (i.e. $A_n \supseteq A_{n+1}$) with $d(A_n) := \sup\{d(x, y) : x, y \in A_n\} \to 0$. By condition 3, A_n is compact (if $d(A_n) < \infty$) and thus $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$.

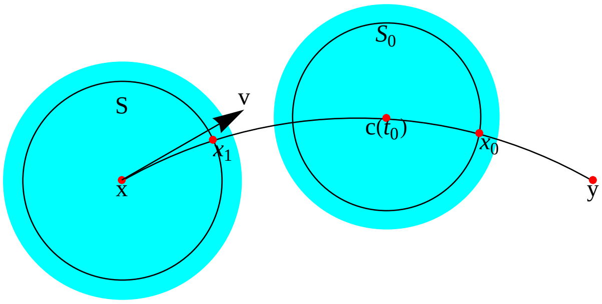
(2 ⇒ 1) Let c be a geodesic parameterized by arc length and $]a, b[$ its maximal domain. W.l.o.g let $b < +\infty$ and consider a sequence $b_n \nearrow b$, then $c(b_n)$ is a Cauchy sequence, because

$$d(c(t_1), c(t_2)) \leq L(c|_{[t_1,t_2]}) = |t_2 - t_1|.$$

By 2 there exists $\lim_{n\to\infty} c(b_n) =: c(b)$. From 13.8 we know that a neighborhood U of $c(b)$ exists and a $\rho > 0$, such that $\exp_x$ is defined for all $x \in U$ and all vectors of length less than ρ. Now we choose n so large that $b - b_n < \rho$ and $c(b_n) \in U$. Then the geodesic with initial velocity $c'(b_n) \in T_{c(b_n)}M$ is defined for all $|t| < \rho$, i.e. c extends beyond b, a contradiction.



(1 ⇒ 4) W.l.o.g. $r := d(x,y) > 0$. We choose $0 < \rho < r$, such that $\exp_x : B_\rho(0) := \{v : g_x(v,v) < \rho^2\} \to M$ is a diffeomorphism onto its image. Let $0 < \rho_1 < \rho$ and $S := \exp_x(\partial B_{\rho_1}(0))$. Since S is compact, an $x_1 \in S$ exists with $d(x_1, y)$ minimal. Let $v \in T_xM$ be defined by $\|v\| = 1$ and $x_1 := \exp_x(\rho_1 v)$.
We **claim** that $\exp_x(rv) = y$, thus $c : t \mapsto \exp_x(tv)$ is a geodesic from x to y with length $r = d(x,y)$, which is minimal.



Suppose this were not the case. Then

$$t_0 := \inf\Big\{t \in [\rho_1, r] : d(c(t), y) \neq r - t\Big\} < r.$$

Since the set of t with $d(c(t),y) \neq r-t$ is open, equality holds for $t := t_0$. Obviously $t_0 > \rho_1$, because every curve from x to y meets the set S and thus the following holds:

$$r = d(x,y) = \min_{s\in S}\big(d(x,s) + d(s,y)\big) \overset{13.11}{=\!=} \rho_1 + d(x_1,y) = \rho_1 + d(c(\rho_1),y).$$

Let S_0 be a geodesic sphere around $c(t_0)$ with radius $\rho_0 < r - t_0$, where we have chosen ρ_0 with $d(c(t_0+\rho_0),y) \neq r - (t_0+\rho_0)$ so small that $\exp_z$ on $\{v \in T_zM : \|v\| < 2\rho_0\}$ is a diffeomorphism for all z with $d(z, c(t_0)) < \rho_0$. Let x_0 be a point on S_0 with minimal distance from y, and let c_0 be the radial geodesic from $c(t_0)$ to x_0. As before

$$d(c(t_0),y) = d(c_0(0),y) = \min_{s\in S_0}\big(d(c_0(0),s) + d(s,y)\big) = \rho_0 + d(x_0,y)$$

and thus $d(x_0,y) = d(c(t_0),y) - \rho_0 = (r - t_0) - \rho_0$. Furthermore,

$$d(x,x_0) \geq d(x,y) - d(x_0,y) = r - (r - t_0 - \rho_0) = t_0 + \rho_0,$$

and the curve $\tilde c$ consisting of the geodesic $c|_{[0,t_0]}$ followed by the geodesic $c_0|_{[0,\rho_0]}$ has length $t_0 + \rho_0$, thus $d(x,x_0) = t_0 + \rho_0$.

For $t_0 - \rho_0 < t_- < t_0 < t_+ := t_0 + \rho_0$ we have

$$t_+ - t_- = L(\tilde c|_{[t_-,t_+]}) \ge d(\tilde c(t_-), \tilde c(t_+)) \ge d(x, x_0) - d(x, \tilde c(t_-)) - d(\tilde c(t_+), x_0)$$
$$\ge (t_0 + \rho_0) - t_- - (\rho_0 - (t_+ - t_0)) = t_+ - t_-,$$

so $\tilde c|_{[t_-,t_+]}$ is a curve of minimal length and thus, by 13.11, is a geodesic and hence $\tilde c|_{[0,t_+]}$ is an extension of $c|_{[0,t_0]}$ and therefore identical to $c|_{[0,t_+]}$, a contradiction, because $c(t_0+\rho_0) = c_0(\rho_0) = x_0$ and hence $r - t_0 - \rho_0 = d(x_0, y) = d(c(t_0+\rho_0), y) \ne r - (t_0 + \rho_0)$.

(1 $\Rightarrow$ 3) Let $A \subseteq M$ be closed and bounded, i.e.

$$\sup\{d(x_0, x_1) : x_0, x_1 \in A\} =: r < \infty.$$

By 4, $A \subseteq \exp_{x_0}\{B_r(0)\} =: B$ for choosen $x_0 \in A$, and B is compact as a continuous image of the compact set $B_r(0)$, so also A is compact. □

13.13 Corollary.

Let (M, g) be a complete Riemannian manifold and let $\xi \in \mathfrak{X}(M)$ be a vector field bounded with respect to g. Then ξ is complete, i.e. has a global flow.

Proof. If $\|\xi(x)\|_g \le R$ for all $x \in M$ and c is a solution curve of ξ, then

$$L(c|_{[a,b]}) = \int_a^b \|c'(t)\|_g \, dt = \int_a^b \|\xi(c(t))\|_g \, dt \le |b - a| \, R$$

So c stays at finite intervals within a bounded, and because of completeness, compact set. This is a contradiction to [**95**, 16.3]. □

13.14 Theorem of Nomitzu-Ozeki.

For each Riemann metric, there is always a conformal equivalent one which is geodesically complete.

Proof. Let (M, g) be a (w.l.o.g. connected) Riemannian manifold, and d the metric associated to g. Let again $B_r(x) := \{y \in M : d(x, y) \le r\}$. Then we put

$$r(x) := \sup\{\rho > 0 : B_\rho(x) \text{ is compact}\} \in (0, +\infty].$$

The triangle inequality for d implies $B_\rho(x_2) \subseteq B_{\rho+d(x_1,x_2)}(x_1)$ and thus $|r(x_1) - r(x_2)| \le d(x_1, x_2)$, so r is continuous. Note, that if $r(x) = +\infty$ for some x, so also for all other $x \in M$, and thus each closed bounded set is compact, thus M is complete by 13.12. We may therefore assume that $r(M) \subseteq \mathbb{R}$. Now, using a partition of unity, we choose a smooth function $f : M \to \mathbb{R}$ with $f(x) \ge \frac{1}{r(x)}$ for all $x \in M$ and consider the conforming equivalent metric $\tilde g := f^2 \, g$.

It remains to show that $\tilde g$ is complete. For this it suffices to prove the inclusion $B^{\tilde g}_{1/3}(x) \subseteq B^g_{r(x)/2}(x)$ for all x, because then, due to the proof of ($3 \Rightarrow 2 \Rightarrow 1$) in 13.12 and the compactness of $B^{\tilde g}_{1/3}(x)$, each geodesic (starting at x) with respect to $\tilde g$ has at least length $\frac{1}{3}$, and thus by pasting together has infinite length. Thus $\tilde g$ is complete.

Let $y \notin B^g_{r(x)/2}(x)$ and $c : [a, b] \to M$ be a smooth curve from x to y, then $L^g(c) = \int_a^b \|c'(t)\|_g \, dt \ge d(x, y) > \frac{r(x)}{2}$ and

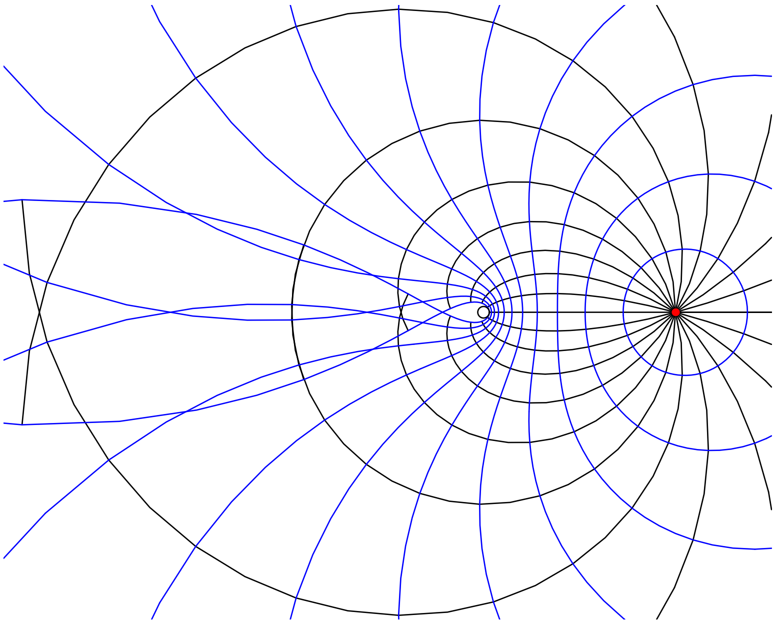
$$L^{\tilde g}(c) = \int_a^b \|c'(t)\|_{\tilde g} \, dt = \int_a^b f(c(t)) \, \|c'(t)\|_g \, dt = \quad \text{(intermediate value theorem)}$$
$$= f(c(\tau)) \int_a^b \|c'(t)\|_g \, dt = f(c(\tau)) \, L^g(c) \ge \frac{L^g(c)}{r(c(\tau))}.$$

Because of $|r(x)-r(c(\tau))| \le d(x,c(\tau)) \le L^g(c)$ we have $r(c(\tau)) \le r(x)+L^g(c)$ and thus

$$L^{\tilde g}(c) \ge \frac{L^g(c)}{r(c(\tau))} \ge \frac{L^g(c)}{r(x)+L^g(c)} > \frac{L^g(c)}{2L^g(c)+L^g(c)} = \frac{1}{3} \qquad \square$$

13.15 Example.

Let $M := \mathbb{R}^2 \setminus \{0\}$. Then M with the Euclidean metric g is not complete (consider antipodal points) and r from the proof of 13.13 is given by $z \mapsto |z|$. The exponential map at one point (e.g., $(1,0)$) in the conformaly equivalent complete metric $\tilde g := f^2 g$ with $f(z) := 1/|z|$ looks like follows:



Note that $h : \mathbb{R}^2 \to \mathbb{R}^2 \setminus \{0\}$, $(t,s) \mapsto e^t(\cos(s),\sin(s))$ is a local isometry from the standard metric to the metric $\tilde g$, thus induces an isometry between the cylinder and $(\mathbb{R}^2 \setminus \{0\}, \tilde g)$. In fact,

$$\begin{aligned} h^*\tilde g_{(t,s)}(v,w) &= \tilde g_{h(t,s)}\big(h'(t,s)\cdot v, h'(t,s)w\big) = \frac{1}{e^{2t}}\big\langle h'(t,s)\cdot v, h'(t,s)w\big\rangle \\ &= \frac{1}{e^{2t}}\Big\langle e^t\begin{pmatrix}\cos(s)\\ \sin(s)\end{pmatrix}v^1 + e^t\begin{pmatrix}-\sin(s)\\ \cos(s)\end{pmatrix}v^2, \\ &\qquad e^t\begin{pmatrix}\cos(s)\\ \sin(s)\end{pmatrix}w^1 + e^t\begin{pmatrix}-\sin(s)\\ \cos(s)\end{pmatrix}w^2\Big\rangle \\ &= \langle v,w\rangle \end{aligned}$$

13.16 Lemma (Divergence via covariant derivative).

Let ξ be a vector field on an oriented Riemannian manifold M. Then

$$\operatorname{div}\xi = \operatorname{trace}(\eta \mapsto \nabla_\eta \xi).$$

Proof. For the divergence from 4.5, which we obtained from the outer derivative by applying the Hodge-Star operator, and which we have also described by means of the Lie derivative of the volume form, the following local formula holds according to 4.5:

$$\operatorname{div}\xi = \frac{1}{\sqrt{G}}\sum_{i=1}^m \frac{\partial}{\partial u^i}\Big(\sqrt{G}\,\xi^i\Big) = \sum_{i=1}^m\Big(\xi^i\,\frac{1}{2G}\frac{\partial}{\partial u^i}G + \frac{\partial}{\partial u^i}\xi^i\Big).$$

For $\nabla\xi$ we have the local formula for $\xi = \sum_i \xi^i g_i$ with respect to $g_i := \frac{\partial}{\partial u^i}$:

$$\begin{aligned}\nabla_{g_i}\xi &\overset{13.4.3}{=\!=} \sum_{j=1}^m\Big(\xi^j\nabla_{g_i}g_j + g_i(\xi^j)g_j\Big) \overset{13.6}{=\!=} \sum_{j=1}^m\Big(\xi^j\sum_{k=1}^m\Gamma^k_{i,j}g_k + g_i(\xi^j)g_j\Big)\\ &= \sum_{j=1}^m\Big(\sum_{k=1}^m\xi^k\Gamma^j_{i,k} + \frac{\partial}{\partial u^i}\xi^j\Big)g_j.\end{aligned}$$

For the trace of $\eta\mapsto\nabla_\eta\xi$ we get

$$\operatorname{trace}\big(\eta\mapsto\nabla_\eta\xi\big) = \sum_{i=1}^m\left(\sum_{k=1}^m\xi^k\Gamma^i_{i,k} + \frac{\partial}{\partial u^i}\xi^i\right) = \sum_{i=1}^m\left(\sum_{k=1}^m\xi^i\Gamma^k_{k,i} + \frac{\partial}{\partial u^i}\xi^i\right).$$

Because of $G := \det((g_{j,k})_{j,k})$ and $\det'(A)(B) = \det(A)\cdot\operatorname{trace}(A^{-1}B)$ we finally obtain:

$$\begin{aligned}\frac{1}{2G}\frac{\partial}{\partial u^i}G &= \frac{1}{2G}G\operatorname{trace}\Big((g^{j,k})_{j,k}\cdot(\tfrac{\partial}{\partial u^i}g_{j,k})_{j,k}\Big) = \frac12\operatorname{trace}\Big(\Big(\sum_l g^{j,l}\cdot\tfrac{\partial}{\partial u^i}g_{l,k}\Big)_{j,k}\Big)\\ &\overset{13.6}{=\!=} \frac12\sum_{k,l}g^{k,l}\,(\Gamma_{i,l,k}+\Gamma_{i,k,l}) \overset{13.6}{=\!=} \sum_k\Gamma^k_{i,k} = \sum_k\Gamma^k_{k,i}. \qquad\square\end{aligned}$$

13.17 Remark.

The Levi-Civita derivative discussed in this section is the most important special case of general covariant derivatives, as described in [**95**, 27.19].

14. Curvatures of Riemannian manifolds

Let two vector fields ξ and η on $\mathbb{R}^n$ be given. Then, for the usual derivative of the vector field ξ in the direction η, which we want to denote here also by $D_\eta\xi : x\mapsto\xi'(x)(\eta(x))$ we have:

$$[D_\xi, D_\eta] := D_\xi\circ D_\eta - D_\eta\circ D_\xi = D_{[\xi,\eta]},$$

because

$$\begin{aligned}(D_\xi\circ D_\eta - D_\eta\circ D_\xi)\Big((\zeta^i)_{i=1}^n\Big) &= \big(\xi(\eta(\zeta^i)) - \eta(\xi(\zeta^i))\big)_{i=1}^n = ([\xi,\eta]\zeta^i)_{i=1}^n\\ &= D_{[\xi,\eta]}\big((\zeta^i)_{i=1}^n\big)\end{aligned}$$

14.1 Theorem (Godazzi-Mainardi equation).

Let M be a hypersurface in $\mathbb{R}^n$ and let ξ, η, ζ be vector fields on $\mathbb{R}^n$ which are tangential to M along M. Then on M one has:

1. Gauss equation: $\Big([\nabla_\xi,\nabla_\eta] - \nabla_{[\xi,\eta]}\Big)\zeta = \langle L\eta,\zeta\rangle L\xi - \langle L\xi,\zeta\rangle L\eta,$
2. Godazzi-Mainardi equation: $\nabla_\xi L\eta - \nabla_\eta L\xi = L[\xi,\eta].$

Proof. Because of $\langle\zeta,\nu\rangle = 0$ we have

$$\nabla_\eta\zeta = D_\eta\zeta - \langle D_\eta\zeta,\nu\rangle\nu = D_\eta\zeta + \langle\zeta, D_\eta\nu\rangle\nu = D_\eta\zeta + \langle\zeta, L\eta\rangle\nu$$

(compare with 13.2) and thus because of the preliminary remark

$$\begin{aligned}
0 &= [D_\xi, D_\eta]\zeta - D_{[\xi,\eta]}\zeta \\
&= D_\xi\Big(\nabla_\eta\zeta - \langle L\,\eta,\zeta\rangle\nu\Big) - D_\eta\Big(\nabla_\xi\zeta - \langle L\,\xi,\zeta\rangle\nu\Big) - D_{[\xi,\eta]}\zeta \\
&= D_\xi\nabla_\eta\zeta - \xi(\langle L\,\eta,\zeta\rangle)\nu - \langle L\eta,\zeta\rangle D_\xi\nu \\
&\quad - D_\eta\nabla_\xi\zeta + \eta(\langle L\,\xi,\zeta\rangle)\nu + \langle L\xi,\zeta\rangle D_\eta\nu \\
&\quad - D_{[\xi,\eta]}\zeta \\
&= \nabla_\xi\nabla_\eta\zeta - \langle L\,\xi,\nabla_\eta\zeta\rangle\nu - \xi(\langle L\,\eta,\zeta\rangle)\nu - \langle L\,\eta,\zeta\rangle L\,\xi \\
&\quad - \nabla_\eta\nabla_\xi\zeta + \langle L\,\eta,\nabla_\xi\zeta\rangle\nu + \eta(\langle L\,\xi,\zeta\rangle)\nu + \langle L\,\xi,\zeta\rangle L\,\eta \\
&\quad - \nabla_{[\xi,\eta]}\zeta + \langle L\,[\xi,\eta],\zeta\rangle\nu.
\end{aligned}$$

The tangential part of this is the Gauss equation:

$$0 = [\nabla_\xi,\nabla_\eta]\zeta - \nabla_{[\xi,\eta]}\zeta - \langle L\,\eta,\zeta\rangle L\,\xi + \langle L\,\xi,\zeta\rangle L\,\eta$$

And the normal part is the Godazzi-Mainardi equation:

$$\begin{aligned}
0 &= -\langle\nabla_\eta\zeta, L\,\xi\rangle - \xi\langle L\,\eta,\zeta\rangle + \langle\nabla_\xi\zeta, L\,\eta\rangle + \eta\langle L\,\xi,\zeta\rangle + \langle L\,[\xi,\eta],\zeta\rangle \\
&= \big\langle -\nabla_\xi(L\,\eta) + \nabla_\eta(L\,\xi) + L\,[\xi,\eta],\zeta\big\rangle. \qquad \square
\end{aligned}$$

14.2 Definition (Riemann curvature).

The Riemann curvature $R : \mathfrak{X}(M)\times\mathfrak{X}(M) \to L(\mathfrak{X}(M),\mathfrak{X}(M))$ of a Riemannian manifold is defined by the left side of the Gauss equation 14.1.1:

$$R(\xi,\eta) := [\nabla_\xi,\nabla_\eta] - \nabla_{[\xi,\eta]}.$$

The motivation for this is that for Riemann surfaces, the right hand side of the Gauss equation 14.1.1 applied to $\zeta := \eta$ and taken in the inner product with ξ yields for orthonormal vectors ξ and η precisely the Gaussian curvature:

$$\big\langle\langle L\,\eta,\eta\rangle L\,\xi - \langle L\,\xi,\eta\rangle L\,\eta,\xi\big\rangle = \langle L\,\eta,\eta\rangle\langle L\,\xi,\xi\rangle - \langle L\,\xi,\eta\rangle\langle L\,\eta,\xi\rangle = \det(L) = K.$$

14.3 Lemma (The Riemann curvature is a tensor field).

The Riemann curvature is a 3-fold co-and 1-fold contravariant tensor field on M, i.e. $R \in \Gamma(T^*M\otimes T^*M\otimes T^*M\otimes TM)$.

(Coordinate free) proof. For this one only has to show that the map $(\xi,\eta,\zeta)\mapsto R(\xi,\eta)(\zeta)$ is in all variables $C^\infty(M,\mathbb{R})$-homogeneous, cf. with the proof of [**95**, 19.10].

$$\begin{aligned}
R(f\xi,\eta) &= [\nabla_{f\xi},\nabla_\eta] - \nabla_{[f\xi,\eta]} \overset{[95,\,17.2.3]}{=\!=} [f\nabla_\xi,\nabla_\eta] - \nabla_{f[\xi,\eta]-\eta(f)\xi} \\
&\overset{13.4.3}{=\!=} (f\nabla_\xi)\nabla_\eta - \nabla_\eta(f\nabla_\xi) - f\nabla_{[\xi,\eta]} + \eta(f)\nabla_\xi \\
&\overset{13.4.3}{=\!=} f\nabla_\xi\nabla_\eta - f\nabla_\eta\nabla_\xi - \eta(f)\nabla_\xi - f\nabla_{[\xi,\eta]} + \eta(f)\nabla_\xi \\
&= f\,\big([\nabla_\xi,\nabla_\eta] - \nabla_{[\xi,\eta]}\big) + 0 = f\,R(\xi,\eta).
\end{aligned}$$

$$
\begin{aligned}
R(\xi,\eta)(f\zeta) &= \big([\nabla_\xi,\nabla_\eta]-\nabla_{[\xi,\eta]}\big)(f\zeta)\\
&\overset{13.4.3}{=\!=\!=} \nabla_\xi\big(f\nabla_\eta\zeta+\eta(f)\zeta\big)-\nabla_\eta\big(f\nabla_\xi\zeta+\xi(f)\zeta\big)-f\nabla_{[\xi,\eta]}\zeta-[\xi,\eta](f)\zeta\\
&\overset{13.4.3}{=\!=\!=} f\nabla_\xi\nabla_\eta\zeta+\xi(f)\nabla_\eta\zeta+\eta(f)\nabla_\xi\zeta+\xi(\eta(f))\zeta\\
&\qquad -f\nabla_\eta\nabla_\xi\zeta-\eta(f)\nabla_\xi\zeta-\xi(f)\nabla_\eta\zeta-\eta(\xi(f))\zeta\\
&\qquad -f\nabla_{[\xi,\eta]}\zeta-\xi(\eta(f))\zeta+\eta(\xi(f))\zeta\\
&= f\left(\nabla_\xi\nabla_\eta-\nabla_\eta\nabla_\xi-\nabla_{[\xi,\eta]}\right)\zeta = f\,R(\xi,\eta)(\zeta). \qquad \square
\end{aligned}
$$

Coordinate proof.

$$
\begin{aligned}
R(X,Y)Z &= \nabla_X(\nabla_Y Z)-\nabla_Y(\nabla_X Z)-\nabla_{[X,Y]}Z\\
&\overset{13.5}{=\!=\!=} \nabla_X\left(\sum_i\left(\sum_k\frac{\partial Z^i}{\partial u^k}Y^k+\sum_{j,k}\Gamma^i_{j,k}Z^jY^k\right)\frac{\partial}{\partial u^i}\right)\\
&\qquad -\nabla_Y\left(\sum_i\left(\sum_k\frac{\partial Z^i}{\partial u^k}X^k+\sum_{j,k}\Gamma^i_{j,k}Z^jX^k\right)\frac{\partial}{\partial u^i}\right)\\
&\qquad -\nabla_{\sum_{i,j}\left(X^i\frac{\partial Y^j}{\partial u^i}-Y^i\frac{\partial X^j}{\partial u^i}\right)\frac{\partial}{\partial u^j}}Z\\
&\overset{13.4.3}{=\!=\!=} \sum_i\left(\sum_k\frac{\partial Z^i}{\partial u^k}Y^k+\sum_{j,k}\Gamma^i_{j,k}Z^jY^k\right)\nabla_X\frac{\partial}{\partial u^i}\\
&\qquad +X\left(\sum_i\left(\sum_k\frac{\partial Z^i}{\partial u^k}Y^k+\sum_{j,k}\Gamma^i_{j,k}Z^jY^k\right)\right)\frac{\partial}{\partial u^i}\\
&\qquad -\sum_i\left(\sum_k\frac{\partial Z^i}{\partial u^k}X^k+\sum_{j,k}\Gamma^i_{j,k}Z^jX^k\right)\nabla_Y\frac{\partial}{\partial u^i}\\
&\qquad -Y\left(\sum_i\left(\sum_k\frac{\partial Z^i}{\partial u^k}X^k+\sum_{j,k}\Gamma^i_{j,k}Z^jX^k\right)\right)\frac{\partial}{\partial u^i}\\
&\qquad -\sum_{i,j}\left(X^i\frac{\partial Y^j}{\partial u^i}-Y^i\frac{\partial X^j}{\partial u^i}\right)\nabla_{\frac{\partial}{\partial u^j}}Z
\end{aligned}
$$

$$
\begin{aligned}
\overset{13.4.2}{=\!=\!=} & \sum_i \Big(\sum_k \frac{\partial Z^i}{\partial u^k} Y^k + \sum_{j,k} \Gamma^i_{j,k} Z^j Y^k \Big) \sum_l X^l \nabla_{\frac{\partial}{\partial u^l}} \frac{\partial}{\partial u^i} \\
& + \sum_i \Big(\sum_k X(\frac{\partial Z^i}{\partial u^k}) Y^k + \sum_k \frac{\partial Z^i}{\partial u^k} X(Y^k) \\
& + \sum_{j,k} X(\Gamma^i_{j,k}) Z^j Y^k + \sum_{j,k} \Gamma^i_{j,k} X(Z^j) Y^k + \sum_{j,k} \Gamma^i_{j,k} Z^j X(Y^k) \Big) \frac{\partial}{\partial u^i} \\
& - \sum_i \Big(\sum_k \frac{\partial Z^i}{\partial u^k} X^k + \sum_{j,k} \Gamma^i_{j,k} Z^j X^k \Big) \sum_l Y^l \nabla_{\frac{\partial}{\partial u^l}} \frac{\partial}{\partial u^i} \\
& - \sum_i \Big(\sum_k Y(\frac{\partial Z^i}{\partial u^k}) X^k + \sum_k \frac{\partial Z^i}{\partial u^k} Y(X^k) \\
& + \sum_{j,k} Y(\Gamma^i_{j,k}) Z^j X^k + \sum_{j,k} \Gamma^i_{j,k} Y(Z^j) X^k + \sum_{j,k} \Gamma^i_{j,k} Z^j Y(X^k) \Big) \frac{\partial}{\partial u^i} \\
& - \sum_{i,j} \Big(X^i \frac{\partial Y^j}{\partial u^i} - Y^i \frac{\partial X^j}{\partial u^i} \Big) \sum_l \Big(Z^l \nabla_{\frac{\partial}{\partial u^j}} \frac{\partial}{\partial u^l} + \frac{\partial Z^l}{\partial u^j} \frac{\partial}{\partial u^l} \Big) \\
\overset{13.6}{=\!=\!=} & \sum_i \Big(\sum_k \frac{\partial Z^i}{\partial u^k} Y^k + \sum_{j,k} \Gamma^i_{j,k} Z^j Y^k \Big) \sum_l X^l \sum_p \Gamma^p_{l,i} \frac{\partial}{\partial u^p} \\
& + \sum_i \Big(\sum_k \sum_p X^p \frac{\partial \frac{\partial Z^i}{\partial u^k}}{\partial u^p} Y^k + \sum_k \frac{\partial Z^i}{\partial u^k} \sum_p X^p \frac{\partial Y^k}{\partial u^p} \\
& + \sum_{j,k} \sum_p X^p \frac{\partial \Gamma^i_{j,k}}{\partial u^p} Z^j Y^k + \sum_{j,k} \Gamma^i_{j,k} \sum_p X^p \frac{\partial Z^j}{\partial u^p} Y^k \\
& + \sum_{j,k} \Gamma^i_{j,k} Z^j \sum_p X^p \frac{\partial Y^k}{\partial u^p} \Big) \frac{\partial}{\partial u^i} \\
& - \sum_i \Big(\sum_k \frac{\partial Z^i}{\partial u^k} X^k + \sum_{j,k} \Gamma^i_{j,k} Z^j X^k \Big) \sum_l Y^l \sum_p \Gamma^p_{i,l} \frac{\partial}{\partial u^p} \\
& - \sum_i \Big(\sum_k \sum_p Y^p \frac{\partial \frac{\partial Z^i}{\partial u^k}}{\partial u^p} X^k + \sum_k \frac{\partial Z^i}{\partial u^k} \sum_p Y^p \frac{\partial X^k}{\partial u^p} \\
& + \sum_{j,k} \sum_p Y^p \frac{\partial \Gamma^i_{j,k}}{\partial u^p} Z^j X^k + \sum_{j,k} \Gamma^i_{j,k} \sum_p Y^p \frac{\partial Z^j}{\partial u^p} X^k \\
& + \sum_{j,k} \Gamma^i_{j,k} Z^j \sum_p Y^p \frac{\partial X^k}{\partial u^p} \Big) \frac{\partial}{\partial u^i} \\
& - \sum_{i,j} \Big(X^i \frac{\partial Y^j}{\partial u^i} - Y^i \frac{\partial X^j}{\partial u^i} \Big) \sum_l \Big(Z^l \sum_p \Gamma^p_{j,l} \frac{\partial}{\partial u^p} + \frac{\partial Z^l}{\partial u^j} \frac{\partial}{\partial u^l} \Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i,k,l,p} X^l\, Y^k\, \frac{\partial Z^i}{\partial u^k}\, \Gamma^p_{l,i}\, \frac{\partial}{\partial u^p} + \sum_{i,j,k,l,p} X^l\, Y^k\, Z^j\, \Gamma^i_{j,k}\, \Gamma^p_{l,i}\, \frac{\partial}{\partial u^p} \\
&\quad + \sum_{i,k,p} X^p\, Y^k\, \frac{\partial^2 Z^i}{\partial u^k \partial u^p}\, \frac{\partial}{\partial u^i} + \sum_{i,k,p} X^p\, \frac{\partial Y^k}{\partial u^p}\, \frac{\partial Z^i}{\partial u^k}\, \frac{\partial}{\partial u^i} \\
&\quad + \sum_{i,j,k,p} X^p\, Y^k\, Z^j\, \frac{\partial \Gamma^i_{j,k}}{\partial u^p}\, \frac{\partial}{\partial u^i} + \sum_{i,j,k,p} X^p\, Y^k\, \frac{\partial Z^j}{\partial u^p}\, \Gamma^i_{j,k}\, \frac{\partial}{\partial u^i} \\
&\quad + \sum_{i,j,k,p} X^p\, \frac{\partial Y^k}{\partial u^p}\, Z^j\, \Gamma^i_{j,k}\, \frac{\partial}{\partial u^i} \\
&\quad - \sum_{i,k,l,p} X^k\, Y^l\, \frac{\partial Z^i}{\partial u^k}\, \Gamma^p_{i,l}\, \frac{\partial}{\partial u^p} - \sum_{i,j,k,l,p} X^k\, Y^l\, Z^j\, \Gamma^i_{j,k}\, \Gamma^p_{i,l}\, \frac{\partial}{\partial u^p} \\
&\quad - \sum_{i,k,p} X^k\, Y^p\, \frac{\partial^2 Z^i}{\partial u^k \partial u^p}\, \frac{\partial}{\partial u^i} - \sum_{i,k,p} \frac{\partial X^k}{\partial u^p}\, Y^p\, \frac{\partial Z^i}{\partial u^k}\, \frac{\partial}{\partial u^i} \\
&\quad - \sum_{i,j,k,p} X^k\, Y^p\, Z^j\, \frac{\partial \Gamma^i_{j,k}}{\partial u^p}\, \frac{\partial}{\partial u^i} - \sum_{i,j,k,p} X^k\, Y^p\, \frac{\partial Z^j}{\partial u^p}\, \Gamma^i_{j,k}\, \frac{\partial}{\partial u^i} \\
&\quad - \sum_{i,j,k,p} \frac{\partial X^k}{\partial u^p}\, Y^p\, Z^j\, \Gamma^i_{j,k}\, \frac{\partial}{\partial u^i} \\
&\quad - \sum_{i,j,l,p} X^i\, \frac{\partial Y^j}{\partial u^i}\, Z^l\, \Gamma^p_{j,l}\, \frac{\partial}{\partial u^p} + \sum_{i,j,l,p} \frac{\partial X^j}{\partial u^i}\, Y^i\, Z^l\, \Gamma^p_{j,l}\, \frac{\partial}{\partial u^p} \\
&\quad - \sum_{i,j,l} X^i\, \frac{\partial Y^j}{\partial u^i}\, \frac{\partial Z^l}{\partial u^j}\, \frac{\partial}{\partial u^l} + \sum_{i,j,l} \frac{\partial X^j}{\partial u^i}\, Y^i\, \frac{\partial Z^l}{\partial u^j}\, \frac{\partial}{\partial u^l} \\
&= \sum_{i,j,k,p} X^p\, Y^k\, Z^j\, \frac{\partial \Gamma^i_{j,k}}{\partial u^p}\, \frac{\partial}{\partial u^i} - \sum_{i,j,k,p} X^k\, Y^p\, Z^j\, \frac{\partial \Gamma^i_{j,k}}{\partial u^p}\, \frac{\partial}{\partial u^i} \\
&\quad + \sum_{i,j,k,l,p} X^l\, Y^k\, Z^j\, \Gamma^i_{j,k}\, \Gamma^p_{l,i}\, \frac{\partial}{\partial u^p} - \sum_{i,j,k,l,p} X^k\, Y^l\, Z^j\, \Gamma^i_{j,k}\, \Gamma^p_{i,l}\, \frac{\partial}{\partial u^p} \\
&\quad + \sum_{i,k,l,p} X^l\, Y^k\, \frac{\partial Z^i}{\partial u^k}\, \Gamma^p_{l,i}\, \frac{\partial}{\partial u^p} - \sum_{i,j,k,p} X^k\, Y^p\, \frac{\partial Z^j}{\partial u^p}\, \Gamma^i_{j,k}\, \frac{\partial}{\partial u^i} \\
&\quad + \sum_{i,j,k,p} X^p\, Y^k\, \frac{\partial Z^j}{\partial u^p}\, \Gamma^i_{j,k}\, \frac{\partial}{\partial u^i} - \sum_{i,k,l,p} X^k\, Y^l\, \frac{\partial Z^i}{\partial u^k}\, \Gamma^p_{i,l}\, \frac{\partial}{\partial u^p} \\
&\quad + \sum_{i,j,k,p} X^p\, \frac{\partial Y^k}{\partial u^p}\, Z^j\, \Gamma^i_{j,k}\, \frac{\partial}{\partial u^i} - \sum_{i,j,l,p} X^i\, \frac{\partial Y^j}{\partial u^i}\, Z^l\, \Gamma^p_{j,l}\, \frac{\partial}{\partial u^p} \\
&\quad - \sum_{i,j,k,p} \frac{\partial X^k}{\partial u^p}\, Y^p\, Z^j\, \Gamma^i_{j,k}\, \frac{\partial}{\partial u^i} + \sum_{i,j,l,p} \frac{\partial X^j}{\partial u^i}\, Y^i\, Z^l\, \Gamma^p_{j,l}\, \frac{\partial}{\partial u^p} \\
&\quad + \sum_{i,k,p} X^p\, \frac{\partial Y^k}{\partial u^p}\, \frac{\partial Z^i}{\partial u^k}\, \frac{\partial}{\partial u^i} - \sum_{i,j,l} X^i\, \frac{\partial Y^j}{\partial u^i}\, \frac{\partial Z^l}{\partial u^j}\, \frac{\partial}{\partial u^l} \\
&\quad - \sum_{i,k,p} \frac{\partial X^k}{\partial u^p}\, Y^p\, \frac{\partial Z^i}{\partial u^k}\, \frac{\partial}{\partial u^i} + \sum_{i,j,l} \frac{\partial X^j}{\partial u^i}\, Y^i\, \frac{\partial Z^l}{\partial u^j}\, \frac{\partial}{\partial u^l} \\
&\quad + \sum_{i,k,p} X^p\, Y^k\, \frac{\partial^2 Z^i}{\partial u^k \partial u^p}\, \frac{\partial}{\partial u^i} - \sum_{i,k,p} X^k\, Y^p\, \frac{\partial^2 Z^i}{\partial u^k \partial u^p}\, \frac{\partial}{\partial u^i}
\end{aligned}
$$

$$= \sum_p \sum_{i,j,k} X^i\, Y^j\, Z^k\, \underbrace{\left(\frac{\partial \Gamma^p_{k,j}}{\partial u^i} - \frac{\partial \Gamma^p_{k,i}}{\partial u^j} + \sum_l \Big(\Gamma^l_{k,j}\, \Gamma^p_{i,l} - \Gamma^l_{k,i}\, \Gamma^p_{l,j}\Big)\right)}_{=:R^p_{i,j,k}} \frac{\partial}{\partial u^p}$$

$$= \sum_p \sum_{i,j,k} X^i\, Y^j\, Z^k\, R^p_{i,j,k} \frac{\partial}{\partial u^p} \quad \square$$

14.4 Remark.

In local coordinates we have:

(1) $$R = \sum_{i,j,k,l} R^l_{i,j,k}\, du^i \otimes du^j \otimes du^k \otimes \tfrac{\partial}{\partial u^l}$$

$$\text{with}\quad R^l_{i,j,k} := du^l\Big(R(\tfrac{\partial}{\partial u^i}, \tfrac{\partial}{\partial u^j})\tfrac{\partial}{\partial u^k}\Big) = \tfrac{\partial}{\partial u^i}\Gamma^l_{j,k} - \tfrac{\partial}{\partial u^j}\Gamma^l_{i,k} + \sum_{p=1}^m (\Gamma^p_{j,k}\Gamma^l_{i,p} - \Gamma^p_{i,k}\Gamma^l_{j,p})$$

Respectively, for $R(\xi,\eta,\zeta,\chi) := \big\langle R(\xi,\eta)\zeta, \chi\big\rangle$:

(2) $$R = \sum_{i,j,k,l} R_{i,j,k,l}\, du^i \otimes du^j \otimes du^k \otimes du^l$$

$$\begin{aligned}\text{with}\quad R_{i,j,k,l} &:= \big\langle R(\tfrac{\partial}{\partial u^i}, \tfrac{\partial}{\partial u^j})\tfrac{\partial}{\partial u^k} \big| \tfrac{\partial}{\partial u^l}\big\rangle = \sum_{p=1}^m R^p_{i,j,k} g_{p,l} \\ &= \tfrac12 \Big(\tfrac{\partial^2}{\partial u^i \partial u^k} g_{l,j} - \tfrac{\partial^2}{\partial u^i \partial u^l} g_{j,k} + \tfrac{\partial^2}{\partial u^j \partial u^l} g_{i,k} - \tfrac{\partial^2}{\partial u^j \partial u^k} g_{l,i}\Big) + \\ &\quad + \sum_{p=1}^m \sum_{q=1}^m g^{p,q} (\Gamma_{i,k,q}\Gamma_{j,l,p} - \Gamma_{j,k,q}\Gamma_{i,l,p}).\end{aligned}$$

Proof. We calculate as follows:

$$\begin{aligned}\sum_{l=1}^m R^l_{i,j,k} \tfrac{\partial}{\partial u^l} &:= R(\tfrac{\partial}{\partial u^i}, \tfrac{\partial}{\partial u^j})\tfrac{\partial}{\partial u^k} := \left(\Big[\nabla_{\frac{\partial}{\partial u^i}}, \nabla_{\frac{\partial}{\partial u^j}}\Big] - \nabla_{\left[\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right]}\right) \tfrac{\partial}{\partial u^k} \\ &\overset{13.6}{=\!=} \nabla_{\frac{\partial}{\partial u^i}} \left(\sum_{l=1}^m \Gamma^l_{j,k} \tfrac{\partial}{\partial u^l}\right) - \nabla_{\frac{\partial}{\partial u^j}} \left(\sum_{l=1}^m \Gamma^l_{i,k} \tfrac{\partial}{\partial u^l}\right) + 0 \\ &\overset{13.4.3}{=\!=} \sum_{l=1}^m \left(\Gamma^l_{j,k} \nabla_{\frac{\partial}{\partial u^i}} \tfrac{\partial}{\partial u^l} + \tfrac{\partial}{\partial u^i}(\Gamma^l_{j,k}) \tfrac{\partial}{\partial u^l}\right) \\ &\quad - \sum_{l=1}^m \left(\Gamma^l_{i,k} \nabla_{\frac{\partial}{\partial u^j}} \tfrac{\partial}{\partial u^l} + \tfrac{\partial}{\partial u^j}(\Gamma^l_{i,k}) \tfrac{\partial}{\partial u^l}\right) \\ &\overset{13.6}{=\!=} \sum_{l=1}^m \left(\Gamma^l_{j,k} \sum_{p=1}^m \Gamma^p_{i,l} \tfrac{\partial}{\partial u^p} + \tfrac{\partial}{\partial u^i}(\Gamma^l_{j,k}) \tfrac{\partial}{\partial u^l}\right) \\ &\quad - \sum_{l=1}^m \left(\Gamma^l_{i,k} \sum_{p=1}^m \Gamma^p_{j,l} \tfrac{\partial}{\partial u^p} + \tfrac{\partial}{\partial u^j}(\Gamma^l_{i,k}) \tfrac{\partial}{\partial u^l}\right) \\ &= \sum_{l=1}^m \left(\sum_{p=1}^m \Big(\Gamma^p_{j,k}\Gamma^l_{i,p} - \Gamma^p_{i,k}\Gamma^l_{j,p}\Big) + \tfrac{\partial}{\partial u^i}\Gamma^l_{j,k} - \tfrac{\partial}{\partial u^j}\Gamma^l_{i,k}\right) \tfrac{\partial}{\partial u^l}.\end{aligned}$$

Coefficient comparison thus provides:

$$R^l_{i,j,k} = \tfrac{\partial}{\partial u^i}\Gamma^l_{j,k} - \tfrac{\partial}{\partial u^j}\Gamma^l_{i,k} + \sum_{p=1}^m \Big(\Gamma^p_{j,k}\Gamma^l_{i,p} - \Gamma^p_{i,k}\Gamma^l_{j,p}\Big).$$

Now we calculate

$$R_{i,j,k,l} := \Big\langle R(\tfrac{\partial}{\partial u^i}, \tfrac{\partial}{\partial u^j})\tfrac{\partial}{\partial u^k} \big| \tfrac{\partial}{\partial u^l}\Big\rangle = \Big\langle \sum_{p=1}^m R^p_{i,j,k}\tfrac{\partial}{\partial u^p} \big| \tfrac{\partial}{\partial u^l}\Big\rangle = \sum_{p=1}^m R^p_{i,j,k} g_{p,l}$$

We have

$$\begin{aligned}\sum_{p=1}^m \tfrac{\partial}{\partial u^i}(\Gamma^p_{j,k})g_{p,l} &= \tfrac{\partial}{\partial u^i}\left(\sum_{p=1}^m \Gamma^p_{j,k} g_{p,l}\right) - \sum_{p=1}^m \Gamma^p_{j,k}\tfrac{\partial}{\partial u^i}(g_{p,l}) \\ &\overset{\boxed{13.6}}{=\!=} \tfrac{\partial}{\partial u^i}(\Gamma_{j,k,l}) - \sum_{p=1}^m \Gamma^p_{j,k}(\Gamma_{i,p,l} + \Gamma_{i,l,p}).\end{aligned}$$

and thus

$$\begin{aligned}R_{i,j,k,l} &= \sum_{p=1}^m R^p_{i,j,k} g_{p,l} \\ &= \sum_{p=1}^m \left(\tfrac{\partial}{\partial u^i}(\Gamma^p_{j,k}) - \tfrac{\partial}{\partial u^j}(\Gamma^p_{i,k}) + \sum_{q=1}^m (\Gamma^q_{j,k}\Gamma^p_{i,q} - \Gamma^q_{i,k}\Gamma^p_{j,q})\right) g_{p,l} \\ &= \tfrac{\partial}{\partial u^i}(\Gamma_{j,k,l}) - \sum_{p=1}^m \Gamma^p_{j,k}(\underbrace{\Gamma_{i,p,l}}_{(1)} + \underbrace{\Gamma_{i,l,p}}_{(2)}) - \tfrac{\partial}{\partial u^j}(\Gamma_{i,k,l}) + \sum_{p=1}^m \Gamma^p_{i,k}(\underbrace{\Gamma_{j,p,l}}_{(3)} + \underbrace{\Gamma_{j,l,p}}_{(4)}) \\ &\quad + \sum_{q=1}^m (\underbrace{\Gamma^q_{j,k}\Gamma_{i,q,l}}_{(1)} - \underbrace{\Gamma^q_{i,k}\Gamma_{j,q,l}}_{(3)}) \\ &\overset{\boxed{13.6}}{=\!=} \tfrac{1}{2}\tfrac{\partial}{\partial u^i}\left(\tfrac{\partial}{\partial u^j} g_{k,l} + \tfrac{\partial}{\partial u^k} g_{l,j} - \tfrac{\partial}{\partial u^l} g_{j,k}\right) - \tfrac{1}{2}\tfrac{\partial}{\partial u^j}\left(\tfrac{\partial}{\partial u^i} g_{k,l} + \tfrac{\partial}{\partial u^k} g_{l,i} - \tfrac{\partial}{\partial u^l} g_{i,k}\right) \\ &\quad + \sum_{p=1}^m \Big(\underbrace{\Gamma^p_{i,k}\Gamma_{j,l,p}}_{(4)} - \underbrace{\Gamma^p_{j,k}\Gamma_{i,l,p}}_{(2)}\Big) \\ &= \tfrac{1}{2}\left(\tfrac{\partial^2}{\partial u^i \partial u^k} g_{l,j} - \tfrac{\partial^2}{\partial u^i \partial u^l} g_{j,k} + \tfrac{\partial^2}{\partial u^j \partial u^l} g_{i,k} - \tfrac{\partial^2}{\partial u^j \partial u^k} g_{l,i}\right) + \\ &\quad + \sum_{p=1}^m \sum_{q=1}^m g^{p,q}\Big(\Gamma_{i,k,q}\Gamma_{j,l,p} - \Gamma_{j,k,q}\Gamma_{i,l,p}\Big). \qquad \square\end{aligned}$$

14.5 Lemma (Symmetry of the Riemann curvature).

The Riemann curvature fulfills the following identities:

1. $R(X,Y)Z + R(Y,X)Z = 0$
2. $\langle R(X,Y)Z, W\rangle + \langle R(X,Y)W, Z\rangle = 0$
3. $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$
4. $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$
5. $(\nabla_Z R)(X,Y,W) + (\nabla_X R)(Y,Z,W) + (\nabla_Y R)(Z,X,W) = 0.$

The equations 4 *and* 5 *are called 1st and 2nd* Bianchi Identities.

Proof.

1 is clear because of the definition $R(X,Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$.

2 is equivalent to $\langle R(X,Y)Z,Z\rangle = 0$ for all X, Y, Z:

$$\begin{aligned}
\langle R(X,Y)Z,Z\rangle &= \underbrace{\langle \nabla_X\nabla_Y Z, Z\rangle}_{X\langle\nabla_Y Z,Z\rangle - \langle\nabla_Y Z,\nabla_X Z\rangle} - \langle\nabla_Y\nabla_X Z,Z\rangle - \langle\nabla_{[X,Y]}Z,Z\rangle \\
&\overset{13.4.5}{=\!=} X\Big(\frac{1}{2}Y\langle Z,Z\rangle\Big) - \langle\nabla_Y Z,\nabla_X Z\rangle - Y\Big(\frac{1}{2}X\langle Z,Z\rangle\Big) + \langle\nabla_X Z,\nabla_Y Z\rangle \\
&\quad - \langle\nabla_{[X,Y]}Z,Z\rangle \\
&= \frac{1}{2}[X,Y]\langle Z,Z\rangle - 0 - \langle\nabla_{[X,Y]}Z,Z\rangle \overset{13.4.5}{=\!=} 0
\end{aligned}$$

4 By 13.4.4, $\nabla_Y Z - \nabla_Z Y = [Y,Z]$ holds and by applying ∇_X we obtain:

$$\nabla_X\nabla_Y Z - \nabla_X\nabla_Z Y - \nabla_{[Y,Z]}X = \nabla_X[Y,Z] - \nabla_{[Y,Z]}X \overset{13.4.4}{=\!=} [X,[Y,Z]]$$

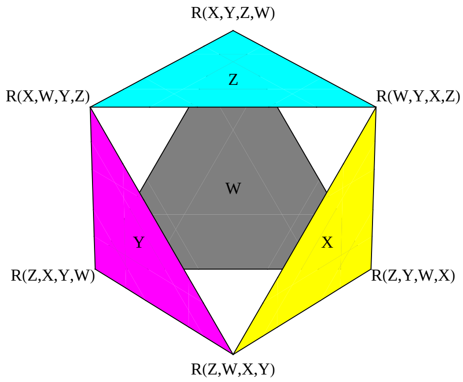
The cyclic expression can now be transformed as follows:

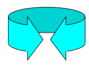
$$\begin{aligned}
R(X,Y)Z + R(Y,Z)X + R(Z,X)Y &= \underbrace{\nabla_X\nabla_Y Z}_{(1)} - \underbrace{\nabla_Y\nabla_X Z}_{(2)} - \underbrace{\nabla_{[X,Y]}Z}_{(3)} \\
&\quad + \underbrace{\nabla_Y\nabla_Z X}_{(2)} - \underbrace{\nabla_Z\nabla_Y X}_{(3)} - \underbrace{\nabla_{[Y,Z]}X}_{(1)} \\
&\quad + \underbrace{\nabla_Z\nabla_X Y}_{(3)} - \underbrace{\nabla_X\nabla_Z Y}_{(1)} - \underbrace{\nabla_{[Z,X]}Y}_{(2)} \\
&= \underbrace{[X,[Y,Z]]}_{(1)} + \underbrace{[Y,[Z,X]]}_{(2)} + \underbrace{[Z,[X,Y]]}_{(3)} \\
&= 0 \qquad \text{(because of the Jacobi identity)}\ .
\end{aligned}$$

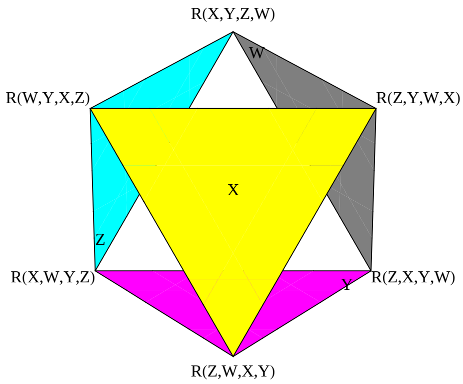
3 follows purely algebraically from 1, 2 and 4:

Consider an octahedron and denote 4 of its faces (intersecting only in vertices) with X, Y, Z, W.

The vertex of the octahedron, e.g. the intersection of the faces X and Y, will be denoted $R(Z,W,X,Y) := \langle R(X,Y)Z,W\rangle$ provided viewed from this vertex the faces X, Z, Y, W follow one another. Because of [1] and [2], it does not matter at which of the two adjacent faces X or Y one starts counting.

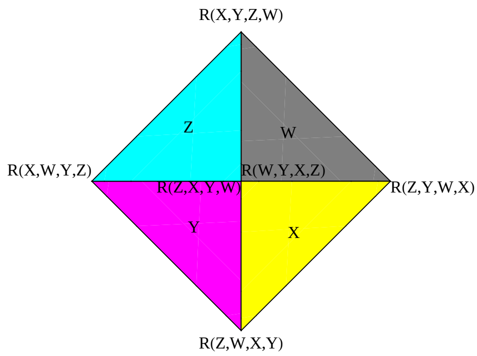






Now note that for each of the triangles X, Y, Z, W, the sum of its vertices is zero due to [4], because when rotated about an axis through the center of a triangle, both its vertices and the remaining three triangles are cyclic permuted.

If one adds these sums for the triangles Z and W together and subtracts those for X and Y, one obtains that double the difference from the corner $W \cap Z$ and the corner $X \cap Y$ is zero, i.e. [3] holds.



In detail this is the following calculation:

$$
\begin{aligned}
(+W)\quad & \underbrace{R(X,Y,Z,W)}_{(1)}+\underbrace{R(Y,Z,X,W)}_{(2)}+\underbrace{R(Z,X,Y,W)}_{(3)}=0\\
(+Z)\quad & \underbrace{R(W,X,Y,Z)}_{(4)}+\underbrace{R(X,Y,W,Z)}_{(1)}+\underbrace{R(Y,W,X,Z)}_{(5)}=0\\
(-Y)\quad & \underbrace{R(Z,W,X,Y)}_{(6)}+\underbrace{R(W,X,Z,Y)}_{(4)}+\underbrace{R(X,Z,W,Y)}_{(3)}=0\\
(-X)\quad & \underbrace{R(Y,Z,W,X)}_{(2)}+\underbrace{R(Z,W,Y,X)}_{(6)}+\underbrace{R(W,Y,Z,X)}_{(5)}=0\\
\Rightarrow\quad & \underbrace{2\,R(X,Y,Z,W)}_{(1)}-\underbrace{2\,R(Z,W,X,Y)}_{(6)}=0
\end{aligned}
$$

5 To make sense of this point, we need to extend ∇_Z to tensor fields. This is done by the product rule, i.e.

$$(\nabla_Z R)(X,Y,W) :=$$
$$= \nabla_Z\Big(R(X,Y)W\Big) - R(\nabla_Z X,Y)W - R(X,\nabla_Z Y)W - R(X,Y)\nabla_Z W$$
$$= \nabla_Z\Big(R(X,Y)W\Big) + R(Y,\nabla_Z X)W - R\Big(X,\nabla_Y Z - [Y,Z]\Big)W - R(X,Y)\nabla_Z W.$$

With $\sum_{\text{cycl.}}$ we denote the cyclic sum with respect to the variables X, Y and Z. Then

$$\sum_{\text{cycl.}} (\nabla_Z R)(X,Y,W) =$$
$$= -\sum_{\text{cycl.}} \nabla_Z\Big(R(X,Y)W\Big) + 0 + \sum_{\text{cycl.}} R(X,[Y,Z])W - \sum_{\text{cycl.}} R(X,Y)\nabla_Z W$$
$$= \sum_{\text{cycl.}} \nabla_Z\Big(\underbrace{[\nabla_X,\nabla_Y]}_{(1)} - \underbrace{\nabla_{[X,Y]}}_{(4)}\Big)W$$
$$+ \sum_{\text{cycl.}} \Big(\underbrace{[\nabla_X,\nabla_{[Y,Z]}]}_{(4)} - \underbrace{\nabla_{[X,[Y,Z]]}}_{(3)}\Big)W - \sum_{\text{cycl.}} \Big(\underbrace{[\nabla_X,\nabla_Y]\nabla_Z}_{(2)} - \underbrace{\nabla_{[X,Y]}\nabla_Z}_{(4)}\Big)W$$
$$= -\overbrace{\nabla_{\sum_{\text{cycl.}}[X,[Y,Z]]}}^{(3)} W + \sum_{\text{cycl.}} \underbrace{\Big(\overbrace{\nabla_Z[\nabla_X,\nabla_Y]}^{(1)} - \overbrace{[\nabla_X,\nabla_Y]\nabla_Z}^{(2)}\Big)}_{\big[\nabla_Z,[\nabla_X,\nabla_Y]\big]} W + \overbrace{0}^{(4)}$$
$$= 0 \qquad \text{, because of the Jacobi identity.} \qquad \square$$

14.6 Corollary (Polarization formula).

For the Riemann curvature one has:

$$4!\, R(X,Y,Z,W) =$$
$$= -R(Z+X,Y+W,Y+W,Z+X) + R(Z+X,Y-W,Y-W,Z+X)$$
$$+ R(Z-X,Y+W,Y+W,Z-X) - R(Z-X,Y-W,Y-W,Z-X)$$
$$+ R(Z+Y,X+W,X+W,Z+Y) - R(Z+Y,X-W,X-W,Z+Y)$$
$$- R(Z-Y,X+W,X+W,Z-Y) + R(Z-Y,X-W,X-W,Z-Y)$$

Proof. We have

(1) $$R(X,Y+Z,Y+Z,X) - R(X,Y-Z,Y-Z,X) =$$
$$= 2\Big(R(X,Y,Z,X) + R(X,Z,Y,X)\Big)$$
$$\overset{14.5.1,\,14.5.2}{=\!=} 2\Big(R(X,Y,Z,X) + R(Z,X,X,Y)\Big)$$
$$\overset{14.5.3}{=\!=} 4\,R(X,Y,Z,X)$$

and

(2) $$R(X+W,Y,Z,X+W) - R(X-W,Y,Z,X-W) =$$
$$= 2\Big(R(X,Y,Z,W) + R(W,Y,Z,X)\Big)$$

So

(3) $R(Y,Z,X,W) =$

$$\overset{14.5.1,\,14.5.2}{=\!=\!=} R(Z,Y,W,X)$$

$$\overset{2}{=\!=} -R(X,Y,W,Z) + \frac{1}{2}\Big(R(Z{+}X,Y,W,Z{+}X) - R(Z{-}X,Y,W,Z{-}X)\Big)$$

$$\overset{14.5.2,\,1}{=\!=\!=} R(X,Y,Z,W)$$
$$+ \frac{1}{8}\Big(R(Z{+}X,Y{+}W,Y{+}W,Z{+}X) - R(Z{+}X,Y{-}W,Y{-}W,Z{+}X)$$
$$- R(Z{-}X,Y{+}W,Y{+}W,Z{-}X) + R(Z{-}X,Y{-}W,Y{-}W,Z{-}X)\Big)$$

(4) $R(Z,X,Y,W) =$

$$\overset{14.5.2}{=\!=\!=} -R(Z,X,W,Y)$$

$$\overset{2}{=\!=} R(Y,X,W,Z) - \frac{1}{2}\Big(R(Z{+}Y,X,W,Z{+}Y) - R(Z{-}Y,X,W,Z{-}Y)\Big)$$

$$\overset{14.5.1,\,14.5.2,\,1}{=\!=\!=\!=} R(X,Y,Z,W)$$
$$- \frac{1}{8}\Big(R(Z{+}Y,X{+}W,X{+}W,Z{+}Y) - R(Z{+}Y,X{-}W,X{-}W,Z{+}Y)$$
$$- R(Z{-}Y,X{+}W,X{+}W,Z{-}Y) + R(Z{-}Y,X{-}W,X{-}W,Z{-}Y)\Big)$$

Hence

$$0 \overset{14.5.4}{=\!=\!=} R(X,Y,Z,W) + \underbrace{R(Y,Z,X,W)}_{3} + \underbrace{R(Z,X,Y,W)}_{4}$$
$$= R(X,Y,Z,W)$$
$$+ R(X,Y,Z,W)$$
$$+ \frac{1}{8}\Big(R(Z+X,Y+W,Y+W,Z+X) - R(Z+X,Y-W,Y-W,Z+X)$$
$$- R(Z-X,Y+W,Y+W,Z-X) + R(Z-X,Y-W,Y-W,Z-X)\Big)$$
$$+ R(X,Y,Z,W)$$
$$- \frac{1}{8}\Big(R(Z+Y,X+W,X+W,Z+Y) - R(Z+Y,X-W,X-W,Z+Y)$$
$$- R(Z-Y,X+W,X+W,Z-Y) + R(Z-Y,X-W,X-W,Z-Y)\Big)$$

and finally

$$4!\,R(X,Y,Z,W) =$$
$$= -R(Z+X,Y+W,Y+W,Z+X) + R(Z+X,Y-W,Y-W,Z+X)$$
$$+ R(Z-X,Y+W,Y+W,Z-X) - R(Z-X,Y-W,Y-W,Z-X)$$
$$+ R(Z+Y,X+W,X+W,Z+Y) - R(Z+Y,X-W,X-W,Z+Y)$$
$$- R(Z-Y,X+W,X+W,Z-Y) + R(Z-Y,X-W,X-W,Z-Y) \qquad \square$$

14.7 Definition.

Let us now further examine the expressions of the form $R(X,Y,Y,X)$ in the polarization formula 14.6. Let

$$X' = a\,X + b\,Y \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$
$$Y' = c\,X + d\,Y$$

Because of the skew symmetry 14.5.1 and 14.5.2 we have

$$R(X',Y',Y',X') = \det(A)\,R(X,Y,Y',X') = \det(A)^2\,R(X,Y,Y,X).$$

The term $|X|^2\,|Y|^2 - \langle X,Y\rangle^2$ has the same transformation behavior, since it measures the square of the area of the parallelogram generated by X and Y, see 9.14. Consequently, the term

$$K(F) := \frac{R(X,Y,Y,X)}{|X|^2\,|Y|^2 - \langle X,Y\rangle^2}$$

is independent of the choosen generators for the 2-dimensional subspace $F := \langle\{X,Y\}\rangle$ of T_pM spanned by X and Y. This number is called the SECTIONAL CURVATURE of F. The polarization formula 14.6 shows that the Riemann curvature can be calculated from the sectional curvature.

14.8 Theorem (Gaussian curvature versus sectional curvature).

For each Riemann surface M, the Gaussian curvature is identical to the sectional curvature (of the entire 2-dimensional tangent space).

Proof.
For hypersurfaces in $\mathbb{R}^3$ we have shown this in 14.2. For abstract Riemann surfaces M and local coordinates (u^1,u^2) on M we have

$$D^2 \cdot K(T_xM) := (E\,G - F^2)\cdot \frac{R\left(\frac{\partial}{\partial u^1},\frac{\partial}{\partial u^2},\frac{\partial}{\partial u^2},\frac{\partial}{\partial u^1}\right)}{|\frac{\partial}{\partial u^2}|^2|\frac{\partial}{\partial u^1}|^2 - |\langle\frac{\partial}{\partial u^1},\frac{\partial}{\partial u^2}\rangle|^2} = R_{1,2,2,1}$$

$$\overset{14.4}{=\!=} \frac{1}{2}\left(\frac{\partial^2}{\partial u^1\partial u^2}g_{1,2} - \frac{\partial^2}{\partial u^1\partial u^1}g_{2,2} + \frac{\partial^2}{\partial u^2\partial u^1}g_{1,2} - \frac{\partial^2}{\partial u^2\partial u^2}g_{1,1}\right)$$
$$+ g^{1,1}\left(\Gamma_{1,2,1}\Gamma_{2,1,1} - \Gamma_{2,2,1}\Gamma_{1,1,1}\right) + g^{1,2}\left(\Gamma_{1,2,2}\Gamma_{2,1,1} - \Gamma_{2,2,2}\Gamma_{1,1,1}\right)$$
$$+ g^{2,1}\left(\Gamma_{1,2,1}\Gamma_{2,1,2} - \Gamma_{2,2,1}\Gamma_{1,1,2}\right) + g^{2,2}\left(\Gamma_{1,2,2}\Gamma_{2,1,2} - \Gamma_{2,2,2}\Gamma_{1,1,2}\right)$$

$$\overset{10.6}{=\!=} \frac{1}{2}\left(F_{1,2} - G_{1,1} + F_{2,1} - E_{2,2}\right)$$
$$+ \frac{G}{D^2}\Big(E_2E_2 - (2F_2 - G_1)E_1\Big) - \frac{F}{D^2}\left(G_1E_2 - G_2E_1\right)$$
$$- \frac{F}{D^2}\Big(E_2G_1 - (2F_2 - G_1)(2F_1 - E_2)\Big) + \frac{E}{D^2}\Big({G_1}^2 - G_2(2F_1 - E_2)\Big)$$
$$= \frac{E}{4D^2}\left(E_2G_2 - 2F_1G_2 + {G_1}^2\right)$$
$$+ \frac{F}{4D^2}\left(E_1G_2 - E_2G_1 - 2E_2F_2 + 4F_1F_2 - 2F_1G_1\right)$$
$$+ \frac{G}{4D^2}\left(E_1G_1 - 2E_1F_2 + {E_2}^2\right) - \frac{1}{2}\left(E_{2,2} - 2F_{1,2} + G_{1,1}\right)$$
$$\overset{9.15}{=\!=} D^2\,K$$

Or calculated differently:

$$R_{i,j,k,l} \overset{\boxed{14.4}}{=\!=} \tfrac12 \Big(\tfrac{\partial^2}{\partial u^i \partial u^k} g_{l,j} - \tfrac{\partial^2}{\partial u^i \partial u^l} g_{j,k} + \tfrac{\partial^2}{\partial u^j \partial u^l} g_{i,k} - \tfrac{\partial^2}{\partial u^j \partial u^k} g_{l,i}\Big) + \sum_{p=1}^m (\Gamma^p_{i,k}\Gamma_{j,l,p} - \Gamma^p_{j,k}\Gamma_{i,l,p})$$

$$\begin{aligned}
R_{1,2,2,1} &\overset{\boxed{14.4}}{=\!=} \tfrac12 \Big(\tfrac{\partial^2}{\partial u^1 \partial u^2} g_{1,2} - \tfrac{\partial^2}{\partial u^1 \partial u^1} g_{2,2} + \tfrac{\partial^2}{\partial u^2 \partial u^1} g_{1,2} - \tfrac{\partial^2}{\partial u^2 \partial u^2} g_{1,1}\Big) \\
&\quad + (\Gamma^1_{1,2}\Gamma_{2,1,1} - \Gamma^1_{2,2}\Gamma_{1,1,1}) + (\Gamma^2_{1,2}\Gamma_{2,1,2} - \Gamma^2_{2,2}\Gamma_{1,1,2}) \\
&\overset{\boxed{10.6}}{=\!=} \tfrac12 (F_{1,2} - G_{1,1} + F_{2,1} - E_{2,2}) \\
&\quad + \frac{E_2\, G - G_1\, F}{2D^2}\cdot\frac{E_2}{2} - \frac{2\,F_2\, G - G_1\, G - G_2\, F}{2D^2}\cdot\frac{E_1}{2} \\
&\quad + \frac{-E_2\, F + G_1\, E}{2D^2}\cdot\frac{G_1}{2} - \frac{-2\,F_2\, F + G_1\, F + G_2\, E}{2D^2}\cdot\frac{2\,F_1 - E_2}{2} \\
&= \frac{E}{4D^2}\,(E_2G_2 - 2F_1G_2 + {G_1}^2) \\
&\quad + \frac{F}{D^2}\,(E_1G_2 - E_2G_1 - 2E_2F_2 + 4F_1F_2 - 2F_1G_1) \\
&\quad + \frac{G}{D^2}\,(E_1G_1 - 2E_1F_2 + {E_2}^2) - \frac12(E_{2,2} - 2F_{1,2} + G_{1,1}) \\
&\overset{\boxed{9.15}}{=\!=} D^2\, K. \qquad \square
\end{aligned}$$

14.9 Definition (Normal coordinates).

Riemannian normal coordinates are defined as the parameterization

$$\varphi : (u^1, \dots, u^m) \mapsto \exp_p\Big(\sum_{i=1}^m u^i X_i\Big)$$

for an orthonormal basis $(X_1, \dots, X_m)$ of T_pM.

14.10 Lemma (Christoffel symbols in normal coordinates).

In Riemannian normal coordinates around p, all Christoffel symbols vanish at p.

Proof. Obviously

$$g_{i,j}(p) := \big\langle \tfrac{\partial}{\partial u^i}, \tfrac{\partial}{\partial u^j}\big\rangle(p) = \langle X_i, X_j\rangle = \delta_{i,j}.$$

The radial geodesics $t \mapsto \exp_p(tX)$ satisfy the geodesic equation

$$\tfrac{d^2u^k}{dt^2} + \sum_{i,j=1}^m \Gamma^k_{i,j} \tfrac{du^i}{dt}\tfrac{du^j}{dt} = 0,$$

and for $u(t) := tX_j$ (i.e. $u^k(t) = \delta^k_j\, t$) we have $\Gamma^k_{j,j}(u(t)) = 0$, so in particular $\Gamma^k_{j,j}(p)$. For $u(t) := t(X_i + X_j)$, it follows analogously $(\Gamma^k_{i,i} + \Gamma^k_{i,j} + \Gamma^k_{j,i} + \Gamma^k_{j,j})(p) = 0$ and because $\Gamma^k_{i,j}$ is symmetric in (i, j), we have $\Gamma^k_{i,j}(p) = 0$ for all i, j, k. $\square$

14.11 Lemma.

Let M be a Riemannian manifold and $F < T_xM$ a 2-dimensional subspace. Then, the sectional curvature $K(F)$ is exactly the Gaussian curvature of the surface which is locally given by $\exp(F)$.

Proof. Because of 14.8, it suffices to show that the Riemann curvature R_N of the surface $N := \exp(F)$ coincides with the restriction of the Riemann curvature R_M of M to N, where N carries the metric induced by M.

For this purpose one chooses Riemannian normal coordinates $\varphi : (u^1, \dots, u^m) \mapsto \exp_p(\sum_{i=1}^m u^i X_i)$ for an orthonormal basis $(X_1, \dots, X_m)$ of the tangent space T_pM such that $\{X_1, X_2\}$ span F and hence N is parameterized by $(t, s) \mapsto \exp_p(tX_1 + sX_2)$. With respect to these coordinates, all Christoffel symbols disappear at p by 14.10. Since the coefficient functions $g_{i,j}$ with $i, j \le 2$ are identical for M and N, this also holds for

$$R_{i,j,k,l} = \frac{1}{2}\left(\frac{\partial^2}{\partial u^j \partial u^l} g_{i,k} + \frac{\partial^2}{\partial u^i \partial u^k} g_{j,l} - \frac{\partial^2}{\partial u^j \partial u^k} g_{i,l} - \frac{\partial^2}{\partial u^i \partial u^l} g_{j,k}\right) + 0. \qquad \square$$

14.12 Theorem (Uncurved spaces).

For a Riemannian manifold are equivalent:

1. $R = 0$.
2. *M is locally isometric to Euclidean space.*
3. *The parallel transport (see 12) is locally path-independent.*

The condition 3 is globally not universally valid as the Möbius strip or the cone with the flat metric shows.

Proof. (1⇒3) By using a chart, we may assume that M is an open neighborhood of 0 in $\mathbb{R}^m$, but with a generic Riemannian metric g. We have to find a vector field X for given initial value X_0, which is parallel along each curve. Because of 13.4.3, it suffices that $\nabla_{\partial_i} X = 0$ for all $i = 1, \dots, m$. First, we find a vector field $u^1 \mapsto X(u^1, 0, \dots, 0)$ parallel to the u^1 axis. For each u^1 we find along the curve $u^2 \mapsto (u^1, u^2, 0, \dots, 0)$ a parallel vector field $u^2 \mapsto X(u^1, u^2, 0, \dots, 0)$ with initial value $X(u^1, 0, \dots, 0)$. And so we get a vector field $(u^1, u^2) \mapsto X(u^1, u^2, 0, \dots, 0)$ along the 2-dimensional surface $\psi : (u^1, u^2) \mapsto (u^1, u^2, 0, \dots, 0)$. It fulfills $\nabla_{\partial_2} X = 0$ along ψ and $\nabla_{\partial_1} X = 0$ along $u^1 \mapsto \psi(u^1, 0)$. Then $\nabla_{\partial_1}\nabla_{\partial_2} X - \nabla_{\partial_2}\nabla_{\partial_1} X = R(\partial_1, \partial_2)X = 0$ holds due to $[\partial_1, \partial_2] = 0$. Thus, $\nabla_{\partial_2}\nabla_{\partial_1} X = 0$, i.e. $\nabla_{\partial_1} X$ is parallel along $u^2 \mapsto \psi(u^1, u^2)$. From $\nabla_{\partial_1} X = 0$ along $u^1 \mapsto \psi(u^1, 0)$ follows $\nabla_{\partial_1} X = 0$ along ψ. Thus X is parallel along all the curves in the 2-surface ψ. Now one continues the above iterative process to obtain the desired parallel vector field X. This shows that the parallel transport is path independent.

(3⇒2) If one chooses the vectors of an orthonormal basis of $T_0\mathbb{R}^m$ as initial value, one obtains parallel vector fields X_i, which by 12.5.1 form an orthonormal basis everywhere. By 13.4.4 $[X_i, X_j] = \nabla_{X_i} X_j - \nabla_{X_j} X_i = 0$ holds. By [**95**, 17.12], the X_i can thus be integrated to obtain a chart φ which satisfies $\partial_i^\varphi = X_i$. In this chart, the Riemann metric has coefficients $\delta_{i,j}$ (by the orthonormality of the X_i), i.e. φ is a local isometry between the flat $\mathbb{R}^m$ and M.

(2⇒1) Since the covariant derivative and thus the Riemannian curvature is an intrinsic quantity, i.e. depends only on the Riemann metric, it suffices to calculate R for the Euclidean space, but there $R = 0$ because of the preliminary remark to 14.1 (where $\nabla_\xi = D_\xi$). $\square$

14.13 Definition (Traces of the Riemann curvature).

The Ricci curvature of a Riemannian manifold is

$$\mathrm{Ricci}(X, Y) := \mathrm{trace}\Big(Z \mapsto R(Z, X)(Y)\Big) = -\,\mathrm{trace}\Big(Z \mapsto R(X, Z)(Y)\Big)$$

and in local coordinates

$$\begin{aligned}
&\operatorname{Ricci}\Big(\sum_i X^i \tfrac{\partial}{\partial u^i}, \sum_j Y^j \tfrac{\partial}{\partial u^j}\Big) = \\
&= \sum_{i,j} X^i Y^j \operatorname{Ricci}\Big(\tfrac{\partial}{\partial u^i}, \tfrac{\partial}{\partial u^j}\Big) \\
&= \sum_{i,j} X^i Y^j \sum_k \underbrace{du^k\Big(R\Big(\tfrac{\partial}{\partial u^k}, \tfrac{\partial}{\partial u^i}\Big)\tfrac{\partial}{\partial u^j}\Big)}_{R^k_{k,i,j}} \\
&\overset{14.4.1}{=\!=} \sum_{i,j} X^i Y^j \sum_k \Big(\tfrac{\partial}{\partial u^k}\Gamma^k_{i,j} - \tfrac{\partial}{\partial u^i}\Gamma^k_{k,j} + \sum_p (\Gamma^p_{i,j}\Gamma^k_{k,p} - \Gamma^p_{k,j}\Gamma^k_{i,p})\Big)
\end{aligned}$$

Note that $\operatorname{Ricci} : T_xM \times T_xM \to \mathbb{R}$ is symmetric because

$$\begin{aligned}
\operatorname{Ricci}\Big(\tfrac{\partial}{\partial u^i}, \tfrac{\partial}{\partial u^j}\Big) &= \sum_k R^k_{k,i,j} \overset{14.4.2}{=\!=} \sum_{k,l} R_{k,i,j,l}\, g^{l,k} \overset{14.5.3}{=\!=} \sum_{k,l} R_{j,l,k,i}\, g^{l,k} \\
&\overset{14.5.1,\,14.5.2}{=\!=\!=} \sum_{l,k} R_{l,j,i,k}\, g^{k,l} = \sum_l R^l_{l,j,i} = \operatorname{Ricci}\Big(\tfrac{\partial}{\partial u^j}, \tfrac{\partial}{\partial u^i}\Big)
\end{aligned}$$

A (pseudo-)Riemannian manifold is called Ricci flat if $\operatorname{Ricci} = 0$.

It is called Einstein manifold if Ricci is proportional to the metric.

The scalar curvature is $S := \operatorname{trace}_g(\xi \mapsto (\operatorname{Ricci}(\xi, \_))^\flat)$, i.e. the trace of the mapping

$$T_xM \xrightarrow[\operatorname{Ricci}^\vee_x]{} (T_xM)^* \xrightarrow[\flat]{\cong} T_xM.$$

In coordinates this is

$$\begin{aligned}
S &= \sum_{i,j} \operatorname{Ricci}\Big(\tfrac{\partial}{\partial u^i}, \tfrac{\partial}{\partial u^j}\Big) g^{i,j} \\
&= \sum_{i,j}\sum_k \Big(\tfrac{\partial}{\partial u^k}\Gamma^k_{i,j} - \tfrac{\partial}{\partial u^i}\Gamma^k_{k,j} + \sum_p (\Gamma^p_{i,j}\Gamma^k_{k,p} - \Gamma^p_{k,j}\Gamma^k_{i,p})\Big)\, g^{i,j}.
\end{aligned}$$

Note that due to the symmetry properties 14.6, these are all non-trivial traces which can be formed from the Riemann curvature.

The completely traceless part of the Riemann curvature (for $m \geq 3$) is called Weyl curvature tensor

$$\begin{aligned}
W &:= R - \underbrace{\frac{1}{m-2}\Big(\operatorname{Ricci} - \frac{S}{2(m-1)} g\Big)}_{\text{Schouten tensor}} \bullet g \\
&= R - \frac{1}{m-2}\Big(\operatorname{Ricci} - \frac{S}{m} g\Big) \bullet g - \frac{S}{2m(m-1)} g \bullet g
\end{aligned}$$

. Here, $k \bullet h$ is the Kulkarni-Nomizu product of two symmetric 2-fold covariant tensors k and h, namely

$$\begin{aligned}
(k \bullet h)(v_1, v_2, v_3, v_4) :&= k(v_1, v_3)\, h(v_2, v_4) + k(v_2, v_4)\, h(v_1, v_3) \\
&\quad - k(v_1, v_4)\, h(v_2, v_3) - k(v_2, v_3)\, h(v_1, v_4)
\end{aligned}$$

14.14 Proposition.

A Riemannian manifold has constant sectional curvature on $G(2, T_pM)$ if and only if for all $X, Y, Z \in T_pM$ the following holds:

$$R(X,Y)Z = K \cdot \Big(g(Y,Z)X - g(X,Z)Y\Big)$$

Proof. ($\Leftarrow$) Let $R(X,Y)Z = K \cdot \Big(g(Y,Z)X - g(X,Z)Y\Big)$ with a constant $K \in \mathbb{R}$. Then

$$\begin{aligned} K\big(\langle\{X,Y\}\rangle\big) &:= \frac{g\Big(R(X,Y)Y, X\Big)}{g(X,X)g(Y,Y) - g(X,Y)^2} \\ &= \frac{g\Big(K \cdot \big(g(Y,Y)X - g(X,Y)Y\big), X\Big)}{g(X,X)g(Y,Y) - g(X,Y)^2} = K \end{aligned}$$

($\Rightarrow$) Let K be the constant sectional curvature. The expression

$$g\Big(K \cdot \big(g(Y,Z)X - g(X,Z)Y\big), W\Big) = K \cdot \Big(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\Big)$$

has the properties 14.5.1 - 14.5.4 and is the same as $g\big(R(X,Y)Y,X\big)$ for $Z = Y$ and $W = X$. Because of 14.6 (where we have only used these properties), it coincides with $g\big(R(X,Y)Z,W\big)$ everywhere. □

14.15 Corollary.

If an m-dimensional Riemannian manifold has constant sectional curvature K, then the Ricci and the scalar curvature are given by:

$$\begin{aligned} \mathrm{Ricci}(X,Y) &= K \cdot (m-1) \cdot g(X,Y) \\ S &= K \cdot (m-1) \cdot m \end{aligned}$$

In particular, it has to be an Einstein manifold, i.e. $\mathrm{Ricci} = \frac{S}{m}\, g$.

Proof.

$$\begin{aligned} \mathrm{Ricci}(X,Y) &= \mathrm{trace}\Big(Z \mapsto R(Z,X)Y\Big) \\ &\overset{14.14}{=\!=} \mathrm{trace}\Big(Z \mapsto K \cdot \big(g(X,Y)Z - g(Z,Y)X\big)\Big) \\ &= K \cdot \big(m \cdot g(X,Y) - g(X,Y)\big) = K \cdot (m-1) \cdot g(X,Y) \\ \text{und} \quad S &= \mathrm{trace}\Big(X \mapsto (\mathrm{Ricci}(X,\_))^\flat\Big) = \mathrm{trace}\Big(X \mapsto \big(K \cdot (m-1) \cdot g(X,\_)\big)^\flat\Big) \\ &= \mathrm{trace}\Big(X \mapsto K \cdot (m-1) \cdot X\Big) = K \cdot (m-1) \cdot m. \quad \square \end{aligned}$$

14.16 Resumee

For plane curves, we interpreted the CURVATURE as the force necessary to hold a mass point with constant scalar velocity on a curve.

For hypersurfaces in $\mathbb{R}^3$, we first got to know the NORMAL CURVATURE of a surface in direction ξ as the curvature of the intersection curve with the plane spanned by the surface normal and ξ. This is at the same time the curvature of the geodesic in direction ξ, see 9.10. The critical points of the normal curvature are the MAIN CURVATURES, and their product is the GAUSS CURVATURE.

For a general Riemannian manifold, the SECTIONAL CURVATURE can be identified with the Gaussian curvature of the 2-dimensional surface parameterized by the exponential mapping. Finally, the Riemann curvature is the tensor field associated with the sectional curvature (i.e., the associated 4-linear mapping).

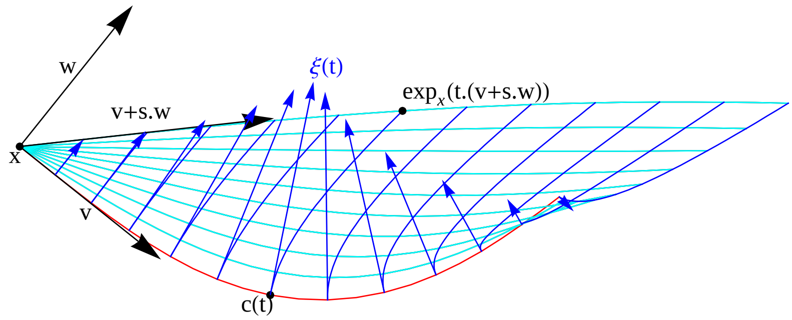
15. Jacobi Fields

15.1 Remark.

Let $c : [0, a] \to M$ be a geodesic in a Riemann surface. It can be written as radial geodesics of the form $c(t) = \exp_x(t\,v)$, where $x := c(0)$ and $v := c'(0)$. We want to discuss neighboring radial geodesics. By 13.8 there is an neighborhood around $[0, a] \times \{v\} \subset \mathbb{R} \times T_xM$ on which exp is well defined. Thus, on the interval $[0, a]$, the radial geodesics, which start at x in a direction near v, exist. Let us now consider the variation $(t, w) \mapsto \exp_x(t(v + w))$ of c for $w \perp v$. For fixed w, the directional derivative

$$\xi(t) := \tfrac{\partial}{\partial s}\big|_{s=0} \exp_x(t(v + sw)) = (T_{tv} \exp_x)(tw)$$

at $(t, 0) \in [0, a] \times T_xM$ in direction $(0, w)$ defines a vector field ξ along c.



We now want to show that the vector field ξ satisfies the so-called JACOBI EQUATION

$$\nabla^2\xi(t) + K(c(t))\,\xi(t) = 0.$$

Since

$$\varphi : (r, \vartheta) \mapsto \exp_x\Big(r\big(\cos(\vartheta)v + \sin(\vartheta)w\big)\Big) \text{ for } |w| = 1 = |v|$$

are geodesic parallel coordinates, i.e. meet $E = 1$, $F = 0$, $G > 0$, the Jacobi equation $K = -\frac{1}{\sqrt{G}} \left(\frac{\partial}{\partial r}\right)^2 \sqrt{G}$ from 9.17 applies.

$$\begin{aligned} \text{For } \xi(t) :&= \tfrac{\partial}{\partial s}\big|_{s=0} \exp_x(t(v + sw)) \\ &= \tfrac{\partial}{\partial \vartheta}\big|_{\vartheta=0} \exp_x(t\cos(\vartheta)v + t\sin(\vartheta)w) = \partial_2\varphi(t, 0) \\ \text{one has } |\xi(t)|^2 &= |\partial_2\varphi(t, 0)|^2 = G(t, 0). \end{aligned}$$

The vector field ξ is normal to c' because the radial geodesics orthogonally intersect the geodesic spheres (because of $F = 0$), so $\xi(t)$ can be written as $\lambda(t)\nu(t)$, where ν is a unit normal field to c' in TM and $\lambda = |\xi| = \sqrt{G}$. Consequently, $\lambda'' + (K \circ c) \cdot \lambda = \lambda'' - \frac{1}{\sqrt{G}}(\frac{\partial}{\partial r})^2\sqrt{G} \cdot \lambda = 0$ holds. Since c is a geodesic, c' is a parallel vector field (see 12.1) along c and the same holds for ν. So for the covariant derivative of ξ we have:

$$\begin{aligned} &\nabla\xi = \nabla(\lambda\,\nu) = \lambda\,\nabla\nu + \lambda'\,\nu = \lambda'\,\nu \\ \Rightarrow\quad &\nabla^2\xi = \nabla(\lambda'\,\nu) = \lambda'\,\nabla\nu + \lambda''\,\nu = \lambda''\,\nu \\ \Rightarrow\quad &\nabla^2\xi + K\,\xi = \lambda''\,\nu + K\,\lambda\,\nu = (\lambda'' + K\,\lambda)\nu = 0 \end{aligned}$$

15.2 Definition (Jacobi fields).

We call a vector field ξ along a geodesic c in a Riemann surface a Jacobi field if it satisfies the Jacobi equation

$$\nabla^2\xi + (K\circ c)\cdot\xi = 0$$

and it is orthogonal to c.

15.3 Lemma.

The Jacobi fields ξ along a geodesic c with initial condition $\xi(0) = 0$ are exactly those vector fields which can be written as $\xi(t) := (T_{tc'(0)}\exp_{c(0)})(tw)$ with $w \in c'(0)^\perp \subset T_{c(0)}M$.

Proof. We have just shown that vector fields of the form $\xi(t) := (T_{tc'(0)}\exp_{c(0)})(tw)$ are Jacobi fields. Now let's calculate their initial values $\xi(0)$ and $\nabla\xi(0)$: Clearly,

$$\xi(0) = (T_{0c'(0)}\exp_{c(0)})(0w) = 0.$$

With respect to the coordinates $(u^1,u^2)\mapsto \exp_x(u^1\,v+u^2\,w)$, we have $u^1(t)=t$, $u^2(t)=0$, $\xi^1(t)=0$ and $\xi^2(t)=t$. Thus,

$$\nabla\xi(0) \overset{13.1}{=\!=} \sum_{k=1}^m\Big(\tfrac{d\xi^k}{dt}+\sum_{i,j}^m\Gamma^k_{i,j}\xi^i\tfrac{du^j}{dt}\Big)\tfrac{\partial}{\partial u^k}\Big|_{t=0}$$

$$= \Big(\tfrac{\partial}{\partial u^2}+\sum_{k=1}^2\Gamma^k_{2,1}\cdot t\cdot 1\cdot\tfrac{\partial}{\partial u^k}\Big)\Big|_{t=0} = \tfrac{\partial}{\partial u^2}\big|_{t=0} = w.$$

Since any orthognal vector field ξ along c is of the form $\xi=\lambda\cdot\nu$ as in 15.1, hence the Jacobi equation translates into the second order linear differential equation $\lambda''+(K\circ c)\,\lambda=0$, which has a unique solution for given initial values $\lambda(0)$ and $\lambda'(0)$. Thus $\xi=(T_{tc'(0)}\exp_{c(0)})(tw)$ for $w:=\nabla\xi(0)$. □

Let M be a complete Riemannian manifold, and let c be an arc length parameterized geodesic in M. We have seen in 10.11 that curves, which ly in geodesic parallel coordinates close to c, can not have a shorter arc length. We now investigate the question of wether we can extend the geodesic polar coordinates around $c(0)$ to coordinates around $c(t)$. For this we need the following

15.4 Definition (Conjugated points).

Let $c: t\mapsto \exp_{c(0)}(tc'(0))$ be a geodesic in M. A point $c(t)$ is called conjugate to $c(0)$ if the differential $T_{tc'(0)}(exp_{c(0)})$ of the exponential map at $tc'(0)\in T_{c(0)}M$ is not an isomorphism from $T_{c(0)}M = T_{tc'(0)}T_{c(0)}M$ to $T_{c(t)}M$.

15.5 Theorem (Conjugated points).

Let c be a geodesic in a complete Riemann surface. Then t.f.a.e.:

1. *$c(t)$ is conjugated to $c(0)$;*
2. *There is a Jacobi vector field $\xi\neq 0$ along c with $\xi(t)=0=\xi(0)$.*

Proof. Let $x:=c(0)$ and $v:=c'(0)$. By definition, $c(t)$ is conjugate to $c(0)$ if and only if $T_{tv}\exp_x: T_xM\to T_{\exp_x(tv)}M$ is not an isomorphism, i.e. $\ker(T_{tv}\exp_x)\neq\{0\}$. Because of $(T_{tv}\exp_x)(v)=c'(t)\neq 0$ and $(T_{tv}\exp_x)(w)\perp c'(t)$ for all $w\perp v$ by 13.10, we have $\ker(T_{tv}\exp_x)\subseteq v^\perp$: In fact, let $w'=a\,v+w$ with $w\perp v$, then $w'\in\ker(T_{tv}\exp_x)\Rightarrow 0=T_{tv}\exp_x(a\,v+w)=a\,c'(t)+T_{tv}\exp_x(w)\Rightarrow a=0$.

And, furthermore, $\xi(t) = 0$, where $\xi : s \mapsto s\, T_{sv} \exp_x(w)$ is the Jacobi field with initial condition $\xi(0) = 0$ and $\nabla\xi(0) = w \in \ker(T_{tv} \exp_x) \subseteq v^\perp$ by 15.3. □

15.6 Corollary.

Let c be a geodesic parameterized by arc length. If c does not contain any conjugate points inside a parameter interval $[t_1, t_2]$, then $L(c) \leq L(c_1)$ holds to each c_1 near c.

The converse is also true, see 15.16.

Proof.

As in 15.1, we consider a chart φ for geodesic polar coordinates at $c(0)$. Because of 15.5, this mapping is a local diffeomorphism in every point of $]t_1, t_2[\times \{0\}$. So, aside from the boundary points, we have geodesic parallel coordinates along c. By 10.11, the length of each curve near c will then be at least as long as that of c. □

15.7 Comparison lemma of Sturm.

Let μ (resp. λ) be a solution of the linear differential equation $\mu''(t) + a(t)\,\mu(t) = 0$ (respectively $\lambda''(t) + b(t)\,\lambda(t) = 0$) with initial value $\mu(0) = 0 = \lambda(0)$ and $\mu'(0) = 1 = \lambda'(0)$. Furthermore, let $a \geq b$ (resp. $\forall\, t : a(t) > b(t)$). Put $t_\mu := \min\{t > 0 : \mu(t) = 0\}$ and $t_\lambda := \min\{t > 0 : \lambda(t) = 0\}$. Then $t_\mu \leq t_\lambda$ and for $0 < t_0 < t_1 < t_\mu$ we have $\mu(t_1)\lambda(t_0) \leq \lambda(t_1)\mu(t_0)$ (or $<$) and $\mu(t_1) \leq \lambda(t_1)$ (resp. $<$).

Proof. By assumption $\mu(t) > 0$ for all $0 < t < t_\mu$ (because of $\mu'(0) = 1$) and $\lambda(t) > 0$ for all $0 < t < t_\lambda$, so $\lambda'(t_\lambda) \leq 0$. Furthermore $\lambda'(t_\lambda) < 0$, otherwise we would have $\lambda = 0$ locally at t_λ. Let $a(t) \geq b(t)$ for all t. Assume $t_\mu > t_\lambda$, then

$$0 = \int_0^{t_\lambda} \mu\,(\lambda'' + b \cdot \lambda) - \lambda\,(\mu'' + a \cdot \mu) = \underbrace{(\mu \cdot \lambda' - \lambda \cdot \mu')\Big|_0^{t_\lambda}}_{=\mu(t_\lambda)\lambda'(t_\lambda)<0} + \underbrace{\int_0^{t_\lambda} (b - a) \cdot \lambda \cdot \mu}_{\leq 0},$$

a contradiction.

Let now $0 < t < t_\mu (\leq t_\lambda)$. Then

$$0 = \int_0^t \mu\,(\lambda''+b\cdot\lambda) - \lambda\,(\mu''+a\cdot\mu) = (\mu\cdot\lambda' - \lambda\cdot\mu')\big|_0^t + \underbrace{\int_0^t (b-a)\cdot\lambda\cdot\mu}_{\leq 0} \leq (\mu\cdot\lambda' - \lambda\cdot\mu')(t)$$

and thus $(\log\circ\lambda)'(t) \geq (\log\circ\mu)'(t)$, hence $\log\circ\frac{\lambda}{\mu}$ is monotonously increasing and thus $\lambda(t_1)\mu(t_0) \geq \mu(t_1)\lambda(t_0)$ for all $0 < t_0 \leq t_1 < t_\mu$. Because of $\lim_{t_0 \to 0} \frac{\lambda(t_0)}{\mu(t_0)} = \frac{\lambda'(0)}{\mu'(0)} = 1$ it follows that $\lambda(t_1) \geq \mu(t_1)$.

The case $a(t) > b(t)$ for all t is treated quite analogously. □

15.8 Corollary.

Let c be a geodesic parameterized by arc length.

1. *If $K(c(t)) \leq K_1$ holds to all t, then no open interval of length $\frac{\pi}{\sqrt{K_1}}$ contains a conjugate point.*
2. *If $K(c(t)) \geq K_0 > 0$ holds to all t, however, there is one conjugate point in each closed interval of length $\frac{\pi}{\sqrt{K_0}}$.*

Here and in the following let $\frac{\pi}{\sqrt{K_1}} = +\infty$ *for* $K_1 \leq 0$.

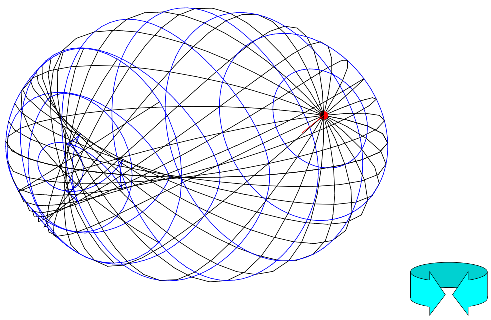
Proof. [1] For $K_1 > 0$ and $b(t) := K(c(t)) \leq K_1 =: a(t)$ we have, by [15.7], for the solution $\xi(t) = \lambda(t)\,\nu(t)$ of the Jacobi equation (see [15.1]) and that of $\mu''(t) + K_1\,\mu(t) = 0$ (i.e. $\mu(t) = \frac{1}{\sqrt{K_1}}\sin(t\sqrt{K_1})$) the relationship $\mu(t_1) \leq \lambda(t_1)$, and thus $\lambda(t) = 0 \Rightarrow t\sqrt{K_1} \geq \pi$.

For $K_1 \leq 0$, w.o.l.g. $K_1 = 0$, we have analogoulsy $\lambda(t) \geq \mu(t) = t > 0$ for all $t > 0$.

The statement [2] is shown quite analogously. □

15.9 Theorem of Bonnet.

If M *is a complete connected Riemann surface and* $K(x) \geq K_0 > 0$ *for all* $x \in M$*, then the geodesic distance of each pair of points is at most* $\frac{\pi}{\sqrt{K_0}}$*. In particular,* M *is compact.*



Proof. By [13.12.4] there is a geodesic of minimal length for every two points. If its length is greater than $\frac{\pi}{\sqrt{K_0}}$, it contains conjugate points by [15.8.2], and by the converse of [15.6], which we will show in [15.16], this geodesic will not be the shortest connection, a contradiction. So their endpoints are at most $\frac{\pi}{\sqrt{K_0}}$ away for one another. In particular, the diameter is

$$d(M) := \sup\{d(x_1, x_2) : x_1, x_2 \in M\} \leq \frac{\pi}{\sqrt{K_0}},$$

and thus M is compact by [13.12]. □

15.10 Lemma.

Let $K(x) \leq K_1$ *for all* $x \in M$ *and* $\rho_1 < \rho := \frac{\pi}{\sqrt{K_1}}$*. Furthermore, let* $c : [0, \rho_1] \to M$ *be an arc-length parameterized geodesic from* $x := c(0)$ *to* $y := c(\rho_1)$*. Let* $v : [s_0, s_1] \to B_\rho(0) \subseteq T_xM$ *be a curve with* $\exp_x(v(s_0)) = x$ *and* $\exp_x(v(s_1)) = y$*. Then* $L(\exp_x \circ v) \geq L(c)$*.*

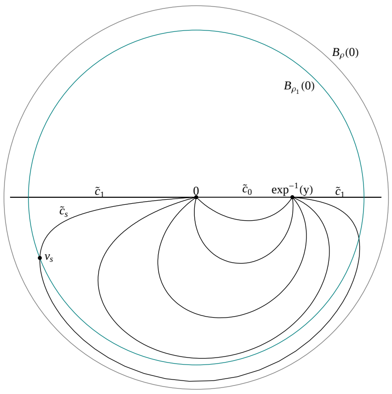
Proof. Because of $K(x) \leq K_1$ for all $x \in M$, $\exp_x : B_\rho(0) \to M$ is a local diffeomorphism by [15.8.1]. Thus, $(\exp_x)^*(g)$ is a Riemann metric on $B_\rho(0)$ and $\exp_x$ is local isometry with respect to it. The polar coordinates on $B_\rho(0)$ thus induce geodesic polar coordinates (see [10.9]) on $B_\rho(0)$ with respect to the metric $(\exp_x)^*(g)$ and thus the result follows from [10.11]. □

15.11 Proposition.

Let $(c_s)_s$ *be a smooth homotopy relative to* $\{0, 1\}$ *between two different geodesics from* x *to* y *with* $L(c_0) \leq L(c_1)$*. If* $K(x) \leq K_1$ *for all* $x \in M$ *then there exists an* $0 \leq s_0 \leq 1$ *with* $L(c_{s_0}) \geq \frac{2\pi}{\sqrt{K_1}} - L(c_0)$

Note that for $K_1 \leq 0$ (and therefore $2\pi/\sqrt{K_1} := +\infty$) this means that different geodesics from x to y can not be homotopic.

Proof. Put $\rho := \frac{\pi}{\sqrt{K_1}}$. Because of 15.8.1, $\exp_x : B_\rho(0) \to M$ is a local diffeomorphism, and thus on each smaller open ball, it is a covering map (because the fibers are finite). W.l.o.g. $L(c_0) < \rho$ (otherwise, already $L(c_0) \geq \frac{2\pi}{\sqrt{K_1}} - L(c_0)$).



Suppose $c_s(t) \in \exp_x(B_{\rho_1}(0))$ for a $\rho_1 < \rho$ and all s and t. Then a lift $(t,s) \mapsto \tilde c_s(t)$ would exist. But since the lift of the geodesic c_1 must be a straight line through 0, this is impossible because of $c_0 \neq c_1$. Thus, the homotopy comes as close as possible to the boundary of $\exp_x(B_\rho(0))$, i.e. for each $\rho_1 < \rho$ there exists an $s \in [0,1]$ s.t. the lift $\tilde c_s : [0,1] \to B_\rho(0)$ exists and contains a point v_s at a distance of ρ_1 from 0. By 15.10 the image of the closed curve, formed from $\tilde c_s$ followed by the reverse straight line $\tilde c_0$, has length $\geq 2\rho_1$ (because the two parts from 0 to v_s have length $\geq \rho_1$), so $L(c_s) \geq 2\rho_1 - L(c_0)$.
Since ρ_1 was arbitrarily close to ρ, the continuity of $s \mapsto L(c_s)$ implies the existence of an s_0 with $L(c_{s_0}) \geq 2\rho - L(c_0) = \frac{2\pi}{\sqrt{K_1}} - L(c_0)$. □

15.12 Theorem [58].

The exponential mapping of each complete connected Riemann surface with $K \leq 0$ is a covering map $\exp_x : T_xM \to M$ for each $x \in M$. Thus, if M is even simply connected, then $\exp_x : T_xM \to M$ is a diffeomorphism and for every two points there is exactly one minimal connecting geodesic.

Proof. Let first M be simply connected. Because of 15.8.1, there are no conjugate points for $K \leq 0$, and thus $\exp_x : T_xM \to M$ is everywhere a local diffeomorphism. By proposition 13.12 of Hopf-Rinow $\exp_x$ is surjective. Now for injectivity. Let $\exp_x(v_0) = \exp_x(v_1) =: p \in M$. Then $c_i(t) := \exp_x(t\, v_i)$ are geodesics connecting x to p. Since M is simply connected, these are homotopic. Because of 15.11 they are identical, i.e. $v_0 = v_1$.

According to 10.11, the radial connecting geodesic is of minimal length.

For general M we consider the universal covering $p : \tilde M \to M$. According to what has just been said, $\exp_{\tilde x} : T_{\tilde x}\tilde M \to \tilde M$ is a diffeomorphism and thus $\exp_x \circ T_{\tilde x}p = p \circ \exp_{\tilde x} : T_{\tilde x}\tilde M \to M$ is a covering map. Since $T_{\tilde x}p : T_{\tilde x}\tilde M \to T_xM$ is a linear isomorphism, $\exp_x : T_xM \to M$ itself is a covering map. □

15.13 Jacobi fields on general Riemannian manifolds.

Since only $\exp_x(t(v+sw))$ for $v,w \in T_xM$ and $t,s \in \mathbb{R}$ was needed for the description of Jacobi fields, they can be defined on general Riemann manifolds as well, where (by 14.11) the Gauss curvature $K(c(t))$ has to be replaced by the sectional curvature $K(\langle\{\xi(t), c'(t)\}\rangle)$ (see 14.7) and $K(c(t))\,\xi(t)$ by $R(\xi(t), c'(t))\,c'(t)$: In fact, for Riemann surfaces and unit normal field ν along c we have

$$R(\nu,c')c' = \langle R(\nu,c')\,c',c'\rangle\, c' + \langle R(\nu,c')\,c',\nu\rangle\, \nu = R(\nu,c',c',c')\,c' + R(\nu,c',c',\nu)\,\nu$$

$$\overset{14.5.2}{=\!=} R(\nu,c',c',\nu)\,\nu \overset{14.1.1}{=\!=} K\,\nu.$$

Hence

$$K(c(t))\,\xi(t) = K(c(t))\,\lambda(t)\,\nu(t) = \lambda(t)\,R(\nu(t),c'(t))\,c'(t) = R(\xi(t),c'(t))\,c'(t).$$

Thus the JACOBI EQUATION for general Riemannian manifolds looks like follows:

$$\nabla^2\xi + R(\xi,c')\cdot c' = 0,$$

The solutions of the Jacobi equation are again called JACOBI FIELDS and these are exactly the derivative in direction of the variation parameter of 1-parameter variations of the geodesic c by geodesics:
In fact, let $\varphi : \mathbb{R}^2 \to M$ be a variation consisting of geodesics $t \mapsto \varphi(s,t)$ (i.e. $0 = \nabla_{\frac{\partial\varphi}{\partial t}} \frac{\partial}{\partial t}\varphi(s,t)$). In the proof of the Gaussian Lemma 13.10 we have shown

$$\nabla_{\frac{\partial\varphi}{\partial s}} \frac{\varphi}{\partial t}\varphi(s,t) = \nabla_{\frac{\partial\varphi}{\partial t}} \frac{\partial}{\partial s}\varphi(s,t).$$

Thus

$$\begin{aligned}
0 &= \nabla_{\frac{\partial\varphi}{\partial s}} \nabla_{\frac{\partial\varphi}{\partial t}} \frac{\partial}{\partial t}\varphi(s,t)\\
&\overset{14.2}{=\!=} R\left(\tfrac{\partial\varphi}{\partial s},\tfrac{\partial\varphi}{\partial t}\right)\cdot \tfrac{\partial}{\partial t}\varphi(s,t) + \nabla_{\frac{\partial\varphi}{\partial t}} \nabla_{\frac{\partial\varphi}{\partial s}} \tfrac{\partial}{\partial t}\varphi(s,t) + \nabla_{[\frac{\partial\varphi}{\partial t},\frac{\partial\varphi}{\partial s}]} \tfrac{\partial}{\partial t}\varphi(s,t)\\
&= R\left(\tfrac{\partial\varphi}{\partial s},\tfrac{\partial\varphi}{\partial t}\right)\cdot \tfrac{\partial}{\partial t}\varphi(s,t) + \nabla_{\frac{\partial\varphi}{\partial t}} \nabla_{\frac{\partial\varphi}{\partial t}} \tfrac{\partial}{\partial s}\varphi(s,t) + 0
\end{aligned}$$

and for $s = 0$, $c(t) := \varphi(0,t)$ and $\xi(t) := \frac{\partial}{\partial s}|_{s=0}\varphi(s,t)$ we obtain the Jacobi equation

$$0 = R(\xi,c')\cdot c' + \nabla^2\xi.$$

The representation in 15.3 for Jacobi fields ξ with $\xi(0) = 0$ via the derivative of $\exp_{c(0)}$ holds exactly as in the 2-dimensional case, and the zeros of these Jacobi fields describe again CONJUGATED points as in 15.5.

15.15 Proposition. Variation of energy.

Let $(c_s)_s$ be a smooth variation with fixed endpoints of a curve $c_0 : [a,b] \to M$, i.e. $c : (s,t) \mapsto c_s(t)$ is smooth from $\mathbb{R}\times[a,b] \to M$ with $s \mapsto c_s(t)$ being constant for $t \in \{a,b\}$. Let $Y_s(t) := \frac{\partial}{\partial s}c_s(t) \in T_{c_s(t)}M$ and let the energy be

$$E(c_s) := \frac{1}{2}\int_a^b g(\dot c_s,\dot c_s) = \frac{1}{2}\int_a^b g_{c_s(t)}\left(\tfrac{\partial}{\partial t}c_s(t), \tfrac{\partial}{\partial t}c_s(t)\right)\,dt.$$

Then

$$\text{(1)}\qquad \frac{d}{ds}E(c_s)\Big|_{s=0} = -\int_a^b g(\nabla_{\dot c_0}\dot c_0, Y_0)$$

and, if c_0 is a geodesic, then $\frac{d}{ds}E(c_s)\big|_{s=0} = 0$ and

$$\text{(2)}\qquad \left(\frac{d}{ds}\right)^2 E(c_s)\Big|_{s=0} = \int_a^b \Big(|\nabla_{\dot c_0}Y_0|_g^2 - \underbrace{K(\langle\{\dot c_0, Y_0\}\rangle)\cdot\left(|Y_0|_g^2 - g(\dot c_0,Y_0)\right)}_{=R(Y_0,\dot c_0,\dot c_0,Y_0)}\Big)$$

Proof.

1. Put $X_s(t) := \frac{\partial}{\partial t}c_s(t)$. Then $X = Tc\cdot\frac{\partial}{\partial t}$ and thus for $Y := Tc\cdot\frac{\partial}{\partial s}$ we have

$$\nabla_Y X - \nabla_X Y \overset{13.4.4}{=\!=} [X,Y] = Tc\cdot\left[\tfrac{\partial}{\partial t},\tfrac{\partial}{\partial s}\right] = 0.$$

Consequently,

$$\begin{aligned}\frac{d}{ds}E(c_s) &= \frac{1}{2}\int_a^b \frac{\partial}{\partial s} g(X,X)\,dt \overset{\boxed{13.4.5}}{=\!=\!=} \int_a^b g(\nabla_Y X, X)\,dt\\ &= \int_a^b g(\nabla_X Y, X)\,dt \overset{\boxed{13.4.5}}{=\!=\!=} \int_a^b \frac{\partial}{\partial t} g(Y,X)\,dt - \int_a^b g(Y,\nabla_X X)\,dt\\ &= g(Y,X)\Big|_a^b - \int_a^b g(Y,\nabla_X X)\,dt\end{aligned}$$

and, because of $Y|_{\mathbb{R}\times\{a,b\}} = 0$ and $X_0(t) := X(0,t) = \dot c_0(t)$, we have

$$\frac{d}{ds}E(c_s)\Big|_{s=0} = 0 - \int_a^b g(Y_0, \nabla_{\dot c_0}\dot c_0)\,dt.$$

2.

$$\begin{aligned}\Big(\frac{d}{ds}\Big)^2 E(c_s) &= \frac{1}{2}\int_a^b \Big(\frac{\partial}{\partial s}\Big)^2 g(X,X)\,dt \overset{\boxed{13.4.5}}{=\!=\!=} \int_a^b \frac{\partial}{\partial s} g(\nabla_Y X, X)\,dt\\ &= \int_a^b \frac{\partial}{\partial s} g(\nabla_X Y, X)\,dt\\ &\overset{\boxed{13.4.5}}{=\!=\!=} \int_a^b \Big(g(\nabla_Y\nabla_X Y, X) + g(\nabla_X Y, \nabla_Y X)\Big)\,dt\\ &= \int_a^b \Big(g(\nabla_X\nabla_Y Y, X) - g([\nabla_X,\nabla_Y]Y, X) + g(\nabla_X Y, \nabla_X Y)\Big)\,dt\\ &= \int_a^b \Big(\frac{\partial}{\partial t} g(\nabla_Y Y, X) - g(\nabla_Y Y, \nabla_X X) +\\ &\qquad\qquad - g(R(X,Y)Y,X) + g(\nabla_X Y, \nabla_X Y)\Big)\,dt\\ &= g(\nabla_Y Y, X)\Big|_a^b +\\ &\quad + \int_a^b \Big(-g(\nabla_Y Y, \nabla_X X) - R(X,Y,Y,X) + g(\nabla_X Y, \nabla_X Y)\Big)\,dt\end{aligned}$$

and for $s = 0$, a geodesic c_0, and $Y_0(t) := Y(0,t) = \frac{\partial}{\partial s}|_{s=0} c_s(t)$ we have:

$$\begin{aligned}\Big(\frac{d}{ds}\Big)^2 E(c_s)\Big|_{s=0} &= \int_a^b \Big(-g(\nabla_{Y_0}Y_0, \underbrace{\nabla_{X_0}X_0}_{=0}) - R(\underbrace{X_0}_{\dot c_0}, Y_0, Y_0, X_0) + g(\underbrace{\nabla_{X_0}Y_0}_{\nabla_{\dot c_0}Y_0}, \nabla_{X_0}Y_0)\Big)\\ &= \int_a^b \Big(|\nabla_{\dot c_0}Y_0|_g^2 - R(\dot c_0, Y_0, Y_0, \dot c_0)\Big).\end{aligned}$$

Finally, according to 14.7, we have

$$R(\dot c_0, Y_0, Y_0, \dot c_0) = K\big(\langle\{\dot c_0, Y_0\}\rangle\big)\cdot\big(|Y_0|_g^2 - g(\dot c_0, Y_0)^2\big)$$

□

The Hessian of E at c_0 (i.e. the symmetric bilinear form $E''(c_0)$ corresponding to the quadratic form $\big(\frac{d}{ds}\big)^2 E(c_s)\Big|_{s=0}$) is thus given by the so-called INDEX FORM

$$I(Y,Z) := \int_a^b \Big(g(\nabla_{\dot c_0}Y, \nabla_{\dot c_0}Z) - R(\dot c_0, Y, Z, \dot c_0)\Big)$$

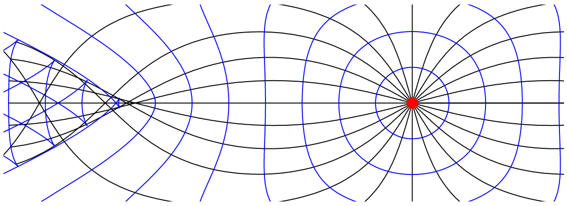
for arbitrary vector fields Y and Z along a geodesic c_0.

We now generalize 15.6 while showing at the same time its converse.

15.16 Theorem.

Let (M,g) be a Riemannian manifold and $c : [a,b] \to M$ be a geodesic. Then the following statements are equivalent:

1. *There are no conjugate points on c.*
2. *c has minimal length among all sufficiently close curves with same end points.*
3. *c has minimal energy among all sufficiently close curves with same end points.*
4. *$E''(c)$ is positive semi-definite, i.e. $\big(\frac{d}{ds}\big)^2|_{s=0}E(c_s) = I(\frac{d}{ds}\big|_{s=0}c_s, \frac{d}{ds}c_s\big|_{s=0}) \geq 0$ for all variations $(c_s)_s$ of c with fixed boundary values.*



Proof. (1⇒2) This is analogous to 15.6, but we need a generalization of Corollary 13.11 instead of 10.11: Let $a = 0$. We put $x := c(0)$ and $v := c'(0)$. Thus $c(t) = \exp_x(t\,v)$ for all $0 \le t \le b$. For each $t \in [0,b]$ there exists $\varepsilon_t > 0$ such that $\exp_x$ restricted to the ball $B_{2\varepsilon_t}(tv)$ is a diffeomorphism onto its image. In particular finitely many of the open intervals $(t-\varepsilon_t, t+\varepsilon_t)$ cover $[0,b]$, lets say for $t_1,\dots,t_N$. Let 2ε be the minimum of the corresponding $\varepsilon_{t_1},\dots,\varepsilon_{t_N}$. So each t is contained in some interval $(t_i-\varepsilon_{t_i}, t_i+\varepsilon_{t_i})$ and thus $\exp_x$ is injective on the ball $B_{2\varepsilon}(tv) \subseteq B_{2\varepsilon_{t_i}}(t_i v)$. Take a partition $0 = b_0 < \dots < b_M = b$ with $b_{i+1} - b_i < \varepsilon$ and let $\bar c$ be a curve (with same end points as c) so close to c that $\bar c([b_i,b_{i+1}]) \subseteq \exp_x(B_\varepsilon(b_i v))$. Thus we have a unique lift $\exp_x|_{B_\varepsilon(b_i v)}^{-1} \circ \bar c$ of $\bar c|_{[b_i,b_{i+1}]}$ into $B_\varepsilon(b_i v)$ for each i and these lifts fit together with end points 0 and bv, since $\bar c(b_{i+1})$ is the unique inverse image of $\exp_x$ on $B_{2\varepsilon}(b_{i+1}v) \supseteq B_\varepsilon(b_{i+1}v) \cup B_\varepsilon(b_i v)$.
Let $t \mapsto r(t)\cdot \bar v(t)$ be the polar decomposition of such a lift. We consider the variation $\varphi(s,t) := \exp_x(\rho\cdot t\cdot \bar v(s))$ by radial geodesics with $\rho := r(b) = |b\,v|$. Then

$$\bar c(t) = \exp_x(r(t)\,\bar v(t)) = \varphi\left(t, \tfrac{r(t)}{\rho}\right)$$

As is the proof of the Gauss lemma 13.10

$$\frac{\partial}{\partial t} g\left(\frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial s}\right) = 0.$$

so $g(\partial_2\varphi, \partial_1\varphi)(s,t) = g(\partial_2\varphi,\partial_1\varphi)(s,0) = g(\rho\,\bar v(s), 0) = 0$ and, as in the proof of 13.11 (with $\gamma(r,t) := \varphi(t, \frac{r}{\rho}) = \exp_x(r\,\bar v(t))$, so $\bar c(t) = \gamma(r(t),t)$), furthermore

$$|\bar c'(t)|^2 \ge |r'(t)|^2$$

with equality if and only if $\bar v$ is constant. Finally

$$L(\bar c) = \int_a^b |\bar c'(t)|\,dt \ge \int_a^b |r'(t)|\,dt \ge \int_a^b r'(t)\,dt = r(b) - r(a) = \rho = L(c)$$

with equality if and only if $\bar v$ is constant ($= v$ because of $r(b)\,\bar v = b\,v$) and $r'(t) \ge 0$, so $\bar c$ is a reparametrization of c.

(2⇒3) The Cauchy-Schwartz inequality yields

$$L(c) = \int_a^b |c'(t)|\,dt \le \left(\int_a^b 1^2\right)^{1/2}\cdot\left(\int_a^b |c'(t)|^2\,dt\right)^{1/2} = \sqrt{b-a}\,\sqrt{2\,E(c)},$$

hence $L(c)^2 \leq 2(b-a)E(c)$ and equality holds if and only if c is parameterized proportional to arc length.

Now, let c_0 be locally of minimal length and w.l.o.g. parametrized by arc length. Let c_s be a variation of c_0 then

$$E(c_0) = \frac{L(c_0)^2}{2(b-a)} \leq \frac{L(c_s)^2}{2(b-a)} \leq E(c_s),$$

Hence c_0 is also of minimal energy locally.

(3⇒4) From $E(c_s) \geq E(c_0)$ for all $s \geq 0$ and hence $\frac{d}{ds}\big|_{s=0} E(c_s) = 0$ we deduce $\left(\frac{d}{ds}\right)^2 |_{s=0} E(c_s) \geq 0$ using 15.15.2.

(4⇒1) Suppose $c(a)$ and $c(t_0)$ were conjugate points on c. Let $Y \neq 0$ be a Jacobi field along c with $Y(a) = 0 = Y(t_0)$ (and hence $\nabla Y(t_0) \neq 0$). Put $V := \chi_{[a,t_0]} Y$ (a continuous and piecewise smooth vector field) and let W be a smooth vector field along c with $W(t_0) = -\nabla Y(t_0)$ and $W(a) = 0 = W(b)$. Let I_1 and I_2 be the index forms of $c|_{[a,t_0]}$ and $c|_{[t_0,b]}$. Then for each vector field Z along c with $Z(a) = 0$:

$$\begin{aligned} 0 &= \int_a^{t_0} g\Big(\nabla^2 Y + R(Y,\dot c)\dot c, Z\Big) = \int_a^{t_0} \Big(g(\nabla^2 Y, Z) + g\big(R(Y,\dot c)\dot c, Z\big)\Big) \\ &= \int_a^{t_0} \Big(\frac{d}{dt} g(\nabla Y, Z) - g(\nabla Y, \nabla Z) + \underbrace{g\big(R(Y,\dot c)\dot c, Z)\big)}_{R(Y,\dot c,\dot c,Z)}\Big) \\ &\overset{14.5.1,\,14.5.2}{=\!=\!=} g(\nabla Y, Z)\Big|_a^{t_0} - I_1(Y,Z). \end{aligned}$$

For $Z := W$ (with $W(a) = 0$ and $W(t_0) = -\nabla Y(t_0)$) we get

$$I(V,W) = I_1(Y,W) + I_2(0,W) = I_1(Y,W) = -|\nabla Y(t_0)|_g^2 < 0.$$

and finally for $Z := Y$ (with $Y|_{\{a,t_0\}} = 0$) we obtain

$$\begin{aligned} I(V+\varepsilon W, V+\varepsilon W) &= I_1(Y,Y) + 2\varepsilon I(V,W) + \varepsilon^2 I(W,W) \\ &= 0 - 2\varepsilon |\nabla Y(t_0)|_g^2 + \varepsilon^2 I(W,W) < 0 \end{aligned}$$

for all small $\varepsilon > 0$. We can approximate $V + \varepsilon W$ by a smooth vector field X, which also vanishes at the boundary points and satisfies $I(X,X) < 0$, a contradiction. □

15.17 Corollary.

Let c be a geodesic parameterized by arc length.

1. *If $K(c(t)) \leq 0$ holds to all t, then there are no conjugate points.*
2. *If $K(c(t)) > K_0$ holds for a constant $K_0 > 0$ and all t, conjugate points will be on each closed interval of length $\frac{\pi}{\sqrt{K_0}}$.*

Proof. 1 Let ξ be a Jacobi field along c with $\xi(0) = 0 \neq \nabla\xi(0)$. Then

$$\begin{aligned} &|\xi(t)|_g^2 > 0 \text{ for all small } t > 0 \\ \text{and}\quad (\tfrac{d}{dt})^2 |\xi(t)|_g^2 &= 2\tfrac{d}{dt} g(\xi(t), \nabla\xi(t)) = 2g(\nabla\xi(t), \nabla\xi(t)) + 2g(\xi(t), \nabla^2\xi(t)) \\ &= 2|\nabla\xi(t)|_g^2 - 2R\big(\xi(t), c'(t), c'(t), \xi(t)\big) \geq 0, \end{aligned}$$

So $t \mapsto |\xi(t)|$ is monotonously increasing and thus $\xi(t) \neq 0$ for all $t \neq 0$.

2 Let ν be a parallel unit vector field along c with $\nu(t) \perp c'(t)$ for all t and consider $\xi(t) := \sin(t\sqrt{K_0})\,\nu(t)$ for $K_0 > 0$. Then $\xi(0) = 0 = \xi(\pi/\sqrt{K_0})$ and

$$\nabla\nu = 0 \;\Rightarrow\; \nabla^2\xi(t) = -K_0 \sin(t\sqrt{K_0})\,\nu(t) \Rightarrow g(\xi(t), \nabla^2\xi(t)) = -K_0 \sin(t\sqrt{K_0})^2$$
$$\Rightarrow R\big(\xi(t), c'(t), c'(t), \xi(t)\big) = \sin(t\sqrt{K_0})^2\, R\big(\nu(t), c'(t), c'(t), \nu(t)\big) > K_0 \sin(t\sqrt{K_0})^2$$

by assumption, and thus (note that $\frac{d}{dt} g(\xi, \nabla\xi) = g(\nabla\xi, \nabla\xi) + g(\xi, \nabla^2\xi)$)

$$I(\xi,\xi) := \int_a^b -g\big(\xi(t), \nabla^2\xi(t)\big) - R\big(\xi(t), c'(t), c'(t), \xi(t)\big)\, dt < 0,$$

hence there are conjugate points on c because of (1⇒4) in 15.16. □

15.18 Remark.

The curvature condition in 15.17.2 can also be replaced by:

$$\operatorname{Ricci}(X,X) > K_0\,(\dim(M) - 1)\,|X|^2 \text{ for a constant } K_0 > 0 \text{ and all } X.$$

Proof. Let $c : [0, L] \to M$ with $L := \frac{\pi}{\sqrt{K_0}}$. Extend c' to an orthonormal basis $c', \nu_1, \dots, \nu_{m-1}$ of parallel vector fields along c and consider $\xi_i(t) := \sin(t\sqrt{K_0})\,\nu_i(t)$. Then $\xi_i(0) = 0 = \xi_i(\pi/\sqrt{K_0})$ and

$$\nabla^2\xi_i(t) = -K_0 \sin(t\sqrt{K_0})\nu(t) \Rightarrow g(\xi_i(t), \nabla^2\xi_i(t)) = -K_0 \sin(t\sqrt{K_0})^2$$

$$\operatorname{Ricci}(c', c') = 0 + \sum_{i=1}^{m-1} R\big(\nu_i, c', c', \nu_i\big)$$

$$\sum_{i=1}^{m-1} R\big(\xi_i(t), c'(t), c'(t), \xi_i(t)\big) = \sum_{i=1}^{m-1} \sin(t\sqrt{K_0})^2 R\big(\nu_i(t), c'(t), c'(t), \nu_i(t)\big)$$
$$= \sin(t\sqrt{K_0})^2 \underbrace{\operatorname{Ricci}(c'(t), c'(t))}_{> K_0\cdot(m-1)}$$

by assumption and thus

$$\sum_{i=1}^{m-1} I(\xi_i, \xi_i) = -\sum_{i=1}^{m-1} \int_0^L g\big(\xi_i(t), \nabla^2\xi_i(t)\big) + R\big(\xi_i(t), c'(t), c'(t), \xi_i(t)\big)\, dt < 0.$$

Hence at least one summand $I(\xi_i, \xi_i) < 0$ and thus, as before, there are conjugated points on c by (1⇒4) in 15.16. □

15.19 Theorem. [26].

The exponential mapping $\exp_x : T_xM \to M$ of complete connected Riemannian manifold M with sectional curvature $K \le 0$ is a covering map for each $x \in M$. So, if M is in addition simply connected, then $\exp_x : T_xM \to M$ is a diffeomorphism and for each two points there is exactly one minimal connecting geodesic.

Proof. By Theorem 13.12 of Hopf-Rinow the mapping $\exp_x : T_xM \to M$ is onto.

Because of $K \le 0$, there are no conjugate points by 15.17.1 and thus $\exp_x : T_xM \to M$ is a local diffeomorphism everywhere. Moreover, $\exp_x : (T_xM, \exp_x^* g) \to (M, g)$ is a local isometry of complete (because at least the geodesics through 0 in T_xM are infinitely long, see the arguments in 15.20) connected Riemann manifolds and thus, by the following lemma 15.20, is a covering map.

If, in addition, M is simply connected, each covering map is a diffeomorphism. □

15.20 Lemma.

Let $(\tilde M, \tilde g)$ and (M, g) be connected Riemannian manifolds of equal dimension and let $f : \tilde M \to M$ be a local isometry everywhere.
If $\tilde M$ is complete, then this also holds for M and, furthermore, $f : \tilde M \to M$ is a covering map and is in particular surjective.

Proof. Claim: *Geodesics can be lifted (uniquely).*
Let $c : I \to M$ be a geodesic with $c(0) = x \in f(\tilde M)$ and $\tilde x \in f^{-1}(x) \subseteq \tilde M$. Since $\tilde M$ is complete, a unique geodesic $\tilde c : \mathbb{R} \to \tilde M$ exists with $\tilde c'(0) = (T_x f)^{-1}(c'(0))$. Since f is a local isometry, $f \circ \tilde c$ is a geodesic with $(f \circ \tilde c)'(0) = c'(0)$, so $c = (f \circ \tilde c)|_I$. In the proof of (1 ⇒ 4) of Theorem 13.12 of Hopf-Rinow we have shown that $\exp_x : T_x M \to M$ is surjective provided the radial geodesics starting at x are infinite long. Consequently, also $\exp_x \circ T_{\tilde x} f = f \circ \exp_{\tilde x}$ and thus f are onto. By the proof of (1 ⇒ 3) in 13.12 we conclude that every bounded closed set in M is compact, and hence M is complete.

For each $x \in M$, let $B_r(0_x) \subseteq T_x M$ be a ball chosen so that $\exp_x : B_r(0_x) \to \exp_x(B_r(0_x)) =: U$ is a diffeomorphism to an open neighborhood U of x and thus every $y \in U$ can be connected with x by a unique (radial) geodesic by 13.11.

Claim: *U is trivializing for f.*
For each $\tilde x \in f^{-1}(x)$ we consider the open set $\tilde U_{\tilde x}$ of all points in $\tilde M$ which can be connected with $\tilde x$ by geodesics in $f^{-1}(U)$. By the previous claim these geodesics are in bijective correspondance via f_* to the geodesics in U starting at x and these are uniquely determined by their endpoint, so $f : \tilde U_{\tilde x} \to U$ is bijective and thus a diffeomorphism.
Let $\tilde y \in f^{-1}(U)$ be arbitrary. Then there is a unique geodesic c in U which connects x to $f(\tilde y)$. So there is a unique geodesic $\tilde c$ in $f^{-1}(U)$ which connects $\tilde y$ with $f^{-1}(x)$. Thus, its associated endpoint $\tilde x := \tilde c(1) \in f^{-1}(x)$ is uniquely determined and $f^{-1}(U)$ is the disjoint union $\bigsqcup_{\tilde x \in f^{-1}(x)} \tilde U_{\tilde x}$, i.e. f is a covering map. □

15.22 Theorem [124].

Let M be a complete connected Riemannian manifold with sectional curvature $K \geq K_0$ (or $\operatorname{Ricci}(X, X) \geq K_0\,(m-1)\,|X|^2$ for all $X \in TM$) for a constant $K_0 > 0$.
Then the geodesic distance of each two points is at most $\frac{\pi}{\sqrt{K_0}}$. In particular, M is compact and the fundamental group $\pi_1(M)$ is finite.

Proof. By 13.12.4 there exists a geodesic of minimal length for every two points. If its length is greater than $\frac{\pi}{\sqrt{K_0}}$, it will contain conjugate points by 15.17.2 (resp. 15.18), and according to 15.16, this geodesic will not be the shortest path, a contradiction. Hence its endpoints are at most $\frac{\pi}{\sqrt{K_0}}$ away from each other. In particular, the diameter is

$$d(M) := \sup\big\{d(x_1, x_2) : x_1, x_2 \in M\big\} \leq \tfrac{\pi}{\sqrt{K_0}},$$

and thus M is compact by 13.12.

Since the universal covering is, by the same reason, compact as well and since it has the fundamental group $\pi_1(M)$ as fibers, this group has to be finite. □

15.23 Sphere theorem. [10], [76] and [18].

Let M be a complete simply connected m-dimensional Riemannian manifold with sectional curvature $1/4 < K \leq 1$. Then M is diffeomorphic to the sphere S^m.

16. The Cartan method of moving frames

16.1 Definition. Connection form ω.

Let (M, g) be an m-dimensional (PSEUDO) RIEMANNIAN MANIFOLD, i.e. g is a (not necessarily positive) definite metric on the manifold M. A local m-FRAME on an open set $U \subseteq M$ is an m-tuple of vector fields s_i on U forming a basis of $T_x M$ pointwise (i.e. for each $x \in U$). It is called $s = (s_1, \dots, s_m)$ ORTHONORMAL-FRAME if $(s_i(x))_i$ is an orthonormal basis of $T_x M$ for each $x \in U$, i.e. $g(s_i, s_j) = \pm\delta_{i,j}$.

Locally orthonormal frames exist, because the symmetric definite bilinear form g_x on $T_x M$ can be expressed in a basis $(e_1, \dots, e_m)$ as

$$g_x\Big(\sum_i v^i e_i, \sum_j v^j e_j\Big) = \sum_{i \le p} v^i w^i - \sum_{i > p} v^i w^i,$$

see [**95**, 4.5]. And by extending the e_i into locally linear independent vector fields and applying the Gram-Schmidt procedure to them, we get an orthonormal frame. If s and s' are two m-frames on U, then $s'_i = \sum_{j=1}^m s_j \cdot h_i^j$ (in short: $s' = s \cdot h$) for a unique $h = (h_i^j)_{i,j=1,\dots m} \in C^\infty(U, GL(m))$.

Let $s = (s_1, \dots, s_m)$ be an m-frame on U and ∇ the Levi-Civita derivative. Then there are uniquely determined $\omega_i^j \in \Omega^1(U)$ with

$$\nabla_\xi s_i = \sum_j s_j \cdot \omega_i^j(\xi), \text{ in short: } \nabla_\xi s = s \cdot \omega(\xi) \text{ or } \nabla s = s \cdot w$$

where $\omega = (\omega_i^j)_{i,j=1,\dots,m} \in \Omega^1(U, L(m, m))$ is called CONNECTION FORM or CONNECTION MATRIX of ∇ with respect to s.

16.2 Lemma. Covariant derivative via the connection form.

Let $\eta = \sum_j s_j \cdot \eta^j$ with $\eta^j \in C^\infty(U, \mathbb{R})$ be a vector field on U. Then

$$\nabla\eta = \sum_k s_k \cdot \Big(\sum_j \omega_j^k \cdot \eta^j + d\eta^k\Big) = s \cdot (\omega \cdot \eta + d\eta).$$

Proof.

$$\nabla_\xi \eta = \nabla_\xi\Big(\sum_j s_j \cdot \eta^j\Big) \overset{\boxed{13.4.3}}{=\!=} \sum_j \Big(\nabla_\xi s_j \cdot \eta^j + s_j \cdot \xi(\eta^j)\Big)$$
$$\overset{\boxed{16.1}}{=\!=} \sum_j \Big(\sum_k s_k \cdot \omega_j^k(\xi) \cdot \eta^j + s_j \cdot d\eta^j(\xi)\Big) = \sum_k s_k \cdot \Big(\sum_j \omega_j^k \cdot \eta^j + d\eta^k\Big)(\xi) \quad \square$$

16.3 Lemma. Transformation behavior of the connection form.

If s and $s' = s \cdot h$ are two m-frames and ω and ω' are the associated connection forms, then

$$h \cdot \omega' = dh + \omega \cdot h.$$

Proof.

$$s \cdot h \cdot \omega' = s' \cdot \omega' = \nabla s' \overset{\boxed{16.1}}{=\!=} \nabla(s \cdot h) \overset{\boxed{13.4.3}}{=\!=} s \cdot dh + \nabla s \cdot h \overset{\boxed{16.1}}{=\!=} s \cdot dh + s \cdot \omega \cdot h$$
$$\Rightarrow \quad h \cdot \omega' = dh + \omega \cdot h. \quad \square$$

16.4 Lemma. Symmetry property of the connection form.

If s is an orthonormal frame, ω is the associated connection form and $\varepsilon_i := g(s_i, s_i) \in \{\pm 1\}$, then:

$$\varepsilon_i\, \omega_k^i + \varepsilon_k\, \omega_i^k = 0.$$

Proof.

$$\begin{aligned}\varepsilon_i \delta_{i,j} = g(s_i, s_j) \Rightarrow 0 = d(g(s_i, s_j)) &\overset{13.4.5}{=\!=} g(\nabla s_i, s_j) + g(s_i, \nabla s_j)\\ &= g\Big(\sum_k s_k \cdot \omega_i^k, s_j\Big) + g\Big(s_i, \sum_k s_k \cdot \omega_j^k\Big)\\ &= \sum_k \omega_i^k\, g(s_k, s_j) + \sum_k \omega_j^k\, g(s_i, s_k) = \varepsilon_j\, \omega_i^j + \varepsilon_i\, \omega_j^i. \quad \square\end{aligned}$$

16.5 Lemma. Curvature via curvature form.

Let s be an orthonormal frame and ω its associated connection form. We put $R(\xi, \eta)s := (R(\xi, \eta)s_i)_{i=1,\dots,m} \in C^\infty(U, L(m, m))$. Then

$$R(\xi, \eta)s = s \cdot (d\omega + \omega \wedge \omega)(\xi, \eta),$$

where

$$\omega \wedge \omega := \Big(\sum_k \omega_k^i \wedge \omega_j^k\Big)_{i,j} \in \Omega^2(U, L(m, m)).$$

Proof.

$$\begin{aligned}R(\xi, \eta)s &= \nabla_\xi \nabla_\eta s - \nabla_\eta \nabla_\xi s - \nabla_{[\xi,\eta]} s\\ &= \nabla_\xi(s \cdot \omega(\eta)) - \nabla_\eta(s \cdot \omega(\xi)) - s \cdot \omega([\xi\, \eta])\\ &= s \cdot d(\omega(\eta))(\xi) + \nabla_\xi s \cdot \omega(\eta) - s \cdot d(\omega(\xi))(\eta) - \nabla_\eta s \cdot \omega(\xi) - s \cdot \omega([\xi, \eta])\\ &= s \cdot \Big(d(\omega(\eta))(\xi) - d(\omega(\xi))(\eta) - \omega([\xi, \eta])\Big) + s \cdot \Big(\omega(\xi) \cdot \omega(\eta) - \omega(\eta) \cdot \omega(\xi)\Big)\\ &\overset{[95,\ 25.9]}{=\!=} s \cdot (d\omega + \omega \wedge \omega)(\xi, \eta). \quad \square\end{aligned}$$

16.6 Definition. Curvature form Ω.

With $\Omega := d\omega + \omega \wedge \omega \in \Omega^2(U, L(m, m))$, we denote the CURVATURE FORM or CURVATURE MATRIX with respect to s.

By 16.5 the 1. STRUCTURE EQUATION OF CARTAN holds:

$$R(\xi, \eta)s_i = \sum_k s_k \cdot \Omega_i^k(\xi, \eta), \text{ in short: } R(s) = s \cdot \Omega.$$

16.7 Lemma. Transformation behavior of the curvature form.

Let s and $s' = s \cdot h$ be two orthonormal frames and Ω and Ω' the corresponding curvature matrices, then:

$$h \cdot \Omega' = \Omega \cdot h.$$

Proof.

$$s \cdot h \cdot \Omega' = s' \cdot \Omega' \overset{\boxed{16.6}}{=\!=} R(s') = R(s \cdot h) \overset{\boxed{14.3}}{=\!=} R(s) \cdot h \overset{\boxed{16.6}}{=\!=} s \cdot \Omega \cdot h$$
$$\Rightarrow h \cdot \Omega' = \Omega \cdot h. \quad \square$$

16.8 Lemma. Symmetry property of the curvature form.

Let s be an orthonormal frame and $\varepsilon_i = g(s_i, s_i) \in \{\pm 1\}$ then $\varepsilon_i\, \Omega^i_j + \varepsilon_j\, \Omega^j_i = 0$.

Proof.

$$\begin{aligned}
\varepsilon_i \Omega^i_j &\overset{\boxed{16.5}}{=\!=} \varepsilon_i\, d\omega^i_j + \sum_k \varepsilon_i\, \omega^i_k \wedge \omega^k_j \overset{\boxed{16.4}}{=\!=} -\varepsilon_j\, d\omega^j_i - \sum_k \varepsilon_k\, \omega^k_i \wedge \omega^k_j \\
&= -\varepsilon_j\, d\omega^j_i - \sum_k \omega^k_i \wedge \varepsilon_k\, \omega^k_j \overset{\boxed{16.4}}{=\!=} -\varepsilon_j \Big(d\omega^j_i - \sum_k \omega^k_i \wedge \omega^j_k \Big) \\
&= -\varepsilon_j \Big(d\omega^j_i + \sum_k \omega^j_k \wedge \omega^k_i \Big) \overset{\boxed{16.5}}{=\!=} -\varepsilon_j \Omega^j_i. \quad \square
\end{aligned}$$

16.9 Definition. Co-frame.

Let $s = (s_i)_{i=1,\dots,m}$ be an m-frame. The dual m-CO-FRAME $r = (r^j)_{j=1,\dots,m}$ in $\Omega^1(U)$ is given by $r^j(x)(s_i(x)) := \delta^j_i$.

16.10 Lemma. Derivation equation for the co-frame.

Let s be an m-frame, r be the associated m-co-frame, and ω be the connection form. Then the 2. STRUCTURE EQUATION OF CARTAN holds:

$$dr^k + \sum_j \omega^k_j \wedge r^j = 0, \textit{ in short: } dr + w \wedge r = 0.$$

Proof. Let η be a vector field on U. Then $\eta = \sum_i s_i \cdot r^i(\eta)$.

$$\begin{aligned}
\nabla_\xi \eta = \nabla_\xi \Big(\sum_j s_j \cdot r^j(\eta) \Big) &\overset{\boxed{13.4.3}}{=\!=} \sum_j \Big(\nabla_\xi s_j \cdot r^j(\eta) + s_j \cdot \xi(r^j(\eta)) \Big) \\
&= \sum_{j,k} s_k \cdot \omega^k_j(\xi) \cdot r^j(\eta) + \sum_k s_k \cdot \xi(r^k(\eta)).
\end{aligned}$$

Consequently:

$$\begin{aligned}
0 &\overset{\boxed{13.4.4}}{=\!=} \nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta] \\
&\overset{\boxed{16.2}}{=\!=} \sum_{j,k} s_k \cdot \underbrace{\Big(\omega^k_j(\xi) \cdot r^j(\eta) - \omega^k_j(\eta) \cdot r^j(\xi) \Big)}_{(\omega^k_j \wedge r^j)(\xi,\eta)} + \\
&\qquad + \sum_k s_k \cdot \underbrace{\Big(\xi(r^k(\eta)) - \eta(r^k(\xi)) - r^k([\xi, \eta]) \Big)}_{dr^k(\xi,\eta)} \\
&= \sum_k s_k \cdot \Big(\sum_j \omega^k_j \wedge r^j + dr^k \Big)(\xi, \eta) \quad \square
\end{aligned}$$

16.11 Cartan's Lemma.

Let $v_1, \dots, v_n$ be linearly independent in a vector space E and $w_1, \dots, w_n \in E$. Then $\sum_i v_i \wedge w_i = 0$ if and only if $w_i = \sum_i a_{i,j} v_j$ with a symmetric matrix $(a_{i,j})_{i,j}$.

Proof. We choose $v_{n+1}, \dots, v_m$ so that $(v_1, \dots, v_n, v_{n+1}, \dots, v_m)$ forms a basis and hence $w_i = \sum_{j=1}^n a_{i,j} v_j + \sum_{k=n+1}^m b_{i,k} v_k$ with certain coefficients $a_{i,j}$ and $b_{i,k}$. Therefore

$$\sum_{i\le n} v_i \wedge w_i = \sum_{i<j\le n} (a_{i,j} - a_{j,i})\, v_i \wedge v_j + \sum_{i\le n<k} b_{i,k}\, v_i \wedge v_k.$$

Since $(v_i \wedge v_j)_{i<j}$ is a basis of $\Lambda^2(E)$, this sum is 0 if and only if $b_{i,k} = 0$ and $a_{i,j} = a_{j,i}$. □

16.12 Lemma.

For orthonormal frames, the connection form ω is uniquely determined by the 2nd structure equation 16.10 of Cartan.

Proof. Let r be the co-frame of an orthonormal frame s. From

$$dr + \omega \wedge r \overset{16.10}{=\!=\!=} 0 \overset{16.10}{=\!=\!=} dr + \omega' \wedge r$$

we deduce for $\sigma := \omega' - \omega$ that $\sigma \wedge r = 0$, i.e. $\sigma^i_k = \sum_j a^i_{j,k} \cdot r^j$ with symmetrical a^i by 16.11. Because of $\varepsilon_j \omega^j_i + \varepsilon_i \omega^i_j = 0$ (with $\varepsilon_i := g(s_i, s_i) \in \{\pm 1\}$) by 16.4 and analogous for ω', the same holds for σ. If we put $b^i_{j,k} := \varepsilon_i a^i_{j,k}$, then

$$0 = \varepsilon_j \sigma^j_i + \varepsilon_i \sigma^i_j = \sum_k (\varepsilon_j a^j_{k,i} + \varepsilon_i a^i_{k,j}) \cdot r^k = \sum_k (b^j_{k,i} + b^i_{k,j}) \cdot r^k,$$

hence $b^j_{k,i} = -b^i_{k,j}$ and $b^i_{j,k} = \varepsilon_i a^i_{j,k} = \varepsilon_i a^i_{k,j} = b^i_{k,j}$ and thus

$$b^i_{j,k} = b^i_{k,j} = -b^j_{k,i} = -b^j_{i,k} = b^k_{i,j} = b^k_{j,i} = -b^i_{j,k},$$

hence $b^i_{j,k} = 0$, i.e. $\sigma = 0$. Therefore $\omega' = \omega$. □

16.13 Remark. Curvatures via curvature form.

Let $(s_i)_i$ be an orthonormal frame and $(r^i)_i$ the associated co-frame. Then we obtain the following representations in terms of the curvature matrix for the Riemann curvature R, the Ricci curvature Ricci and the scalar curvature S by definition 14.13:

$$R^l_{i,j,k} := r^l\Big(R(s_i, s_j)\, s_k\Big) \overset{16.6}{=\!=\!=} r^l\Big(\sum_p s_p\, \Omega^p_k(s_i, s_j)\Big) = \Omega^l_k(s_i, s_j)$$

$$R_{i,j,k,l} := g\Big(R(s_i, s_j)\, s_k, s_l\Big) \overset{16.6}{=\!=\!=} g\Big(\sum_p s_p\, \Omega^p_k(s_i, s_j), s_l\Big) = \varepsilon_l\, \Omega^l_k(s_i, s_j)$$

$$R = \sum_{i,j,k,l} \underbrace{r^l(R(s_i, s_j) s_k)}_{=:R^l_{i,j,k}}\, r^i \otimes r^j \otimes r^k \otimes s_l = \sum_{i,j,k,l} \Omega^l_k(s_i, s_j)\, r^i \otimes r^j \otimes r^k \otimes s_l$$

$$\mathrm{Ricci} = \sum_{i,j} \underbrace{\mathrm{spur}\Big(Z \mapsto R(Z, s_i)(s_j)\Big)}_{=\mathrm{Ricci}(s_i,s_j)=:\mathrm{Ricci}_{i,j}}\, r^i \otimes r^j = \sum_{i,j} \sum_k \underbrace{r^k(R(s_k, s_i) s_j)}_{=R^k_{k,i,j} = \Omega^k_j(s_k, s_i)}\, r^i \otimes r^j$$

$$S = \sum_i \mathrm{Ricci}_{i,i} = \sum_{i,k} \Omega^k_i(s_k, s_i)$$

16.14 Proposition.

Let (s_i) be an orthonormal-frame. Then (compare this with 14.15)

$$\mathrm{Ricci}_{i,i} := \mathrm{Ricci}(s_i, s_i) = \underbrace{g(s_i, s_i)}_{=:\varepsilon_i} \cdot \sum_{j \neq i} K\Big(\langle\{s_i, s_j\}\rangle\Big)$$

Conversely, for $\dim(M) = 3$*, the sectional curvature can be determined from the Ricci curvature because the system of equations*

$$\sum_{j\neq i} K\Big(\langle\{s_i, s_j\}\rangle\Big) = \varepsilon_i \, \mathrm{Ricci}_{i,i} \quad \textit{for } i \in \{1,2,3\}$$

has a unique solution $\Big(K\big(\langle\{s_1,s_2\}\rangle\big), K\big(\langle\{s_2,s_3\}\rangle\big), K\big(\langle\{s_3,s_1\}\rangle\big)\Big)$.

Proof.

$$\mathrm{Ricci}(s_i, s_i) \overset{16.13}{=\!=} \sum_j R^j_{j,i,i} \overset{16.13}{=\!=} \sum_j \underbrace{\varepsilon_j^{-1}}_{\pm 1} R_{j,i,i,j} = \sum_j g(s_j, s_j) \cdot g\Big(R(s_j, s_i)s_i, s_j\Big)$$

$$\overset{14.7}{=\!=} 0 + \sum_{j\neq i} g(s_j, s_j) \cdot K\big(\langle\{s_j, s_i\}\rangle\big) \cdot \Big(g(s_i,s_i) g(s_j,s_j) - \underbrace{g(s_j,s_i)^2}_{=0}\Big)$$

$$= g(s_i, s_i) \cdot \sum_{j\neq i} K\big(\langle\{s_i, s_j\}\rangle\big). \quad \square$$

16.15 Examples.

1. **The 2-sphere S^2.**
   Let $f : (0, 2\pi) \times (-\pi, \pi) \to S^2$ be the parameterization according to spherical coordinates, i.e. $f(\varphi, \vartheta) := \big(\cos(\vartheta)\cos(\varphi), \cos(\vartheta)\sin(\varphi), \sin(\vartheta)\big)$. Then

$$\begin{aligned} df^1 &= -\cos(\vartheta)\sin(\varphi)\, d\varphi - \sin(\vartheta)\cos(\varphi)\, d\vartheta \\ df^2 &= \cos(\vartheta)\cos(\varphi)\, d\varphi - \sin(\vartheta)\sin(\varphi)\, d\vartheta \\ df^3 &= \cos(\vartheta)\, d\vartheta \end{aligned}$$

   and consequently the metric in the coordinates (φ, ϑ) is given by

$$f^*\Big(\sum_{i=1}^3 dx^i \otimes dx^i\Big) = \sum_i df^i \otimes df^i = \cos(\vartheta)^2\, d\varphi \otimes d\varphi + d\vartheta \otimes d\vartheta$$

   Thus, $s_1 := \frac{\partial}{\partial\vartheta}$, $s_2 := \frac{1}{\cos(\vartheta)} \frac{\partial}{\partial\varphi}$ is an orthonormal frame with $\varepsilon_1 = 1 = \varepsilon_2$ and associated orthonormal co-frame $r^1 := d\vartheta$, $r^2 := \cos(\vartheta)\, d\varphi$.
   For this we have $dr^1 = 0$ and $dr^2 = -\sin(\vartheta)\, d\vartheta \wedge d\varphi = -\tan(\vartheta)\, r^1 \wedge r^2$.
   We obtain the connection form ω because of 16.12 from the 2nd structure equation 16.10 of Cartan:
   Because of 16.4 we have $\omega^1_1 = 0 = \omega^2_2$ and $\omega^2_1 = -\omega^1_2$, so

$$\begin{aligned} 0 &\overset{16.10}{=\!=} dr^1 + \omega^1_1 \wedge r^1 + \omega^1_2 \wedge r^2 = 0 + 0 + \omega^1_2 \wedge r^2 \\ &\Rightarrow \omega^1_2 = 0\, r^1 + b(\varphi,\vartheta)\, r^2 \\ 0 &\overset{16.10}{=\!=} dr^2 + \omega^2_1 \wedge r^1 + \omega^2_2 \wedge r^2 = -\tan(\vartheta)\, r^1 \wedge r^2 - b(\varphi,\vartheta)\, r^2 \wedge r^1 + 0 \\ &\Rightarrow b(\varphi,\vartheta) = \tan(\vartheta) \\ \Rightarrow -\omega^2_1 &= \omega^1_2 = \tan(\vartheta)\, r^2 = \sin(\vartheta)\, d\varphi \end{aligned}$$

For the curvature form $\Omega := d\omega + \omega\wedge\omega$ we thus get

$$\Omega^1_1 = \Omega^2_2 = 0 \quad \text{because of } \boxed{16.8}, \text{ and}$$
$$-\Omega^2_1 = \Omega^1_2 = d\omega^1_2 + \omega^1_1\wedge\omega^1_2 + \omega^1_2\wedge\omega^2_2 = d\Big(\sin(\vartheta)d\varphi\Big) + 0 + 0$$
$$= \cos(\vartheta)\,d\vartheta\wedge d\varphi = r^1\wedge r^2.$$

The 1st structural equation 16.6 of Cartan yields the sectional curvature (=Gaussian curvature) as

$$K(T_pS^2) = g(R(s_1,s_2)s_2,s_1) = R_{1,2,2,1} \overset{16.13}{=\!=} \varepsilon_1\,\Omega^1_2(s_1,s_2) = 1.$$

2. **The Poincaré half-plane.**
The metric of the half-plane $H_+ := \{(x,y)\in\mathbb{R}^2 : y>0\}$ is given by $g = \frac{1}{y^2}(dx\otimes dx + dy\otimes dy)$. An orthonormal frame is $s_1 := y\,\frac{\partial}{\partial y}$ and $s_2 := y\,\frac{\partial}{\partial x}$ with co-frame $r^1 = \frac{1}{y}dy$ and $r^2 = \frac{1}{y}dx$. Because of $dr^1 = 0$, $dr^2 = -\frac{1}{y^2}dy\wedge dx = -r^1\wedge r^2$ the 2nd structural equation 16.10 of Cartan yields

$$\omega^1_1 = 0 = \omega^2_2,\quad \omega^1_2 = -\omega^2_1 = \frac{1}{y}\,dx = r^2$$

and consequently

$$\Omega^1_1 = 0 = \Omega^2_2,\quad \Omega^1_2 = -\Omega^2_1 = \frac{1}{y^2}dx\wedge dy = -r^1\wedge r^2$$

The sectional curvature (=Gauss curvature) is thus

$$K(T_pH_+) = \varepsilon_1\,\Omega^1_2(s_1,s_2) = -1.$$

3. **The 3-sphere S^3.**
Generalized spherical coordinates are

$$f(\varphi,\vartheta,\tau) := (\cos\tau\,\cos\vartheta\,\cos\varphi, \cos\tau\,\cos\vartheta\,\sin\varphi, \cos\tau\,\sin\vartheta, \sin\tau).$$

The metric $g := f^*\Big(\sum_{i=1}^4 dx^i\otimes dx^i\Big) = \sum_{i=1}^4 df^i\otimes df^i\Big)$ is

$$g = \cos(\tau)^2\cos(\vartheta)^2\,d\varphi\otimes d\varphi + \cos(\tau)^2\,d\vartheta\otimes d\vartheta + d\tau\otimes d\tau$$

An orthonormal co-frame is

$$r^1 := d\tau,\quad r^2 := \cos(\tau)\,d\vartheta,\quad r^3 := \cos(\tau)\cos(\vartheta)\,d\varphi$$

For the connection form we get (because of $dr + \omega\wedge r = 0$)

$$\omega = \begin{pmatrix} 0 & \sin(\tau)\,d\vartheta & \sin(\tau)\cos(\vartheta)\,d\varphi \\ -\sin(\tau)\,d\vartheta & 0 & \sin(\vartheta)\,d\varphi \\ -\sin(\tau)\cos(\vartheta)\,d\varphi & -\sin(\vartheta)\,d\varphi & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \tan(\tau)\,r^2 & \tan(\tau)\,r^3 \\ -\tan(\tau)\,r^2 & 0 & \frac{\tan(\vartheta)}{\cos(\tau)}\,r^3 \\ -\tan(\tau)\,r^3 & -\frac{\tan(\vartheta)}{\cos(\tau)}\,r^3 & 0 \end{pmatrix}$$

and for the curvature form $\Omega := d\omega + \omega\wedge\omega$

$$\Omega = \begin{pmatrix} 0 & -\cos\tau\,d\vartheta\wedge d\tau & -\cos\tau\cos\vartheta\,d\varphi\wedge d\tau \\ \cos\tau\,d\vartheta\wedge d\tau & 0 & -\cos^2\tau\cos\vartheta\,d\varphi\wedge d\vartheta \\ \cos\tau\cos\vartheta\,d\varphi\wedge d\tau & \cos^2\tau\cos\vartheta\,d\varphi\wedge d\vartheta & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & r^1\wedge r^2 & r^1\wedge r^3 \\ r^2\wedge r^1 & 0 & r^2\wedge r^3 \\ r^3\wedge r^1 & r^3\wedge r^2 & 0 \end{pmatrix}$$

So

$$R^l_{i,j,k} \overset{\boxed{16.14}}{=\!=\!=} \Omega^l_k(s_i,s_j) = (r^l\wedge r^k)(s_i,s_j)$$
$$= \delta^l_i\delta^k_j - \delta^k_i\delta^l_j = g\Big(g(s_j,s_k)s_i - g(s_i,s_k)s_j, s_l\Big),$$

and, by 14.14, the sectional curvature K is constant to 1, and because of 14.15 furthermore $\mathrm{Ricci}(X,Y) = K\cdot(m-1)\cdot g(X,Y) = 2\,g(X,Y)$, and finally the scalar curvature $S = K\cdot(m-1)\cdot m = 6$.

4. **The hyperbolic space.**
The HYPERBOLIC SPACE is $H^+ := \{(x^1,x^2,x^3)\in\mathbb{R}^3 : x^1>0\}$ with the metric

$$g = \frac{1}{(x^1)^2}\Big(dx^1\otimes dx^1 + dx^2\otimes dx^2 + dx^3\otimes dx^3\Big)$$

An orthonormal co-frame is

$$r^1 := \frac{1}{x^1}\,dx^3,\ r^2 := \frac{1}{x^1}\,dx^2,\ r^3 := \frac{1}{x^1}\,dx^1$$

For the connection form we get (because of $dr+\omega\wedge r=0$)

$$\omega = \begin{pmatrix} 0 & 0 & -\frac{1}{x^1}\,dx^3 \\ 0 & 0 & -\frac{1}{x^1}\,dx^2 \\ \frac{1}{x^1}\,dx^3 & \frac{1}{x^1}\,dx^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -r^1 \\ 0 & 0 & -r^2 \\ r^1 & r^2 & 0\end{pmatrix}$$

and for the curvature form $\Omega := d\omega + \omega\wedge\omega$

$$\Omega = \begin{pmatrix} 0 & \frac{1}{(x^1)^2}\,dx^2\wedge dx^3 & \frac{1}{(x^1)^2}\,dx^1\wedge dx^3 \\ -\frac{1}{(x^1)^2}\,dx^2\wedge dx^3 & 0 & \frac{1}{(x^1)^2}\,dx^1\wedge dx^2 \\ -\frac{1}{(x^1)^2}\,dx^1\wedge dx^3 & -\frac{1}{(x^1)^2}\,dx^1\wedge dx^2 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -r^1\wedge r^2 & -r^1\wedge r^3 \\ -r^2\wedge r^1 & 0 & -r^2\wedge r^3 \\ -r^3\wedge r^1 & -r^3\wedge r^2 & 0 \end{pmatrix}$$

As before, it now follows that the sectional curvature is constant to -1, the Ricci curvature is $\mathrm{Ricci} = -2\,g$ and the scalar curvature $S=-6$.

5. **Space forms**
SPACE FORMS are complete Riemannian manifolds with constant sectional curvature. A joint form of the metric is

$$g = \frac{1}{1-\kappa\rho^2}\,d\rho\otimes d\rho + \rho^2\cdot\Big(d\vartheta\otimes d\vartheta + \sin^2(\vartheta)\,d\varphi\otimes d\varphi\Big)$$

where $\rho>0$, $\kappa\rho^2<1$, $|\vartheta|<\pi/2$ and $|\varphi|<\pi$. The co-frame is

$$r^1 := \rho\,d\vartheta,\ r^2 := \sin(\vartheta)\rho\,d\varphi,\ r^3 := \frac{1}{\sqrt{1-\kappa\rho^2}}\,d\rho$$

and the connection form (because of $dr+\omega\wedge r=0$)

$$\omega = \begin{pmatrix} 0 & -\cos(\vartheta)\,d\varphi & \sqrt{1-\kappa\rho^2}\,d\vartheta \\ \cos(\vartheta)\,d\varphi & 0 & \sin(\vartheta)\sqrt{1-\kappa\rho^2}\,d\varphi \\ -\sqrt{1-\kappa\rho^2}\,d\vartheta & -\sin(\vartheta)\sqrt{1-\kappa\rho^2}\,d\varphi & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -\frac{\cot(\vartheta)}{\rho}\,r^2 & \frac{\sqrt{1-\kappa\rho^2}}{\rho}\,r^1 \\ \frac{\cot(\vartheta)}{\rho}\,r^2 & 0 & \frac{\sqrt{1-\kappa\rho^2}}{\rho}\,r^2 \\ -\frac{\sqrt{1-\kappa\rho^2}}{\rho}\,r^1 & -\frac{\sqrt{1-\kappa\rho^2}}{\rho}\,r^2 & 0 \end{pmatrix}.$$

Furthermore, the curvature form is $\Omega := d\omega + \omega \wedge \omega$

$$\Omega = \begin{pmatrix} 0 & \kappa \sin(\vartheta)\rho^2\, d\vartheta \wedge d\varphi & -\frac{\kappa\rho}{\sqrt{1-\kappa\rho^2}}\, d\rho \wedge d\vartheta \\ -\kappa \sin(\vartheta)\rho^2\, d\vartheta \wedge d\varphi & 0 & -\frac{\kappa \sin(\vartheta)\rho}{\sqrt{1-\kappa\rho^2}}\, d\rho \wedge d\varphi \\ \frac{\kappa\rho}{\sqrt{1-\kappa\rho^2}}\, d\rho \wedge d\vartheta & \frac{\kappa \sin(\vartheta)\rho}{\sqrt{1-\kappa\rho^2}}\, d\rho \wedge d\varphi & 0 \end{pmatrix}$$
$$= \kappa \cdot \begin{pmatrix} 0 & r^1 \wedge r^2 & r^1 \wedge r^3 \\ r^2 \wedge r^1 & 0 & r^2 \wedge r^3 \\ r^3 \wedge r^1 & r^3 \wedge r^2 & 0 \end{pmatrix}.$$

Thus, as before, the sectional curvature is constant to κ, the Ricci curvature is $\mathrm{Ricci} = 2\,\kappa\, g$ and the scalar curvature $S = 6\kappa$.

Similar to 11.13, it can be shown that each simply connected complete m-dimensional Riemannian manifold with constant sectional curvature K is isometric isomorphic to $\mathbb{R}^m$ with the flat metric in case $K = 0$, to $S^m \subseteq \mathbb{R}^{m+1}$ in case $K = 1$ and to hyperbolic space $\mathbb{R}^+ \times \mathbb{R}^{m-1}$ in case $K = -1$.

6. **The Schwarzschild Metric.**
The general theory of relativity is described by a LORENTZIAN MANIFOLD (that is, a pseudo-Riemannian manifold with LORENTZIAN METRIC, i.e. with signature $(-,+,+,+)$ (or equivalent $(+,-,-,-)$) for which the EINSTEIN FIELD EQUATION

$$\mathrm{Ricci} - \frac{1}{2} S\, g = T$$

holds, where g is a Lorentz metric, S is the scalar curvature, and T is the ENERGY-MOMENTUM TENSOR, which is described by the mass distribution. For a C^∞-manifold with a given energy-momentum tensor, this is a partial differential equation for the metric. In the special case $T = 0$ one speaks of the VACUUM EQUATION. Even for the vacuum equation, only a few explicit local solutions are known. One is the SCHWARZSCHILD METRIC, which results in the rotation symmetric time-independent case:

$$g = -h(\rho)\, dt \otimes dt + \frac{1}{h(\rho)}\, d\rho \otimes d\rho + \rho^2\, d\vartheta \otimes d\vartheta + \rho^2 \sin(\vartheta)^2 d\varphi \otimes d\varphi$$

with $h(\rho) := 1 - \frac{2M}{\rho}$ for $\rho > 2M$. Here $\rho := 2M$ is called the SCHWARZSCHILD RADIUS. This metric can be used in the exterior of slowly rotating isolated stars or black holes. An orthonormal framework for this metric is thus

$$r^1 := \rho\, d\vartheta, \quad r^2 := \rho \sin(\vartheta)\, d\varphi, \quad r^3 := \frac{1}{\sqrt{h(\rho)}}\, d\rho, \quad r^4 := \sqrt{h(\rho)}\, dt$$

For the connection form we obatin (because of $dr + \omega \wedge r = 0$):

$$\omega = \begin{pmatrix} 0 & \cos(\vartheta)\, d\varphi & \sqrt{h}\, d\vartheta & 0 \\ -\cos(\vartheta)\, d\varphi & 0 & -\sqrt{h} \sin(\vartheta)\, d\varphi & 0 \\ \sqrt{h}\, d\vartheta & \sqrt{h} \sin(\vartheta)\, d\varphi & 0 & \frac{M}{\rho^2}\, dt \\ 0 & 0 & -\frac{M}{\rho^2}\, dt & 0 \end{pmatrix}$$

and for the curvature form $\Omega := d\omega + \omega\wedge\omega$

$$\Omega = M\begin{pmatrix} 0 & -\frac{2\sin\vartheta}{\rho}\,d\vartheta\wedge d\varphi & -\frac{1}{\sqrt{h}\rho^2}\,d\rho\wedge d\vartheta & -\frac{\sqrt{h}}{\rho^2}\,dt\wedge d\vartheta \\ \frac{2\sin\vartheta}{\rho}\,d\vartheta\wedge d\varphi & 0 & -\frac{\sin\vartheta}{\sqrt{h}\rho^2}\,d\rho\wedge d\varphi & -\frac{\sqrt{h}\sin\vartheta}{\rho^2}\,dt\wedge d\varphi \\ \frac{1}{\sqrt{h}\rho^2}\,d\rho\wedge d\vartheta & \frac{\sin\vartheta}{\sqrt{h}\rho^2}\,d\rho\wedge d\varphi & 0 & \frac{2}{\rho^3}\,dt\wedge d\rho \\ \frac{\sqrt{h}}{\rho^2}\,dt\wedge d\vartheta & \frac{\sqrt{h}\sin\vartheta}{\rho^2}\,dt\wedge d\varphi & -\frac{2}{\rho^3}\,dt\wedge d\rho & 0 \end{pmatrix}$$

$$= \frac{M}{\rho^3}\begin{pmatrix} 0 & -2\,r^1\wedge r^2 & r^1\wedge r^3 & r^1\wedge r^4 \\ -2\,r^2\wedge r^1 & 0 & r^2\wedge r^3 & r^2\wedge r^4 \\ r^3\wedge r^1 & r^3\wedge r^2 & 0 & -2\,r^3\wedge r^4 \\ r^4\wedge r^1 & r^4\wedge r^2 & -2\,r^4\wedge r^3 & 0 \end{pmatrix}$$

The coefficients of the Riemann metric in the associated orthonormal frame $(s_k)_k$ are $R^i_{k,l,j} = \Omega^i_j(s_k, s_l)$ by 16.13. From the form of Ω it follows that $R^k_{k,l,j} = 0$ for $j \neq l$, Thus $\mathrm{Ricci}_{l,j} = \sum_k R^k_{k,l,j} = 0$. A look at the columns of Ω shows that $\mathrm{Ricci}_{j,j} = \sum_k R^k_{k,j,j} = \frac{M}{\rho^3}(-2+1+1) = 0$, so the Schwarzschild metric is Ricci-flat by 16.13, i.e. $\mathrm{Ricci} = 0$.

7. **The Friedmann-Robertson-Walker Metric (s).**
The Friedmann-Robertson-Walker metric describes an isotropic (that is, without distinguished directions) homogeneous universe and is given by

$$g = dt\otimes dt - h(t)^2\Big(\frac{1}{1-\kappa\rho^2}\,d\rho\otimes d\rho + \rho^2\cdot\big(d\vartheta\otimes d\vartheta + \sin^2(\vartheta)\,d\varphi\otimes d\varphi\big)\Big)$$

with orthonormal co-frame

$$r^1 := dt,\ r^2 := h(t)\rho\,d\vartheta,\ r^3 := h(t)\rho\sin(\vartheta)\,d\varphi,\ r^4 := \frac{h(t)}{\sqrt{1-\kappa\rho^2}}\,d\rho$$

with connection form (because of $dr + \omega\wedge r = 0$)

$$\omega = \begin{pmatrix} 0 & -h'\rho\,d\vartheta & -h'\rho\sin\vartheta\,d\varphi & -\frac{h'}{\sqrt{1-\kappa\rho^2}}\,d\rho \\ h'\rho\,d\vartheta & 0 & -\cos\vartheta\,d\varphi & \sqrt{1-\kappa\rho^2}\,d\vartheta \\ h'\rho\sin\vartheta\,d\varphi & \cos\vartheta\,d\varphi & 0 & \sin\vartheta\sqrt{1-\kappa\rho^2}\,d\varphi \\ \frac{h'}{\sqrt{1-\kappa\rho^2}}\,d\rho & -\sqrt{1-\kappa\rho^2}\,d\vartheta & -\sin\vartheta\sqrt{1-\kappa\rho^2}\,d\varphi & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\frac{h'(t)}{h(t)}\,r^2 & -\frac{h'(t)}{h(t)}\,r^3 & -\frac{h'(t)}{h(t)}\,r^4 \\ \frac{h'(t)}{h(t)}\,r^2 & 0 & -\frac{\cot(\vartheta)}{h(t)\rho}\,r^3 & \frac{\sqrt{1-\kappa\rho^2}}{h(t)\rho}\,r^2 \\ \frac{h'(t)}{h(t)}\,r^3 & \frac{\cot(\vartheta)}{h(t)\rho}\,r^3 & 0 & \frac{\sqrt{1-\kappa\rho^2}}{h(t)\rho}\,r^3 \\ \frac{h'(t)}{h(t)}\,r^4 & -\frac{\sqrt{1-\kappa\rho^2}}{h(t)\rho}\,r^2 & -\frac{\sqrt{1-\kappa\rho^2}}{h(t)\rho}\,r^3 & 0 \end{pmatrix}$$

and curvature form $\Omega := d\omega + \omega\wedge\omega$

$$\Omega = \begin{pmatrix} 0 & -\frac{h''(t)}{h(t)}\,r^1\wedge r^2 & -\frac{h''(t)}{h(t)}\,r^1\wedge r^3 & -\frac{h''(t)}{h(t)}\,r^1\wedge r^4 \\ -\frac{h''(t)}{h(t)}\,r^2\wedge r^1 & 0 & \frac{\kappa-h'(t)^2}{h(t)^2}\,r^2\wedge r^3 & \frac{\kappa-h'(t)^2}{h(t)^2}\,r^2\wedge r^4 \\ -\frac{h''(t)}{h(t)}\,r^3\wedge r^1 & \frac{\kappa-h'(t)^2}{h(t)^2}\,r^3\wedge r^2 & 0 & \frac{\kappa-h'(t)^2}{h(t)^2}\,r^3\wedge r^4 \\ -\frac{h''(t)}{h(t)}\,r^4\wedge r^1 & \frac{\kappa-h'(t)^2}{h(t)^2}\,r^4\wedge r^2 & \frac{\kappa-h'(t)^2}{h(t)^2}\,r^4\wedge r^3 & 0 \end{pmatrix}$$

Thus, as before, $R^m_{m,l,j} = 0$ is for $l \neq j$ and thus also $\mathrm{Ricci}_{l,j} = 0$. The non-vanishing coefficients of the Riemann's curvature tensor are (up to symmetries)

$$R^1_{1,2,2} = R^1_{1,3,3} = R^1_{1,4,4} = -\frac{h''(t)}{h(t)},$$

$$R^2_{2,3,3} = R^2_{2,4,4} = R^3_{3,4,4} = \frac{\kappa - h'(t)^2}{h(t)^2},$$

the Ricci curvature is

$$\text{Ricci} = \begin{pmatrix} -\frac{3h''}{h} & 0 & 0 & 0 \\ 0 & \frac{2\left(\kappa - h'^2\right)}{h^2} - \frac{h''}{h} & 0 & 0 \\ 0 & 0 & \frac{2\left(\kappa - h'^2\right)}{h^2} - \frac{h''}{h} & 0 \\ 0 & 0 & 0 & \frac{2\left(\kappa - h'^2\right)}{h^2} - \frac{h''}{h} \end{pmatrix}$$

and the scalar curvature

$$S = \frac{6(\kappa - h'(t)^2)}{h(t)^2} - \frac{6h''(t)}{h(t)}$$

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