Advanced Functional Analysis
Locally Convex Spaces and Spectral Theory

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This is the preliminary english version of the script for my homonymous lecture course in the Sommer Semester 2019. It was translated from the german original using a pre and post processor (written by myself) for google translate. Due to the limitations of google translate – see the following article by Douglas Hofstadter www.theatlantic.com/.../551570 – heavy corrections by hand had to be done afterwards. However, it is still a rather rough translation which I will try to improve during the semester.

The contents of this lecture course are choosen according to the curriculum of the Master's program: locally convex vector spaces as well as bounded and unbounded operators on Hilbert spaces. These two topics are only loosely related to each other and this dichotomy is reflected in these lecture notes.

The first part deals with an introduction to the theory of locally convex spaces. In addition to the basic concepts and constructions, we will discuss generalizations of the central propositions of Banach-space theory and discuss the duality theory.

The second part revolves around the spectral theory of bounded and unbounded operators. I followed closely the chapters VII - X in [5].

These lecture notes are the result of a combination of lecture notes for lectures I have given in the years since 1991.

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Teil I

Locally Convex Spaces

1. Seminorms

In this chapter we will introduce the adequate notion of distance on vector spaces and discuss its elementary properties.

1.1 Basics

1.1.1 Motivation and definitions.

All vector spaces we are going to consider will have as BASE FIELD $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$.

Distance functions d on vector spaces E should additionally be TRANSLATION INVARIANT, i.e. $d(x,y) = d(a+x, a+y)$ is fulfilled for all $x, y, a \in E$. Then $d(x,y) = d(0, y-x) =: p(y-x)$ (if we choose $a := -x$), so $d : E \times E \to \mathbb{R}$ is already determined by the mapping $p : E \to \mathbb{R}$.

The triangle inequality $d(x,z) \leqslant d(x,y) + d(y,z)$ for d translates into the

$$\text{SUBADDITIVITY:} \quad p(x+y) \leqslant p(x) + p(y).$$

Regarding the scalar multiplication we should probably require $d(\lambda x, \lambda y) = \lambda d(x,y)$ for $\lambda > 0$, i.e.

$$\mathbb{R}^+\text{-HOMOGENEITY:} \quad p(\lambda x) = \lambda\, p(x) \text{ for all } \lambda \in \mathbb{R}^+ := \{t \in \mathbb{R} : t > 0\} \text{ and } x \in E.$$

Note that this has $p(0) = p(2 \cdot 0) = 2\, p(0)$ and hence $p(0) = 0$ as consequence, so also the homogeneity $p(0\, x) = p(0) = 0 = 0\, p(x)$ for $\lambda := 0$ holds. However, we can not expect the homogeneity for all $\lambda \in \mathbb{K}$, because then p would be linear: In fact,

$$\begin{aligned} p(x) + p(y) \geqslant p(x+y) = p(-((-x)+(-y))) &\stackrel{?}{=} -p((-x)+(-y)) \\ \geqslant -(p(-x)+p(-y)) &= p(x) + p(y). \end{aligned}$$

A function $p : E \to \mathbb{R}$ is called SUBLINEAR if it is subadditive and $\mathbb{R}^+$-homogeneous. Note that this is the case if and only if

$$p(0) = 0 \text{ and } p(x + \lambda \cdot y) \leqslant p(x) + \lambda\, p(y)\ \forall x, y \in E\ \forall \lambda > 0.$$

Related to subadditivity is convexity: A function $p : E \to \mathbb{R}$ is called CONVEX (see [20, 4.1.16]) if

$$p\big(\lambda\, x + (1-\lambda)\, y\big) \leqslant \lambda\, p(x) + (1-\lambda)\, p(y) \text{ for all } 0 \leqslant \lambda \leqslant 1 \text{ and all } x, y \in E,$$

so the function lies below each of its chords. By induction this is equivalent to

$$p\big(\sum_{i=1}^n \lambda_i\, x_i\big) \leqslant \sum_{i=1}^n \lambda_i\, p(x_i) \text{ for all } n \in \mathbb{N},\ x_i \in E \text{ and } \lambda_i > 0 \text{ with } \sum_{i=1}^n \lambda_i = 1.$$

For twice-differentiable functions $f : \mathbb{R} \to \mathbb{R}$ one shows in analysis (see [20, 4.1.17]) that these are convex if and only if $f'' \geqslant 0$ holds:

($\Leftarrow$) From $f'' \geqslant 0$ follows the Mean Value Theorem that f' is monotonously increasing, because $\frac{f'(x_1)-f'(x_0)}{x_1-x_0} = f''(\xi) \geqslant 0$ for some ξ between x_0 and x_1. So let $x_0 < x_1$, $0 < \lambda < 1$ and $x = x_0 + \lambda(x_1 - x_0)$. Again by the Mean Value Theorem, $\xi_0 \in [x_0, x]$ and $\xi_1 \in [x, x_1]$ exist with $f(x) - f(x_0) = f'(\xi_0)\,(x - x_0)$ and $f(x_1) - f(x) = f'(\xi_1)\,(x_1 - x)$, so

$$\begin{aligned}
\lambda\, f(x_1) + (1-\lambda)\, f(x_0) - f(x) &= \\
&= (1-\lambda)\,\big(f(x_0) - f(x)\big) + \lambda\,\big(f(x_1) - f(x)\big)\\
&= (1-\lambda)\, f'(\xi_0)\,(x_0 - x) + \lambda\, f'(\xi_1)(x_1 - x)\\
&= (1-\lambda)\, f'(\xi_0)\,\big(-\lambda(x_1 - x_0)\big) + \lambda\, f'(\xi_1)\,\big((1-\lambda)\,(x_1 - x_0)\big)\\
&= \lambda\,(1-\lambda)\,\Big(f'(\xi_1) - f'(\xi_0)\Big)\,(x_1 - x_0) \geqslant 0,
\end{aligned}$$

i.e. f is convex.

($\Rightarrow$) Let f be convex. Then for $x_0 < x < x_1$ with $\lambda := \frac{x-x_0}{x_1-x_0}$ resp. $\lambda := \frac{x_1-x}{x_1-x_0}$:

$$\frac{f(x) - f(x_0)}{x - x_0} \leqslant \frac{f(x_1) - f(x_0)}{x_1 - x_0} \leqslant \frac{f(x_1) - f(x)}{x_1 - x}.$$

Thus $f'(x_0) \leqslant \frac{f(x_1)-f(x_0)}{x_1-x_0} \leqslant f'(x_1)$, i.e. f' is increasing monotonously. Thus, we have $f''(x_0) = \lim_{x_1 \searrow x_0} \frac{f'(x_1)-f'(x_0)}{x_1-x_0} \geqslant 0$.

In the definition of “sublinearly” we may replace “subadditive” equivalently by “convex”:

($\Leftarrow$) We put $\lambda := \frac{1}{2}$ and get

$$p(x+y) = 2\,p\left(\frac{x+y}{2}\right) \leqslant 2\,\left(\frac{1}{2}\,p(x) + \frac{1}{2}\,p(y)\right) = p(x) + p(y).$$

($\Rightarrow$) Then

$$p\Big(\lambda\, x + (1-\lambda)\, y\Big) \leqslant p(\lambda\, x) + p((1-\lambda)\, y) = \lambda\, p(x) + (1-\lambda)\, p(y).$$

The symmetry $d(x,y) = d(y,x)$ of d translates into the SYMMETRY: $p(x) = p(-x)$ for all $x \in E$. Together with the $\mathbb{R}^+$-homogeneity, this is therefore equivalent to the following homogeneity: $p(\lambda\, x) = |\lambda|\, p(x)$ for $x \in E$ and $\lambda \in \mathbb{R}$.

A function $p : E \to \mathbb{R}$ is called SEMINORM (for short SN) if it is subadditive and POSITIVELY HOMOGENEOUS, i.e. $p(\lambda\, x) = |\lambda|\, p(x)$ holds for $x \in E$ and $\lambda \in \mathbb{K}$.

A seminorm is therefore a sublinear mapping which fullfills additionally $p(\lambda x) = p(x)$ for all $x \in E$ and $|\lambda| = 1$. Note that multiplication with a complex number of absolute value 1 is usually interpreted as a rotation.

Every seminorm p fulfills $p \geqslant 0$, because $0 = p(0) \leqslant p(x) + p(-x) = 2\,p(x)$.

A seminorm p is called NORM if additionally $p(x) = 0 \Rightarrow x = 0$ holds. A NORMED SPACE is a vector space together with a norm, cf. [**22**, 5.4.2].

1.2 Important norms

1.2.1 Definition. ∞-norm.

The SUPREMUM or ∞-norm is defined by

$$\|f\|_\infty := \sup\{|f(x)| : x \in X\},$$

where $f : X \to \mathbb{K}$ is a bounded function on a set X, cf. [**20**, 2.2.5].

The distance d, which we looked at in application [18, 1.3] on the vector space $C(I,\mathbb{R})$, was just given by $d(u_1,u_2) := \|u_1-u_2\|_\infty$, see also [20, 4.2.8]

1.2.2 Examples.

The following vector spaces are normed spaces with respect to the ∞-norm:

1. For each set X the space $B(X)$ of all bounded functions $X \to \mathbb{K}$;
2. For each compact space X the space $C(X)$ of all continuous functions $X \to \mathbb{K}$;
3. For each topological space X the space $C_b(X)$ of all bounded continuous functions $X \to \mathbb{K}$;
4. For each locally compact space X the space $C_0(X)$ of all continuous functions $X \to \mathbb{K}$ vanishing at ∞, i.e. those functions $f : X \to \mathbb{K}$ for which there is a compact set $K \subseteq X$ for each $\varepsilon > 0$, s.t. $|f(x)| < \varepsilon$ for all $x \notin K$;
5. If you use (roughly speeking) the maximum of the ∞-norms of the derivatives, then for each compact manifold M also the space $C^n(M)$ of the n-times continuously differentiable functions $M \to \mathbb{K}$ becomes a normed space;

On the other hand, we can not use reasonable norms on any of the following spaces:

6. $C(X)$ for general non-(pseudo-)compact X,
7. The space $C^\infty(M)$ of the smooth functions for manifolds M,
8. $C^n(M)$ for non compact manifolds M,
9. The space $H(G)$ of holomorphic (i.e., complex differentiable) functions for domains $G \subseteq \mathbb{C}$.

1.2.3 The variation norm.

Let $f : I \to \mathbb{K}$ be a function and $\mathcal{Z} = \{0 = x_0 < \dots < x_n = 1\}$ a partition of $I = [0,1]$. Then one denotes the variation of f on $\mathcal{Z}$ by

$$V(f,\mathcal{Z}) := \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|,$$

cf. [22, 6.5.11]. The (total) variation of a function is

$$V(f) := \sup_{\mathcal{Z}} V(f,\mathcal{Z}).$$

With $BV(I)$ we denote the space of all functions with bounded variation, i.e. those functions f for which $V(f) < \infty$ holds. It is easy to verify that $BV(I)$ is a vector space, and V is a seminorm on $BV(I)$ which vanishes exactly on the constant functions.

1.2.4 Definition. p norm.

For $1 \leqslant p < \infty$, the p-norm is defined by

$$\|f\|_p := \left(\int_X |f(x)|^p dx\right)^{\frac{1}{p}},$$

where $|f|^p : X \to \mathbb{K}$ is an integrable function. For $p = 2$ this is a continuous analogue of the Euclidean norm

$$\|x\|_2 := \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

for $x \in \mathbb{R}^n$ or $x \in \mathbb{C}^n$ (here the absolute value in $|x_i|^2$ is necessary).

The formula $\langle f|g\rangle := \int_X f(x)\,\overline{g(x)}\,dx$ generalizes the inner product $\langle .|.\rangle$ on $\mathbb{K}^n$.

Clearly $\|fg\|_1 \leqslant \|f\|_\infty \cdot \|g\|_1$ holds. In order to use the inner product for measureing angles, the inequality of Cauchy-Schwarz $\|fg\|_1 \leqslant \|f\|_2 \cdot \|g\|_2$ is necessary, see [**18**, 6.2.1]. A common generalization is the

1.2.5 Hölder inequality.

$$|\langle f|g\rangle| \leqslant \|fg\|_1 \leqslant \|f\|_p \cdot \|g\|_q \text{ for } \frac{1}{p}+\frac{1}{q}=1 \text{ with } 1\leqslant p,q\leqslant\infty$$

See [**23**, 5.36].

$$\text{resp. } \int |fg| \leqslant \left(\int |f|^p\right)^{\frac{1}{p}} \left(\int |g|^q\right)^{\frac{1}{q}}$$

Proof. Let first $\|f\|_p = 1 = \|g\|_q$. Then $|f(x)\,g(x)| \leqslant \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q}$, because log is concave (i.e. $-\log$ is convex, because $\log''(x) = -\frac{1}{x^2} < 0$) and thus $\log(a^{1/p}\cdot b^{1/q}) = \frac{1}{p}\log a + \frac{1}{q}\log b \leqslant \log(\frac{1}{p}a+\frac{1}{q}b)$ for $a := |f(x)|^p$ and $b := |g(x)|^q$, i.e. $a^{\frac{1}{p}}\cdot b^{\frac{1}{q}} \leqslant \frac{1}{p}a+\frac{1}{q}b$. By integration we get

$$\|f\,g\|_1 = \int |f\,g| \leqslant \frac{\|f\|_p^p}{p} + \frac{\|g\|_q^q}{q} = \frac{1}{p}+\frac{1}{q} = 1.$$

Let $\alpha := \|f\|_p$ and $\beta := \|g\|_q$ be arbitrary (unequal to 0). Then we can apply the first part on $f_0 := \frac{1}{\alpha}f$ and $g_0 := \frac{1}{\beta}g$ and get

$$\frac{1}{\alpha\,\beta}\|f\,g\|_1 = \|f_0\,g_0\|_1 \leqslant 1 \Rightarrow \|f\,g\|_1 \leqslant \|f\|_p \cdot \|g\|_q.$$

The remaining inequality $|\langle f|g\rangle| = |\int f\,\bar g| \leqslant \int |f|\,|\bar g| = \|f\,g\|_1$ is obvious.

□

1.2.6 Minkowski inequality.

$$\|f+g\|_p \leqslant \|f\|_p + \|g\|_p, \text{ i.e. } \|\_\|_p \text{ is a seminorm}$$

See [**20**, 2.2.4], [**21**, 2.72], [**23**, 5.37].

Proof. With $\frac{1}{p}+\frac{1}{q}=1$ we have

$$\begin{aligned}
\|f+g\|_p^p &= \int |f+g|^p \leqslant \int |f|\,|f+g|^{p-1} + \int |g|\,|f+g|^{p-1} \\
&\leqslant \|f\|_p \cdot \|(f+g)^{p-1}\|_q + \|g\|_p \cdot \underbrace{\|(f+g)^{p-1}\|_q}_{(\int |f+g|^{(p-1)q})^{1/q}} \qquad \text{(Hölder Inequality)} \\
&= (\|f\|_p + \|g\|_p)\cdot \|f+g\|_p^{p/q} \qquad \text{since } q = \frac{p}{p-1} \qquad \Rightarrow
\end{aligned}$$

$$\|f+g\|_p = \|f+g\|_p^{p(1-\frac{1}{q})} \leqslant \|f\|_p + \|g\|_p. \quad \square$$

1.2.7 Examples.

1. The space $C(I)$ of all continuous functions is a normed space with respect to the p-norm.
2. On the space $R(I)$ of all Riemann-integrable functions, however, the p-norm is not a norm but only a seminorm, since a function f which vanishes except at most finitely many points, nevertheless fulfills $\|f\|_p = 0$.

3. Also ℓ^p is a normed space, where ℓ^p denotes the space of sequences $n \mapsto x_n \in \mathbb{K}$, which are p-summable, i.e. for which $\sum_{n=1}^\infty |x_n|^p < \infty$ holds. This space can be identified (via $f(t) := x_n$ for $n \leqslant t < n+1$) with left-continuous staircase functions $f : \{t : t \geqslant 0\} \to \mathbb{K}$ having jumps in at most points in $\mathbb{N}$.

1.3 Elementary properties of seminorms

1.3.1 Lemma. Reverse triangle inequality.

Each seminorm $p : E \to \mathbb{R}$ fulfills the REVERSE TRIANGLE INEQUALITY*:*

$$|p(x_1) - p(x_2)| \leqslant p(x_1 - x_2).$$

Proof. The following applies:

$$\begin{aligned} p(x_1) \leqslant p(x_1 - x_2) + p(x_2) &\Rightarrow p(x_1) - p(x_2) \leqslant p(x_1 - x_2) \\ \text{and } p(-x) = p(x) &\Rightarrow p(x_2) - p(x_1) \leqslant p(x_2 - x_1) = p(x_1 - x_2) \\ &\Rightarrow |p(x_1) - p(x_2)| \leqslant p(x_1 - x_2) \quad \square \end{aligned}$$

We now want to give a more geometric description of seminorms p. The idea is to examine the level surfaces $p^{-1}(c)$.

1.3.2 Definition. Balls.

Let $p : E \to \mathbb{R}$ be a mapping and $c \in \mathbb{R}$. Then we put

$$p_{<c} := \{x : p(x) < c\} \quad \text{and} \quad p_{\leqslant c} := \{x : p(x) \leqslant c\},$$

and call this (if p is sublinear) the OPEN and the CLOSED p-BALL AROUND 0 WITH RADIUS c. .

1.3.3 Lemma. Balls of sublinear mappings.

For each sublinear mapping $0 \leqslant p : E \to \mathbb{R}$ and $c > 0$, $p_{\leqslant c}$ and $p_{<c}$ are convex absorbing subsets of E. We have $p_{\leqslant c} = c \cdot p_{\leqslant 1}$ as well as $p_{<c} = c \cdot p_{<1}$, and further $p(x) = c \cdot \inf\{\lambda > 0 : x \in \lambda \cdot p_{\leqslant c}\}$.

So we may recover the mapping p from the unit ball $p_{\leqslant 1}$.

A set $A \subseteq E$ is called CONVEX (see [**22**, 5.5.17]), if $\sum_{i=1}^n \lambda_i x_i \in A$ follows from $\lambda_i \geqslant 0$ with $\sum_{i=1}^n \lambda_i = 1$ and $x_i \in A$. It suffices to asssume this for $n = 2$, because for $n < 2$ it is obvious and from $n = 2$ it follows for all $n > 2$ by induction:

$$\sum_{i=1}^{n+1} \lambda_i x_i = \lambda_{n+1} x_{n+1} + (1 - \lambda_{n+1}) \sum_{i=1}^{n} \frac{\lambda_i}{1 - \lambda_{n+1}} x_i.$$

A set A is called ABSORBENT if $\forall x \in E\ \exists \lambda > 0 : x \in \lambda \cdot A$.

Proof. For $c > 0$ we have:

$$\begin{aligned} p_{\leqslant c} &= \{x : p(x) \leqslant c\} = \left\{x : p\left(\frac{x}{c}\right) = \frac{1}{c} p(x) \leqslant 1\right\} \\ &= \{c\, y : p(y) \leqslant 1\} = c \cdot \{y : p(y) \leqslant 1\} = c \cdot p_{\leqslant 1} \end{aligned}$$

and analogously for $p_{<c}$.

The convexity of $p_{\leqslant c} = p^{-1}\{\lambda : \lambda \leqslant c\}$ and $p_{<c} = p^{-1}\{\lambda : \lambda < c\}$ immediately follows from the easy-to-see property that inverse images of intervals, being unbounded from below, under convex functions are convex.

To see that $p_{\leqslant c} = c \cdot p_{\leqslant 1}$ is absorbent for $c > 0$, it is sufficient to put $c = 1$: Let $x \in E$ be arbitrary. If $p(x) = 0$, then $x \in p_{\leqslant 1}$. Otherwise, $x \in p(x) \cdot p_{\leqslant 1}$ holds because $x = p(x) \cdot y$, where $y := \frac{1}{p(x)} x$ and $p(y) = p(\frac{1}{p(x)} x) = \frac{1}{p(x)} p(x) = 1$.

Hence also the superset $p_{<c} \supseteq p_{\leqslant c/2}$ is absorbent.

Because of following equivalences for $\lambda > 0$ we have $p(x) = \inf\{\lambda > 0 : x \in \lambda \cdot p_{\leqslant 1}\}$:

$$x \in \lambda \cdot p_{\leqslant 1} = p_{\leqslant \lambda} \Leftrightarrow p(x) \leqslant \lambda,$$

hence

$$\inf\{\lambda > 0 : x \in \lambda\, p_{\leqslant 1}\} = \inf\{\lambda > 0 : \lambda \geqslant p(x)\} = p(x). \quad \square$$

1.3.4 Lemma. Balls of seminorms.

For each seminorm $p : E \to \mathbb{R}$ and $c > 0$, $p_{<c}$ and $p_{\leqslant c}$ are absorbent and absolutely convex and

$$p(x) = \inf\Big\{\lambda > 0 : x \in \lambda \cdot p_{\leqslant 1} = p_{\leqslant \lambda}\Big\}.$$

A subset $A \subseteq E$ is called BALANCED, if for all $x \in A$ and $|\lambda| = 1$ also $\lambda \cdot x \in A$ holds.

More generally, a subset $A \subseteq E$ is called ABSOLUTELY CONVEX if it follows from $x_i \in A$ and $\lambda_i \in \mathbb{K}$ with $\sum_{i=1}^n |\lambda_i| = 1$ that $\sum_{i=1}^n \lambda_i\, x_i \in A$ holds.

Sublemma.

A set A is absolutely convex if and only if it is convex and balanced.

Proof. ($\Rightarrow$) is clear, because every convex combination is also an absolutely convex combination and for $|\lambda| = 1$ also $\lambda\, x$ is an absolutely convex combination. Note that for this it is sufficient to have absolutely convexity for $n = 2$, because that for $n = 1$ it follows from $\lambda_1\, x_1 = \lambda_1\, x_1 + 0\, x_1$.

($\Leftarrow$) Let $\sum_{i=1}^n |\lambda_i| = 1$, then

$$\sum_{i=1}^n \lambda_i\, x_i = \sum_{\lambda_i \neq 0} \lambda_i\, x_i = \sum_{\lambda_i \neq 0} |\lambda_i|\, \frac{\lambda_i}{|\lambda_i|} x_i \in A,$$

holds because of $\left|\frac{\lambda_i}{|\lambda_i|}\right| = 1$ and therefore, because of the balancedness $\frac{\lambda_i}{|\lambda_i|} x_i \in A$, and therefore, because of the convexity, also $\sum_{\lambda_i \neq 0} |\lambda_i|\, \frac{\lambda_i}{|\lambda_i|} x_i \in A$ holds. $\square$

This proof shows that even for "absolutely convex" it is enough to ask this for the case $n = 2$.

Proof of the lemma 1.3.4. Because of the previous lemma and the sublemma, only balancing is to be shown, and this is obvious because of the positive homogeneity of p. $\square$

1.3.5 Definition. Minkowski functional.

We now want to construct from sets A related seminorms p. For this we define the MINKOWSKI FUNCTIONAL p_A:

$$x \mapsto p_A(x) := \inf\{\lambda > 0 : x \in \lambda \cdot A\} \in \mathbb{R} \cup \{+\infty\} \text{ for each } x \in E.$$

Then $p_A(x) < \infty$ holds if and only if x lies in the cone $\{\lambda \in \mathbb{R} : \lambda > 0\} \cdot A$ generated by A.

1.3.6 Lemma. From balls to seminorms.

Let A be convex and absorbent. Then the Minkowski functional of A is a well-defined sublinear mapping $p := p_A \geqslant 0$ on E, and for $\lambda > 0$ we have:

$$p_{<\lambda} \subseteq \lambda \cdot A \subseteq p_{\leqslant\lambda}.$$

If A is also absolutely convex, then p is a seminorm.

So we can recover the set A almost from the function p.

Proof. Since A is absorbent, the cone is $\{\lambda : \lambda > 0\} \cdot A = E$. So p is finite on E.

Furthermore, $0 \in A$ holds, because $\exists \lambda > 0 : 0 \in \lambda A$ and thus $0 = \frac{0}{\lambda} \in A$ holds.

The function p is $\mathbb{R}^+$-homogeneous, because for $\lambda > 0$ we have:

$$\begin{aligned} p(\lambda x) &= \inf \{\mu > 0 : \lambda x \in \mu A\} \\ &= \inf\Big\{\mu > 0 : x \in \frac{\mu}{\lambda} A\Big\} = \inf\{\lambda \nu > 0 : x \in \nu A\} = \lambda \inf\{\nu > 0 : x \in \nu A\} \\ &= \lambda\, p(x). \end{aligned}$$

($p_{<\lambda} \subseteq \lambda \cdot A$) Let $p(x) = \inf\{\mu > 0 : x \in \mu A\} < \lambda$. Then there is a $0 < \mu \leqslant \lambda$ with $x \in \mu A = \lambda \frac{\mu}{\lambda} A \subseteq \lambda A$, because $0 \in A$ and thus $\frac{\mu}{\lambda} a = (1 - \frac{\mu}{\lambda}) 0 + \frac{\mu}{\lambda} a \in A$ for all $a \in A$.

($\lambda \cdot A \subseteq p_{\leqslant\lambda}$) If $x \in \lambda A$, then by definition of p it is clear that $p(x) \leqslant \lambda$, i.e. $x \in p_{\leqslant\lambda}$.

The function p is subadditive because

$$\begin{aligned} p(x) < \lambda,\ p(y) < \mu &\Rightarrow x \in \lambda A, y \in \mu A \\ &\Rightarrow x + y \in \lambda A + \mu A \overset{!}{=} (\lambda + \mu) A \Rightarrow p(x + y) \leqslant \lambda + \mu \\ &\Rightarrow p(x + y) \leqslant \inf\{\lambda + \mu : p(x) < \lambda,\ p(y) < \mu\} = p(x) + p(y), \end{aligned}$$

holds, since for convex sets A and $\lambda_i > 0$ we have $\sum_{i=1}^n \lambda_i A = (\sum_{i=1}^n \lambda_i) A$: In fact, $x_i \in A$ implies $\sum_i \lambda_i x_i = \sum_i \lambda \cdot \frac{\lambda_i}{\lambda} x_i = \lambda \cdot \sum_i \frac{\lambda_i}{\lambda} x_i \in (\sum_i \lambda_i) \cdot A$, where $\lambda := \sum_{i=1}^n \lambda_i$, and thus $\sum_i \frac{\lambda_i}{\lambda} x_i$ is a convex combination. Conversely, $x \in A$ implies $(\sum_{i=1}^n \lambda_i) x = \sum_i \lambda_i x \in \sum_i \lambda_i A$.

If A is additionally absolutely convex then p is a seminorm, because $p(\lambda x) = p(x)$ holds for all $|\lambda| = 1$ since A is balanced, so $\lambda A = A$ is fullfilled. □

1.3.7 Lemma. Comparison of seminorms.

For each two sublinear mappings $p, q \geqslant 0$:

$$p \leqslant q \Leftrightarrow p_{\leqslant 1} \supseteq q_{\leqslant 1} \Leftrightarrow p_{<1} \supseteq q_{<1}.$$

Proof. ($1 \Rightarrow 3$) The following holds:

$$x \in q_{<1} \Rightarrow p(x) \leqslant q(x) < 1 \Rightarrow x \in p_{<1}.$$

($3 \Rightarrow 2$) The following holds:

$$\begin{aligned} x \in q_{\leqslant 1} &\Rightarrow q(x) \leqslant 1 \\ &\Rightarrow \forall \lambda > 1 : q\left(\frac{x}{\lambda}\right) = \frac{1}{\lambda} q(x) \leqslant \frac{1}{\lambda} 1 < 1 \\ &\Rightarrow \frac{x}{\lambda} \in q_{<1} \subseteq p_{<1} \Rightarrow \frac{1}{\lambda} p(x) = p\left(\frac{x}{\lambda}\right) < 1 \Rightarrow p(x) < \lambda \\ &\Rightarrow p(x) \leqslant \inf\{\lambda : \lambda > 1\} = 1 \\ &\Rightarrow x \in p_{\leqslant 1} \end{aligned}$$

(2 ⇒ 1) The following holds:

$$\begin{aligned}\text{Let } \lambda > 0 \text{ be s.t. } 0 \leqslant q(x) < \lambda \Rightarrow q\left(\frac{x}{\lambda}\right) &= \frac{1}{\lambda}\, q(x) \leqslant \frac{\lambda}{\lambda} = 1 \\ \Rightarrow \frac{x}{\lambda} &\in q_{\leqslant 1} \subseteq p_{\leqslant 1} \\ \Rightarrow p\left(\frac{x}{\lambda}\right) &\leqslant 1, \text{ i.e. } p(x) \leqslant \lambda \\ \Rightarrow p(x) &\leqslant \inf\{\lambda : \lambda > q(x)\} = q(x) \quad \square\end{aligned}$$

1.4 Seminorms versus topology

1.4.1 Topologies generated by seminorms.

Motivation: The seminorms provide us, as in Analysis, with balls, which we want to use for questions of convergence and continuity. For this the notion of a topology has been developed:

In Analysis, we call $O \subseteq \mathbb{R}$ open if there is an δ-neighborhood $U \subseteq O$ for each $a \in O$ (i.e. a set $U := \{x : |x - a| < \delta\}$ with $\delta > 0$).

This definition can be transfered almost literally to normed spaces (E, p):
$O \subseteq E$ is called OPEN $:\Leftrightarrow \forall a \in O\ \exists \delta > 0$: $\{x : p(x-a) < \delta\} \subseteq O$. Note that

$$\{x : p(x-a) < \delta\} = a + p_{<\delta} = a + \delta \cdot p_{<1},$$

because $p(x-a) < \delta \Leftrightarrow x = a + y$ with $y := x - a \in p_{<\delta}$.

But important function spaces do not have a reasonable norm. For example, we can no longer consider the supremum norm on $C(\mathbb{R}, \mathbb{R})$. But for each compact interval $K \subseteq \mathbb{R}$ we may consider the supremum p_K on K, i.e. $p_K(f) := \sup\{|f(x)| : x \in K\}$.

We call $O \subseteq E$ OPEN with respect to a given family $\mathcal{P}_0$ of seminorms on a vector space E, if

$$\forall a \in O\ \exists n \in \mathbb{N}\ \exists p_1, \dots, p_n \in \mathcal{P}_0,\ \exists \varepsilon > 0 : \{x : p_i(x-a) < \varepsilon \text{ for } i = 1, \dots, n\} \subseteq O.$$

The family $\mathcal{O} := \{O : O \subseteq E \text{ ist open}\}$ defines then a topology on E, the so-called TOPOLOGY GENERATED BY $\mathcal{P}_0$ (unions of the so defined open sets are obviously open again and the same applies for intersections of finitely many open sets, because the union of finitely many sets, each consists of finite many seminorms, is finite and the minimimum of the finitely many $\varepsilon > 0$ is positive). Generally, a TOPOLOGY (see [**26**, 1.1.1]) $\mathcal{O}$ on a set X is a set $\mathcal{O}$ of subsets of X, which fullfills the following two conditions:

1. If $\mathcal{F} \subseteq \mathcal{O}$, then the union $\bigcup \mathcal{F} = \bigcup_{O \in \mathcal{F}} O$ belongs to $\mathcal{O}$;
2. If $\mathcal{F} \subseteq \mathcal{O}$ is finite, the intersection $\bigcap \mathcal{F} = \bigcap_{O \in \mathcal{F}} O$ is also in $\mathcal{O}$.

Note that $\bigcup \emptyset = \emptyset$ and $\bigcap \emptyset := X$. The subsets O of X, which belong to $\mathcal{O}$, are also called OPEN SETS of the topology in the general case. A TOPOLOGICAL SPACE is a set together with a topology.

The above construction is a general principle. One calls a subset $\mathcal{O}_0 \subseteq \mathcal{O}$ SUBBASIS OF A TOPOLOGY $\mathcal{O}$, if $\forall a \in O \in \mathcal{O}\ \exists \mathcal{F} \subseteq \mathcal{O}_0$, finite: $a \in \bigcap \mathcal{F} \subseteq O$, cf. [**26**, 1.1.6]. In order to construct a topology $\mathcal{O}$ it is sufficient to specify a set $\mathcal{O}_0$ of subsets of X, and then to designate $\mathcal{O}$ as the set of all $O \subseteq X$ for which there is a finite subset $\mathcal{F} \subset \mathcal{O}_0$ with $x \in \bigcap \mathcal{F} \subseteq O$ for each of the points $x \in O$. One says, that the topology $\mathcal{O}$ is generated by the sub-basis $\mathcal{O}_0$.

The topology generated by $\mathcal{P}_0$ is just the topology generated by sub-basis $\mathcal{O}_0 := \{a + p_{<\varepsilon} : a \in E,\ p \in \mathcal{P}_0,\ \varepsilon > 0\}$.
($\subseteq$) The topology generated by $\mathcal{P}_0$ is obviously coarser or equal to that generated by the sub-basis $\mathcal{O}_0$, because all we have to do is to set all $a_i = a$ and $\varepsilon_i = \varepsilon$.
($\supseteq$) In fact, let $O \subseteq E$ be open in the latter topology, i.e. $\forall a \in O\ \exists \mathcal{F} \subseteq \mathcal{O}_0$, finite: $a \in \bigcap \mathcal{F} \subseteq O$. So $\exists a_1, \dots, a_n \in E$, $p_1, \dots, p_n \in \mathcal{P}_0$ and $\varepsilon_1, \dots, \varepsilon_n > 0$ with

$$a \in \{x \in E : p_i(x - a_i) < \varepsilon_i \text{ for } i = 1, \dots, n\} \subseteq O.$$

If we put now $\varepsilon := \min\{\varepsilon_i - p_i(a - a_i) : i = 1, \dots, n\}$, i.e.

$$\begin{aligned} a &\in \{x \in E : p_i(x - a) < \varepsilon \text{ for } i = 1, \dots, n\} \\ &\subseteq \{x \in E : p_i(x - a_i) \leqslant p_i(x - a) + p_i(a - a_i) < \varepsilon_i \text{ for } i = 1, \dots, n\} \subseteq O. \end{aligned}$$

By a NEIGHBORHOOD U of a point a in a topological space X, one understands a subset $U \subseteq X$ for which an open set $O \in \mathcal{O}$ exists with $a \in O \subseteq U$.
A NEIGHBORHOOD(SUB)BASIS $\mathcal{U}$ of a point a in a topological space X is a set $\mathcal{U}$ of neighborhoods U of a such that for each neighborhood O, a set (finitely many sets) $U_i \in \mathcal{U}$ exists (exist), so that $\bigcap_i U_i \subseteq O$, cf. [**26**, 1.1.7].

As in Analysis, a mapping $f : X \to Y$ between topological spaces is called CONTINUOUS at $a \in X$, if the inverse image of each neighborhood (in a neighborhood basis) of $f(a)$ there is a neighborhood of a, cf. [**26**, 1.2.4]. It is called continuous, if it is continuous in each point $a \in X$, that is the case if and only if the inverse image of each open set is open. It is easy to see that it is sufficient to check this condition for the elements of a sub-basis.

Each seminorm $p \in \mathcal{P}_0$ is continuous for the topology generated by $\mathcal{P}_0$, because if $a \in E$ and $\varepsilon > 0$, then $p(a + p_{<\varepsilon}) \subseteq \{t : |t - p(a)| < \varepsilon\}$, since $x \in p_{<\varepsilon} \Rightarrow |p(a + x) - p(a)| \leqslant p(x) < \varepsilon$. But also the addition $+ : E \times E \to E$ is continuous, because $(a_1 + p_{<\varepsilon}) + (a_2 + p_{<\varepsilon}) \subseteq (a_1 + a_2) + p_{<2\varepsilon}$. In particular, the translations $x \mapsto a + x$ are homeomorphisms.

The scalar multiplication $\cdot : \mathbb{K} \times E \to E$ is continuous. For $\lambda \in \mathbb{K}$ and $a \in E$: $\{\mu \in \mathbb{K} : |\mu - \lambda| < \delta_1\} \cdot \{x : p(x - a) < \delta_2\} \subseteq \{z : p(z - \lambda \cdot a) < \varepsilon\}$ if $\delta_1 < \frac{\varepsilon}{2p(a)}$ and $\delta_2 < \frac{\varepsilon}{2}(|\lambda| + \frac{\varepsilon}{2p(a)})^{-1}$, since

$$\begin{aligned} p(\mu \cdot x - \lambda \cdot a) &= p((\mu - \lambda) \cdot x + \lambda \cdot (x - a)) \\ &\leqslant |\mu - \lambda| \cdot p(x) + |\lambda| \cdot p(x - a) \\ &\leqslant \delta_1 \cdot (p(a) + p(x - a)) + |\lambda| \cdot \delta_2 \\ &\leqslant \delta_1 \cdot (p(a) + \delta_2) + |\lambda| \cdot \delta_2 = \delta_1 \cdot p(a) + \delta_2 \cdot (\delta_1 + |\lambda|) \\ &\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\left(|\lambda| + \frac{\varepsilon}{2p(a)}\right)^{-1} \cdot \left(|\lambda| + \frac{\varepsilon}{2p(a)}\right) = \varepsilon. \end{aligned}$$

In particular, the homothetics $x \mapsto \lambda \cdot x$ are homeomorphisms for $\lambda \neq 0$.

So the topology generated by $\mathcal{P}_0$ turns E into a TOPOLOGICAL VECTOR SPACE, i.e. a vector space together with a topology with respect to which the addition and the scalar multiplication are continuous. Moreover, E is even a LOCALLY CONVEX VECTOR SPACE, i.e. there exists a 0-neighborhood basis consisting of (absolutely) convex sets (namely, $\bigcap_{i=1}^n (p_i)_{<\varepsilon}$), or a sub-basis consisting of (absolutely) convex sets (namely, $p_{<\varepsilon}$).

1.4.2 Lemma. Continuity of seminorms.

1. *A seminorm $p : E \to \mathbb{R}$ on a topological vector space E is continuous if and only if $p_{<1}$ (or, equivalently, $p_{\leqslant 1}$) is a 0-neighborhood.*

2. *A seminorm* $p : E \to \mathbb{R}$ *is continuous in the topology generated by* $\mathcal{P}_0$ *if and only if* $\exists p_1, \dots, p_n \in \mathcal{P}_0,\ \lambda > 0$*:* $p \leqslant \lambda \cdot \max\{p_1, \dots, p_n\}$*.*

Proof.
$\boxed{1}$ ($\Rightarrow$) Since p is continuous, $0 \in p^{-1}\{t : t < 1\} = p_{<1}$ is open.
($\Leftarrow$)

$$a \in a + \varepsilon \cdot p_{<1} = \{x : p(x - a) < \varepsilon\} \subseteq p^{-1}\{t : |t - p(a)| < \varepsilon\}.$$

$\boxed{2}$ ($\Rightarrow$) If p is continuous, then $p_{<1}$ is a 0-neighborhood, so $p_1, \dots, p_n \in \mathcal{P}_0$ and $\varepsilon > 0$ exist with

$$\begin{aligned} p_{<1} &\supseteq \bigcap_{i=1}^{n} (p_i)_{<\varepsilon} = \bigcap_{i=1}^{n} \varepsilon\,(p_i)_{<1} = \varepsilon \bigcap_{i=1}^{n} (p_i)_{<1} = \varepsilon\,(\max\{p_1, \dots, p_n\})_{<1} \\ &= (\max\{p_1, \dots, p_n\})_{<\varepsilon} = q_{<1}, \end{aligned}$$

where $q := \frac{1}{\varepsilon} \cdot \max\{p_1, \dots, p_n\}$. Thus $p \leqslant q := \frac{1}{\varepsilon} \cdot \max\{p_1, \dots, p_n\}$ holds by $\boxed{1.3.7}$.
($\Leftarrow$) With p_i also $q := \lambda \cdot \max\{p_1, \dots, p_n\}$ is continuous, and thus $p_{<1} \supseteq q_{<1}$ is a 0-neighborhood, i.e. p continuous by $\boxed{1}$. □

1.4.3 Summary.

Let $\mathcal{P}_0$ *be a family of seminorms on a vector space* E*. Then the balls* $a + p_{<\varepsilon} := \{x \in E : p(x - a) < \varepsilon\}$ *with* $p \in \mathcal{P}_0$*,* $\varepsilon > 0$ *and* $a \in E$ *form a sub-basis of a locally convex topology. This so-called topology generated by* $\mathcal{P}_0$ *is the coarsest topology (i.e. with the fewest open sets) on* E*, for which all seminorms* $p \in \mathcal{P}_0$ *as well as all translations* $x \mapsto a + x$ *with* $a \in E$ *are continuous. With respect to this topology, a seminorm* p *on* E *is continuous if and only if there are finite many seminorms* $p_i \in \mathcal{P}_0$ *and one* $K > 0$*, s.t.*

$$p \leqslant K \max\{p_1, \dots, p_n\}.$$

1.4.4 Definition. Seminormed space.

By a SEMINORMED SPACE we therefore understand a vector space E together with a **set** $\mathcal{P}$ of seminorms, which are just the continuous seminorms of the topology generated by it, that is, with $p_1, p_2 \in \mathcal{P}$ also every seminorm $p \leqslant p_1 + p_2$ is in $\mathcal{P}$.

A set $\mathcal{P}_0 \subseteq \mathcal{P}$ is called SUB-BASIS OF THE SEMINORMED SPACE $(E, \mathcal{P})$, if it generates the same topology as $\mathcal{P}$, that is for any seminorm p in $\mathcal{P}$ finite many $p_1, \dots, p_n \in \mathcal{P}_0$ exist as well as a $\lambda > 0$ with $p \leqslant \lambda \cdot \max\{p_1, \dots, p_n\}$.

For any family $\mathcal{P}_0$ of seminorms on E, we get a uniquely determined seminormed space, which has $\mathcal{P}_0$ as sub-basis of its seminorms, by using the family $\mathcal{P}$ of, with respect to the topology generated by $\mathcal{P}_0$, continuous seminorms:

$$\mathcal{P} := \Big\{p \text{ is a seminorm on } E : \exists \lambda > 0\ \exists p_1, \dots, p_n \in \mathcal{P}_0 \text{ with } p \leqslant \lambda \cdot \max\{p_1, \dots, p_n\}\Big\}.$$

By the SEMINORMS OF THE SO OBTAINED SEMINORMED SPACE we understand all seminorms belonging to the generating family $\mathcal{P}_0$. We would actually have to say "seminorms of the given sub-basis of the seminormed space", but that's too long for us.

By a COUNTABLY SEMINORMED SPACE we mean a seminormed space which has a countable sub-basis $\mathcal{P}_0$ of seminorms. We may then assume that $\mathcal{P}_0 = \{p_n : n \in \mathbb{N}\}$ and the sequence $(p_n)_n$ is monotone increasing and will eventually dominate any continuous seminorm p, that is there is an $n \in \mathbb{N}$ with $p \leqslant p_n$. To achieve this, replace the p_n with $n \cdot \max\{p_1, \dots, p_n\}$.

1.4.5 Definition. Convex hull.

The convex hull $\langle A\rangle_{\mathrm{kv}}$ of a subset $A \subseteq E$ is the smallest convex subset of E which includes A.

1.4.6 Lemma. Convex hull.

Let $A \subseteq E$. Then the convex hull of A exists and is given by

$$\begin{aligned}\langle A\rangle_{kv} &= \bigcap\{K : A \subseteq K \subseteq E,\ K \text{ is convex }\}\\ &= \Big\{\sum_{i=1}^{n} \lambda_i a_i : n \in \mathbb{N}, a_i \in A, \lambda_i \geqslant 0, \sum_{i=1}^{n} \lambda_i = 1\Big\}.\end{aligned}$$

Proof. The set $\mathcal{A} := \{K : A \subseteq K \subseteq E,\ K \text{ ist convex}\}$ is not empty, because $E \in \mathcal{A}$. Consequently there exists $\bigcap \mathcal{A}$ and obviously is itself convex and thus the minimal element in $\mathcal{A}$, i.e. $\langle A\rangle_{\mathrm{kv}} = \bigcap \mathcal{A}$.

For the second description of the convex hull note that the set $A_0 := \{\sum_{i=1}^n \lambda_i a_i : n \in \mathbb{N}, a_i \in A, \lambda_i \geqslant 0, \sum_{i=1}^n \lambda_i = 1\}$ obviously includes A. It is convex, because let $x_j \in A_0$, i.e. $x_j = \sum_{i=1}^{n_j} \lambda_{i,j}\, a_{i,j}$ for $n_j \in \mathbb{N}$, $a_{i,j} \in A$, $\lambda_{i,j} \geqslant 0$ with $\sum_{i=1}^{n_j} \lambda_{i,j} = 1$. Then for $\mu_j \geqslant 0$ with $\sum_{j=1}^m \mu_j = 1$ we have:

$$\sum_{j=1}^{m} \mu_j\, x_j = \sum_{j=1}^{m} \mu_j \sum_{i=1}^{n_j} \lambda_{i,j}\, a_{i,j} = \sum_{\substack{i,j\\ i\leqslant n_j}} \mu_j\, \lambda_{i,j}\, a_{i,j}$$

$$\text{with} \qquad \sum_{i\leqslant n_j} \mu_j\, \lambda_{i,j} = \sum_{j=1}^{m} \mu_j \sum_{i=1}^{n_j} \lambda_{i,j} = \sum_{j=1}^{m} \mu_j\, 1 = 1.$$

Since A_0 is clearly contained in every set $K \in \mathcal{A}$, $\langle A\rangle_{\mathrm{kv}} = A_0$ holds. □

1.4.7 Definition. Absolutely-convex hull.

The absolutely convex hull $\langle A\rangle_{\mathrm{akv}}$ of a subset $A \subseteq E$ is the smallest absolutely convex subset of E that contains A, thus is the intersection of all these sets.

1.4.8 Lemma. Absolutely-convex hull.

Let $A \subseteq E$. Then the absolutely convex hull is given by

$$\langle A\rangle_{akv} = \langle\{\lambda : |\lambda| = 1\} \cdot A\rangle_{kv},$$

so it is the convex hull of the balanced hull $\{\lambda : |\lambda| = 1\} \cdot A$.

Proof. It is only to be shown that the convex hull of a balanced set A is itself balanced. So let $|\mu| = 1$ and $\sum_{i=1}^n \lambda_i\, a_i \in \langle A\rangle_{\mathrm{kv}}$, then

$$\mu \cdot \sum_{i=1}^{n} \lambda_i\, a_i = \sum_{i=1}^{n} \lambda_i\, \mu\, a_i \in \langle A\rangle_{\mathrm{kv}}, \text{ since } \mu \cdot a_i \in A. \quad \square$$

1.4.9 Lemma.

Each locally convex vector space E has a 0-neighborhood base of absolutely convex sets.

Proof. Let U be a convex 0-neighborhood. This is open without restriction of generality, because its interior is also convex(!). Since the scalar multiplication $\{\lambda \in \mathbb{K} : |\lambda| = 1\} \times E \to E$ is continuous and $0 \cdot \lambda = 0$ holds, there exists a neighborhood $V_\lambda \subseteq \mathbb{K}$ of λ for each $|\lambda| = 1$ and a convex 0-neighborhood $U_\lambda \subseteq E$ with $V_\lambda \cdot U_\lambda \subseteq U$.

Since $\{\lambda \in \mathbb{K} : |\lambda| = 1\}$ is compact, finitely many exist $\lambda_1, \dots, \lambda_n$ with $\{\lambda \in \mathbb{K} : |\lambda| = 1\} \subseteq \bigcup_{i=1}^n V_{\lambda_i}$. Let $U_0 := \bigcap_{i=1}^n U_{\lambda_i}$. Then U_0 is a convex 0-neighborhood and $U_0 \subseteq U_1 := \{\lambda \in \mathbb{K} : |\lambda| = 1\} \cdot U_0 \subseteq U$. The convex hull of the balanced set U_1 is thus an absolutely convex 0-neighborhood in U by 1.4.8. □

1.4.10 Remarks.

The topology of each locally convex vector space is generated by the set $\mathcal{P}$ of all continuous seminorms:
($\supseteq$) If O is open in the topology generated by $\mathcal{P}$, then for every $a \in O$ finitely many $p_1, \dots, p_n \in \mathcal{P}$ and $\varepsilon > 0$ exist with $\bigcap_{i=1}^n \big(a + \varepsilon \cdot (p_i)_{<1}\big) = \big\{x : p_i(x-a) < \varepsilon\ \forall i = 1, \dots, n\big\} \subseteq O$, so O is also in the original topology open since the $(p_i)_{<1}$ are 0-neighborhoods.
($\subseteq$) Conversely, let the latter be fulfilled, i.e. by 1.4.9 there exists an absolutely convex 0-neighborhood U with $U \subseteq O - a$ for each $a \in O$. Then $p := p_U$ is a continuous seminorm, because $p_{\leqslant 1} \supseteq U$ is also a 0-neighborhood. Consequently, $a + p_{<1} \subseteq a + U \subseteq O$ holds, so O is also open in the topology generated by the continuous seminorms.

Since we only have to use the Minkowski functionals of a 0-neighborhood basis in this argument, the following holds:

The topology of each locally convex vector space is already generated by the Minkowski functionals of a 0-neighborhood basis consisting of absolutely convex sets.

1.4.11 Corollary. Special 0-neighborhood basis.

Each locally convex vector space E has a 0-neighborhood basis consisting of closed absolutely convex sets.

Proof. This is obvious because $(p_U)_{\leqslant 1/2} \subseteq U$ is closed. □

1.4.12 Summary.

Let E be a locally convex vector space and $\mathcal{U}$ a 0-neighborhood sub-basis consisting of absolutely convex sets. Then the family $\{p_U : U \in \mathcal{U}\}$ is a sub-basis of that seminormed space, whose seminorms are exactly those being continuous with respect to the given topology, these are exactly those seminorms q for which $q_{\leqslant 1}$ is a 0-neighborhood.

So we have a bijection between seminormed spaces and locally convex vector spaces, and can work with topology or with seminorms on a fixed vector space as needed.

1.5 Convergence and continuity

1.5.1 Definition. Convergent sequence.

A sequence $(x_i)_i$ CONVERGES towards a in a topological space X if and only if for each neighborhood U (of a sub-basis) of a an index i_U exists, such that $x_i \in U$ for all $i \geqslant i_U$, cf. [**26**, 1.1.11].

1.5.2 Lemma. Convergent sequences.

A sequence (x_i) converges in the underlying topology of a locally convex space with sub-basis $\mathcal{P}_0$ towards a if and only if $p(x_i - a) \to 0$ for all $p \in \mathcal{P}_0$.

Proof. (⇒) Since for $a \in E$ the translation $y \mapsto y - a$ is continuous, $x_i - a \to a - a = 0$, and thus also $p(x_i - a) \to p(0) = 0$ for each continuous seminorm p.

(⇐) Let U be a neighborhood of a. Then there are finitely many seminorms $p_j \in \mathcal{P}_0$ and a $\varepsilon > 0$ with $a + \bigcap_{j=1}^n (p_j)_{<\varepsilon} \subseteq U$. Since $p_j(x_i - a) \to 0$, for each j there exists an i_j with $p_j(x_i - a) < \varepsilon$ for $i \geqslant i_j$. Let I be greater than all the finitely many i_j. Then $x_i \in a + \bigcap_{j=1}^n (p_j)_{<\varepsilon}$ for $i \geqslant I$ and thus also in U, i.e. $x_i \to a$. □

1.5.3 Lemma. Sequentially continuous mapping.

A mapping $f : E \to X$ of a countably seminormed space E into a topological space X is continuous if and only if it is sequentially continuous, i.e. for each convergent sequence $x_i \to a$ also the image sequence $f(x_i) \to f(a)$ converges.

See [**20**, 3.1.3].

Proof. (⇒) is clear, because of the above description 1.5.2 of the convergent sequences.

(⇐) indirectly: Suppose $f^{-1}(U)$ is not a neighborhood of a for a neighborhood U of $f(a)$. Let $\{p_n : n \in \mathbb{N}\}$ be a countable sub-basis of the seminorms of E. Then for each n there is an $x_n \in E$ with $p_k(x_n - a) < \frac{1}{n}$ for all $k \leqslant n$ and $f(x_n) \notin U$. So $p_k(x_n - a) \to 0$ for $n \to \infty$, and thus also $x_n \to a$ according to the above lemma 1.5.2. But since $f(x_n) \notin U$, this is a contradiction to the sequential continuity of f. □

1.5.4 Definition. Net.

Since the above lemma does not hold for non-countably seminormed spaces , we extend the notion of a sequence to:
A net (generalized sequence or Moore-Smith sequence, see [**26**, 3.4.1]) is a mapping $x : I \to X$, where I is a directed index set, i.e. a set together with a relation $\prec$, which is transitive and has for any two elements i_1 and i_2 in I also a $i \in I$ with $i_1 \prec i$ and $i_2 \prec i$, see also [**26**, 3.4.1]. Exactly, as for sequences, one defines the convergence of nets and shows thus also the first of the two lemmas from above. Regarding the second lemma we have

1.5.5 Lemma. Continuity via nets.

A mapping $f : E \to X$ from a locally convex space to a topological space is continuous if and only if for each convergent net $x_i \to a$ the image net $f(x_i) \to f(a)$. See [**26**, 3.4.3].

Proof. (⇒) is obvious, because if U is a $f(a)$-neighborhood and $x_i \to a$, then $\exists i_0\ \forall i \geqslant i_0 : x_i \in f^{-1}(U)$, i.e. $f(x_i) \in U$, that is $f(x_i) \to f(a)$.

(⇐) Let $\mathcal{U}$ be a neighborhood basis of a. Then we use as index set $I := \{(U, u) : U \in \mathcal{U}, u \in U\}$ with the order $(U, u) \prec (U', u') \Leftrightarrow U \supseteq U'$ and as net on it the mapping $x : (U, u) \mapsto u$. Then, clearly, the net x converges to a, so by assumption also $f \circ x$ towards $f(a)$, i.e. for each $f(a)$-neighborhood V exists an index (U_0, u_0), s.t. $f(u) \in V$ for all $U \subseteq U_0$ and $u \in U$. So $f(U_0) \subseteq V$, that means f is continuous. □

1.5.6 Definition. Separatedness.

A locally convex space is called separated (or also Hausdorff, see [**26**, 3.4.4]), if the limits of convergent sequences (or nets) are unique, this is the case if and only if $p(x) = 0$ for all $p \in \mathcal{P}_0$ implies $x = 0$:

($\Leftarrow$) Let x_i be a net converging to x' and x''. Then $x_i - x'$ converges towards 0 and also towards $x'' - x'$. Because of the continuity of p, $p(x_i - x')$ converges to $p(0) = 0$ and also to $p(x'' - x')$. Because of the uniqueness of the limits in $\mathbb{K}$, $p(x'' - x') = 0$ holds for all p, and thus, by assumption, $x'' - x' = 0$.
($\Rightarrow$) Let $p(x) = 0$ for all p. Then the constant sequence (net) with value x converges to both 0 and x, hence, by assumption, $x = 0$.

We are going to use the abbreviation LCS for separated locally convex spaces.

1.6 Normable spaces

1.6.1 Definition. Normable spaces and bounded sets.

One calls a separated lcs, which has a sub-basis consisting of a single (semi-)norm, NORMABLE.

A set $B \subseteq E$ is called BOUNDED if and only if $p(B)$ is bounded for all $p \in \mathcal{P}_0$, cf. [**20**, 2.2.9]. That's exactly the case when it gets absorbed by all 0-neighborhoods, i.e. $\forall$ 0-neighborhood $U\ \exists K > 0 : B \subseteq K \cdot U$:
($\Leftarrow$) Let p be a continuous seminorm, then $p_{\leqslant 1}$ is a 0-neighborhood, so by assumption there is an $K > 0$ with $B \subseteq K \cdot p_{\leqslant 1} = p_{\leqslant K}$, i.e. p is bounded on B by K.

($\Rightarrow$) Let U be a 0-neighborhood. Then there are finitely many seminorms $p_i \in \mathcal{P}_0$ and an $\varepsilon > 0$ with $\bigcap_{i=1}^n (p_i)_{\leqslant\varepsilon} \subseteq U$. For each p_i there is a $K_i > 0$ with $|p_i(B)| \leqslant K_i$, so $B \subseteq \bigcap_{i=1}^n (p_i)_{\leqslant K_i} \subseteq \bigcap_{i=1}^n (p_i)_{\leqslant K\,\varepsilon} = K \cdot U$, where $K := \frac{1}{\varepsilon} \cdot \max\{K_1, \dots, K_n\}$.

1.6.2 Theorem of Kolmogoroff.

A separated lcs is normable if and only if it has a bounded zero-neighborhood.

Proof. ($\Rightarrow$) Let p be a norm generating the structure. Then $U := p_{\leqslant 1}$ is a 0-neighborhood. For any continuous seminorm q there exists an $K > 0$ with $q \leqslant K \cdot p$, and thus q is bounded on U by K. So U is bounded.

($\Leftarrow$) Let U be a bounded zero neighborhood. Then there is a continuous seminorm with $p_{\leqslant 1} \subseteq U$. Now let q be any seminorm. Since U is bounded, there is a $K > 0$ with $|q(U)| \leqslant K$. So $p_{\leqslant 1} \subseteq U \subseteq q_{\leqslant K} = (\frac{1}{K} q)_{\leqslant 1}$ and therefore $p \geqslant \frac{1}{K} q$, that is $q \leqslant K \cdot p$. Thus, $\{p\}$ is a sub-basis of the seminorms of E and p is even a norm. $\square$

1.6.3 Example. The pointwise convergence of continuous functions.

The pointwise convergence on $C(I, \mathbb{R})$ can not be a normed space.

Proof. A sub-basis of seminorms for pointwise convergence is given by $f \mapsto |f(x)|$ for $x \in I$. Suppose there is a bounded zero neighborhood B. Then finitely many points $x_1, \dots x_n \in I$ and a $\varepsilon > 0$ exist, s.t. $B := \{f : |f(x_i)| < \varepsilon \text{ for } i = 1, \dots, n\}$ is bounded. Let $x_0 \notin \{x_1, \dots, x_n\}$. Then the seminorm $q : f \mapsto |f(x_0)|$ is not bounded on B, because certainly there exists a (polynomial) f which vanishes on $\{x_1, \dots, x_n\}$, but not on x_0, and thus $K \cdot f \in B$, but $q(K \cdot f) = K \cdot f(x_0) \to \infty$ for $K \to \infty$. $\square$

Analogously one shows that the uniform convergence on compact sets in the space $C(\mathbb{R}, \mathbb{R})$ is not normable but yields a countably seminormed space. And similarly for the uniform convergence in each derivative on $C^\infty(I, \mathbb{R})$.

2. Linear mappings and completeness

In this chapter we examine the basic properties of linear mappings as well as the notion of completeness and its relevance for power series. In particular, we apply this to prove the inverse function theorem and the Weierstrass approximation theorem, as well as for solving linear differential equations.

2.1 Continuous and bounded mappings

2.1.1 Lemma. Continuity of linear mappings.

For a linear mapping $f : E \to F$ between lcs's are equivalent:

1. *f is continuous;*

$\Leftrightarrow$ 2. *f is continuous at 0;*

$\Leftrightarrow$ 3. *For each (continuous) SN q of F, $q \circ f$ is a continuous SN of E.*

Proof. ($\boxed{1} \Rightarrow \boxed{3}$) q a continuous SN, f continuous linear $\Rightarrow q \circ f$ is a continuous SN.

($\boxed{3} \Rightarrow \boxed{2}$) Let U be a 0-neighborhood of $0 = f(0)$ in F, without restriction of generality $U = \bigcap_i \{y : q_i(y) < \varepsilon\}$ for SN's $q_1, \dots, q_n$ of F. Then $f^{-1}(U) = \bigcap_i \{x : q_i(f(x)) < \varepsilon\} = \bigcap_i (q_i \circ f)_{<\varepsilon}$ is open in E.

($\boxed{2} \Rightarrow \boxed{1}$) We have $f(x) = f(x - a) + f(a)$, i.e. $f = T_{f(a)} \circ f \circ T_{-a}$, where the translations T_{-a} and $T_{f(a)}$ are continuous and the middle f is continuous at 0, hence also the composition f is continuous at $(T_{-a})^{-1}(0) = a$. □

2.1.2 Lemma. Continuity of multi-linear mappings.

An n-linear mapping $f : E_1 \times \dots \times E_n \to F$ between lcs's is continuous if and only if it is continuous at 0.

Proof. Let first $n = 2$. For $a_i \in E_i$ and any neighborhood $f(a_1, a_2) + W$ of $f(a_1, a_2)$ with absolutely convex W, 0-neighborhoods U_i exist in E_i with $f(U_1 \times U_2) \subseteq \frac{1}{3} W$, because of the continuity of f at 0. Now choose a $0 < \rho < 1$ with $\rho\, a_i \in U_i$ for $i = 1, 2$. Then $f((a_1 + \rho U_1) \times (a_2 + \rho U_2)) \subseteq f(a_1, a_2) + W$, because $u_i \in U_i$ is

$$\begin{aligned} f\big(a_1 + \rho\, u_1, a_2 + \rho\, u_2\big) - f\big(a_1, a_2\big) &= \underbrace{f(a_1, \rho\, u_2)}_{= f(\rho\, a_1, u_2)} + \underbrace{f(\rho\, u_1, a_2)}_{= f(u_1, \rho\, a_2)} + \underbrace{f(\rho\, u_1, \rho\, u_2)}_{= \rho^2 f(u_1, u_2)} \\ &\subseteq \frac{1}{3} W + \frac{1}{3} W + \frac{1}{3} W \subseteq W. \end{aligned}$$

For $n > 2$, choose $U_1, \dots, U_n$ analogously with $(2^n - 1)\, f(U_1 \times \dots \times U_n) \subseteq W$. □

2.1.3 Definition. Bounded linear mappings.

A linear mapping is called BOUNDED if the image of each bounded set is bounded. Warning: In the literature this notation is sometimes also used for the non-equivalent

property to be bounded on some 0-neighborhood!
Note that bounded subsets of an LCS can not contain any ray $a + \mathbb{R}^+ \cdot v$ for $v \neq 0$, since otherwise $t \mapsto p(a + t\, v)$ would be bounded on $\mathbb{R}^+$, say by $K_p > 0$, for each seminorm p of E, hence $t\, p(v) = p(tv) \leqslant p(a + tv) + p(-a) \leqslant K_p + p(a)$ for all $t > 0$ by 1.3.1, hence $p(v) = 0$, i.e. $v = 0$.

Consequently, a linear mapping $f : E \to F$ is bounded as mapping from the set E to F (i.e. $f(E) \subseteq F$ is bounded), only if is the 0-map, because $f(E)$ would then be a bounded linear subspace, and thus $f(E) = \{0\}$.

2.1.4 Lemma. Bounded linear mappings.

For linear mappings $f : E \to F$ between lcs's the following implications hold:

1. *f is continuous;*

$\Rightarrow$ 2. *f is sequentially continuous;*

$\Rightarrow$ 3. *f is bounded.*

Proof. (1 $\Rightarrow$ 2) holds even for non-linear f by 1.5.5.

(2 $\Rightarrow$ 3) Suppose $f(B)$ is not bounded for some bounded set $B \subseteq E$. Then there is a seminorm q of F and a sequence $b_n \in B$, s.t. $0 < \lambda_n := q(f(b_n)) \to \infty$. The sequence $\frac{1}{\lambda_n} b_n$ then converges to 0 (see the following lemma), so because of the sequential continuity also $f(\frac{1}{\lambda_n} b_n) = \frac{1}{\lambda_n} f(b_n)$ and thus also $q(\frac{1}{\lambda_n} f(b_n)) = \frac{1}{\lambda_n} q(f(b_n)) = 1$, a contradiction. □

Now the question arises of the validity of the converse to the implications in 2.1.4. For (1 $\Leftarrow$ 2) we have already answered this positively in 1.5.3 for countably seminormed spaces.
For (2 $\Leftarrow$ 3) we need some relationship between bounded and convergent sequences. A simple fact is the following.

2.1.5 Lemma. Mackey-convergence.

Let $\{y_n : n \in \mathbb{N}\} \subseteq E$ be bounded in an lcs and $\rho_n \to 0$ in $\mathbb{R}$. Then $\rho_n y_n \to 0$.

Proof. By applying seminorms this is reduced to the corresponding result for $\mathbb{R}$. Or directly: Let U be an absolutely convex 0-neighborhood. Then $\{y_n : n \in \mathbb{N}\} \subseteq K \cdot U$ for some $K > 0$ and thus $\rho_n\, y_n \in U$ for all $|\rho_n| \leqslant \frac{1}{K}$, so for almost all n. □

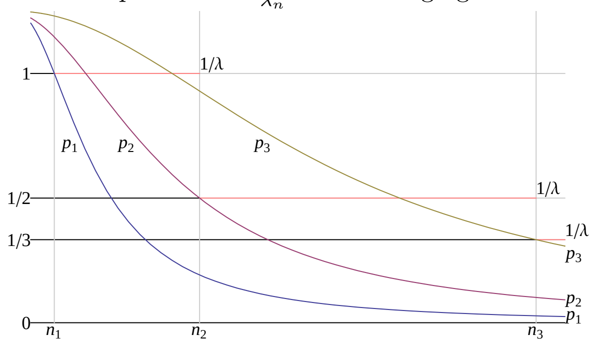
In order to be able to deduce at least sequential continuity from boundedness, it would be helpfull if the converse were true, i.e. if we could write any convergent sequence $(x_n)_n$ in E as a product of a bounded sequence $(y_n)_n$ in E and a 0-sequence ρ_n in $\mathbb{R}$. A sequence (x_n), for which this holds, is called Mackey 0-sequence or Mackey-convergent towards 0, so if $\exists 0 \leqslant \lambda_n \to \infty$, s.t. $\{\lambda_n x_n : n \in \mathbb{N}\}$ is bounded.

Each Mackey 0-sequence $(x_n)_n$ converges to 0 by Lemma 2.1.5 applied to $y_n := \lambda_n x_n$. For normable spaces, the converse implication also holds, because $x_n \to 0$ implies $0 \leqslant \lambda_n \to \infty$, where $\lambda_n := \frac{1}{\|x_n\|}$ for $x_n \neq 0$ and $\lambda_n := n$ otherwise, and obviously $\{\lambda_n x_n : n \in \mathbb{N}\}$ is bounded in the norm by 1. More generally, this also holds for countably seminormed spaces:

2.1.6 Lemma.

In countably seminormed spaces E, each sequence converging to 0 is even Mackey-convergent to 0.

Proof. Let $\{p_k : k \in \mathbb{N}\}$ be a monotonously increasing sub-basis of E and $x_n \to 0$ a 0-sequence. The idea is to define for the countable many zero sequences $(p_k(x_n))_n$ for $k \in \mathbb{N}$ another zero sequence $n \mapsto \frac{1}{\lambda_n} > 0$ converging slower towards 0.



From $p_k(x_n) \to 0$ for $n \to \infty$ follows the existence of $n_k \in \mathbb{N}$ with $p_i(x_n) \leqslant \frac{1}{k}$ for all $n \geqslant n_k$ and all $i \leqslant k$. Without loss of generality $k \mapsto n_k$ is strictly monotonously increasing. We define $\lambda_n := k$ for $n_{k+1} > n \geqslant n_k$. Then, $n \mapsto \lambda_n$ is monotonously increasing, $\lambda_n \to \infty$, and for $n \geqslant n_k$, $p_k(\lambda_n x_n) = \lambda_n p_k(x_n) = j\, p_k(x_n) \leqslant j\, p_j(x_n) \leqslant j \frac{1}{j} = 1$, where $j \geqslant k$ is selected to be $n_{j+1} > n \geqslant n_j$. □

2.1.7 Corollary. Bornologicity of metrizable lcs.

Every countably seminormed space is bornological. Even more holds: Multilinear bounded mappings on countably seminormed spaces are continuous.

Where an lcs is called BORNOLOGICAL, if each bounded linear mapping on it is continuous.
In 4.2.5 we will give examples of lcs's that are not bornological.

Proof. Because of 1.5.3, we only need to show the sequential continuity (at 0) of each bounded m-linear mapping f. Let $x_n \to 0$. By Lemma 2.1.6 there exists a sequence $\lambda_n \to \infty$, so that $\lambda_n x_n$ is bounded. Then, by assumption $f(\lambda_n x_n) = \lambda_n^m f(x_n)$ is also bounded, and thus $f(x_n)$ is a (Mackey) 0-sequence by 2.1.5. □

2.1.8 Lemma. Continuity in normed spaces.

For linear mappings $f : E \to F$ between normed spaces are equivalent:

1. *f is continuous;*
⇔ 2. *f is* Lipschitz, *i.e.* $\exists K > 0 : \|f(x) - f(y)\| \leqslant K \cdot \|x - y\|$;
⇔ 3. $\|f\| < \infty$.

The OPERATOR NORM $||f||$ *on f is defined as follows (cf.* [22, 5.4.10]*)*

$$\|f\| := \sup\Big\{\|f(x)\| : \|x\| \leqslant 1\Big\} = \sup\Big\{\|f(x)\| : \|x\| = 1\Big\} = \sup\left\{\tfrac{\|f(x)\|}{\|x\|} : x \neq 0\right\}$$
$$= \inf\Big\{K : \|f(x)\| \leqslant K\|x\| \text{ for all } x\Big\}$$

If f is multi-linear, then f is continuous if and only if

$$\|f\| := \sup\left\{\frac{\|f(x_1,\dots,x_n)\|}{\|x_1\|\dots\|x_n\|} : x_i \neq 0\right\} < \infty.$$

Proof. ($\boxed{1}$ $\Leftrightarrow$ $\boxed{3}$) f is continuous $\overset{\boxed{2.1.7}}{\Leftrightarrow}$ f is bounded on bounded sets (without restriction of generality on $\{x : \|x\| \leqslant 1\}$, since $f(B) \subseteq c \cdot f(\{x : \|x\| \leqslant 1\})$ for $B \subseteq c \cdot \{x : \|x\| \leqslant 1\}$) $\Leftrightarrow \sup\{\|f(x)\| : \|x\| \leqslant 1\} =: \|f\| < \infty$.

The following applies:

$$\begin{aligned}\sup\{\|fx\| : \|x\| = 1\} &\leqslant \sup\{\|fx\| : \|x\| \leqslant 1\} \qquad \text{(because more elements)}\\ &\leqslant \sup\left\{\frac{\|fx\|}{\|x\|} : x \neq 0\right\} \qquad \text{(because } \|fx\| \leqslant \frac{\|fx\|}{\|x\|} \text{ for } ||x|| \leqslant 1)\\ &\leqslant \sup\{\|fx\| : \|x\| = 1\} \qquad \text{(because } \frac{\|fx\|}{\|x\|} = \|f(\frac{1}{\|x\|}x)\|),\end{aligned}$$

so equality holds everywhere. Furthermore:

$$\begin{aligned}\inf\Big\{K : \|fx\| \leqslant K \cdot \|x\| \text{ for all } x\Big\} &= \inf\left\{K : \frac{\|fx\|}{\|x\|} \leqslant K \text{ for all } x \neq 0\right\}\\ &= \inf\left\{K : \sup\left\{\frac{\|fx\|}{\|x\|} : x \neq 0\right\} \leqslant K\right\}\\ &= \sup\left\{\frac{\|fx\|}{\|x\|} : x \neq 0\right\}.\end{aligned}$$

The mapping f is Lipschitz $\Leftrightarrow \left\{\frac{\|fz\|}{\|z\|} : z \neq 0\right\} = \left\{\frac{\|fx-fy\|}{\|x-y\|} : x \neq y\right\}$ is bounded.

The statement for multilinear mappings f is shown analogously. $\square$

2.1.9 Corollary. Operator norm.

Let E and F be normed spaces, then the set

$$L(E,F) := \{f : E \to F \mid f \text{ is linear and bounded}\}$$

is a normed space with respect to the pointwise vector operations and the operator norm as defined in $\boxed{2.1.8}$. *Furthermore:* $\|\operatorname{id}_E\| = 1$ *and* $\|f \circ g\| \leqslant \|f\| \cdot \|g\|$.

Proof. The following applies:

$$\begin{aligned}\forall x : \|(f+g)x\| \leqslant \|fx\| + \|gx\| \leqslant (\|f\| + \|g\|)\,\|x\| &\Rightarrow \|f+g\| \leqslant \|f\| + \|g\|\\ \forall x : \|(\lambda f)x\| = |\lambda|\,\|fx\| &\Rightarrow \|\lambda f\| = |\lambda|\,\|f\|\\ \forall x : \|(f \circ g)x\| \leqslant \|f\|\,\|g\|\,\|x\| &\Rightarrow \|f \circ g\| \leqslant \|f\|\,\|g\|. \quad \square\end{aligned}$$

Attention $\|f \circ g\| \neq \|f\| \cdot \|g\|$, e.g. $f(x,y) := (x,0)$ and $g(x,y) := (0,y)$.

2.1.10 Definition. Normed algebra.

A normed algebra is a normed space A along with a bilinear mapping $\bullet : A \times A \to A$, which is associative, has a unit 1 and satisfies $\|1\| = 1$ as well as $\|a \bullet b\| \leqslant \|a\| \cdot \|b\|$. One of the most important examples is $L(E,E) =: L(E)$ for normed spaces E.

2.2 Completeness

2.2.1 Definition. Completeness.

An lcs E is called sequentially complete if every Cauchy sequence converges. It is called complete when every Cauchy net converges. A net (or sequence) x_i is called Cauchy if $x_i - x_j \to 0$ for $i,j \to \infty$, i.e.

$$\forall \varepsilon > 0\ \forall p\ \exists i_0\ \forall i,j \succ i_0 : p(x_i - x_j) < \varepsilon.$$

A BANACH SPACE is a normed space that is (sequentially) complete.
A (sequentially) complete countably seminormed space is called FRÉCHET SPACE.

2.2.2 Lemma. Fréchet-spaces.

For each countably seminormed space and each everywhere positive $\lambda \in \ell^1$ are equivalent

1. *It is complete;*
2. $\Leftrightarrow$ *It is sequentially complete;*
3. $\Leftrightarrow$ *Any absolutely convergent series converges;*
4. $\Leftrightarrow$ *For each bounded sequence (b_n) the series $\sum_n \lambda_n b_n$ converges;*
5. $\Leftrightarrow$ *Each Cauchy sequence has a convergent subsequence.*

A series $\sum_n x_n$ is called ABSOLUTELY CONVERGENT if for each continuous seminorm p the series $\sum_n p(x_n)$ converges (absolutely) in $\mathbb{R}$.

Proof. ($\boxed{1} \Rightarrow \boxed{2}$) is trivial.

($\boxed{2} \Rightarrow \boxed{3}$) Let $\sum_n x_n$ be absolutely convergent, then the partial sums of $\sum_n x_n$ form a Cauchy sequence, for $p(\sum x_n) \leqslant \sum p(x_n)$, hence $\sum_n x_n$ converges by $\boxed{2}$.

($\boxed{3} \Rightarrow \boxed{4}$) Let the sequence (b_n) be bounded and (λ_n) be absolutely summable. Then $\sum_n \lambda_n b_n$ is absolutely summable, because $\sum_n p(\lambda_n b_n) \leqslant \|\lambda\|_1 \cdot \|p \circ b\|_\infty$. So this series converges by $\boxed{3}$.

($\boxed{4} \Rightarrow \boxed{5}$) Let $\{p_n : n \in \mathbb{N}\}$ be a monotonously increasing sub-basis of seminorms. Let (x_i) be a Cauchy sequence. Then:

$$\begin{aligned}
&\forall k\ \exists i_k\ \forall i,j \geqslant i_k : p_k(x_i - x_j) \leqslant \lambda_k \qquad \text{(without loss of generality } i_k \leqslant i_{k+1})\\
\Rightarrow\quad & p_n\left(\frac{1}{\lambda_k}(x_{i_{k+1}} - x_{i_k})\right) \leqslant p_k\left(\frac{1}{\lambda_k}(x_{i_{k+1}} - x_{i_k})\right) \leqslant 1 \text{ for } n \leqslant k\\
\Rightarrow\quad & \frac{1}{\lambda_k} y_k \text{ is bounded, where } y_k := x_{i_{k+1}} - x_{i_k}\\
\overset{\boxed{4}}{\Rightarrow}\quad & x_{i_j} = x_{i_0} + \sum_{k<j} \lambda_k \frac{1}{\lambda_k} y_k \text{ converges.}
\end{aligned}$$

($\boxed{5} \Rightarrow \boxed{1}$) Let (x_i) be a Cauchy net and (p_n) a increasing sub-basis of seminorms. Then:

$$\begin{aligned}
&\forall k\ \exists i_k\ \forall i,j \succ i_k : p_k(x_i - x_j) \leqslant \frac{1}{k} \qquad \text{(without loss of generality } i_{k+1} \succ i_k)\\
\Rightarrow\quad & x_{i_k} \text{ is a Cauchy sequence}\\
\overset{\boxed{5}}{\Rightarrow}\quad & \text{a convergent subsequence } (x_{i_{k_l}})_l \text{ exists. Let } x_\infty := \lim_l x_{i_{k_l}} \text{ and } n \leqslant k\\
\Rightarrow\quad & p_n(x_i - x_\infty) \leqslant p_k(x_i - x_\infty) \leqslant \underbrace{p_k(x_i - x_{i_k})}_{\leqslant \frac{1}{k} \text{ for } i \succ i_k} + \underbrace{p_k(x_{i_k} - x_\infty)}_{=\lim_l p_k(x_{i_k} - x_{i_{k_l}}) \leqslant \frac{1}{k}} \leqslant \frac{2}{k}. \qquad \square
\end{aligned}$$

2.2.3 Lemma. Completeness of the space of bounded mappings.

Let X be a set and E a (sequentially) complete lcs. Then the space

$$B(X,E) := \{f : X \to E \mid f(X) \text{ is bounded in } E\},$$

being seminormed by the family $f \mapsto \|q \circ f\|_\infty = \sup\{q(f(x)) : x \in X\}$ where q runs through the seminorms of E, is also (sequentially) complete.
Its locally convex topology is that of uniform convergence.
Subsets $\mathcal{B} \subseteq B(X,E)$ are bounded if and only if they are uniformly bounded, i.e. $\mathcal{B}(X) = \{f(x) : f \in \mathcal{B}, x \in X\} \subseteq E$ is bounded.

We will write $B(X)$ instead of $B(X,\mathbb{K})$. See also [20, 4.2.9].

Proof. Let f_i be a Cauchy net in $B(X,E)$. The point evaluations $\mathrm{ev}_x : B(X,E) \to E$, $f \mapsto f(x)$ are continuous (because of $q(\mathrm{ev}_x(f)) = q(f(x)) \leqslant \|q \circ f\|_\infty$ this follows from 2.1.1 and 1.4.3) and linear, hence $f_i(x)$ is a Cauchy net in E for each $x \in X$, and thus converges. Let $f(x) := \lim_i f_i(x)$, then for each continuous seminorm p on E:

$$\begin{aligned} p\big(f_i(x) - f(x)\big) &\leqslant p\big(f_i(x) - f_j(x)\big) + p\big(f_j(x) - f(x)\big) \\ &\leqslant \underbrace{\|p \circ (f_i - f_j)\|_\infty}_{<\varepsilon \text{ for } i,j > i_0(\varepsilon,p)} + \underbrace{p\big(f_j(x) - f(x)\big)}_{<\varepsilon \text{ for } j > i_0(x,\varepsilon,p)} \leqslant 2\varepsilon \end{aligned}$$

for $i > i_0(\varepsilon, p)$ (and j selected depending on x). So $f_i \to f$ with respect to the supremum norm constructed using p.

In case that was too short, again in more detail: Let $\varepsilon > 0$.

$$\begin{aligned} (f_i) \text{ is Cauchy } &\Rightarrow \exists i_0\, \forall i,j > i_0 : \|p \circ (f_i - f_j)\|_\infty < \frac{\varepsilon}{2} \\ f_j \to f \text{ pointwise } &\Rightarrow \forall x\, \exists j_0\, \forall j > j_0 : p(f_j(x) - f(x)) < \frac{\varepsilon}{2} \\ &\Rightarrow \exists i_0\, \forall x\, \exists j_0 > i_0\, \forall i > i_0\, \forall j > j_0 : \\ &\qquad p\big(f_i(x) - f(x)\big) \leqslant \|p \circ (f_i - f_j)\|_\infty + p\big(f_j(x) - f(x)\big) < \varepsilon \\ &\Rightarrow \exists i_0\, \forall i > i_0\, \forall x : p\big(f_i(x) - f(x)\big) < \varepsilon \\ &\Rightarrow \exists i_0\, \forall i > i_0 : \|p \circ (f_i - f)\|_\infty \leqslant \varepsilon \end{aligned}$$

Furthermore,

$$p(f(x)) \leqslant p\big(f(x) - f_i(x)\big) + p(f_i(x)) \leqslant \|p \circ (f - f_i)\|_\infty + \|p \circ f_i\|_\infty < \infty,$$

hence f belongs to $B(X,E)$.

The statement about the bounded subsets $\mathcal{B} \subseteq B(X,E)$ is proved as follows:
A set $\mathcal{B} \subseteq B(X,E)$ is bounded exactly when $\{\|q \circ f\|_\infty : f \in \mathcal{B}\}$ is bounded for each seminorm q of E, so $\{q(f(x)) : x \in X,\ f \in \mathcal{B}\} \subseteq \mathbb{R}$ is bounded, i.e. $\mathcal{B}(X) := \{f(x) : f \in \mathcal{B}, x \in X\}$ is bounded in E. □

2.2.4 Lemma. Subspaces of complete spaces.

Let E be a (sequentially) complete lcs, F a linear subspace with the restrictions $p|_F$ of the seminorms p of E as a sub-basis. If F is (sequentially) closed in E, then F is also (sequentially) complete

We will show in 3.1.4 that in this situation the subspace F carries the trace topology of E. A subset Y of a topological space X is called CLOSED, respectively SEQUENTIALLY CLOSED, if with each net, resp. sequence, $(y_i)_i$ in Y, which converges in X, also the limit belongs to Y. It is easy to show that a subset is closed exactly when its complement is open.

Proof. If (y_i) is Cauchy in F, i.e. $p|_F(y_i - y_j) \to 0$ for each SN p of E, then it is Cauchy in E, hence converges in E because of the completeness of E. Let $y_\infty \in E$ be its limit, then $y_\infty \in F$ because of the closedness of F and $p|_F(y_i - y_\infty) = p(y_i - y_\infty) \to 0$, thus y_i converges to y_∞ in F. □

2.2.5 Corollary. Subspaces of the space of bounded mappings.

The spaces $C(X)$ for compact X, as well as $C_b(X) := C(X) \cap B(X)$ and $C_0(X) := \{f \in C(X) : \forall \varepsilon > 0\ \exists K \subseteq X \text{ compact } \forall x \notin K : |f(x)| \leqslant \varepsilon\}$ for general topological spaces X, are all complete with respect to the supremum norm.

Proof. All we have to do is to show the sequentially closedness of the above subspaces of $B(X)$, which follows from the fact that the limit of any uniformly convergent sequence of continuous functions is continuous, cf. [**20**, 4.2.8]:
Let $f_n \to f_\infty$ be uniformly convergent and f_n be continuous for all $n \in \mathbb{N}$. For $\varepsilon > 0$ and $x_0 \in X$ choose $n \in \mathbb{N}$ with $\|f_n - f_\infty\|_\infty < \frac{\varepsilon}{3}$, as well as, because of the continuity of f_n, a neighborhood U of x_0 with $|f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}$ for all $x \in U$. Then we have for all $x \in U$:

$$|f_\infty(x) - f_\infty(x_0)| \leqslant |f_\infty(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f_\infty(x_0)| < 3\frac{\varepsilon}{3}.$$

If $f_n \in C_0(X)$, then also $f_\infty \in C_0(X)$, because for $\varepsilon > 0$ there exists a n_0 with $\|f_n - f_\infty\|_\infty < \varepsilon$ for all $n \geqslant n_0$ and, because $f_{n_0} \in C_0$, there exists a compact $K \subseteq X$ with $|f_{n_0}(x)| \leqslant \varepsilon$ for $x \notin K$. So

$$|f_\infty(x)| \leqslant \|f_\infty - f_{n_0}\| + |f_{n_0}(x)| < 2\varepsilon \text{ for all } x \notin K.$$

Usually, $C_0(X)$ is only considered for locally compact X, because in points $x_0 \in X$ without compact neighborhood each function $f \in C_0(X)$ must vanish: If $f(x_0) \neq 0$, then we choose a compact set K with $|f(x)| \leqslant \frac{1}{2}|f(x_0)|$ for all $x \notin K$ and thus $K \supseteq \{x : |f(x)| > \frac{1}{2}|f(x_0)|\}$ would be a neighborhood of x_0.

Each locally compact space X has a one-point compactification $X_\infty = X \cup \{\infty\}$ (see [**26**, 2.2.5]) and $C_0(X)$ can then also be described as $C_0(X) = \{f \in C(X) : \lim_{x\to\infty} f(x) = 0\} \cong \{f \in C(X_\infty) : f(\infty) = 0\}$. □

2.2.6 Example. Completeness of the space of the functions with bounded variation.

$BV(I, \mathbb{R})$ is a Banach space.

The variation seminorm V has as kernel

$$\operatorname{Ker} V := \{f : V(f) = 0\} = \{f : f \text{ is constant}\}.$$

To get a separated space we have the following options:

- We add another seminorm to V, e.g. the supremum norm or even just $f \mapsto |f(0)|$, which recognizes constant non-vanishing mappings. Equivalent, we can also consider the sum or the maximum of V with the additional seminorm and get a normed space.
- We shrink the space of functions with bounded variation to $BV(I, \mathbb{R}) := \{f : I \to \mathbb{R} : V(f) < \infty \text{ und } f(0) = 0\}$ in order to get rid of the constants unequal 0.
- We factor out the kernel of the seminorm V and get a vector space of equivalence classes of functions with the seminorm induced by V as norm.

Since $\operatorname{Ker}(V)$ is 1-dimensional, it does not really matter which of the 3 options we pick, for other seminorms (see [**18**, 4.11.7]) this is not the case anymore.

Proof. Let $BV(I, \mathbb{R}) := \{f : I \to \mathbb{R} : f(0) = 0 \text{ und } V(f) < \infty\}$ and $(f_n)_n$ be a Cauchy sequence in $BV(I, \mathbb{R})$. Because of $|f(x)| \leqslant V(f, Z) \leqslant V(f)$ with $Z = \{0, x, 1\}$ and thus $\|f\|_\infty \leqslant V(f)$, the inclusion $BV(I, \mathbb{R}) \hookrightarrow B(I, \mathbb{R})$ is continuous and thus $f_n \to f_\infty$ converges uniformly. Furthermore, the convergence is also with respect to V, because

$$\begin{aligned}V(f_n - f_\infty, Z) &\leqslant V(f_n - f_m, Z) + V(f_m - f_\infty, Z)\\ &\leqslant V(f_n - f_m) + \sum_k |(f_m - f_\infty)(x_k)| + \sum_k |(f_m - f_\infty)(x_{k-1})| < 2\varepsilon,\end{aligned}$$

for all $n > n(\varepsilon)$ provided $m > n(\varepsilon)$ was selected in dependence of Z so that $|f_m(x_k) - f_\infty(x_k)| \leqslant \frac{\varepsilon}{2|Z|}$ for all subdivision points x_k of Z.

Because of $V(f_\infty) \leqslant V(f_\infty - f_n) + V(f_n)$ we have $f_\infty \in BV(I, \mathbb{R})$. □

2.2.7 Corollary. Completeness of the space of bounded linear mappings.

Suppose E and F are locally convex spaces. Then the set $L(E, F) := \{f : E \to F \mid f$ is linear and bounded$\}$ is a locally convex space with respect to the pointwise vector operations and the seminorms of the form $f \mapsto \|q \circ f|_B\|_\infty$ with all bounded $B \subseteq E$ and all SN's q of F, see also 3.1.1 and 3.1.3. Its locally convex topology is thus that of uniform convergence on bounded sets in E. If F is (sequentially) complete, so is $L(E, F)$.

Note that $L(E, F)$ is a countably seminormed space if F is one and, in addition, a countable sub-basis of bounded sets exists in E, that is, a set $\mathcal{B}$ of bounded sets, s.t. each bounded set B is included in a union of finitely many sets from $\mathcal{B}$.

Proof. Completeness: Let $(f_i) \in L(E, F)$ be a Cauchy net. For each $x \in E$, the sequence $f_i(x)$ converges towards some $f(x) \in F$. Furthermore, for every bounded $A \subseteq E$, the net $f_i|_A$ is a Cauchy net in $B(A, F)$, thus converges to an $f_A \in B(A, F)$ by 2.2.3. Since this also has to hold pointwise for $x \in A$, we have $f_A(x) = f(x)$. The mapping f is bounded because $f(A) = f_A(A)$ is bounded. It is linear because f_i converges pointwise towards f. Finally, $f_i \to f$ in $L(E, F)$ because for each A the restrictions on A converge in $B(A, F)$. □

If $F = \mathbb{K}$, then we denote with $E' := L(E, \mathbb{K})$ the space of all bounded linear functionals on E and with E^* the subspace of all continuous linear functionals on E.
If $f : E \to F$ is a bounded (resp. continuous) linear operator, we denote with $f^* : F' \to E'$ (resp. $f^* : F^* \to E^*$) the adjoint operator given by $f^*(\ell)(x) := \ell(f(x))$.

2.2.8 Remark. Completeness of the space of the continuous functions.

Analogously, it is shown that $C(X, F)$ is (sequentially) complete, if it is supplied with the topology of uniform convergence on compact sets, i.e. the seminorms $f \mapsto \|p \circ f|_K\|_\infty$ for compact $K \subseteq X$ and continuous seminorms p of F, and F is (sequentially) complete and X is a Kelley space, i.e. is a Hausdorff space where each set $A \subseteq X$, for which $A \cap K \subseteq K$ is closed for all compact $K \subseteq X$, itself is closed, because then a mapping $f : X \to F$ is continuous if and only if it is the restrictions $f|_K : K \to F$ for all compact $K \subseteq X$.

Obviously, the limit of a net of continuous functions is continuous on all compact sets, and because X is Kelley, it is continuous on X.

3. Constructions

3.1 General initial structures

3.1.1 Motivational examples.

For compact spaces X we have made the space of the continuous functions $C(X, \mathbb{R})$ by means of the supremum norm into a Banach space in 1.2.2 and 2.2.5. This is no longer possible for non-compact X, as continuous functions on X need not be bounded. But for every compact set $K \subseteq X$ we can define a seminorm $\|\_\|_K$ by $\|f\|_K := \|f|_K\|_\infty$. By means of the family of these seminorms for all compact $K \subseteq X$, we have made $C(X, \mathbb{R})$ an lcs in 2.2.5.

Similarly, we proceeded in 2.2.7 with $L(E, F)$, by considering restriction mapping $\mathrm{ins}^* : f \mapsto f|_A$, $L(E, F) \to B(A, F)$ for each bounded set $A \subseteq E$ and as seminorms q on $L(E, F)$ the compositions $f \mapsto \|q \circ f|_A\|_\infty$ for the seminorms F.

We now want to tease out the essentials from these constructions. The starting point is a vector space $E := C(X, \mathbb{R})$ (or $L(E, F)$) and a family of linear mapping $f_K : E \to E_K$ with values in lcs's $E_K := C(K, \mathbb{R})$ (or $B(A, F)$). The f_K are in our case given by $f_K : g \mapsto g|_K$. The goal now is to be able to make the space E as canonically as possible into a locally convex space by means of this data.

3.1.2 Theorem on initial structures.

Given a point-separating family of linear mapping $f_k : E \to E_k$ on a vector space E into lcs's E_k.
The set

$$\mathcal{P}_0 := \bigcup_k \{p \circ f_k : p \text{ seminorm of } E_k\}$$

is a sub-basis of the coarsest structure of an lcs's E, s.t. every f_k is continuous.
We call this structure, the INITIAL STRUCTURE *with respect to the mappings f_k.*
With this structure, E has the following universal property:
A linear mapping $f : F \to E$ from an lcs F to E is continuous if and only if all of the composites $f_k \circ f$ are.

$$\begin{tikzcd} E \arrow[r, "f_k"] & E_k \\ F \arrow[u, dotted, "f"] \arrow[ur, "f_k \circ f"'] & \end{tikzcd}$$

Furthermore:
The topology of E is the initial one with respect to the family of mappings f_k, i.e. it is the coarsest so that all f_k are continuous.
For nets (x_i) in E and $x \in E$ we have: $x_i \to x$ in $E \Leftrightarrow \forall k$: $f_k(x_i) \to f_k(x)$ in E_k.
Subsets $B \subseteq E$ are bounded in $E \Leftrightarrow \forall k$: $f_k(B)$ is bounded in E_k.
If the family of mappings f_k is finite and the E_k are normable then so is E.
If this family is countable and the E_k are countably seminormed then so is E.

Proof.

Sub-basis of the coarsest structure. The f_k are continuous if and only if $p \circ f_k$ is a continuous seminorm on E for all (continuous) seminorms of E_k; consequently, the locally convex topology on E generated by $\mathcal{P}_0$ has the smallest family of seminorms such that all f_k are continuous.

Initial topology. Since all f_k are continuous with respect to the topology generated by the seminorms, the initial topology is coarser or equal to it.
Conversely, $U = a + q_{<\varepsilon}$ is an element of the sub-basis of the topology generated by the seminorms $q \in \mathcal{P}_0$. Then $q = p \circ f_k$ for some k and some (continuous) seminorm p of E_k. Thus, $U = a + q_{<\varepsilon} = \{x : p(f_k(x-a)) < \varepsilon\} = \{x : f_k(x) - f_k(a) \in p_{<\varepsilon}\} = (f_k)^{-1}(f_k(a) + p_{<\varepsilon})$ is open (being an inverse image) in the initial topology with respect to the f_k.

Universal property. For linear mappings $f : F \to E$, the following holds:

$$\begin{aligned} & f \text{ is continuous} \\ \Leftrightarrow\;& \forall q \in \mathcal{P}_0 : q \circ f \text{ is continuous} \\ \Leftrightarrow\;& \forall k\, \forall p \text{ seminorm of } E_k : p \circ f_k \circ f \text{ is continuous} \\ \Leftrightarrow\;& \forall k : f_k \circ f \text{ is continuous.} \end{aligned}$$

Convergent nets. For nets (x_i) in E and $x \in E$, the following holds:

$$\begin{aligned} & x_i \to x \text{ in } E \\ \Leftrightarrow\;& \forall q \in \mathcal{P}_0 : q(x_i - x) \to 0 \\ \Leftrightarrow\;& \forall k\, \forall p \text{ seminorm of } E_k : p(f_k(x_i) - f_k(x)) = (p \circ f_k)(x_i - x) \to 0 \\ \Leftrightarrow\;& \forall k : f_k(x_i) \to f_k(x) \text{ in } E_k. \end{aligned}$$

Bounded sets. For subsets $B \subseteq E$ the following holds:

$$\begin{aligned} & B \text{ is bounded in } E \\ \Leftrightarrow\;& \forall q \in \mathcal{P}_0 : q(B) \text{ is bounded in } \mathbb{K} \\ \Leftrightarrow\;& \forall k\, \forall p \text{ seminorm of } E_k : p(f_k(B)) = (p \circ f_k)(B) \text{ is bounded in } \mathbb{K} \\ \Leftrightarrow\;& \forall k : f_k(B) \text{ is bounded in } E_k. \end{aligned}$$

Separatedness. Let $q(x) = 0$ for all $q \in \mathcal{P}_0$, i.e. $p(f_k(x)) = 0$ for all k and all (continuous) seminorms p of E_k. Because E_k is separated, $f_k(x) = 0$ for all k. Because the f_k separate points, $x = 0$.

Cardinality of a sub-basis. By construction, the sub-basis of the seminorms of E is countable provided those of the E_k are and the index set of the k is countable. If the index set finite and all E_k are normable, the sub-basis of E is finite. If $\mathcal{P}_0 := \{p_1, \dots, p_N\}$ is a finite sub-basis of the seminorms of E, then $\{\max\{p_1, \dots, p_N\}\}$ is a sub-basis, and thus E normalizable. □

3.1.3 Examples of initial structures.

On several spaces E (e.g. $C(X, F)$ and $L(X, F)$) of functions $f : X \to F$, we have considered the structure of the uniform convergence on certain subsets $K \subseteq X$. This topology is exactly the initial topology induced by the restriction mappings $\mathrm{incl}_K^* : E \to B(K, F)$. A subset $A \subseteq E$ of functions is thus bounded exactly when $\mathrm{incl}_K^*(A) \subseteq B(K, F)$ is bounded, so $\{f(x) : f \in A, x \in K\}$ is bounded in F by 2.2.3, i.e. A is uniformly bounded on the sets K.

Somewhat more general, also the structure of $C^p(U,\mathbb{R}^m)$ is of this form, where one has to consider the derivatives followed by restriction mappings

$$C^p(U,\mathbb{R}^m) \to C^{p-j}(U,L(\mathbb{R}^n,\dots,\mathbb{R}^n;\mathbb{R}^m)) \to B(K,\mathbb{R}^{n^j\cdot m}).$$

So a subset of $A \subseteq C^p(U,\mathbb{R}^m)$ is bounded exactly when each derivative is uniformly bounded on compact sets.

3.1.4 Corollary. Structure of subspaces.

Let F be a linear subspace of an lcs E. We provide F with the initial structure with respect to the inclusion $\iota : F \hookrightarrow E$.

- *The continuous seminorms on F are exactly the restrictions of those on E.*
- *The topology of F is the trace topology induced by E on F.*
- *A subset of F is bounded if and only if it is so in E.*
- *A subspace of a (sequentially) complete space is (sequentially) complete if and only if it is (sequentially) closed.*

Proof.

Extending continuous seminorms. Let q be a continuous seminorm of F and let $U_0 := q_{<1}$ be its open unit ball. By 1.4.2.2 there are finitely many $q_i \in \mathcal{P}_0 := \{p|_F : p \text{ is SN of } E\}$ and some $K > 0$ with $q \leqslant K \cdot \max\{q_1,\dots,q_N\}$. Let p_i be continuous seminorms of E with $(p_i)|_F = q_i$ and put $p := K \cdot \max\{p_1,\dots,p_N\}$. Then p is a continuous seminorm on E and $q \leqslant p|_F$ holds. For the open unit ball $U_1 := p_{<1}$ we have $U_1 \cap F = (p|_F)_{<1} \subseteq q_{<1} = U_0$ by 1.3.7. Let now U be the absolutely convex hull of $U_0 \cup U_1$. Since U_0 and U_1 are themselves absolutely convex, we have

$$U = \Big\{(1-t)u_0 + tu_1 : u_0 \in U_0, u_1 \in U_1, 0 \leqslant t \leqslant 1\Big\} = \bigcup_{0\leqslant t\leqslant 1} U_t,$$

where $U_t := \{(1-t)u_0 + tu_1 : u_0 \in U_0, u_1 \in U_1\}$.
Since U_1 is open, $U_t = \bigcup_{u_0\in U_0}(1-t)u_0 + t\,U_1$ is also open in E for $t \neq 0$.

We now want to show that $U_0 \subseteq \bigcup_{0<t\leqslant 1} U_t$ and hence $U = \bigcup_{0<t\leqslant 1} U_t$. As a side result, we obtain that U is open. Let $u_0 \in U_0 \subseteq F$. Since U_0 is open in F and $U_1 \cap F$ is a 0-neighborhood in F, a small $0 < t \leqslant 1$ exists, s.t. $t\,u_0 \in U_1 \cap F$ and $(1+t)\,u_0 \in U_0$. Thus $u_0 = (1-t)\,(1+t)\,u_0 + t\,t\,u_0 \in U_t$ for this $t > 0$.

Furthermore, $U_0 = U \cap F$ holds: On the one hand $U_0 \subseteq U$ and $U_0 \subseteq F$. On the other hand, let $u \in U \cap F$, then $u \in U_t$ for some $0 < t \leqslant 1$, i.e. $u = (1-t)u_0 + tu_1$ with $u_0 \in U_0$ and $u_1 \in U_1$. So $u_1 = \frac{1}{t}(u - (1-t)u_0) \in U_1 \cap F \subseteq U_0$ and, since U_0 is convex, $u = (1-t)u_0 + tu_1 \in U_0$ holds.

Now let $\tilde{q}$ be the Minkowski functional p_U of U (see 1.3.5). Then $\tilde{q}|_F$ is the Minkowski functional of $U \cap F = U_0 = q_{<1}$ and this matches with q by 1.3.3.

The statement about the trace topology and bounded subsets follows directly from Theorem 3.1.2.

Completeness of closed subspaces. We have already shown this in 2.2.4.

Closedness of complete subspaces. Let x_i be a net (sequence) in x converging towards E in F, then x_i is a Cauchy net (Cauchy-sequence) in E, and thus the net $x_{i,j} := x_i - x_j$ converges to 0 in E. By 3.1.2 it also converges in F, which means x_i is a Cauchy net in F. Since F is assumed to be (sequentially) complete, x_i converges towards some y in F, and because the inclusion is continuous, also in E. Since E is separated, the two limits x and y must coincide, thus $x = y \in F$. □

3.1.5 Subspaces of the Banach space $B(X)$, see Lemma 2.2.3.

Let X be a topological space. Thus $C_b(X) := C(X) \cap B(X)$ is a closed subspace of $B(X)$ and hence itself a Banach space.

Furthermore, $C_0(X)$ itself is a closed subspace of $C_b(X)$, and thus a Banach space.

3.2 Products

3.2.1 Corollary. The structure of products.

Let E_k be lcs's and $E = \prod_k E_k$ be their Cartesian PRODUCT, *provided with the initial structure with respect to the projections $\mathrm{pr}_k : E \to E_k$ to the individual factors.*

- *Then the topology of E is the product topology.*
- *The convergence is the coordinate (or componentwise) convergence.*
- *A set B is bounded in E if and only if it is contained in a product $\prod_k B_k$ of bounded sets $B_k \subseteq E_k$.*
- *Any product of (sequentially) complete spaces is (sequentially) complete.*
- *A product of bornological space is again bornological if it does not consist of too many factors; More precisely, if the index set is smaller than the first measurable cardinal number. Whether such cardinal numbers exist depends on the set theory used.*

Proof.

Product topology and convergence. The product topology is by definition the coarsest topology, s.t. the projections $\mathrm{pr}_k : E \to E_k$ are continuous, so this is the topology of the lcs E by Theorem 3.1.2. Likewise, the statement about convergence follows from this theorem. A basis of this topology is given by the products $\prod_k U_k$ with $U_k \subseteq E_k$ open and $U_k = E_k$ apart from finite many indices k.

Bounded sets. A set $B \subseteq E$ is bounded by 3.1.2 if and only if $B_k := \mathrm{pr}_k(B) \subseteq E_k$ is bounded for all k. Since always $B \subseteq \prod_j B_j$, the desired statement follows, because $\mathrm{pr}_k(\prod_j B_j) = B_k$ shows that $\prod_j B_j$ is bounded.

Completeness. Let x_i be a Cauchy net in E, then the k-th coordinate of the x_i forms a Cauchy net in E_k, by the continuity and linearity of pr_k, and thus converges in E_k. Then, according to the description of the convergence, x_i converges towards the point $x \in E$ whose k-th coordinate is just $\lim_i \mathrm{pr}_k(x_i)$.

Bornologicity. The proof of this statement follows from the following Theorem 3.2.3 together with Remark 3.2.4. □

3.2.2 Definition. Ulam-measures.

A Ulam measure on a set J is a $\{0,1\}$-valued measure on the power set $\mathcal{P}(J)$, i.e. a mapping $\mu : \mathcal{P}(J) \to \{0,1\}$ satisfying $\mu(\bigsqcup_{n\in\mathbb{N}} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ for all pairwise disjoint sets $A_n \subseteq J$.
It is called non-trivial if $\mu \neq 0$, but $\mu(\{j\}) = 0$ for all $j \in J$.

Obviously, a Ulam measure μ is uniquely determined by $\mathcal{F} := \mu^{-1}(1)$. For $\mu \neq 0$ this is a filter on J, because

- $\emptyset \notin \mathcal{F}$, since $\mu(\emptyset) = \mu(\emptyset \bigcup \emptyset) = 2 \cdot \mu(\emptyset)$ and hence $\mu(\emptyset) = 0$.

- Let $A \in \mathcal{F}$ and $A \subseteq B \subseteq J$. Then
$$1 \geqslant \mu(B) = \mu(B \backslash A) + \mu(A) \geqslant \mu(A) = 1,$$
hence $\mu(B) = 1$, i.e. $B \in \mathcal{F}$.
- Let us assumed indirectly that $A, B \in \mathcal{F}$ and $A \cap B \notin \mathcal{F}$, then
$$1 = \mu(A) = \mu(A \backslash A \cap B) + \mu(A \cap B) = \mu(A \backslash A \cap B)$$
and thus $\mu(A \cup B) = \mu(A \backslash A \cap B) + \mu(B) = 2 \notin \{0, 1\}$.

Moreover, $\mathcal{F}$ is even an ULTRAFILTER, i.e. a maximal filter with respect to inclusion (or equivalently, $A \subseteq J \Rightarrow$ either $A \in \mathcal{F}$ or $A^c := J \backslash A \in \mathcal{F}$): Otherwise $A \notin \mathcal{F}$ and $A^c \notin \mathcal{F}$ and then $\mu(J) = \mu(A) + \mu(A^c) = 0 + 0$ gives a contradiction to $\mu(J) \neq 0$.

Furthermore, $\mathcal{F}$ is a δ-FILTER, i.e. $A_n \in \mathcal{F}$ implies $\bigcap_{n \in \mathbb{N}} A_n \in \mathcal{F}$: Otherwise, $\bigcup_{n \in \mathbb{N}} A_n^c = (\bigcap_{n \in \mathbb{N}} A_n)^c \in \mathcal{F}$ and $A_n^c \notin \mathcal{F}$ for all $n \in \mathbb{N}$ is a contradiction to the σ-(sub)additivity, i.e. $\mu(A_n^c) = 0$ but $\mu(\bigcup_{n \in \mathbb{N}} A_n^c) \neq 0$, because if $B_n := A_n^c$ and $C_n := B_n \backslash \bigcup_{k<n} B_k$, then $\bigcup_n B_n = \bigsqcup_n C_n$ and $\mu(C_n) \leqslant \mu(B_n) = 0$, but $\mu(\bigcup_n C_n) \neq 0$.

Conversely, if $\mathcal{F}$ is a δ-ultrafilter on J, then $0 \neq \mu := \chi_{\mathcal{F}} : \mathcal{P}(J) \to \{0, 1\}$ is a Ulam measure: Namely, let $A_n \subseteq J$ be pairwise disjoint. Due to the obvious monotony of μ, we only have to show that $\mu(\bigsqcup_{n \in \mathbb{N}} A_n) = 1$ implies the existence of a (unique) $n \in \mathbb{N}$ with $\mu(A_n) = 1$. If $\mu(A_n) = 1$ would hold for at least two n, then these would satisfy $A_n \in \mathcal{F}$ and thus also their empty intersection. So let us assume indirectly that $\mu(A_n) = 0$ for all $n \in \mathbb{N}$, hence $A_n \notin \mathcal{F}$ and thus $A_n^c \in \mathcal{F}$. Because of the δ-filter property, we would have $(\bigcup_{n \in \mathbb{N}} A_n)^c = \bigcap_{n \in \mathbb{N}} A_n^c \in \mathcal{F}$, i.e. $\bigcup_{n \in \mathbb{N}} A_n \notin \mathcal{F}$. Hence $\mu(\bigcup_{n \in \mathbb{N}} A_n) = 0$, a contradiction.

Note that the *a Ulam measure μ is non-trivial if and only if* $\bigcap \mathcal{F} = \emptyset$:
It suffices to show $j \in \bigcap \mathcal{F} \Leftrightarrow \mu(\{j\}) = 1$:
$\mu(\{j\}) = 1 \Rightarrow \{j\} \in \mathcal{F} \Rightarrow j \in A$ for all $A \in \mathcal{F}$, otherwise $\emptyset = \{j\} \cap A \in \mathcal{F}$. And vice versa, $\mu(\{j\}) = 0$ implies $j \notin A := \{j\}^c \in \mathcal{F}$, hence $j \notin \bigcap \mathcal{F}$.

Moreover, $\bigcap \mathcal{F} \neq \emptyset \Leftrightarrow \exists! j \in J : \mathcal{F} = \{A \subseteq J : j \in A\}$: Let $j \in \bigcap \mathcal{F}$. Since $j \notin \{j\}^c$ we have $\{j\} \in \mathcal{F}$ by the ultrafilter property, hence $\mathcal{F} = \{A \subseteq J : j \in A\}$.

A cardinal number is called MEASURABLE if a non-trivial Ulam measure exists on it. If measurable cardinal numbers exist, then, by results of [**36**] and [**16**] and [**28**], the smallest measurable cardinal number m is INACCESSIBLE, i.e. $\aleph_0 < m$, furthermore, $c < m \Rightarrow 2^c < m$, as well as $k < m$ and $c_i < m$ for all $i \in k \Rightarrow \sum_{i \in k} c_i < m$, as the following arguments show.

1. Sublemma.

Let m be the smallest measurable cardinal and μ be a Ulam measure on m. Then μ is k-additive for each cardinal $k < m$.

Proof. Let $\{A_i : i \in k\}$ be a family of pairwise disjoint subsets of m with $k < m$. and $\mu(\bigsqcup_{i \in k} A_i) \neq \sum_{i \in k} \mu(A_i)$. Obviously the set $k' := \{i \in k : \mu(A_i) > 0\}$ has to be countable, since otherwise there is some $\varepsilon > 0$ such that $\{i : \mu(A_i) < \varepsilon\}$ is infinite, contradicting the σ-additivity. Hence
$$\sum_{i \in k} \mu(A_i) = \sum_{i \in k'} \mu(A_i) = \mu(\bigsqcup_{i \in k'} A_i) < \infty, \text{ hence } \sum_{i \in k \backslash k'} \mu(A_i) = 0,$$
$$\text{whereas } \mu(\bigsqcup_{i \in k \backslash k'} A_i) = \mu(\bigsqcup_{i \in k} A_i) - \mu(\bigsqcup_{i \in k'} A_i) = \mu(\bigsqcup_{i \in k} A_i) - \sum_{i \in k} \mu(A_i) \neq 0.$$

We define a measure μ' on k by

$$\mu'(B) := \mu(\bigsqcup_{i\in B} A_i) \text{ for } B \subseteq k$$

This is obviously a Ulam-measure, since for any countable family of pairwise disjoint $B_i \subseteq k$, we have

$$\sum_{i\in\mathbb{N}} \mu'(B_i) = \sum_{i\in\mathbb{N}} \mu(\bigsqcup_{j\in B_i} A_j) = \mu\Big(\bigsqcup_{i\in\mathbb{N}} \bigsqcup_{j\in B_i} A_j\Big) = \mu\Big(\bigsqcup_{j\in\bigsqcup_{i\in\mathbb{N}} B_i} A_j\Big) = \mu'(\bigsqcup_{i\in\mathbb{N}} B_i).$$

And it is non-trivial, since $\mu'(\{i\}) = \mu(A_i) = 0$. A contradiction to the minimality of m. □

2. Subcorollary.

Let m be the smallest measurable cardinal and $k < m$, then $2^k < m$.

Proof. Suppose $2^k \geqslant m$. By 1 it suffices to show that each measure μ on m, which is k-additive for all $k < m$, is trivial. Such a μ induces a measure on the superset $2^k \geqslant m$. For each ordinal $l \leqslant k$ and $f \in 2^l$ let

$$U(f,l) := \Big\{g \in 2^k : g(j) = f(j)\, \forall j < l \text{ (i.e. } j \in l)\Big\}.$$

Thus $U(f,l) = \{f\}$ for $l = k$. For $l < k$ and $i \in 2 := \{0,1\}$ let

$$U^i(f,l) := \{g \in U(f,l) : g(l) = i\}.$$

Then $U(f,l) = U^0(f,l) \sqcup U^1(f,l)$.

By transfinite induction and succesive extension we will construct an element $f \in 2^k$ with $\mu(U(f|_l,l)) = 1$ for all $l \leqslant k$, and hence $\mu(\{f\}) = \mu(U(f,k)) = 1$, a contradiction.

Note that $f|_0 = \emptyset$ and $U(f|_0,0) = 2^k$, hence $\mu(U(f|_0,0)) = 1$. Thus there is an $i \in 2$ such that $\mu(U^i(f|_0,0)) = 1$, and we put $f(0) := i$.

Let now $0 < l \leqslant k$. If l is a limit ordinal, then by induction we have f already on $\bigcup_{j<l} j = l$ such that $\mu(U(f|_j,j)) = 1$ forall $j < l$. Since $U(f|_l,l) = \bigcap_{j<l} U(f|_j,j)$ the l-additivity implies $\mu(U(f|_l,l)) = 1$. Thus there is an $f(l) := i \in 2$ such that $\mu(U^i(f|_l,l)) = 1$.

Otherwise, l is a successor ordinal, i.e. $l = j+1$ for some $j < l \leqslant k$ and by induction hypothesis we have $f \in 2^j$ with $\mu(U(f,j)) = 1$ and have defined $f(j)$ with $\mu(U^{f(j)}(f|_j,j)) = 1$. Since $U(f|_l,l) = U^{f(j)}(f|_j,j)$ there is again an $f(l) := i \in 2$ such that $\mu(U^i(f|_l,l)) = 1$. □

3. Subcorollary.

Let m be the smallest measurable cardinal, and $c_i < m$ for alle $i \in k < m$. Then $\sum_{i\in k} c_i < m$.

Proof. Otherwise, $m \leqslant \sum_{i\in k} c_i$, thus there are disjoint C_i with $|C_i| < m$ and $m = \bigsqcup_{i\in k} C_i$. Let μ be a non-trivial Ulam-measure on m. By assumption $\mu(\{i\}) = 0$ for all $i \in m$, hence $\mu(C_i) = 0$ by 1 and hence $\mu(m) = \sum_{i\in k} \mu(C_i) = 0$ again by 1, thus $\mu = 0$. □

3.2.3 Theorem. Bounded functionals on products.

For sets J, the following statements are equivalent:

1. *All bounded linear functionals on $\mathbb{R}^J := \prod_{j\in J} \mathbb{R}$ are continuous;*
2. *The only algebra homomorphisms $\mathbb{R}^J \to \mathbb{R}$ are pr_j for $j \in J$;*

3. *All Ulam measures on J are trivial,*
   i.e. the cardinality of J is less than the smallest measurable cardinal.

($\boxed{1}\Leftrightarrow\boxed{3}$) is due to [**29**] and ($\boxed{2}\Leftrightarrow\boxed{3}$) is due to [**12**].

Proof. ($\boxed{1}\Leftarrow\boxed{3}$) Let $f : \mathbb{R}^J \to \mathbb{R}$ be linear and bounded. We have to show that f is continuous.

The set $A := \{j \in J : f(e^j) \neq 0\}$ is finite, where e^j is the j-th unit vector in $\mathbb{R}^J$, so all the coordinates are 0 except the j-th which is 1: Otherwise, pairwise distinct $j_n \in A$ exist for $n \in \mathbb{N}$ with $f(e^{j_n}) \neq 0$. Then $\{\frac{n}{f(e^{j_n})} e^{j_n} : n \in \mathbb{N}\}$ is bounded in $\mathbb{R}^J$, but $f(\frac{n}{f(e^{j_n})} e^{j_n}) = n$ is unbounded, a contradiction to the boundedness of f. Thus $g : x \mapsto f(x \cdot \chi_A)$, $\mathbb{R}^J \twoheadrightarrow \mathbb{R}^A \hookrightarrow \mathbb{R}^J \to \mathbb{R}$ is continuous (by $\boxed{3.4.6.3}$) and $h := f - g$ is bounded, linear, and vanishes on $\mathbb{R}^{(J)} := \{x \in \mathbb{R}^J : \{j : x_j \neq 0\} \text{ ist finite}\}$.

It suffices to show $h = 0$. Let us assume indirectly $h \neq 0$. We consider filters contained in $\mathcal{H} := \{I \subseteq J : h_I := h|_{\mathbb{R}^I} \neq 0\}$. The set $\{J\}$ such a filter and the union of a linearly ordered set of such filters is again such a filter. Thus, according to Zorn's lemma, there is a maximal filter $\mathcal{F}$ contained in $\mathcal{H}$.

This maximal filter $\mathcal{F}$ is an ultrafilter: Let $I \subseteq J$.
If $I \cap A \notin \mathcal{H}$ and $I^c \cap B \notin \mathcal{H}$ for some $A, B \in \mathcal{F}$, then $C := A \cap B \in \mathcal{F} \subseteq \mathcal{H}$, but $h_C = h_{I\cap C} + h_{I^c \cap C} = 0$, so $C \notin \mathcal{H}$, a contradiction.
Thus, $I \cap A \in \mathcal{H}$ for all $A \in \mathcal{F}$, or $I^c \cap A \in \mathcal{H}$ for all $A \in \mathcal{F}$. Consider the filter $\mathcal{F}' := \{A' \subseteq J : \exists A \in \mathcal{F} \text{ with } I \cap A \subseteq A'\}$ generated by the trace of $\mathcal{F}$ to I. Then $\mathcal{F} \subseteq \mathcal{F}' \subseteq \mathcal{H}$ and $I \in \mathcal{F}' = \mathcal{F}$ by maximality. So $I \in \mathcal{F}$ or $I^c \in \mathcal{F}$.

The filter $\mathcal{F}$ is a δ-filter (thus defines a Ulam measure):
Let $A_n \in \mathcal{F}$ be arbitrary and $A_\infty := \bigcap_{n\in\mathbb{N}} A_n$.
Suppose $A_\infty \cap A = \varnothing$ for some $A \in \mathcal{F}$. Since $B_n := A \cap \bigcap_{k\leqslant n} A_k \in \mathcal{F} \subseteq \mathcal{H}$ there exists a $b^n \in \mathbb{R}^{B_n} \subseteq \mathbb{R}^J$ with $|h(b^n)| \geqslant n$. Because of $B_{n+1} \subseteq B_n$ and $\bigcap_n B_n = \varnothing$, each $i \in J$ is only in a finite number of B_n's, so $b^n_i = 0$ for all but finitely many n and thus $\{b^n : n \in \mathbb{N}\} \subseteq \mathbb{R}^J$ is bounded, but $\{h(b^n) : n \in \mathbb{N}\}$ is unbounded, a contradiction.
Thus, $A_\infty \cap A \neq \varnothing$ for all $A \in \mathcal{F}$ and (as before) $A_\infty \in \mathcal{F}$, because $\{A_\infty \cap A : A \in \mathcal{F}\}$ generates a filter $\mathcal{F}' \supseteq \mathcal{F}$ containing A_∞ and $\mathcal{F}$ is an ultrafilter.

Since Ulam measures are trivial on J by assumption (i.e. $\bigcap \mathcal{F} \neq \varnothing$), an i exists with $\{i\} \in \mathcal{F} \subseteq \mathcal{H}$, i.e. $h(e^i) \neq 0$, a contradiction to $h|_{\mathbb{R}^{(J)}} = 0$.

($\boxed{1}\Rightarrow\boxed{2}$) Let $f : \mathbb{R}^J \to \mathbb{R}$ be an algebra homomorphism. For each $x \in \mathbb{R}^j$ there is an i with $f(x) = x_i$, otherwise $x - f(x) \cdot 1$ is invertible, so $0 \neq f(x - f(x) \cdot 1) = f(x) - f(x) \cdot f(1) = 0$ is a contradiction. Therefore f is monotonous, since to $x, y \in \mathbb{R}^J$ an $i \in J$ exists with $f(x) = x_i$ and $f(y) = y_i$, otherwise consider $(x - f(x))^2 + (y - f(y))^2$. Finally, f is bounded, because let $\mathcal{B} \subseteq \mathbb{R}^J$ be bounded and $f(\mathcal{B})$ be unbounded. Then we find $x^n \in \mathcal{B}$ with $|f(x^n)| > 2^n$ and by replacing x^n with $(x^n)^2 \in \mathcal{B}^2$ we may assume $x^n \geqslant 0$. Hence $x^\infty := \sum_n \frac{1}{2^n} x^n \in \mathbb{R}^J$ converges (cf. $\boxed{2.2.2}$) and

$$
\begin{aligned}
f(x^\infty) &= f\Big(\sum_{n\leqslant N} \frac{1}{2^n} x^n + \sum_{n>N} \frac{1}{2^n} x^n\Big) = \sum_{n\leqslant N} \frac{1}{2^n} f(x^n) + f\Big(\sum_{n>N} \frac{1}{2^n} x^n\Big) \\
&\geqslant \sum_{n\leqslant N} \frac{1}{2^n} f(x^n) + 0 \geqslant \sum_{n\leqslant N} 1 \geqslant N,
\end{aligned}
$$

because of the monotonicity of f, a contradiction.

By (1) the mapping f is continuous, so depends only on finitely many coordinates(!) and thus is a point evaluation, because for $i \neq j$ we have $0 = f(e^i \cdot e^j) = f^i \cdot f^j$ with $f^i := f(e^i)$, and hence only one of them can be non-zero.

(2⇒3) Suppose there were a non-trivial Ulam measure $\mu : \mathcal{P}(J) \to \{0,1\}$. Then $\mathcal{F} := \mu^{-1}(1)$ is a δ-ultrafilter with $\bigcap \mathcal{F} = \emptyset$. For given $x \in \mathbb{R}^J$ we consider the (image) filter $\mathcal{F}_x$ on $\mathbb{R}$ generated by the sets $x(I) := \{x_i : i \in I\}$ with $I \in \mathcal{F}$. This is a δ-ultrafilter(!) on $\mathbb{R}$ and, since $\mathbb{R}$ only allows trivial Ulam measures μ_x (because $|\mathbb{R}| = 2^{\aleph_0}$ is accessible!), there is a (unique) $f_x \in \mathbb{R}$ with $\{f_x\} \in \mathcal{F}_x$, i.e. $\exists A_x \in \mathcal{F}$ with $x(A_x) \subseteq \{f_x\}$, and for all $A \in \mathcal{F}$ with $A \subseteq A_x$ we have $\emptyset \neq x(A) \subseteq x(A_x) \subseteq \{f_x\}$, i.e. $x(A) = \{f_x\}$.

The mapping $f : x \mapsto f_x$ is an algebra homomorphism:
For $x, y \in \mathbb{R}^J$, there exist $A_x, A_y \in \mathcal{F}$ with $\{f_x\} = x(A_x)$ and $\{f_y\} = y(A_y)$ and thus $C := A_x \cap A_y \in \mathcal{F}$ and $x(C) = \{f_x\}$ and $y(C) = \{f_y\}$, so that

$$f_{x\cdot y} \in (x\cdot y)(C) \subseteq x(C)\cdot y(C) = \{f_x\}\cdot\{f_y\} = \{f_x \cdot f_y\}, \text{ i.e. } f(x\cdot y) = f(x)\cdot f(y).$$

Furthermore, $f_1 \in 1(J) = \{1_i : i \in J\} = \{1\}$, hence $f(1) = 1$.

Because of 2, $f = \mathrm{pr}_j$ for some $j \in J$. But, because of $\bigcap \mathcal{F} = \emptyset$, there is an $A \in \mathcal{F}$ with $j \notin A$. Thus $1 = \mathrm{pr}_j(e^j) = f(e^j) = f_{e^j} \in e^j(A) = \{0\}$ is a contradiction. □

3.2.4 Remark. Bornologicity of function spaces.

We will show later (see also [**14**, S.281]) that $\mathbb{R}^I$ is bornological (or, equivalently, just all bounded linear functionals on $\mathbb{R}^I$ are continuous, i.e. the cardinality of I is not measurable), if and only if $\prod_{i\in I} E_i$ is bornological for all bornological spaces E_i.

More generally, a completely regular topological space X (rather than a discrete set) is called REAL-COMPACT if the only algebra homomorphisms $C(X,\mathbb{R}) \to \mathbb{R}$ are the point evaluations. This is the case if and only if it is a closed subspace of a power, i.e. a product of the form $\mathbb{R}^J$, see [**26**, 2.5.2]. Thus, by 3.2.3 a discrete space is real-compact if and only if its cardinality is not measurable.

According to a theorem of [**30**] and [**34**], the space $C(X,\mathbb{R})$ is bornological for a completely regular space X if and only if X is real-compact.

This can be generalized to some vector valued cases: Due to a theorem of [**32**], the space $C(X,E)$ is bornological for countably seminormed spaces E and completely regular X if and only if X is real-compact.

Susanne Dierolf gave an example mentioned in [**33**], that $C(\mathbb{N}_\infty, \mathbb{R}^{(J)})$ is not bornological for uncountable J, although the 1-point compactification $\mathbb{N}_\infty$ of $\mathbb{N}$ is compact and thus real-compact and $\mathbb{R}^{(J)}$ is bornological, see 3.3.2.

3.2.5 Initial structures as subspaces of products.

Assume a point separating family of linear mapping $f_k : E \to E_k$ is given on a vector space E with values in lcs's E_k. Then the initial structure on E is just given by the embedding of E into the product $\prod_k E_k$, which maps $x \in E$ to $(f_k(x))_k \in \prod_k E_k$.

For a topological Hausdorff space X, the space $C(X,\mathbb{K})$ can be considered as a subspace of the product $\prod_K C(K,\mathbb{K})$, with K running through the compact subsets of X. The topology of $C(X,\mathbb{K})$ is then of course that of the uniform convergence on compact sets $K \subseteq X$. Note that this subspace is closed provided X is a Kelley space, i.e. a set A is closed in X, when its intersection $A \cap K$ is closed in K, for all compact $K \subseteq X$. If there is a countable basis of the compact sets of X, i.e. a countable family of compact sets K_n, so that each compact subset of X is contained in some K_n, then $C(X,\mathbb{K})$ is a countably seminormed space. If X is a locally compact and σ-compact (i.e. a union of countably many compact subsets) the $C(X,\mathbb{K})$ is a Fréchet space, i.e. a complete countably seminormed lcs.

Let $G \subseteq \mathbb{C}$ be open, then the space of the holomorphic functions $H(G,\mathbb{C})$ is a closed(!) subspace of $C(G,\mathbb{C})$ and thus itself a Fréchet space.

If $I \subset \mathbb{R}$ is a compact interval, then the space $C^\infty(I,\mathbb{R})$ of the smooth functions can be embedded by $f \mapsto (f^{(n)})_{n\in\mathbb{N}}$ as (because of [**20**, 4.2.11]) closed subspace in $\prod_{n\in\mathbb{N}} C(I,\mathbb{R})$. Thus, $C^\infty(I,\mathbb{R})$ is a Fréchet space. Its topology is that of uniform convergence in each derivative separately. More generally, for each open set $X \subseteq \mathbb{R}^m$, the space $C^\infty(X,\mathbb{R})$ can be made into a Fréchet space.

3.3 General final structures

3.3.1 Convergent power series as motivational example.

We now want to make the space E of the locally convergent power series into an lcs. A power series $\sum_{n=0}^\infty a_n z^n$ is uniquely determined by its coefficients a_n, and addition and scalar multiplication of convergent power series corresponds to addition and scalar multiplication of their coefficients. So E obviously identifies with $\{(a_n) \in \mathbb{C}^\mathbb{N} : \limsup_{n\to\infty} |a_n|^{1/n} < \infty\}$.

A first approach would be to provide E with the initial structure as the subspace of the product $\mathbb{C}^\mathbb{N} := \prod_{n\in\mathbb{N}} \mathbb{C}$, but unfortunately it is not closed, because the polynomials (= finite sequences) are dense in $\mathbb{C}^\mathbb{N}$ (proof!). This structure is therefore too coarse and on the other hand $(a_n)_n \mapsto \limsup_{n\to\infty} |a_n|^{1/n}$ is not a seminorm. But if we consider the linear subspace E_r of the power series with convergence radius $1/(\limsup_{n\to\infty} |a_n|^{1/n}) > r$ for $r > 0$, then we have a suitable norm, namely $(a_n)_n \mapsto \sup\{|a_n|\, r^n : n \in \mathbb{N}\}$. So we can write E as union $\bigcup_{r>0} E_r$ of normed spaces E_r. Now we want to make E into a (complete) lcs by means of the family of inclusions $f_r : E_r \to E$ in the most natural way possible. In particular, the mapping $f_r : E_r \to E$ should be continuous, i.e. for a continuous seminorms q on E, the composition $q \circ f_r$ should be a continuous seminorm on E_r.

3.3.2 Theorem on final structures.

Let $f_k : E_k \to E$ be a family of linear mappings of lcs's into a vector space E. The vector space E provided with the set

$$\mathcal{P} := \Big\{ p \text{ is a seminorm on } E : \ \forall k \text{ the seminorm } p \circ f_k \text{ is continuous on } E_k \Big\}$$

is the not necessarily separated locally convex space that carries the finest structure, s.t. each $f_k : E_k \to E$ is continuous. We call this structure the FINAL STRUCTURE *with respect to the family of mappings f_k.*
With this structure, E has the following universal property: A linear mapping $f : E \to F$ into a locally convex space F is continuous if and only if all of its compositions are $f \circ f_k : E_k \to F$.
If all E_k are bornological, so is E.

In general, neither the topology, nor the convergence, nor the bounded sets, nor the separatedness have a direct description similar to that of initial structures.

Proof.

Finest structure. The mappings $f_k : E_k \to E$ are continuous if and only if each continuous SN of E belongs to $\mathcal{P}$. So it remains to show that $\mathcal{P}$ describes a locally convex space. Let q be a seminorm on E, for which finite many $q_i \in \mathcal{P}$ exist and an $R > 0$, s.t. $q \leqslant R \cdot \max\{q_1, \dots, q_N\}$. Then the same inequality also holds to the compositions of q and q_i with f_k, so $q \circ f_k$ is a continuous seminorm on E_k, and thus q belongs to $\mathcal{P}$, so E together with $\mathcal{P}$ is a locally convex space by Lemma 1.4.2, and the structure is the finest, s.t. all f_k are continuous. This implies also the desired universal property by means of 2.1.1.

$$\begin{tikzcd}
E_k \arrow[r, "f_k"] \arrow[dr, "p\circ f_k"'] & E \arrow[r, "f"] \arrow[d, "p"] & F \arrow[dl, "q"] \\
 & \mathbb{R} &
\end{tikzcd}$$

Bornologicity. If $f : E \to F$ is a bounded linear mapping, then $f \circ f_k$ is also bounded, because continuous mappings (like f_k) are bounded, according to Lemma 2.1.4. Since E_k was assumed to be bornological, $f \circ f_k : E_k \to F$ is continuous. Due to the universal property, f is continuous. □

Regarding the other properties that are not necessarily inherited, we restrict our considerations to special cases.

3.3.3 Corollary. Quotient spaces.

Let E be an lcs and F a linear subspace. We provide the QUOTIENT SPACE $E/F := \{x + F : x \in E\}$ *of the cosets $x + F$ of F in E with the final structure with respect to the canonical projection $\pi : x \mapsto x + F$, $E \to E/F$. Then we have:*

- *The lcs E/F carries the quotient topology, that is the finest topology, s.t. $\pi : E \to E/F$ is continuous. Furthermore, π is open.*
- *The quotient space E/F is separated exactly when F is closed in E.*
- *The continuous seminorms on E/F are precisely the mappings $\tilde{q} : x + F \mapsto \inf\{q(x + y) : y \in F\}$, where q runs through the continuous seminorms of E.*
- *If E is normable (or countably seminormed lcs) and F is closed, then E/F is also normable (or countably seminormed lcs).*

Regarding completeness we unfortunately have no general statement, but see 3.5.3.

Proof.

Continuous seminorms of E/F. To each seminorm q on E we define a new seminorm q_F by $q_F(x) := \inf\{q(x+y) : y \in F\}$. This infimum exists since $q(x+y) \geqslant 0$. We have that q_F is a seminorm, because for $\lambda \neq 0$ we have

$$\begin{aligned}
q_F(\lambda x) = \inf\big\{q(\lambda x + y) : y \in F\big\} &= \inf\big\{q(\lambda(x + \tfrac{1}{\lambda} y)) : y \in F\big\} \\
&= \inf\Big\{|\lambda|\, q(x + z) : z \in \frac{1}{\lambda} F = F\Big\} = |\lambda|\, q_F(x)
\end{aligned}$$

and the subadditivity of q_F follows from

$$\begin{aligned}
q_F(x_1 + x_2) &= \inf\Big\{q(x_1 + x_2 + y) : y \in F = F + F\Big\} \\
&= \inf\Big\{q(x_1 + x_2 + y_1 + y_2) : y_1 \in F, y_2 \in F\Big\} \\
&\leqslant \inf\Big(\{q(x_1 + y_1) : y_1 \in F\} + \{q(x_2 + y_2) : y_2 \in F\}\Big) \\
&= \inf\big\{q(x_1 + y_1) : y_1 \in F\big\} + \inf\big\{q(x_2 + y_2) : y_2 \in F\big\} \\
&= q_F(x_1) + q_F(x_2).
\end{aligned}$$

Furthermore, $(q_F)_{<1} \subseteq q_{<1} + F$ (in fact, even equality holds, and thus q_F is the Minkowski functional of $q_{<1} + F$), because $1 > q_F(x) = \inf\{q(x+y) : y \in F\} \Rightarrow \exists y \in F : q(x+y) < 1$, so $x = (x+y) + (-y)$ with $x + y \in q_{<1}$ and $-y \in -F = F$. If q is continuous, also q_F is continuous, because $q_F \leqslant q$. Since q_F is constant on the cosets $x + F$ by construction, q_F factors to a seminorm $\tilde{q}$ on E/F, which is also continuous by construction of the final structure. Conversely, if $\tilde{q}$ is any continuous seminorm on E/F, then $q := \tilde{q} \circ \pi$ is continuous seminorm on E, which is constant on cosets $x + F$. So $q_F = q$ and $\tilde{q}$ is the seminorm on E/F which is associated (by the above construction) to q.

The statement about the cardinality of a sub-basis is now evident.

Quotient topology and openness of π. A set $V \subseteq E/F$ is by definition open in the quotient topology if and only if $\pi^{-1}(V)$ is open in E. We now show the equality of the topologies and the openness of π.

If U is open in E, then $\pi^{-1}(\pi(U)) = U + F = \bigcup_{y \in F} U + y$ is open in E, and thus $V := \pi(U)$ open in the quotient topology.

If $V \subseteq E/F$ is open in the quotient topology, then $U := \pi^{-1}(V)$ is open in E. We have to show that V is a neighborhood of each $y \in V$ in the topology generated by the seminorms. Then $y = \pi(x)$ and w.l.o.g. $x = 0$ because the topologies under consideration are all translation invariant. There is a continuous seminorm q on E with $q_{<1} \subseteq U$ and thus $(q_F)_{<1} \subseteq q_{<1} + F \subseteq U + F = U$ holds. Then $\tilde{q}_{<1} \subseteq V$, because

$$1 > \tilde{q}(x + F) = q_F(x) \Rightarrow x \in U = \pi^{-1}(V) \Rightarrow x + F = \pi(x) \in \pi(\pi^{-1}(V)) \subseteq V,$$

and thus V is a 0-neighborhood in the topology generated by the seminorms.

Conversely, if $V \subseteq E/F$ is open in the topology generated by the seminorms, then $U := \pi^{-1}(V) \subseteq E$ is open in E and thus V is open in the quotient topology.

Separatedness. Let E/F be separated. Then

$$\{0\} = \bigcap \{q^{-1}(0) : q \text{ is seminorm of } E/F\},$$

thus $\{0\} \subseteq E/F$ is closed, and hence $F = \pi^{-1}(0) \subseteq E$ is closed.
Conversely, let $F \subseteq E$ be closed. Then $E \backslash F$ is open and, since π is an open mapping, also $\pi(E \backslash F) = E/F \backslash \{0\}$ is open. So $\{0\}$ is closed in E/F. Thus, E/F is separated because $q(y) = 0$ for all SN's q has as consequence that the constant sequence 0 converges to y and, since $\{0\}$ is closed, $y = 0$ follows. □

3.3.4 Kernel of a seminorm.

If $p : E \to \mathbb{R}$ is a seminorm of an lcs E, then the kernel $F := \operatorname{Ker}(p) := p^{-1}(0)$ of p is a closed linear subspace, because $p(x) = 0 = p(y)$ implies $p(\lambda x) = |\lambda| p(x) = |\lambda| 0 = 0$ and $0 \leqslant p(x+y) \leqslant p(x) + p(y) = 0 + 0$ and $p_F = p$, because $p(x) - 0 = p(x) - p(-y) \leqslant p(x+y) \leqslant p(x) + p(y) = p(x) + 0$ for $y \in \operatorname{Ker}(p)$. Thus, $E_p := E/\operatorname{Ker}(p)$ is a normed space with respect to $\tilde{p}$.

Thus, *each lcs E is embeddable as a subspace in the product $\prod_p E_p$,*

with p running through the seminorms of E.
The embedding is given by $x \mapsto (x + \operatorname{Ker}(p))_p$. It is injective because E is separated. And E carries the initial structure with respect to this embedding since the $\tilde{p} \circ \mathrm{pr}_p : \prod_q E_q \to E_p \to \mathbb{R}$ form a sub-basis of seminorms of the product.

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d, "p"'] \arrow[dr, "\pi_p"] & \prod_q E_q \arrow[d, "\mathrm{pr}_p"] \\ \mathbb{R} & E_p \arrow[l, "\tilde{p}"] \end{tikzcd}$$

3.4 Finite dimensional lcs

3.4.1 Lemma. 1-dimensional lcs's.

Let E be a 1-dimensional lcs and $0 \neq a \in E$, then the mapping $f : \mathbb{K} \to E$, $t \mapsto t\,a$ is an isomorphism of lcs's (i.e. a linear homeomorphism). Any linear isomorphism of E with $\mathbb{K}$ is thus a homeomorphism.

Proof. Since $\{a\}$ is a basis of the vector space E, the mapping f is bijective, and each linear isomorphism $f : \mathbb{K} \to E$ looks like this with $a := f(1)$. Because the scalar multiplication is continuous, f is continuous. Since E is separated, there is a seminorm q with $q(a) \geqslant 1$. Then $|f^{-1}(ta)| = |t| = \frac{q(ta)}{q(a)} \leqslant q(ta)$, i.e. $|f^{-1}| \leqslant q$, so f^{-1} is also continuous. □

3.4.2 Lemma. Continuous functionals.

Let E be an lcs and $f : E \to \mathbb{K}$ a linear functional. Then:

1. *f is continuous;*

$\Leftrightarrow$ 2. *$|f| : x \mapsto |f(x)|$ is a continuous seminorm;*

$\Leftrightarrow$ 3. *The kernel* $\operatorname{Ker}(f)$ *is closed.*

If, on the other hand, f is not continuous, then $\operatorname{Ker}(f)$ *is dense in E.*

Proof. ($\boxed{1} \Rightarrow \boxed{2}$) Obvious, because $|\_|$ is a continuous norm on $\mathbb{K}$.

($\boxed{2} \Rightarrow \boxed{3}$) Obvious, because $\operatorname{Ker}(f) = \operatorname{Ker}(|f|)$.

($\boxed{3} \Rightarrow \boxed{1}$) It suffices to consider the case $f \neq 0$. Then $f : E \to \mathbb{K}$ is surjective. Since $F := \operatorname{Ker}(f)$ is closed, E/F is an lcs by 3.3.3. Because $f|_F = 0$, the function f factors over $\pi : E \to E/F$ to a linear mapping $\tilde{f} : E/F \to \mathbb{K}$.

$$\begin{tikzcd} E \arrow[rr, two heads, "\pi"] \arrow[dr, "f"'] & & E/\ker(f) \arrow[dl, dotted, "\tilde{f}", "\cong"'] \\ & \mathbb{K} & \end{tikzcd}$$

Since f is surjective, the same holds for $\tilde{f}$. Moreover, $\tilde{f}$ is injective, because $0 = \tilde{f}(\pi(x)) = f(x) \Rightarrow x \in \operatorname{Ker}(f) \Rightarrow \pi(x) = 0$. So $\tilde{f}$ is an isomorphism of lcs's by Lemma 3.4.1. Consequently, $f = \tilde{f} \circ \pi$ is continuous as a composition of continuous mappings.

Let now f be not continuous, so $\operatorname{Ker}(f)$ is not closed. Let $a \in \overline{\operatorname{Ker}(f)} \setminus \operatorname{Ker}(f)$. Without loss of generality $f(a) = 1$. The mapping $\operatorname{Ker}(f) \times \mathbb{K} \to E$, $(x, t) \mapsto x + t\,a$ is continuous, linear and its image is contained in the linear subspace $\overline{\operatorname{Ker}(f)}$. However, it is even onto, hence $\overline{\operatorname{Ker}(f)} = E$, because $E \ni y \mapsto (y - f(y)\,a, f(y)) \in \operatorname{Ker}(f) \times \mathbb{K}$ is obviously right-inverse to it. □

3.4.3 Examples of linear discontinuous functionals.

Let $E := C([0,1], \mathbb{K})$ with the 1-norm. Then $\mathrm{ev}_0 : E \to \mathbb{K}$ is linear, but not bounded (= continuous) and $\operatorname{Ker}(\mathrm{ev}_0) = \{f \in E : f(0) = 0\}$ is thus dense, because we easily find piecewise affine functions $f_n \geqslant 0$ with $\int f_n = 1$ and $f_n(0) = n$.

Similarly, $\sum : E \to \mathbb{K}$ is linear and not continuous (= bounded), where E is the space of the finite sequences with the ∞-norm and $\sum : x \mapsto \sum_{n=1}^{\infty} x_n$.

However, in order to find discontinuous linear functionals E on Banach spaces, one needs the axiom of choice. If one adds instead the axiom, that every subset of $\mathbb{R}$ is

Lebesgue measurable, to set theory (see [35]), then every linear mapping between Banach spaces is continuous (see [8]).

3.4.4 Corollary. Subspaces of co-dimension 1.

Let F be a closed subspace of an lcs E of co-dimension 1 (i.e. $\exists a \in E\backslash F$, s.t. the vector space E is generated by $F \cup \{a\}$).
Then $F \times \mathbb{K} \cong E$ holds, where the isomorphism is given by $(y, \lambda) \mapsto y + \lambda\, a$.
In partcular, there is a continuous linear functional f with $\ker f = F$.

Proof. The mapping $(y, \lambda) \mapsto y + \lambda\, a$ is clearly continuous. It is surjective since the vector space E is generated by $F \cup \{a\}$; and it is injective, because $y + \lambda\, a = 0$ with $\lambda \neq 0 \Rightarrow a = -\frac{1}{\lambda} y \in F$, a contradiction.

$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[r, two heads] \arrow[dr, dotted, "f"'] & E/F \arrow[d, "\cong"'] \\ & & \mathbb{K} \end{tikzcd}$$

Now to the inverse map. For this we define a linear functional $f : E \to \mathbb{K}$ by $f(y + \lambda\, a) := \lambda$. The kernel of f is F, hence is closed. Thus, f and also the desired inverse mapping $E \ni x \mapsto (x - f(x)\, a, f(x)) \in F \times \mathbb{K}$ is continuous. □

3.4.5 Theorem of Tychonoff on finite dimensional lcs's.

For every lcs E, the following statements are equivalent:

1. *E is finite dimensional.*
2. $\Leftrightarrow$ *$E \cong \mathbb{K}^n := \prod_{k=1}^n \mathbb{K}$ for some $n \in \mathbb{N}$. More precisely: Each linear isomorphism $E \cong \mathbb{K}^n$ is also an isomorphism of lcs's.*
3. $\Leftrightarrow$ *E is locally compact.*
4. $\Leftrightarrow$ *E has a precompact 0-neighborhood.*

A topological space is called LOCALLY COMPACT if every point has a neighborhood basis consisting of compact sets. For a Hausdorff space it is sufficient to find a compact neighborhood for each point. And for an lcs this is equivalent to the existence of a compact 0-neighborhood!

A subset K of an lcs is called PRECOMPACT if a finite set of F exists for each 0-neighborhood U with $K \subseteq U + F = \bigcup_{y \in F} U + y$, i.e. each 'uniform' open covering has a finite subcovering.

Proof. ($\boxed{1} \Rightarrow \boxed{2}$) We show by means of induction, with respect to the dimension n, that every linear bijection $\mathbb{K}^n \to E$ is already a homeomorphism:
(n = 1) was already shown in Lemma $\boxed{3.4.1}$.
(n + 1) Let $f : \mathbb{K}^{n+1} \to E$ be a linear bijection. Obviously, there is a natural topological isomorphism $k : \mathbb{K}^n \times \mathbb{K} \cong \mathbb{K}^{n+1}$. Let now $e^{n+1} := k(0, 1) \in \mathbb{K}^{n+1}$. Then $f \circ k|_{\mathbb{K}^n} : \mathbb{K}^n \to f(\mathbb{K}^n) =: F$ is a linear bijection onto an lcs. So, by induction, $f \circ k|_{\mathbb{K}^n} : \mathbb{K}^n \to F$ is a homeomorphism. Since $\mathbb{K}^n = \prod_{i=1}^n \mathbb{K}$ is complete, the same is true for F, and thus F is closed in E, by Corollary $\boxed{3.1.4}$. According to Corollary $\boxed{3.4.4}$, $h : (y, \lambda) \mapsto y + \lambda\, f(e^{n+1})$, $F \times \mathbb{K} \to E$, is a homeomorphism and thus also $f = h \circ (f \circ k|_{\mathbb{K}^n} \times \mathrm{id}_{\mathbb{K}}) \circ k^{-1} : \mathbb{K}^{n+1} \cong \mathbb{K}^n \times \mathbb{K} \to F \times \mathbb{K} \cong E$.

($\boxed{2} \Rightarrow \boxed{3}$) Is a direct sequence of the Theorem of Bolzano-Weierstrass (see [20, 3.3.4]) because then the unit cube in $\mathbb{R}^n$ is a compact 0-neighborhood.

($\boxed{3} \Rightarrow \boxed{4}$) Each compact set K is precompact as $\{U + x : x \in K\}$ represents an open covering.

($\boxed{4} \Rightarrow \boxed{1}$) Let U be a precompact (absolutely convex) 0-neighborhood. For the 0-neighborhood $\frac{1}{2}U$ there exists a finite set F, s.t. $U \subseteq F + \frac{1}{2}U$ and we may replace

F by the generated finite-dimensional subspace, which we again denote F. We now want to show that F is equal to E. Since F is finite-dimensional, F is complete because of (1 $\Rightarrow$ 2), so F is closed by Corollary 3.1.4. Now let's look at the canonical projection $\pi : E \twoheadrightarrow E/F$. The precompact set U is also bounded: For each (absolutely convex) 0-neighborhood W there is a finite set A with $U \subseteq A + W$, and since W is absorbent and A is finite we find a $K > 0$, s.t. $A \subseteq K \cdot W$, so $U \subseteq (K+1)W$. Thus, $V := \pi(U)$ is a bounded 0-neighborhood in E/F, so E/F is normable by Theorem 1.6.2, the family $\frac{1}{2^n}V$ is a 0-neighborhood basis and thus $\bigcap_n \frac{1}{2^n}V = \{0\}$. Furthermore we have $V = \pi(U) \subseteq \pi(F + \frac{1}{2}U) = 0 + \frac{1}{2}\pi(U) = \frac{1}{2}V$. From this we obtain by means of induction $V \subseteq \frac{1}{2^n}V$ and thus $V \subseteq \bigcap_{n\in\mathbb{N}} \frac{1}{2^n}V = \{0\}$. Since V must be absorbent as 0-neighborhood, $E/F = \{0\}$, i.e. $F = E$. □

3.4.6 Corollary.

1. *On $\mathbb{K}^m$, all norms and more generally all point-separating sets $\mathcal{P}_0$ of seminorms are equivalent (that is, generate the same topology).*
2. *Let F be a finite dimensional subspace of some lcs E. Then F is closed and consequently E/F separated (see also 5.1.7).*
3. *If $f : E \to F$ is a linear mapping of a finite dimensional lcs E into an lcs F, then f is continuous.*
4. *If F is a closed subspace of an lcs E and F has finite co-dimension in E, i.e. E/F is finite-dimensional, then E as lcs is isomorphic to $F \times (E/F)$.*

Proof. 1 Let p be a norm on $\mathbb{K}^m$, then according to Theorem 3.4.5 of Tychonoff $(\mathbb{K}^m, p)$ is topologically isomorphic to $(\mathbb{K}^m, \|\_\|_\infty)$, so the norm p is equivalent to the ∞-norm. Consequently, any two norms are equivalent.

2 Since F is isomorphic to $\mathbb{K}^m$ and $\mathbb{K}^m$ is complete, F is also complete, and thus closed in E.

3 Without loss of generality $E = \mathbb{K}^m$. Each linear f can be written as $f(x) = \sum_{k=1}^n \mathrm{pr}_k(x)\, f(e_k)$, where e_k are the standard unit vectors of $\mathbb{K}^n$. Since the projections pr_k are, by construction of the product, continuous, also f is continuous.

4 We consider the canonical projection $\pi : E \to E/F$. Since it is surjective, there exists a linear right-inverse f (We choose inverse images in E under π of a basis in the finite dimensional space E/F). Since E/F is separated (F is closed), f is continuous by 3. Now the desired isomorphism $E \to F \times (E/F)$ is given by $x \mapsto (x - f(\pi(x)), \pi(x))$. Its inverse is $(y, z) \mapsto y + f(z)$. □

3.5 Metrizable lcs

3.5.1 Lemma. Products of metric spaces.

Let E_n be normed spaces. Then the topology of $E := \prod_{n\in\mathbb{N}} E_n$ is metrizable.

Proof. We define a metric d on product E by the pointwise convergent series

$$d(x,y) := \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot \frac{\|x_n - y_n\|}{1 + \|x_n - y_n\|}.$$

This is well-defined, since $\frac{\|x_n - y_n\|}{1+\|x_n - y_n\|} \leqslant 1$, and $(\frac{1}{2^n})_n$ is summable, so by the Hölder inequality the inner product $d(x,y) = \langle (\frac{1}{2^n})_n | (\frac{\|x_n-y_n\|}{1+\|x_n-y_n\|})_n \rangle$ exists. The

Δ-inequality holds, because $t \mapsto \frac{t}{1+t} = \frac{1}{1+1/t}$ is monotonously growing and thus the estimate

$$\frac{\alpha}{1+\alpha} + \frac{\beta}{1+\beta} = \frac{\alpha+\beta+2\alpha\beta}{1+\alpha+\beta+\alpha\beta} \geqslant \frac{\alpha+\beta+\alpha\beta}{1+\alpha+\beta+\alpha\beta} \geqslant \frac{\alpha+\beta}{1+\alpha+\beta} \geqslant \frac{\gamma}{1+\gamma}.$$

holds for $\gamma \leqslant \alpha+\beta$.

If $\|x_i - y_i\| \leqslant \frac{1}{2^{n+1}}$ for all $i \leqslant n$, then $d(x,y) \leqslant \frac{1}{2^{n-1}}$, because

$$d(x,y) = \sum_i \frac{1}{2^i} \cdot \frac{\|x_i - y_i\|}{1+\|x_i-y_i\|} \leqslant \sum_{i \leqslant n} \frac{1}{2^i} \cdot \|x_i - y_i\| + \sum_{i>n} \frac{1}{2^i} \cdot 1 \leqslant 2 \cdot \frac{1}{2^{n+1}} + \frac{1}{2^n} \cdot 1 = \frac{1}{2^{n-1}}.$$

Conversely, let $d(x,y) \leqslant \frac{1}{2^n(n+1)}$, then $\|x_i - y_i\| \leqslant \frac{1}{n}$ for all $i \leqslant n$, because

$$\begin{aligned}
&\frac{1}{2^i} \cdot \frac{\|x_i - y_i\|}{1+\|x_i - y_i\|} \leqslant d(x,y) \leqslant \frac{1}{2^n(n+1)} \leqslant \frac{1}{2^i(n+1)} \\
&\Rightarrow \frac{\|x_i - y_i\|}{1+\|x_i - y_i\|} \leqslant \frac{1}{n+1} \\
&\Rightarrow n \cdot \|x_i - y_i\| \leqslant 1.
\end{aligned}$$

Thus d generates the same topology as the sub-basis $\{\|\operatorname{pr}_n(x)\| : n \in \mathbb{N}\}$ of seminorms. $\square$

3.5.2 Corollary. Characterization of metrizable lcs's.

Let E be an lcs. Then the topology of E is metrizable if and only if E is a countably seminormed lcs. For such lcs's, a translation invariant metric generating the topology is complete if and only if it is complete as locally convex topology. A Fréchet space is nothing else but a complete metrizable lcs.

Proof. Let E be metrizable. Then the sets $U_n := \{x : d(x,0) < \frac{1}{n}\}$ with $n \in \mathbb{N}$ form a 0-neighborhood basis. Consequently there are continuous seminorms p_n with $(p_n)_{<1} \subseteq U_n$. These p_n form a sub-basis: Namely, if p is a continuous seminorm, then $p_{<1}$ is a 0-neighborhood, so an n exists with $(p_n)_{<1} \subseteq U_n \subseteq p_{<1}$, hence $p_n \geqslant p$ by 1.3.7.

Conversely, if $\{p_n : n \in \mathbb{N}\}$ is a sub-basis of the seminorms of E, then E may be considered as subspace of the product $\prod_n E_n$ as in 3.3.4, where E_n is the normed space resulting from E by factoring out the kernel of p_n. According to Lemma 3.5.1, this product is metrizable, and so is the subspace since it carries the trace topology by 3.1.4.

Completeness. We only have to show that a sequence $(x_n)_n$ is Cauchy with respect to the metric if and only if it is so with respect to the seminorms. However, since the metric is translation invariant, the former means that for each $\varepsilon > 0$, the difference $x_n - x_m \in U_\varepsilon := \{y : d(y,0) < \varepsilon\}$ for n and m sufficiently large. Since the U_ε form a 0-neighborhood basis, as well as the balls $p_{<\varepsilon}$, this is equivalent to the inequality $p(x_n - x_m) < \varepsilon$ for n and m being sufficiently large for all p and all $\varepsilon > 0$. $\square$

3.5.3 Lemma. Quotients of Fréchet spaces.

Suppose F is a closed subspace of a Fréchet space E, then E/F is a Fréchet space, and every convergent sequence in E/F has a convergent lift.

Proof.

Lifts of convergent sequences. Let $y_n \to y = \pi(x)$ in E/F and let $p_k \leqslant p_{k+1}$ be a countable basis of the seminorms of E. So $\tilde{p}_k(y_n - y) \to 0$, i.e. $\exists n_k \in \mathbb{N}$

$\forall n \geqslant n_k$: $\tilde{p}_k(y_n - y) = \inf\{p_k(x' - x) : \pi(x') = y_n\} < \frac{1}{k}$. Without loss of generality, $k \mapsto n_k$ is strictly monotonously increasing. For $n_k \leqslant n < n_{k+1}$ we thus may choose $x_n \in \pi^{-1}(y_n)$ with $p_k(x_n - x) < \frac{1}{k}$. This lifted sequence converges to x, since for $\varepsilon > 0$ and seminorm p_j we find $k \geqslant j$ with $\frac{1}{k} \leqslant \varepsilon$ and for each $n \geqslant n_k$ there exists a $k' \geqslant k$ with $n_{k'} \leqslant n < n_{k'+1}$ and hence

$$p_j(x_n - x) \leqslant p_{k'}(x_n - x) < \frac{1}{k'} \leqslant \frac{1}{k} \leqslant \varepsilon.$$

Completeness. Let $\lambda_n := \frac{1}{4^n}$ and $(y_n)_{n\in\mathbb{N}}$ be bounded in E/F. Because of Lemma 2.2.2, it suffices to show that $\sum_n \lambda_n y_n$ converges to E/F. But since $\frac{1}{2^n} y_n \to 0$ in E/F, by the first part there exists a convergent and thus bounded sequence $x_n \in E$ with $\pi(x_n) = \frac{1}{2^n} y_n$. Since E is complete the series $\sum_n \frac{1}{2^n} x_n$ converges in E, and because of the continuity of π the same holds for the series $\sum_n \pi(\frac{1}{2^n} x_n) = \sum_n \frac{1}{4^n} y_n$. □

3.6 Coproducts

3.6.1 Lemma. Structure of coproducts.

Let E_k be lcs's. By the coproduct *or* direct sum *of the E_k we understand the vector space*

$$E := \coprod_k E_k := \Big\{x = (x_k)_k \in \prod_k E_k : x_k = 0 \textit{ for all but finitely many } k\Big\}$$

provided with the final structure with respect to the injections $\mathrm{inj}_k : E_k \to E$, which map $x \in E_k$ to the point $\mathrm{inj_k}(x)$, whose k-th component is x and all others are 0. The coproduct is an lcs.
A sub-basis of the seminorms of E is formed by the seminorms $p(x) := \sum_k p_k(x_k)$, where the p_k are the seminorms of E_k. Note that the sum makes sense since only finite many summands are 0.
A set is bounded in $\coprod_i E_i$ if it is already contained and bounded in a finite partial sum.
The coproduct of (sequentially) complete space is (sequentially) complete.
The inclusion $\coprod_k E_k \to \prod_k E_k$ is continuous. And if the index set is finite, then the coproduct will coincide with the product.
If the index set is countable, then the seminorms $p(x) := \sup\{p_k(x_k) : k\}$, with arbitrary seminorms p_k of E_k, form a sub-basis.

Proof.

Sub-basis of seminorms. For each k, let p_k be a continuous seminorm on E_k. Then $p(x) := \sum_k p_k(x_k)$ is a well-defined seminorm on E. The composition with inj_k is $p \circ \mathrm{inj}_k = p_k$ and thus continuous, so also p is continuous by the construction of the final structure.

Conversely, let p be a continuous seminorm on E. Then $p_k := p|_{E_k} = p \circ \mathrm{inj}_k$ is one on E_k, and $p(x) = p(\sum_k \mathrm{inj}_k(x_k)) \leqslant \sum p_k(x_k)$. So these seminorms form a sub-basis for E.

Separation is now clear.

Countable index set. Since $p(x) := \sup_k p_k(x_k) \leqslant \sum_k p_k(x_k)$, this p is a continuous seminorm. Conversely, because of the Hölder inequality

$$\sum_k p_k(x_k) = \sum_k \frac{1}{2^k}(2^k p_k)(x_k) \leqslant \sup\{2^k p_k(x_k) : k\} \cdot \sum_k \frac{1}{2^k},$$

So the suprema generate the same continuous seminorms as the sums do.

Finite index sets. In case of a finite index set, we have $\max\{p_1, \dots, p_N\}$ as a basis and this is also a basis for the product.

Continuous inclusion in the product. The projections $\mathrm{pr}_j : \coprod_k E_k \to E_j$ are continuous because of the final structure, since the compositions with inj_k are the identity for $j = k$ and 0 otherwise. Because of the universal property the inclusion $(\mathrm{pr}_k)_k : \coprod_k E_k \to \prod_k E_k$ is also continuous.

Boundedness. A set that is bounded in a finite partial sum, is also bounded in the total sum, since the inclusion is continuous.
Conversely, let B be bounded in E. Note first that any finite partial sum $\coprod_{k\in K} E_k$ is a locally convex subspace of $\prod_k E_k$, because $(\mathrm{pr}_k)_{k\in K} : \prod_k E_k \to \prod_{k\in K} E_k = \coprod_{k\in K} E_k$ provides a continuous linear right inverse to the inclusion $\coprod_{k\in K} E_k \to \coprod_k E_k \to \prod_k E_k$. It suffices to show that $K := \{k : \mathrm{pr}_k(B) \neq \{0\}\}$ is finite, because $B \subseteq \prod_k \mathrm{pr}_k(B)$. Suppose K would be infinite. We choose a countable subset of K that we can identify with $\mathbb{N}$. For each $k \in \mathbb{N}$ we choose a matching point $b^k \in B$ with $b^k_k \neq 0$. Since E_k is separated, a continuous seminorm p_k exists on E_k with $p_k(b^k_k) = k \in \mathbb{N}$. For the $k \notin \mathbb{N}$ we choose $p_k = 0$. Let $p(x) := \sum_k p_k(x_k)$. Then p is a continuous seminorm on E, and thus $p(B)$ is bounded, in contradiction to $k = p_k(b^k_k) \leqslant p(b^k) \in p(B)$ for all $k \in \mathbb{N}$.

Completeness. We show sequential completeness first. Let (x^n) be a Cauchy sequence. As such, it is bounded, i.e. contained in a finite partial sum. Since this partial sum forms a locally convex subspace of $\prod_k E_k$, the sequence is a Cauchy sequence in this finite sum = product, and thus it is convergent in the finite product by 3.2.1 and therefore also in E.

Now the completeness: Let (x^i) be Cauchy in $\coprod_k E_k$. Then (x^i_j) is Cauchy for each j, so x^i converges coordinatewise towards some $x^\infty \in \prod_k E^k$. We have $x^\infty \in \coprod_k E_k$: In fact let $K := \{k : x^\infty_k \neq 0\}$. Choose for $k \in K$ a continuous seminorm p_k on E_k with $p_k(x^\infty_k) > 1$, put $p_k := 0$ for $k \notin K$, and let $p(x) := \sum_k p_k(x_k)$. Then there is an i_0 with $p_k(x^i_k - x^j_k) \leqslant p(x^i - x^j) \leqslant 1$ for $i, j > i_0$ and all k. Consequently, also $p_k(x^i_k - x^\infty_k) \leqslant 1$ for $i > i_0$ and all k. Because of $x^i \in \coprod_k E_k$, we have $x^i_k = 0$ for almost all k, so $p_k(x^\infty_k) \leqslant 1$ for almost all k, hence the carrier K of x^∞ is finite.

Finally, $x^i \to x^\infty$ converges in $\coprod_k E_k$, because let p be a seminorm of the specified sub-basis and let $\varepsilon > 0$, then $\sum_k p_k(x^i_k - x^j_k) =: p(x^i - x^j) \leqslant \varepsilon$ for all $i, j > i_0$. Hence for given $i > i_0$ let K be the finite set $\{k : x^i_k - x^\infty_k \neq 0\}$, then $p(x^i - x^\infty) = \sum_{k\in K} p_k(x^i_k - x^\infty_k) = \lim_j \sum_{k\in K} p_k(x^i_k - x^j_k) \leqslant \varepsilon$, i.e. $x^i \to x^\infty$ with respect to the structure of $\coprod_k E_k$. □

3.6.2 Bornological vector spaces.

Let E carry the final structure with respect to a family of linear mapping $f_k : E_k \to E$ whose images generate the vector space E. Then E *can also be represented as quotient of the coproduct* $\coprod_k E_k$*:*
Namely, let F be the kernel of linear mapping $\sum_k f_k : \coprod_k E_k \twoheadrightarrow E$, which maps $x = (x_k)_k$ to $\sum_k f_k(x_k)$. This mapping is surjective, because the images $f_k(E_k)$

generate the vector space E by assumption, and it is continuous because of the final structure. Consequently we obtain a bijective (and because of the final structure of the quotient) continuous mapping $(\coprod_k E_k)/F \to E$. This is even a homeomorphism since E carries the final structure with respect to the mapping f_k.

$$\begin{tikzcd} & E_j \arrow[r, "f_j"] \arrow[d, hook, "\mathrm{inj}_j"'] & E \\ F \arrow[r, hook] & \coprod_k E_k \arrow[r, two heads] \arrow[ur, "\sum_k f_k"] & (\coprod_k E_k)/F \arrow[u, dotted, tail, two heads] \end{tikzcd}$$

Let now E be an arbitrary lcs. For each bounded absolutely convex set B we may consider the linear subspace E_B of E generated by B. Since B is by construction absorbent in E_B, the Minkowski functional p_B is a seminorm on E_B. It is even a norm, because $0 = p_B(x) = \inf\{\lambda > 0 : x \in \lambda B\} \Rightarrow \exists\lambda_n \to 0$ with $\frac{1}{\lambda_n}x \in B$, so $x = \lambda_n \frac{1}{\lambda_n}x \to 0$ by 2.1.5 and consequently $x = 0$. Furthermore, the inclusion $E_B \rightarrowtail E$ is bounded on the open unit ball $\subseteq B$, so it is even continuous, because E_B is normed (and thus bornological by 2.1.7).

An lcs E carries the final structure with respect to all these inclusions $E_B \rightarrowtail E$, if and only if E is bornological:

($\Rightarrow$) Namely, if $f : E \to F$ is a bounded linear mapping, then $f|_{E_B} : E_B \to E \to F$ is a bounded linear mapping on a normed space, i.e. continuous by 2.1.7. If E carries the final structure with respect to subspaces E_B, f is continuous, i.e. E bornological.

($\Leftarrow$) Conversely, let E be bornological. The final structure on E with respect to the mappings $E_B \rightarrowtail E$ is always finer or equal to the one given on E. So let's consider the identity f from E with the given structure to E with the final one. Let $B \subseteq E$ be bounded and without loss of generality absolutely convex. Then the inclusion $E_B \rightarrowtail E$ is continuous and hence bounded with respect to the final structure on E. Thus, $f(B)$ is bounded, i.e. f is a bounded linear mapping, and since E is assumed to be bornological, f is continuous. So the two structures coincide.

Consequently, the *bornological vector spaces are exactly the quotients of coproducts of normed spaces.* Compare this with the dual description of lcs's in 3.3.4.

3.6.3 Test functions and distributions.

A partial differential operator (PDO) is an operator of the form

$$D = \sum_{\alpha_k \in \mathbb{N}^m} a_\alpha \partial^\alpha,$$

where ∂^α denotes the iterated partial derivative of order $\alpha = (\alpha_1, \dots, \alpha_m)$. We restrict our considerations to the case of constant coefficents $a_\alpha \in \mathbb{K}$. Solving the associated partial differential equation (PDE)

$$D(u) = f$$

amounts in finding for given functions f corresponding functions u.

The idea is, that the solution operator $G : f \mapsto u$ should be a kind of integral operator, i.e. have the form

$$G(f) : x \mapsto \int_{\mathbb{R}^m} \gamma(x, y)\, f(y)\, dy$$

for some (integral kernel) $\gamma : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$. This is the continuous pendent to the matrix representation $G(f)(i) = \sum_j \gamma_{i,j}\, f_j$ of a linear mapping G.

Since D obviously commtes with partial derivatives, the same is to be expected for the solution operator G. Partial integration yields that $\partial_1\gamma + \partial_2\gamma = 0$ and hence $\gamma(x,y)$ depends only on the difference $x-y$, i.e. G could be writen as convolution operator

$$G(f) = \gamma \star f : x \mapsto \int_{\mathbb{R}^m} \gamma(x-y)\, f(y)\, dy.$$

That G is inverse to D gives

$$f = D(G(f)) = D(\gamma \star f) = D(\gamma) \star f,$$

since partial derivatives commute with convolution. Thus $\delta := D(\gamma)$ should be a neutral element for the convolution of functions. However such a function cannot exist: Otherwise $f : y \mapsto \delta(-y) \cdot y^2$ would yield $0 = f(0) = (\delta \star f)(0) = \int_{\mathbb{R}^m} \delta(-y)^2\, y^2\, dy$, hence $\delta(-y) = 0$ for almost all $y \neq 0$.

Nevertheless $G : f \mapsto \gamma \star f$ should be a linear mapping between spaces of functions. Since

$$(\gamma \star f)(x) = \int_{\mathbb{R}^m} \gamma(x-y)\, f(y)\, dy = \int_{\mathbb{R}^m} (S \circ T_x)(\gamma)(y)\, f(y)\, dy,$$

where S defnoted the reflection $g \mapsto (y \mapsto g(-y))$ and T_x the translation $g \mapsto (y \mapsto g(y-x))$, it would be enough to determine $f \mapsto (\gamma \star f)(0) = \int_{\mathbb{R}^m} \gamma(-y)\, f(y)\, dy$, which seems to be a linear functional on some space of functions f. One calls such a functional a DISTRIBUTION.

We have to figure out on which functions the distributions should act on, and with respect to which topology they should be continuous. Of course, we want the notion of distributions to be an extension of that of the functions, so at least we should be able to think of continuous functions $g \in C(\mathbb{R}^m, \mathbb{R})$ as distributions by $g(f) := \langle g|f\rangle := \int_{\mathbb{R}^m} g(y)\, f(y)\, dy$. But for the integral to make sense, the product $g \cdot f$ must approach 0 sufficiently fast. Since g may grow arbitrarily fast, f must even have compact support. As a first candidate for the space of test functions f, the space of the continuous functions with compact support comes to ones mind. On it we already met two structures, namely as subspace of the Fréchet space $C(\mathbb{R}^m, \mathbb{R})$, and as subspace of the Banach space $B(\mathbb{R}^m, \mathbb{R})$. Is the linear functional $f \mapsto \langle g|f\rangle$ continuous for any continuous function g? In particular we may choose $g = 1$, Then $\langle g|f_n\rangle = \int_{\mathbb{R}^m} f_n$, and for the convergence of $\langle g|f_n\rangle$ the uniform convergence (on compact sets) of f_n is not enough. Because of $\int_{\mathbb{R}^m} |f| \leqslant \mathrm{volume}(\mathrm{supp}(f)) \cdot \|f\|_\infty$, only those sequence should converge in the test space which converges uniformly and their supports are contained in a fixed compact set. Let $C_K(\mathbb{R}^m, \mathbb{K})$ be the space of the continuous functions from $\mathbb{R}^m$ to $\mathbb{K}$, which have support within the compact set $K \subseteq \mathbb{R}^m$. Then $C_K(\mathbb{R}^m, \mathbb{K})$ provided with the uniform convergence is a closed subspace of the space $C_b(\mathbb{R}^m, \mathbb{K})$ of continuous bounded functions and thus a Banach space. The space $C_c(\mathbb{R}^m, \mathbb{K})$ of all continuous functions with compact support is then the union of these Banach spaces $C_K(\mathbb{R}^m, \mathbb{R})$ where K runs through all compact sets or just a basis of the compact sets (i.e. each compact set is contained in one of the sets in the basis). So we may consider the final topology on it.

We then have to verify that the convergent sequences are really those that already converge in one step $C_K(\mathbb{R}^m, \mathbb{K})$, and that sequential continuity suffices.

Since we want to use distributions for solving differential equations, they have to be differentiable. If two functions g and f are differentiable, then $\langle \partial_i g|f\rangle = -\langle g|\partial_i f\rangle$ as is shown by means of partial integration. So for a distribution f we could define the partial derivative $\partial_i g$ by $\partial_i g(f) := -g(\partial_i f)$. Hence our test functions should be even smooth, and we need to do the same construction for $C_c^\infty(\mathbb{R}^m, \mathbb{R}) = \bigcup_K C_K^\infty(\mathbb{R}^m, \mathbb{R})$. The lcs $C_c^\infty(\mathbb{R}^m, \mathbb{R})$ defined in this way is also denoted $\mathcal{D}$. The corresponding notation for the Fréchet space $C^\infty(\mathbb{R}^m, \mathbb{R})$ is $\mathcal{E}$.

3.7 Strict inductive limits

3.7.1 Lemma. Structure of strict inductive limits.

Let a vector space E be given, which can be written as a union of an ascending sequence E_n of linear subspaces. Furthermore, suppose that the E_n are lcs in such a way that E_n is a closed locally convex subspace of E_{n+1} for all $n \in \mathbb{N}$.
The space E with the final locally convex structure with respect to all inclusions $E_n \to E$ is called STRICT INDUCTIVE LIMIT *of the E_n and one writes $E = \varinjlim_n E_n$. We call the E_n the* STEPS *of the inductive limit.*
Each seminorm of any E_n has a continuous extension to E.
Each E_n is a closed locally convex subspace of E. The space E is separated.
A set is bounded in E if and only if it is contained in some step and bounded there.
If all E_n are (sequentially) complete, then so is E.

Proof.

Continuation of the seminorms. Let p_n be a seminorm of E_n. Since E_n is a subspace of E_{n+1}, there is a continuous extension p_{n+1} on E_{n+1} by 3.1.4. By induction we obtain a sequence of successive extensions p_k to E_k for $k \geqslant n$. Let $p_k := p_n|_{E_k}$ for $k < n$ and let $p := \bigcup_k p_k$. Then p is a seminorm on E and the trace on each step E_k is p_k. So p is continuous by the definition of the final structure.

It immediately follows that E is **separated**.

Steps as closed subspaces of E. Since by the previous claim the continuous seminorms of E_n are just the restrictions of the continuous SN's of E, each step E_n carries the trace topology of E. Let $i \mapsto x_i$ be a net in E_n, which converge towards x_∞ in E. Because of $E = \bigcup_k E_k$, there is a $k \geqslant n$ with $x_\infty \in E_k$. Since $E_k \supseteq E_n$ is a topological subspace of E, the net x_i converges in E_k towards x_∞. By assumption, however, E_n is closed in E_k and thus is $x_\infty \in E_n$, i.e. E_n is closed in E.

Boundedness. Let $B \subseteq E$ be a bounded set. Because of the previous claim, it suffices to show that B is contained in some step (it is bounded there automatically). Suppose $B \nsubseteq E_n$ for each $n \in \mathbb{N}$. We may choose $b_1 \in B\backslash E_1$ and n_1 with $b_1 \in E_{n_1}$. Recursively we obtain a strictly monotonously increasing sequence (n_k) and $b_k \in E_{n_k} \cap (B\backslash E_{n_{k-1}})$. Let p_1 be a continuous seminorm on E_{n_1} with $p_1(b_1) = 1$, which is possible because $b_1 \notin E_1$ so $b_1 \neq 0$. We are looking inductively for continuous seminorms p_k on E_{n_k}, with $p_k|_{E_{n_{k-1}}} = p_{k-1}$ and $p_k(b_k) = k$: For this we consider the subspace F of E_{n_k} generated by $E_{n_{k-1}}$ and b_k. Since $b_k \notin E_{n_{k-1}}$, $(x, \lambda) \mapsto x+\lambda\, b_k$ by 3.4.4 is an isomorphism $E_{n_{k-1}} \times \mathbb{K} \cong F$. On F we define the continuous seminorm q by $q(x + \lambda\, b_k) := p_{k-1}(x) + k \cdot |\lambda|$. By 3.1.4 there is a continuous seminorm p_k on E_{n_k}, which extends q. Let, finally, $p := \bigcup_k p_k$. Then p is a continuous seminorm on E and $p(b_k) = k$, a contradiction to the boundedness of B.

Sequential completeness. Let x_n be a Cauchy sequence in E. Then $\{x_n : n \in \mathbb{N}\}$ is bounded, thus included in some E_n by what we have shown above. Since E_n is a locally convex subspace of E, x_n is a Cauchy sequence in it, thus converges towards an x_∞ in E_n, hence also in E.

Completeness. Since a Cauchy net $(x_i)_i$ is not necessarily bounded, we can not conclude, as for sequences, that almost the entire net is already contained in one step. But we now show that this is almost the case:

Claim: $\exists n\, \forall U$ *abs.conv. 0-neighborhood* $\forall i\, \exists j \succ i\, \exists u \in U : x_j + u \in E_n$.
Suppose this were not the case, i.e. $\forall n\, \exists U_n\, \exists i_n\, \forall j \succ i_n : (x_j + U_n) \cap E_n = \emptyset$.

Without loss of generality $2U_{n+1} \subseteq U_n$. The set $U := \bigcup_n \sum_{i=0}^n U_i \cap E_i$ is an absolutely convex 0-neighborhood, because $U \cap E_n \supseteq U_n \cap E_n$. and hence the restriction of the Minkowski functional of U to E_n is a continuous seminorm. Thus an i exists, s.t. $x_j - x_k \in U$ for all $j, k > i$. Let $n \in \mathbb{N}$ and $j > i$ be choosen such that $x_i \in E_n$ and $j > i_n$. Then there exists an m (without loss of generality $m \geqslant n$) and $u_k \in U_k \cap E_k$ for each $k \leqslant m$ with $x_i - x_j = \sum_{k=0}^m u_k$. Then $x_i - \sum_{k=1}^n u_k = x_j + \sum_{k=n+1}^m u_k \in E_n \cap (x_j + U_n)$, because of $2U_{k+1} \subseteq U_k$. This is a contradiction to $(x_j + U_n) \cap E_n = \emptyset$.

We now consider the net $(i, U) \mapsto x_j + u \in E_n$, where $j > i$ and $u \in U$ are chosen as in the claim and we use as index set the product of the original one and a 0-neighborhood basis. This net is Cauchy in E_n, because for every 0-neighborhood V exists an absolutely convex 0-neighborhood U with $U + (U - U) = 3U \subseteq V$ and an i such that for all $i', i'' > i$ and all $U', U'' \subseteq U$ with corresponding j', j'', $u' \in U'$, and $u'' \in U''$ we have $x_{j'} - x_{j''} \in U$ and thus $x_{j'} + u' - x_{j''} - u'' = (x_{j'} - x_{j''}) + u' - u'' \in 3U \subseteq V$. Thus this new net converges towards an $x_\infty \in E_n$. We claim that the orginal net converges to x_∞ as well: For each 0-neighborhood W let V be choosen so that $3V \subseteq W$. Then there exists an i and a U (without loss of generality $U \subseteq V$) with $x_k - x_j \in V$ for all $k, j > i$ and $x_j + u - x_\infty \in V$ for the corresponding $j > i$ and $u \in U$. Thus $x_k - x_\infty = x_k - x_j + x_j - x_\infty \in V + (V - u) \subseteq V + V - V \subseteq W$ for all $k > i$.

For a proof by means of filter see [**14**, S.86]. □

3.7.2 Example. The space of test functions.

We may now consider the space $C_c^\infty(\mathbb{R}^n, \mathbb{R})$ of the smooth functions with compact support as strict inductive limit $\mathcal{D} := \varinjlim_K C_K^\infty(\mathbb{R}^n, \mathbb{R})$ of the steps $C_K^\infty(\mathbb{R}^n, \mathbb{R}) := \{f \in C^\infty(\mathbb{R}^n, \mathbb{R}) : \operatorname{Trg} \subseteq K\}$, where K runs through a basis of the compact sets, e.g. $(\{x : |x| \leqslant k\})_{k \in \mathbb{N}}$. This space is complete by 3.7.1 and bornological by 3.3.2 because the $C_K^\infty(\mathbb{R}^n, \mathbb{R})$ are Fréchet spaces as closed subspaces of the Fréchet space $C^\infty(\mathbb{R}^n, \mathbb{R})$. The continuous (= bounded = sequentially continuous, see 2.1.4) linear functionals on $\mathcal{D}$ are called DISTRIBUTIONS.

A “discrete” version is the space $\mathbb{K}^{(\mathbb{N})} = \varinjlim_n \mathbb{K}^n$ of the finite sequences.

3.8 Completion

We now want to tackle the problem of what we can do when a space turns out to be incomplete.

3.8.1 Definition. Completion.

By the COMPLETION of an lcs E, we understand a complete lcs $\tilde{E}$ together with a continuous linear mapping $\iota : E \to \tilde{E}$, which has the following universal property:
For each continuous linear mapping $f : E \to F$ into a complete lcs F, there exists a unique continuous linear mapping $\tilde{f} : \tilde{E} \to F$ with $\tilde{f} \circ \iota = f$.

$$\begin{tikzcd} E \arrow[r, "\iota"] \arrow[dr, "f"'] & \tilde{E} \arrow[d, dotted, "\tilde{f}"] \\ & F \end{tikzcd}$$

3.8.2 Remark. Uniqueness of the completion.

The completion of any lcs$E$ is unique up to isomorphisms. Namely let $\iota_i : E \to E^i$ for $i = 1, 2$ be two completions of E. Then there are unique continuous linear maps $\tilde{\iota}_1 : E^1 \to E^2$ and $\tilde{\iota}_2 : E^2 \to E^1$ with $\tilde{\iota}_2 \circ \iota_1 = \iota_2$ and $\tilde{\iota}_1 \circ \iota_2 = \iota_1$. So $\tilde{\iota}_2 \circ \tilde{\iota}_1 \circ \iota_2 = \iota_2 = \mathrm{id} \circ \iota_2$, and because of the uniqueness of $\tilde{f}$ also $\tilde{\iota}_2 \circ \tilde{\iota}_1 = \mathrm{id}$.

3.8.3 Lemma. Neighborhood basis of completion.

Let E be a dense subspace of an lcs $\tilde{E}$.

- *The continuous seminorms of $\tilde{E}$ are exactly the unique extensions of those of E.*
- *If $\mathcal{U}$ is a 0-neighborhood basis of E, then the closures $\{\bar{U} : U \in \mathcal{U}\}$ in $\tilde{E}$ form a 0-neighborhood basis of $\tilde{E}$.*
- *Each continuous linear mapping $f : E \to F$ into a complete lcs F has a unique continuous linear extension $\tilde{f} : \tilde{E} \to F$.*
- *In addition, if $\tilde{E}$ complete, then $E \hookrightarrow \tilde{E}$ is a completion of E.*

Proof.

Seminorms. By 3.1.4, each continuous seminorm p of E has an extension $\tilde{p}$ to $\tilde{E}$. Since E is dense in $\tilde{E}$, $\tilde{p}$ is uniquely determined.

0-neighborhood basis. It is enough to show $\tilde{p}_{\leqslant 1} \subseteq \overline{p_{\leqslant 1}}$ (Then we even have equality, because $p_{\leqslant 1} \subseteq \tilde{p}_{\leqslant 1}$ and thus $\overline{p_{\leqslant 1}} \subseteq \overline{\tilde{p}_{\leqslant 1}} = \tilde{p}_{\leqslant 1}$). So let $\tilde{p}(\tilde{x}) \leqslant 1$. Since E sits densely in $\tilde{E}$, there exists a net (x_i) in E which converges to $\tilde{x}$ (consider as index set $\{(V, x) : V$ ist neighborhood von $\tilde{x}$ und $x \in V \cap E\}$ with the ordering $(V, x) \prec (V', x') :\Leftrightarrow V \supseteq V'$ and as net the mapping $(U, x) \mapsto x$). In case $\tilde{p}(\tilde{x}) < 1$, we have $x_i \in \tilde{p}_{\leqslant 1} \cap E = p_{\leqslant 1}$ finally, i.e. $\tilde{x} \in \overline{p_{\leqslant 1}}$. Otherwise, $p(x_i) \neq 0$ for all sufficiently large i and thus $y_i := \frac{x_i}{p(x_i)} \in p_{\leqslant 1}$ and $y_i \to \frac{\tilde{x}}{\tilde{p}(\tilde{x})} = \tilde{x}$.

Continuous extensions. Let $f : E \to F$ be continuous linear and $\tilde{x} \in \tilde{E}$ be arbitrary. Since E is dense in $\tilde{E}$, there is a net (x_i) in E which converges to $\tilde{x}$ in $\tilde{E}$. Since $\tilde{f}$ should be continuous, $\tilde{f}(\tilde{x}) = \tilde{f}(\lim_i x_i) = \lim_i \tilde{f}(x_i) = \lim_i f(x_i)$ must hold. So there is at most one continuous extension $\tilde{f}$, and this has to be given by $\tilde{f}(\tilde{x}) = \lim_i f(x_i)$. Since x_i is a Cauchy net and f is uniformly continuous (by linearity), the same holds for $f(x_i)$, and thus $f(x_i)$ converges because F is complete.

We define $\tilde{f}(\tilde{x})$ as this limit and have to show that it does not depend on the choice of the net. Let therefore x_j be a second net in E, which converges towards $\tilde{x}$. We consider as an index set the product $I \times J$ with the product ordering, i.e. $(i, j) \succ (i', j') :\Leftrightarrow (i \succ i')\&(j \succ j')$ and as net the mapping $(i, j) \mapsto x_{i,j} := x_i - x_j$. This net converges now towards $\lim_i x_i - \lim_j x_j = \tilde{x} - \tilde{x} = 0$, thus the image net $f(x_{i,j}) = f(x_i) - f(x_j)$ converges towards $f(0) = 0$, on the other hand its limit is just $\lim_{i,j} f(x_{i,j}) = \lim_i f(x_i) - \lim_j f(x_j)$, which means that the limit $\tilde{f}(\tilde{x})$ is unique.

The extension $\tilde{f}$ is linear: Let $\tilde{x}$ and $\tilde{y}$ in $\tilde{E}$, then nets x_i and y_j exist in E with $x_i \to \tilde{x}$ and $y_j \to \tilde{y}$. So:

$$
\begin{aligned}
\tilde{f}(\tilde{x} + \lambda\tilde{y}) &= \tilde{f}\big(\lim_i x_i + \lambda \lim_j y_j\big) = \tilde{f}\big(\lim_{i,j}(x_i + \lambda y_j)\big) \\
&= \lim_{i,j} \tilde{f}(x_i + \lambda y_j) = \lim_{i,j} f(x_i + \lambda y_j) = \lim_{i,j} f(x_i) + \lambda f(y_j) \\
&= \lim_i f(x_i) + \lambda \lim_j f(y_j) = \tilde{f}(\tilde{x}) + \lambda \tilde{f}(\tilde{y}).
\end{aligned}
$$

The extension $\tilde{f}$ is continuous:

(Proof by means of seminorms) Let q be a continuous seminorm on F. Then $q \circ f$ is one on E, so by 3.1.4 there is a continuous seminorm $\widetilde{q \circ f}$ on $\tilde{E}$, which extends

$q \circ f$. We have $\widetilde{q \circ f} = q \circ \tilde{f}$, since

$$(\widetilde{q \circ f})(\tilde{x}) = (\widetilde{q \circ f})(\lim_i x_i) = \lim_i (\widetilde{q \circ f})(x_i) = \lim_i (q \circ f)(x_i)$$
$$= q(\lim_i f(x_i)) = q(\tilde{f}(\lim x_i)) = (q \circ \tilde{f})(\tilde{x})$$

(Proof by means of 0-neighborhoods) Namely, let V be a closed 0-neighborhood of F and U one of E with $f(U) \subseteq V$. Then $\bar{U}$ is a 0-neighborhood in $\tilde{E}$ with $\tilde{f}(\bar{U}) \subseteq \overline{f(U)} \subseteq \bar{V} = V$ (Namely, let $\tilde{x} \in \bar{U}$, then there is a net x_i in $U \subseteq E$ which converges to $\tilde{x}$. So $\tilde{f}(\tilde{x}) = \lim_i f(x_i) \in \bar{V} = V$). □

3.8.4 Theorem. Existence of the completion.

Each lcs E has a completion $\iota : E \to \tilde{E}$, which is unique up to isomorphisms. If E is normable (or metrizable) then the same holds for $\tilde{E}$.

Proof. We first deal with the case that E is a normed space. So we are looking for a complete space in which E can be embedded isometrically as a subspace. By 2.2.7, the dual space $E' := L(E, \mathbb{K})$ is always complete, hence also the bidual $E'' := (E')'$. Now let's consider the mapping $\iota : E \to E''$, given by $\iota(x) = \mathrm{ev}_x : x' \mapsto x'(x)$. This is clearly well-defined, linear and continuous, because

$$\|\iota(x)\| := \sup\{\underbrace{|\iota(x)(x')|}_{|x'(x)| \leqslant \|x'\| \cdot \|x\|} : \|x'\| = 1\} \leqslant \|x\|.$$

Remains to show that ι is isometric. All it takes is to find for each $x \in E$ an $x' \in E'$ with $x'(x) = \|x\|$ and $\|x'\| = 1$. Geometrically this means that an affine closed hyperplane H exists which contains x and is disjoint from the open ball $\{y : \|y\| < \|x\|\}$, i.e. is tangential to the unit sphere at x:
($\Rightarrow$) The affine hyperplane $H := \{y : x'(y) = \|x\|\}$ satisfies $x \in H$ and $\|x\| = x'(y) \leqslant \|x'\| \, \|y\| = \|y\|$ for each $y \in H$.
($\Leftarrow$) Conversely, let H be such a closed affine hyperplane, i.e. by 3.4.4 there exists $0 \neq x' \in E'$ and $c \in \mathbb{K}$ with $H = \{y : x'(y) = c\}$. Since $0 \notin H$ we have $c \neq 0$ and thus without loss of generality $c = \|x\|$. Since $x \in H$ we have $\|x\| = x'(x) \leqslant \|x'\| \, \|x\|$, i.e. $1 \leqslant \|x'\|$. Suppose $1 < \|x'\| = \sup\{|x'(z)| : \|z\| = 1\}$. Then there exists a z with $\|z\| = 1$ and $x'(z) > 1$, hence $y := \frac{\|x\|}{x'(z)} z \in H$ but $\|y\| = \frac{\|x\|}{x'(z)} < \|x\|$, a contradiction.

The existence of such a hyperplane will be shown in 5.2.2 (see also 5.1.10) by means of the theorem of Hahn-Banach.

As $\tilde{E}$ we now take the closure of image $\iota(E)$ in E''. Then ι is an embedding from E onto the dense subspace $\iota(E)$ of the Banach space $\tilde{E}$, and thus is a completion by Lemma 3.8.3.

Now the case of a general lcs E. By 3.3.4, E can be considered as the subspace of a product of normed space E_p. This, in turn, can be understood as the subspace of the product of the completions $\widetilde{E_p}$ of the factors. So E is a subspace of a complete lcs. For $\tilde{E}$ we may now take the closure of E in this product. □

3.9 Complexification

3.9.1 Lemma. Complex vector spaces.

A vector space E over $\mathbb{R}$ is a vector space over $\mathbb{C}$ if and only if an $\mathbb{R}$-linear mapping $I : E \to E$ exists which satisfies $I^2 = -\mathrm{id}$.

Proof. If E is a vector space over $\mathbb{C}$, then I is given by $I(x) := i\,x$. Conversely, we define $(a + i\,b) \cdot x := a \cdot x + b \cdot I(x)$ and thus obtain a vector space over $\mathbb{C}$. □

3.9.2 Corollary. Complex locally convex spaces.

An lcs E over $\mathbb{R}$ is an lcs over $\mathbb{C}$ if and only if there is a continuous $\mathbb{R}$-linear mapping $I : E \to E$ that satisfies $I^2 = -\operatorname{id}$.

Proof. As seminorms of the complex vector space E we use the positively homogeneous (with respect to scalars in $\mathbb{C}$) seminorms of the real lcs.
If p is a seminorm of the real lcs and $\lambda = a + i\,b \in \mathbb{C}$, we define another seminorm $p_\lambda : x \mapsto p(\lambda\, x)$ of the real lcs. If E is a complex lcs, then the complex scalar multiplication and in particular I is continuous, hence $q := p \circ I$ is a continuous seminorm of the real lcs. We have

$$\begin{aligned} p_\lambda(x) &= p((a + i\,b)x) = p(a\,x + b\,I(x)) \leqslant |a|\,p(x) + |b|\,q(x) \\ &\leqslant |a + i\,b|\,\sqrt{p(x)^2 + q(x)^2} \leqslant |a + i\,b|\,(p(x) + q(x)). \end{aligned}$$

Thus $p_{\mathbb{C}}(x) := \sup\{p_\lambda(x) : |\lambda| = 1\}$ defines a seminorm of the complex vector space with $p \leqslant p_{\mathbb{C}} \leqslant p + q$. Consequently these seminorms of the complex vector space define the same topology as the seminorms of the real lcs. □

3.9.3 Remark. Complexification.

We are now trying to produce a complex vector space from any real one. Note that the complex vector spaces of complex-valued functions which belong to some real vector space of real-valued functions, usually consist of pairs of functions of the real vector space, namely the real and imaginary parts of the complex-valued function. So, in general, we define the COMPLEXIFICATION $E_{\mathbb{C}}$ of a real vector space E as $E_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} E = E \times E$, and write the elements $(x, y) \in E_{\mathbb{C}}$ as $x + i\,y$. The multiplication with $z = a + ib \in \mathbb{C}$ is then defined by $z \cdot (z' \otimes w) := (zz') \otimes w$, i.e. $(a + ib) \cdot (x + iy) := (ax - by) + i(ay + bx)$. Obviously, this makes $E_{\mathbb{C}}$ into a complex vector space and the mappings $\iota : E \to E_{\mathbb{C}}$, $x \mapsto x + i\,0$ as well as $\mathfrak{Re} : E_{\mathbb{C}} \to E$, $(x + iy) \mapsto x$ are $\mathbb{R}$-linear.

The usual sub-basis of seminorms on the real lcs $E \times E$ like $(x, y) \mapsto p(x) + p(y)$, like $(x, y) \mapsto \sqrt{p(x)^2 + p(y)^2}$, or like $(x, y) \mapsto \max\{p(x), p(y)\}$, are not seminorms for the complex vector space. To obtain such we consider the continuous seminorms $p_z(w) := p(\mathfrak{Re}(zw))$ for $|z| = 1$ and seminorms p of E and then define $p_{\mathbb{C}} := \sup\{p_z : |z| = 1\}$. We have that $p_{\mathbb{C}}$ is a well-defined real seminorm on $E_{\mathbb{C}}$, because by the Hölder inequality for $z = a + i\,b$ we have

$$\begin{aligned} p_z(x + iy) = p\Big(\mathfrak{Re}\big((a + ib)(x + iy)\big)\Big) = \\ = p(ax - by) \leqslant |a|p(x) + |b|p(y) \leqslant |z|\sqrt{p(x)^2 + p(y)^2}. \end{aligned}$$

It is even a complex seminorm, because $p_{\mathbb{C}}(z\,w) = p_{\mathbb{C}}(w)$ obviously holds for all $|z| = 1$. Moreover, $\max\{p(x), p(y)\} \leqslant p_{\mathbb{C}}(x + iy) \leqslant p(x) + p(y)$, hence these seminorms generate the topology of the product=coproduct.

Thus we can use as generating seminorms on $E_{\mathbb{C}}$ the family of all $p_{\mathbb{C}}$, where p runs through the continuous seminorms of E.

3.9.4 Proposition. Universality of the complexification.

Complexifying $E \mapsto E_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} E := E \times E$ provides the following isomorphisms for vector spaces E and G over $\mathbb{R}$ as well as F over $\mathbb{C}$:

1. *First universal property:*
   $L_{\mathbb{C}}(E_{\mathbb{C}}, F) \cong L_{\mathbb{R}}(E, F)$, $h \mapsto h \circ \iota$, $\Big(f^{\mathbb{C}} : x + i\,y \mapsto f(x) + i\,f(y)\Big) \leftarrow\!\shortmid f$.

   The real-linear mappings $f : E \to F$ in each complex vector space F correspond in a bijective manner to the complex-linear mappings $f^{\mathbb{C}} : E_{\mathbb{C}} \to F$ by virtue of $f^{\mathbb{C}} \circ \iota = f$.

$$\begin{tikzcd} E_{\mathbb{C}} \arrow[dr, "f^{\mathbb{C}}"'] & & E \arrow[ll, "\iota"'] \arrow[dl, "f"] \\ & F & \end{tikzcd}$$

2. *Second universal property:*
   $L_{\mathbb{C}}(F, E_{\mathbb{C}}) \cong L_{\mathbb{R}}(F, E)$, $h \mapsto \mathfrak{Re} \circ h$, $\Big(f_{\mathbb{C}} : x \mapsto f(x) - if(ix)\Big) \leftarrow\!\shortmid f$.

   The real-linear mappings $f : F \to E$ on each complex vector space F correspond in a bijective manner to the complex-linear mappings $f_{\mathbb{C}} : F \to E_{\mathbb{C}}$ by virtue of $\mathfrak{Re} \circ f_{\mathbb{C}} = f$.

$$\begin{tikzcd} E_{\mathbb{C}} \arrow[rr, "\mathfrak{Re}"] & & E \\ & F \arrow[ul, "f_{\mathbb{C}}"] \arrow[ur, "f"'] & \end{tikzcd}$$

3. $L_{\mathbb{R}}(E, G)_{\mathbb{C}} \cong L_{\mathbb{R}}(E, G_{\mathbb{C}})$, $f + i\,g \mapsto \Big(x \mapsto f(x) + i\,g(x)\Big)$, $(\mathfrak{Re} \circ h, \mathfrak{Im} \circ h) \leftarrow\!\shortmid h$.
4. $L_{\mathbb{R}}(E, G)_{\mathbb{C}} \cong L_{\mathbb{R}}(E_{\mathbb{C}}, G)$,
   $f + i\,g \mapsto \Big(x + i\,y \mapsto f(x) - g(y)\Big)$, $(h \circ \iota, -h \circ I \circ \iota) \leftarrow\!\shortmid h$.

All these isomorphisms are $\mathbb{C}$-linear with respect to the complex structures given on $L_{\mathbb{R}}(F, E)$ by $i \cdot f := f \circ I$ and on $L_{\mathbb{R}}(E, F)$ by $i \cdot f := I \circ f$.
For lcs's all isomorphisms are also homeomorphisms when we provide $E_{\mathbb{C}}$ with the product structure.
If all spaces are Banach spaces, however, only the isomorphisms in 2 and 3 are isometries.

Proof. 1 Obviously, the specified mappings are continuous, linear, and the composition on $L_{\mathbb{R}}(E, F)$ is the identity. Likewise it is so on $L_{\mathbb{C}}(E_{\mathbb{C}}, F)$, because $h(x+i\,y) = h(x) + i\,h(y) = (h \circ \iota)(x) + i\,(h \circ \iota)(y)$.

2 Let $f : F \to E$ be a $\mathbb{R}$-linear mapping. If a $\mathbb{C}$-linear mapping $f_{\mathbb{C}} : F \to E_{\mathbb{C}}$ exists with $\mathfrak{Re} \circ f_{\mathbb{C}} = f$, then $\mathfrak{Im} \circ f_{\mathbb{C}} = -\mathfrak{Re} \circ i \circ f_{\mathbb{C}} = -\mathfrak{Re} \circ f_{\mathbb{C}} \circ i = -f \circ i$ since $\mathfrak{Re}(i(x+iy)) = -\mathfrak{Im}(x+iy)$. So $f_{\mathbb{C}}$ is uniquely defined and given by $f_{\mathbb{C}}(x) = \mathfrak{Re}\, f_{\mathbb{C}}(x) + i\,\mathfrak{Im}\, f_{\mathbb{C}}(x) = f(x) - if(ix)$. In fact, this defines a $\mathbb{C}$-linear mapping $f_{\mathbb{C}}$, because it is obviously $\mathbb{R}$-linear and $f_{\mathbb{C}}(ix) = f(ix) - if(iix) = f(ix) + if(x) = i(f(x) - if(ix)) = i\,f_{\mathbb{C}}(x)$.

That the universal property is also valid for continuous and for bounded linear mappings can be seen as follows:
We have $p \circ \mathfrak{Re} \leqslant p_{\mathbb{C}}$, i.e. $\mathfrak{Re} : E_{\mathbb{C}} \to E$ is continuous, and conversely

$$(p_{\mathbb{C}} \circ f_{\mathbb{C}})(z) = p_{\mathbb{C}}(f(z) - if(iz)) \leqslant \sqrt{p(f(z))^2 + p(f(iz))^2},$$

hence $f_{\mathbb{C}}$ is continuous provided f is so.
The bijection $\mathfrak{Re}_* : L_{\mathbb{C}}(F, E_{\mathbb{C}}) \to L_{\mathbb{R}}(F, E)$ is a topological linear isomorphism because it is continuous and $\mathbb{R}$-linear and its inverse map is given by $f \mapsto f - i \cdot f \cdot i$. It is also $\mathbb{C}$-linear if we consider $L_{\mathbb{R}}(F, E)$ as a complex vector space via $i \cdot f : x \mapsto f(i\,x)$, because $(i \cdot \mathfrak{Re}_*(f))(x) = \mathfrak{Re}_*(f)(i\,x) = \mathfrak{Re}(f(i\,x)) = \mathfrak{Re}(i\,f(x)) = \mathfrak{Re}((i\,f)(x)) = (\mathfrak{Re}_*(i\,f))(x)$.

3 Obviously, the mappings given are continuous linear and inverse to each other.

4 The isomorphism $L_{\mathbb{R}}(E_{\mathbb{C}}, G) \cong L_{\mathbb{R}}(E, G)_{\mathbb{C}}$ of complex lcs's is given by:
$h \mapsto \big(x \mapsto h(x), x \mapsto -h(i\,x)\big)$ with inverse $(f, g) \mapsto \big((x + i\,y) \mapsto (f(x) - g(y))\big)$, because one composition results in $h : (x + iy) \mapsto h(x) + h(i\,y) = h(x + i\,y)$ and the

other in $(f, g) = \big(x \mapsto f(x), x \mapsto -(-g(x))\big)$. The inverse mapping is also complex-linear, because $i\cdot(f, g) = (-g, f)$ is mapped to $(x, y) \mapsto -g(x)-f(y) = f(-y)-g(x)$.

The statement about isometries is shown in $\boxed{3.9.6.2}$ and $\boxed{3.9.6.3}$. □

3.9.5 Corollary. Complexification of spaces of linear mappings.

For real vector space E and G we obtain:

$$\begin{tikzcd}
L_{\mathbb{C}}(G_{\mathbb{C}}, E_{\mathbb{C}}) \arrow[r, equal, "\sim", "\boxed{1}"'] \arrow[d, equal, "\sim"', "\boxed{2}"] & L_{\mathbb{R}}(G, E_{\mathbb{C}}) \arrow[d, equal, "\sim"', "\boxed{3}"] \\
L_{\mathbb{R}}(G_{\mathbb{C}}, E) \arrow[r, equal, "\sim", "\boxed{4}"'] & L_{\mathbb{R}}(G, E)_{\mathbb{C}} \arrow[ul, dotted]
\end{tikzcd}$$

The diagonal isomorphism is given by

$$f + i\,g \mapsto \Big(x + i\,y \mapsto \big(f(x) - g(y)\big) + i\,\big(f(y) + g(x)\big)\Big).$$

For the dual space of any complex vector space F we have:

$$L_{\mathbb{R}}(F, \mathbb{R}) \cong L_{\mathbb{C}}(F, \mathbb{C}).$$

Proof.

$$\begin{aligned}
f + i\,g &\xmapsto{\boxed{4}} \Big(x + i\,y \mapsto \big(f(x) - g(y)\big)\Big) \\
&\xmapsto{\boxed{2}} \Big(x + i\,y \mapsto \big(f(x) - g(y)\big) + i\,\big(f(y) + g(x)\big)\Big) \\
f + i\,g &\xmapsto{\boxed{3}} \Big(x \mapsto f(x) + i\,g(x)\Big) \\
&\xmapsto{\boxed{1}} \Big(x + i\,y \mapsto \big(f(x) - g(y)\big) + i\,\big(f(y) + g(x)\big)\Big) \quad \square
\end{aligned}$$

3.9.6 Remarks. Isometric natural isomorphisms.

1. *The complexification of $\mathbb{R}$ is isometric to $\mathbb{C}$:*
The complex norm $\|x + i\,y\|_{\mathbb{C}}$ to $\|(x, y)\|_\infty$ is, by the Cauchy-Schwarz inequality [**18**, 6.2.1], *given by*

$$\begin{aligned}
\|x + i\,y\|_{\mathbb{C}} &:= \sup\{\|\mathfrak{Re}((a + i\,b)\cdot(x + i\,y))\|_\infty : |a + i\,b| = 1\} = \\
&= \sup\{|a\,x - b\,y| : |a + i\,b| = 1\} = \|(x, y)\|_2.
\end{aligned}$$

2. *The canonical isomorphism $L_{\mathbb{C}}(F, E_{\mathbb{C}}) \cong L_{\mathbb{R}}(F, E)$ of* $\boxed{3.9.4.2}$ *is an isometry for normed spaces: Because for absolutely convex bounded sets $B \subseteq F$ we have*

$$\begin{aligned}
\sup\{p_{\mathbb{C}}(f_{\mathbb{C}}(x)) : x \in B\} &= \sup\{p(\mathfrak{Re}(\lambda\, f_{\mathbb{C}}(x))) : |\lambda| = 1,\ x \in B\} \\
&= \sup\{p(\mathfrak{Re}(f_{\mathbb{C}}(\lambda\,x))) : |\lambda| = 1,\ x \in B\} \\
&= \sup\{p(f(y)) : y = \lambda x \in B\}.
\end{aligned}$$

3. *The canonical isomorphism $B(X,G)_{\mathbb{C}} \cong B(X,G_{\mathbb{C}})$ is an isometry, thus also for C, ℓ^∞, c_0 and $L_{\mathbb{R}}(E,\_)$ (this is 3.9.4.3): Let p be a seminorm on G and $h \in B(X,G)_{\mathbb{C}}$, then*
$$\begin{aligned}\sup\{p_{\mathbb{C}}(h(x)) : x \in X\} &= \sup\Big\{p(\Re e(\lambda\, h(x))) : x \in X,\ |\lambda| = 1\Big\}\\ &= \sup\Big\{\sup\{p(\Re e(\lambda\, h)(x)) : x \in X\} : |\lambda| = 1\Big\}.\end{aligned}$$
4. *The canonical isomorphism $\ell^p(I,G)_{\mathbb{C}} \cong \ell^p(I,G_{\mathbb{C}})$ is not an isometry for $1 \leqslant p < \infty$: We choose $I := 2$ and $G := \mathbb{R}$ and consider $(1,0)+i(0,1) \in \ell^p(2,\mathbb{R})_{\mathbb{C}}$. The norm in $\ell^p(2,\mathbb{C})$ is then $\|(1,i)\|_p = 2^{\frac{1}{p}}$, while the one in $\ell^p(2,\mathbb{R})_{\mathbb{C}}$ is*
$$\begin{aligned}\|(1,0)+i(0,1)\|_{\mathbb{C}} &:= \sup\Big\{\|\Re e(a+i\,b, -b+i\,a)\|_p : |a+i\,b| = 1\Big\}\\ &\leqslant \sup\{\|(a,-b)\|_p : |a+i\,b| = 1\}\\ &= \max\{1, (2\frac{1}{2^{p/2}})^{1/p}\} = \max\{1, 2^{\frac{1}{p}-\frac{1}{2}}\} < 2^{\frac{1}{p}}.\end{aligned}$$
5. *The complexified norm $\|\_\|_{\mathbb{C}}$ of a real Hilbert space $(E,\|\_\|)$ is not a Hilbert space norm: indeed, for $x = (1,0)$ and $y = (0,i)$ in $\ell^2(2,\mathbb{R})_{\mathbb{C}}$, the parallelogram equality does not hold since $\|x\|_{\mathbb{C}} = 1 = \|y\|_{\mathbb{C}}$ but*
$$\|x \pm y\|_{\mathbb{C}} = \sup\Big\{\|\Re e(a+i\,b, \pm(i\,a-b))\|_2 : |a+i\,b| = 1\Big\} = 1.$$
6. *For normed spaces, the canonical isomorphisms $L_{\mathbb{C}}(G_{\mathbb{C}},E_{\mathbb{C}}) \cong L_{\mathbb{R}}(G,E_{\mathbb{C}})$, $L_{\mathbb{R}}(G_{\mathbb{C}},E) \cong L_{\mathbb{R}}(G,E)_{\mathbb{C}}$ and $L_{\mathbb{R}}(G,E)_{\mathbb{C}} \cong L_{\mathbb{C}}(G_{\mathbb{C}},E_{\mathbb{C}})$ are not isometries: It is enough to show this for the middle one because of 3.9.5. Let $E := \mathbb{R}$, $G := \ell^2(2)$, $f := \mathrm{pr}_1$ and $g := \mathrm{pr}_2$. By 3.9.4.4 we have*
$$\begin{aligned}\|f+i\,g\|_{\mathbb{C}} &:= \sup\{\|\Re e((a+i\,b)\,(f+i\,g))\| : |a+ib| = 1\}\\ &= \sup\{|a\,f(x) - b\,g(x)| : \|x\|_2 = 1,\ |a+ib| = 1\}\\ &= \sup\{|a\,x_1 - b\,x_2| : \|(x_1,x_2)\|_2 = 1,\ \|(a,b)\|_2 = 1\} \leqslant 1\\ \|x+i\,y \mapsto f(x)-g(y)\| &:= \sup\{|f(x)-g(y)| : \|x+i\,y\|_{\mathbb{C}} = 1\}\\ &= \sup\{|x_1 - y_2| : \|x+i\,y\|_{\mathbb{C}} = 1\} \geqslant 2,\end{aligned}$$
by 4 provided we choose $x = (1,0)$ and $y = (0,-1)$.
7. *Not every complex lcs is the complexification of a real lcs: In [1], a complex Banach space was constructed that is not $\mathbb{C}$ isomorphic to its complex conjugate $\bar{E}$ (i.e. E with the scalar multiplication $\bullet$ given by $\lambda \bullet x := \bar{\lambda}\cdot x$). So, if $E \cong F \otimes_{\mathbb{R}} \mathbb{C}$ then also $\bar{E} \cong \overline{F \otimes_{\mathbb{R}} \mathbb{C}} \cong F \otimes_{\mathbb{R}} \mathbb{C} \cong E$, where the isomorphism in the middle is given by $x \otimes \lambda \mapsto x \otimes \bar{\lambda}$.*
For vector spaces, on the other hand, this is true, because after choosing a basis, we can interpret them as a complexification of the subspace of real linear combinations.

4. Baire property

In this chapter, we use the Baire property and its generalizations to detect the continuity of certain linear mappings.

4.1 Baire spaces

4.1.1 Measurable sets.

A σ-ALGEBRA $\mathcal{A}$ on a set X is a subset of the power set of X with the following properties:

1. $\varnothing \in \mathcal{A}$;
2. $A \in \mathcal{A} \Rightarrow X \backslash A \in \mathcal{A}$;
3. $\mathcal{F} \subset \mathcal{A}$, countable $\Rightarrow \bigcup \mathcal{F} \in \mathcal{A}$.

The pair $(X, \mathcal{A})$ is then called a MEASURE SPACE.

Furthermore, one still needs a MEASURE μ on $(X, \mathcal{A})$, i.e. a mapping $\mu : \mathcal{A} \to [0, +\infty]$, which is σ-additive, i.e. $\mathcal{F} \subseteq \mathcal{A}$ countable and pairwise disjunct $\Rightarrow \mu(\bigcup \mathcal{F}) = \sum_{A \in \mathcal{F}} \mu(A)$.

Now define the space of the ELEMENTARY FUNCTIONS as the space generated by χ_A with $A \in \mathcal{A}$ and $\mu(A) < \infty$.

A function $f : X \to \mathbb{R}$ is called MEASURABLE if $f^{-1}(U) \in \mathcal{A}$ for all open $U \subseteq \mathbb{R}$. Since each open set $U \subseteq \mathbb{R}$ is a countable union of open intervals, each open interval (a, b) is the intersection of $(-\infty, b) \cap (a, +\infty)$, and $(a, +\infty) = \bigcup_{n \in \mathbb{N}} \mathbb{R} \backslash (-\infty, a + \frac{1}{n})$, it suffices that $f_{<c} \in \mathcal{A}$ for all c. On the other hand, of course, $f^{-1}(A) \in \mathcal{A}$ for every Borel set $A \subseteq \mathbb{R}$ (see 4.1.3).

A function is ELEMENTARY if it is measurable and takes only finitely many values.

4.1.2 Theorem. Pointwise limits of elementary functions.

Each measurable function $f : X \to [0, +\infty]$ is the pointwise limit of a monotonically increasing sequence of elementary functions. If f is bounded, then the convergence is uniformly. The measurable functions are the pointwise limits of sequences of elementary functions. The space of the measurable functions is closed under pointwise limits of sequences . It is a vector space and closed under $\sup$, $\inf$, $\liminf$, $\limsup$ *and composition with continuous (or even Borel-measurable) functions.*

Proof. Let f_n be measurable, and $f := \sup_n f_n$ everywhere finite. Then f is measurable, because $f_{\leqslant c} = \bigcap_n (f_n)_{\leqslant c}$. Furthermore, $\limsup_n f_n = \inf_n \sup_{k \geqslant n} f_k$ and $\liminf_n f_n = \sup_n \inf_{k \geqslant n} f_k$ measurable. So also $\lim_n f_n$ is measurable.

Let now f be measurable. Since $f = f^+ - f^-$ with $f^+ = \max(f, 0) \geqslant 0$ and $f^- = \max(-f, 0) \geqslant 0$, we may assume that $f \geqslant 0$. Then

$$f_n := \begin{cases} \frac{k}{n} & \text{if } \frac{k}{n} \leqslant f(x) < \frac{k+1}{n} \text{ with } k < n^2 \\ n & \text{if } f(x) \geqslant n. \end{cases}$$

an elementary function (Attention: $\mu(f^{-1}(a)) \nless \infty$). And $(f_n)_n$ converges pointwise from below towards f. $\square$

4.1.3 Definition. Borel and Baire σ-algebra.

Let X be a topological space. The σ-algebra generated by the open (or equivalent closed) sets is called Borel σ-algebra in the extended sense. The σ-algebra generated by the compact sets is called Borel σ-algebra.

The Borel sets are exactly those Borel sets in the extended sense, which are contained in a countable union compact sets. I.e. for σ-compact spaces the Borel sets coincide with the Borel sets in the extended sense.

By the Baire σ-algebra we mean the smallest σ-algebra, s.t. all continuous real-valued functions are measurable, i.e. is generated by the inverse images $f^{-1}(U)$ of the open sets $U \subseteq \mathbb{R}$ under all $f \in C(X, \mathbb{R})$. The Baire sets are the elements of Baire σ-algebra.

A function is called Baire-measurable (or Baire for short) if it is measurable with respect to Baire σ-algebra.

A Borel measure is a measure on the σ-algebra of the Borel sets, which is finite on the compact sets.
A Baire measure is a measure on the σ-algebra of the Baire sets, which is finite on the compact Baire sets.

4.1.4 Theorem. Baire σ-algebra.

Let X be a locally compact σ-compact space. Then the Baire σ-algebra is generated by the compact G_δ-sets.
If X is in addition metrizable, the Borel and Baire sets are the same.
The Baire-measurable functions are the elements of the sequential closure of the set of continuous functions (with compact support) with respect to pointwise convergence.

A G_δ-set is a subset that is a countable intersection of open sets.

Proof. (compact-$G_\delta \subseteq$ Baire sets) Let K be a compact G_δ set, so $K = \bigcap_n U_n$ with open U_n. By the Lemma of Urysohn (see [**26**, 1.3]), there are continuous functions $f_n : X \to [0, 1]$ with $f_n|_K = 1$ and $f_n|_{X \setminus U_n} = 0$. The sequence $g_n := \min\{f_1, \dots, f_n\} \in C(X, [0, 1])$ converges then pointwise and monotonously decreasing towards χ_K, because for each $x \notin K$ there exists an n with $x \notin U_n$, i.e. $f_n(x) = 0$. Thus, χ_K is a Baire-measurable function by $\boxed{4.1.2}$, and $K := \chi_K^{-1}(1)$ is a Baire set.

(Baire sets $\subseteq \langle \text{kp-}G_\delta \rangle_{\sigma\text{-algebra}}$) Since the Baire σ-algebra is generated by the inverse images of the $[c, +\infty)$ intervals with respect to all continuous functions (see $\boxed{4.1.1}$), we only need to show that $f^{-1}[c, +\infty)$ belongs to the σ algebra generated by the compact G_δ sets . These inverse images are clearly closed G_δ. Since X was assumed to be σ-compact, compact sets exist K_n with $X = \bigcup_n K_n$. Because local compactness and Urysohn's lemma (see [**26**, 1.3.1]), we find $g_n \in C_c(X, [0, 1])$ with $g_n|_{K_n} = 1$. Thus, however, $f^{-1}[c, +\infty) = \bigcup_n f_{\geqslant c} \cap (g_n)_{\geqslant 1}$ and $f_{\geqslant c} \cap (g_n)_{\geqslant 1} = (h_n)_{\geqslant 0}$ is a compact G_δ set, where $h_n := \min\{f - c, g_n - 1\}$.

($\overline{C_c}^{\text{Flg}} \subseteq$ Baire functions) The subset of Baire-measurable functions is sequencially closed with respect to pointwise convergence according to 4.1.2. Thus, the sequential closure of the continuous functions (with compact support) is included in the Baire-measurable functions.

($\overline{C_c}^{\text{Flg}} \supseteq$ Baire functions) let us now consider those sets A, for which the characteristic function χ_A lies in the sequential closure of the continuous functions with compact support. These form a σ-algebra $\mathcal{A}$, as the pointwise limit of χ_{A_n} is again in the sequential closure. The compact G_δ sets K are included in $\mathcal{A}$ because by the first part of the proof χ_K is the pointwise limit of a sequence of continuous functions (with compact support). Thus the Baire σ-algebra is included in $\mathcal{A}$, and hence the elementary Baire functions are in the sequential closure of the continuous functions (with compact support). But since every measurable function is the pointwise limit of a sequence of elementary functions (see 4.1.2), the same holds for all Baire-measurable functions.

If X is metrizable, then each closed set A is a G_δ set, because $A = \bigcap_n U_n$, where $U_n := \left\{x : \sup\{d(x,a) : a \in A\} < \frac{1}{n}\right\}$. □

4.1.5 Definition. Meager and nowhere dense sets.

A subset $M \subseteq X$ of a topological space X is called NOWHERE DENSE if no point in X has a neighborhood U in which M is dense (i.e. $U \subseteq \overline{M}$), in short, when the interior of the closure of M is empty, see [26, 3.2.1].

A subset is called MEAGER if it is a countable union nowhere dense sets. This is exactly the case if it is contained in the countable union of closed sets with empty interior, see [26, 3.2.1].

Proof. ($\Rightarrow$) Let $M = \bigcup_n N_n$, then $M \subseteq \bigcup_n \overline{N_n}$.

($\Leftarrow$) Let $M \subseteq \bigcup_n A_n$, then $M = \bigcup_n (M \cap A_n)$ and $\overline{M \cap A_n} \subseteq \overline{A_n}$. □

Warning: Meager is not a property of the topological space M but depends essentially on the surrounding space X: For example, $\{0\}$ is nowhere dense in $\mathbb{R}$, but of course no meager in itself. However:

4.1.6 Lemma. Meager in subspaces.

If M is nowhere dense or meager in X then the same is true in each space Y which contains X as topological subspace.

Proof. Let M be nowhere dense in X. Suppose M is not nowhere dense in Y, i.e. there exists an open set $U \neq \emptyset$ in Y with $U \subseteq \overline{M}^Y$. Then $U \cap X \subseteq \overline{M}^Y \cap X = \overline{M}^X$ and since $U \cap X$ is open in X and M is nowhere dense in X we have $U \cap X = \emptyset$. However, since M is dense in $\overline{M}^Y$, its intersection with the non-empty open set $U \subseteq \overline{M}^Y$ is not empty, a contradiction.

The statement for meager sets obviously follows. □

4.1.7 Theorem of Osgood.

Any set of real-valued continuous functions which is pointwise bounded on a non-meager set X is uniformly bounded on an open non-empty subset.

See [26, 3.2.2]

Proof. Let $\mathcal{F}$ set the set of real-valued continuous functions on X. Let

$$A_{f,k} := \{x \in X : |f(x)| \leq k\}.$$

Then $A_{f,k}$ is closed, and therefore also the set $A_k := \bigcap_{f\in\mathcal{F}} A_{f,k}$ of points on which the f's are uniformly bounded by k. By assumption, X is not meager and clearly $X = \{x : \sup\{|f(x)| : f \in \mathcal{F}\} < \infty\} = \bigcup_{k\in\mathbb{N}} A_k$, therefore there is an $k \in \mathbb{N}$ and an open non-empty set U with $U \subseteq A_k$, i.e. $\mathcal{F}$ is uniformly bounded by k on U. □

4.1.8 Theorem of Baire.

If a sequence of continuous real-valued functions converges on a topological space X pointwise, then the set of points where the limit function is discontinuous is meager.
.

See [26, 3.2.3]

Proof. Let a sequence of continuous functions $f_n \in C(X,\mathbb{R})$ converge pointwise towards a function $f : X \to \mathbb{R}$.

Let $A_{k,\varepsilon} := \{x \in X : |f(x) - f_k(x)| \leqslant \varepsilon\}$ and $A_\varepsilon := \bigcup_k (A_{k,\varepsilon})^o$ be the set of those points where f is locally approximated by a f_k up to ε. Then both $A_{k,\varepsilon}$ and A_ε are increasing in ε.

We claim that f is continuous in every point from $\bigcap_{\varepsilon>0} A_\varepsilon$ (and even equality holds). If $a \in \bigcap_{\varepsilon>0} A_\varepsilon$, then $a \in A_\varepsilon$ is for each $\varepsilon > 0$, and thus for each $\varepsilon > 0$ there is an $k \in \mathbb{N}$ with $a \in (A_{k,\varepsilon})^0$, i.e. there is a neighborhood $U(a)$ with $|f(x) - f_k(x)| \leqslant \varepsilon$ for all $x \in U(a)$s. Since f_k is continuous we can choose $U(a)$ so small that $|f_k(x) - f_k(a)| \leqslant \varepsilon$ for all $x \in U(a)$. Thus, $|f(x) - f(a)| \leqslant |f(x) - f_k(x)| + |f_k(x) - f_k(a)| + |f_k(a) - f(a)| \leqslant 3\varepsilon$ holds for all $x \in U(a)$, i.e. f is continuous at a.

So it remains to show that $X\backslash\bigcap_{\varepsilon>0} A_\varepsilon$ is meager. Let $F_{k,\varepsilon} := \{x \in X : \forall n : |f_k(x) - f_{k+n}(x)| \leqslant \varepsilon\}$. Then $F_{k,\varepsilon}$ is closed, since the f_i are continuous, and $X = \bigcup_{k\in\mathbb{N}} F_{k,\varepsilon}$, because the sequence of the f_i converges pointwise. Furthermore, $F_{k,\varepsilon} \subseteq A_{k,\varepsilon}$ because f_i converges pointwise towards f. So also the interior of $F_{k,\varepsilon}$ is included in that of $A_{k,\varepsilon}$, and therefore: $\bigcup_k (F_{k,\varepsilon})^o \subseteq \bigcup_k (A_{k,\varepsilon})^o = A_\varepsilon$. For each closed set A, $A\backslash A^o$ is closed and nowhere dense, so

$$X\backslash A_\varepsilon \subseteq X\backslash\bigcup_k (F_{k,\varepsilon})^o = \bigcup_l (F_{l,\varepsilon}\backslash\bigcup_k (F_{k,\varepsilon})^o) =$$

$$= \bigcup_l \bigcap_k (F_{l,\varepsilon}\backslash(F_{k,\varepsilon})^o) \subseteq \bigcup_{l=k} (F_{k,\varepsilon}\backslash(F_{k,\varepsilon})^o)$$

is meager, and so is $\bigcup_{n\in\mathbb{N}} (X\backslash A_{1/n}) = X\backslash\bigcap_{\varepsilon>0} A_\varepsilon$. □

4.1.9 Definition. Baire spaces.

A topological space X is called Baire if one of the following equivalent conditions holds (see [26, 3.2.3]):

1. Complements of meager subsets are dense,
   i.e. M meager in $X \Rightarrow \overline{X\backslash M} = X$ (or $M^o = \emptyset$),
2. A_n closed, $A_n^o = \emptyset \Rightarrow (\bigcup_{n\in\mathbb{N}} A_n)^o = \emptyset$;
3. O_n open, $\overline{O_n} = X \Rightarrow \overline{(\bigcap_{n\in\mathbb{N}} O_n)} = X$.

Proof. ($\boxed{1} \Rightarrow \boxed{2}$) A_n closed, $A_n^o = \emptyset \Rightarrow M := \bigcup_n A_n$ meager $\Rightarrow M^o = \emptyset$.

($\boxed{2} \Leftrightarrow \boxed{3}$) A_n open, $A_n^o = \emptyset \Leftrightarrow O_n := X\backslash A_n$ open, $\overline{O_n} = X$. And $(\bigcup_{n\in\mathbb{N}} A_n)^o = (\bigcup_{n\in\mathbb{N}} X\backslash O_n)^o = (X\backslash\bigcap_{n\in\mathbb{N}} O_n)^o = X\backslash\overline{\bigcap_n O_n}$.

($\boxed{2} \Rightarrow \boxed{1}$) M meager $\Rightarrow M = \bigcup_n N_n$ with $\overline{N_n}^o = \emptyset$. $A_n := \overline{N_n} \Rightarrow M^o \subseteq (\bigcup_n A_n)^o = \emptyset$. □

4.1.10 Lemma. Baire locally convex spaces.

A locally convex space is Baire if and only if it is not meager in itself.

Proof. ($\Rightarrow$) This direction holds for any topological space $X \neq \emptyset$, because let X be a Baire space which is meager in itself, then the complement $\emptyset = X \setminus X$ would be dense by 4.1.9.1, i.e. $X = \emptyset$.

($\Leftarrow$) So let E be a locally convex space that is not meager in itself. Suppose E is not Baire, i.e. by 4.1.9.2 $\exists A_n$, A_n closed, $A_n^o = \emptyset$ and $\exists x : x \in (\bigcup_n A_n)^o$, i.e. $\bigcup_n A_n$ is a neighborhood of x and thus $U := \bigcup_n (A_n - x) = (\bigcup_n A_n) - x$ is a neighborhood of 0, hence absorbent. This makes

$$E = \bigcup_{k \in \mathbb{N}} k\,U = \bigcup_{k,n} k(A_n - x),$$

meager because of $(A_n - x)^o = A_n^o - x = \emptyset$. □

4.1.11 Baire-Hausdorff Category Theorem.

Every complete metric space is Baire.
Each (locally-)compact topological space is Baire, see [**26**, 3.2.4].

There are Baire metrizable lcs's which are not complete, see [**14**, S.97].

Proof for complete metric spaces. Let M be meager, i.e. contained in $\bigcup_{n=1}^\infty A_n$ for closed sets A_n with empty interiour. By 4.1.9.1 we have to show that the complement $X \setminus M =: M^c$ is dense in X. So let $U_0 := \{x : d(x, x_0) < r_0\}$ be an open neighborhood of some point $x_0 \in X$ with radius $r_0 > 0$. We construct inductively open balls $U_n := \{x : d(x, x_n) < r_n\}$ with center $x_n \in U_{n-1} \setminus A_n$ and radius $0 < r_n < \frac{r_{n-1}}{2}$ such that $\overline{U_n} \subseteq U_{n-1} \setminus A_n$. This is possible, since by assumption A_n^c is dense and U_{n-1} is an open neighborhood of x_{n-1}, hence an x_n exists in $U_{n-1} \cap A_n^c$ and we may choose the radius $0 < r_n < \frac{r_{n-1}}{2}$ such that $\overline{U_n} = \{x : d(x, x_n) \leqslant r_n\}$ is contained in this open set.

The sequence $(x_n)_n$ is Cauchy, since for $k' > k > n$ we have

$$d(x_{k'}, x_k) \leqslant \sum_{j=k+1}^{k'} d(x_j, x_{j-1}) \leqslant \sum_{j=k}^{k'-1} r_j < \sum_{j=k}^{\infty} \frac{r_n}{2^{j-n}} \leqslant r_n.$$

Let $x_\infty := \lim_n x_n$. Since $x_n \in U_{n-1} \subseteq U_m \subseteq \overline{U_m}$ for all $n > m$, and hence $x_\infty \in \overline{U_m} \subseteq U_0 \setminus A_m$ for all $m > 0$, i.e. $x_\infty \in U_0 \cap \bigcap_m A_m^c \subseteq U_0 \cap M^c$. □

4.1.12 Corollary of Weierstrass.

There are continuous functions on $[-1, 1]$ *that are nowhere differentiable.*

See [**26**, 3.2.5]

Proof. We consider $C([-1, 1], \mathbb{R})$ as a subspace of $C(\mathbb{R}, \mathbb{R})$

$$f \mapsto \tilde{f} \left(: x \mapsto \begin{cases} f(-1) & \text{for } x < -1 \\ f(x) & \text{for } |x| \leqslant 1 \\ f(1) & \text{for } x > 1 \end{cases} \right)$$

Let $M_n := \{f \in C([-1, 1], \mathbb{R}) : \exists t \in [-1, 1]\ \forall 0 < |h| \leqslant 1 : |\frac{\tilde{f}(t+h) - \tilde{f}(t)}{h}| \leqslant n\}$. Then M_n is closed in $C([-1, 1], \mathbb{R})$ (because, if $f_k \in M_n$ with $f_k \to f_\infty$, then there are $|t_k| \leqslant 1$ and without loss of generality t_k converging towards t_∞, which guarantees $f_\infty \in M_n$). Furthermore, M_n is nowhere dense, because otherwise M_n contains a neighborhood of a polynomial by the approximation theorem of Weierstrass. That can not be, because there are arbitrarily close curves, with anywhere arbitrarily large increase (add to the polynomial a small sawtooth curve with sufficiently large slope). So $\bigcup_n M_n$ is meager and contains all the continuous functions that are differentiable in at least one point. □

4.1.13 Remark. Consequences for Baire lcs.

The theorem 4.1.8 of Baire garantees in particular for Fréchet spaces E (because of 4.1.11) that for each pointwise convergent sequence of continuous linear functionals $f_n : E \to \mathbb{R}$, the limit function f is a continuous linear functional. In fact, according to the theorem of Baire, f has to be continuous in the points of a dense set, and thus at least in one point. But, as f clearly has to be linear, this garantees the continuity everywhere.

The Theorem 4.1.7 of Osgood gives us in particular for Fréchet spaces E, that every pointwise bounded family $\mathcal{F}$ of continuous linear functionals is $f : E \to \mathbb{R}$ equi-continuous (see 4.2.2) and thus bounded in $L(E, \mathbb{R})$: In fact, according to the theorem of Osgood, there exists a non-empty open set O on which $\mathcal{F}$ is uniformly bounded (by K). Let $\varepsilon > 0$. We choose an $a \in O$, then for all $x \in O - a$ we have

$$\begin{aligned}|f(x)| &\leqslant |f(x+a)| + |f(-a)| \\ &\leqslant \sup\{|f(y)| : y \in O, f \in \mathcal{F}\} + \sup\{|f(-a)| : f \in \mathcal{F}\} \\ &\leqslant K + K_{-a}.\end{aligned}$$

Thus $\mathcal{F}(U) \subseteq [-\varepsilon, \varepsilon]$ for the 0-neighborhood $U := \frac{\varepsilon}{K+K_{-a}}(O - a)$.

Unfortunately, every (strictly) inductive limit of a truely increasing sequence of Fréchet or, in particular, of Banach spaces is not Baire, because the closed steps have empty interior, otherwise they would be absorbent and thus equal to the whole space.

4.2 Uniform boundedness

Consequently, we should generalize these two continuity results from 4.1.13 further. Let $\mathcal{F}$ be a pointwise bounded family of continuous linear mappings $f : E \to F$. We look for conditions such that each such family is equi-continuous, i.e. for each (closed) 0-neighborhood V in F the set

$$U := \Big\{x \in E : f(x) \in V \text{ for all } f \in \mathcal{F}\Big\} = \bigcap_{f \in \mathcal{F}} f^{-1}(V)$$

is a 0-neighborhood in E. This set is itself closed and absolutely convex as intersection of closed absolutely convex sets. And it is absorbent, because for $x \in E$ we have that $\mathcal{F}(x) := \{f(x) : f \in \mathcal{F}\}$ is bounded in F, so there is an $K > 0$, with $\mathcal{F}(x) \subseteq K \cdot V$, and thus $x \in K \cdot U$. Consequently, we define:

4.2.1 Definition. Barreled spaces.

A subset U of an lcs E is called a BARREL (german: Tonne), if it is closed, absolutely convex, and absorbent.

An lcs E is called BARRELED (german: tonneliert) if each barrel is a 0-neighborhood; this is exactly the case if each seminorm with closed unit ball is continuous, because the barrels are exactly the unit balls of such seminorms: Let A be a barrel, then the Minkowski functional p from A to 1.3.6 is a seminorm with $p_{<1} \subseteq A \subseteq p_{\leqslant 1}$. Since A is assumed to be closed $A = p_{\leqslant 1}$: In fact, let $1 = p(x) = \inf\{\lambda > 0 : x \in \lambda A\}$, then

$\lambda_n \searrow 1$ and $a_n \in A$ exist with $x = \lambda_n\, a_n$ and thus $x = \lim_{n\to\infty} \frac{x}{\lambda_n} = \lim_{n\to\infty} a_n \in A$. The converse, that closed unit balls of seminorms are barrels, is obvious.

So we proved the implication $(1 \Rightarrow 3)$ of the following theorem:

4.2.2 Uniform Boundedness Principle.

Let E be a barreled lcs and F an arbitrary lcs. Then for each set $\mathcal{F}$ of continuous linear mappings $f : E \to F$ the following statements are equivalent

1. *$\mathcal{F}$ is pointwise bounded,*
   i.e. for each $x \in E$ the set $\mathcal{F}(x)$ is bounded in F.

$\Leftrightarrow$ 2. *$\mathcal{F}$ is bounded in $L(E, F)$,*
   i.e. for each bounded $B \subseteq E$, the set $\mathcal{F}(B)$ is bounded in F (see 3.1.3).

$\Leftrightarrow$ 3. *$\mathcal{F}$ is equi-continuous,*
   i.e. for each 0-neighborhood V of F there exists a 0-neighborhood U of E with $f(U) \subseteq V$ for all $f \in \mathcal{F}$.

Proof. We have already shown (1 $\Rightarrow$ 3) in 4.1.13, because $\bigcap_{f\in\mathcal{F}} f^{-1}(V)$ is a barrel by 1.
The implications (1 $\Leftarrow$ 2 $\Leftarrow$ 3) hold in general:

(2 $\Leftarrow$ 3) We have to show that $\mathcal{F}(B)$ is bounded in F for each bounded $B \subseteq E$. So let V be a 0-neighborhood. Since $\mathcal{F}$ is equi-continuous, there exists a 0-neighborhood U of E with $f(U) \subseteq V$ for all $f \in \mathcal{F}$. Since B is bounded, a $K > 0$ exists with $B \subseteq K \cdot U$, and thus $\mathcal{F}(B) \subseteq \mathcal{F}(K \cdot U) \subseteq K \cdot V$, i.e. $\mathcal{F}(B)$ is bounded.

(1 $\Leftarrow$ 2) is obvious, since single points are bounded sets. □

4.2.3 The converse implication also holds.

I.e. a space with the equivalence of the properties from 4.2.3 is barreled: Let U be a barrel. Then $\{x' \in E^* : |x'(U)| \leqslant 1\}$ is a pointwise bounded set in E^*. In fact, U is absorbent, and thus is equi-continuous by assumption, i.e. there exists a 0-neighborhood $V \subseteq E$, s.t. $|x'(V)| \leqslant 1$ for all $x' \in E^*$ with $|x'(U)| \leqslant 1$. It would therefore be enough to show that $V \subseteq U$. For this we need the Lemma 5.2.4 of Mazur, which is a corollary of the theorem of Hahn-Banach: If $x \notin U$, a closed absolutely convex set, then there exists a $x' \in E^*$ with $|x'(x)| > 1$ and $|x'(U)| \leqslant 1$.

Those lcs's E, for which the Uniform Boundedness Principle for countable sets $\mathcal{F}$ holds, are called $\aleph_0$-barreled, see [**14**, S.252]. The dual space of each metrizable lcs's has this property, but it is not always barreled.

4.2.4 Lemma. Heritability of barreledness.

Every Baire lcs is barreled.
Barreledness is inherited by final structures and products.

Proof. Let A be a barrel in a Baire lcs E, then $E = \bigcup_{n\in\mathbb{N}} n \cdot A$, and thus there is an $n \in \mathbb{N}$ with $n \cdot A^o = (n \cdot A)^o \neq \emptyset$. So there is an $a \in A^0$. Then $-a \in A^0$ and thus $0 = \frac{1}{2}a - \frac{1}{2}a \in A^0$, i.e. A is a 0-neighborhood.

Let $f_i : E_i \to E$ be a final family and all E_i be barreled. Let $q : E \to \mathbb{R}$ be a seminorm with closed unit ball, then the same holds for $q \circ f_i$, because $(q \circ f_i)_{\leqslant 1} = (f_i)^{-1}(q_{\leqslant 1})$. Thus $q \circ f_i$ is continuous, and so is q.

With respect to products see [**14**, S.223]. □

4.2.5 Corollary. Pointwise convergence is not bornological.

The dual space E^* of each barreled lcs E, which has bounded set B contained in no finite dimensional subspace, is not bornological with respect to the topology of pointwise convergence.
For example, this is satisfied for each infinite dimensional Banach space E.

Proof. Let $B \subseteq E$ be bounded. Then the polar $B^o := \{x' \in E^* : \forall x \in B : |x'(x)| \leqslant 1\}$ is an absolutely convex 0-neighborhood in E^* and thus BORNIVOROUS (i.e. absorbs bounded sets) in E^*. Due to the Uniform Boundedness Principles, the bounded sets in E^* are exactly those which are bounded with respect to the topology of pointwise convergence. So if this latter structure were bornological, then B^o would be one of its 0-neighborhoods, i.e. a finite set $A \subseteq E$ would exist with $A^o \subseteq B^o$. According to the bipolar theorem 5.4.7, we would have $B \subseteq (B^o)_o \subseteq (A^o)_o = \langle A \rangle_{\text{closed,abs.conv.}}$, i.e. it would be contained in a finite dimensional subspace, a contradiction to the assumption. □

4.2.6 Banach-Steinhaus Theorem.

The pointwise limit of a sequence of continuous linear mappings from a barreled lcs E to an lcs F is a continuous linear mapping. I.e. for complete F, the space $LC(E,F) := L(E,F) \cap C(E,F)$, of the continuous linear mappings, is sequentially complete with respect to pointwise convergence (but not necessarily complete).

Proof. Let $f_n : E \to F$ be continuous linear mappings, such that f_n converges pointwise towards f. Then f is obviously linear and $\{f_n : n \in \mathbb{N}\}$ is pointwise bounded. So by the Uniform Boundedness Principle 4.2.2 it is equi-continuous, i.e. for each (closed) 0-neighborhood V there exists a 0-neighborhood U with $f_n(U) \subseteq V$ for all n. Then $f(U) \subseteq \overline{V} = V$ also holds, i.e. f is continuous. □

4.2.7 Corollary. Scalarly boundedness.

Every scalarly bounded set is bounded.

A set $B \subset E$ is called SCALARLY BOUNDED if $x'(B) \subseteq \mathbb{K}$ is bounded for all continuous linear functionals $x' \in E^*$.

Proof. Let E be first a normed space, then $\iota : E \to E''$ is an isometry onto the subspace $\iota(E)$ by the theorem of Hahn-Banach (see 5.1.10, compare with the proof of 3.8.4, or directly with 5.1.10). The set $\iota(B)$ is pointwise bounded, because $x'(B)$ is bounded for all $x' \in E'$. Since E' is a Banach space, $\iota(B)$ is bounded in $L(E', \mathbb{K})$ by the Uniform Boundedness Principle 4.2.2, so $B \subseteq E$ is bounded because ι is an isometry.

Now let $B \subset E$ be scalarly bounded in some lcs E. We have to show that $p(B)$ is bounded for each continuous seminorm p of E. Let $N := \ker(p)$. Then $E_p := E/N$ is a normed space, with respect to the seminorm $\tilde{p}$ with $\tilde{p} \circ \pi = p$, where $\pi : E \to E_p$ is the natural quotient mapping. We have that $\pi(B)$ is scalarly bounded in the normed space E_p, because $\tilde{\ell}(\pi(B)) = (\tilde{\ell} \circ \pi)(B)$ is bounded for each continuous linear functional $\tilde{\ell}$ on E_p. So $\pi(B)$ is bounded in the norm by the first part of the proof, i.e. $p(B) = \tilde{p}(\pi(B))$ is bounded. □

4.2.8 Corollary. Separately continuous bilinear mappings.

Let E_1 and E_2 be metrizable lcs's and E_2 be barreled. Then each bilinear separately continuous mapping $f : E_1 \times E_2 \to F$ with values in any lcs F is continuous.

This result also holds for barreled spaces with a countable basis of bornology, see [**14**, S.338].

Proof. Since E_1 and E_2 are metrizable lcs's, it suffices by 2.1.7 to show that f is bounded. So let $B_i \subseteq E_i$ be bounded for $i \in \{1,2\}$. We consider the mapping $\check{f} : E_1 \to L(E_2, F)$, $\check{f}(x_1) : x_2 \mapsto f(x_1, x_2)$. This is well-defined, since $f(x_1, \_)$ is linear and continuous by assumption. It is also linear because $f(\_, x_2)$ is linear. Furthermore, $\check{f}(B_1)$ is pointwise bounded in $L(E_2, F)$ because $\check{f}(B_1)(x_2) = f(B_1 \times \{x_2\})$ for $x_2 \in E_2$. Since E_2 is barreled, $f(B_1 \times B_2) = \check{f}(B_1)(B_2) \subseteq F$ is bounded. □

4.2.9 Discontinuous but separatedly continuous natural bilinear forms.

For any lcs E we consider the obviously bilinear evaluation mapping $\mathrm{ev} : E^* \times E \to \mathbb{K}$, $(x', x) \mapsto x'(x)$. It is bounded, because if $A \subseteq E^*$ and $B \subseteq E$ are both bounded, then $A(B)$ is bounded by the structure of $E^* \subseteq E' = L(E, \mathbb{K})$.
Suppose ev were continuous. Then 0-neighborhoods $V \subseteq E^*$ and $U \subseteq E$ would have to exist with $|x'(x)| \leqslant 1$ for all $x' \in V$ and $x \in U$. Since V as 0-neighborhood is absorbent, there exists a $k > 0$ with $x' \in k \cdot V$ for each $x' \in E^*$, and hence x' is bounded on U by k. Thus U is scalarly bounded and by 4.2.7 even bounded in E, hence E has to be normable by 1.6.2.
Note that for the arguments above it was not essential that we use the usual structure on E^*, but this holds for any topological vector space structure. This indicates that continuity is a too strong condition for nonlinear mappings, because the most natural bilinear mapping is not continuous. Taking this remark into account, a calculus has been developed for mappings between lcs's, see [**27**].

Let's look at the simplest special case of non-normable spaces $E = \mathbb{R}^{(\mathbb{N})} := \coprod_{\mathbb{N}} \mathbb{R}$ or $E = \mathbb{R}^{\mathbb{N}} := \prod_{\mathbb{N}} \mathbb{R}$. Because of the universal property of the final structure, $(\mathbb{R}^{(\mathbb{N})})^* = \mathbb{R}^{\mathbb{N}}$ as vector space, where the action of $x = (x_n)_n \in \mathbb{R}^{\mathbb{N}}$ to $y = (y_n)_n \in \mathbb{R}^{(\mathbb{N})}$ is given by $\mathrm{ev}(x, y) = \sum_n x_n\, y_n$. Since each bounded set in $\mathbb{R}^{(\mathbb{N})}$ is bounded in some finite dimensional $\mathbb{R}^N$, also the topology on $(\mathbb{R}^{(\mathbb{N})})^*$ is just that of $\mathbb{R}^{\mathbb{N}}$.

On the other hand, the dual space of $\mathbb{R}^{\mathbb{N}}$ is just $\mathbb{R}^{(\mathbb{N})}$ with the above evaluation map, because for continuous linear $x' : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ there exists a 0-neighborhood, i.e. an $N \in \mathbb{N}$ and an $\varepsilon > 0$, s.t. $x'(\{x \in \mathbb{R}^{\mathbb{N}} : |x_n| < \varepsilon \text{ for all } n \leqslant N\}) \subseteq [-1, 1]$. Let $p = \mathrm{inkl}^* : \mathbb{R}^{\mathbb{N}} \twoheadrightarrow \mathbb{R}^N$ and $i : \mathbb{R}^N \to \mathbb{R}^{\mathbb{N}}$, $x \mapsto (x, 0)$. Then p and i are continuous and linear and $|x'(k\cdot(x-(i\circ p)(x)))| \leqslant 1$ for all $k > 0$ and thus $x'(x) = x'(i(p(x))) = (i^*(x') \circ p)(x)$, where $i^*(x') = x' \circ i \in (\mathbb{R}^N)' \cong \mathbb{R}^N$, so $(\mathbb{R}^{\mathbb{N}})'$ is identifiable with the union $\bigcup_{N\in\mathbb{N}} \mathbb{R}^N = \mathbb{R}^{(\mathbb{N})}$. This is even a linear homeomorphism: A typical 0-neighborhood in $(\mathbb{R}^{\mathbb{N}})'$ is given by the polar B^o of $B = \{x \in \mathbb{R}^{\mathbb{N}} : |x_i| \leqslant \mu_i\}$ for some sequence $\mu_i > 0$, hence

$$B^o = \Big\{x' \in \mathbb{R}^{(\mathbb{N})} : \Big|\sum_i x'_i \mu_i \frac{x_i}{\mu_i}\Big| \leqslant 1 \;\forall x \in B\Big\} = \Big\{x' \in \mathbb{R}^{(\mathbb{N})} : \underbrace{\sum_i \mu_i |x'_i|}_{=:p(x')} \leqslant 1\Big\},$$

where p is a typical seminorm of $\mathbb{R}^{(\mathbb{N})}$.

The evaluation map is bounded and thus separately continuous (since both factors are bornological): In fact, if $A \subseteq \mathbb{R}^{\mathbb{N}}$ and $B \subseteq \mathbb{R}^{(\mathbb{N})}$ are bounded, then $B \subseteq \mathbb{R}^N$ is bounded for some N and thus the finitely many non-vanishing coordinates of $y \in B$ and the corresponding ones of $x \in A$ are bounded and hence also $\mathrm{ev}(x, y) = \sum_{n=0}^{\infty} x_n\, y_n = \sum_{n=0}^{N} x_n\, y_n$ is bounded.

The evaluation map is however not continuous, because if there were 0-neighborhoods $V \subseteq \mathbb{R}^{\mathbb{N}}$ and $U \subseteq \mathbb{R}^{(\mathbb{N})}$ with $\mathrm{ev}(V \times U) \subseteq [-1, 1]$, then V can only control finitely many coordinates, i.e. there is an n with $\mathbb{R} \cdot e_n \in V$. But since $\varepsilon > 0$ exists with $\varepsilon \cdot e_n \in U$, we would have $1 \geqslant |\,\mathrm{ev}(k\, e_n, \varepsilon\, e_n)| = k \cdot \varepsilon$ for all k, a contradiction.

4.2.10 Counterexample concerning the Uniform Boundedness Theorem.

Let E be the subspace of the finite sequences in the Banach space ℓ^∞, and $f_n : (x_k)_{k=1}^\infty \mapsto \sum_{k \leqslant n} x_k$. Then $\{f_n : n \in \mathbb{N}\} \subseteq L(E, \mathbb{R})$ is pointwise bounded, but not bounded in $L(E, \mathbb{R})$, because $\|f_n\| := \sup\{|\sum_{k \leqslant n} x_k| : (x_k)_{k=1}^\infty \in E \text{ and } \forall k : |x_k| \leqslant 1\} = n$. Thus E is not barreled.

4.2.11 Lemma. Automatic boundedness of adjoint mappings.

Let $T : E \to F$, $S : F' \to E'$ both be linear with $y'(Tx) = S(y')(x)$, then T and S are bounded linear mappings.

Proof. Let $B \subseteq E$ be bounded. Then $y'(TB) = S(y')(B)$ is bounded, i.e. TB is scalarly bounded, thus TB is bounded by the corollary in 4.2.7. Furthermore, if $A \subseteq F'$ is bounded, then $(SA)(B) = A(TB)$ is bounded in $\mathbb{K}$, i.e. SA is bounded in E'. □

4.3 Closed and open mappings

We have seen that by the Banach Steinhaus Theorem 4.2.6 the Baire property has the continuity of certain linear mappings as consequence. We want to work that out even further. Let $f : E \to F$ be a mapping. The GRAPH of f is the set $\mathrm{graph}(f) := \{(x, y) \in E \times F : f(x) = y\}$. The graph is closed if and only if $\mathrm{graph}(f) \ni (x_i, y_i) \to (x_\infty, y_\infty) \Rightarrow (x_\infty, y_\infty) \in \mathrm{graph}(f)$, i.e. the existence of the limits $\lim_i x_i$ and $\lim_i f(x_i)$ implies the equality $f(\lim_i x_i) = \lim_i f(x_i)$. Clearly this condition is formally weaker than the continuity of f, where the existence of the 2nd limit is not presupposed. Nevertheless, we show the converse implication under suitable assumptions:

4.3.1 Closed Graph Theorem.

Let E be a Baire lcs, F a Fréchet space, and $f : E \to F$ a linear mapping whose graph is closed in $E \times F$. Then f is continuous.

Proof. We choose a 0-neighborhood basis $(V_n)_n$ of F consisting of closed and absolutely convex sets with $2\, V_n \subseteq V_{n-1}$ and let $A_n := f^{-1}(V_n)$. For each n we have $E = \bigcup_{k \in \mathbb{N}} k \cdot A_n$. Since E is presumed to be Baire, $\overline{A_n}$ contains a point x such that $x + U_n \subseteq \overline{A_n}$ is for a 0-neighborhood U_n of E. But then $U_n = (x + U_n) - x \subseteq (x + U_n) - (x + U_n) \subseteq 2\,\overline{A_n} \subseteq \overline{A_{n-1}}$ holds.

We claim that $f(U_{n+1}) \subseteq V_{n-1}$ (hence f is continuous). Let $x \in U_{n+1} \subseteq \overline{A_n} \subseteq A_n + U_{n+2}$, i.e. there is an $x_0 \in A_n$ with $x - x_0 \in U_{n+2}$, and recursively we find $x_k \in A_{n+k}$ with $x - \sum_{i=0}^k x_i \in U_{n+2+k}$. Then $\sum_k f(x_k)$ satisfies the Cauchy condition, because $\sum_{i=k}^{k+p} f(x_i) \in \sum_{i=k}^{k+p} V_{n+i} \subseteq \sum_{j=0}^p 2^{-j} V_{n+k} \subseteq V_{n+k-1}$. Since F is complete, $y := \sum_{k=0}^\infty f(x_k)$ exists and is in V_{n-1} because V_{n-1} is closed.

If E is in addition metrizable, we may assume that the U_n form a 0-neighborhood basis of E, thus $\sum_k x_k$ converges to x. The closedness of the graph then yields $f(x) = y \in V_{n-1}$.

In the general case of a Baire space E, we take any two symmetric (closed) 0-neighborhoods U and V in E and F. Since $x - \sum_{i=0}^k x_i \in U_{n+2+k} \subseteq \overline{A_{n+1+k}} \subseteq A_{n+1+k} + U$, there exists an $a_k \in A_{n+1+k}$, with $x - \sum_{i=0}^k x_i \in a_k + U$, i.e. $x - \big(a_k + \sum_{i=0}^k x_i\big) \in U$. Then $f(a_k) \in V_{n+1+k}$ is a 0-sequence, hence $y - f\big(a_k + \sum_{i=0}^k x_i\big) = \big(y - \sum_{i=0}^k f(x_i)\big) - f(a_k) \in V$ for sufficiently large k. Therefore $(x, y) + U \times V$ meets the graph of f at least at the point $a_k + \sum_{i=0}^k x_i$. Since the graph is closed, $f(x) = y$ holds. □

4.3.2 Remark. Webbed spaces.

One can summarize the essential property of sets V_n in F more abstractly. For this one calls a mapping V on the set of finite sequences of natural numbers into the absolutely convex subsets of an lcs's F, a COMPLETING WEB if

1. $V(\emptyset) = F$;
2. For each finite sequence $\mathbf{k} := (k_1, \dots, k_n)$ and each k_{n+1} the inclusion $2\,V(\mathbf{k}, k_{n+1}) \subseteq V(\mathbf{k})$ holds;
3. For each finite sequence $\mathbf{k} := (k_1, \dots, k_n)$ every point in $V(\mathbf{k})$ is absorbed by $\bigcup_{k_{n+1}\in\mathbb{N}} V(\mathbf{k}, k_{n+1})$;
4. And for each infinite sequence $(k_1, k_2, \dots)$ and $x_n \in V(k_1, \dots k_n)$ the series $\sum_n x_n$ converges.

A lcs F is called WEBBED if it has a completing web V.

4.3.3 Lemma. Heritability of webbed spaces.

Every Fréchet space E is webbed.

Sequentially closed subspaces, countable products, separated quotients and countable coproducts of webbed spaces are webbed.

The closed graph theorem also holds for functions from Baire into webbed spaces.

The Fréchet spaces are exactly the Baire webbed lcs's.

Proof. Each Fréchet space E is webbed: To see this we only have to take a 0 neighborhood basis V_n as above and define $V(k_1, \dots, k_n) := V_n$.
For subspaces, the trace is a complete web, and for quotients the image of such is again one (see [14, S.90]).
For the remaining heritabilities see [14, S.91].
The above proof of the closed graph theorem can be transferred directly to webbed spaces F by [6] with the following changes (see [14, S.92]): We inductively choose $k_n \in \mathbb{N}$ so that $V_n := V(k_1, \dots, k_n)$ does not have meager inverse image $A_k := f^{-1}(V_n)$. This is possible because of property 4.3.2.3 of webs. Now, one shows, as in the proof of 4.3.1, the existence of 0-neighborhoods $U_n \subseteq \overline{A_{n-1}}$ with $f(U_n) \subseteq \overline{V_{n-1}}$, showing the continuity of f.
For the last statement, see [14, S.94]. □

4.3.4 Remark.

Usually, the closed graph theorem is formulated more technically by specifying only linear mappings $f : G \to F$ with closed graphs in $E \times F$ defined on a non-meager subspace $G \subseteq E$. However, this version follows immediately from the above, because G is then not even meager in itself by 4.1.6, thus is Baire by 4.1.10 and the graph is then also closed in $G \times F$, so the theorem 4.3.1 applicable, where we need only the weaker assumptions that G is Baire and the graph is closed in $G \times F$.

4.3.5 Open Mapping Theorem.

Let E be webbed, F a Baire lcs and $f : E \to F$ linear and surjective with closed graph. Then f is an open mapping, i.e. the image of each open subset is open.

Proof. If f were bijective, we could use simply apply 4.3.1 to f^{-1}.
In general, we consider the diagram:

$$\begin{tikzcd}
\mathrm{graph}(f) \arrow[r, hook] & E \times F & \\
\mathrm{Ker}(f) \arrow[u, hook] \arrow[r, hook] \arrow[d, equal] & E \arrow[u, hook, "\mathrm{inj}_1"'] \arrow[r, "f"] \arrow[dr, "\pi"'] & F \\
N & & E/N \arrow[u, dotted, two heads, "\tilde f"']
\end{tikzcd}$$

Since f has closed graph, the kernel $N := \mathrm{Ker}(f) = \mathrm{inj}_1^{-1}(\mathrm{graph}\, f)$ of f is closed. Thus, with E also E/N is webbed by 4.3.3. We now consider the bijective mapping $\tilde f : E/N \to F$, $[x] \mapsto f(x)$. If f has closed graph, the same holds for $\tilde f$, because $\pi \times F : E \times F \to (E/N) \times F$ is a quotient map (since open), and $(\pi \times F)^{-1}(\mathrm{graph}\, \tilde f) = \mathrm{graph}\, f$. Thus the inverse map $\tilde f^{-1}$ of $\tilde f$ has closed graph in $F \times (E/N)$, since the reflection $(E/N) \times F \to F \times (E/N)$ is an isomorphism. Consequently, according to the Closed Graph Theorem 4.3.1, the mapping $\tilde f^{-1} : F \to E/N$ is continuous, i.e. $\tilde f$ is open, and thus also $f = \tilde f \circ \pi$ is an open mapping. □

4.3.6 Corollary. Quotient maps of Fréchet spaces.

Let E be a Fréchet space and $f : E \to F$ a continuous linear mapping with non-meager image $f(E)$ in F.
Then $f : E \to F$ is surjective and even a quotient mapping, i.e. $F \cong E/\mathrm{Ker}(f)$.

Proof. In particular, $f(E)$ is not meager in itself by 4.1.6, so it is Baire by 4.1.10 and thus $f : E \to f(E)$ is an open (by 4.3.5) and continuous surjective mapping, hence a quotient map. Thus, $f(E) \cong E/\mathrm{Ker}(f)$ is also a Fréchet space, hence complete and therefore closed in F. If $f(E) \neq F$, then $f(E)$ would be nowhere dense (because 0-neighborhoods are absorbent), a contradiction to the fact that $f(E)$ was assumed to be not meager. □

4.3.7 Corollary. Inverse functions between Fréchet spaces.

The inverse of a bijective continuous linear mapping between Fréchet spaces is continuous. □

We now want to examine continuity of linear mappings with values in spaces smooth functions.

4.3.8 Corollary. Scalar continuity.

Let E be a Baire lcs, F a webbed space and $\mathcal{F}$ a point separating family of continuous linear functionals on F. If $g : E \to F$ is a linear mapping, all of whose compositions $f \circ g : E \to F \to \mathbb{K}$ with $f \in \mathcal{F}$ are continuous, then g is continuous.

Proof. We can use the Closed Graph Theorem 4.3.1 because we only have to show that $g(x) = y$ follows from $x_i \to x$ and $g(x_i) \to y$. Since the $f \in \mathcal{F}$ are continuous, $f(g(x)) = (f \circ g)(\lim_i x_i) = \lim_i (f \circ g)(x_i) = f(\lim_i g(x_i)) = f(y)$ is. And since the $f \in \mathcal{F}$ are point separating, we have $g(x) = y$. □

4.3.9 Examples.

Clearly, the previous corollary also holds if E itself is not necessarily Baire, but carries the final structure of Baire spaces.
In particular, this can be applied for the point evaluations instead of $\mathcal{F}$ on the Fréchet spaces $C^n(U)$, $C^\infty_K(U)$ and $\mathcal{E}$; as well as the strict inductive limits $C_c(X)$, $C^n_c(U)$ and $\mathcal{D}$ of Fréchet spaces instead of E.

This way we easily verify that the mappings from [**18**, 4.9] and [**18**, 4.13.4]

1. T_x, S, $\partial^\alpha : \mathcal{D} \to \mathcal{D}$;
2. $f \cdot (\_) : \mathcal{D} \to \mathcal{D}$ for $f \in \mathcal{E}$;
3. $\varphi \star (\_) : \mathcal{D} \to \mathcal{E}$ for $\varphi \in \mathcal{D}'$ (see [**18**, 4.13.5]);
4. $\varphi \star (\_) : \mathcal{D} \to \mathcal{D}$ for $\varphi \in \mathcal{E}'$

are continuous, and that the initial structure of $C(U)$ and $C^\infty(U)$ on $H(U)$ is identical. In fact,

$$\begin{aligned}
(\mathrm{ev}_x \circ T_y)(f) &= f(x-y) = \mathrm{ev}_{x-y}(f);\\
(\mathrm{ev}_x \circ S)(f) &= f(-x) = \mathrm{ev}_{-x}(f);\\
(\mathrm{ev}_x \circ \partial^\alpha)(f) &= \partial^\alpha f(x);\\
\mathrm{ev}_x(g \cdot f) &= g(x) \cdot f(x) = (g(x)\,\mathrm{ev}_x)(f);\\
\mathrm{ev}_x(\varphi \star f) &= \varphi(T_x(S(f))) = (\varphi \circ T_x \circ S)(f).
\end{aligned}$$

In the case where the target space is $\mathcal{D}$, also the Closed Graph Theorem 4.3.1 for the Fréchet spaces $C^\infty_K(\mathbb{R}^m)$ instead of the webbed space $\mathcal{D}$ can be used, provided we keep track of the support: For example, $\operatorname{Trg}(\varphi \star f) \subseteq \operatorname{Trg}\varphi + \operatorname{Trg} f$ holds.

4.3.10 Remark.

The Closed Graph Theorem 4.3.1 has the Uniform Boundedness Principle 4.2.2 for linear functionals on Baire spaces as easy consequence: Let $\mathcal{F} \subset E^*$ be pointwise bounded. Then the mapping $\iota : E \to B(\mathcal{F}, \mathbb{K})$, $x \mapsto (f \mapsto f(x))$ is a well-defined linear mapping. The composition with $\mathrm{ev}_f : B(\mathcal{F}, \mathbb{K}) \to \mathbb{K}$ is just f, so continuous. Thus it follows that ι is continuous, because $B(\mathcal{F}, \mathbb{K})$ is a Banach space, and thus there exists a 0-neighborhood U with $|\mathcal{F}(U)| = |\iota(U)(\mathcal{F})| \subseteq [0, 1]$.

5. The Theorem of Hahn Banach

This chapter discusses the richness of the space of the continuous linear functionals on locally convex spaces and the geometric separation properties that follow. We will apply this to determine some dual spaces and also to questions of complex analysis.

5.1 Extension theorems

Our first goal is to find as many linear functionals ℓ as possible, which should of course be continuous, i.e. satisfy $|\ell| \leqslant q$ for a (fixed) seminorm q. Absolute values are difficult to evaluate and linear functionals and seminorms are hard to compare. However, we have already introduced a common generalization, namely sublinear functionals in 1.1.1. Thus, we first turn to the inequality $\ell \leqslant q$ for sublinear q.

5.1.1 Lemma. Minimal sublinear functions are linear.

A function on a real vector space E is minimal among the sublinear functions $E \to \mathbb{R}$ if and only if it is linear.

Proof. ($\Leftarrow$) Let $\ell : E \to \mathbb{R}$ be linear and $q : E \to \mathbb{R}$ sublinear and $q \leqslant \ell$. Then:

$$\begin{aligned} 0 = \ell(x) + \ell(-x) \geqslant q(x) + q(-x) \geqslant q(0) = 0 \Rightarrow q(x) = -q(-x) \\ \Rightarrow \ell(x) \geqslant q(x) = -q(-x) \geqslant -\ell(-x) = \ell(x) \Rightarrow q(x) = \ell(x). \end{aligned}$$

($\Rightarrow$) Let $p : E \to \mathbb{R}$ be minimal among the sublinear functions.
Suppose p is not additive, then $a, b \in E$ exist with $p(a+b) < p(a)+p(b)$. We are now trying to find a smaller sublinear function. Obviously, $x \mapsto p(x+a)-p(a)$ is convex and at the point b less than p. In order to obtain $\mathbb{R}^+$-homogeneity we consider $p_a(x) := \inf_{t>0}(p(x + t\,a) - t\,p(a))$. Because of $-p(-x) \leqslant p(x + t\,a) - t\,p(a)$, this definition makes sense. Furthermore, $p(x + t\,a) - t\,p(a) \leqslant p(x)$, i.e. $p_a \leqslant p$ and $p_a(b) \leqslant p(a + b) - p(a) < p(b)$.
The function p_a is $\mathbb{R}^+$-homogeneous, because for $\lambda > 0$ we have:

$$\begin{aligned} p_a(\lambda\,x) &= \inf_{t\geqslant 0}\Big(p(\lambda\,x + t\,a) - t\,p(a)\Big) = \inf_{t\geqslant 0}\Big(p(\lambda\,(x + \tfrac{t}{\lambda}\,a)) - t\,p(a)\Big) \\ &= \inf_{t\geqslant 0}\lambda\,\Big(p(x + \tfrac{t}{\lambda}\,a) - \tfrac{t}{\lambda}\,p(a)\Big) = \lambda\cdot\inf_{s\geqslant 0}\Big(p(x + s\,a) - s\,p(a)\Big) = \lambda\cdot p_a(x). \end{aligned}$$

With $x \mapsto p(x + ta) - p(ta)$ also p_a is convex, a contradiction to minimality.
From the additivity and the $\mathbb{R}^+$-homogeneity follows also the $\mathbb{R}$-linearity, because $p(-x) + p(x) = p(0) = 0$ implies that p is odd. $\square$

5.1.2 Corollary. Existence of linear minorants.

Let $p : E \to \mathbb{R}$ be a sublinear function on a real vector space E. Then there exists a linear $f : E \to \mathbb{R}$ with $f \leqslant p$.

Proof. We apply Zorn's Lemma to the set

$$\mathcal{S} := \{q \leqslant p : q \text{ is sublinear}\}.$$

Let $\mathcal{L}$ be a linearly ordered subset of $\mathcal{S}$. Then $\inf_{q\in\mathcal{L}} q =: q_\infty$ is a lower bound of $\mathcal{L}$: In fact, q_∞ is **well-defined**, otherwise an $x \in E$ would exist with $\mathcal{L}(x)$ unbounded from below. But then there would be $q_n \in \mathcal{L} \subseteq \mathcal{S}$, s.t. $q_n(x) \leqslant -n$ and $q_n \leqslant q_{n-1}$ without loss of generality, consequently,

$$0 = q_n(0) \leqslant q_n(x) + q_n(-x) \leqslant -n + q_0(-x) \Rightarrow \forall n : q_0(-x) \geqslant n,$$

would be a contradiction.
The infimum q_∞ is **sublinear** as infimum of sublinear functions.
So we may apply Zorn's Lemma (or, as google translated it, the lemma of anger) and get a minimal element $q \in \mathcal{S}$, which has to be linear according to the Lemma 5.1.1. □

5.1.3 Theorem of Hahn and Banach.

Let $q : E \to \mathbb{R}$ be a sublinear function on a vector space E over $\mathbb{R}$ and $f : F \to \mathbb{R}$ be a linear function on a subspace F of E such that $f \leqslant q|_F$. Then there is an extension $\tilde f : E \to \mathbb{R}$ (i.e. $\tilde f|_F = f$), which is linear and satisfies $\tilde f \leqslant q$ on E.

Proof. We consider $\tilde q : x \mapsto \inf_{y\in F}(q(x + y) - f(y))$. Similar to the proof of 5.1.1, it follows that $\tilde q$ is well-defined (because $q(x + y) - f(y) \geqslant -q(-x) + q(y) - f(y) \geqslant -q(-x)$), sublinear, and $\tilde q \leqslant q$ (put $y := 0$).
By Corollary 5.1.2 there is a linear $\tilde f : E \to \mathbb{R}$ with $\tilde f \leqslant \tilde q \leqslant q$.
For $x \in F$ we have $\tilde f(x) \leqslant \tilde q(x) \leqslant q(x - x) - f(-x) = f(x)$. Thus $\tilde f|_F = f$, because as linear function $f : F \to \mathbb{R}$ has to be minimal by 5.1.1. □

5.1.4 Corollary.

Let E be a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and F a linear subspace. Let q be a seminorm on E and $f : F \to \mathbb{K}$ a linear function that satisfies $|f| \leqslant q|_F$.
Then there is an extension $\tilde f : E \to \mathbb{K}$ (i.e. $\tilde f|_F = f$), which is linear and satisfies $|\tilde f| \leqslant q$ on E.

Proof. First for $\mathbb{K} = \mathbb{R}$: Let q be a seminorm and $|f| \leqslant q|_F$. By 5.1.3 there is a linear $\tilde f : E \to \mathbb{R}$ with $\tilde f \leqslant q$. But this implies $|\tilde f| \leqslant q$, because $-\tilde f(x) = \tilde f(-x) \leqslant q(-x) = q(x)$.

Now, if the scalar field is $\mathbb{C}$, then consider $f_\mathbb{R} := \mathfrak{Re} f$. We have $f_\mathbb{R} \leqslant |f| \leqslant q|_F$. So, according to what we have shown above, there is a $\mathbb{R}$-linear extension $\widetilde{f_\mathbb{R}} : E \to \mathbb{R}$ with $\widetilde{f_\mathbb{R}} \leqslant q$. Let $\tilde f$ be the $\mathbb{C}$-linear function $x \mapsto \widetilde{f_\mathbb{R}}(x) - i\,\widetilde{f_\mathbb{R}}(i\,x)$ given by the second universal property 3.9.4.2 for the complexification $\mathbb{C}$ of $\mathbb{R}$. Then $\tilde f|_F = f$ and $\mathfrak{Re}(\tilde f) = \widetilde{f_\mathbb{R}} \leqslant q$. For $x \in E$, let $r\,e^{i\vartheta} = \tilde f(x)$ be the polar representation with $r \geqslant 0$. Then $\mathbb{R} \ni |\tilde f(x)| = r = \tilde f(e^{-i\vartheta}x) = \widetilde{f_\mathbb{R}}(e^{-i\vartheta}x) \leqslant q(e^{-i\vartheta}x) = q(x)$. □

5.1.5 Corollary.

Let E be an lcs and F a linear subspace of E. Each continuous linear functional $f : F \to \mathbb{K}$ has a continuous linear extension $\tilde f : E \to \mathbb{K}$.
If E is normed, then there is such an $\tilde f$, which additionally fulfills $\|\tilde f\| = \|f\|$.

For bounded linear functions, this theorem is generally wrong.

Proof. Since f is continuous, $|f|$ is a continuous seminorm on F. By 3.1.4 there is an extension to a continuous seminorm q on E. By 5.1.4 there is an extension of f to a linear functional $\tilde{f}: E \to \mathbb{K}$, which fulfills $|\tilde{f}| \leqslant q$ and is thus continuous.

If, in addition, E is normed. Then we may choose $x \mapsto \|f\| \cdot \|x\|$ for q. So $|\tilde{f}(x)| \leqslant \|f\| \cdot \|x\|$ holds, i.e. $\|f\| = \|\tilde{f}|_F\| \leqslant \|\tilde{f}\| \leqslant \|f\|$. Consequently, the desired equality holds. □

5.1.6 Corollary. Dual vectors.

*Let E be an lcs and $\{x_1, \dots, x_n\}$ linearly independent and $\ell_i \in \mathbb{K}$.
Then there exists an $\ell \in E^*$ with $\ell(x_i) = \ell_i$ for all $i \in \{1, \dots, n\}$.*

Proof. Let F be the linear subspace generated by $\{x_1, \dots, x_n\}$. A unique linear functional can be defined on it by $\ell(x_i) := \ell_i$. This functional is continuous by 3.4.6.3. By 5.1.5, a continuous extension ℓ to E exists, and this has also the desired properties. □

5.1.7 Corollary. Complements of finite dimensional subspaces.

*Every finite dimensional subspace of an lcs has a topological complement.
Compare this with 3.4.6.4 in case of finite codimension.*

Proof. Let F be an n-dimensional subspace of E. We choose a basis $\{e_1, \dots, e_n\}$ of F. By 5.1.6 there exist $\ell_k \in E^*$ with $\ell_k(e_j) = \delta_{k,j}$ for all $k, j \in \{1, \dots, n\}$. Thus $p(x) := \sum_{k=1}^n \ell_k(x)\, e_k$ defines a continuous linear mapping $p: E \to F$ satisfying $p|_F = \mathrm{id}$. This provides a decomposition $E \cong F \oplus \ker p$, where the isomorphism is given by $y + z \leftarrow\!\shortmid (y, z)$ and $x \mapsto (p(x), x - p(x))$. □

5.1.8 Corollary. The functionals are points-separating.

On each lcs, the continuous linear functionals are points-separating.

Moreover, let F be a closed linear subspace in an lcs E and $a \in E \backslash F$. Then there is a $\ell \in E^*$ with $\ell|_F = 0$ and $\ell(a) = 1$.

If E is normed, then $\ell \in E^*$ can be choosen s.t. $\|\ell\| = 1/d(a, F)$.

If q is a seminorm of E with $q|_F = 0$, then $\ell \in E^*$ can be choosen s.t. $|\ell| \leqslant q$ and $\ell(a) = q(a)$ instead of $\ell(a) = 1$.

Proof. We define a functional ℓ on $F_a := \{x + t\,a : x \in F, t \in \mathbb{K}\}$ by $\ell(x + t\,a) := t$, i.e. with $\ell|_F = 0$ and $\ell(a) = 1$. By 3.4.4, $F_a \cong F \times \mathbb{K}$ and therefore ℓ is continuous and linear on F_a, hence by 5.1.5 there is a continuous linear extension $\tilde{\ell}$ to E.

In particular, the continuous linear functionals are point-separating, because for $a_1 \neq a_2$ we have $a := a_1 - a_2 \notin F := \{0\}$, hence they can be separated by an $\ell \in E^*$.

If E is normed, then $\|\ell\| \leqslant 1/d(a, F)$, because $|\ell(x + ta)| \cdot d(a, F) \leqslant |t| \cdot \|a - (-\frac{x}{t})\| = \|x + ta\|$. Even equality holds, because there are $x_n \in F$ with $\|a - x_n\| \to d(a, F)$, and thus $1 = \ell(a - x_n) \leqslant \|\ell\| \cdot \|a - x_n\| \to \|\ell\| \cdot d(a, F) \leqslant 1$. By 5.1.5 the extension $\tilde{\ell}$ can be choosen s.t. $\|\tilde{\ell}\| = \|\ell\| \leqslant \frac{1}{d(a,F)}$).

Finally let q be a seminorm of E with $q|_F = 0$, then we define $\ell : F_a \to \mathbb{K}$ by $\ell(x + t\,a) := t\,q(a)$, so $\ell(a) = q(a)$ and $|\ell| \leqslant q$, because $|\ell(x + t\,a)| = |t|\,q(a) = q(t\,a) = q(x + t\,a)$. Thus, we can choose the extension $\tilde{\ell}$ by 5.1.4 so that $|\tilde{\ell}| \leqslant q$. □

5.1.9 Corollary. The closure as intersection of kernels.

If E is an lcs and F is a linear subspace, then the closure of F is given by

$$\overline{F} = \bigcap\{\ker \ell : \ell \in E^*, \ell|_F = 0\}.$$

See 5.2.3 for a generalization.

Proof.
($\subseteq$) Obviously, $\overline{F} \subseteq \ker \ell$ for all continuous linear functional $\ell \in E^*$ with $\ell|_F = 0$.

($\supseteq$) Conversely, if $a \notin \overline{F}$, then there is a continuous linear functional $\ell : E \to \mathbb{K}$ with $\ell(a) = 1$ and $\ell(F) = 0$ by 5.1.8. Consequently, $a \notin \bigcap\{\ker \ell : \ell \in E^*, \ell|_F = 0\}$. □

5.1.10 Corollary. Isometric embedding in the bidual.

Let E be normed and $x \in E$, then $\|x\| = \max\big\{|\ell(x)| : \ell \in E^*, \|\ell\| = 1\big\} = \|\delta(x)\|$, i.e. $\delta : E \to E^{**}$ is an isometry.

Proof. $\|\delta(x)\| = \sup\{\underbrace{|\delta(x)(\ell)|}_{|\ell(x)|} : \ell \in E^*, \|\ell\| = 1\}$

($\geqslant$) is valid because $|\ell(x)| \leqslant \|\ell\| \cdot \|x\|$.
($\leqslant$) holds, because by 5.1.8 an $\ell \in E^*$ exists with $\|\ell\| = 1/d(x,0) = 1/\|x\|$ and $\ell(x) = 1$. We replace this ℓ with $\|x\| \cdot \ell$ and thus get $\|\ell\| = 1$ and $\ell(x) = \|x\|$. □

5.1.11 Corollary. The operator norm of the adjoint.

Let $T : E \to F$ be bounded and linear between normed spaces. Then $\|T^*\| = \|T\|$.

Proof. We have

$$\begin{aligned}
\|T^*\| &= \sup\{\|T^*(y^*)\| : \|y^*\| = 1\} = \sup\{\sup\{|T^*(y^*)(x)| : \|x\| = 1\} : \|y^*\| = 1\} \\
&= \sup\{\underbrace{|T^*(y^*)(x)|}_{|y^*(T(x))|} : \|x\| = 1, \|y^*\| = 1\} \\
&= \sup\{\sup\{|\delta(T(x))(y^*)| : \|y^*\| = 1\} : \|x\| = 1\} = \sup\{\|\delta(T(x))\| : \|x\| = 1\} \\
&\overset{\boxed{5.1.10}}{=\!=\!=} \sup\{\|T(x)\| : \|x\| = 1\} = \|T\|. \quad \square
\end{aligned}$$

5.1.12 Corollary. Separability of the dual space.

If the dual space of a normed space is separable, then the space itself is separable.

The converse does not hold, as the example $(\ell^1)' = \ell^\infty$ shows, see 5.3.1.

Proof. Let $D^* \subseteq E^*$ be a countable dense subset. For each $x^* \in D^*$ we choose an $x \in E$ with $\|x\| = 1$ and $|x^*(x)| \geqslant \frac{\|x^*\|}{2}$. Let D be the set of these x's for all x^* in D^*. We claim that the linear subspace generated by D is dense. Because of 5.1.9 it suffices to show that every $x^* \in E^*$, which vanishes on D, is already 0. So let x^* be such a functional. Since D^* is dense in E^*, there exists a sequence $x_n^* \in D^*$ with $\|x_n^* - x^*\| \to 0$. Let x_n be the corresponding sequence in D. Then

$$\begin{aligned}
\|x_n^* - x^*\| &= \sup\{|(x_n^* - x^*)(x)| : \|x\| = 1\} \\
&\geqslant |(x_n^* - x^*)(x_n)| = |x_n^*(x_n)| \geqslant \tfrac{1}{2}\|x_n^*\|,
\end{aligned}$$

hence x_n^* converges to 0, i.e. $x^* = 0$. □

5.2 Separation theorems

5.2.1 Separation theorems for convex sets.

Let A and B be disjoint convex not empty subsets of a real lcs E. Then there exists a continuous linear functional $f : E \to \mathbb{R}$ and a $\gamma \in \mathbb{R}$, s.t. for all $a \in A$ and all $b \in B$ the following holds:

1. *If A is open, $f(a) < \gamma \leqslant f(b)$ holds;*
2. *If A and B are open, $f(a) < \gamma < f(b)$ holds;*
3. *If A is closed and B is compact, then $f(a) < \gamma < f(b)$ holds.*

Hence the affine hyperplane $\{x \in E : f(x) = \gamma\}$ separates the two sets, meaning that they are on different sides of it.

Proof. 1 The set $U := A - B \neq \emptyset$ is open, convex, and $0 \notin U$. We choose $u \in U$ and put $V := U - u$ with associated Minkowski functional $q := q_V$ (which is sublinear by 1.3.6). Let further $F := \{t\,u : t \in \mathbb{R}\}$ and $f : F \to \mathbb{R}$ be given by $f(t\,u) := -t$ (well-defined, since $u \neq 0$). Then $f|_U < 0$, because $f(U) \subseteq \mathbb{R}$ is convex, $-1 = f(u) \in f(U)$ and $0 \notin f(U)$. Consequently, $f \leqslant q|_F$ by 1.3.7, because for $v \in F$ with $q(v) < 1$ we have $v \in V = U - u$, hence $0 > f(u + v) = f(v) - 1$, i.e. $f(v) < 1$. By Theorem 5.1.3 of Hahn-Banach there exists an extension to a linear functional on E (which we denote again by f) with $f \leqslant q$. Since $W := V \cap -V$ is a 0-neighborhood, $f(w) \leqslant q(w) \leqslant 1$ and $-f(w) = f(-w) \leqslant q(-w) \leqslant 1$ for all $w \in W$, we deduce that f is continuous. For $x \in U$ we have $x - u \in V \subseteq q_{\leqslant 1}$ and thus $1 \geqslant q(x - u) \geqslant f(x - u) = f(x) + 1$, i.e. $f(x) \leqslant 0$. Thus, $f(a - b) \leqslant 0$, i.e. $f(a) \leqslant \gamma := \inf f(B) \leqslant f(b)$. Now if A is open, then also $f(A)$ and thus $f(a) < \gamma$ for all $a \in A$.

2 If, in addition, B is open, then, by analogous arguments, $f(b) > \gamma$ for all $b \in B$.

3 If A is closed, there is an open absolutely convex 0-neighborhood U_y for each $y \notin A$, so that $A \cap (y + 3\,U_y) = \emptyset$. Since B is compact, there are finitely many $y_i \in B$, so that $B \subseteq \bigcup_i (y_i + U_i)$ with $U_i := U_{y_i}$. Because of $(y_i + 2\,U_i) \cap (A + U_i) = \emptyset$, the two open convex sets $B + U = \bigcup_i y_i + U_i + U \subseteq \bigcup_i y_i + 2U_i$ and $A + U \subseteq A + U_i$ are disjoint, when $U := \bigcap_i U_i$. So the claim follows from (2). □

5.2.2 Corollary. Separation of a point from a convex set.

Let E be an lcs, U a non-empty convex open subset, and F a linear subspace that does not intersect U. Then there is a closed hyperplane $H \supseteq F$, which does not intersect U.

Proof. Let's first assume $\mathbb{K} = \mathbb{R}$. By 5.2.1.1 for $A := U$ and $B := F$ we have the existence of $f \in E^*$ and $\gamma \in \mathbb{R}$ with $f(a) < \gamma \leqslant f(b)$ for all $a \in A$ and $b \in B$. Since $b := 0 \in F$ we have $\gamma \leqslant 0$ and therefore $U \cap \operatorname{Ker}(f) = \emptyset$. Furthermore, $F \subseteq \operatorname{Ker}(f)$, because $f(y) \neq 0$ implies $f(y) < 0$ or $f(y) > 0$ and thus $f(-y) < 0$, but then $f(ty) < \gamma$ for a suitably chosen multiple, thus $ty \notin F$.

Let now $\mathbb{K} = \mathbb{C}$. By the first case, there is an $\mathbb{R}$-linear $f : E \to \mathbb{R}$ with $f(x) < 0$ for $x \in U$ and $f|_F = 0$. Then $\tilde{f} : x \mapsto f(x) - i\,f(i\,x)$ is $\mathbb{C}$-linear, with $0 \notin \tilde{f}(U)$ and $F \subseteq \operatorname{Ker}(\tilde{f})$ (note that $\operatorname{Ker}(\tilde{f}) \subseteq \operatorname{Ker}(f)$). □

5.2.3 Corollary. The closure as intersection of half-spaces.

The closed convex hull of a subset of a real lcs is the intersection of all half-spaces that contain it, cf. 5.1.9.

A HALF-SPACE is a subset of a vector space of the form $\{x : f(x) \leqslant \gamma\}$ with a $f \in E^*$ and $\gamma \in \mathbb{R}$.

Proof. This follows as 5.1.9 using 5.2.1.3 or 5.2.4 instead of 5.1.8:
In fact, half-spaces are obviously closed and convex, so the closed convex hull of A is included in this intersection. Let conversely b be not in the closed convex hull of A. Then by 5.2.1.3 there is a $\gamma \in \mathbb{R}$ and a continuous linear functional $f : E \to \mathbb{R}$ with $f(a) < \gamma < f(b)$ for all $a \in A$. So A is in the half-space $\{x : f(x) \leqslant \gamma\}$ but b, so b is not in the intersection of these. □

Next, a generalization of 5.1.8.

5.2.4 Lemma of Mazur.

Let $A \subseteq E$ be a closed convex subset of an lcs E over $\mathbb{K}$ and $b \in E \setminus A$.

1. *If $\mathbb{K} = \mathbb{R}$ and $0 \in A$, then there is a continuous linear functional $f : E \to \mathbb{K}$ with $f(b) > 1$ and $f(a) \leqslant 1$ for all $a \in A$.*
2. *If A is absolutely convex, then there is a continuous linear functional $f : E \to \mathbb{K}$ with $f(b) > 1$ and $|f(a)| \leqslant 1$ for all $a \in A$.*

Proof. 1 By 5.2.1.3 for the compact set $B := \{b\}$ there is an $f \in E^*$ and a $\gamma \in \mathbb{R}$ with $f(a) < \gamma < f(b)$ for all $a \in A$. Because of $0 \in A$, we have $0 = f(0) < \gamma$ and thus $g := \frac{1}{\gamma} f : E \to \mathbb{R}$ is the desired functional with $g(a) < 1 < g(b)$ for all $a \in A$.

2 If $\mathbb{K} = \mathbb{R}$, then this follows from the first part, because with $a \in A$ also $-a \in A$ and thus $-f(a) = f(-a) \leqslant 1$, altogether $|f(a)| \leqslant 1$.

Let now $\mathbb{K} = \mathbb{C}$. By what we have just shown, there exists a continuous $\mathbb{R}$-linear $f : E \to \mathbb{R}$ with $|f(a)| \leqslant 1 < f(b)$ for all $a \in A$. The 2π-periodic function $t \mapsto f(e^{i\,t} b)$ assumes its maximum at some point τ and there its derivative $f(i\,e^{i\,\tau} b)$ has to vanish. Now let's consider the $\mathbb{C}$-linear continuous functional

$$\tilde{f} : x \mapsto f(e^{i\,\tau} x) - i\, f(ie^{i\tau} x).$$

We have $\tilde{f}(b) = f(e^{i\,\tau} b) - i\,0 \geqslant f(b) > 1$ and for $a \in A$ let $\tilde{f}(a) = r\, e^{i\,\vartheta}$ be the polar representation. Then $0 \leqslant r = |\tilde{f}(a)| = e^{-i\,\vartheta}\,\tilde{f}(a) = \tilde{f}(e^{-i\,\vartheta} a) = f(e^{i\,\tau}\, e^{-i\,\vartheta}\, a) - i\,0 \leqslant 1$ since $e^{i\,(\tau-\vartheta)} a \in A$. □

5.3 Dual spaces of important examples

5.3.1 Lemma. The dual space of ℓ^p.

Let $1 \leqslant p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $(\ell^p)' = \ell^q$. Furthermore, $(c_o)' = \ell^1$. Note in particular that $c_0 \neq \ell^\infty = (\ell^1)' = (c_0)''$.

We will show in 5.5.2 that c_0 can not be a dual space of a Banach space.

Proof. The $\iota : \ell^q \to (\ell^p)'$, given by $x \mapsto (y \mapsto \langle x, y \rangle)$, is a well-defined mapping with $\|\iota(x)\| \leqslant \|x\|$ because of Hölder's inequality.

$$\begin{tikzcd} \ell^q \arrow[rr, hook, "\iota"] \arrow[dr, hook, "\mathrm{incl}"'] & & (\ell^p)' \arrow[dl, "(\mathrm{ev}_{e_k})_k"] \\ & \mathbb{K}^{\mathbb{N}} & \end{tikzcd}$$

We now show the surjectivity: Let $\lambda \in (\ell^p)'$. If an $x \in \ell^q$ exists with $\iota(x) = \lambda$, then we would have $x_k = \iota(x)(e^k) = \lambda(e^k)$. So we define $x_k := \lambda(e^k)$. There are $\lambda_n \in (\ell^p)'$ given by

$$\lambda_n(y) := \lambda(y|_{\{1,\dots,n\}}) = \lambda\Big(\sum_{k\leqslant n} y_k\, e^k\Big) = \sum_{k\leqslant n} y_k\, x_k.$$

Then $\lambda_n \to \lambda$ converges pointwise, since $\sum_k y_k\, e^k \to y$ converges in ℓ^p (or in c_0). So $\lambda \in (\ell^p)^*$ by the Banach-Steinhaus Theorem 4.2.6 and

$$\lambda(y) = \lim_{n\to\infty} \lambda_n(y) = \lim_{n\to\infty} \sum_{k\leqslant n} x_k\, y_k = \sum_{k=0}^{\infty} x_k\, y_k =: \iota(x)(y).$$

Thus, $|\sum_k x_k\, y_k| \leqslant \|\lambda\|\,\|y\|_p$ holds. For fixed n we define $y \in \ell^p$ by $y_k := \bar{x}_k\,|x_k|^{q-2}$ in case $x_k \neq 0$ and $k \leqslant n$, and 0 otherwise. We have $|y_k|^p = |x_k|^q$ and thus

$$\sum_{k\leqslant n} |x_k|^q = \sum_{k\leqslant n} x_k\, y_k = \sum_{k=0}^{\infty} x_k\, y_k \leqslant \|\lambda\|\,\|y\|_p = \|\lambda\|\,\Big(\sum_{k\leqslant n} |x_k|^q\Big)^{1/p}$$

So $\|x\|_q \leqslant \|\lambda\|$ and $x \in \ell^q$. □

5.3.2 Generalization. The dual space of L^p.

For $1 \leqslant p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$: $L^q(X) = (L^p(X))^*$ (For $p = 1$ only if X is σ-finite).

For a proof, see e.g. [**5**, S.381].

5.3.3 Corollary. The dual space of $C([0,1])$.

The continuous functionals on $C([0,1])$ are exactly the Riemann-Stieltjes integrals with functions of bounded variation as integrator.

Recall from analysis that, in analogy to Riemann-sums, the Riemann-Stieltjes sum of a function f with respect to another function g, a decomposition $Z := \{0 = t_1 < \dots < t_n = 1\}$, and an intermediate vector $\xi = \{\xi_1, \dots, \xi_n\}$ with $t_{i-1} < \xi_i < t_i$, are given by

$$R_g(f, Z, \xi) := \sum_{i=1}^{n} f(\xi_i)\cdot(g(t_i) - g(t_{i-1})).$$

The function f is called Riemann-Stieltjes integrable with respect to g with integral $\int_0^1 f\,dg$, if the limit $\int_0^1 f\,dg := \lim_{|Z|\to 0} R_g(f, Z, \xi)$ exists, where $|Z| := \max\{|t_i - t_{i-1}| : 1 \leqslant i \leqslant n\}$.

Proof. It can be easily shown (see [**22**, 6.5.14]) that for continuous f and any function g of bounded variation $V(g)$ (see 1.2.3) the Riemann-Stieltjes integral $\int_0^1 f\,dg$ exists and satisfies $|\int_0^1 f\,dg| \leqslant \|f\|_\infty \cdot V(g)$. Consequently, $g \mapsto (f \mapsto \int_0^1 f\,dg)$ is a bounded linear mapping with norm less than or equal to 1.

Conversely, let now ℓ be a continuous linear functional on $C([0,1])$. We have to find a function g, with $\ell(f) = \int_0^1 f\,dg$ for all continuous f. Note that $\int_0^1 \chi_{[0,s]}\,dg = g(s) - g(0)$. Since the Riemann-Stieltjes integral remains unchanged, if one adds to g a constant, e.g. adding $-g(0)$, we may assume that $g(0) = 0$, and it is suggestive to define g by $g(s) := \ell(\chi_s)$ with $\chi_s := \chi_{[0,s]}$. Unfortunately, this definition does not make sense for the time being because χ_s is not continuous. However, according to Theorem 5.1.5 of Hahn-Banach, we may assume that ℓ has been extended norm preserving to $B([0,1])$.

Claim: g is of bounded variation.
Let $0 = t_0 < \dots < t_n = 1$ be a partition of $[0,1]$, then we define $f_k := e^{-i\varphi_k}$,

where $g(t_k) - g(t_{k-1}) = r_k\, e^{i\varphi_k}$. Finally, f is the step function that has value f_k on $(t_{k-1}, t_k]$, i.e. $f = \sum_{k=1}^n f_k(\chi_{t_k} - \chi_{t_{k-1}})$. Then $f \in B([0,1])$ with $\|f\|_\infty \leqslant 1$ is

$$\|\ell\| \geqslant |\ell(f)| = \Big|\sum_{k=1}^n f_k(g(t_k) - g(t_{k-1}))\Big| = \sum_{k=1}^n \big|g(t_k) - g(t_{k-1})\big|$$

and thus $\|\ell\| \geqslant V(g)$.

Claim: For $f \in C([0,1])$ we have $\ell(f) = \int_0^1 f\, dg$.
Let $Z := \{0 = t_0 < \cdots < t_n = 1\}$ be a partition and $\xi = \{\xi_1, \dots, \xi_n\}$ be an intermediate vector. With $f_Z \in B([0,1])$ we denote $f_Z := \sum_{k=1}^n f(\xi_k)(\chi_{t_k} - \chi_{t_{k-1}})$. Then $f = \lim_{|Z|\to 0} f_Z$ in $B([0,1])$ and because ℓ is continuous we obtain

$$\begin{aligned}\ell(f) &= \ell\Big(\lim_{|Z|\to 0} f_Z\Big) = \lim_{|Z|\to 0} \ell(f_Z) = \lim_{|Z|\to 0} \ell\Big(\sum_{k=1}^n f(\xi_k)\,(\chi_{t_k} - \chi_{t_{k-1}})\Big)\\ &= \lim_{|Z|\to 0} \sum_{k=1}^n f(\xi_k)\,(g(t_k) - g(t_{k-1})) = \int_0^1 f\, dg. \quad \square\end{aligned}$$

The mapping $BV([0,1]) \to C([0,1])'$, however, is not injective, even if one requests $g(0) = 0$, see [2, S.121]: To force injectivity, you can request $g(0) = 0$ and $g(x) = g(x+) := \lim_{t\searrow 0} g(x)$ for all $0 < x < 1$.

5.3.4 Representation Theorem of Riesz. The dual space of $C(K)$.

Let K be a compact space. Then the mapping $\mu \mapsto (f \mapsto \int_K f\, d\mu)$ is an isometric isomorphism from the space of the Baire measures onto $C(K)'$.

Recall 4.1.3.

Without proof. It is easy to see that this mapping is an isometry. Difficult is to show surjectivity, see [14, S.139].

A regular Borel measure μ is a signed measure μ (i.e. a σ-additive mapping) on the Borel set algebra, which is regular, i.e.

$$\begin{aligned}|\mu|(A) &= \sup\{|\mu(K)| : K \subseteq A, K \text{ compact}\}\\ &= \inf\{|\mu(U)| : U \supset A, A \text{ open Borel-measurable}\},\end{aligned}$$

where the (positive) measure $|\mu|$ is defined by

$$|\mu|(A) := \sup\Big\{\sum_n |\mu(A_n)| : A_n \in \mathcal{A}, A = \bigcup_n A_n. A_n \text{ pairwise disjoint}\Big\}$$

The variation norm is defined by $\|\mu\| := |\mu|(X)$.

On compact spaces, the Baire measures are in bijective correspondence to the regular Borel measures, i.e. they can be uniquely extended from the Baire sets (see 4.1.3) to the Borel sets (see 4.1.3).

5.3.5 Corollary. The dual space of $C(X)$.

The dual space of $C(X)$ for completely regular X consists of all the regular Borel measures with support in compact subsets of X.

Proof. For each $\mu \in C(X)^*$ there is a compact $K \subseteq X$ and a $C > 0$ with $|\mu(f)| \leqslant C\,\|f|_K\|_\infty$. Then, μ factors to $\tilde\mu \in C(K)^*$ via $\mathrm{incl}^* : C(X) \to C(K)$ (by virtue of $\tilde\mu(f) := \mu(\tilde f)$, where $\tilde f \in C(X)$ is any continuous extension of $f \in C(K)$), so it is given by 5.3.4 by a regular Borel measure on K. $\square$

5.3.6 Runge's Approximation Theorem.

Let $K \subset \mathbb{C}$ compact and $A \subseteq \mathbb{C}_\infty \backslash K$ a set that meets every connected component of $\mathbb{C}_\infty \backslash K$. If f is holomorphic in a neighborhood of K then there are rational functions with poles in A which converge uniformly on K towards f.

With $\mathbb{C}_\infty$ we denote the Riemann sphere, i.e. the one-point compactification $\mathbb{C} \cup \{\infty\}$ of the plane $\mathbb{C}$, see [**19**, 2.16,2.22]

Proof. We denote with $R_A(K) := \{\frac{p}{q}|_K : p, q \text{ sind polynomials}, q^{-1}(0) \subseteq A\}$ the set of all rational functions on K with poles in A.
Let $E := \{f|_K : f \text{ is holomorphic on a neighborhood of K}\}$ be the subspace of $C(K)$ formed by those functions which possess a holomorphic extension to a neighborhood of K. We have to show that the closure of $R_A(K)$ contains the space E. Because of 5.1.9, it suffices to show that every $\mu \in C(K)^*$ vanishing on $R_A(K)$ vanishes on all E (According to Riesz's representation theorem 5.3.5, such a μ is given by a regular signed Borel measure).

So let $f|_K$ be in E with $f : U \to \mathbb{C}$ holomorphic on an open set U containing the K. According to the Cauchy integral formula (see [**19**, 3.28]) there are finitely many C^1 curves (in fact, line segments) c_k in $U \backslash K$, such that

$$f(z) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{c_k} \frac{f(w)}{w - z}\, dw$$

for all $z \in K$ (see 6.21). So

$$\mu(f) = \sum_{k=1}^{n} \frac{1}{2\pi i} \mu\Big(z \mapsto \int_{c_k} \frac{f(w)}{w - z}\, dw\Big) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{c_k} f(w)\, \underbrace{\mu\Big(z \mapsto \frac{1}{w - z}\Big)}_{=:-\tilde{\mu}(w)}\, dw.$$

5.3.7 Sublemma.

Let $\mu \in C(K, \mathbb{C})^*$ with $K \subseteq \mathbb{C}$ compact. Then a holomorphic function $\tilde{\mu} : \mathbb{C}_\infty \backslash K \to \mathbb{C}$ is given by

$$\tilde{\mu}(w) := \mu\Big(z \mapsto \frac{1}{z - w}\Big)$$

with derivatives

$$\frac{\tilde{\mu}^{(n)}(w)}{n!} = \mu\Big(z \mapsto \frac{1}{(z - w)^{n+1}}\Big) \text{ for } w \in \mathbb{C} \backslash K$$
$$\frac{\tilde{\mu}^{(n)}(\infty)}{n!} = -\mu\Big(z \mapsto z^{n-1}\Big) \text{ for } n > 0$$

Proof. Let the continuous $r : (\mathbb{C} \backslash K) \times K \to \mathbb{C}$ be defined by $(w, z) \mapsto \frac{1}{z-w}$, and thus $\check{r} : w \mapsto (r_w : z \mapsto r(w, z))$ is a continuous mapping $\mathbb{C} \backslash K \to C(K, \mathbb{C})$ (see [**26**, 2.4.5]). Then also $\tilde{\mu} = \mu \circ \check{r}$ is continuous. The mapping $\tilde{\mu} : \mathbb{C} \backslash K \to \mathbb{C}$ is even holomorphic, because

$$\frac{\tilde{\mu}(w') - \tilde{\mu}(w)}{w' - w} = \mu\Big(z \mapsto \frac{1}{(z - w')(z - w)}\Big) \to \mu(r_w^2) \text{ for } w' \to w,$$

so $\tilde{\mu}'(w) = \mu(r_w^2)$. Inductively one shows $\tilde{\mu}^{(n)}(w) = n!\, \mu(r_w^{n+1})$.

Because of $r_w \to 0$ for $w \to \infty$, $\tilde{\mu}$ is extendable continuously to $\mathbb{C}_\infty \backslash K$ by $\tilde{\mu}(\infty) := 0$, and thus, according to Riemann's theorem [**19**, 3.31] on removable singularities, it is holomorphic on $\mathbb{C}_\infty\setminus K$. As Taylor development of $\tilde\mu$ at ∞ - i.e. that of $w \mapsto \tilde\mu(\frac{1}{w})$ at 0 - we obtain:

$$\begin{aligned}\tilde\mu(w) &= \mu\Big(z\mapsto \frac{1}{z-w}\Big) = \frac{1}{w}\,\mu\Big(z\mapsto \Big(1-\frac{z}{w}\Big)^{-1}\Big)\\ &= -\frac{1}{w}\sum_{n=0}^\infty \mu\Big(z\mapsto \Big(\frac{z}{w}\Big)^n\Big) = -\sum_{n=0}^\infty \frac{1}{w^{n+1}}\,\mu(z\mapsto z^n).\end{aligned}$$

Hence we have for the derivative

$$\frac{1}{n!}\,\tilde\mu^{(n)}(\infty) = -\mu\Big(z\mapsto z^{n-1}\Big) \quad \square$$

Now we are able to complete the proof of Runge's Theorem 5.3.6:
Because of $\mu|_{R_A(K)} = 0$, the Taylor development of $\tilde\mu$ is 0 for each $a \in A$, and since $\tilde\mu$ is holomorphic and A meets all the components of $\mathbb{C}_\infty\setminus K$, $\tilde\mu = 0$ on $\mathbb{C}_\infty\setminus K$ and thus $\mu(f) = -\sum_{k=1}^n \frac{1}{2\pi i}\int_{c_k} f(w)\,\tilde\mu(w)\,dw = 0$. $\square$

5.3.8 Corollary. The polynomials lie dense.

If K is compact and $\mathbb{C}\setminus K$ is connected, then each function being holomorphic on a neighborhood of K can be approximated by a sequence of polynomials uniformly on K.

Proof. For $A := \{\infty\}$, the rational function with poles in A are just the polynomials by the fundamental theorem of algebra (see [**19**, 1.8]). $\square$

5.3.9 Theorem. Dual space of $H(U)$.

Let $U \subseteq \mathbb{C}$ be open. The dual space of the Fréchet space $H(U)$ can be identified with $H_0(\mathbb{C}_\infty\setminus U)$, the space of the germs of holomorphic functions f on $\mathbb{C}_\infty\setminus U$ with $f(\infty) = 0$.

A germ of a function on K is an equivalence class of functions locally defined around K, where "equivalent" means that they conicide on a neighborhood of K.

Proof. Let $[g] \in H_0(\mathbb{C}_\infty\setminus U)$, i.e. g is holomorphic on a neighborhood W of the compact set $\mathbb{C}_\infty\setminus U$. Without loss of generality, the boundary of W is parameterized by finite many C^1-curves c_k, see 6.21, and g still holomorphic on it. Then

$$\mu_g(f) := \int_{\partial W} f(z)\,g(z)\,dz = \sum_k \int_{c_k} f(z)\,g(z)\,dz$$

defines a continuous linear functional on $C(U) \supseteq H(U)$. This definition depends only on the germ $[g]$ of g, because if W_1 is a smaller neighborhood of $\mathbb{C}_\infty\setminus U$ with C^1 parameterizable boundary in W, then both g and f are holomorphic on $\overline{W}\setminus W_1$ and thus the integral of $f\cdot g$ over the boundary $\partial(W\setminus\overline{W_1})$ vanishes by the Cauchy Integral Theorem 6.20, but this is just the difference $\int_{\partial W} f\cdot g - \int_{\partial W_1} f\cdot g$.

Conversely, let $\mu \in H(U)^*$ and because of the Theorem of Hahn-Banach, w.l.o.g., $\mu \in C(U,\mathbb{C})^*$. Then the support of μ is a compact subset $K \subseteq U$, i.e. $\mu \in C(K,\mathbb{C})^*$. The mapping $\tilde\mu : \mathbb{C}_\infty\setminus K \to \mathbb{C}$ is holomorphic by the above sublemma 5.3.7 and because of the Cauchy integral formula we have (like in the proof of Runge's theorem 5.3.6)

$$\mu(f) = -\sum_k \frac{1}{2\pi i}\int_{c_k} f(w)\,\tilde\mu(w)\,dw \text{ for } f \in H(U),$$

So μ is given by an "inner product" with $\tilde\mu \in H_0(\mathbb{C}_\infty\setminus K)$. $\square$

5.4 Introduction to duality theory

5.4.1 Definition. Annihilators.

Let E be an lcs and let F be a subspace. With F^o we denote the ANNIHILATOR of F in E^*, i.e. $F^o := \{\ell \in E^* : \ell|_F = 0\}$. If E is a Hilbert space, we can identify E^* with E by [**18**, 6.2.10]. The set F^o then coincides via $\iota : E \to E^*$, $x \mapsto (y \mapsto \langle x, y\rangle)$ with the orthogonal complement $F^\perp$ of F, because

$$x \in F^\perp \Leftrightarrow \forall y \in F : 0 = \langle y, x\rangle = \iota(x)(y) \Leftrightarrow \iota(x)|_F = 0 \Leftrightarrow \iota(x) \in F^o.$$

If G is a subspace of E^*, then we denote with G_o the ANNIHILATOR of G in E, i.e.

$$\begin{aligned} G_o &:= \{x \in E : \ \forall g \in G : 0 = g(x) = \delta(x)(g)\} = \bigcap\{\ker g : g \in G\} \\ &= \{x \in E : \delta(x)|_G = 0\} = \{x : \delta(x) \in G^o\} = \delta^{-1}(G^o), \end{aligned}$$

where now $\delta : F \to F^{**}$ is the canonical injection.

5.4.2 Corollary. The closure as the bi-annihilator.

If E is an lcs and F is a subspace, then its closure is $\overline{F} = (F^o)_o$.

Proof. From 5.1.9 follows:

$$\overline{F} = \bigcap\{\ker \ell : \ell|_F = 0\} = \bigcap\{\ker \ell : \ell \in F^o\} = (F^o)_o. \quad \square$$

5.4.3 Corollary. The kernel of the adjoint.

Let $T : E \to F$ be a continuous linear mapping between lcs's.
Then $(\operatorname{img} T)^o = \ker(T^*)$ holds. Furthermore, $\overline{\operatorname{img} T} = (\ker T^*)_o$.

Proof. The first equation holds since $y' \in (\operatorname{img} T)^o \Leftrightarrow \forall x : 0 = y'(Tx) = T^*(y')(x)$ $\Leftrightarrow T^*(y') = 0$, i.e. $y' \in \ker T^*$.
From 5.4.2 follows $\overline{\operatorname{img} T} = ((\operatorname{img} T)^o)_o = (\ker T^*)_o$. $\square$

5.4.4 Corollary. The dual space of quotients and subspaces.

Let F be a closed linear subspace of an lcs E. Then natural continuous linear bijections $E^*/F^o \to F^*$ and $(E/F)^* \to F^o$ exist. For normed E these are isometries.

Proof. We dualize the sequence $F \xhookrightarrow{\iota} E \xrightarrow{\pi} E/F$ and get:

$$\begin{tikzcd} & & \operatorname{Ker}\iota^* = F^o \arrow[dl, hook', "\text{incl}"'] \\ F^* & E^* \arrow[l, two heads, "\iota^*"'] \arrow[dl, two heads] & (E/F)^* \arrow[l, tail, "\pi^*"] \arrow[u, dotted, "(2)"'] \\ E^*/F^o \arrow[u, dotted, "(1)"] & & \end{tikzcd}$$

Since π is surjective, π^* is injective and by the Extension Theorem 5.1.5 $\iota^* : E^* \to F^*$ is surjective. Because $\operatorname{Ker}\iota^* = F^o$, there exists a uniquely determined continuous linear bijective map $(1) : E^*/F^o \to F^*$ given by $x^* + F^o \mapsto \iota^*(x^*) = x^*|_F$.

Because of $\iota^* \circ \pi^* = (\pi \circ \iota)^* = 0$, there is a unique determined continuous linear mapping $(2) : (E/F)^* \to F^o$ given by $\ell \mapsto \pi^*(\ell) = \ell \circ \pi$. Since π^* is injective, (2) is injective and also surjective, because every $y^* \in F^o \subseteq E^*$ vanishes on F and thus factorizes to an $\ell \in (E/F)^*$ with $y^* = \ell \circ \pi = \pi^*(\ell)$.

If E is now normed, then with π and ι also π^* and ι^* are contractions and thus also the two vertical mappings. For $y^* \in F^*$, there is an $x^* \in E^*$ with $\|x^*\| = \|y^*\|$

and $\iota^*(x^*) = y^*$ by 5.1.5. Thus, $\|x^* + F^o\| \leqslant \|x^*\| = \|y^*\| = \|\iota^*(x^*)\|$, i.e. (1) is an isometry. The same holds for (2) since π^* is an isometry because of $|\ell(x+F)| = |\ell(\pi(x+y))| = \|\pi^*(\ell)(x+y)| \leqslant \|\pi^*(\ell)\|\,\|x+y\|$ for all $y \in F$ and thus $\|\ell\| \leqslant \|\pi^*(\ell)\|$ for $\ell \in (E/F)^*$. □

5.4.5 Definition. Dual pairing.

A DUAL PAIRING is a bilinear mapping $\langle\_,\_\rangle : E \times F \to \mathbb{K}$ on the product of two vector spaces, which is not degenerated, i.e. $\forall x : \langle x, y\rangle = 0$ implies $y = 0$ and similarly for the variables exchanged.

So we may, for example, consider the elements $y \in F$ via $\langle\_, y\rangle$ as linear functionals on E. By the WEAK TOPOLOGY $\sigma(E, F)$ on E we understand the initial topology with respect to all of these functionals $x \mapsto \langle x, y\rangle$ for $y \in F$.

A basis of seminorms is given by the functions $x \mapsto |\langle x, y\rangle|$ with $y \in F$.

We say that a structure of an lcs E is COMPATIBLE with the dual pairing $\langle E, F\rangle$, if F is the space of the continuous linear functionals with respect to this structure, or, more precisely, the natural mapping $F \to E^*$, $y \mapsto \langle\_, y\rangle$, is a well-defined bijection.

The topology $\sigma(E, F)$ is called weak because it is the weakest compatible topology:

5.4.6 Lemma. Compatibility of the weak topology.

Let $\langle E, F\rangle$ be a dual pairing. Then the vector space F is isomorphic to the space E^* of all linear functionals, which are continuous for the weak topology $\sigma(E, F)$ on E. More specific, the natural mapping $\iota : F \to E^*$, $y \mapsto \langle\_, y\rangle$ is a bijection.

Proof. The mapping ι is clearly well-defined, linear and injective because of the non-degeneracy assumption. So all that remains to show is the surjectivity. Let $x^* : E \to \mathbb{K}$ be a linear functional on E which is continuous with respect to $\sigma(E, F)$, i.e. there exist $y_1, \dots, y_n \in F$ with $|x^*(x)| \leqslant p(x) := \max\{|\langle x, y_i\rangle| : i = 1, \dots, n\}$. Let $\ell_i := \iota(y_i)$ and $\ell := (\ell_1, \dots, \ell_n) : E \to \mathbb{K}^n$. Then $\ker(\ell) = \bigcap_{i \leqslant n} \ker \ell_i \subseteq \ker x^*$ and hence x^* factors uniquely as linear functional over $\ell : E \to \ell(E) \subseteq \mathbb{K}^n$. This factorization can be extended from the subspace $\ell(E)$ to a linear functional $\mu : \mathbb{K}^n \to \mathbb{K}$:

$$\begin{tikzcd}
\ker \ell \arrow[r, hook] \arrow[d, hook, "\text{incl}"'] & E \arrow[r, two heads, "\ell"] \arrow[d, equal] & \ell(E) \arrow[r, hook, dotted] \arrow[d, dotted] & \mathbb{K}^n \arrow[d, dotted, "\mu"] \\
\ker x^* \arrow[r, hook] & E \arrow[r, two heads, "x^*"] & x^*(E) \arrow[r, hook] & \mathbb{K}
\end{tikzcd}$$

Such a μ is of the form $\mu(x_1, \dots, x_n) = \sum_{i=1}^n \mu_i\, x_i$ for some scalars $\mu_i \in \mathbb{K}$. So $x^* = \mu \circ \ell = \sum_{i=1}^n \mu_i\, \ell_i = \iota\Big(\sum_{i=1}^n \mu_i\, y_i\Big) \in \iota(F)$. □

5.4.7 Bipolar Theorem.

Let $\langle E, F\rangle$ be a dual pairing, and $A \subseteq E$. Then $(A^o)_o$ is the $\sigma(E, F)$-closure of the absolutely convex hull of A. Where $A^o := \{y \in F : |\langle x, y\rangle| \leqslant 1$ for alle $x \in A\}$ is the POLAR *of A; and analogously $B_o := \{x \in E : |\langle x, y\rangle| \leqslant 1$ for all $y \in B\}$ for $B \subseteq F$.*

Note that the polar A^o defined here agrees for linear subspaces A with the annihilator A^o defined in 5.4.1, because $\forall a \in A : |\langle a, y\rangle| \leqslant 1 \Leftrightarrow \forall a \in A\ \forall t > 0 : t \cdot |\langle a, y\rangle| = |\langle t \cdot a, y\rangle| \leqslant 1$, i.e. $\langle a, y\rangle = 0$.

Proof. ($\supseteq$) Obviously, the polar $(A^o)_o$ is $\sigma(E, F)$-closed, absolutely convex, and contains A.

($\subseteq$) Suppose $x \in E$ is not in the $\sigma(E,F)$-closure of the absolutely convex hull of A. By the Lemma 5.2.4 of Mazur, there is an $y \in F$ with $y(x) > 1$ and $|y(z)| \leqslant 1$ for all z in (the closure of the absolutely convex hull of) A. So $y \in A^o$ and $x \notin (A^o)_o$. □

5.4.8 Lemma.
The closure of convex sets with respect to compatible topologies.

Let $A \subseteq E$ be convex and closed for a structure compatible with the dual pairing $\langle E, F\rangle$. Then A is also closed for any other such structure.

Proof. In case $\mathbb{K} = \mathbb{R}$, we have that A is the intersection of the half-spaces containing A by 5.2.3. Since this only involves the continuous linear functionals, A is closed with respect to any compatible topology.

In case $\mathbb{K} = \mathbb{C}$, the real part of the dual pairing $\langle \_, \_\rangle : E \times F \to \mathbb{C}$ provides a pairing $\langle \_, \_\rangle_{\mathbb{R}} : E \times F \to \mathbb{R}$ as real vector spaces, because $\langle x, y\rangle = \mathfrak{Re}(\langle x, y\rangle) + i\,\mathfrak{Im}(\langle x, y\rangle) = \langle x, y\rangle_{\mathbb{R}} - i\,\mathfrak{Re}(i\langle x, y\rangle) = \langle x, y\rangle_{\mathbb{R}} - i\langle x, i\,y\rangle_{\mathbb{R}}$. A structure on E as complex lcs is compatible with the complex pairing if and only if it is with the real part, because the $\mathbb{C}$-linear mapping $\iota : E \to L_{\mathbb{C}}(F, \mathbb{C})$, $x \mapsto \langle x, \_\rangle$ is surjective by 3.9.4.2 if and only if $\mathfrak{Re} \circ \iota : E \to L_{\mathbb{C}}(F, \mathbb{C}) \xrightarrow{\cong} L_{\mathbb{R}}(F, \mathbb{R})$, $x \mapsto \mathfrak{Re}(\langle x, \_\rangle)$ is so. So everything follows from the real case. □

5.4.9 Theorem of Mackey.

A subset of an lcs E is bounded if and only if it is bounded with respect to some (each) topology τ being compatible with the dual pairing $\mathrm{ev} : E \times E^* \to \mathbb{K}$.

Proof. We have shown in 4.2.7 by means of the Theorem of Hahn-Banach and the Uniform Boundedness Principle for Banach spaces that a set is bounded if and only if it is bounded under all continuous linear functionals. This does not depend on the compatible topology. □

5.4.10 Remark. Topologies of uniform convergence.

Let X be a set, F an lcs and $\mathcal{B}$ a family of subsets of X. By the topology of uniform convergence on the sets $B \in \mathcal{B}$ on the space of all mappings $X \to F$ being bounded on the sets in $\mathcal{B}$, one understands the topology generated by seminorms $f \mapsto \|(p \circ f)|_B\|_\infty$, with B runs through $\mathcal{B}$ and p runs through the seminorms of F.

In particular, if $X = E$ is an lcs over $\mathbb{K}$ and $F = \mathbb{K}$, and $\mathcal{B}$ is a set of bounded sets in E which is closed under homotheties, i.e. $\lambda B \in \mathcal{B}$ with $B \in \mathcal{B}$ and $\lambda > 0$, then the polars $B^o := \{x^* \in E^* : \forall x \in B : |x^*(x)| \leqslant 1\}$ with $B \in \mathcal{B}$ form a 0-neighborhood subbasis of the topology of uniform convergence on the sets $\mathcal{B}$. If, in addition, $\mathcal{B}$ is closed under unions, this is a 0-neighborhood basis.

In 5.1.10 we have shown that the canonical mapping $\delta : E \to E^{**}$ for normed spaces E is an isometric embedding. We now want to examine to what extent this translates to general lcs.

For the usual topology on $L(E^*, \mathbb{K})$ of uniform convergence on bounded subsets $B \subseteq E^*$ the sets B^o form a 0-neighborhood basis. Continuity of δ would mean that $\delta^{-1}(B^o) = B_o$ would have to be a 0-neighborhood and thus $B \subseteq (B_o)^o$ would be equi-continuous. At least for barreled E this is the case because of the Uniform Boundedness Principles 4.2.2.

We now show that, when we use the topology of uniform convergence on each equi-continuous subset $B \subseteq E^*$ on $(E^*)^*$, the mapping $\delta : E \to (E^*)^*$ is always an embedding lcs's.

5.4.11 Corollary. Embedding in the bidual.

The topology on any lcs E is that of uniform convergence on equi-continuous subsets of E^*, i.e. the natural mapping $E \to E^{**}$ is an embedding, provided we supply the target space with the uniform convergence on equi-continuous subsets of E^*.

Note that this natural mapping is not always continuous with respect to the usual topology of uniform convergence on bounded sets, but is obviously bounded.

Proof. Let U be a closed absolutely convex 0-neighborhood in E. By 5.4.8, U is also $\sigma(E, E^*)$-closed, so $(U^o)_o = U$ by the Bipolar Theorem 5.4.7. Since U^o is clearly equi-continuous, $U = (U^o)_o$ is a 0-neighborhood of E with respect to uniform convergence on equi-continuous sets.

Conversely, let $V = A_o = \delta^{-1}(A^o)$ be a typical 0-neighborhood of E for the topology of uniform convergence on equi-continuous sets $A \subseteq E^*$. Then there is a closed absolutely convex 0-neighborhood U in E with $A \subseteq U^o$. Thus, $V = A_o \supseteq (U^o)_o = U$, i.e. V is a 0-neighborhood of E. □

5.4.12 Theorem of Alaoğlu-Bourbaki.

Each equi-continuous subset of E^* is relatively compact with respect to $\sigma(E^*, E)$.

Proof. We have to show this only for polars U^o of 0-neighborhoods U. We consider the dual pairing $\langle E, G\rangle$, where G consists of all linear (not necessarily continuous) functionals. Let us denote the polar with respect to this pairing by $^\bullet$. Then $U^\bullet \subseteq G$ is closed and bounded (since U is absorbent) with respect to $\sigma(G, E)$. The natural mapping $\delta : G \to \prod_E \mathbb{K}$, $y \mapsto (\langle x, y\rangle)_{x\in E}$ is linear, injective, has a closed subspace as image (the pointwise limit of linear mappings is linear) and is initial by definition of the weak topology $\sigma(G, E)$. The image of $U^\bullet$ is therefore compact because of the Theorem of Tychonov (products of compact spaces are compact, see [**26**, 2.1.13]) and thus $U^\bullet$ itself is $\sigma(G, E)$-compact. Because of $E^* \subseteq G$, we have $U^o \subseteq U^\bullet$ and even equality is true, because $y \in U^\bullet$ is continuous ($y^{-1}(\{t : |t| \leqslant \varepsilon\}) \supseteq \varepsilon U$). So U^o is compact with respect to $\sigma(G, E)$. But since $\sigma(G, E)$ induces on E^* the topology $\sigma(E^*, E)$, everything is shown. □

5.4.13 Corollary. Normed spaces as subspaces of $C(K)$.

The closed unit ball K in the dual space E^* of a normed spacees E is $\sigma(E^*, E)$-compact. Thus, E is isometrically isomorphic to a subspace of $C(K)$, with an embedding being given by $\delta : E \to E^{**} \to C(K)$, $x \mapsto (x^* \mapsto x^*(x))$. □

In 7.10, cf. 6.43, we will characterize the Banach algebras of the form $C(K)$ with compact K.

By 3.4.5 the unit ball is compact in the norm topology if and only if E is finite dimensional. Thus for each infinite dimensional normed E the topology $\sigma(E^*, E)$ is strictly coarser than the norm topology.

5.4.14 Definition. Mackey topology.

Let $\langle E, F\rangle$ be a dual pairing. Then the Mackey topology $\mu(E, F)$ on E is the topology of the uniform convergence on the $\sigma(F, E)$-compact, absolutely convex sets in F.

5.4.15 Theorem of Mackey-Arens.

A topology on E is compatible with the dual pairing $\langle E, F\rangle$ if and only if it lies between the weak topology $\sigma(E, F)$ and the Mackey topology $\mu(E, F)$.

Proof. We first show the compatibility of $\mu(E,F)$. Let $\ell : E \to \mathbb{K}$ continuous linear functional with respect to $\mu(E,F)$. So there is a $\sigma(F,E)$-compact absolutely convex set $K \subseteq F$ with $|\ell(K_o)| \leqslant 1$. We consider as in 5.4.12 the dual pairing $\langle E,G\rangle$, where $G \supseteq F$ denotes the space of all linear functionals on E. Since $\sigma(G,E)$ induces on F the topology $\sigma(F,E)$, $K \subseteq F \subseteq G$ is also $\sigma(G,E)$-compact and thus closed. From the bipolar theorem it follows that $K = (K_\bullet)^\bullet$ where $^\bullet$ denotes the polar with respect to $\langle G,E\rangle$. Obviously $K_o = K_\bullet$ and because of $|\ell(K_o)| \leqslant 1$ we have $\ell \in (K_o)^\bullet = (K_\bullet)^\bullet = K \subseteq F$, i.e. the $\mu(E,F)$-dual of E is included in F.
The converse inclusion immediately follows from the fact, that each $y \in F$ is continuous even with respect to $\sigma(E,F)$ and therefore also with respect to $\mu(E,F)$.

Let τ be any compatible topology on E. Since all $y \in F$ are thus continuous functionals with respect to τ, it is finer than the weak topology $\sigma(E,F)$.
On the other hand, let U be a 0-neighborhood in E with respect to τ. Because of 5.4.11, we may assume that $U = K_o$ with $K \subseteq F$ τ-equi-continuous absolutely convex. Because of the Theorem 5.4.12 of Alaoğlu-Bourbaki the set K is $\sigma(F,E)$-compact, and thus $U = K_o$ is a 0-neighborhood with respect to the Mackey topology $\mu(E,F)$. □

5.4.16 Remark. Topologies on the dual space.

For each lcs E we consider the dual pairing $E \times F \to \mathbb{K}$ with $F := E^*$ and the following types of subsets $B \subseteq E^*$ which in addition are assumed to be closed and absolutely convex:

1. The absolutely convex hulls of finite subsets;
2. The equi-continuous ones;
3. The $\sigma(F,E)$-compact ones;
4. The Banach discs;
5. The sets being uniformly bounded on bounded subsets of E, i.e. the bounded sets in $L(E,\mathbb{K})$;
6. The sets being bounded on each point in E, i.e. the $\sigma(F,E)$-bounded ones.

A set $B \subseteq F$ is called Banach disk if it is absolutely convex, $\sigma(F,E)$-bounded and the normed space F_B (see 3.6.2) is complete.

Lemma.

Let $\mathcal{A}$ and $\mathcal{B}$ be two families of bounded subsets of E that are invariant by formation of subsets, absolutely convex hulls, closures, and twofold sums (and thus finite unions and homotheties). Then the induced topologies on F of uniform convergence on these sets are the same if and only if $\mathcal{A} = \mathcal{B}$ holds.

Proof. For $B \in \mathcal{B}$, B^o is a 0-neighborhood of the associated topology, so an $A \in \mathcal{A}$ exists with $A^o \subseteq B^o$ and thus $B \subseteq (B^o)_o \subseteq (A^o)_o = \langle A\rangle_{\text{closed,abs.conv.}} \in \mathcal{A}$, hence $B \in \mathcal{A}$. □

The corresponding topologies on E of uniform convergence on the respective sets in F have as neighborhood basis of 0 just the ($\sigma(E,F)$-closed absolutely convex) polars of the sets listed. So these topologies are

1. The weak topology $\sigma(E,F)$ by definition;
2. The original topology from E to 5.4.11;
3. The Mackey topology $\mu(E,F)$ by definition;
4. This has no common name;
5. The one with the bornivorous (see 4.2.5) barrels as 0-neighborhood basis;

6. The one with the barrels (see 4.2.1) as 0-neighborhood basis.

For the last two topologies, we use the following:
B_o absorbs A $\Leftrightarrow$ $\langle A, B\rangle$ is bounded, i.e. B is uniformly bounded on A:
In fact, $A \subseteq K\, B_o \Leftrightarrow |\langle A, B\rangle| \leqslant K$.
Therefore, the polars of the sets in (6) and (5) are just the barrels, resp. the bornivorous barrels:
The polar B_o of a set B, being bounded on all finite/bounded sets, absorbs all these sets by what we have just shown. Conversely, for each (bornivorous) barrel $A = (A^o)_o$ (by 5.4.7) the polar A^o is bounded on finite (bounded) sets what we have just shown.

We now want to show that the mentioned topologies are successively stronger in the given order, or equivalently that the corresponding inclusions of the underlying families of closed absolutely convex sets hold. For (1) $\Rightarrow$ (2) and (5) $\Rightarrow$ (6) this is trivial, (2) $\Rightarrow$ (3) is the Theorem 5.4.12 of Alaoğlu-Bourbaki. The remaining implications (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are shown in the following two results:

5.4.17 Lemma.

Each $\sigma(E, F)$-compact absolutely convex set is a Banach disk.

Proof. Let $(x_n)_n$ be a Cauchy sequence in E_B. Then $\sup_n p_B(x_n) < \infty$ and thus there is a $K > 0$ with $x_n \in K\, B$ for all n. Since $K\, B$ is also $\sigma(E, F)$-compact, there exists a $\sigma(E, F)$-accumulation point $x_\infty \in K\, B$ of $(x_n)_n$. For $\varepsilon > 0$ we have $p_B(x_m - x_n) < \varepsilon$ for sufficiently large n and m and therefore $x_m \in x_n + \varepsilon\, B$. Because $x_n + \varepsilon\, B$ is also $\sigma(E, F)$-closed and x_∞ is an accumulation point of $(x_m)_m$, we have $x_\infty \in x_n + \varepsilon\, B$, and thus $p_B(x_\infty - x_n) \leqslant \varepsilon$ for these n. So $x_n \to x_\infty$ converges in E_B. $\square$

5.4.18 Banach-Mackey Theorem.

Each barrel absorbs each Banach disk.
Moreover, Banach disks in $F = E^*$ are uniformly bounded on bounded sets in E.

Proof. Let $B \subseteq E$ be a Banach disk, meaning that B is absolutely convex, $\sigma(E, F)$-bounded and the normed space $E_B := \langle B\rangle_{\mathrm{VR}}$, considered with the Minkowski functional $p_B : E_B \to \mathbb{R}$, is complete. Let $\iota : E_B \rightarrowtail E$ be the natural linear inclusion.

Furthermore, let $A \subseteq E$ be a barrel, i.e. absolutely convex, $\sigma(E, F)$-closed and absorbent. Then the Minkowski functional p_A on $\langle A\rangle_{\mathrm{VR}} = E$ is a well-defined seminorm. Let E_A be the quotient space $E/\ker(p_A)$ and $\pi : E \twoheadrightarrow E_A$ the canonical linear surjection. The seminorm p_A factorizes over $\pi : E \twoheadrightarrow E_A$ to a norm $E_A \to \mathbb{R}$ and this we can uniquely extend to the norm $\widetilde{p_A}$ on the completion $\widetilde{E_A}$.

Obviously, $A \subseteq \pi^{-1}(\pi A) \subseteq (p_A)_{\leqslant 1}$. Moreover, equality holds, because $1 \geqslant p_A(x) = \inf\{\lambda > 0 : x \in \lambda\, A\}$ implies the existence of a sequence $\lambda_n \searrow 1$ with $x \in \lambda_n\, A$ and thus $A \ni \frac{1}{\lambda_n} x \to x$. Since A is closed with respect to $\sigma(E, F)$, we finally get $x \in A$.

Let us now show the continuity and thus the boundedness of the composition

$$E_B \xrightarrow{\iota} (E, \sigma(E, F)) \xrightarrow{\pi} E_A \hookrightarrow \widetilde{E_A}.$$

By 4.3.8 it sufficies to find a point-separating family of continuous linear functionals $\tilde{\ell}$ on $\widetilde{E_A}$ for which the composition $\tilde{\ell} \circ \pi \circ i : E_B \to \mathbb{K}$ is continuous.

Each $y \in A^o \subseteq F$ satisfies $\{x \in E : |\langle x, y\rangle| \leqslant 1\} \supseteq A = (p_A)_{\leqslant 1}$ and thus $|x \mapsto \langle x, y\rangle| \leqslant p_A$ for the associated linear functional. Thus this functional factorizes over $\pi : E \twoheadrightarrow E_A$ to a contraction $E_A \to \mathbb{K}$ and thus has a continuous extension $\tilde{y} : \widetilde{E_A} \to \mathbb{K}$. The composition $\tilde{y} \circ \pi \circ \iota = y \circ \iota : E_B \to \mathbb{K}$ is continuous (= bounded) because B is $\sigma(E, F)$-bounded.

Remains to show that these $\tilde{y}$ act point-separating on $\widetilde{E_A}$. Let $0 \neq \tilde{x} \in \widetilde{E_A}$, i.e. $\widetilde{p_A}(\tilde{x}) > 0$. Then there is an $x \in E$ with $\widetilde{p_A}(\tilde{x} - \pi(x)) < \frac{1}{2}\widetilde{p_A}(\tilde{x}) =: \delta > 0$. We therefore have $p_A(x) = \widetilde{p_A}(\pi(x)) > \delta$. By the Lemma 5.2.4.2 of Mazur there is a $y \in A^o \subseteq F$ with $y(\frac{x}{\delta}) > 1$. The associated $\tilde{y} : \widetilde{E_A} \to \mathbb{K}$ thus fulfills $|\tilde{y}| \leqslant \widetilde{p_A}$ and $\tilde{y}(\pi(x)) = y(x) > \delta$. So

$$|\tilde{y}(\tilde{x})| \geqslant |\tilde{y}(\pi(x))| - |\tilde{y}(\tilde{x} - \pi(x))| \geqslant |y(x)| - \widetilde{p_A}(\tilde{x} - \pi(x)) > \delta - \delta = 0$$

The second part of the theorem is shown as follows: Let $B \subseteq F$ be a Banach disk and $C \subseteq E$ bounded. Then C is pointwise bounded on F by 4.2.7 and thus $C^o \subseteq F$ is a barrel by 5.4.16. Because of the first part, a $K > 0$ exists with $B \subseteq K\, C^o$, i.e. B is bounded on C by K. □

5.4.19 Remark.

For $\delta : E \to (E^*)^*$ being continuous with respect to the topology of the uniform convergence on sets $B \subseteq E^*$ we need, by what has been shown in 5.4.10, that the $B_o = \delta^{-1}(B^o)$ are 0-neighborhoods and hence $B \subseteq (B_o)^o$ are equi-continuous.

Moreover, δ is an embedding under this assumption:

5.4.20 Corollary. Barreledness and bidual.

The topology of any lcs E is that of uniform convergence on pointwise bounded sets of E^* if and only if E is barreled.

It is that of uniform convergence on all bounded sets of $E^* \subseteq L(E, \mathbb{K})$, if and only if E is infra-barreled.

In both cases, it is also equal to $\mu(E, E^*)$.

An lcs is called INFRA-BARRELED or also QUASI-BARRELED if every bornivorous barrel is a 0-neighborhood. Note that (by the following lemma) obviously all bornological as well as all barreled lcs's are infra-barreled.

Related to this is also the notion ULTRA-BORNOLOGICAL, i.e. when each absolutely convex set, which absorbs Banach-disks, is a 0-neighborhood. Obviously, ultra-bornological spaces are bornological and according to 5.4.18 they are also barreled.

$$\begin{tikzcd}
 & \text{abs.conv, absorbs Ban.disks} & \\
\text{closed, abs.conv, absorbing} \arrow[ur, hook, "\boxed{5.4.18}"] & & \text{abs.conv, bornivorous} \arrow[ul, hook'] \\
 & \text{closed, abs.conv, bornivorous} \arrow[ul, hook] \arrow[ur, hook'] &
\end{tikzcd}$$

Proof. Because of 5.4.16.5 and 5.4.16.6, the (bornivorous) barrels form a zero neighborhood basis of the said topologies of uniform convergence, which concide with the (weaker) original one of E precisely if those barrels are 0-neighborhoods, i.e. the space is (infra-) barreled.

Since $\mu(E, E^*)$ lies between the topology of E and that of the uniform convergence on the bounded sets by 5.4.16, equality holds in these cases. □

Lemma.

An lcs E is bornological if and only if all bornivorous absolutely convex subsets are 0-neighbpprhoods.

Proof. ($\Leftarrow$) Let $f : E \to F$ be a bounded linear mapping and V be an absolutely convex 0-neighborhood in F. Then $f^{-1}(V)$ is absolutely convex and bornivorous, since for each bounded set B there is some $K > 0$ with $f(B) \subseteq K V$, i.e. $B \subseteq K f^{-1}(V)$. Thus $f^{-1}(V)$ is a 0-neighborhood and hence f is continuous.
($\Rightarrow$) Let U be absolutely convex and bornivorous. Then the linear subspace E_U generated by U is E and we may consider the corresponding Minkowski functional p_U and form the normed space $F := E/\ker(p_U)$ with norm $\tilde{p}_U$. The natural linear map $\pi : E \to F$ is bounded, since for each bounded $B \subseteq E$ there is some $K > 0$ with $B \subseteq K U$ and hence p_U is bounded on B by K, i.e. $\tilde{p}_U(\pi(B))$ is bounded. Since E is assumed to be bornological, the map π is continuous, and hence $U \supseteq (p_U)_{<1} = \pi^{-1}((\tilde{p}_U)_{<1})$ is a 0-neighborhood. □

5.4.21 Definition. Reflexivity.

A lcs E is called (reflexive) semireflexive if the canonical mapping $\iota : E \to E^{**}$ is surjective (is a topological isomorphism).

5.4.22 Proposition. Semireflexivity.

[**14**, S.227] *For lcs's E are equivalent:*

1. *E is semireflexive;*
2. *$(E^*, \mu(E^*, E))$ is barreled;*
3. *Each bounded set is $\sigma(E, E^*)$-relative-compact;*
4. *$(E, \sigma(E, E^*))$ is* quasi-complete*, meaning every bounded and closed subset is complete.*

Proof.

(1⇔2) Since, by 5.4.15, $\mu(E^*, E)$ is the finest topology on E^* with dual space E and the natural topology of uniform convergence on the bounded sets in E is finer ($\sigma(E, E^*)$-compact sets are obviously bounded), E is semireflexive if and only if these two topologies coincide. By 5.4.20 applied to $(E^*, \mu(E^*, E))$, this is exactly the case when $\mu(E^*, E)$ is barreled, because by 4.2.7 the pointwise(=scalarly) bounded sets of $(E^*, \mu(E^*, E))^* = E$ are just the bounded sets and the topology of uniform convergence on them is the natural topology on E^*.

(1⇔3) The two topologies considered in (1⇔2) coincide by the Lemma in 5.4.16 if and only the bounded closed absolutely convex sets are $\sigma(E, E^*)$-compact. Since the closed absolutely convex hull of each bounded set is obviously bounded, this condition is equivalent to 3.

(3⇔4) The bounded sets in E are bounded in $\prod_{E^*} \mathbb{K}$, so relatively compact there, and thus pre-compact in E with respect to $\sigma(E, E^*)$. Precompact sets are compact if and only if they are complete, see [**26**, 3.5.9]. □

5.4.23 Proposition. Reflexivity.

[**14**, S.227] *For lcs's E are equivalent:*

1. *E is reflexive;*
2. *E is semi-reflexive and infra-barreled;*
3. *Each bounded set is $\sigma(E, E^*)$-relative-compact and E is infra-barreled;*

4. *E is semi-reflexive and barreled.*

Proof. (1⇔2) since $E \to E^{**}$ is an embedding if and only if E is infra-barreled by 5.4.20.

(1⇒4) If E is reflexive, then E is even barreled: For this we have to show that all barrels in E are bornivorous and by 5.4.16 this is exactly the case when all $\sigma(E^*, E)$-bounded subsets A are bounded in E^*, i.e. are uniformly bounded on bounded (absolutely convex) subsets B. Because of 5.4.22.4 we may assume that B is $\sigma(E, E^*)$-complete and thus E_B is a Banach space (namely, let (x_n) be a Cauchy sequence in E_B, then (w.l.o.g.) $x_n \in B$. Then (x_n) is also $\sigma(E, E^*)$-Cauchy, hence $\sigma(E, E^*)$-convergent towards $x_\infty \in E$. For each $\varepsilon > 0$ and sufficiently large n and m we have $x_n - x_m \in \varepsilon B$, hence $x_n - x_\infty \in \varepsilon B$, i.e. $x_n \to x_\infty$ in E_B). Now we consider the natural inclusion $\iota_B : E_B \rightarrowtail E$ and obtain, by the uniform boundedness principle 4.2.2, that $(\iota_B)^*(A) \subseteq (E_B)^*$ is bounded, i.e. A is bounded on B.

(4⇒3⇒2) follows from 5.4.22. □

5.5 Compact sets revisited

5.5.1 Theorem of Krein-Milman.

Let K be a compact convex subset of an lcs. Then K is the closed convex hull of its extremal points

$$\begin{aligned}\operatorname{Ext}(K) &:= \{a \in K : K\setminus\{a\} \textit{ is convex }\}\\ &= \{a \in K :\ \forall x, y \in K\ \ \forall 0 < t < 1 : a = tx + (1-t)y \Rightarrow x = a = y\}.\end{aligned}$$

Proof. We may assume, without loss of generality, that $K \neq \emptyset$. The two descriptions of extremal points are equivalent because $K\setminus\{a\}$ is convex if and only if all $x, y \in K$ with $x \neq a$, $y \neq a$, and all $0 < t < 1$ are: $t\,x + (1-t)\,y \neq a$, or equivalent: $t\,x + (1-t)\,y = a \Rightarrow x = a$ oder $y = a$. Because of $t\,x + (1-t)\,y = a$, however, $x = a$ and $y = a$ are equivalent.

The essential part of the proof consists in proving that $\operatorname{Ext}(K)$ is not empty. For this, we call in addition a subset $A \subseteq K$ extremal in K, if

$$\forall x, y \in K\ \ \forall 0 < t < 1 : t\,x + (1-t)\,y \in A \Rightarrow x, y \in A.$$

Any one-point set $\{a\}$ is extremal if and only if a is an extremal point. Let

$$\mathcal{E} := \{A \subseteq K : A \text{ is extremal in } K, \text{ closed (=compact) and convex}\}.$$

There are extremal points. Obviously, $\mathcal{E}$ is closed under forming intersections. We now want to apply Zorn's Lemma to $\mathcal{E}_0 := \mathcal{E}\setminus\{\emptyset\}$. The finite intersections of each linearly ordered subset $\mathcal{L} \subseteq \mathcal{E}_0$ are not empty, so because of the finite intersection property of compact sets (i.e. if each finite intersection is not empty, then so is the whole intersection) the entire intersection is in $\mathcal{E}_0$. According to the Lemma of Zorn, there is (for each $B \in \mathcal{E}_0$) a minimal element $A \in \mathcal{E}_0$ (with $A \subseteq B$). We claim that A is a singleton. Let $x, y \in A$. If $x \neq y$, then by 5.1.6 there is a continuous linear functional $f : E \to \mathbb{R}$ with $f(x) \neq f(y)$.

Claim. *If $A \in \mathcal{E}_0$ and $f \in E^*$ then $A_f := A \cap f^{-1}(\sup f(A)) \in \mathcal{E}_0$:*
Since f is continuous and A is compact, the supremum $M := \sup f(A)$ is obtained, so the closed set A_f is not empty. It is convex since f is linear and A is convex.

Remains to show that A_f is extremal in A. Let $x, y \in A$ and $0 < t < 1$ with $z = tx + (1-t)y \in A_f$. Because of $f(x), f(y) \leqslant M$ we have

$$\begin{aligned} M &= f(z) = t\, f(x) + (1-t)\, f(y) \\ \Rightarrow t\, f(x) &= M - (1-t)\, f(y) \geqslant (1-(1-t))\, M = t\, M \geqslant t\, f(x) \\ \Rightarrow f(x) &= M \text{ and analogously } f(y) = M \Rightarrow x, y \in A_f. \end{aligned}$$

Since A_f is extremal in the extremal subset A of K, it is so in K.

Due to the minimality of A, $A = A_f$ follows. This is a contradiction because f is not constant on $\{x, y\} \subseteq A$.

Now let B be the closed convex hull of $\operatorname{Ext}(K)$. Obviously, $\operatorname{Ext}(K) \subseteq B \subseteq K$ holds. Assuming $B \neq K$, then there is an $a \in K \backslash B$ and thus a continuous linear $f : E \to \mathbb{R}$ with $f(b) < f(a)$ for all $b \in B$ by 5.2.4, so $B \cap K_f = \emptyset$. Because of $f \in E^*$ and $K \in \mathcal{E}_0$ we obtain $K_f \in \mathcal{E}_0$, as shown above, and by the first part an extremal point $b \in K_f$ of K exists, i.e. $b \in \operatorname{Ext}(K) \cap K_f \subseteq B \cap K_f = \emptyset$, a contradiction. □

5.5.2 Corollary.

Neither c_0 nor $L^1(\mathbb{R})$ are dual spaces of normed spaces.

Proof. If a Banach space E is topologically isomorphic to the dual space of a normed space F, its closed unit ball must be contained in a multiple of the dual ball of F. So it is an $\sigma(E,F)$-closed subset of the $\sigma(E,F)$-compact (by the Theorem 5.4.12 of Alaoğlu-Bourbaki) dual ball. So it is itself $\sigma(E,F)$-compact, and has extremal points according to the Theorem 5.5.1 of Krein-Milman. However, this is not the case for c_0 or $L^1(\mathbb{R})$:

Let $x = (x_k)_k \in c_0$ with $\|x\|_\infty \leqslant 1$. Then there is a k with $|x_k| < 1$ and by choosing an $\varepsilon > 0$ with $|x_k| + \varepsilon \leqslant 1$ we have for the two points

$$x^\pm : j \mapsto \begin{cases} x_j & \text{for } j \neq k \\ x_k \pm \varepsilon & \text{for } j = k \end{cases}$$

$x = \frac{1}{2}(x^+ + x^-)$, $x^+ \neq x \neq x^-$ and $\|x^\pm\| \leqslant 1$. So x is not an extremal point.

Let $[f] \in L^1(\mathbb{R})$ with $\|f\|_1 \leqslant 1$. Without loss of the generality $\|f\|_1 \neq 0$. Then there is a measurable subset X_0 in $\mathbb{R}$ with $0 < \int_{X_0} |f| < \|f\|_1$. Then the analog inequality holds for $X_1 := \mathbb{R}\backslash X_0$. Now $t_i := \|f|_{X_i}\| / \|f\| > 0$ and $t_i\, f_i := f \cdot \chi_{X_i}$ for $i = 0, 1$. Then $\|f_i\|_1 = \|f\|_1$, $f_0 \neq f \neq f_1$, $f = t_0\, f_0 + t_1\, f_1$ and $t_0 + t_1 = 1$. So f is not an extremal point. □

Another important theorem about compact convex sets is the following

5.5.3 Fixed-point Theorem by Brouwer-Schauder-Tychonoff.

Let K be a non-empty compact convex subset of an lcs E and $f : K \to K$ a continuous mapping. Then f has a fixed-point $x \in K$.

Proof. In algebraic topology (see also [**11**] or [**17**, 9.2] or [**24**, 7.6.13] or exercise [**25**, 7.63]), Brouwer's fixed-point theorem tells us that this holds for finite dimensional E.

Now for lcs's E: Compare this with the exercises [**25**, 7.65] and [**25**, 7.66]. We show the existence of a fixed-point under the weaker assumption that $K \subseteq E$ is closed, convex and non-empty, $f : K \to K$ is continuous and $f(K)$ is relatively compact. For each closed absolutely convex 0-neighborhood U there exists a finite set $M_U \subseteq \overline{f(K)} \subseteq K$ with $\overline{f(K)} \subseteq M_U + U$. Furthermore, there exists a continuous partition $\{h_U^y : y \in M_U\}$ of unity with respect to the metric p_U, which is subordinate to the covering $\{y + U : y \in M_U\}$, e.g. $g_U^y : x \mapsto \max\{0, 1 - p_U(x - y)\}$ and $h_U^y := g_U^y / \sum_{z \in M_U} g_U^z$. Then $f_U := \sum_{y \in M_U} (h_U^y \circ f) \cdot y$ is a continuous mapping into the convex hull K_U of M_U and

$$\begin{aligned} p_U(f(x) - f_U(x)) &= p_U\Big(\sum_{y \in M_U} h_U^y(f(x)) \cdot (f(x) - y)\Big) \\ &\leqslant \sum_{f(x) \in y + U} h_U^y(f(x)) \cdot p_U(f(x) - y) \leqslant \sum_{y \in M_U} h_U^y(f(x)) = 1. \end{aligned}$$

According to Brouwer's fixed-point theorem, $f_U : K_U \to K_U \subseteq \overline{f(K)} \cap \langle M_U \rangle_{\mathrm{VR}}$ has a fixed-point $x_U \in K_U$.

The set $\{x - f(x) : x \in K\}$ is closed, because if $\lim_i x_i - f(x_i) = z$, then $i \mapsto f(x_i)$ has an accumulation value $y \in \overline{f(K)}$ and thus $x := z + y$ is an accumulation value of $i \mapsto x_i$. Therefore, $x \in K$ and $x - f(x) = z$, because f is continuous.

Let us assume f has no fixed-point, then 0 would not be in the closed set $\{x - f(x) : x \in K\}$, so there would be an absolutely convex closed 0-neighborhood U with $x - f(x) \notin U$ for all $x \in K$. Because of $x_U - f(x_U) = (f_U - f)(x_U) \in U$ this is a contradiction. □

5.5.4 Fixed-point Theorem of Kakutani. [31] **and** [4]**.**

Let $K \subseteq \mathbb{R}^m$ be a non-empty, convex and compact subset, and $f : K \to 2^K \cong \mathcal{P}(K)$ a convex-valued mapping with closed graph $\{(x, y) : y \in f(x)\} \subseteq K \times K$ and $f(x) \neq \emptyset$ for all $x \in K$.
Then f has a fixed-point, i.e. $\exists x \in K \colon x \in f(x)$.

Proof. Since the graph of f is closed, $f(x) \cong \{x\} \times f(x) = \operatorname{graph}(f) \cap \{x\} \times K$ is closed. Furthermore, f is semicontinuous from above, i.e. U open $\Rightarrow \{x : f(x) \subseteq U\}$ open, otherwise there would be a net $x_i \to x_\infty$ with $f(x_\infty) \subseteq U$ and $y_i \in f(x_i) \subseteq K$ with $y_i \notin U$. Since K is compact, (y_i) has an accumulation point y_∞ and, since the graph is closed, $y_\infty \in f(x_\infty) \subseteq U$, hence $y_i \in U$ for some i, a contradiction.

Since K is (pre)compact there exists for each absolutely convex 0-neighborhood U a finite set $M_U \subseteq K$ with $K \subseteq M_U + U$ and thus as in the proof of 5.5.3 a subordinate partition $\{h_U^x : x \in M_U\}$ of unity. For $x \in M_U$ we choose $y_x \in f(x)$ and thus define a continuous mapping $f_U : K \to K$ by $f_U(z) := \sum_{x \in M_U} h_U^x(z)\, y_x$ which has a fixed-point $x_U \in K$ by 5.5.3. In particular, for U we can use the balls with radius $\frac{1}{n}$ and denote the corresponding f_U with f_n and M_U with M_n. The sequence of the associated fixed-points $x_n \in K$ has an accumulation point x_∞. We show that x_∞ is a fixed-point of f. Since f is semicontinuous from above, there is for each $\varepsilon > 0$ an open δ-neighborhood $U_\delta(x_\infty)$ of x_∞ , s.t. $f(x) \subseteq f(x_\infty) + U_\varepsilon$ for all $x \in U_\delta(x_\infty) \cap K$.

Claim: $f_n(U_{\delta - 1/n}(x_\infty) \cap K) \subseteq f(x_\infty) + U_\varepsilon$ for $1/n < \delta$:
Let $z \in U_{\delta - 1/n}(x_\infty) \cap K$, i.e. $\|z - x_\infty\| < \delta - \frac{1}{n}$. Because of $K \subseteq M_U + U$, there exists z for $x \in M_n$ with $\|z - x\| < \frac{1}{n}$. For each such $x \in M_U$ (with $z \in x + U$) $\|x - x_\infty\| \leqslant \|x - z\| + \|z - x_\infty\| < \delta$, i.e. $x \in U_\delta(x_\infty) \cap K$ and thus $y_x \in f(x) \subseteq f(x_\infty) + U_\varepsilon$. Since this holds for all $x \in M_U$ with $h_U^x(z) \neq 0$ (i.e. $z \in x + U$) we have $f_n(z) = \sum_{x \in M_U} h_U^x(z)\, y_x \in f(x_\infty) + U_\varepsilon$.

For sufficiently large n we have $x_n \in U_{\delta/2}(x_\infty) \cap K$ and thus $x_n = f_n(x_n) \in f(x_\infty) + U_\varepsilon$. So the accumulation point $x_\infty \in f(x_\infty) + U_{2\varepsilon}$ for each $\varepsilon > 0$.

Suppose $x_\infty \notin f(x_\infty)$. Then $\rho := d(x_\infty, f(x_\infty)) > 0$, i.e. $x_\infty \notin f(x_\infty) + U_\rho$ for a sufficiently small $\rho > 0$, a contradiction. □

5.5.5 Fixed-point Theorem of Kakutani for locally convex spaces. [7] **and** [9]**.**

Let $K \subseteq E$ be a non-empty, convex and compact subset of an lcs E and $f : K \to 2^K \cong \mathcal{P}(K)$ a convex-valued mapping with closed graph and $f(x) \neq \emptyset$ for all $x \in K$s.
Then f has a fixed-point, i.e. $\exists x \in K$: $x \in f(x)$.

Proof. Let $\mathcal{U}$ a 0-neighborhood basis of absolutely convex closed sets. For $U \in \mathcal{U}$ let $K_U := \{x \in K : x \in f(x) + U\} = \{x \in K : \exists y \in f(x) : x - y \in U\}$.

The set K_U is closed, because $\Delta_U := \{(x, y) : x - y \in U\}$ is a closed neighborhood of the diagonal in $K \times K$ and thus $\mathrm{pr}_1(\Delta_U \cap \mathrm{graph}(f)) = K_U$ is compact, hence closed.

We have $K_U \neq \emptyset$: For a finite $M_U \subseteq K$ we have $K \subseteq M_U + U$. Let A be the convex hull of M_U and $f_A : A \to 2^A$ given by $x \mapsto (f(x) + U) \cap A$. Then, f_A satisfies the assumptions of 5.5.4 (because of $K \subseteq M_U + U$, $(f(x) + U) \cap A$ is not empty and $\mathrm{graph}(f_A) = (\mathrm{graph}(f) + \{0\} \times U) \cap (A \times A)$ is closed because $\mathrm{graph}(f)$ is compact and $\{0\} \times U$ is closed), and thus there exists an $x \in (f(x) + U) \cap K$, i.e. $K_U \neq \emptyset$.

The family K_U has the finite intersection property (by monotonicity), so there exists $x_0 \in \bigcap_U K_U$. Suppose $x_0 \notin f(x_0)$, i.e. $\exists U$: $x_0 \notin f(x_0) + U$, a contradiction to $x_0 \in K_U$. □

Remark.

Obviously, Kakutani's Fixed-point Theorem 5.5.5 conversely implies the Fixed-point Theorem 5.5.3 by Brouwer-Schauder-Tychonoff. The former has among others applications in the form of a minimax theorem in game theory and thus in mathematical economics.

5.5.6 Lemma. Approximability of linear functionals.

Let E be an lcs, $A \subseteq E$ absolutely convex and $f : E \to \mathbb{K}$ linear. Then $f|_A$ is continuous if and only if $\forall \varepsilon > 0\ \exists x^* \in E^*\ \forall x \in A$: $|\langle f - x^*, x\rangle| \leqslant \varepsilon$.

Proof. ($\Leftarrow$) is obvious because the uniform limit of continuous functions is continuous.

($\Rightarrow$) Let $F := \langle A\rangle_{\mathrm{vs}}$ be the linear span of A supplied with the Minkowski functional q_A as seminorm. Let $\varepsilon > 0$. Since $f|_A$ is continuous, there exists an absolutely convex 0-neighborhood $U \subseteq E$ with $|\langle f, y\rangle| < \varepsilon$ for all $y \in A \cap U$, i.e. $\max\{q_A, q_U\}_{<1} \subseteq (\frac{1}{\varepsilon}|f|)_{<1}$ and thus $|\langle f, y\rangle| \leqslant \varepsilon \max\{q_A, q_U\}(y) \leqslant \varepsilon\,(q_A(y) + q_U(y))$ for all $y \in F$ by 1.3.7. We put $\varphi := \varepsilon\, q_A$ and $\psi := \varepsilon\, q_U$. For $(x, y) \in E \times F$ we thus have

$$\begin{aligned}-\psi(x) &\leqslant \psi(-y) + \varphi(-y) - \langle f, -y\rangle - \psi(x) = \psi(y) + \varphi(y) + \langle f, y\rangle - \psi(x)\\ &\leqslant \psi(x - y) + \langle f, y\rangle + \varphi(y)\end{aligned}$$

and therefore $p : x \mapsto \inf\{\psi(x - y) + \langle f, y\rangle + \varphi(y) : y \in F\}$ is well-defined and satisfies both $p(x) \leqslant \psi(x) = \varepsilon\, q_U(x)\ \forall x \in E$ and $p(y) \leqslant \langle f, y\rangle + \varepsilon\, q_A(y)\ \forall y \in F$. Since p is sublinear, there exists a linear $x^* : E \to \mathbb{K}$ with $x^* \leqslant p$ by 5.1.2. Due to the above inequalities, $x^* \in E^*$ and $\langle x^* - f, y\rangle \leqslant \varepsilon\ \forall y \in A$ and since A is balanced also $\langle f - x^*, y\rangle = \langle x^* - f, -y\rangle \leqslant \varepsilon$ holds for all $y \in A$. This proves the theorem in case $K = \mathbb{R}$.

Let now $\mathbb{K} = \mathbb{C}$. For a linear function $f : E \to \mathbb{C}$ being continuous on A we have $f(x) = f_{\mathbb{R}}(x) - i\, f_{\mathbb{R}}(i\, x)$, where $f_{\mathbb{R}} := \Re e \circ f : E \to \mathbb{R}$. Because of the real case, there is a continuous $\mathbb{R}$-linear $x^* : E \to \mathbb{R}$ with $|\langle f_{\mathbb{R}} - x^*, x\rangle| \leqslant \varepsilon$ for all $x \in A$.

Let $\tilde{x}^* : x \mapsto x^*(x) - i\, x^*(i\, x)$. Then $\tilde{x}^* : E \to \mathbb{C}$ is continuous and $\mathbb{C}$-linear with $|\langle f - \tilde{x}^*, x\rangle| \leqslant \sqrt{\varepsilon^2 + \varepsilon^2} = \sqrt{2}\,\varepsilon$. □

5.5.7 Proposition. Grothendieck's Completion Theorem.

The completion of an lcs E can be described as

$$\hat{E} := \Big\{ f : E^* \to \mathbb{K} \text{ linear } : f|_{U^o} \text{ is } \sigma(E^*, E)\text{-continuous } \forall\, 0\text{-neighborhoods } U \subseteq E \Big\}$$

supplied with the topology of the uniform convergence on the polars U^o.

Proof. We will use 3.8.3.

($\hat{E}$ is complete) because uniform limits of continuous functions are continuous.

($E \subseteq \hat{E}$) Because of $E \cong (E^*, \sigma(E^*, E))^* \subseteq \hat{E}$, we can think of E as a linear subspace of $\hat{E}$, and, by 5.4.11, E carries the topology of uniform convergence on $U^o \subseteq E^*$ by virtue of this embedding, i.e. the trace topology induced by $\hat{E}$.

(E is dense in $\hat{E}$) Let $f \in \hat{E}$. Then $f : F := E^* \to \mathbb{K}$ is linear. For each (absolutely convex) 0-neighborhood U in E, the set $A := U^o$ is absolutely convex in E^*. For each $\varepsilon > 0$ there is by Lemma 5.5.6 an $x^* \in F^* = (E^*, \sigma(E^*, E))^* \cong_{\mathrm{vs}} E$, with $|\langle f - x^*, x\rangle| \leqslant \varepsilon$ for all $x \in A$, i.e. f can be approximated in the topology of the uniform convergence on the U^o by $x^* \in E$, i.e. E is dense in $\hat{E}$. □

Teil II

Spectral Theory

6. Spectral and Representation Theory for Banach Algebras

Preliminary remarks

The goal of spectral theory is to find to a given linear operator T a representation which is as explicit and invariant as possible. In the 1-dimensional situation, each linear operator $T : \mathbb{K} \to \mathbb{K}$ is a multiplication operator of the form $T : x \mapsto \lambda \cdot x$, where the slope λ is given by $\lambda := T(1)$. In the finite-dimensional case, an analogue would be the matrix representation obtained by choosing a basis, and, in the infinite-dimensional case, the representation as an integral operator by an integral kernel. On the one hand, these representations are as explicit as possible, but on the other hand they are not invariant under change of basis (rotations). An invariant approach is to find as many non-trivial linear subspaces (i.e. EIGENSPACES) on which T acts as multiplication by some $\lambda \in \mathbb{K}$ (the corresponding EIGENVALUE).

The eigenspace for the eigenvalue λ is therefore given by the kernel of $T - \lambda \cdot \mathrm{id}$. And this kernel is non-trivial iff $T - \lambda \cdot \mathrm{id}$ is not injective, which for finite dimensional E is equivalent to being not invertible, i.e. $\det(T - \lambda \cdot \mathrm{id}) = 0$. So the eigenvalues λ are the zeros of the characteristic polynomial $x \mapsto \det(T - x \cdot \mathrm{id})$.

The existence of sufficiently many such subspaces should now mean that the operator is already uniquely given by the restrictions to these subspaces. In linear algebra we learn that this is achievable for normal operators on complex finite-dimensional Hilbert spaces, i.e. any such operator is diagonalizable. So up to the isomorphism $E \cong \mathbb{C}^{\dim E}$, given by $(x_k)_k \mapsto \sum_k x_k e_k$, where $(e_k)_k$ is a basis of eigenvectors, T acts as multiplication operators $(x_k)_k \mapsto (\lambda_k x_k)_k$. Since for normal operators we may choose the e_k to form an orthonormal system, we have $T(x) = \sum_k \lambda_k \langle x, e_k \rangle e_k$, where λ_k denotes the eigenvalues corresponding to e_k.

But what about infinite-dimensional spaces? For self adjoint compact operators T on Hilbert spaces, we have seen in [**18**, 6.5.4] that the eigenvalues form a sequence λ_k for which there exists an orthonormal basis of eigenvectors e_k, and $T(x) = \sum_k \lambda_k \langle x, e_k \rangle e_k$. This is even true for normal compact operators, see 8.24.

Examples of operators being not compact.

1. The left-shift operator $T : \ell^2(\mathbb{N}, \mathbb{C}) \to \ell^2(\mathbb{N}, \mathbb{C})$ is defined by $T : (x_k)_{k \geqslant 0} \mapsto (x_{k+1})_{k \geqslant 0}$. The equation $T(x) = \lambda x$ is in coordinates the system of equations $(x_{k+1} = \lambda x_k)_{k \geqslant 0}$. The only possible solution is $x = (\lambda^k x_0)_{k \geqslant 0}$ which is in ℓ^2 for $|\lambda| < 1$ and thus λ is an eigenvalue. For $|\lambda| \geqslant 1$ and $x_0 \neq 0$, $x \notin \ell^2$, i.e. λ, is not an eigenvalue. So, the set of eigenvalues is the open unit disk in $\mathbb{C}$, and thus no longer countable, hence T is not representable as series like above.
Since $1 - S$ is invertible with inverse $\sum_{k=0}^{\infty} S^k$ provided S is a linear operator with $\|S\| < 1$ (cf. 6.2.1), we have that $\lambda - T = \lambda(1 - \frac{1}{\lambda} T)$ is invertible for each $|\lambda| > \|T\| = 1$ (cf. 6.25). Moreover, the set of invertible operators is open (see

6.2.2), hence $\lambda - T$ is not invertible $\Leftrightarrow$ $|\lambda| \leqslant 1$. So we see that for $|\lambda| = 1$ the operator $\lambda - T$ is injective but not invertible.

2. The adjoint operator $T^* : \ell^2(\mathbb{N}, \mathbb{C}) \to \ell^2(\mathbb{N}, \mathbb{C})$ to T is the right-shift operator $T^* : (x_0, x_1, \dots) \mapsto (0, x_0, x_1, \dots)$, because

$$\langle T^*(x), y\rangle = \sum_{k=1}^{\infty} x_{k-1} \cdot \overline{y_k} = \sum_{k=0}^{\infty} x_k \cdot \overline{y_{k+1}} = \langle x, T(y)\rangle.$$

Since T^* is an isometry, it follows from $T^*x = \lambda x$ for an $x \neq 0$ that $|\lambda| = 1$ and thus from $0 = \lambda x_0$, $x_0 = \lambda x_1$, ... recursively that $x_k = 0$ for all k. So T^* has no eigenvalues at all.
As before, it follows that for each $|\lambda| > 1$, the mapping $\lambda - T^*$ is invertible. Let us assume that $\lambda - T^*$ is invertible for some $|\lambda| \leqslant 1$ and let S be its inverse. Then S^* is an inverse of $(\lambda - T^*)^* = \overline{\lambda} - T$, a contradiction to what was said about T. Hence $\lambda - T^*$ is not invertible $\Leftrightarrow$ $|\lambda| \leqslant 1$. Note however that T is not normal, because

$$T \circ T^* = \mathrm{id} \neq (\mathrm{id} - \mathrm{pr}_0) = T^* \circ T.$$

3. Next, consider the unitary (right-)shift operator $T : \ell^2(\mathbb{Z}, \mathbb{C}) \to \ell^2(\mathbb{Z}, \mathbb{C})$ defined by $T : (x_k)_{k\in\mathbb{Z}} \mapsto (x_{k-1})_{k\in\mathbb{Z}}$. Then again only λ with $|\lambda| = 1$ might be eigenvalues. But no such λ can be an eigenvalue, because the equation $T(x) = \lambda\, x$ is equivalent to the system $(x_{k-1} = \lambda\, x_k)_{k\in\mathbb{Z}}$. Hence $|x_{k-1}| = |x_k|$ for all k and thus $x \notin \ell^2$ for $x \neq 0$. Thus T has no eigenvalues at all.
Obviously T is invertible with inverse T^{-1} being the left-shift. Moreover, $\lambda - T$ is invertible for each $|\lambda| > 1 = \|T\|$ as before and also for each $|\lambda| < 1$, because

$$(\lambda - T)^{-1} = ((\lambda T^{-1} - \mathrm{id})T)^{-1} = T^{-1}\,(\lambda T^{-1} - \mathrm{id})^{-1}$$

On the other hand, for $|\lambda| = 1$, the mapping $\lambda - T$ is not invertible, because the standart unit vector e_0 is not in the image: Let $(\lambda - T)(x) = e_0$, then $\lambda\, x_k - x_{k-1} = 0$ for $k \neq 0$. So $|x_k| = |x_{k-1}|$ for $k \neq 0$ and thus $x = 0$, a contradiction to $\lambda\, x_0 - x_{-1} = 1$.
The Fourier series development $\mathcal{F} : L^2([-\pi, \pi], \mathbb{C}) \xrightarrow{\cong} \ell^2(\mathbb{Z}, \mathbb{C})$ from [**18**, 6.3.8] conjugates the operator T into the multiplication operator M_f with $f : x \mapsto e^{ix}$, because in [**18**, 5.4.4] we have shown $\mathcal{F}(M_f\, g) = T(\mathcal{F}g)$. The unit circle S^1 consists exactly of those $\lambda \in \mathbb{C}$ for which $\lambda - T$ is not invertible and T is up to the isomorphism $\mathcal{F}$ a multiplication operator on $L^2(S^1, \mathbb{C}) \cong L^2([-\pi, \pi], \mathbb{C})$.

So we see that the notion eigenvalue in infinite dimensions is too strict. The (in finite dimensions equivalent) condition "$\lambda - T$ is not invertible" seems to be more suitable. Such an λ is called a SPECTRAL VALUE of T, and the set of all spectral values is denoted the SPECTRUM $\sigma(T)$.

In case the lcs E on which the operator T acts is not normable, even this notion is a too weak one and there is no reasonable spectral theory for operators on lcs's:

4. Consider for example the space E of all $(x_k)_{k\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$, for which $x_k = 0$ for k sufficiently small. We provide E with the strictly inductive limit structure $\varinjlim_{n\to-\infty} E_n \cong \mathbb{C}^{(\mathbb{Z}_-)} \times \mathbb{C}^{\{0\}} \times \mathbb{C}^{\mathbb{Z}_+}$ with steps $E_n := \{(x_k)_{k\in\mathbb{Z}} : x_k = 0 \text{ for } k < n\} \cong \mathbb{C}^{\mathbb{N}}$. Let T be the left-shift $(x_k)_{k\in\mathbb{Z}} \mapsto (x_{k+1})_{k\in\mathbb{Z}}$, which is obviously a continuous linear bijective operator because $T|_{E_n} : E_n \to E_{n-1}$ is an isomorphism and by the closed graph theorem the inverse of $T|_{E_n}$ is continuous as well.
Note that $E^* = \mathbb{C}^{\mathbb{Z}_-} \times \mathbb{C}^{\{0\}} \times \mathbb{C}^{(\mathbb{Z}_+)} \cong E$, via the reflection $(x_k)_k \mapsto (x_{-k})_k$, and the right-shift T^* corresponds to the left-shift T under this isomorphism, i.e. T is self adjoint with respect to the pairing $\langle\_,\_\rangle : E \times E \to \mathbb{K}$, $(x, y) \mapsto \sum_k x_k y_{-k}$.
Now $T - \lambda \cdot \mathrm{id}$ is invertible for all $\lambda \in \mathbb{C}$, because for $y \in E_n$ the equation $T(x) - \lambda \cdot x = y$ has a unique solution $x \in E_{n+1} \subset E$. It can be recursively

calculated from $x_{k+1} = \lambda\, x_k + y_k$, since $x_{k+1} = \lambda\, x_k$ and thus $x_{k+1} = 0$ holds for $k < n$ and $x_{n+k} = \sum_{j=0}^{k-1} \lambda^{j-1}\, y_{n+j}$. Hence the spectrum of T is empty.

In contrast to eigenvalues, one immediately sees that for the above definitions of spectral values and spectrum of T, the vectors in E play no essential role. It suffices to be able to form the expressions $T - \lambda \cdot \mathrm{id}$ in order to question the invertibility of these expressions. For the former, T should be in a vector space, and for the latter, this vector space should be an algebra with unit. In order to be able to control invertibility well, the absolutely convergent geometric series $\sum_{k=0}^{\infty} T^k$ should converge, i.e. T should lie in a Banach algebra. So we will develop spectral theory for elements of abstract Banach algebras (see [**18**, 3.2.9]). Let's recall the most important examples:

6.1 Examples.

1. For each Banach space E, $L(E) := L(E,E)$ is a Banach algebra with 1 with respect to the composition as multiplication, see [**18**, 3.2.9]
2. For each compact space X, $C(X,\mathbb{K})$ is a commutative Banach algebra with 1 with respect to the pointwise multiplication. More generally, this also holds for the space $B(X,\mathbb{K})$ of the bounded functions on a set X, see 2.2.3.
3. Thus, the Banach space $L^\infty(X,\Omega,\mu)$ is, for each σ-finite measurable space (X,Ω,μ), also a commutative Banach algebra with 1 with respect to the pointwise operations, see [**18**, 4.12.3].
4. Furthermore, $\ell^1(\mathbb{N})$ and $\ell^1(\mathbb{Z})$ are commutative Banach algebras with 1 with respect to convolution.

6.2 Remark about the invertibility in a Banach algebra.

We have shown in [**18**, 3.3.1] the following facts for the invertible elements $a \in \mathrm{Inv}(A)$ of Banach Algebras A with unit 1:

1. For $\|a-1\| < 1$ we have $a \in \mathrm{Inv}(A)$ and $a^{-1} = \sum_{k=0}^{\infty}(1-a)^k$, the absolutely convergent geometric series.
2. If $a_0 \in \mathrm{Inv}(A)$ and $\|a-a_0\| < \frac{1}{\|a_0^{-1}\|}$ then by (1) also $a = (a\, a_0^{-1})\, a_0 \in \mathrm{Inv}(A)$; in particular, $\mathrm{Inv}(A)$ is open in A.
3. If $a_1\, a_2 = a_2\, a_1 \in \mathrm{Inv}(A)$, then $a_1, a_2 \in \mathrm{Inv}(A)$.
   This holds in every semigroup, because $a_1\, a_2$ is invertible with inverse $b := (a_1\, a_2)^{-1}$. Then $a_1\, a_2\, b = 1 = b\, a_1\, a_2 = b\, a_2\, a_1$, so $r := a_2\, b$ is a right inverse to a_1 and $l := b\, a_2$ is a left inverse to a_1, thus $r = l\, a_1\, r = l$, i.e. $r = l$ is the (unique) two-sided inverse to a_1.
4. The mapping $\mathrm{inv} : \mathrm{Inv}(A) \to \mathrm{Inv}(A)$, $a \mapsto a^{-1}$, is (complex-)differentiable and its derivative is $\mathrm{inv}'(a)(h) = -a^{-1}\, h\, a^{-1}$.
   One obtains the derivative by differentiating the implicit equation $a^{-1}\, a = 1$: Let us denote with $\mathrm{mult} : A \times A \to A$ the bilinear multiplication. Then, by differentiating of $1 = \mathrm{mult} \circ (\mathrm{inv}, \mathrm{id})$ at the point $a \in \mathrm{Inv}(A)$ in the direction h, we obtain

$$\begin{aligned}
0 &= \partial_1 \mathrm{mult}(\mathrm{inv}(a), \mathrm{id}(a))\,(\mathrm{inv}'(a)(h)) + \partial_2 \mathrm{mult}(\mathrm{inv}(a), \mathrm{id}(a))\,(\mathrm{id}'(a)(h)) \\
&= \mathrm{mult}(\mathrm{inv}'(a)(h), \mathrm{id}(a)) + \mathrm{mult}(\mathrm{inv}(a), \mathrm{id}(h)) \\
&= \mathrm{inv}'(a)(h) \cdot a + a^{-1} \cdot h
\end{aligned}$$

and thus $\mathrm{inv}'(a)(h) = \mathrm{inv}'(a)(h) \cdot a \cdot a^{-1} = -a^{-1} \cdot h \cdot a^{-1}$. That inv is differentiable with this derivative can also be calculated directly as follows:

$$\begin{aligned}\frac{\|(a+h)^{-1} - a^{-1} + a^{-1}\,h\,a^{-1}\|}{\|h\|} &= \frac{\left\|a^{-1}\Big((1+h\,a^{-1})^{-1} - 1 + h\,a^{-1}\Big)\right\|}{\|h\|}\\ &\leqslant \|a^{-1}\| \sum_{k\geqslant 2} \frac{\|(h\,a^{-1})^k\|}{\|h\|}\\ &\leqslant \|a^{-1}\|\,\|h\| \sum_{k\geqslant 0} (\|h\|\,\|a^{-1}\|)^k\,\|a^{-1}\|^2\\ &\leqslant \|h\|\,\|a^{-1}\|^3\,\frac{1}{1-\|h\|\,\|a^{-1}\|} \to 0 \text{ for } h\to 0\end{aligned}$$

Before introducing the spectral theory of Banach algebras, let us consider what we can do if the algebra in question does not satisfy all the axioms of a Banach algebra.

6.3 Completion

Examples of incomplete algebras.

1. The polynomials on a compact subset $K \subseteq \mathbb{R}$ constitute, with respect to the ∞ norm, a non-complete sub-algebra of $C(K)$.
2. The continuous functions on $\mathbb{R}$ with compact support form a non-complete Banach algebra with respect to the 1-norm and convolution. Likewise the continuous functions on S^1.
3. The finite-dimensional operators on a Hilbert space H form an incomplete subalgebra of $L(H)$.

Proposition.

Let A be a normed algebra, i.e. a normed space with an algebra structure $\bullet$, so that $\|x \bullet y\| \leqslant \|x\| \cdot \|y\|$. Then there is a (up to isomorphy) a unique Banach algebra $\widetilde{A}$ and an isometric embedding $\iota : A \to \widetilde{A}$ (i.e. $\forall x \in A : \|\iota(x)\| = \|x\|$) with the following universal property:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr, "f"'] & \widetilde{A} \arrow[d, dotted, "\exists!\ \tilde{f}"] \\ & B \end{tikzcd}$$

where f and $\tilde{f}$ are continuous algebra homomorphisms and B is a complete algebra.

Proof. Let A be a normed algebra. Then, by 3.8.4, there is a Banach space $\widetilde{A}$ with the universal extension property for continuous linear mappings. We now want to extend the multiplication $\mu : A \times A \to A$ to a mapping $\widetilde{\mu} : \widetilde{A} \times \widetilde{A} \to \widetilde{A}$. For this we consider the associated mapping $\check{\mu} : A \to L(A, A)$. The natural isometric mapping $\iota : A \to \widetilde{A}$ provides us with an isometry $L(A, E) \cong L(\widetilde{A}, E)$ for each Banach space E. Therefore we obtain an isometric embedding $L(A, A) \xrightarrow{\iota_*} L(A, \widetilde{A}) \cong L(\widetilde{A}, \widetilde{A})$. I.e. we may consider $\check{\mu}$ as a continuous mapping (contraction) from A to $L(\widetilde{A}, \widetilde{A})$. By the universal property this has an extension $\widetilde{\check{\mu}} : \widetilde{A} \to L(\widetilde{A}, \widetilde{A})$. The associated mapping $\widetilde{\mu} : \widetilde{A} \times \widetilde{A} \to \widetilde{A}$ is then the desired multiplication on $\widetilde{A}$, since all necessary (continuous) equations hold on the dense subspace $A \times A$ and thus everywhere.

Note that the essential point is, that multi-linear continuous mappings $E_1 \times \ldots \times E_n \to F$ are uniquely extendable to such on $\widetilde{E_1} \times \ldots \times \widetilde{E_n} \to \widetilde{F}$.

Now to the universal property.
Since we know that $\|f\| = \|\tilde f\|$, we only have to show that $\tilde f$ is multiplicative:

$$\begin{aligned}
\tilde f(\tilde\mu(\tilde a, \tilde b)) &= \tilde f(\tilde\mu(\lim_n a_n, \lim_m b_m)) = \tilde f(\lim_{n,m} \tilde\mu(a_n, b_m)) = \lim_{n,m} \tilde f(\tilde\mu(a_n, b_m)) \\
&= \lim_{n,m} f(\mu(a_n, b_m)) = \lim_{n,m} f(a_n) \cdot f(b_m) = \lim_n f(a_n) \cdot \lim_m f(b_m) \\
&= \lim_n \tilde f(a_n) \cdot \lim_m \tilde f(b_m) = \tilde f(\lim_n a_n) \cdot \tilde f(\lim_m b_m) \\
&= \tilde f(\tilde a) \cdot \tilde f(\tilde b). \quad \square
\end{aligned}$$

Remark.

The completion in the above examples is:

1. The Banach algebra of all continuous functions according to the Theorem [18, 3.4.1] of Weierstrass;
2. The Banach algebra L^1 with the convolution, since the C_c functions are dense, see [18, 4.13.9];
3. The compact operators according to [18, 6.4.8].

6.4 Adjunction of a unit

Examples of algebras without unit.

1. $L^1(\mathbb{R})$ and $L^1(S^1)$ with the convolution. The unit would be the delta distribution.
2. The algebra of compact operators on an infinite-dimensional Hilbert space. The unit would be the identity.
3. For each locally compact space X the algebra $C_0(X)$, of at ∞ vanishing continuous functions. The unit would be the constant function 1.

Proposition.

Let A be a Banach algebra without 1. Then there is a (up to isomorphy unique) Banach algebra A_1 with unit and an isometric embedding $\iota : A \to A_1$ with the following universal property:

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[dr, "f"'] & A_1 \arrow[d, dotted, "\exists!\ f_1"] \\
& B
\end{tikzcd}$$

where f and f_1 are continuous algebra homomorphisms, B is a Banach algebra with unit, and f_1 respects the units.

Proof. Let A be a Banach algebra (not necessary with 1). Let $A_1 := A \oplus \mathbb{K}$. The multiplication is defined by $(a \oplus \lambda) \bullet (b \oplus \mu) := (a \bullet b + \mu a + \lambda b) \oplus \lambda\mu$. Then it is easy to calculate that A_1 is an algebra with $1 = 0 \oplus 1$, and $\iota : A \to A_1$, $a \mapsto a \oplus 0$ is an algebra homomorphism. We define a norm on A_1 by $\|a\oplus\lambda\| := \|a\| + |\lambda|$. Then $\|1\| = \|0\| + |1| = 1$ and

$$\begin{aligned}\|(a\oplus\lambda)\bullet(b\oplus\mu)\| &= \|(a\bullet b + \mu a + \lambda b)\oplus\lambda\mu\| = \|a\bullet b + \mu a + \lambda b\| + |\lambda\mu|\\ &\leqslant \|a\|\cdot\|b\| + |\mu|\cdot\|a\| + |\lambda|\cdot\|b\| + |\lambda|\cdot|\mu|\\ &= (\|a\| + |\lambda|)\cdot(\|b\| + |\mu|)\\ &= \|a\oplus\lambda\|\cdot\|b\oplus\mu\|.\end{aligned}$$

Now to the universal property:
An f_1 making the diagram commutative must satisfy $f_1(a\oplus\lambda) = f_1(a) + \lambda\cdot f_1(1) = f(a) + \lambda$. And the f_1 defined by it is multiplicative, because:

$$\begin{aligned}f_1((a\oplus\alpha)\cdot(b\oplus\mu)) &= f_1((a\,b + \lambda\,b + \mu\,a)\oplus\lambda\mu) = f(ab + \lambda b + \mu a) + \lambda\mu\\ &= f(a)\,f(b) + \lambda\,f(b) + \mu\,f(a) + \lambda\mu = (f(a) + \lambda)\,(f(b) + \mu)\\ &= f_1(a\oplus\alpha)\cdot f_1(b\oplus\mu).\end{aligned}$$

Since ι is an isometry, $\|f\| = \|f_1\circ\iota\| \leqslant \|f_1\|\cdot\|\iota\| = \|f_1\|$ holds. On the other hand, $\|f_1\| = \sup\{\|f(a)+\lambda\| : \|a\oplus\lambda\| \leqslant 1\} \leqslant \sup\{\|f\|\,\|a\| + |\lambda| : \|a\| + |\lambda| \leqslant 1\} \leqslant \max\{\|f\|, 1\}$. So f is a contraction (or continuous) if and only if f_1 is it. Note, however, that $\|f\| = \|f_1\|$ does not apply: Let e.g. $f = 0$, then $f_1 = \mathrm{pr}_2$ and $\|f_1\| = 1$. $\square$

Remark.

With respect to the above examples:

1. A Banach algebra with 1, which includes $L^1(G)$, is the algebra of the regular Borel measures on G with convolution, see [**5**, 193]. This can be identified with $C_0(G)^*$ because of Ries's Theorem 5.3.4. The convolution corresponds to the mapping $(\mu,\nu)\mapsto(f\mapsto(\mu\otimes\nu)(f\circ m))$, where $m: G\times G\to G$ denotes the multiplication and $\mu\otimes\nu$ is the extension from $(f,g)\mapsto\mu(f)\,\nu(g)$ to $C_0(G\times G)\supseteq C_0(G)\times C_0(G)$.
2. The operators of the form $1+K$ with compact K are the so-called Fredholm operators, see [**5**, Chapt.XI] and 8.26.
3. The algebra $C_0(X)_1$ consists of those continuous functions f on X, for which $\lim_{x\to\infty} f(x)$ exists, these are exactly the restrictions of continuous functions on the 1-point compactification X_∞ of X, i.e. $C_0(X)_1 \cong C(X_\infty)$.

Next, let's examine how much the continuity condition $\|x\bullet y\| \leqslant \|x\|\cdot\|y\|$ can be weakend.

6.5 Proposition (Submultiplicity).

Let A be a Banach space and an associative algebra with 1, s.t. the multiplication $\mu: A\times A\to A$ is separately continuous. Then there is an equivalent norm that turns A into a Banach algebra. On elements x with $\|x\bullet y\| \leqslant \|x\|\cdot\|y\|$ for all y it coincides with the given norm.

Proof. Without restriction of generality $\|1\| = 1$, otherwise replace $\|\_\|$ with $\frac{1}{\|1\|}\|\_\|$. We have that μ is continuous by 4.2.8, i.e. $\|\mu\| := \sup\{\|x\bullet y\| : \|x\| \leqslant 1, \|y\| \leqslant 1\} < \infty$. We consider the mapping $L: A\to L(A,A)$, which assigns to each $x\in A$ the left multiplication $L_x: A\to A$, $y\mapsto x\bullet y$. Because of $\|L_x\| = \sup\{\|x\bullet y\| : \|y\| \leqslant 1\} \leqslant \|\mu\|\cdot\|x\|$, L has values in $L(A,A)$ and is a continuous linear mapping $A\to L(A,A)$. For each Banach space A, however, $L(A,A)$ is a Banach algebra (see [**18**, 3.2.9]). The mapping L is also an algebra homomorphism, because $L_{x_1\bullet x_2}(y) =$

$(x_1 \bullet x_2) \bullet y = x_1 \bullet (x_2 \bullet y) = (L_{x_1} \circ L_{x_2})(y)$. Futhermore, $\|L_x\| = \sup\{\|x \bullet y\| : \|y\| \leqslant 1\} \geqslant \|x \bullet 1\| = \|x\|$ because $\|1\| = 1$. So L is a homeomorphism of A onto its image A_0 in $L(A, A)$, i.e. A_0 is also complete and thus closed in $L(A, A)$ and thus $L : A \to A_0$ is a topological algebra isomorphism onto the Banach algebra A_0. Note that this the norm $\|\_\|$ can be replaced by the equivalent but submultiplicative norm $x \mapsto \|L_x\| := \sup\{\|x \bullet y\| : \|y\| \leqslant 1\}$.

If the inequality $\|x \bullet y\| \leqslant \|x\| \cdot \|y\|$ is valid for all y for a $x \in A$, then its norm is not changed because it follows $\|L_x\| \leqslant \|x\|$ and $\|x\| \leqslant \|L_x\|$ holds always. □

6.6 Complexification of real Banach algebras

Examples of real algebras.

1. For each compact space X, $C(X; \mathbb{R})$ is a real commutative Banach algebra.
2. For every real Banach space E, $L(E)$ is a real Banach algebra.

In 3.9.3 we discussed the complexification $E_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} E \cong E \times E$ of real Banach spaces E. The multiplication of $z = x + i\,y \in \mathbb{C}$ with $w = u + i\,v := (u, v) \in E_{\mathbb{C}}$ was given by $(x + i\,y)(u + i\,v) := (x\,u - y\,v) + i\,(x\,v + y\,u)$ and the norm by

$$p_{\mathbb{C}}(w) := \max\{\|\Re e(z\,w)\| : |z| = 1\} = \max\{\|x\,u - y\,v\| : x^2 + y^2 = 1\}.$$

In 3.9.4 we had two universal properties that told us that for every complex Banach space G the maps

$$\begin{aligned} \Re e_* &: L_{\mathbb{C}}(G, E_{\mathbb{C}}) \to L_{\mathbb{R}}(G, E) \\ \iota^* &: L_{\mathbb{C}}(E_{\mathbb{C}}, G) \to L_{\mathbb{R}}(E, G) \end{aligned}$$

are topological linear isomorphisms, and the former even an insometry. In the sequence we had in 3.9.5 for real Banach spaces a commutative diagram of topological linear isomorphisms:

$$\begin{tikzcd}
 & L_{\mathbb{C}}(E_{\mathbb{C}}, F_{\mathbb{C}}) \arrow[dl, "\Re e_*"'] \arrow[dr, "\iota^*"] \arrow[dd, dotted, "\cong"] & \\
L_{\mathbb{R}}(E_{\mathbb{C}}, F) \arrow[dr, "\cong"] & & L_{\mathbb{R}}(E, F_{\mathbb{C}}) \arrow[dl, "\cong"'] \\
 & L_{\mathbb{R}}(E, F)_{\mathbb{C}} &
\end{tikzcd}$$

$$\begin{tikzcd}
 & z \mapsto h(z) - i\,h(i\,z) & & & & (x + i\,y \mapsto h(x) + i\,h(y)) & \\
h \arrow[ur] \arrow[dr] & & (x + i\,y \mapsto f(x) - g(y)) & & (x \mapsto f(x) + i\,g(x)) & & h \arrow[ul] \arrow[dl] \\
 & h \circ \iota - i\,h \circ I \circ \iota & & f + i\,g \arrow[ul] \arrow[ur] & & \Re e \circ h - i \Re e \circ I \circ h &
\end{tikzcd}$$

The mappings going to the lower left are isometries and the diagonal isomorphism $L_{\mathbb{R}}(E, F)_{\mathbb{C}} \xrightarrow{\cong} L_{\mathbb{C}}(E_{\mathbb{C}}, F_{\mathbb{C}})$ is given by $f + i\,g \mapsto (x + i\,y \mapsto (f(x) - g(y)) + i\,(f(y) + g(x)))$.

Proposition (Complexification).

Let A be a real Banach algebra (with 1). Then there is a (up to isomorphy a unique) complex Banach algebra $A_{\mathbb{C}}$ (with 1 and) with the following universal property:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr, "f"'] & A_{\mathbb{C}} \arrow[d, dotted, "\exists!\ f_{\mathbb{C}}"] \\ & B \end{tikzcd}$$

where B is any complex Banach algebra, f is a continuous $\mathbb{R}$-algebra homomorphism (which preserves 1), and $f_{\mathbb{C}}$ is a continuous $\mathbb{C}$-algebra homomorphism (which preserves 1).

Proof. Obviously, $A_{\mathbb{C}}$ as a vector space should just be the complexification of the real Banach space A. We now need to extend the multiplication $\mu : A \times A \to A$ to a bilinear mapping $\mu_{\mathbb{C}} : A_{\mathbb{C}} \times A_{\mathbb{C}} \to A_{\mathbb{C}}$. So we also need the universal property of the complexification of a Banach space for continuous bilinear mappings. For this we again consider the linear contraction $\check{\mu} : A \to L_{\mathbb{R}}(A, A) \subseteq L_{\mathbb{R}}(A, A)_{\mathbb{C}} \cong L_{\mathbb{C}}(A_{\mathbb{C}}, A_{\mathbb{C}})$, with $x_1 \mapsto (x_2 \oplus i\, y_2 \mapsto \mu(x_1, x_2) \oplus i\, \mu(x_1, y_2))$. Because of the universal property, this has a complex-linear extension $(\check{\mu})_{\mathbb{C}} : A_{\mathbb{C}} \to L_{\mathbb{C}}(A_{\mathbb{C}}, A_{\mathbb{C}})$, which is given by:

$$x_1 \oplus i\, y_1 \mapsto \Big(x_2 \oplus i\, y_2 \mapsto \big(\mu(x_1, x_2) - \mu(y_1, y_2)\big) \oplus i \,\big(\mu(x_1, y_2) + \mu(y_1, x_2)\big)\Big).$$

The associated mapping $\mu_{\mathbb{C}} : A_{\mathbb{C}} \times A_{\mathbb{C}} \to A_{\mathbb{C}}$,

$$(x_1 \oplus i\, y_1, x_2 \oplus i\, y_2) \mapsto \big(\mu(x_1, x_2) - \mu(y_1, y_2)\big) \oplus i \,\big(\mu(x_1, y_2) + \mu(y_1, x_2)\big)$$

is then the desired multiplication. The following simple calculation shows the associativity (and obviously $1 \in A \subset A_{\mathbb{C}}$ is a unit):

$$\begin{aligned}
&\Big((x_1 \oplus i\, y_1) \bullet (x_2 \oplus i\, y_2)\Big) \bullet (x_3 \oplus i\, y_3) \\
&= \Big((x_1x_2 - y_1y_2) \oplus i\, (x_1y_2 + y_1x_2)\Big) \bullet (x_3 \oplus i\, y_3) \\
&= \Big((x_1x_2 - y_1y_2)x_3 - (x_1y_2 + y_1x_2)y_3\Big) \oplus i \,\Big((x_1x_2 - y_1y_2)y_3 + (x_1y_2 + y_1x_2)x_3\Big) \\
&= (x_1x_2x_3 - x_1y_2y_3 - y_1x_2y_3 - y_1y_2x_3) \oplus i\, (x_1x_2y_3 + x_1y_2x_3 + y_1x_2x_3 - y_1y_2y_3).
\end{aligned}$$

Note that $A_{\mathbb{C}}$ is commutative if A is it.

The norm $p_{\mathbb{C}}$ defined in 3.9.3 is generally not submultiplicative. Let $A = \mathbb{R}^2$ with the multiplication of $\mathbb{C} \cong \mathbb{R}^2$ and the Euclidean norm. Then for $w := \binom{1}{0} \oplus i \binom{0}{1} \in A_{\mathbb{C}}$ the identity

$$w \bullet w = \left(\begin{pmatrix}1\\0\end{pmatrix}^2 - \begin{pmatrix}0\\1\end{pmatrix}^2\right) \oplus 2\, i \begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\1\end{pmatrix} = 2 \left(\begin{pmatrix}1\\0\end{pmatrix} \oplus i \begin{pmatrix}0\\1\end{pmatrix}\right) = 2\, w$$

holds and since

$$p_{\mathbb{C}}(w) := \max\left\{\left\|\begin{pmatrix}x\\-y\end{pmatrix}\right\| : x^2 + y^2 = 1\right\} = 1$$

we obtain a contradiction:

$$p_{\mathbb{C}}(w \bullet w) = 2\, p_{\mathbb{C}}(w) = 2 > 1 = p_{\mathbb{C}}(w)^2.$$

Therefore, none of the remaining isomorphisms in the rhombic diagram can be an isometry either. For, if one of them were an isometry, then also all others because of the commutativity, and therefore $\check{\mu} : A \to L_{\mathbb{C}}(A_{\mathbb{C}}, A_{\mathbb{C}})$ would be a contraction and thus also $(\check{\mu})_{\mathbb{C}} : A_{\mathbb{C}} \to L_{\mathbb{C}}(A_{\mathbb{C}}, A_{\mathbb{C}})$ one, i.e. $\|\mu_{\mathbb{C}}\| \leqslant 1$, i.e. $p_{\mathbb{C}}$ were submultiplicative.

However, we are able to find an equivalent submultiplication extension of the norm from A to $A_{\mathbb{C}}$. Namely let $\|\text{-}\|_{\mathbb{C}}$ be the equivalent submultiplicative norm for $p_{\mathbb{C}}$

existing by 6.5. It agrees with $p_\mathbb{C}$ on A and thus with $p := \|\_\|$, because for $a \in A \subseteq A_\mathbb{C}$, $w \in A_\mathbb{C}$ and $|z| = 1$ we have

$$\begin{aligned} p_z(a\,w) &:= p(\Re e(z\,a\,w)) = p(a\,\Re e(z\,w)) \leqslant p(a)\,p(\Re e(z\,w)) \\ &\leqslant p_\mathbb{C}(a)\,p_\mathbb{C}(w) \end{aligned}$$

and thus $p_\mathbb{C}(a \cdot w) \leqslant p_\mathbb{C}(a) \cdot p_\mathbb{C}(w)$.

Now to the universal property: Let $f_\mathbb{C}$ be the unique $\mathbb{C}$-linear extension. Then $f_\mathbb{C}$ is also an algebra homomorphism, because

$$\begin{aligned} & f_\mathbb{C}((u_1 \oplus i\,v_1) \bullet (u_2 \oplus i\,v_2)) \\ &\quad = f_\mathbb{C}((u_1u_2 - v_1v_2) \oplus i\,(u_1v_2 + v_1u_2)) \\ &\quad = f(u_1u_2 - v_1v_2) + i\,f(u_1v_2 + v_1u_2) \\ &\quad = f(u_1)f(u_2) - f(v_1)f(v_2) + i\,f(u_1)f(v_2) + i\,f(v_1)f(u_2) \\ &\quad = (f(u_1) + i\,f(v_1)) \cdot (f(u_2) + i\,f(v_2)) \\ &\quad = f_\mathbb{C}(u_1 \oplus i\,v_1) \cdot f_\mathbb{C}(u_2 \oplus i\,v_2). \quad \square \end{aligned}$$

Remark.

The complexifications of the above examples are obviously the following:

$$\begin{aligned} C(X,\mathbb{R})_\mathbb{C} &\cong C(X,\mathbb{C}) \\ L_\mathbb{R}(E,E)_\mathbb{C} &\cong L_\mathbb{C}(E_\mathbb{C},E_\mathbb{C}) \end{aligned}$$

From now on, we can assume that all Banach algebras are over $\mathbb{C}$, have a unit, and satisfy $\|a \cdot b\| \leqslant \|a\| \cdot \|b\|$ and $\|1\| = 1$.

Let us return to spectral theory. As we have already indicated, we give the following

6.7 Definition.

Let A be a Banach algebra with 1 and $a \in A$. Then one calls the set

$$\sigma_A(a) := \sigma(a) := \{\lambda \in \mathbb{C} : \lambda\,1 - a \text{ is not invertible in } A\}$$

the SPECTRUM of a. The complement

$$\rho(a) := \mathbb{C}_\infty \backslash \sigma(a) = \{\infty\} \cup (\mathbb{C}\backslash\sigma(a)),$$

in the 1-point compactification $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ of $\mathbb{C}$ is called RESOLVENT SET of a and the mapping

$$r_a : \rho(a) \to A, \quad \lambda \mapsto \begin{cases} (\lambda\,1 - a)^{-1} & \text{for } \lambda \neq \infty \\ 0 & \text{for } \lambda = \infty \end{cases}$$

is called RESOLVENT FUNCTION of a. Note that the definition $r_a(\infty) := 0$ is reasonable because of

$$\begin{aligned} \|r_a(\lambda)\| &= \|(\lambda\,1 - a)^{-1}\| = \frac{1}{|\lambda|}\left\|(1 - \frac{1}{\lambda}a)^{-1}\right\| \\ &= \frac{1}{|\lambda|}\left\|\sum_{n=0}^\infty \left(\frac{1}{\lambda}a\right)^n\right\| \leqslant \frac{1}{|\lambda|}\sum_{n=0}^\infty \left\|\frac{1}{\lambda}a\right\|^n \\ &= \frac{1}{|\lambda|}\,\frac{1}{1 - \|\frac{1}{\lambda}a\|} = \frac{1}{|\lambda| - \|a\|} \to 0 \text{ for } |\lambda| \to \infty \end{aligned}$$

Examples.

1. Let $A = C(X,\mathbb{C})$. Then $f \in A$ is invertible if and only if $0 \notin f(X)$. Consequently, $\sigma(f) = \{\lambda \in \mathbb{C} : 0 \in (\lambda - f)(X)\} = \{\lambda \in \mathbb{C} : \lambda \in f(X)\} = f(X)$.

2. Let $A = L(E) := L(E, E)$. Then $a \in A$ is invertible by the open mapping theorem if and only if a is bijective. So $\sigma(a) := \{\lambda \in \mathbb{C} : \lambda \,\mathrm{id} - a$ is not bijective$\}$.

We want to prove the holomorphy of $r_a : \mathbb{C}_\infty \supseteq \rho(a) \to A$. For this and for the following we need some tools from Complex Analysis.

Recap from complex analysis

In this section we summarize the required results from complex analysis (cf. [19]). Let F be a sequentially complete lcs. The classical theorems refer to the case $F = \mathbb{C}$ and we will first outline the proofs for this case. We will sketch how to get the vector-valued results at the end of this section.

6.8 Differential forms and line integrals.

Let E and F be lcs's and $U \subseteq E$ be open. An F-valued 1-FORM on $U \subseteq E$ is a mapping $\omega : E \supseteq U \to L(E, F)$ (see [22, 6.5.3]).

If ω is continuous, $c : [a, b] \to U$ is a C^1-curve, and F is sequentially complete, then the LINE INTEGRAL is well-defined by the vector-valued Riemann integral

$$\int_c \omega := \int_a^b \omega(c(t))(c'(t))\, dt \in F$$

(see [22, 6.5.6]). This is invariant under reparametrizations of c and for normed spaces E and F

$$\Big\| \int_c \omega \Big\|_F \leqslant (b-a) \cdot \sup_{t \in [a,b]} \|\omega(c(t))\|_{L(E,F)} \cdot \sup_{t\in[a,b]} \|c'(t)\|_E.$$

holds. As is well known, this definition can be extended to RECTIFIABLE CURVES in normed spaces using the vector-valued RIEMANN-STIELTJES INTEGRAL, and then $\| \int_c \omega\| \leqslant (b-a)\cdot \sup\{\|\omega(c(t))\| : t \in [a, b]\} \cdot V(c)$, where

$$V(c) := \sup\Big\{ \sum_{k=1}^n \|c(t_k) - c(t_{k-1})\| : a = t_0 \leqslant t_1 \leqslant \dots \leqslant t_n = b \Big\}$$

is the TOTAL VARIATION of c (see [22, 6.5.10]).

Each differentiable mapping $f : E \supseteq U \to F$ between Banach spaces E and F has as derivative $f' : E \supseteq U \to L(E, F)$ (see [22, 6.1.4]) a 1-form which is also denoted by df and is called the TOTAL DIFFERENTIAL of f. If f is affine, df is constant.

Because of the Theorem of Schwarz (see [22, 6.3.11]), this differential form satisfies the following symmetry condition for $f \in C^2$:

$$(df)'(x)(v)(w) = f''(x)(v, w) = f''(x)(w, v) = (df)'(x)(w)(v),$$

i.e. df is closed in the following sense: A continuously differentiable 1-form ω is called CLOSED if its OUTER DERIVATIVE $d\omega : E \supseteq U \to L(E, L(E, F)) \cong L(E, E; F)$, defined by $d\omega(x)(v, w) = \omega'(x)(v)(w) - \omega'(x)(w)(v)$, vanishes. Instead of "$\omega$ is closed" one also says that the INTEGRABILITY CONDITION $\omega'(x)(v)(w) = \omega'(x)(w)(v)$ holds.

Conversely, for star shaped or, more generally, for simple connected sets U, one can show that each closed 1-form $\omega : U \to L(E, F)$ is EXACT, i.e. a differentiable mapping $f : E \supseteq U \to F$ exists with $df = \omega$ (see [22, 6.5.4]).

As a consequence, the line integral of closed 1-forms is locally independent on the curve and therefore coincides on homotopic curves, where two curves c_0 and c_1 are called HOMOTOPIC if a continuous mapping $H : [0, 1] \times [a, b] \to U$ exists with $H(j, t) = c_j(t)$ for all $j \in \{0, 1\}$ and all $t \in [a, b]$.

6.9 Holomorphic functions.

A mapping $f : \mathbb{C} \supseteq U \to F$ is called $\mathbb{C}$-differentiable or (more frequently) holomorphic if the following limit exists for all $z \in U$

$$f'(z) := \lim_{\mathbb{C}\ni w\to 0} \frac{f(z+w)-f(z)}{w} \in F.$$

We will write $H(U,F)$ (and $H(U)$ in case $F = \mathbb{C}$) for the vector space of all holomorphic mappings $f : U \to F$. If $f : \mathbb{C} \supseteq U \to F$ is holomorphic, then f is also $\mathbb{R}$-differentiable as mapping $f_{\mathbb{R}}$ from $U \subseteq \mathbb{R}^2$ into the real vector space $F_{\mathbb{R}}$ and the $\mathbb{R}$-derivative $(f_{\mathbb{R}})'(z) \in L_{\mathbb{R}}(\mathbb{R}^2, F_{\mathbb{R}})$ is then even $\mathbb{C}$-linear and coincides with $w \mapsto f'(z)\cdot w$ (where we let the skalar multiplication act from the right), because

$$\lim_{\|w\|\to 0} \frac{\|f(z+w)-f(z)-f'(z)\cdot w\|_F}{\|w\|_{\mathbb{C}}} = \lim_{w\to 0}\left\|\frac{f(z+w)-f(z)}{w} - f'(z)\right\|_F = 0.$$

But the converse implication also holds (see [19, 2.5]):
The $\mathbb{C}$-linearity of the derivative $(f_{\mathbb{R}})'(z) \in L_{\mathbb{R}}(\mathbb{R}^2, F)$ of a $\mathbb{R}$-differentiable mapping $f : \mathbb{C} \supseteq U \to F$ means that $(f_{\mathbb{R}})'(z)$ is given by multiplication $w = 1\cdot w \mapsto (f_{\mathbb{R}})'(z)(1\cdot w) = (f_{\mathbb{R}})'(z)(1)\cdot w$. If we put $f'(z) := (f_{\mathbb{R}})'(z)(1) \in F$ then

$$\begin{aligned}0 &= \lim_{\|w\|\to 0} \frac{\|f(z+w)-f(z)-(f_{\mathbb{R}})'(z)(w)\|}{\|w\|} = \lim_{\|w\|\to 0}\left\|\frac{f(z+w)-f(z)-f'(z)\cdot w}{w}\right\| \\ &= \lim_{w\to 0}\left\|\frac{f(z+w)-f(z)}{w} - f'(z)\right\|, \text{ hence } f'(z) = \lim_{w\to 0}\frac{f(z+w)-f(z)}{w}.\end{aligned}$$

For $F = \mathbb{C}$ we can also describe the $\mathbb{C}$-linearity of the derivative in real coordinates as follows: To do this, we decompose f into real and imaginary part, i.e. $f = g + i\,h$, and $w = (u,v) = u + i\,v$. Then,

$$(f_{\mathbb{R}})'(z) = \begin{pmatrix} \frac{\partial g}{\partial x}(z) & \frac{\partial g}{\partial y}(z) \\ \frac{\partial h}{\partial x}(z) & \frac{\partial h}{\partial y}(z)\end{pmatrix} =: \begin{pmatrix} a & b \\ c & d\end{pmatrix}$$

is $\mathbb{C}$-linear if and only if

$$\begin{pmatrix} bu - av \\ du - cv\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\cdot i\begin{pmatrix} u \\ v\end{pmatrix} = i\begin{pmatrix} a & b \\ c & d\end{pmatrix}\cdot\begin{pmatrix} u \\ v\end{pmatrix} = \begin{pmatrix} -cu - dv \\ au + bv\end{pmatrix}$$

holds for all $u + iv \in \mathbb{C}$, i.e. (by means of coefficient comparison) iff $d = a$ and $c = -b$ holds. These are exactly the Cauchy-Riemann differential equations (see [19, 2.6])

$$\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y}, \quad \frac{\partial g}{\partial y} = -\frac{\partial h}{\partial x}.$$

If $f : \mathbb{C} \supseteq U \to F$, then $\omega : U \to L(\mathbb{C}, F)$, defined by $\omega(z) := f(z)\cdot dz$, is an $F_{\mathbb{R}}$-valued 1-form, where dz denotes the (constant) derivative of the $\mathbb{C}$-linear function id $: z \mapsto z$. Here the multiplication $f(z)\cdot dz$ is given by the mapping $F \times L(\mathbb{C},\mathbb{C}) \to L(\mathbb{C}, F)$, $(y, T) \mapsto (z \mapsto T(z)\cdot y)$. With slight abuse of notation one uses the same symbol dz for the 1-form $U \to L(\mathbb{C},\mathbb{C})$ and its value id $\in L(\mathbb{C},\mathbb{C})$. If f is holomorphic, then the 1-form $z \mapsto f(z)\,dz$ is closed, because its (real) derivative at the point z is given by $v \mapsto (w \mapsto f'(z)\cdot v\cdot w)$, and hence is symmetric in v and w (see [19, 3.5]).

Let dx and dy denote the (constant) derivatives of the $\mathbb{R}$-linear functions $\mathfrak{Re} : z = x + i\,y \mapsto x$ and $\mathfrak{Im} : x + i\,y \mapsto y$, i.eȧ basis in the real vector space $L_{\mathbb{R}}(\mathbb{C},\mathbb{R})$. Then obviously $dz = dx + i\,dy$ and analogously $d\bar z = dx - i\,dy$, where $d\bar z$ denotes the derivative of $z \mapsto \bar z$. So $\{dz, d\bar z\}$ is also a basis of the complex vector space

$L_\mathbb{R}(\mathbb{C},\mathbb{R})_\mathbb{C} \cong L_\mathbb{R}(\mathbb{C},\mathbb{C})$ which is equivalent to the standard basis $\{dx, dy\}$. For each $\mathbb{R}$-differentiable $f : \mathbb{C} \supseteq U \to \mathbb{C}$ we have

$$df(z) = \frac{\partial f}{\partial x}(z)\,dx + \frac{\partial f}{\partial y}(z)\,dy.$$

Consequently, $df(z)$ must have also a matrix representation with respect to the basis $\{dz, d\bar z\}$. And one denotes the corresponding coefficients in analogy by $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar z}$, i.e.

$$df = \frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial \bar z}\,d\bar z.$$

Because of $2\,dx = dz + d\bar z$ and $2i\,dy = dz - d\bar z$, we can also easily calculate these coefficients (the so-called Wirtinger derivatives):

$$\begin{aligned} df &= \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \\ &= \frac{\partial f}{\partial x}\,\frac{dz + d\bar z}{2} + \frac{\partial f}{\partial y}\,\frac{dz - d\bar z}{2i} \\ &= \underbrace{\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\,\frac{\partial f}{\partial y}\right)}_{=\frac{\partial f}{\partial z}} dz + \underbrace{\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\,\frac{\partial f}{\partial y}\right)}_{=\frac{\partial f}{\partial \bar z}} d\bar z, \end{aligned}$$

that is

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \bar z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right). \end{aligned}$$

Since dz is $\mathbb{C}$-linear and $d\bar z$ is conjugated $\mathbb{C}$-linear, f is holomorphic if and only if $\frac{\partial}{\partial \bar z} f = 0$ (see [**19**, 2.11]). Likewise, f is ANTI-HOLOMORPHIC, i.e. $\overline{f}$ holomorphic, if $\frac{\partial}{\partial z} f = 0$, because

$$d\bar f = \frac{\partial \bar f}{\partial z}\,dz + \frac{\partial \bar f}{\partial \bar z}\,d\bar z = \overline{\left(\frac{\partial f}{\partial \bar z}\right)}\,dz + \overline{\left(\frac{\partial f}{\partial z}\right)}\,d\bar z.$$

6.10 Cauchy Integral Theorem.

If $f : \mathbb{C} \supseteq U \to F$ is holomorphic and c_0 and c_1 are two curves $I \to U$ being homotopic in U relative $\partial I = \{0,1\}$ (i.e. the homotopy satisfies $H(j,t) = c_j(t)$ in addition to $H(s,k) = c_j(k)$ for all $j,k \in \{0,1\}$ and all t and s), then

$$\int_{c_0} f(z)\,dz = \int_{c_1} f(z)\,dz.$$

In particular, if $c : S^1 \to U$ is a closed curve, which is homotopic in U to a constant curve (i.e. is called 0-HOMOTOPIC), then $\int_c f(z)\,dz = 0$.

See [**19**, 3.18] and [**19**, 3.23].

Proof. The first part is a consequence of the closedness of the 1-form $z \mapsto f(z)\,dz$.

For the second part, note that from a (free) homotopy H between c and a constant curve konst_x a homotopy relatively $\{0,1\}$ of c with the concatenation of the curves $c_1 : t \mapsto H(t,1)$, the constant curve konst_x and the reversed curve $t \mapsto H(1-t,1)$, can be constructed. So $\int_c f(z)\,dz = \int_{c_1} f(z)\,dz - \int_{c_1} f(z)\,dz = 0$. □

6.11 Winding number.

If c is a closed C^1-curve in $\mathbb{C}\backslash\{z\}$, then

$$\operatorname{ind}_c(z) := \frac{1}{2\pi i}\int_c \frac{1}{w-z}\,dw$$

is called the winding number (or revolution number) of c at z, see [19, 3.24]. For a circle $c : t \mapsto z + r\,e^{2\pi i\,t}$ with center z and radius r we obviously get

$$\begin{aligned}\operatorname{ind}_c(z) &= \frac{1}{2\pi i}\int_c \frac{1}{w-z}\,dw = \frac{1}{2\pi i}\int_0^1 \frac{2\pi i\,r\,e^{2\pi i\,t}}{r\,e^{2\pi i\,t}}\,dt\\ &= \int_0^1 1\,dt = 1.\end{aligned}$$

Since $w \mapsto \frac{1}{w-z}$ is holomorphic on $\mathbb{C}\backslash\{z\}$, this integral is homotopy invariant and therefore constant for z varying in a connected component of $\mathbb{C}\backslash c(S^1)$: In fact

$$\operatorname{ind}_c(z_s) = \frac{1}{2\pi i}\int_c \frac{1}{w-z_s}\,dw = \frac{1}{2\pi i}\int_{c_s} \frac{1}{w}\,dw$$

for each curve $s \mapsto z_s$ in $\mathbb{C}\backslash c(S^1)$, where $c_s(t) := c(t) - z_s$ describes a homotopy.

For a closed curve c, which in $\mathbb{C}\backslash\{z\}$ is homotopic to the k-fold traversed circle, consequently $\operatorname{ind}_c(z) = k$ holds. In Algebraic Topology (see [17, 2.17]) it is shown that the winding number is a topological invariant, which means is well-defined even for closed continuous curve, is homotopy invariant, and has values in $\mathbb{Z} \subseteq \mathbb{C}$. Furthermore, it is shown that every closed curve in $\mathbb{C}\backslash\{z\}$ is homotopic to the $\operatorname{ind}_c(z)$-fold traversed unit circle with center z.

6.12 Cauchy Integral Formulas.

Let $f : \mathbb{C} \supseteq U \to F$ be holomorphic, K a closed disc in U and z in the interior of K. Then

$$f(z) = \frac{1}{2\pi i}\int_{\partial K} \frac{f(w)}{w-z}\,dw,$$

where ∂K denotes the positively parameterized (i.e. $\operatorname{ind}_{\partial K}(z) = +1$) boundary of K.

Furthermore, f is infinitely often $\mathbb{C}$-differentiable and

$$f^{(p)}(z) = \frac{p!}{2\pi i}\int_{\partial K} \frac{f(w)}{(w-z)^{p+1}}\,dw.$$

holds for each $p \in \mathbb{N}$.

See [19, 3.28].

Proof. Let $g(w) := \frac{f(w)-f(z)}{w-z}$. Then g is holomorphic on $U\backslash\{z\}$ and bounded on K since f is differentiable at z. According to the Cauchy Integral Theorem 6.11 $\int_{\partial K} g = \int_{\partial K_\varepsilon} g$, where K_ε is a disc of radius $\varepsilon > 0$ at z. Now use $\|\int_{K_\varepsilon} g\| \leqslant 2\pi\varepsilon\|g|_K\|_\infty \to 0$ for $\varepsilon \to 0$ to obtain get $0 = \int_{\partial K} g = \int_{\partial K} \frac{f(w)}{w-z}\,dw - f(z)\cdot 2\pi i$.

That f is often infinitely differentiable follows by interchanging the derivative with the integral:

$$\begin{aligned}f^{(p)}(z) &= \left(\frac{d}{dz}\right)^p \frac{1}{2\pi i}\int_{\partial K} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{\partial K} f(w)\left(\frac{d}{dz}\right)^p \frac{1}{w-z}\,dw\\ &= \frac{p!}{2\pi i}\int_{\partial K} \frac{f(w)}{(w-z)^{p+1}}\,dw. \quad\square\end{aligned}$$

6.13 Cauchy Estimate.

Let $f : \mathbb{C} \supseteq U \to F$ be holomorphic and K be a disc with radius r and center z in U. Then:

$$\left\| \frac{f^{(n)}(z)}{n!} \right\| \leqslant \frac{\|f|_{\partial K}\|_\infty}{r^n}.$$

In particular, the Taylor series at the point z of f is uniformly convergent on K to f.

See [**19**, 3.30].

Proof. The inequality follows by estimating the integral, and the absolute and uniform convergence of the Taylor series, by considering a slightly larger disc K_R with radius $R > r$ as follows:

$$\Big\|\sum_k \frac{w^k}{k!} f^{(k)}(z)\Big\| \leqslant \sum_k |r|^k \frac{\|f^{(k)}(z)\|}{k!} \leqslant \|f|_{K_R}\|_\infty \sum_k \Big(\frac{r}{R}\Big)^k.$$

That the Tayler series converges to f uniformly on K follows by the integral formula $\boxed{6.12}$ of Cauchy:

$$f(z) \overset{\boxed{6.12}}{=\!=} \frac{1}{2\pi i} \int_{\partial K_R} \frac{f(w)}{w - z}\, dw = \frac{1}{2\pi i} \int_{\partial K_R} \sum_{k=0}^\infty \frac{f(w)}{(w - z_0)^{k+1}} (z - z_0)^k\, dw$$
$$= \sum_{k=0}^\infty (z - z_0)^k \frac{1}{2\pi i} \int_{\partial K_R} \frac{f(w)}{(w - z_0)^{k+1}}\, dw \overset{\boxed{6.12}}{=\!=} \sum_{k=0}^\infty (z - z_0)^k \frac{f^{(k)}(z_0)}{k!} \qquad \square$$

6.14 Identity Theorem.

Let $f : \mathbb{C} \supseteq U \to F$ be holomorphic on the open connected set U and vanishing on a convergent not finally constant sequence in U. Then $f = 0$.

See [**19**, 4.7].

Proof. By means of induction this follows for (the coefficients of a) convergent power series at the limit point of the sequence, so f is 0 locally around the limit point. Hence a maximal open connected set $W \subseteq U$ exists on which f vanishes. However, it must also be closed in U and thus agree with U. $\square$

6.15 Removable Singularity.

Let $z \in U$ and $f : U\backslash\{z\} \to F$ be holomorphic and near z be locally bounded. Then f has a holomorphic extension to all of U.

See also [**19**, 3.31].

Proof. Let K be a disc around z in U on which f is bounded. Let $z' \in K\backslash\{z\}$. As in the proof of the Cauchy Integral Formula $\boxed{6.12}$, it is shown that for the function $w \mapsto \frac{f(w)-f(z')}{w-z'}$, which is bounded on K and is holomorphic on $U\backslash\{z, z'\}$, we have: $0 = \int_{\partial K} \frac{f(w)-f(z')}{w-z'}\, dw = \int_{\partial K} \frac{f(w)}{w-z'}\, dw - f(z')\, 2\pi i$. The integral on the right side is holomorphic with respect to z' in the interior of K, so the same holds for f. $\square$

6.16 Theorem of Liouville.

Let $f : \mathbb{C} \to F$ be holomorphic and bounded, then f is constant.

See [**19**, 3.42].

Proof. By $\boxed{6.13}$ $|f'(z)| \leqslant \frac{\|f\|_\infty}{r}$ for all $r > 0$ and all $z \in \mathbb{C}$, so $f' = 0$ and thus f is constant. $\square$

6.17 Maximum Modulus Principle.

Let $f : \mathbb{C} \supseteq U \to F$ be holomorphic and not constant on the open and connected set U. Then $z \mapsto \|f(z)\|$ does not attain its supremum.

See [19, 3.41].

Proof. Let $F = \mathbb{C}$. Suppose there is a maximum at $z_0 \in U$, i.e. $|f(z)| \leqslant |f(z_0)|$ for all $z \in U$. We first show that this implies the constancy of $z \mapsto |f(z)|$. Assuming this were not the case, then there would be a $z_1 \in U$ with $|f(z_0)| > |f(z_1)|$. Since U is connected, we can connect z_0 wich z_1 by a curve $t \mapsto z_t$. We choose t_0 maximal with $|f(z_{t_0})| = |f(z_0)|$. Then there are arbitrary close to z_{t_0} points z_t with $|f(z_0)| > |f(z_t)|$. We choose a circle $K \subseteq U$ at z_{t_0} whose periphery contains such a point z_{t_1}. Then $|f(z_{t_1})| < |f(z_0)|$ and $|f(z)| \leqslant |f(z_0)|$ for all $z \in \partial K$. From the Cauchy Integral Formula 6.12 we obtain $|f(z_{t_0})| < |f(z_0)|$, a contradiction.

If the constant $|f|$ is 0 we are done. Otherwise, by differentiating the constant $|f|^2$ we obtain.

$$0 = \frac{\partial}{\partial \bar z}(f \cdot \bar f)(z) = \frac{\partial f(z)}{\partial \bar z} \cdot \bar f(z) + f(z) \cdot \frac{\partial \bar f(z)}{\partial \bar z} = 0 + f(z) \cdot \overline{\left(\frac{\partial f}{\partial z}\right)}$$

Since $|f| \neq 0$, we conclude $0 = \frac{\partial f}{\partial z} = f'(z)$, hence f is constant. □

6.18 Differentiable structure of $\mathbb{C}_\infty$, holomorphy at ∞.

In order to be able to speak about differentiability of functions such as r_a on open subsets of $\mathbb{C}_\infty$, we have to provide $\mathbb{C}_\infty$ with a differentiable structure (see [19, 2.18,2.19]). We identify $\mathbb{C}_\infty$ with the unit sphere $S^2 := \{(y,t) \in \mathbb{C}\times\mathbb{R} : |y|^2 + t^2 = 1\}$ in $\mathbb{C}\times\mathbb{R} = \mathbb{R}^3$. The embedding of $\mathbb{C}$ in S^2 is given by the inverse to the stereographic projection with the North Pole $N := (0,0,1) \in S^2$ as center onto the equatorial plane $\mathbb{C} \times \{0\} \cong \mathbb{C}$. The North Pole itself corresponds to the point $\infty \in \mathbb{C}_\infty$. The basic proportionality theorem (or intercept theorem) $z : 1 = y : (1-t)$ shows that the stereographic projection is given by

$$\mathbb{C} \times \mathbb{R} \supset S^2\backslash\{N\} \ni (y,t) \mapsto z = \frac{1}{1-t} y \in \mathbb{C} \cong \mathbb{C} \times \{0\}$$

and its inverse is

$$\varphi_+ : \mathbb{C} \ni z \mapsto \frac{1}{|z|^2+1}(2z, |z|^2 - 1) \in \mathbb{C}\times\mathbb{R},$$

because the second intersection point of the sphere with the straight line $t \mapsto z + t(N - z)$ through N and z is given by the solution $t = \frac{|z|^2-1}{|z|^2+1}$ of the equation $1 = \|t\,N + (1-t)\,z\|^2 = t^2 + (1-t)^2|z|^2$.

So this provides one "chart" for S^2. We can also define another chart now around N by analogously using the inverse φ_- of the stereographic projection $(y,t) \mapsto (y,-t) \mapsto \frac{1}{1+t} y$ with respect to the South Pole $S := -N$.

Using these charts we transfer the definition of differentiability to functions $f : S^2 \supseteq U \to F$ by requesting that the two compositions $f \circ \varphi_j : \mathbb{C} \supseteq \varphi_j^{-1}(U) \to U \to F$ for $j \in \{+,-\}$ are differentiable. It should be checked, however, that for points $(x,t) \in S^2$ in temperate latitudes, i.e. those in $\varphi_+(\mathbb{C}) \cap \varphi_-(\mathbb{C})$, the differentialiability of $f \circ \varphi_+$ at $\varphi_+^{-1}(x,t)$ is equivalent to that of $f \circ \varphi_-$ at $\varphi_-^{-1}(x,t)$. Because of $f \circ \varphi_- = (f \circ \varphi_+) \circ (\varphi_+^{-1} \circ \varphi_-)$, it is enough to show that the chart change $\varphi_+^{-1} \circ \varphi_- : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$ is differentiable. This is given by

$$z \mapsto \frac{1}{|z|^2+1}(2z, -(|z|^2 - 1)) \mapsto \frac{1}{1 - \frac{1-|z|^2}{1+|z|^2}} \frac{2z}{|z|^2+1} = \frac{z}{|z|^2} = \frac{1}{\bar z}.$$

This is the reflection at the unit circle, as can also be easily seen by means of elementary geometrical considerations. This mapping is smooth and anti-holomorphic, so we should compose the second chart yet with the conjugation $\mathbb{C} \to \mathbb{C}$, $z \mapsto \overline{z}$ to get the holomorphic mapping $z \mapsto \frac{1}{z}$ as a new chart change.

In summary, this means that a mapping $f : \mathbb{C}_\infty \supseteq U \to F$ is called holomorphic if both $f|_\mathbb{C} : \mathbb{C} \cap U \to F$ and $z \mapsto f(\frac{1}{z})$ is holomorphic $\{z \in \mathbb{C} : \frac{1}{z} \in U\} \to F$. See also [**19**, 2.18].

6.19 Chains and cycles.

Since we want to use not only discs but general compact sets $K \subseteq \mathbb{C}$, we have to replace closed curves with something more general, namely so-called 1-CHAINS, i.e. formal linear combinations $c := \sum_j k_j\, c_j$ of curves $c_i : [0,1] \to U$ with coefficients $k_i \in \mathbb{Z}$. The set of all 1-chains forms an Abelian group (all mappings $C([0,1],U) \to \mathbb{Z}$ with finite support) with respect to the componentwise addition. The BOUNDARY ∂c of a 1-chain is a 0-chain, i.e. a formal linear combination of points, which is defined as follows $\partial c := \sum_j k_j\,(c_j(1) - c_j(0))$. A 1-chain c is called CYCLE if $\partial c = 0$. This is in particular the case when all c are closed curves. The subset formed by all cycles is a subgroup of 1-chains. One extends the line integral of 1-forms ω to 1-chains c by linearity, i.e.

$$\int_c \omega = \sum_j k_j \int_{c_j} \omega$$

and defines the winding number of 1-cycles c again by

$$\operatorname{ind}_c(z) := \frac{1}{2\pi i} \int_c \frac{1}{w-z}\, dw$$

for all $z \notin \operatorname{img}(c) := \bigcup_j c_j[0,1]$.

A 1-cycle c is called 0-HOMOLOGOUS in U if $\operatorname{ind}_c(z) = 0$ for all $z \notin U$. Two cycles c_1 and c_2 are called HOMOLOGOUS in U if $c_1 - c_2$ is 0-homologous, i.e. $\operatorname{ind}_{c_1}(z) = \operatorname{ind}_{c_2}(z)$ for all $z \notin U$.

Note that two closed curves that are homotopic in U are also homologous because of the homotopy invariance of the winding number. The converse implication does not hold, since homotopy is not commutative. Let us now generalize Cauchy's Integral Theorem 6.10 and Cauchy's Integral Formula 6.12.

6.20 Generalized Cauchy Integral Theorem and Integral Formula.

Let $f : \mathbb{C} \supseteq U \to F$ be holomorphic. For any two homologous cycles c_1 and c_2 in U we have

$$\int_{c_1} f(z)\, dz = \int_{c_2} f(z)\, dz.$$

If c is a 0-homologous cycle in U, then

$$f(z)\, \operatorname{ind}_c(z) = \frac{1}{2\pi i} \int_c \frac{f(w)}{w-z}\, dw \text{ for all } z \in U \setminus \operatorname{img}(c).$$

Proof. First we prove the second part. For this we consider the mapping $\varphi : (z,w) \mapsto \frac{f(w)-f(z)}{w-z}$ for $z \neq w$ and $\varphi : (z,z) \mapsto f'(z)$. We have that $\varphi : U \times U \to F$ is continuous (and indeed even holomorphic, according to Hartogs' Theorem and

Theorem 6.15 on removable singularities). For $z \in U$ let $h(z) := \frac{1}{2\pi i}\int_c \varphi(z,w)\,dw$. In particular, for $z \in U \setminus \operatorname{img}(c)$, we get

$$\begin{aligned} h(z) &= \frac{1}{2\pi i}\int_c \frac{f(w)}{w-z}\,dw - \frac{f(z)}{2\pi i}\int_c \frac{1}{w-z}\,dw \\ &= \frac{1}{2\pi i}\int_c \frac{f(w)}{w-z}\,dw - f(z)\,\operatorname{ind}_c(z). \end{aligned}$$

So we have to show that $h = 0$. It is easy to see that $h : U \to F$ can be holomorphically extended to $\mathbb{C}$ by

$$h(z) := \frac{1}{2\pi i}\int_c \frac{f(w)}{w-z}\,dw \text{ for } z \in U_1 := \{z \notin \operatorname{img}(c) : \operatorname{ind}_c(z) = 0\} \supseteq \mathbb{C}\setminus U.$$

Since this integral goes to 0 for $z \to \infty$, h is bounded and thus according to the Theorem 6.16 of Liouville identical to $h(\infty) = 0$.

Now for the first part. It suffices to show that $\int_c f(z)\,dz = 0$ for the 0-homologous cycle $c := c_1 - c_2$. For $z \in U \setminus \operatorname{img}(c)$ let $f_z(w) := (w-z)\,f(w)$. Then, by the second part,

$$0 = f_z(z)\,\operatorname{ind}_c(z) = \frac{1}{2\pi i}\int_c \frac{f_z(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_c f(w)\,dw \quad \square$$

6.21 Lemma. Captureing holes.

Let $U \subseteq \mathbb{C}$ be open and $K \subseteq U$ compact. Then, a 1-cycle $c = \sum_j c_j$ of smooth closed curves c_j exists in $U\setminus K$ so that $\operatorname{ind}_c(z) \in \{0,1\}$ holds for all $z \notin \operatorname{img}(c)$. Let the interior and exterior of c be defined by

$$\begin{aligned} \operatorname{inn}(c) &:= \{z \notin \operatorname{img}(c) : \operatorname{ind}_c(z) = 1\} \\ \operatorname{out}(c) &:= \{z \notin \operatorname{img}(c) : \operatorname{ind}_c(z) = 0\}. \end{aligned}$$

Then $K \subseteq \operatorname{inn}(c) \subseteq U$, or equivalently, $\mathbb{C}\setminus U \subseteq \operatorname{out}(c) \subseteq \mathbb{C}\setminus K$.
Such a cycle is called a JORDAN SYSTEM.

Proof. Let $0 < 2\delta < d(K, \mathbb{C}\setminus U)$. We consider straight lines parallel to the axes with distance δ between them. Let $R_1, \dots, R_m$ be those (finite many) squares (with side length δ) which meet (the compact set) K. The boundary ∂R_j of R_j is a broken line which we orient positively.

For $z \in R_j$ we have $d(z,K) < \sqrt{2}\delta$ and thus $R_j \subseteq U$. Let $c_1, \dots, c_n$ be those edges that belong to exactly one of the R_i. Then $\sum_{k=1}^n \int_{c_k} \omega = \sum_{j=1}^m \int_{\partial R_j} \omega$ for each continuous 1-form ω on $\bigcup_{j=1}^m \partial R_j$, because the other edges belong to two of the R_i with opposite orientation.

We have that the image of c_k is included in $U\setminus K$, otherwise the two adjacent squares would meet K and thus be in U, a contradiction to the choice of c_k.

For $f \in H(U)$ and $z \in K\setminus \bigcup_j \partial R_j$, $w \mapsto \frac{1}{2\pi i}\frac{f(w)}{w-z}\,dw$ is a continuous 1-form on $\bigcup \partial R_j$ and thus

$$\sum_{k=1}^n \frac{1}{2\pi i}\int_{c_k} \frac{f(w)}{w-z}\,dw = \sum_{j=1}^m \frac{1}{2\pi i}\int_{\partial R_j} \frac{f(w)}{w-z}\,dw.$$

We have

$$\frac{1}{2\pi i}\int_{\partial R_j} \frac{f(w)}{w-z}\,dw = \begin{cases} 0 & \text{for } z \notin R_j \\ f(z) & \text{for } z \in R_j \end{cases}$$

by the Cauchy Integral Formula 6.12. Since z is an inner point in exactly one of the R_j, we have

$$f(z) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{c_k} \frac{f(w)}{w-z}\, dw.$$

Since both sides are continuous for $z \in K$, this equation holds for all of K.

If we investigate the intersection of K with the 4 squares with a common vertices, we see that $c := \sum_j c_j$ is a cycle, hence a finite sum of closed polygons.

For $z \in K$ we have $1 = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{c_k} \frac{1}{w-z}\, dw = \operatorname{ind}_c(z)$, i.e. $K \subseteq \operatorname{inn}(c)$.

If $z \notin U$, then $\int_{\partial R_j} \frac{1}{w-z}\, dw = 0$ for all j and thus $\operatorname{ind}_c(z) = 0$, i.e. $\operatorname{inn}(c) \subseteq U$. □

To obtain these theorems from complex analysis for vector-valued functions, one can successfully use the following lemma.

6.22 Lemma.

Let F be a sequentially complete lcs. Then $f : \mathbb{C} \supseteq U \to F$ is holomorphic if and only if $\ell \circ f : \mathbb{C} \supseteq U \to F \to \mathbb{C}$ is holomorphic for all $\ell \in F^*$.

Proof. ($\Rightarrow$) is obvious, because $\ell \in F^*$, as linear continuous mapping, commutes with limits and difference quotient formation.

($\Leftarrow$) The following holds:

$$\begin{aligned}
\ell\left(\frac{f(z)-f(0)}{z} - \frac{f(w)-f(0)}{w}\right) &= \frac{(\ell\circ f)(z)-(\ell\circ f)(0)}{z} - \frac{(\ell\circ f)(w)-(\ell\circ f)(0)}{w}\\
&= \int_0^1 (\ell\circ f)'(t\,z) - (\ell\circ f)'(t\,w)\, dt\\
&= (z-w)\int_0^1\int_0^1 t\,(\ell\circ f)''(tw + ts(z-w))\, ds\, dt.
\end{aligned}$$

Since $\ell \circ f$ is holomorphic, $\ell \circ f$ is 2 times continuously differentiable and thus the integrand is uniformly bounded for $t, s \in [0,1]$ and z, w near 0. So also the integral is bounded locally in z and w near 0, and thus

$$\frac{1}{z-w}\left(\frac{f(z)-f(0)}{z} - \frac{f(w)-f(0)}{w}\right)$$

is scalarly bounded and even bounded by 4.2.7. So the net $\frac{f(z)-f(0)}{z} - \frac{f(w)-f(0)}{w} \to 0$ converges for $w, z \to 0$, i.e. $w \mapsto \frac{f(w)-f(0)}{w}$ is a Cauchy net and consequently converges (since each subsequence converges), i.e. f is holomorphic. □

By means of this lemma, all of the above mentioned results from complex analysis can be transferred to the vector-valued case.
E.g., for the Theorem 6.16 of Liouville this goes as follows: Let $f : \mathbb{C} \to F$ be holomorphic and bounded. Then $\ell \circ f : \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded, i.e. according to the classical theorem constant, for all $\ell \in F^*$. Since these ℓ are point-separating, f itself is constant.
For the Cauchy integral theorem 6.10 and the integral formula 6.12, and 6.20, note:

$$\ell\left(\int_c f\right) = \int_c \ell \circ f \text{ and } \ell \circ f' = (\ell \circ f)'.$$

Let now A be a complex Banach algebra with 1.

6.23 Lemma.

For $a \in A$:

1. *If $\lambda \in \rho(a)$, then* $\operatorname{dist}(\lambda, \sigma(a)) \geqslant \|(\lambda - a)^{-1}\|^{-1}$.
2. *For $\lambda, \mu \in \rho(a)$, we have the resolvent equality:*

$$\frac{r_a(\lambda) - r_a(\mu)}{\lambda - \mu} = -r_a(\lambda)\, r_a(\mu) = -r_a(\mu)\, r_a(\lambda).$$

Proof. (1) Let $\lambda \in \rho(a)$ and $|\mu| < \|(\lambda - a)^{-1}\|^{-1}$. Then $\lambda + \mu \in \rho(a)$ and thus $\operatorname{dist}(\lambda, \sigma(a)) \geqslant \|(\lambda - a)^{-1}\|^{-1}$ holds, because $\lambda + \mu - a$ is invertible by 6.2.2 since $\|(\lambda + \mu - a) - (\lambda - a)\| = |\mu| < \|(\lambda - a)^{-1}\|^{-1}$.

(2) With $x := \lambda - a$ and $y := \mu - a$

$$\begin{aligned} r_a(\lambda) - r_a(\mu) &= x^{-1} - y^{-1} = x^{-1}\,(y - x)\, y^{-1} \\ &= (\lambda - a)^{-1}\,(\mu - \lambda)\,(\mu - a)^{-1} = (\mu - \lambda)\,(\lambda - a)^{-1}\,(\mu - a)^{-1} \\ &= (\mu - \lambda)\, r_a(\lambda)\, r_a(\mu). \quad \square \end{aligned}$$

6.24 Theorem.

Let $a \in A$. Then the spectrum $\sigma(a)$ of a is compact and non-empty. The resolvent function is holomorphic from the open subset $\rho(a)$ of the Riemannian sphere $\mathbb{C}_\infty$ to A.

Proof. For $|\lambda| > \|a\|$: $\lambda 1 - a = \lambda(1 - \frac{1}{\lambda} a)$ and $\|1 - (1 - \frac{1}{\lambda} a)\| = \|\frac{1}{\lambda} a\| = \frac{\|a\|}{|\lambda|} < 1$, hence $1 - \frac{1}{\lambda} a$ is invertible by 6.2.1, and so is $\lambda 1 - a = \lambda(1 - \frac{1}{\lambda} a)$, i.e. $\lambda \in \rho(a)$. So $\sigma(a) \subseteq \{\lambda : |\lambda| \leqslant \|a\|\}$ and is therefore bounded.

We have $\rho(a) \cap \mathbb{C} := \{\lambda \in \mathbb{C} : \lambda 1 - a \in \operatorname{Inv}(A)\}$. Since the affine mapping $\lambda \mapsto \lambda 1 - a$ is continuous, its inverse image of the open set $\operatorname{Inv}(A)$ is also open. So $\rho(a) \cap \mathbb{C}$ is open in $\mathbb{C}$.

Consequently, $\sigma(a) = \mathbb{C} \backslash (\rho(a) \cap \mathbb{C})$ is closed and bounded in $\mathbb{C}$, i.e. compact.

So $\sigma(a)$ is also compact in $\mathbb{C}_\infty$, and thus $\rho(a) = \mathbb{C}_\infty \backslash \sigma(a)$ is open in $\mathbb{C}_\infty$.

The mapping $\lambda \mapsto (\lambda 1 - a) \mapsto (\lambda 1 - a)^{-1}$ is, considered as composition of an affine mapping with a (by 6.2.4) complex-differentiable mapping, a complex differentiable mapping $r_a : \rho(a) \cap \mathbb{C} \to \operatorname{inv}(A) \subseteq A$ and, by the Chain Rule, we obtain for the derivative:

$$r_a'(\lambda) = \operatorname{inv}'(\lambda 1 - a) \cdot 1 = -(\lambda - a)^{-1}\, 1\, (\lambda - a)^{-1} = -(\lambda - a)^{-2}.$$

If one does not want to use the complex differentiability of the inversion, then this can also easily be calculated by means of resolvent equation 6.23.2.

For the holomorphy at ∞ we have to study the mapping $z \mapsto \frac{1}{z} \mapsto r_a(\frac{1}{z})$ near 0. For $z \neq 0$ this is holomorphic because $\rho(a)$ is a neighborhood of ∞ and because of $\lim_{z \to \infty} r_a(z) = 0 =: r_a(\infty)$ (see 6.7) we have that r_a is holomorphic at 0 by 6.15. Directly one sees this also from the fact that for $\|z\, a\| < 1$ (i.e. for $|z| < \frac{1}{\|a\|}$) this mapping can be developed into a convergent power series:

$$\begin{aligned} r_a\left(\frac{1}{z}\right) &= \left(\frac{1}{z} - a\right)^{-1} = \left(\frac{1}{z}(1 - z\, a)\right)^{-1} = z\,(1 - z\, a)^{-1} \\ &= z \sum_{k=0}^{\infty} (z\, a)^k = \sum_{k=0}^{\infty} z^{k+1}\, a^k. \end{aligned}$$

It only remains to show that the spectrum is not empty:

Otherwise, $r_a : \mathbb{C}_\infty \to A$ would be a holomorphic (hence bounded) function on the whole $\mathbb{C}_\infty$ and thus according to the theorem 6.16 of Liouville constant. Because of $r_a(\infty) = 0$, we would have $r_a = 0 \notin \operatorname{Inv}(A)$, a contradiction. □

6.25 Lemma and Definition.

The SPECTRAL RADIUS $r(a)$ *of* $a \in A$ *is*

$$r(a) := \max\{|z| : z \in \sigma(a)\}.$$

We have:

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

Proof. Since $r_a : \rho(a) \to A$ is holomorphic by 6.24, $z \mapsto r_a(\frac{1}{z})$ is holomorphic for $\frac{1}{z} \in \rho(a)$, so the Taylor series $\sum_{k=0}^\infty z^{k+1} a^k$ of this function converges in the interior of the largest disc contained in $\{z : \frac{1}{z} \in \rho(a)\}$. This has by definition radius $\inf\{|z| : \frac{1}{z} \notin \rho(a)\} = \frac{1}{\sup\{|w|:w\in\sigma(a)\}} = \frac{1}{r(a)}$. Since this power series is divergent for $|z| > \frac{1}{\overline{\lim}_{n\to\infty} \sqrt[n]{\|a^n\|}}$ (moreover, the radius of convergence is $\overline{\lim}_{n\to\infty} \sqrt[n]{\|a^n\|}$), we have $\frac{1}{r(a)} \leqslant \frac{1}{\overline{\lim}_{n\to\infty} \sqrt[n]{\|a^n\|}}$, i.e. $r(a) \geqslant \overline{\lim}_{n\to\infty} \sqrt[n]{\|a^n\|}$ (and even equality holds).

It remains to show that this limit superior is even a limit. By means of the inequality $\|a^{n+m}\| \leqslant \|a^n\|\,\|a^m\|$ one can show this directly, see [**11**, 169]. Another proof goes as follows:

For $z \in \sigma(a)$ we have $z - a \notin \operatorname{Inv}(A)$. Since $z^n - a^n = (z-a)\,(z^{n-1} + z^{n-2}\,a + \cdots + z\,a^{n-2} + a^{n-1})$ and the two factors commute with each other, $z^n - a^n \notin \operatorname{Inv}(A)$ by 6.2.3, i.e. $|z|^n \leqslant \|a^n\|$ by 6.2.1 and thus $|z| \leqslant \|a^n\|^{1/n}$. Thus $r(a) \leqslant \inf_n \|a^n\|^{1/n} \leqslant \underline{\lim}_n \|a^n\|^{1/n} \leqslant r(a)$. □

Functional Calculus

Remark.

In finite-dimensional spectral theory, the algebra $\{p(T) : p$ ist ein polynomial$\}$ plays an important role for operators T. Just think of the Theorem of Cayley-Hamilton and the role that the minimal polynomial plays. In infinite dimensions polynomials will probably not suffice. The most obvious generalization is convergent power series. We have shown in [**18**, 3.2.10] that the convergence for all $|z| < R$ of a power series $f(z) := \sum_{k=0}^\infty f_k z^k$ with $f_k \in \mathbb{C}$ coefficient also implies the convergence of the series $f(a) := \sum_{k=0}^\infty f_k\, a^k$ in A for all $a \in A$ with $\|a\| < R$. So this works if the radius of convergence is greater than $\|a\|$. However, the series $f(a) := \sum_{k=0}^\infty f_k\, a^k$ will converge (absolutely) by the root test (see [**20**, 2.5.10]) even if the radius of convergence is greater than $\lim_{n\to\infty} \|a^n\|^{1/n} = r(a)$ (in fact, $\overline{\lim}_k \sqrt[k]{|f_k|\,\|a^k\|} < 1 \Leftrightarrow r(a) = \lim_k \|a^k\|^{1/k} < \frac{1}{\overline{\lim}_k |f_k|^{1/k}}$). Under these assumptions, $z \mapsto f(z)$ has to be a holomorphic function on an open disc containing $\sigma(a)$.

We now want to try to define $f(a)$ also for functions f that are holomorphic on an arbitrary neighborhood of $\sigma(a)$. We can no longer use the power series expansion, because it only needs to converge in the interior of the largest disk in the domain of f. To get a definition of $f(a)$ also in this case, we first give another description of $f(a)$ for power series f with radius of convergence $R > \|a\|$. According to the Cauchy Integral Formula $\boxed{6.20}$,

$$f(z) = \frac{1}{2\pi i}\int_c \frac{f(w)}{w-z}\,dw,$$

holds where c is a parameterized circle with radius $r < R$. Thus we expect that

$$f(a) = \frac{1}{2\pi i}\int_c f(w)\,(w-a)^{-1}\,dw,$$

where the integral makes sense, since the circle c has values in $\rho(a)$ and thus $(w-a)^{-1}$ is well-defined for all $w \in \operatorname{img}(c)$.

Because of the Cauchy Integral Formula $\boxed{6.20}$ $\frac{1}{2\pi i}\int_c \frac{w^k}{w-z}\,dw = z^k$, hence analogously we should have $\frac{1}{2\pi i}\int_c w^k(w-a)^{-1}\,dw = a^k$. This is indeed the case, because

$$\begin{aligned}\frac{1}{2\pi i}\int_c w^k(w-a)^{-1}\,dw &= \frac{1}{2\pi i}\int_c w^{k-1}\left(1-\frac{1}{w}a\right)^{-1}dw = \frac{1}{2\pi i}\int_c w^{k-1}\sum_{j=0}^\infty \frac{1}{w^j}a^j\,dw\\ &= \frac{1}{2\pi i}\sum_{j=0}^\infty \underbrace{\left(\int_c w^{k-(j+1)}\,dw\right)}_{2\pi i\delta_k^j} a^j = a^k\end{aligned}$$

So

$$\begin{aligned}f(a) &= \frac{1}{2\pi i}\int_c f(w)\,(w-a)^{-1}\,dw = \frac{1}{2\pi i}\int_c \sum_{k=0}^\infty f_k w^k\,(w-a)^{-1}\,dw\\ &= \sum_{k=0}^\infty f_k\,\frac{1}{2\pi i}\int_c w^k\,(w-a)^{-1}\,dw = \sum_{k=0}^\infty f_k\,a^k\end{aligned}$$

This definition of $f(a)$ as line integral now also makes sense if c is not necessarily a circle, but is any 1-chain c in $\rho(a) \cap U$ and $f \in H(U)$. So we define as follows:

6.26 Definition.

Let $a \in A$ and $f : U \to \mathbb{C}$ be holomorphic on an open neighborhood U of $K := \sigma(a)$ in $\mathbb{C}$. Then put

$$f(a) := \frac{1}{2\pi i}\int_c f(w)\,(w-a)^{-1}\,dw \in A,$$

for some Jordan cycle c as in $\boxed{6.21}$.

Lemma.

This definition does not depend on the choice of the 1-chain c.

Proof. Let $c = \sum_{j=1}^n c_j$ and $d = \sum_{j=1}^m d_j$ be two Jordan cycles as in Lemma $\boxed{6.21}$. With c_{n+j} for $j \in \{1,\dots,m\}$ we denote the reversely parametrized curve d_j. For $z \notin U\backslash\sigma(a)$ either $z \notin U$ or $z \in \sigma(a)$. In the first case $\sum_{j=1}^{n+m}\operatorname{ind}_{c_j}(z) = \operatorname{ind}_c(z) - \operatorname{ind}_d(z) = 0-0 = 0$ and in the second $\sum_{j=1}^{n+m}\operatorname{ind}_{c_j}(z) = \operatorname{ind}_c(z)-\operatorname{ind}_d(z) = 1-1 = 0$. So $\Gamma := \sum_{j=1}^{n+m} c_j$ is a cycle of closed curves in $U\backslash\sigma(a)$ and for all $z \notin U\backslash\sigma(a)$ we have $\operatorname{ind}_\Gamma(z) = 0$ and since $w \mapsto f(w)\,(w-a)^{-1}$ is holomorphic on $U\backslash\sigma(a)$, it follows from Cauchy's Integral Theorem $\boxed{6.20}$ that

$$0 = \int_\Gamma f(w)\,(w-a)^{-1}\,dw = \int_c f(w)\,(w-a)^{-1}\,dw - \int_d f(w)\,(w-a)^{-1}\,dw. \qquad \square$$

6.27 Germs

As we have just seen, $f(a)$ does not depend on the selection of the Jordan cycle c in $U\backslash\sigma(a)$, hence $f_1(a) = f_2(a)$ in case f_1 and f_2 coincide on some U neighborhood of $K := \sigma(a)$. So we need the following

Definition.

Let $K \subseteq \mathbb{C}$ be compact. Under a HOLOMORPHIC GERM on K we understand an equivalence class of holomorphic functions f defined on open neighborhoods $U \subseteq \mathbb{C}$ of K. The equivalence relation is given as follows: $f_1 : U_1 \to \mathbb{C}$ and $f_2 : U_2 \to \mathbb{C}$ are called equivalent if an open neighborhood $U \subseteq U_1 \cap U_2$ of K exists with $f_1|_U = f_2|_U$. With $H(K) := H(K, \mathbb{C})$ we denote the set of all holomorphic germs on K. This is a $\mathbb{C}$-algebra when we define the algebra operations via the representatives.

The mapping $H(U, \mathbb{C}) \to H(K)$, $f \mapsto [f]$, is injective, provided each connected component of U contains at least one point from K, because then it follows from the uniqueness theorem 6.14 that any two holomorphic functions on U that coincide on a neighborhood of K are already identical. We can assume without loss of generality that all considered neighborhoods U have this property, and thus that the Fréchet space $H(U, \mathbb{C})$ is a linear subspace of $H(K, \mathbb{C})$. By definition, $H(K)$ is the union of these subspaces, and we can therefore provide $H(K)$ with the final structure.

6.28 Theorem (Holomorphic Functional Calculus).

For $a \in A$ the mapping $[f] \mapsto f(a)$ given by 6.26 defines the uniquely determined continuous algebra homomorphism $H(\sigma(a)) \to A$, which maps id to a, i.e. extends the evaluation $\sum_k f_k z^k \mapsto \sum_k f_k a^k$ of polynomials.

Proof. First the existence statement:
According to the above lemma, $f(a) := \frac{1}{2\pi i} \int_c f(w)\,(w-a)^{-1}\,dw$ is well-defined and does not depend on the choice of c and the representative of the germ f.

Obviously, $f \mapsto f(a)$ is linear.

We show that this is also an algebra homomorphism. Let f and g be two holomorphic functions defined on an open $U \supseteq \sigma(a)$. Let Λ be a fitting Jordan cycle in U and Γ be such a cycle in $\mathrm{inn}(\Lambda)$. Then:

$$\begin{aligned}
f(a)\,g(a) &= -\frac{1}{4\pi^2}\left(\int_\Gamma f(w)\,(w-a)^{-1}\,dw\right)\left(\int_\Lambda g(z)\,(z-a)^{-1}\,dz\right)\\
&= -\frac{1}{4\pi^2}\int_\Gamma\int_\Lambda f(w)\,g(z)\,(w-a)^{-1}\,(z-a)^{-1}\,dz\,dw\\
&\overset{6.23.2}{=\!=} -\frac{1}{4\pi^2}\int_\Gamma\int_\Lambda f(w)\,g(z)\,\frac{r_a(w)-r_a(z)}{z-w}\,dz\,dw\\
&= -\frac{1}{4\pi^2}\int_\Gamma f(w)\left(\int_\Lambda \frac{g(z)}{z-w}\,dz\right)(w-a)^{-1}\,dw+\\
&\quad+\frac{1}{4\pi^2}\int_\Lambda g(z)\left(\int_\Gamma \frac{f(w)}{z-w}\,dw\right)(z-a)^{-1}\,dz.
\end{aligned}$$

For all $z \in \mathrm{img}(\Lambda) \subseteq \mathrm{out}(\Gamma)$, according to Cauchy's theorem 6.20, $\int_\Gamma \frac{f(w)}{z-w}\,dw = 0$. For all $w \in \mathrm{img}(\Gamma) \subseteq \mathrm{inn}(\Lambda)$, $\int_\Lambda \frac{g(z)}{z-w}\,dz = 2\pi i\,g(w)$ holds, so

$$f(a)\,g(a) = \frac{1}{2\pi i}\int_\Gamma f(w)\,g(w)\,(w-a)^{-1}\,dw = (f\,g)(a).$$

Now to the continuity. We only have to show that $f \mapsto f(a)$, $H(U) \to A$ is continuous, or, because $H(U)$ is a Fréchet space with respect to the uniform convergence on each compact subset of U, that this mapping is bounded. Let $\mathcal{F} \subseteq H(U)$ be bounded. Then $\mathcal{F}$ is uniformly bounded on the image of c, so there is a constant K with $\|f|_{\mathrm{img}(c)}\|_\infty \leqslant K$ for all $f \in \mathcal{F}$. Furthermore, $r_a(\mathrm{img}(c))$ is compact, so bounded and consequently there is a constant K_1 with $\|(w-a)^{-1}\| \leqslant K_1$ for all $w \in \mathrm{img}(c)$. Therefore, $\|f(a)\| \leqslant \frac{1}{2\pi} K\, K_1\, L(c)$, and thus $\{f(a) : f \in \mathcal{F}\}$ is bounded.

Let $f(z) = \sum_k f_k\, z^k$ be a polynomial, or more generally a power series that converges on a neighborhood of $\sigma(a)$. Then $f(a) = \sum_{k=0}^\infty f_k\, a^k$, as we have shown above.

Now for the uniqueness statement:
Let τ be such an algebra-homomorphism. As algebra-homomorphism which maps id to a, $\tau(f) = f(a)$ holds for all polynomials $f \in \mathbb{C}[z]$.
Let next $f = \frac{p}{q}$ be a rational function with poles outside $\sigma(a)$. So we may assume that q is a polynomial not vanishing on $\sigma(a)$, and thus $\frac{1}{q} \in H(\sigma(a))$. But then $1 = \tau(1) = \tau(q\frac{1}{q}) = \tau(q)\,\tau(\frac{1}{q})$ holds, so $\tau(\frac{1}{q}) = \tau(q)^{-1}$ and therefore $\tau(\frac{p}{q}) = \tau(p)\cdot\tau(q)^{-1} = p(a)\cdot q(a)^{-1} = \frac{p}{q}(a) = f(a)$ holds.
Let finally $f \in H(\sigma(a))$ be arbitrary, i.e. w.l.o.g. $f \in H(U,\mathbb{C})$ for some open neighborhood U of $\sigma(a)$. Let $K \subseteq U$ be a compact set containing $\sigma(a)$ in the interior. According to Runge's Approximation Theorem [5.3.6], there exists a sequence of rational functions f_n with poles outside K, which uniformly converges to K towards f. Then the germs $[f_n]$ converge towards that of f, and the continuity statement implies $f(a) = \lim f_n(a) = \lim \tau(f_n) = \tau(f)$. □

6.29 Spectral Mapping Theorem.

For $f \in H(\sigma(a))$ the equation $\sigma(f(a)) = f(\sigma(a))$ holds.

Proof. Let $f \in H(U)$ with open $U \supseteq \sigma(a)$.
($\supseteq$) For given $z \in \sigma(a)$ we have that

$$g : w \mapsto \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{for } w \neq z \\ f'(z) & \text{for } w = z \end{cases}$$

is a holomorphic function on U. Suppose $f(z) \notin \sigma(f(a))$. Then $(z-a)\,g(a) = f(z) - f(a)$ would be invertible and since the two factors commute with each other, also $z-a$ would be invertible by [6.2.3], i.e. $z \notin \sigma(a)$, a contradiction.

($\subseteq$) Conversely, let $z \notin f(\sigma(a))$. Then $g : w \mapsto (z - f(w))^{-1}$ is a holomorphic function on the neighborhood $U \setminus f^{-1}(z)$ of $\sigma(a)$ with $1 = g(a)\,(z - f(a))$. So $z - f(a)$ would be invertible by [6.2.3], i.e. $z \notin \sigma(f(a))$. □

6.30 Lemma.

Let A be a Banach algebra and $a, b \in A$. Then $\sigma(a\,b) \cup \{0\} = \sigma(b\,a) \cup \{0\}$.

Proof. We have to show that $\lambda - a\,b \in \mathrm{inv}(A) \Leftrightarrow \lambda - b\,a \in \mathrm{inv}(A)$ for all $\lambda \neq 0$. Without loss of generality, $\lambda = 1$ and $1 - a\,b$ are invertible with $u := (1 - a\,b)^{-1}$. We claim that $1 - b\,a$ is invertible and $(1 - b\,a)^{-1} = 1 + b\,u\,a$:

$$\begin{aligned}(1-b\,a)\,(1+b\,u\,a) &= 1 - b\,a + b\,u\,a - b\,a\,b\,u\,a = 1 + b\,(-1 + u - a\,b\,u)\,a \\ &= 1 + b\,((1-a\,b)\,u - 1)\,a = 1 \\ (1+b\,u\,a)\,(1-b\,a) &= 1 + b\,u\,a - b\,a - b\,u\,a\,b\,a = 1 + b\,(u - 1 - u\,a\,b)\,a \\ &= 1 + b\,(u\,(1-a\,b) - 1)\,a = 1. \quad \square\end{aligned}$$

6.31 Definition. Commutant.

We denote the set of elements, which commute with all b in a set $B \subseteq A$, as COMMUTANT $B^k := \{x \in A : x\,b = b\,x \;\forall b \in B\}$ of B. In algebra one calls this also the CENTRALIZER of B in A.

We have that $B \mapsto B^k$ is an antitone mapping on the power set of A and $B_1 \subseteq B_2^k \Leftrightarrow B_2 \subseteq B_1^k$, because both sides mean $\forall b_1, b_2 : b_1 \in B_1, b_2 \in B_2 \Rightarrow b_1\,b_2 = b_2\,b_1$.

Thus, $B \subseteq (B^k)^k =: B^{kk}$ because of $B^k \subseteq B^k$.

In addition, $B^k = B^{kkk}$ always holds, since $B \subseteq B^{kk}$ implies $B^k \supseteq (B^{kk})^k$ and, on the other hand, $B^k \subseteq (B^k)^{kk}$ holds.

Note that B^k is a closed (with respect to any topology for which the multiplication is separately continuous) subalgebra of A for each subset $B \subseteq A$, because $x_1\,x_2\,b = x_1\,b\,x_2 = b\,x_1\,x_2$.

Furthermore $B^k = B_1^k$, where B_1 denotes the closure of the subalgebra generated by B in a topology with respect to which the multiplication is separately continuous.

Obviously, B is commutative if and only if $B \subseteq B^k$ holds. Thus, for commutative B also B^{kk} is commutative, because $B \subseteq B^k \Rightarrow B^{kk} \subseteq B^k \Rightarrow B^{kk} \subseteq B^{kkk} = (B^{kk})^k$.

6.32 Corollary.

For $f \in H(\sigma(a))$ we have that $f(a)$ commutes with all $b \in A$ commuting with a, i.e. $f(a) \in \{a\}^{kk}$. Moreover, $\{f(a) : f \in H(\sigma(a))\}^k = \{a\}^k$.

Proof. Because of Runge's Approximation Theorem 5.3.6, $\{a\}^k = \{f(a) : f \in H(\sigma(a))\}^k$ (for polynomials f this is obvious. It follows easily (c.f. 6.28) that this also holds for rational functions with poles outside $\sigma(a)$) and thus $f(a) \in \{a\}^{kk}$ for all $f \in H(\sigma(a))$. □

In the finite-dimensional case one uses the decomposition of the characteristic polynomial into prime factors to obtain a direct sum decomposition (diagonal block description) of the operator. We can now transfer this to elements of a Banach algebra. However, since we do not have a space available for these elements to operate on and hence we can not restrict the summands to invariant subspaces, the spectrum of the summands contains 0.

6.33 Corollary.

Let $a \in A$ and $\sigma(a) = K_1 \sqcup K_2$ a decomposition into closed disjoint sets. Then there is an idempotent $e \in \{a\}^{kk}$ (i.e. $e^2 = e$) and for $a_1 := a\,e$ and $a_2 := a\,(1-e)$ we have $a = a_1 + a_2$, $a_1\,a_2 = 0 = a_2\,a_1$, and $\sigma(a_j) = K_j \cup \{0\}$ for $j \in \{1,2\}$.

Proof. The idea of the proof is to first show this for the inverse image $\mathrm{id} \in H(\sigma(a))$ under the algebra homomorphism $H(\sigma(a)) \to \{a\}^{kk} \subseteq A$ from 6.28 and then apply this homomorphism. For $j \in 1,2$, let U_j be two disjoint open neighborhoods of K_j. Then the characteristic function $\chi_{U_1} \in H(U_1 \cup U_2)$. So $e := \chi_{U_1}(a) \in A$ is well-defined. By 6.32, e commutes with all b commuting with a, in particular, with a itself. Moreover, e is idempotent because of $\chi_{U_1}^2 = \chi_{U_1}$. Furthermore, $1-e = (1-\chi_{U_1})(a) = \chi_{U_2}(a)$. We have $1 = e + (1-e)$ and $e(1-e) = e - e^2 = e - e = 0$, so all claimed equations hold for $a_1 := a\,e = e\,a$ and $a_2 := a\,(1-e) = (1-e)\,a$. Moreover $\sigma(a_j) = \sigma((\mathrm{id}\cdot\chi_{U_j})(a)) = (\mathrm{id}\cdot\chi_{U_j})(K_1 \sqcup K_2) = \mathrm{id}(K_j) \cup \{0\} = K_j \cup \{0\}$ by the Spectral Mapping Theorem 6.29. □

Dependency of the spectrum on the algebra

Let A be a Banach algebra and B a Banach subalgebra with $a \in B$. Then obviously $\rho_B(a) \subseteq \rho_A(a)$ and thus $\sigma_A(a) \subseteq \sigma_B(a)$. We now want to investigate to what extent the two spectra can be different. First, a rather typical example.

6.34 Example for the dependence of the spectrum on the algebra.

Let $A := C(\partial\mathbb{D}, \mathbb{C})$ and B the Banach subalgebra generated by the identity $a : z \mapsto z$. Then $\sigma_A(a) = \partial\mathbb{D}$ and $\sigma_B(a) = \overline{\mathbb{D}}$:

By 6.7.1 we have $\sigma_A(a) = a(\partial\mathbb{D}) = \partial\mathbb{D}$. Because $\|a\|_\infty = 1$, $\sigma_B(a) \subseteq \overline{\mathbb{D}}$. Suppose $\sigma_B(a) \subset \overline{\mathbb{D}}$. Then there is a $\lambda \in \overline{\mathbb{D}}$ and a $b \in B$ with $(\lambda - a)\, b = 1$, i.e. $(\lambda - z)\, b(z) = 1$ for all $z \in \partial\mathbb{D}$. Since $b \in B$, there exists a sequence of polynomials b_n which converges on $\partial\mathbb{D}$ uniformly towards b. By the Maximum Modulus Principle 6.17, the b_n form a Cauchy sequence in $C(\overline{\mathbb{D}})$, thus converge uniformly towards a $\tilde{b} \in C(\overline{\mathbb{D}})$, which is thus holomorphic on $\mathbb{D}$ and coincides with b on $\partial\mathbb{D}$. In the same way, we obtain $(\lambda - z)\, b_n(z) - 1 \to 0$ uniformly for $z \in \overline{\mathbb{D}}$, so $(\lambda - z)\tilde{b}(z) = 1$ holds for all $z \in \overline{\mathbb{D}}$. For $z := \lambda$, we therefore get the contradiction $0 = (\lambda - \lambda)\, \tilde{b}(\lambda) = 1$. So $\sigma_B(a) = \overline{\mathbb{D}}$ holds.

6.35 Definition.

Let $K \subseteq \mathbb{C}$ be compact. Then the polynomial convex hull $\hat{K}$ of K is defined by:

$$\widehat{K} := \{z \in \mathbb{C} : |p(z)| \leqslant \|p|_K\|_\infty \forall p \in \mathbb{C}[z]\},$$

i.e. the set of all points on which no polynomial attains larger absolute values than on K. The set K is called POLYNOMIAL CONVEX if $K = \widehat{K}$.

The complement $\mathbb{C}_\infty \backslash K$ has as open subset of $\mathbb{C}_\infty$ only countable many components: Namely the unbounded component in $\mathbb{C}$ (i.e. the component in $\mathbb{C}_\infty$ which contains ∞) together with the bounded components in $\mathbb{C}$, the so-called HOLES of K.

Lemma.

Let $K \subseteq \mathbb{C}$ be compact. Then, the complement $\mathbb{C}\backslash\widehat{K}$ of $\widehat{K}$ is the unbounded component of the complement $\mathbb{C}\backslash K$ of K. So $\widehat{K}$ is obtained by filling in all holes of K. And K is polynomial convex if and only if the complement of K is connected.

Proof. Let $\mathbb{C}\backslash K = U_\infty \sqcup \bigsqcup_{k\neq\infty} U_k$ be the partition into the connected components. Let U_∞ be the unbounded component and $L := \mathbb{C}\backslash U_\infty = K \sqcup \bigsqcup_{k\neq\infty} U_k$.

We claim $L \subseteq \widehat{K}$:
Because of $L = K \sqcup \bigsqcup_{k\neq\infty} U_k$ and $K \subseteq \widehat{K}$ it is enough to show $U_k \subseteq \widehat{K}$ for $k \neq \infty$. According to the Maximum Modulus Principle 6.17 it is enough to show $\partial U_k \subseteq K$, so let $x \in \partial U_k = \overline{U_k}\backslash U_k \subseteq \mathbb{C}\backslash U_k$. Since $x \notin U_j$ also for $j \neq k$ (since U_j is open and disjoint to U_k), we conclude that $x \in K$ holds.

Suppose $L \subset \widehat{K}$:
Let $z \in \widehat{K}\backslash L$. Then $w \mapsto \frac{1}{w-z}$ is a holomorphic function on a neighborhood of L. Since $\mathbb{C}\backslash L = U_\infty$ is connected, there exists a sequence of polynomials p_n with $\sup_{w\in L} |p_n(w) - \frac{1}{w-z}| \to 0$ by Runge's Approximation Theorem 5.3.8. Let $q_n : w \mapsto (w - z)\, p_n(w)$. Since $z \in \widehat{K}$ we obtain

$$\begin{aligned} 1 = |0-1| = |q_n(z) - 1| &\leqslant \sup\{|q_n(w) - 1| : w \in \widehat{K}\} = \sup\{|q_n(w) - 1| : w \in K\} \\ &\leqslant \sup\{|q_n(w) - 1| : w \in L\} \to 0, \end{aligned}$$

a contradiction. □

6.36 Theorem.

Let A be a Banach algebra, B a Banach subalgebra, and $a \in B$. Then $\sigma_B(a)$ is obtained by completely filling up some holes of $\sigma_A(a)$. In particular:

1. $\sigma_B(a) \supseteq \sigma_A(a)$.
2. $\partial\sigma_B(a) \subseteq \partial\sigma_A(a)$.
3. $\widehat{\sigma_B(a)} = \widehat{\sigma_A(a)}$.
4. *If B is generated as Banach algebra by a, then $\sigma_B(a) = \widehat{\sigma_A(a)}$.*

Proof. (1) is obvious because an inverse to $z - a$ in B is also one in A.

(2) Let $z \in \partial\sigma_B(a)$ and suppose $z \notin \sigma_A(a)$, i.e. $(z-a)^{-1} \in A$ exists. Since $z \in \partial\sigma_B(a)$, there exists a sequence $z_n \notin \sigma_B(a)$ (i.e. $(z_n - a)^{-1} \in B$ exists) with $z_n \to z$ and, by continuity of the inversion for A, $(z_n - a)^{-1} \to (z-a)^{-1}$. Since B is closed, we have $(z-a)^{-1} \in B$, i.e. $z \notin \sigma_B(a) \supseteq \partial\sigma_B(a)$, a contradiction. Thus $z \in \partial\sigma_A(a)$, because the interior of $\sigma_A(a)$ has be in the interior of $\sigma_B(a) \supseteq \sigma_B(a)$ and thus in $\mathbb{C}\backslash\partial\sigma_B(a)$.

(3) $\widehat{\sigma_B(a)} \supseteq \widehat{\sigma_A(a)}$ holds because of $\sigma_B(a) \supseteq \sigma_A(a)$. Suppose $\exists z_0 \in U_\infty := \widehat{\sigma_B(a)} \cap (\mathbb{C}_\infty\backslash\widehat{\sigma_A(a)})$. Let $z : [0,1] \to U_\infty \subseteq \mathbb{C}_\infty$ be a curve in the, by lemma in 6.35, unbounded connected component U_∞ connecting z_0 with ∞ and $t_0 := \sup\{t : z(t) \in \widehat{\sigma_B(a)} = \sigma_B(a) \sqcup \bigsqcup_{k\neq\infty} U_k\}$. Then $z(t_0) \in \sigma_B(a)$ and is not in the interior of $\sigma_B(a)$, hence $z_0 \in \partial\sigma_B(a) \subseteq \partial\sigma_A(a) \subseteq \sigma_A(a)$, a contradiction.

(4) By 3, $\sigma_B \subseteq \widehat{\sigma_B} = \widehat{\sigma_A}$ always holds. Suppose there were an $z \in \widehat{\sigma_A}\backslash\sigma_B$. Then $(z-a)^{-1} \in B \subseteq A$. Since B is the closure of the polynomials in a, there exists a sequence of polynomials p_n with $p_n(a) \to (z-a)^{-1}$. Let $q_n : w \mapsto (z-w)\, p_n(w)$. Then $q_n(a) = (z-a)p_n(a) \to (z-a)\,(z-a)^{-1} = 1$ holds. By the Spectral Mapping Theorem 6.29 we have $\sigma_A\big((q_n - 1)(a)\big) = (q_n - 1)\big(\sigma_A(a)\big)$ and thus, by 6.25,

$$\begin{aligned}\|q_n(a) - 1\| \geqslant r_A(q_n(a) - 1) &:= \sup\Big\{|w| : w \in \sigma_A(q_n(a) - 1) = q_n(\sigma_A(a)) - 1\Big\}\\ &= \sup\Big\{|q_n(w) - 1| : w \in \sigma_A(a)\Big\} \geqslant |q_n(z) - 1| = 1,\end{aligned}$$

since $z \in \widehat{\sigma_A(a)}$, a contradiction to $q_n(a) \to 1$.

Remains to show that in general $\sigma_B(a)$ is obtained by completely filling up some holes of $\sigma_A(a)$:
Let U be a hole of σ_A. Then $U = U_1 \sqcup U_2$, where $U_1 := U \cap \sigma_B = U \cap (\sigma_B\backslash\partial\sigma_B)$, because $\partial\sigma_B \subseteq \partial\sigma_A \subseteq \sigma_A \subseteq \mathbb{C}\backslash U$ by 2, and $U_2 := U \cap \rho_B$. Thus, U_1 and U_2 are open and disjoint. Since U is connected as a hole, one of the two sets is empty, so the hole U is completely contained in σ_B or in the complement ρ_B. □

Commutative Banach algebras

We now want to develop a duality theory for Banach algebras A. Instead of the linear functionals we should probably use Banach algebra homomorphisms $A \to \mathbb{C}$. So we start by studying algebra homomorphisms. Since continuity of linear functionals can be described by closedness of the kernel by 3.4.2, we should in particular study the kernels of algebra homomorphisms.

6.37 Definition (Ideals).

A subset $I \subseteq A$ of a (Banach) algebra A is called an IDEAL if I is a linear subspace and with $i \in I$ and $a \in A$ also $i\,a \in I$ and $a\,i \in I$.

An ideal is called TRUE IDEAL, if $I \neq A$, or equivalent if $1 \notin I$, or further equivalent $\operatorname{inv}(A) \cap I = \emptyset$: The directions ($\Leftarrow$) are obvious. Conversely, let $i \in I$ be invertible in A and $a \in A$ arbitrary, then $a = i^{-1}\,i\,a \in I$.

The kernel of each algebra homomorphism is obviously a true ideal (because of $f(1) = 1$), and conversely, each ideal $I \subset A$ of an algebra A defines an algebra structure on A/I such that the canonical map $\pi : A \to A/I$ with kernel I is an algebra homomorphism: For the projection $\pi : A \to A/I$ to become an algebra homomorphism, one has to define the multiplication in A/I by $(a+I)\cdot(b+I) := a \cdot b + I$. Since I is an ideal, this definition makes sense, because $(a+i)\cdot(b+j) = a\cdot b + a\cdot j + i\cdot b + i\cdot j \in a\cdot b + A\cdot I + I\cdot A + I\cdot I \subseteq a\cdot b + I$ for $i, j \in I$.

An ideal I in A is called maximal if it is maximal among all true ideals with respect to the inclusion.

Lemma.

The maximal ideals of a commutative algebra are exactly the kernels of surjective algebra homomorphisms with values in divisional algebras (i.e. where each element unequal to 0 is invertible).

Proof. Let $f : A \twoheadrightarrow B$ be a surjective algebra homomorphism (between not necessary commutative algebras) and let every $0 \neq b \in B$ be invertible. Then $\ker f$ is a maximal ideal, because if $I \supset \ker f$ is an ideal, then it is easy to see that $f(I) \neq \{0\}$ is an ideal in B, thus contains an invertible element $b = f(i)$ with $i \in I$. Let $f(a) = b^{-1}$. Then $f(1 - a\,i) = 0$, i.e. $1 \in \ker f + a\,i \subseteq I$, so $I = A$.

Conversely, let $I \subset A$ be a maximal ideal. And let $\pi : A \to A/I$ be the canonical mapping. Furthermore, let $0 \neq b \in A/I$. Then there is an $a \in A \backslash I$ with $\pi(a) = b$. Let I_a be the ideal generated by I and a. Because of the commutativity $I_a = I + A\,a$. The maximality of I implies $1 \in I_a$, i.e. there are $i \in I$ and $a' \in A$ with $1 = i + a'\,a$, hence $1 = 0 + \pi(a')\,b$ in A/I and thus b is invertible. □

6.38 Theorem of Gelfand-Mazur.

Let A be a Banach algebra with $\operatorname{inv}(A) = A\backslash\{0\}$ (i.e. a division algebra). Then $A = \{\lambda 1 : \lambda \in \mathbb{C}\} \cong \mathbb{C}$.

Proof. Let $a \in A$. Then $\sigma(a) \neq \emptyset$. Let $z \in \sigma(a)$, i.e. $z\,1 - a$ has no inverse, thus $z\,1 - a = 0$, i.e. $a \in \mathbb{C}\,1$. □

6.39 Proposition. Automatic continuity.

Let A be a Banach algebra and $f : A \to \mathbb{C}$ be an algebra homomorphism. Then f is continuous and $\|f\| = 1$.

Proof. Since $f(1) = 1$ we only have to show that $|f(a)| \leqslant \|a\|$ for all $a \in A$: Suppose $|f(a)| > \|a\|$, then $f(a)\cdot 1 - a$ is invertible and hence also $f(f(a)\cdot 1 - a) = 0$, a contradiction. □

6.40 Lemma.

Let A be an Abelian Banach algebra. Then there is a bijection

$$\begin{aligned}\operatorname{Alg}(A, \mathbb{C}) &\rightleftarrows \big\{I : \ I \textit{ is a maximal ideal in } A\big\}\\ f &\mapsto \ker(f)\end{aligned}$$

Here, the algebra homomorphism $f : A \to \mathbb{C}$ associated to a maximal ideal I is determined by $f(a) \cdot 1 = \pi(a)$, where π denotes the canonical projection $A \to A/I$.

Proof. ($\mapsto$) is well-defined by the lemma in 6.37.

($\leftarrow\!\shortmid$) Let now I be a maximal ideal. Then $\operatorname{Inv}(A) \cap I = \emptyset$ and, since $\operatorname{inv}(A)$ is open, also $\operatorname{Inv}(A) \cap \bar{I} = \emptyset$. Since $\bar{I}$ obviously is an ideal, it follows from the maximality that $I = \bar{I}$, i.e. I is closed.

Claim. *For every closed true ideal $I \lhd A$ the Banach space A/I is a Banach algebra.*
By 6.37, $(a + I) \cdot (b + I) := a \cdot b + I$ defines a multiplication that makes A/I into an algebra and $\pi : A \to A/I$ into an algebra homomorphism. The quotient norm is submultiplicative, because

$$\begin{aligned}
\|(a + I) \cdot (b + I)\| &= \|a \cdot b + I\| \\
&= \inf\{\|a \cdot b + i\| : i \in I\} \\
&\leqslant \inf\{\|a \cdot b + k\| : k = a \cdot j + i \cdot b + i \cdot j \text{ with } i, j \in I\} \\
&= \inf\{\|(a + i) \cdot (b + j)\| : i, j \in I\} \\
&\leqslant \inf\{\|a + i\| \cdot \|b + j\| : i, j \in I\} \\
&= \inf\{\|a + i\| : i \in I\} \cdot \inf\{\|b + j\| : j \in I\} \\
&= \|a + I\| \cdot \|b + I\|
\end{aligned}$$

We have $\|1 + I\| = \inf\{\|1 + i\| : i \in I\} \leqslant \|1 + 0\| = 1$. Suppose $\|1 + I\| < 1$. Then $\|1 + I\| = \|(1 + I)^2\| \leqslant \|1 + I\|^2 < \|1 + I\|$ would be a contradiction.

Since I is maximal and A is Abelian, A/I is a division algebra by 6.37, and thus $A/I = \mathbb{C} \cdot 1 \cong \mathbb{C}$ by 6.38. So $f : A \to A/I \cong \mathbb{C}$ is the required algebra homomorphimus with $f(a) \cdot 1 = \pi(a)$.

Obviously, the two mappings are inverse to each other, because on the one hand $\ker(f) = \ker(\pi) = I$ and on the other hand, two algebra homomorphisms f_1 and f_2 of $A \to B$ haveing the same kernel are identical, because $f_2(a) = f_2(a - f_1(a)\,1) + f_2(f_1(a)\,1) = f_1(a)\,f_2(1) = f_1(a)$, since $a - f_1(a)\,1 \in \ker(f_1) = \ker(f_2)$. $\square$

6.41 Lemma. Abelization of Banach algebras.

Let A be a Banach algebra. With A' we denote the closed ideal of A generated by $\{ab - ba : a, b \in A\}$. Then $A_{Abel} := A/A'$ is a commutative Banach algebra and the natural projection $A \to A_{Abel}$ is a Banach algebra homomorphism with the following universal property: To each Banach algebra homomorphism $f : A \to B$ with values in a commutative Banach algebra B exists a unique Banach algebra homomorphism $f_{Abel} : A_{Abel} \to B$ which makes following diagram commutative:

$$\begin{tikzcd}
A \arrow[rr, "\pi"] \arrow[dr, "f"'] & & A_{Abel} \arrow[dl, dotted, "\exists!"', "f_{Abel}"] \\
& B &
\end{tikzcd}$$

Proof. We have shown in the proof of 6.40 that A/A' is a Banach algebra, because A' is a closed ideal. Obviously, A/A' is commutative, because $(a + A')\,(b + A') - (b + A')\,(a + A') = (ab - ba) + A' \subseteq A'$. So let B be a commutative Banach algebra and $f : A \to B$ be a Banach algebra homomorphism. Then $f(ab - ba) = f(a)\,f(b) - f(b)\,f(a) = 0$ and, as f is continuous, also $A' \subseteq \ker f$. So f factors to a unique continuous linear mapping $f_{Abel} := \tilde{f} : A/A' \to B$. We have that $\tilde{f}$ is an algebra homomorphism, because $\tilde{f}((a+A')\,(b+A')) = \tilde{f}(a\,b+A') = f(a\,b) = f(a)\,f(b) = \tilde{f}(a+A')\,\tilde{f}(b+A')$. □

6.42 From $\mathrm{Alg}(A,\mathbb{C})$ back to A

We now want to find out to which extent one can recover the algebra A from the set $\mathrm{Alg}(A,\mathbb{C})$ of all algebra homomorphisms $A\to\mathbb{C}$.
Since all of these homomorphisms factor over the Abelization, we can at most recapture Abelian Banach algebras from their $\mathbb{C}$-valued homomorphisms. Let's look first at our typical example $A := C(X,\mathbb{C})$ of a commutative Banach algebra and try to describe the algebra homomorphisms $A\to\mathbb{C}$ as explicitly as possible. Obviously, every $x\in X$ defines such a homomorphism $\mathrm{ev}_x : A\to\mathbb{C}$ by $\mathrm{ev}_x(f) = f(x)$. This assignment $\delta : x\mapsto \mathrm{ev}_x$ is injective, since the continuous functions $f : X\to\mathbb{C}$ on compact spaces X are point separating (a special case of the Lemma of Urysohn).

The mapping $\delta : X\to \mathrm{Alg}(A,\mathbb{C})$ is onto:
Let $\varphi : A\to\mathbb{C}$ be an algebra homomorphism. We are searching for a point $x\in X$ with $\varphi(f) = f(x)$ for all $f\in A$. Let $I := \ker\varphi$. For each $f\in I$ we consider the closed zero set $f^{-1}(0) = \{x\in X : f(x) = 0\}$. This is not empty, otherwise f would be invertible in A, i.e. $1\in I$. This family of zero sets has the finite intersection property, because $f^{-1}(0)\cap g^{-1}(0) = (f\bar{f}+g\bar{g})^{-1}(0)$ and with $f,g\in I$ also $f\bar{f}+g\bar{g}$ is in the ideal. Since X is compact, $\bigcap_{f\in I} f^{-1}(0)\neq\emptyset$. So let $x\in f^{-1}(0)$ for all $f\in I$. For any $f\in A$ we have $f-\varphi(f)\,1\in I = \ker(\varphi)$ and thus $0 = (f-\varphi(f)\,1)(x) = f(x)-\varphi(f)$, i.e. $\varphi = \mathrm{ev}_x$.

Thus we can identify the points of X with the $\mathbb{C}$-valued algebra homomorphisms on $A := C(X,\mathbb{C})$. If we want to recover the algebra A, then we have to provide $\mathrm{Alg}(A,\mathbb{C})$ with a Hausdorff topology such that the mapping $X\to\mathrm{Alg}(A,\mathbb{C})$ is continuous (then it is automatically a homeomorphism since X is compact). So $x_i\to x$ should imply $\mathrm{ev}_{x_i}\to\mathrm{ev}_x$ in $\mathrm{Alg}(A,\mathbb{C})$. Pointwise at $f\in A$ this is valid, because $\mathrm{ev}_{x_i}(f) = f(x_i)\to f(x)$.

We have thus shown the following:

Proposition.

Let X be a compact Hausdorff space and $A := C(X,\mathbb{C})$. If we consider $\mathrm{Alg}(A,\mathbb{C})$ with the topology the pointwise convergence, i.e. as subspace of $\mathbb{C}^A = \prod_{a\in A}\mathbb{C}$, then the mapping $\delta : X\to\mathrm{Alg}(A,\mathbb{C}) = \mathrm{Alg}(C(X,\mathbb{C}),\mathbb{C})$ is a homeomorphism. □

More generally, a completely regular topological space is called a REAL-COMPACT, if this mapping $\delta : X\to\mathrm{Alg}(A,\mathbb{C}) = \mathrm{Alg}(C(X,\mathbb{C}),\mathbb{C})$ is a homeomorphism, see [**26**, 2.5].

Consequently, for the Banach algebra $A := C(X,\mathbb{C})$ we obtain an isomorphism

$$\delta^* : C(\mathrm{Alg}(A,\mathbb{C}),\mathbb{C})\cong C(X,\mathbb{C}) = A.$$

Note that $(\delta^*)^{-1} : A\to C(\mathrm{Alg}(A,\mathbb{C}),\mathbb{C})$ is given by $\delta : a\mapsto \mathrm{ev}_a(: \varphi\mapsto\varphi(a))$, because

$$(\delta^*\circ\delta)(f)(x) = \delta^*(\delta(f))(x) = \delta(f)(\delta(x)) = \delta(x)(f) = f(x)$$

We want to generalize this as far as possible to arbitrary (commutative) Banach algebras A. For this we supply the so-called SPECTRUM $\sigma(A) := \mathrm{Alg}(A,\mathbb{C})$ of A again with the topology of pointwise convergence. If we can prove the compactness of $\sigma(A)$, then $C(\sigma(A),\mathbb{C})$ is a Banach algebra with respect to the topology of uniform convergence and $\delta : A \to C(\sigma(A), \mathbb{C})$, $a \mapsto \mathrm{ev}_a(: \varphi \mapsto \varphi(a))$ is a well-defined algebra homomorphism, which we will now examine in more detail.

6.43 Gelfand's Representation Theorem.

Let A be a commutative Banach algebra. Then its spectrum $\sigma(A) := \mathrm{Alg}(A, \mathbb{C}) =: X$ is a compact Hausdorff space with respect to the topology of pointwise convergence. The GELFAND TRANSFORMATION

$$\mathcal{G} = \delta : A \to C(X, \mathbb{C}) = C\big(\mathrm{Alg}(A, \mathbb{C}), \mathbb{C}\big), \quad a \mapsto \mathrm{ev}_a(: \varphi \mapsto \varphi(a))$$

is a Banach algebra homomorphism with the radical of A as its kernel

$$\ker(\mathcal{G}) = \mathrm{Rad}(A) := \bigcap \{I : I \text{ is a maximal ideal of } A\}.$$

For $a \in A$ the identities $\sigma_A(a) = \sigma_{C(X,\mathbb{C})}(\mathcal{G}(a))$ and $\|\mathcal{G}(a)\|_\infty = r(a)$ hold.

Proof. Obviously, $X := \mathrm{Alg}(A, \mathbb{C})$ is closed in $\mathbb{C}^A$, because $X \ni \varphi_i \to \varphi$ implies $\varphi(a\, b) = \lim \varphi_i(a\, b) = \lim \varphi_i(a)\, \varphi_i(b) = \lim_i \varphi_i(a) \lim_j \varphi_j(b) = \varphi(a)\, \varphi(b)$ and similarly one shows the linearity of φ. Furthermore, X is bounded in $\mathbb{C}^A$, because $|\operatorname{pr}_a(\varphi)| = |\varphi(a)| \leqslant \|a\|$ for $a \in A$ and $\varphi \in X$ by 6.39. Hence, by Tychonoff's Theorem, X is compact.

The mapping $\mathcal{G}$ has values in $C(X, \mathbb{C})$, because $\varphi_i \to \varphi$ in $X \subseteq \mathbb{C}^A$ implies that $\mathcal{G}(a)(\varphi_i) = \varphi_i(a) \to \varphi(a) = \mathcal{G}(a)(\varphi)$.

Obviously, $\mathcal{G}$ is an algebra homomorphism since $\mathrm{ev}_\varphi \circ \mathcal{G} = \varphi$ is one for all $\varphi \in X$.

For the kernel of $\mathcal{G}$, the following holds:

$$\begin{aligned} a \in \ker \mathcal{G} &\Leftrightarrow 0 = \mathcal{G}(a) \Leftrightarrow \forall \varphi \in X : 0 = \mathcal{G}(a)(\varphi) = \varphi(a) \\ &\Leftrightarrow a \in \bigcap_{\varphi \in X} \ker \varphi = \bigcap_I I = \mathrm{Rad}(A), \end{aligned}$$

where the last intersection is over all maximal ideals I of A.

Now to the statement $\sigma_A(a) = \sigma_{C(X,\mathbb{C})}(\mathcal{G}(a))$ about the spectra for $a \in A$:
Note that $\sigma_{C(X,\mathbb{C})}(\mathcal{G}(a)) = \{\mathcal{G}(a)(\varphi) = \varphi(a) : \varphi \in X\}$ holds by 6.7.1.
($\supseteq$) Let $z = \varphi(a) \in \sigma_{C(X,\mathbb{C})}(\mathcal{G}(a))$, then $\varphi(a)\, 1 - a \in \ker \varphi$ and thus is not invertible, i.e. $z = \varphi(a) \in \sigma_A(a)$.
($\subseteq$) Now let $z \in \sigma_A(a)$, i.e. $z\, 1 - a$ is not invertible. Then the ideal $A \cdot (z\, 1 - a)$ generated by $z\, 1 - a$ is a true ideal. Thus, according to the Lemma of Zorn, there is a maximal ideal I containing $z\, 1 - a$. Let $\varphi : A \to \mathbb{C}$ be the algebra homomorphism with kernel I. Then $0 = \varphi(z\, 1 - a) = z - \varphi(a)$, i.e. $z \in \sigma_{C(X,\mathbb{C})}(\mathcal{G}(a))$.

Consequently, we obtain the following estimate for the norms:

$$\begin{aligned} \|\mathcal{G}(a)\|_\infty &:= \sup\{|\mathcal{G}(a)(\varphi)| = |\varphi(a)| : \varphi \in X\} \\ &= \sup\{|z| : z \in \sigma_{C(X,\mathbb{C})}(\mathcal{G}(a)) = \sigma_A(a)\} = r(a) \leqslant \|a\|. \quad \square \end{aligned}$$

A commutative Banach algebra is called SEMISIMPLE if $\mathrm{Rad}(A) = \{0\}$, i.e. the Gelfand homomorphism is injective.

Because of $\sigma(a) = \sigma(\mathcal{G}(a)) = \{\varphi(a) : \varphi \in \sigma(A)\}$ the mapping $\mathrm{ev}_a : \sigma(A) \to \sigma(a)$ is onto and by definition of the topology on $\sigma(A)$ it is also continuous, because $\varphi_i \to \varphi$ pointwise implies that $\mathrm{ev}_a(\varphi_i) = \varphi_i(a) \to \varphi(a) = \mathrm{ev}_a(\varphi)$. Since $\sigma(A) = \mathrm{Alg}(A, \mathbb{C})$ is compact, $\mathrm{ev}_a : \sigma(A) \to \sigma(a)$ is a quotient mapping.

6.44 Proposition.

Let A be a Banach algebra generated by some $a \in A$ as Banach algebra, i.e. $\{p(a) : p \in \mathbb{C}[z]\}$ is dense in A.

Then the mapping

$$\mathrm{ev}_a : \sigma(A) := \mathrm{Alg}(A, \mathbb{C}) \to \sigma(a)$$

is a homeomorphism and the diagram to the right commutes.

$$\begin{tikzcd} C(\sigma(A),\mathbb{C}) & C(\sigma(a),\mathbb{C}) \arrow[l, "(ev_a)^*"', "\cong"] \\ A \arrow[u, "\mathcal{G}"] & H(\sigma(a)). \arrow[l, "\mathrm{ev}_a"] \arrow[u, "(\_)|_{\sigma(a)}"'] \end{tikzcd}$$

Proof. The quotient mapping $\mathrm{ev}_a : \sigma(A) \to \sigma(a)$ is in addition injective and thus a homeomorphism, because for $\varphi_j \in \mathrm{Alg}(A, \mathbb{C})$ with $\varphi_1(a) = \varphi_2(a)$ we have $\varphi_1(p(a)) = \varphi_2(p(a))$ for all polynomials $p \in \mathbb{C}[z]$ and, since the set $\{p(a) : p \in \mathbb{C}[z]\}$ is dense in A by assumption, $\varphi_1 = \varphi_2$ holds.

Since all arrows in the diagram are continuous algebra homomorphisms and $\mathbb{C}\backslash\sigma(a)$ is connected by 6.36, i.e. $\mathbb{C}[z]$ is dense in $H(\sigma(a))$ by 5.3.8, it suffices to prove the commutativity of the diagram on $\mathrm{id} : z \mapsto z$:

$$(\mathrm{ev}_a)^*(\mathrm{id}|_{\sigma(a)})(\varphi) = (\mathrm{id} \circ \mathrm{ev}_a)(\varphi) = \mathrm{id}(\varphi(a)) = \varphi(a) = \mathcal{G}(a)(\varphi) = \mathcal{G}(\mathrm{id}(a))(\varphi). \quad \square$$

Example.

Let

$$A := \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{C} \right\}.$$

be the 2-dimensional commutative Banach subalgebra of $L(\mathbb{C}^2)$ which is generated by $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The only eigenvalue of T is 0, so $\sigma(A) \cong \sigma(T) = \{0\}$ by 6.44. So there is a unique algebra homomorphism $\varphi : A \to \mathbb{C}$ and it suffices $\varphi(T) = 0$. One can see this directly as well: Let $\varphi \in \sigma(A)$, hence an algebra homomorphism $A \to \mathbb{C}$. Then $\varphi(T)^2 = \varphi(T^2) = \varphi(0) = 0$ and thus

$$\varphi \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \varphi(a\,1 + b\,T) = a.$$

Therefore the only maximal ideal in A is $\ker(\varphi) = \mathbb{C} \cdot T$ and hence $\mathrm{Rad}(A) = \ker(\varphi) \neq \{0\}$, i.e. A *is not semisimple*. Moreover, $\mathcal{G} : A \to C(\sigma(A), \mathbb{C}) \cong \mathbb{C}$ is the mapping

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mapsto \varphi \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = a.$$

Example.

A continuous generalization of the last example is given as follows. Let

$$(Kf)(x) := \int_0^1 k(x,y)\, f(y)\, dy = \int_0^x k(x,y)\, f(y)\, dy$$

with measurable integral kernel $k \in L^\infty([0,1]^2)$ and $k(x,y) = 0$ for $x < y$. Then $K : L^2[0,1] \to L^2[0,1]$ is a so-called Volterra operator with norm $\|K\| \leqslant \|k\|_\infty$ and furthermore $\|K^n\| \leqslant \frac{1}{n!}\|k\|_\infty^n$. For all this see [**18**, 3.5.5]. Consequently, $\|K^n\|^{1/n} \leqslant \frac{\|k\|_\infty}{\sqrt[n]{n!}} \to 0$. Thus, the spectral radius $r(K)$ equals 0, and hence $\sigma(K) = \{0\}$, i.e. the Banach algebra generated by K has exactly one maximal ideal (namely the closure of $\{p(K) : p \in \mathbb{C}[z]$ und $p(0) = 0\}$) by 6.44 and is therefore not semisimple.

Example.

The Gelfand homomorphism $\mathcal{G}$ is generally not onto:
Let A be the closure of the polynomials in $C(\partial\mathbb{D},\mathbb{C})$, i.e. the Banach subalgebra of $C(\partial\mathbb{D},\mathbb{C})$ generated by the identity $a : z \mapsto z$. Then according to Proposition 6.44 $\sigma(A) = \mathrm{Alg}(A,\mathbb{C}) \cong \sigma_A(a) = \overline{\mathbb{D}}$ by 6.34. If $\mathcal{G}$ would have dense image in $C(\sigma(A),\mathbb{C})$, then also the composite with $\mathbb{C}[z] \subseteq H(\sigma_A(a)) \to A$, and by 6.44 also $\mathbb{C}[z] \subseteq H(\overline{\mathbb{D}}) \to C(\overline{\mathbb{D}},\mathbb{C})$, which is not the case (uniform limits of sequences of polynomials must be holomorphic on $\mathbb{D}$).

As a first application of the Gelfand transformation we prove the existence of the Stone-Čech compactification:

6.45 Stone-Čech compactification.

For each topological space X there exists a compact space βX, the so-called Stone-Čech compactification *and a continuous mapping $\delta : X \to \beta X$ with the following universal property:*

$$\begin{tikzcd} X \arrow[rr, "\delta"] \arrow[dr, "f"'] & & \beta X \arrow[dl, dotted, "\exists!\ \tilde f"] \\ & K & \end{tikzcd}$$

where K is compact and both f and $\tilde f$ are continuous.

Proof. Let $A := C_b(X,\mathbb{C})$ be the Banach algebra of bounded continuous functions on X with the ∞-norm and pointwise operations. Let $\beta X := \mathrm{Alg}(A,\mathbb{C})$. The mapping $\delta_X : X \to \beta X$, $x \mapsto \mathrm{ev}_x$ is continuous according to the definition of the topology of pointwise convergence on $\mathrm{Alg}(A,\mathbb{C})$.

Let now K be any compact space. By 6.42, $\delta : K \to \mathrm{Alg}(C(K,\mathbb{C}),\mathbb{C})$ is a homeomorphism. Each continuous $f : X \to K$ induces an algebra homomorphism $f^* : C(K,\mathbb{C}) \to C(X,\mathbb{C})$, $g \mapsto g \circ f$ and, since K is compact, it has values in the subalgebra $C_b(X,\mathbb{C})$.

By dualizing again we obtain a continuous mapping $f^{**} : \mathrm{Alg}(C_b(X,\mathbb{C}),\mathbb{C}) \to \mathrm{Alg}(C(K,\mathbb{C}),\mathbb{C})$ and thus a continuous mapping $\tilde f : \mathrm{Alg}(C_b(X,\mathbb{C}),\mathbb{C}) \to K$ with $\delta \circ \tilde f = f^{**}$. This fulfills $\tilde f \circ \delta = f$, because

$$\begin{tikzcd} X \arrow[r, "\delta"] \arrow[d, "f"] & \mathrm{Alg}(C_b(X,\mathbb{C}),\mathbb{C}) \arrow[d, "f^{**}"] \arrow[dl, dotted, "\tilde f"'] \\ K \arrow[r, "\delta", "\cong"'] & \mathrm{Alg}(C(K,\mathbb{C}),\mathbb{C}) \end{tikzcd}$$

$$\begin{aligned}(\delta \circ \tilde f \circ \delta)(x)(h) &= (f^{**} \circ \delta)(x)(h) = f^{**}(\delta(x))(h) = \delta(x)(f^*(h)) = (f^*(h))(x) \\ &= h(f(x)) = \delta(f(x))(h) = (\delta \circ f)(x)(h).\end{aligned}$$

For the uniqueness of $\tilde f$, it is enough to show the denseness of the image of $\delta : X \to \beta X$. Let $\varphi \in \beta X = \mathrm{Alg}(C_b(X,\mathbb{C}),\mathbb{C})$. A typical neighborhood of φ is described by

$$U := \big\{\psi : |(\psi - \varphi)(f_i)| < \varepsilon \text{ for } 1 \leqslant i \leqslant n\big\}$$

with finite many $f_1, \dots, f_n \in C_b(X,\mathbb{C})$ and given $\varepsilon > 0$. We have to find an $x \in X$ with $\mathrm{ev}_x \in U$. Consider the function

$$f := \sum_{i=1}^n |f_i - \varphi(f_i)\,1|^2 = \sum_{i=1}^n (f_i - \varphi(f_i)\,1)\,\overline{(f_i - \varphi(f_i)\,1)} \in C_b(X,\mathbb{C}).$$

Obviously, $\varphi(f) = 0$. Suppose $\mathrm{ev}_x \notin U$ for all $x \in X$ and hence $f(x) \geqslant \varepsilon^2$ for all $x \in X$ and thus also $\frac{1}{f} \in C_b(X,\mathbb{C})$, i.e. $f \in \ker(\varphi)$ is invertible, a contradiction. $\square$

7. Representation theory for C^*-algebras

Basics about C^*-algebras

We now want to find those commutative Banach algebras A for which the Gelfand homomorphism $\mathcal{G} : A \to C(\mathrm{Alg}(A, \mathbb{C}), \mathbb{C})$ from 6.43 is an isomorphism.
Note that the pointwise conjugation $C(X, \mathbb{C}) \to C(X, \mathbb{C})$, $f \mapsto \overline{f}$ defines an involution, i.e. a conjugated linear isometry whose square is the identity and which satisfies $\overline{f \cdot g} = \overline{f} \cdot \overline{g}$. Because of $\overline{f} \cdot f = |f|^2$, we have in addition $\|\overline{f} \cdot f\| = \|f\|^2$ for the ∞-norm.

More generally, the conjugation on $L^\infty(X, \mathcal{A}, \mu)$ for σ-finite measure spaces $(X, \mathcal{A}, \mu)$ also has these properties.

7.1 Definition.

A C^*-ALGEBRA is a Banach algebra A along with an INVOLUTION, i.e. a conjugated linear mapping $(\_)^* : A \to A$, with $(a \cdot b)^* = b^* \cdot a^*$ and $(a^*)^* = a$, which additionally satisfies $\|a\|^2 \leqslant \|a^* \cdot a\|$. One also says for the last condition that $\|\_\|$ is a $*$-NORM. If A has a 1 then $1^* = 1$, because $1^* = 1^* \cdot 1 = 1^* \cdot 1^{**} = (1^* \cdot 1)^* = 1^{**} = 1$.

A algebra homomorphism between C^*-algebras which intertwines with their involutions $*$ is called $*$-HOMOMORPHISM. We will show in 7.28 that continuity is automatical.

For each complex Hilbert space $H \neq \{0\}$ the Banach algebra $L(H)$ with the adjoint $(\_)^* : L(H) \to L(H)$ is a non-commutative C^*-algebra:

$$\|fx\|^2 = \langle fx, fx\rangle = \langle f^* fx, x\rangle \leqslant \|f^* fx\| \cdot \|x\| \leqslant \|f^* f\| \cdot \|x\|^2 \Rightarrow \|f\|^2 \leqslant \|f^* f\|.$$

7.2 Lemma.

Let A be a C^*-algebra (possibly without 1) and $a \in A$. Then

$$\begin{aligned} \|a^*\| = \|a\| &= \max\big\{\|a\,x\| : \|x\| \leqslant 1\big\} = \max\big\{\|x\,a\| : \|x\| \leqslant 1\big\} \\ \textit{and} \quad \|a^* \cdot a\|^2 &= \|a\|^2 = \|a \cdot a^*\|. \end{aligned}$$

Proof. We have $\|a\|^2 \leqslant \|a^* a\| \leqslant \|a^*\|\,\|a\|$, hence $\|a\| \leqslant \|a^*\|$. If we replace a by $b := a^*$ then we get $\|a^*\| \leqslant \|a^{**}\| = \|a\|$ and $\|a\|^2 = \|a^* \cdot a\|$. Moreover,

$$\|a \cdot a^*\| = \|b^* \cdot b\| = \|b\|^2 = \|a^*\|^2 = \|a\|^2.$$

Let $\alpha := \sup\{\|a\,x\| : \|x\| \leqslant 1\} \leqslant \sup\{\|a\|\,\|x\| : \|x\| \leqslant 1\} = \|a\|$. For $x := \frac{1}{\|a\|} a^*$ we have $\|x\| = 1$ by the first part and $\|ax\| = \frac{1}{\|a\|}\|a \cdot a^*\| = \|a\|$, thus $\|a\| = \alpha$ and the supremum is a maximum. □

7.3 Corollary (Adjunction of a unit).

Let A be an C^*-algebra without 1, then $A_1 := \{L_a + \lambda \cdot \mathrm{id} : a \in A, \lambda \in \mathbb{C}\}$ with $L_a : b \mapsto a\,b$ defines a subalgebra of $L(A)$, which, with respect to $(L_a + \lambda \cdot \mathrm{id})^* :=$

$L_{a^*} + \bar\lambda \cdot \mathrm{id}$, *is a C^*-algebra and the canonical mapping $\iota : A \to A_1$, $a \mapsto L_a$ is an isometry with the following universal property:*

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr, "f"'] & A_1 \arrow[d, dotted, "\exists!\ f_1"] \\ & B \end{tikzcd}$$

where f and f_1 are $*$-homomorphisms, B is a C^*-algebra with 1, and f_1 preserves the unit.

Compare this with 6.4. However, the norm defined there is not a $*$ norm.

Proof. We have to show that the operator norm turns A_1 into a C^*-algebra. That A_1 is an algebra is clear because of $(L_a + \lambda \cdot \mathrm{id})\,(L_b + \mu \cdot \mathrm{id}) = L_{a\,b+\lambda b+\mu\, a} + \lambda\mu \cdot \mathrm{id}$. The star defined by $(L_a + \lambda \cdot \mathrm{id})^* := L_{a^*} + \bar\lambda \cdot \mathrm{id}$ is an involution on A_1. So we only have to verify the C^*-condition.
Let $a \in A$ and $\lambda \in \mathbb{C}$. For each $\varepsilon > 0$ there is an $x \in A$ with $\|x\| = 1$ and

$$\begin{aligned}\|L_a + \lambda \cdot \mathrm{id}\,\|^2 - \varepsilon^2 &\leqslant \|a\,x + \lambda\,x\|^2 = \|(a\,x + \lambda\,x)^*\,(a\,x + \lambda\,x)\| \\ &= \|(x^* a^* + \bar\lambda\,x^*)\,(a\,x + \lambda\,x)\| \\ &\leqslant \|x^*\|\,\|(a^* + \bar\lambda)\,(a + \lambda)\,x\| \\ &\leqslant 1\,\|(L_a + \lambda \cdot \mathrm{id})^*(L_a + \lambda \cdot \mathrm{id})\|\,1 \\ &\leqslant \|(L_a + \lambda \cdot \mathrm{id})^*\| \cdot \|L_a + \lambda \cdot \mathrm{id}\,\|.\end{aligned}$$

The universal property follows immediately, as a $*$-homomorphism $f : A \to B$ has as its only possible 1-preserving extension $\tilde f(L_a + \lambda \cdot \mathrm{id}) = f(a) + \lambda \cdot 1$. This extension is indeed an algebra homomorphism because of the above expression for the product. It is also a $*$-homomorphism, due to $\tilde f((L_a + \lambda \cdot \mathrm{id})^*) = \tilde f(L_{a^*} + \bar\lambda \cdot \mathrm{id}) = f(a^*) + \bar\lambda \cdot 1 = f(a)^* + (\lambda \cdot 1)^* = (\tilde f(L_a + \lambda \cdot \mathrm{id}))^*$. □

7.4 Definition.

Let A be a C^*-algebra and $a \in A$.
The element a is called HERMITIAN (or SELF ADJOINT) $:\Leftrightarrow a = a^*$.
The element a is called NORMAL $:\Leftrightarrow a^* a = a\,a^*$.
The element a is called UNITARY $:\Leftrightarrow a^* a = 1 = a\,a^*$.

Example.

For $a \in A := C(X, \mathbb{C})$ with compact X the following holds:

1. a is automatically normal.
2. a is Hermitian $\Leftrightarrow \sigma(a) = a(X) \subseteq \mathbb{R}$.
3. a is unitary $\Leftrightarrow \sigma(a) = a(X) \subseteq S^1$.

7.5 Lemma.

Let H be a Hilbert space. Then the continuous linear operators $T \in L(H)$ correspond in a bijective and isometric manner to the continuous sesqui-linear forms $b : H \times H \to \mathbb{C}$ by virtue of the relation

$$b(x, y) = \langle Tx, y\rangle\ \forall x, y \in H.$$

Moreover, T is self adjoint if and only if b is conjugate-symmetric, and T is positive if and only if b is positive.

Proof. Let $\overline{H}$ be the Hilbert space conjugate to H, i.e. it differs from H only in the definition of scalar multiplication $\lambda \bar\cdot a := \overline{\lambda} \cdot a$. According to Riesz's Theorem

[**18**, 6.2.9], $\iota : H \to \overline{H}^*$, $x \mapsto \langle x, \_\rangle$ is a surjective $\mathbb{C}$-linear isometry, and hence also $\iota_* : L(H,H) \to L(H, \overline{H}^*) \cong L(H, \overline{H}; \mathbb{C})$. The latter space is just that of the continuous sesqui-linear forms on H. Via this isometry the $T \in L(H,H)$ correspond to $b : H \times H \to \mathbb{C}$ given by $b(x,y) := \langle Tx, y\rangle$. Thus T is self adjoint if and only if $b(x,y) = \langle Tx, y\rangle = \langle x, Ty\rangle = \overline{\langle Ty, x\rangle} = \overline{b(y,x)}$, i.e. b is conjugated symmetric; and similarly for positivity. □

7.6 Proposition.

Let $b : H \times H \to \mathbb{C}$ be sesqui-linear. Then the following holds:

1. *The parallelogram equation:*
$$b(x+y, x+y) + b(x-y, x-y) = 2\Big(b(x,x) + b(y,y)\Big)\ \forall x, y \in H.$$
2. *The polarization equation:*
$$\begin{aligned} 4\, b(x,y) &= b(x+y, x+y) - b(x-y, x-y) \\ &\quad + i\, b(x+i\,y, x+i\,y) - i\, b(x-i\,y, x-i\,y), \end{aligned}$$
that means b is already uniquely determined by its values on the diagonal $\{(x,x) : x \in H\}$.
3. $b = 0 \Leftrightarrow \forall x \in H : b(x,x) = 0$.
4. *b is conjugated symmetric* $\Leftrightarrow \forall x \in H : b(x,x) \in \mathbb{R}$.
5. *If b is positive (i.e. $b(x,x) \geqslant 0$ for all $x \in H$), then the Cauchy Schwarz inequality holds:*
$$|b(x,y)|^2 \leqslant b(x,x)\, b(y,y)\ \forall x, y \in H.$$

Note that (3) implies that an operator $B \in L(H)$ is the 0 operator if and only if the associated sesqui-linear form b vanishes on the diagonal, i.e. $\forall x \in H : Bx \perp x$. In the real case this is obviously wrong!

Proof. (1) follows by expanding the left hand side, as was shown in [**18**, 6.2.2].

(2) follows by expansion using the sesqui-linearity.

(3) follows immediately from the polarization equation (2).

(4) The sesqui-linear form $(x,y) \mapsto b(x,y) - \overline{b(y,x)}$ vanishes by (3) if and only if it vanishes on (x,x) for all x, i.e. $b(x,x) \in \mathbb{R}$ for all x.

(5) That's what we have shown in [**18**, 6.2.1]. □

7.7 Proposition.

Let H be a Hilbert space and $a \in L(H)$, then:

1. *a is Hermitian* $\Leftrightarrow \forall x \in H : \langle ax, x\rangle \in \mathbb{R}$.
2. *a is normal* $\Leftrightarrow \forall x \in H : \|ax\| = \|a^* x\|$.
3. $a^* a = 1 \Leftrightarrow \forall x \in H : \|ax\| = \|x\|$
$\Leftrightarrow \forall x, y \in H : \langle ax, ay\rangle = \langle x, y\rangle$, *i.e. a is an isometry.*
4. *a is unitary $\Leftrightarrow$ a is a surjective isometry.*

Proof. (1) The operator a is Hermitian if and only if the conjugated linear form $b(x,y) := \langle ax, y\rangle$ is conjugated symmetric by 7.5. This is the case by 7.6.4 if and only if $b(x,x) = \langle ax, x\rangle$ is real for all x.

(2) By 7.6.3 we have that a is normal, i.e. $b := a^* a - a\, a^* = 0$, exactly if $0 = \langle bh, h\rangle = \langle (a^* a - a\, a^*)h, h\rangle = \|ah\|^2 - \|a^* h\|^2$ for all $h \in H$.

($\boxed{3}$) We have $a^*a = 1 \Leftrightarrow \forall x, y \in H : \langle x, y\rangle = \langle a^*a\, x, y\rangle = \langle a\, x, a\, y\rangle$ and because of the polarization-equation, resp. $\boxed{7.6.3}$, this is equivalent to $\forall x \in H : \|x\|^2 = \|a\, x\|^2$.

($\boxed{4}$) ($\Rightarrow$) $a\, a^* = 1$ implies directly the surjectivity of a.
($\Leftarrow$) $a^*a = 1$ implies $a\, a^*a = a = 1\, a$ and thus $a\, a^* = 1$ by the surjectivity of a. $\square$

7.8 Lemma.

Let A be a C^*-algebra and $a \in A$.

1. *If a is invertible, then so is a^* and $(a^*)^{-1} = (a^{-1})^*$ holds. More generally, $\sigma(a^*) = \overline{\sigma(a)}$ for all $a \in A$.*
2. *We have a unique decomposition $a = \Re e(a) + i\, \Im m(a)$, where $\Re e(a) := \frac{a+a^*}{2}$ and $\Im m(a) := \frac{a-a^*}{2i}$ are Hermitian.*
3. *The element a is normal $\Leftrightarrow \Re e(a)\, \Im m(a) = \Im m(a)\, \Re e(a)$.*
4. *If a is Hermitian, then $\|a\| = r(a)$.*

Proof. ($\boxed{1}$) Applying the involution to $a^{-1}\, a = 1 = a\, a^{-1}$ yields $a^*\, (a^{-1})^* = 1 = (a^{-1})^*\, a^*$. Thus, $\lambda - a$ is invertible if and only if $\overline{\lambda} - a^* = (\lambda - a)^*$ is it.

($\boxed{2}$) Let $a = a_1 + i\, a_2$ be a decomposition into Hermitian elements a_1 and a_2. Then $a^* = a_1 - i\, a_2$ and thus $a_1 = \Re e(a)$ and $a_2 = \Im m(a)$.
On the other hand obviously $a = \Re e(a) + i\, \Im m(a)$ and $(\Re e(a))^* = \frac{a^*+a^{**}}{2} = \Re e(a)$ as well as $(\Im m(a))^* = \frac{a^*-a^{**}}{-2i} = \Im m(a)$.

($\boxed{3}$) We have $a^* = \Re e(a) - i\, \Im m(a)$, hence

$$a^*a = (\Re e(a))^2 - i\, \Im m(a)\, \Re e(a) + i\, \Re e(a)\, \Im m(a) + (\Im m(a))^2 \text{ and}$$
$$a\, a^* = (\Re e(a))^2 + i\, \Im m(a)\, \Re e(a) - i\, \Re e(a)\, \Im m(a) + (\Im m(a))^2.$$

Thus $a^*a = a\, a^* \Leftrightarrow \Im m(a)\, \Re e(a) = \Re e(a)\, \Im m(a)$.

($\boxed{4}$) For Hermitian a the equation $\|a^2\| = \|a^*a\| = \|a\|^2$ holds and thus by induction $\|a^{2^n}\| = \|a\|^{2^n}$. Hence $r(a) = \lim_n \|a^n\|^{1/n} = \lim_n \|a^{2^n}\|^{1/2^n} = \|a\|$. $\square$

Spectral Theory of Abelian C^*-Algebras

We now want to study the Gelfand homomorphism for C^*-algebras. For this we first have to study the $\mathbb{C}$-valued algebra homomorphisms.

7.9 Lemma.

Let A be an C^*-algebra and $f : A \to \mathbb{C}$ an algebra homomorphism. Then f is a $*$-homomorphism.

Proof. We first show that f preserves self-adjointness. So let $a^* = a \in A$ and $t \in \mathbb{R}$. Because of $\|f\| = 1$ by $\boxed{6.39}$ we have

$$\begin{aligned}|f(a + i\, t)|^2 &\leqslant \|a + i\, t\|^2 = \|(a + it)^*(a + it)\| = \|(a - i\, t)\, (a + i\, t)\| \\ &= \|a^2 + t^2\| \leqslant \|a\|^2 + t^2.\end{aligned}$$

If $f(a) = \alpha + i\, \beta$ is the decomposition in real and imaginary parts, we obtain:

$$\|a\|^2 + t^2 \geqslant |f(a + i\, t)|^2 \geqslant |\alpha + i\, (\beta + t)|^2 = \alpha^2 + (\beta + t)^2 = \alpha^2 + \beta^2 + 2\beta t + t^2,$$

hence $\|a\|^2 \geqslant \alpha^2 + \beta^2 + 2\beta t$. If $\beta \neq 0$ then $t \to \pm\infty$ yields a contradiction. Thus $\beta = 0$, i.e. $f(a) = \alpha \in \mathbb{R}$.

Now let $a \in A$ be arbitrary. Since $f(\mathfrak{Re}(a))$ and $f(\mathfrak{Im}(a))$ are real by what has been shown above, we have

$$\begin{aligned} f(a^*) &\overset{7.8.2}{=\!=} f(\mathfrak{Re}(a) - i\,\mathfrak{Im}(a)) = f(\mathfrak{Re}(a)) - i\,f(\mathfrak{Im}(a)) = \overline{f(\mathfrak{Re}(a)) + i\,f(\mathfrak{Im}(a))} \\ &= \overline{f(\mathfrak{Re}(a) + i\,\mathfrak{Im}(a))} = \overline{f(a)}. \quad \square \end{aligned}$$

7.10 Theorem of Gelfand-Naimark.

The Gelfand homomorphism $\mathcal{G} : A \to C(\mathrm{Alg}(A, \mathbb{C}), \mathbb{C})$ is a ($*$-)isomorphism for exactly those Banach algebras A, which can be made into a commutative C^*-algebra by some involution.

Proof. ($\Rightarrow$) If $\mathcal{G}$ is an isomorphism of Banach algebras, we can use it to pull back the involution $f \mapsto \bar{f}$ of $C(\mathrm{Alg}(X, \mathbb{C}), \mathbb{C})$ to A and thus make A into a commutative C^*-algebra.

($\Leftarrow$) Conversely, let A be a commutative C^*-algebra. Then $\mathcal{G}(a^*)(f) = f(a^*) = \overline{f(a)} = \overline{\mathcal{G}(a)(f)}$ holds for all $f \in \mathrm{Alg}(A, \mathbb{C})$ by 7.9, so $\mathcal{G}$ is a $*$-homomorphism.

By 6.43 we have $\|\mathcal{G}(a)\|_\infty = r(a) \leqslant \|a\|$ for all $a \in A$ and for Hermitian elements a we have equality by 7.8.4. In particular, $\|\mathcal{G}(a)\|_\infty^2 = \|\mathcal{G}(a)^*\,\mathcal{G}(a)\|_\infty = \|\mathcal{G}(a^*a)\|_\infty = \|a^*a\| = \|a\|^2$ for all $a \in A$, i.e. $\mathcal{G}$ is an isometry and thus injective.

Since $\mathcal{G}$ has as isometry closed image, it is sufficient for surjectivity to show the denseness of the image. The subalgebra $\mathcal{G}(A)$ of $C(X, \mathbb{C})$ with $X := \mathrm{Alg}(A, \mathbb{C})$ contains the constants and is closed under conjugation. It is also points-separating: Let $\varphi_1 \neq \varphi_2$ be in $X = \mathrm{Alg}(A, \mathbb{C})$, then by definition there is an $a \in A$ with $\mathcal{G}(a)(\varphi_1) = \varphi_1(a) \neq \varphi_2(a) = \mathcal{G}(a)(\varphi_2)$. Thus $\mathcal{G}(A)$ is dense by the Theorem [**18**, 3.4.3] of Stone-Weierstraß. $\square$

Résumé.

So one can calculate with elements of any C^*-algebra as if they were continuous functions on a compact space, as long as one stays inside a commutative subalgebra.

7.11 Remark.

For each set X, the space $A := B(X, \mathbb{C})$ of all bounded $\mathbb{C}$-valued functions on X is a commutative C^*-algebra, thus is by 7.10 isomorphic to $C(\sigma(A), \mathbb{C})$ via Gelfand homomorphism. The spectrum $\sigma(A)$ is the Stone-Čech compactification βX of the discrete space X because $A = B(X, \mathbb{C}) = C_b(X, \mathbb{C})$ and thus $\beta X = \sigma(C_b(X, \mathbb{C})) = \sigma(A)$ by 6.45.

In particular, $\sigma(\ell^\infty) = \sigma(B(\mathbb{N}, \mathbb{C})) = \beta\mathbb{N}$, cf. [**26**, 2.1.15,2.1.16].

7.12 Proposition.

Let A be generated as C^*-algebra by a normal $a \in A$. Then the following diagram is commutative.

$$\begin{tikzcd} C(\mathrm{Alg}(A, \mathbb{C}), \mathbb{C}) & C(\sigma(a), \mathbb{C}) \arrow[l, "(ev_a)^*"', "\cong"] \\ A \arrow[u, "\mathcal{G}", "\cong"'] & \mathbb{C}[z, \bar{z}]. \arrow[l, "ev_a"'] \arrow[u, "(\text{-})|_{\sigma(a)}"'] \end{tikzcd}$$

Proof. Since a is normal, the dense subalgebra $\{p(a, a^*) : p \in \mathbb{C}[z, \bar{z}]\}$ is commutative and thus also A itself. So by 7.10, $\mathcal{G}$ is an isomorphism.

That $\mathrm{ev}_a : \mathrm{Alg}(A,\mathbb{C}) \to \sigma(a)$ is a homeomorphism can be seen as in the proof of 6.44 (Attention: A need not be generated as Banach algebra by a):
Because of the remark after 6.43, $\mathrm{ev}_a : \sigma(A) \to \sigma(a)$ is surjective, and from $\varphi_1(a) = \varphi_2(a)$ follows $\varphi_1(p(a,a^*)) = p(\varphi_1(a),\varphi_1(a)^*) = p(\varphi_2(a),\varphi_2(a)^*) = \varphi_2(p(a,a^*))$ for all $p \in \mathbb{C}[z,\bar z]$ and finally $\varphi_1 = \varphi_2$, since $\{p(a,a^*) : p \in \mathbb{C}[z,\bar z]\}$ is dense in A.

Since all occurring mappings are $*$-homomorphisms, and $\mathbb{C}[z,\bar z]$ is generated by the identity as $*$-algebra, it suffices to check the commutativity on $\mathrm{id} : z \mapsto z$, this already happened in Proposition 6.44 □

In contrast to Banach algebras, the spectrum of an element of a C^*-algebra does not depend on the algebra:

7.13 Proposition.

Let A be a C^*-algebra, B be a C^*-subalgebra, and $b \in B$. Then $\sigma_B(b) = \sigma_A(b)$.

Proof. Let's start with a Hermitian $b \in B$ and let $C^*(b)$ be the Banach subalgebra of B generated by b. Since this is an Abelian C^*-algebra, $\sigma_{C^*(b)}(b) = \{\varphi(b) : \varphi \in \mathrm{Alg}(C^*(b),\mathbb{C})\} \subseteq \mathbb{R}$ by 6.43 and 7.9. By Theorem 6.36 we have

$$\sigma_B(b) \subseteq \sigma_{C^*(b)}(b) \overset{\sigma\subseteq\mathbb{R}}{=\!=} \partial\sigma_{C^*(b)}(b) \subseteq \partial\sigma_B(b) \subseteq \sigma_B(b)$$

and thus $\sigma_B(b) = \sigma_{C^*(b)}(b)$. The same works for A, so $\sigma_B(b) = \sigma_{C^*(b)}(b) = \sigma_A(b)$.

Now let $b \in B$ be arbitrary. It remains to show that the invertibility of b in A implies the invertibility in B, i.e. $\mathrm{Inv}(B) = \mathrm{Inv}(A) \cap B$. So let $a\,b = 1 = b\,a$ for some $a \in A$. Then $(b^*b)(aa^*) = b^*(ba)a^* = b^*a^* = (ab)^* = 1^* = 1$ and analogously $(aa^*)(b^*b) = 1$. Since b^*b is Hermitian and invertible in A, it follows from the first part that b^*b is also invertible in B and because of the uniqueness of the inverse, $a\,a^*$ is in B. So $a = a\,1 = a(a^*b^*) = (aa^*)b^* \in B$. □

Corollary.

Let $a \in A$ be normal. Then $\|a\| = r(a)$ holds.

Proof. Because the C^*-algebra $C^*(a)$ generated by a is commutative, $\|a\| = \|\mathcal{G}(a)\|_\infty = r(a)$ by 7.10 and 6.43. □

7.14 Definition.

Let A be a C^*-algebra and $a \in A$ normal. Then we define by means of 7.12 and 7.13 a $*$-isometry $\rho : C(\sigma(a),\mathbb{C}) \to C^*(a) \subseteq A$, called FUNCTION(AL) CALCULUS for a, by the composite

$$\begin{tikzcd} C(\mathrm{Alg}(C^*(a),\mathbb{C}),\mathbb{C}) & C(\sigma_{C^*(a)}(a),\mathbb{C}) \arrow[l, "(ev_a)^*"', "\cong"] \arrow[d, equal] \\ C^*(a) \arrow[u, "\mathcal{G}", "\cong"'] & C(\sigma_A(a),\mathbb{C}) \arrow[l, dotted, "\rho"] \end{tikzcd}$$

where $C^*(a)$ denotes the (commutative) C^*-subalgebra of A generated by a.

Theorem (Function Calculus).

Let A be a C^*-algebra and $a \in A$ normal. Then the function calculus is the unique $*$-isometry $\rho : C(\sigma(a), \mathbb{C}) \cong C^*(a) \subseteq A$ which extends the Riesz function calculus from $\boxed{6.28}$, i.e. the following diagram commutes.

$$\begin{tikzcd} H(\sigma(a)) \arrow[rr, "(\_)|_{\sigma(a)}"] \arrow[dr, "\text{Riesz}"'] & & C(\sigma(a), \mathbb{C}) \arrow[dl, "\rho"] \\ & A & \end{tikzcd}$$

Proof. Since ρ was obtained by composing C^*-isomorphisms, ρ is also a (not necessarily surjective) $*$-isometry. Due to Proposition $\boxed{7.12}$, ρ coincides with the Riesz calculus on the subspace $\mathbb{C}[z]$ of polynomials. Since the Riesz function calculus is uniquely defined by $\boxed{6.28}$, the triangle commutes.

Now to the uniqueness. Let $\rho : C(\sigma(a), \mathbb{C}) \to A$ be any $*$-homomorphism that extends the Riesz calculus. For each $f \in C(\sigma(a), \mathbb{C})$ there exists, according to Theorem [**18**, 3.4.1] of Stone-Weierstraß, a sequence of polynomials $f_n : \mathbb{R}^2 \to \mathbb{C}$ which converges uniformly on $\sigma(a)$ towards f. We have $\mathbb{C}[\mathfrak{Re}(z), \mathfrak{Im}(z)] \cong \mathbb{C}[z, \overline{z}]$, by $\mathfrak{Re}(z) = \frac{z+\overline{z}}{2}$ and $\mathfrak{Im}(z) = \frac{z-\overline{z}}{2i}$. On $\mathrm{id} : z \mapsto z$ the Riesz function calculus and hence ρ is given by $\rho(\mathrm{id}) = a$ and thus ρ is uniquely determined as $*$-homomorphism on the $*$-algebra $\mathbb{C}[z, \overline{z}]$ generated by id. Because of continuity, ρ is uniquely determined on $C(\sigma(a), \mathbb{C})$. $\square$

Corollary.

Let A be a C^*-algebra and $a \in A$ normal.

1. *a is Hermitian $\Leftrightarrow \sigma(a) \subseteq \mathbb{R}$.*
2. *a is unitary $\Leftrightarrow \sigma(a) \subseteq S^1$.*

This generalizes the example in $\boxed{7.4}$.

Proof. Since a is normal, we have the $*$-homomorphism $\rho : C(\sigma(a), \mathbb{C}) \xrightarrow{\cong} C^*(a) \subseteq A$. Thus:

($\boxed{1}$) $\rho(\mathrm{id}) = a = a^* = \rho(\overline{\mathrm{id}}) \Leftrightarrow \mathrm{id} = \overline{\mathrm{id}}$ on $\sigma(a)$, i.e. $\sigma(a) \subseteq \mathbb{R}$.

($\boxed{2}$) $\rho(\overline{\mathrm{id}})\, \rho(\mathrm{id}) = a^* a = 1 = \rho(1) \Leftrightarrow |\,\mathrm{id}\,|^2 = \overline{\mathrm{id}}\, \mathrm{id} = 1$ on $\sigma(a)$, i.e. $\sigma(a) \subseteq S^1$. $\square$

7.15 Spectral Mapping Theorem.

Let A be a C^*-algebra and $a \in A$ normal. Then for each $f \in C(\sigma(a), \mathbb{C})$ the equation

$$\sigma(f(a)) = f(\sigma(a)).$$

Proof. Let $\rho : C(\sigma(a), \mathbb{C}) \xrightarrow{\cong} C^*(a) \subseteq A$ be the function calculus $f \mapsto f(a)$. Since ρ is an $*$-isomorphism,

$$\sigma(f(a)) = \sigma_A(\rho(f)) \overset{\boxed{7.13}}{=\!=} \sigma_{C^*(a)}(\rho(f)) \overset{\boxed{7.14}}{=\!=} \sigma(f) \overset{\boxed{6.7.1}}{=\!=} f(\sigma(a)). \quad \square$$

7.16 Corollary.

Let $a \in A$ be normal and $f \in C(\sigma(a), \mathbb{C})$. Then $f(a)$ is in the double commutant $\{a, a^*\}^{kk}$ of $\{a, a^*\}$. Equivalently, $\{a, a^*\}^k = \{f(a) : f \in C(\sigma(a), \mathbb{C})\}^k$.

Cf. $\boxed{6.32}$ and $\boxed{8.15}$.

Proof. According to the Theorem [**18**, 3.4.1] of Stone-Weierstraß the subalgebra $\{p(a, a^*) : p \in \mathbb{C}[z, \overline{z}]\}$ generated by $\{a, a^*\}$ is (because a is normal) dense in

$\{f(a) : f \in C(\sigma(a), \mathbb{C})\}$, so $\{a, a^*\}^k = \{f(a) : f \in C(\sigma(a), \mathbb{C})\}^k$ by the remarks in 6.31, and thus $f(a) \in \{a, a^*\}^{kk}$ for all $f \in C(\sigma(a), \mathbb{C})$. □

Applications to Hermitian elements

We will now give some applications of the function calculus to normal elements of C^*-algebras.

7.17 Definition.

We denote with $\mathfrak{Re}(A) := \{a \in A : a = a^*\}$ the linear subspace of all Hermitian elements. We have seens in 7.8 that $A = \mathfrak{Re}(A) \oplus i \cdot \mathfrak{Re}(A)$.

An $a \in A$ is called POSITIVE and we write $a \geqslant 0$ if a is Hermitian and $\sigma(a) \subseteq [0, +\infty)$. The set of positive elements will be denoted A_+. An $f \in C(X, \mathbb{C})$ is positive if and only if $\forall x \in X : f(x) \geqslant 0$, because $\sigma(f) = f(X)$ by 6.7.1.

We write $a \geqslant b$ for Hermitian elements a and b if $a - b \geqslant 0$.
For $a \in \mathfrak{Re}(A)$ and $f, g \in C(\sigma(a), \mathbb{R})$ with $f \geqslant g$ we have $f(a) \geqslant g(a)$, because $\sigma(f(a) - g(a)) = \sigma((f - g)(a)) = (f - g)(\sigma(a)) \in \mathbb{R}_+$ by 7.15. In particular, $\|a\| \geqslant a$, because by $\sigma(a) \subseteq [-\|a\|, \|a\|]$ we have $\|a\| \geqslant \mathrm{id}|_\sigma(a)$.

7.18 Proposition (Positive and negative parts).

Let $a \in \mathfrak{Re}(A)$. Then there are unique elements a_+, $a_- \in A_+$ with $a = a_+ - a_-$ and $a_+ a_- = 0 = a_- a_+$.

Proof. The idea is to play this back to $a \in C(X)$.
Existence: Let $\mathrm{id}_\pm(t) := \max\{\pm t, 0\}$. Then $\mathrm{id}_\pm \in C(\mathbb{R}, \mathbb{C})$ with $\mathrm{id} = \mathrm{id}_+ - \mathrm{id}_-$ and $\mathrm{id}_+ \mathrm{id}_- = 0$. The Spectral Mapping Theorem 7.15 implies $a_\pm := \mathrm{id}_\pm(a) \geqslant 0$ and $a = \mathrm{id}(a) = (\mathrm{id}_+ - \mathrm{id}_-)(a) = a_+ - a_-$ as well as $a_+ a_- = \mathrm{id}_+(a)\, \mathrm{id}_-(a) = (\mathrm{id}_+ \mathrm{id}_-)(a) = 0(a) = 0$.

Uniqueness: Let $a = b_+ - b_-$ be a second decomposition with $b_\pm \geqslant 0$ and $b_+ b_- = 0 = b_- b_+$. The Banach subalgebra generated by $\{a_+, a_-, b_+, b_-\}$ is a commutative C^*-algebra, because $a b_+ = (b_+ - b_-) b_+ = b_+ b_+ = b_+(b_+ - b_-) = b_+ a$ and thus $a_\pm b_+ = b_+ a_\pm$ by 7.16. And analogously for $a_\pm b_- = b_- a_\pm$. By 7.10 this subalgebra is isomorphic to $C(X, \mathbb{C})$ for some compact space X and there the decomposition of $\mathbb{R}$-valued functions into positive and negative parts is unique, i.e. $b_\pm = a_\pm$. □

7.19 Proposition (Roots).

Let $a \in A_+$ and $1 \leqslant n \in \mathbb{N}$, then there is a unique element $\sqrt[n]{a} \in A_+$ with $a = (\sqrt[n]{a})^n$.

Proof. As in the proof of 7.18 we use the function calculus 7.14 to define $\sqrt[n]{a}$ by $\sqrt[n]{a} := f(a)$ with $f : t \mapsto \sqrt[n]{t}$ and, because of 7.16, $f(a)$ commutes with each other "n-th root" b of a since these commute with $b^n = a$. Because of Theorem 7.10 of Gelfand-Naimark and the uniqueness of n-th positive root for $0 \leqslant a \in C(\sigma(a), \mathbb{C})$, the uniqueness of $\sqrt[n]{a}$ follows. □

7.20 Lemma.

For $a \in \mathfrak{Re}(A)$ are equivalent:

1. $a \geqslant 0$;
2. $\|t - a\| \leqslant t$ *for all* $t \geqslant \|a\|$;
3. $\|t - a\| \leqslant t$ *for some* $t \geqslant \|a\|$.

This description avoids the spectrum, which behaves complicated on sums and products.

Proof. (1 $\Rightarrow$ 2) Let $a \geqslant 0$ and $t \geqslant \|a\|$, then $0 \leqslant t - s \leqslant t$ for all $s \in \sigma(a) \subseteq [0, \|a\|]$. Consequently, via function calculus 7.14, $\|t - a\| = \|t - \mathrm{id}\|_\infty \leqslant \|t\|_\infty = t$.

(2 $\Rightarrow$ 3) is trivial.

(3 $\Rightarrow$ 1) Because of $a = a^*$, $C^*(a)$ is Abelian, and hence by 7.14 isomorphic to $C(X, \mathbb{C})$ where $X := \sigma(a)$. Thus, by assumption, $|t - s| \leqslant t$ for some $t \geqslant \|a\|$ and all $s \in \sigma(a) \subseteq \mathbb{R}$. No such s can be negative, otherwise we would have $|t - s| \geqslant t - s > t$. □

Corollary.

The set A_+ of all positive elements of a C^*-algebra is a closed cone.

Here we understand by a cone K a convex subset $K \subseteq A$, which satisfies $\lambda a \in K$ for $0 \neq a \in K$ and $\lambda \in \mathbb{R}$ if and only if $\lambda \geqslant 0$.

Proof. We first show that A_+ is closed. So let $a_n \in A_+$ with $a_n \to a$. Then, because of the continuity of $*$, also a is Hermitian. And $\|a_n - \|a_n\|\| \leqslant \|a_n\|$ implies $\|a - \|a\|\| \leqslant \|a\|$, i.e. $a \geqslant 0$ by 7.20.

If $a \in A_+$ and $\lambda \geqslant 0$, then obviously $\lambda\, a \in A_+$ by 7.15. Furthermore $A_+ \cap (-A_+) = \{0\}$, because $a \in A_+$ implies $a = a^*$ and $\sigma(a) \subseteq [0, +\infty)$ and $a \in -A_+$ implies $\sigma(a) \subseteq (-\infty, 0]$. So $\sigma(a) = \{0\}$ and $\|a\| = r(a) = 0$ by 7.8.4, i.e. $a = 0$.
So if $\lambda a \in A_+$ with $\lambda < 0$, then $\lambda a \in A_+ \cap (-A_+) = \{0\}$ because of $-\lambda a \in A_+$, i.e. $a = 0$.

It remains to show that with $a, b \in A_+$ also $a + b \in A_+$:
We have $\big\|\|a\| + \|b\| - (a+b)\big\| \leqslant \big\|\|a\| - a\big\| + \big\|\|b\| - b\big\| \leqslant \|a\| + \|b\|$ and $\|a\| + \|b\| \geqslant \|a+b\|$, so by 3 also $a + b \geqslant 0$. □

Remark.

For a, $b \in A_+$ we have $a\,b \in A_+ \Leftrightarrow a\,b = b\,a$:
In fact, $a\,b \in \mathfrak{Re}(A) \Leftrightarrow a\,b = (a\,b)^* = b^*\,a^* = b\,a$. And under these equivalent conditions, according to function calculus, w.l.o.g. $a, b \in C(X, \mathbb{R}_+)$ and thus also $a\,b \geqslant 0$.

7.21 Corollary.

Let $a_i \in A_+$ for $i \in \{1, \ldots, n\}$ with $a_1 + \cdots + a_n = 0$. Then $a_i = 0$ for all i.

Proof. By 7.20 we have $-a_1 = a_2 + \cdots + a_n \geqslant 0$. So $a_1 \in A_+ \cap (-A_+) = \{0\}$ and, because of symmetry, all $a_i = 0$. □

7.22 Corollary.

For $a \in A$ are equivalent:

1. $a \geqslant 0$;
2. $a = b^2$ *for some* $b \in \mathfrak{Re}(A)$;
3. $a = x^* x$ *for some* $x \in A$.

Proof. (1 $\Rightarrow$ 2) is 7.19 for $n = 2$.

(1 $\Leftarrow$ 2) Let $b \in \mathfrak{Re}(A)$ and $a = b^2$. Because of the Spectral Mapping Theorem $\sigma(a) = \sigma(b^2) = \sigma(b)^2 \subseteq \{t^2 : t \in \mathbb{R}\} = [0, +\infty)$, so $a \in A_+$.

(2 $\Rightarrow$ 3) is obvious by $x := b$.

(3 $\Rightarrow$ 1) So let $a = x^*x$ with $x \in A$. Then obviously $a^* = a$. Let $a = a_+ - a_-$ be the decomposition in positive and negative parts by 7.18 . We have to show: $a_- = 0$. Let $x\sqrt{a_-} = b + i\,c$ be the decomposition into real and imaginary parts by 7.8.2 . Then $(x\sqrt{a_-})^*(x\sqrt{a_-}) = (b - i\,c)\,(b + i\,c) = b^2 + c^2 + i(b\,c - c\,b)$ but also $(x\sqrt{a_-})^*(x\sqrt{a_-}) = \sqrt{a_-}\,x^*x\,\sqrt{a_-} = \sqrt{a_-}\,(a_+ - a_-)\,\sqrt{a_-} = -(a_-)^2$. The uniqueness of decomposition in real and imaginary parts implies thus: $b\,c = c\,b$ and $b^2 + c^2 + (a_-)^2 = 0$. Because of (2 $\Rightarrow$ 1) we have $b^2, c^2, (a_-)^2 \geqslant 0$ and thus $(a_-)^2 = 0$ by 7.21 . Finally the positive element $a_- = 0$ because of the uniqueness of the root. □

Corollary.

Let H be a Hilbert space and $a \in L(H)$.
Then a is positive if and only if $\langle ax, x\rangle \geqslant 0$ for all $x \in H$.

Cf. 7.7 .

Proof. ($\Rightarrow$) If $a \geqslant 0$, then $a = b^*b$ for some $b \in L(H)$ by 7.22 . So $\langle ax, x\rangle = \langle b^*b\,x, x\rangle = \langle b\,x, b\,x\rangle = \|b\,x\|^2 \geqslant 0$.

($\Leftarrow$) By 7.7.1 we have $a = a^*$ and it remains to show $\sigma(a) \subseteq [0, +\infty)$. For $t < 0$

$$\begin{aligned}\|(a-t)h\|^2 &= \|ah\|^2 - t\langle ah, h\rangle - t\langle h, ah\rangle + t^2\|h\|^2\\ &= \|ah\|^2 + 2(-t)\langle ah, h\rangle + t^2\|h\|^2 \geqslant 0 + 0 + t^2\|h\|^2.\end{aligned}$$

holds. Thus $\ker(a-t) = \{0\}$, the image $\mathrm{img}(a-t)$ is closed and a continuous inverse b to $a - t$ is uniquely determined on it. We extend this by $b|_{(\mathrm{img}(a-t))^\perp} = 0$ and get $b \circ (a-t) = 1$ and thus $1 = (b \circ (a-t))^* = (a-t)^* \circ b^* = (a-t) \circ b^*$. So $a - t$ has both a left and a right inverse and is thus invertible (see 6.2.3), i.e. $t \notin \sigma(a)$. □

7.23 Proposition.

For the elements of each C^*-algebra, the following holds:

1. *$a \leqslant b$ implies $x^*ax \leqslant x^*bx$.*
2. *$0 \leqslant a \leqslant b$ and a invertible implies b invertible and $0 \leqslant \frac{1}{b} \leqslant \frac{1}{a}$.*

Proof. (1) We have $b - a \geqslant 0$ and thus $\exists y : b - a = y^*y$ by 7.22 . Hence, $x^*bx - x^*ax = x^*(b-a)x = (yx)^*(yx) \geqslant 0$, i.e. $x^*bx \geqslant x^*ax$.

(2) Playing everything back to continuous functions on $\sigma(b) \subseteq [0, \|b\|]$ shows the following special commutative cases:

3. If $b \geqslant 0$ is invertible, then $\frac{1}{b} \geqslant 0$ and $\sqrt{b}$ is invertible;
4. If $b \geqslant 1$, then b is invertible and $\frac{1}{b} \leqslant 1$.

Because of $0 \leqslant b - a$ we have $0 \leqslant (\frac{1}{\sqrt{a}})^*(b-a)\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}}b\frac{1}{\sqrt{a}} - 1 =: b_1 - 1$ by 3 and 1 . So $b_1 \geqslant 1$ and is invertible with $\frac{1}{b_1} \leqslant 1$ by 4 . Then $b = \sqrt{a}\,b_1\sqrt{a}$ is also invertible and $0 \leqslant \frac{1}{b} = \frac{1}{\sqrt{a}}\frac{1}{b_1}\frac{1}{\sqrt{a}} = (\frac{1}{\sqrt{a}})^*\frac{1}{b_1}\frac{1}{\sqrt{a}} \leqslant (\frac{1}{\sqrt{a}})^*1\frac{1}{\sqrt{a}} = \frac{1}{a}$ by 1 and 3 . □

7.24 Proposition (Polar decomposition).

Let H_1 and H_2 be two Hilbert spaces and $a \in L(H_1, H_2)$. Then there is a unique positive $|a| \in L(H_1)$ and a unique partial isometry $u \in L(H_1, H_2)$ with $a = u \circ |a|$ and $\ker u = (\operatorname{img}|a|)^\perp$.

Furthermore: $\ker a = \ker |a| = \ker u$, $\overline{\operatorname{img} a} = \overline{\operatorname{img} u}$, *and* $\overline{\operatorname{img}|a|} = (\ker |a|)^\perp$.

A $u \in L(H_1, H_2)$ is called a PARTIAL ISOMETRY if $u|_{\ker(u)^\perp}$ is an isometry. The subspace with $\operatorname{ini} u := (\ker u)^\perp$ on which u acts isometrically, is called INITIAL SPACE of u. The space $\operatorname{fin} u := \operatorname{img} u = \overline{\operatorname{img} u}$ is called FINAL SPACE of u.

The positive element $|a|$ is also defined for a in an abstract C^*-algebra by $|a| := \sqrt{a^* a}$ by 7.19.

Proof. Existence: We define $|a| := \sqrt{a^* a} \geqslant 0$. For $h \in H_1$ we have

$$\|ah\|^2 = \langle ah, ah\rangle = \langle a^* ah, h\rangle = \langle |a|^2 h, h\rangle = \langle |a|h, |a|h\rangle = \big\||a|h\big\|^2.$$

$$\begin{tikzcd}
H_1 \arrow[rr, dotted, "u"] & & H_2 \\
\operatorname{img}|a| \arrow[u, hook] \arrow[rr, dotted, "u", "\cong"'] & & \operatorname{img} a \arrow[u, hook] \\
& H_1 \arrow[ul, "|a|"] \arrow[ur, "a"'] &
\end{tikzcd}$$

Therefore, $\ker |a| = \ker a$ and the mapping $u : \operatorname{img}|a| \to \operatorname{img} a$, given by $u(|a|h) := ah$, is a well-defined isometry. Hence may be extended to an isometry $u : \overline{\operatorname{img}|a|} \to \overline{\operatorname{img} a}$. And, if we put $u|_{\ker a} = 0$ also to a partial isometry with $a = u\,|a|$ because $(\ker a)^\perp = (\ker |a|)^\perp = \overline{\operatorname{img}|a|}$ by 5.4.3.

Thus $\ker |a| = \ker a = \ker u$ and $\overline{\operatorname{img} a} = \overline{\operatorname{img} u}$.

Uniqueness: Let $a = w\,p$ with $0 \leqslant p \in L(H_1)$ and partial isometry $w \in L(H_1, H_2)$ with $\ker w = (\operatorname{img} p)^\perp$.
We claim that $w^* w$ is the orthogonal projection onto $\operatorname{ini} w := (\ker w)^\perp$:
We have a surjective isometry $w_1 := w|_{(\ker w)^\perp} : \operatorname{ini} w \to \operatorname{fin} w$, so $w_1^* w_1 = 1$ holds by 7.7.4. With respect to the orthogonal decompositions $H_1 := \operatorname{ini} w \oplus \ker w$ and $H_2 := \operatorname{fin} w \oplus (\operatorname{fin} w)^\perp$, we have

$$w = \begin{pmatrix} w_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad w^* = \begin{pmatrix} w_1^* & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and thus} \quad w^* w = \begin{pmatrix} w_1^* w_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is the orthogonal projection onto $\operatorname{ini} w := (\ker w)^\perp = (\operatorname{img} p)^{\perp\perp} = \overline{\operatorname{img} p}$.
Now $a^* a = p w^* w p = p^2$, i.e. $p = |a|$ because of the uniqueness of the positive root $|a| := \sqrt{a^* a}$ by 7.19. Furthermore, $w|a| = wp = a = u|a|$, i.e. $w = u$ holds on $\operatorname{img}|a| = \operatorname{img} p$, and $\overline{\operatorname{img} p} = (\operatorname{img} p)^{\perp\perp} = (\ker w)^\perp = (\ker u)^\perp$, thus $w = u$ holds because $\ker u = (\operatorname{img}|a|)^\perp = (\operatorname{img} p)^\perp = \ker w$. □

Ideals and quotients of C^*-algebras

Our goal is also to handle non-commutative C^*-algebras A. According to the Theorem 7.10 of Gelfand-Naimark we can describe the commutative ones completely by their algebra-homomorphisms $f : A \to \mathbb{C}$. However, for general A, the algebra homomorphisms $f : A \to \mathbb{C}$ factor over the Abelization $A \twoheadrightarrow A/A' = A_{\mathrm{Abel}}$, thus provide too little information about A. Instead, we should discuss algebra homorphisms $f : A \to B$ into more general C^*-algebras B (such as $B = L(H)$) instead of $\mathbb{C}$, and thus ideals $I := \ker(f)$, which are not necessarily maximal (see 6.40).

7.25 Lemma. Closed ideals are invariant under function calculus.

Let I be a closed (one-sided) ideal of a C^*-algebra A.
If $a \in I$ is Hermitian and $f \in C(\sigma(a), \mathbb{C})$ with $f(0) = 0$, then $f(a) \in I$.
In particular, a_+, a_-, $|a|$ and $\sqrt{|a|}$ are I.

Proof. Without loss of the generality $I \neq A$. Then $0 \in \sigma(a)$, because by 6.37 $a \in I$ must not be invertible. Since a is assumed to be Hermitian, $\sigma(a) \subseteq \mathbb{R}$ holds. Let now f_n be a sequence of polynomials converging on $\sigma(a)$ uniformly towards f. Since $f_n(0) \to f(0) = 0$, we may replace f_n by $f_n - f_n(0)$ and thus assume, without loss of generality, that $f_n(0) = 0$, i.e. $g_n : t \mapsto \frac{f_n(t)}{t}$ is a polynomial and thus $f_n(a) = a \cdot g_n(a) \in I$. Since I is closed, $f(a) \in I$.

All the elements a_+, a_-, $|a|$ and $\sqrt{|a|}$ are represented by means of function calculus as $f(a)$ with $f(0) = 0$ and thus belong to I by the first part. □

7.26 Theorem (Approximating unit).

Let I be an ideal in a C^*-algebra A. Then there is a monotonously increasing net $j \mapsto u_j$ in I with $0 \leqslant u_j \leqslant 1$ and $\|a\, u_j - a\| \to 0$ for each $a \in I$.

Proof. Let $\mathcal{J} := \{j : \emptyset \neq j \subseteq I, j \text{ finite}\}$ be the index set for the net partially ordered by inclusion. For $j \in \mathcal{J}$ let $v_j := \sum_{x\in j} x^* x \geqslant 0$. Obviously, $v_j \in I$ and for $j \subseteq j'$ we have $v_{j'} - v_j = \sum_{x \in j' \setminus j} x^* x \geqslant 0$, i.e. $v_j \leqslant v_{j'}$.

Let $u_j := v_j\big(\frac{1}{|j|} + v_j\big)^{-1} = f_{1/|j|}(v_j)$, where $f_t(s) := \frac{s}{t+s}$ for $s \geqslant 0$ and $t > 0$. Since $0 \leqslant f_t(s) \leqslant 1$ we have $0 \leqslant u_j \leqslant 1$ and $u_j \in I$ since I is an ideal. If $0 < t' \leqslant t$ and $0 \leqslant u \leqslant u'$, then $f_t(u') \leqslant f_{t'}(u')$ and $f_t(u) \leqslant f_t(u')$, because on the one hand $f_t(s) \leqslant f_{t'}(s)$ for all $s \geqslant 0$, i.e. $f_t(u') \leqslant f_{t'}(u')$, and on the other hand $t \leqslant t + u \leqslant t + u'$ and thus $\frac{1}{t+u'} \leqslant \frac{1}{t+u}$ by 7.23.2 and consequently $f_t(u) = u\frac{1}{t+u} = 1 - t\frac{1}{t+u} \leqslant 1 - t\frac{1}{t+u'} = u'\frac{1}{t+u'} = f_t(u')$. All in all, $u_j \leqslant u_{j'}$ for $j \subseteq j'$.

Remains to show the convergence. Since

$$u_j - 1 = v_j \left(\frac{1}{|j|} + v_j\right)^{-1} - \left(\frac{1}{|j|} + v_j\right)\left(\frac{1}{|j|} + v_j\right)^{-1} = -\frac{1}{|j|}\left(\frac{1}{|j|} + v_j\right)^{-1}.$$

we obtain

$$\begin{aligned}\sum_{x\in j}(x(u_j - 1))^*(x(u_j - 1)) &= (u_j - 1)\Big(\sum_{x\in j} x^* x\Big)(u_j - 1) = (u_j - 1)v_j(u_j - 1)\\ &= \frac{1}{|j|^2}\underbrace{v_j\left(\frac{1}{|j|} + v_j\right)^{-2}}_{g_{1/|j|}(v_j)} \qquad \text{with } g_t(s) := \frac{s}{(t+s)^2}.\end{aligned}$$

The derivative g_t' at s is $1(t+s)^{-2} - 2s(t+s)^{-3} = \frac{t-s}{(t+s)^3}$. So the maximum is attained at $s = t$ and $g_t(s) \leqslant g_t(t) = \frac{1}{4t}$ for $s \geqslant 0$ and $t > 0$. For $a \in j$, therefore, $(a(u_j - 1))^*(a(u_j - 1)) \leqslant \sum_{x\in j}(x(u_j - 1))^*(x(u_j - 1)) = \frac{1}{|j|^2} g_{1/|j|}(v_j) \leqslant \frac{1}{4|j|}$. So $\|a(u_j - 1)\|^2 = \|(a(u_j - 1))^*(a(u_j - 1))\| \leqslant \frac{1}{4|j|}$ and hence $\|au_j - a\| \to 0$. □

Corollary.

Let I be a closed ideal of a C^*-algebra A.
Then I is $*$-closed, i.e. $a \in I \Rightarrow a^* \in I$.

Proof. Let $a \in I$. Because of Theorem 7.26, there exists a net $u_j \in I$ with $0 \leqslant u_j \leqslant 1$ and $\|u_j^*\, a^* - a^*\| = \|a\, u_j - a\| \to 0$. Since $u_j \geqslant 0$ we have $u_j = u_j^*$ and thus $u_j^*\, a^* = u_j\, a^* \in I$ and hence also $a^* \in I$. □

Lemma.

Let I be a closed ideal of a C^*-algebra A and $j \mapsto u_j$ an approximating unit. Then $\|a+I\|_{A/I} = \lim_j \|a - au_j\|_A$ for each $a \in A$.

Proof. Because of $u_j \in I$ also $au_j \in I$ and thus $\|a - au_j\| \geqslant \inf\{\|a-y\| : y \in I\} =: \|a+I\|$. Hence $\inf_j \|a - au_j\| \geqslant \|a+I\|$.

Let $y \in I$, then $\|yu_j - y\| \to 0$ and thus

$$\begin{aligned}\overline{\lim_j}\, \|a - au_j\| &= \overline{\lim_j}(\|a-au_j\| - \|yu_j - y\|) \leqslant \overline{\lim_j}\, \|a - au_j - yu_j + y\| \\ &= \overline{\lim_j}\, \|(a+y) - (a+y)u_j\| \leqslant \|a+y\| \cdot \overline{\lim_j}\, \|1-u_j\| \leqslant \|a+y\|,\end{aligned}$$

since $0 \leqslant 1-u_j \leqslant 1 \Rightarrow ||1-u_j|| \leqslant 1$: In fact, $w \in A$ and $\lambda \in \mathbb{R}$ with $0 \leqslant w \leqslant \lambda \Rightarrow \lambda - \sigma(w) = \sigma(\lambda - w) \subseteq \mathbb{R}^+ \Rightarrow \sigma(w) \subseteq (-\infty, \lambda] \cap \mathbb{R}^+ = [0,\lambda] \Rightarrow \lambda \geqslant r(w) = \|w\|$. Thus $\overline{\lim}_j \|a - au_j\| \leqslant \|a+I\| := \inf\{\|a+y\| : y \in I\}$.

Hence, $\lim_j \|a - au_j\| = \|a+I\|$. □

7.27 Proposition.

Let I be a closed ideal in a C^*-algebra A. Then A/I is an C^*-algebra and $\pi : A \to A/I$ is a $*$-homomorphism.

Proof. We already know that A/I is a Banach algebra, see the claim in 6.40. Since I is $*$-closed by the corollary in 7.26, $*$ induces an involution on A/I by $(a+I)^* := a^* + I$.

To prove the C^*-property of the quotient norm we use the lemma in 7.26: For $y \in I$ we have

$$\begin{aligned}\|a+I\|^2 &= \lim_j \|a - au_j\|^2 = \lim_j \| \underbrace{(a-au_j)^*(a-au_j)}_{=(1-u_j)a^*a(1-u_j)} \| \\ &= \lim_j \|(1-u_j)(a^*a + y)(1-u_j)\| \quad (\text{because } \|y(1-u_j)\| \to 0) \\ &\leqslant \|a^*a + y\| \quad (\text{because } \|1-u_j\| \leqslant 1) \\ \Rightarrow \|a+I\|^2 &\leqslant \inf_{y \in I} \|a^*a+y\| = \|a^*a + I\| = \|(a+I)^*(a+I)\|. \quad \square\end{aligned}$$

7.28 Theorem.

Let $f : A \to B$ be a $*$-homomorphism between C^*-algebras. Then f is continuous with $\|f\| = 1$ and its image $\operatorname{img}(f)$ is closed. If, in addition, f is injective, then f is an isometry.

Proof. ($\|f\| = 1$) For $a \in A$ we have $\sigma(f(a)) \subseteq \sigma(a)$, because $b(a-\lambda) = 1 = (a-\lambda)b$ implies $f(b)\,(f(a)-\lambda) = (f(a)-\lambda)\,f(b)$, i.e. $\rho(a) \subseteq \rho(f(a))$. So $r(f(a)) \leqslant r(a)$. If we apply this to the Hermitian element a^*a, we obtain, because of 7.8.4 and because $f(\mathfrak{Re}A) \subseteq \mathfrak{Re}B$: $\|f(a)\|^2 = \|f(a)^*f(a)\| = \|f(a^*a)\| = r(f(a^*a)) \leqslant r(a^*a) = \|a^*a\| = \|a\|^2$. So $\|f\| \leqslant 1$. Since f preserves the unit, $\|f\| = 1$ holds.

Let now f be injective and a be Hermitian. Then also $f(a)$ is Hermitian with $\sigma(f(a)) \subseteq \sigma(a) \subseteq \mathbb{R}$ and the nearby diagram commutes because of the naturality of the function calculus 7.14 (in fact, $f^{**} \circ \mathcal{G} = \mathcal{G} \circ f$ and $\mathrm{ev}_a \circ f^* = \mathrm{ev}_{f(a)}$). So also incl^* is injective and thus, according to the Lemma of Urysohn, $\sigma(f(a)) = \sigma(a)$. Consequently, by 7.8.4, $\|a\| = r(a) = r(f(a)) = \|f(a)\|$.

$$\begin{tikzcd}
A \arrow[r, tail, "f"] & B \\
C^*(a) \arrow[u, hook] \arrow[r, tail, "f"] & C^*(f(a)) \arrow[u, hook] \\
C(\sigma(a)) \arrow[u, two heads, tail, "\rho"', "\cong"] \arrow[r, "\mathrm{incl}^*"] & C(\sigma(f(a))) \arrow[u, two heads, tail, "\rho"', "\cong"]
\end{tikzcd}$$

Let $a \in A$ be arbitrary now. Then $\|a\|^2 = \|a^*a\| = \|f(a^*a)\| = \|f(a)^*f(a)\| = \|f(a)\|^2$, i.e. f is an isometry.

Finally, let f be arbitrary. By 7.27, f then induces an injective $*$-homomorphism $A/_{\ker f} \rightarrowtail B$ which is an isometry by the previous part. Thus $\mathrm{img}(f)$ is closed. $\square$

7.29 The closed ideals of $C(X, \mathbb{C})$.

Let X be a topological space. We consider the two mappings

$$\{A : A \subseteq X\} \overset{\Phi}{\underset{\Psi}{\rightleftarrows}} \{I : I \subseteq C(X,\mathbb{C})\}, \qquad \begin{cases} A \mapsto \{f \in C(X,\mathbb{C}) : f|_A = 0\} \\ \{x \in X : f(x) = 0 \; \forall f \in I\} \leftarrow\!\shortmid I \end{cases}$$

These describe a Galois connection, i.e. they are antiton between the two sets $\{A : A \subseteq X\}$ and $\{I : I \subseteq C(X,\mathbb{C})\}$ being partially ordered by inclusion, and satisfy

$$\begin{aligned} I \subseteq \Phi(A) &\Leftrightarrow \forall f \in I : f \in \Phi(A), \text{ i.e. } f|_A = 0 \\ &\Leftrightarrow \forall f \in I \; \forall a \in A : f(a) = 0 \\ &\Leftrightarrow \forall a \in A \; \forall f \in I : f(a) = 0 \\ &\Leftrightarrow \forall a \in A : a \in \Psi(I) \\ &\Leftrightarrow A \subseteq \Psi(I). \end{aligned}$$

Each Galois connection induces a bijection between the image of Φ and image of Ψ given by $\Psi : \mathrm{img}(\Phi) \to \mathrm{img}(\Psi)$ with inverse $\Phi : \mathrm{img}(\Psi) \to \mathrm{img}(\Phi)$:
From the above equivalence immediately follows $I \subseteq \Phi(\Psi(I))$ and $A \subseteq \Psi(\Phi(A))$ for all I and A, and hence $\Phi(A) \subseteq \Phi(\Psi(\Phi(A))) \subseteq \Phi(A)$ for $I := \Phi(A)$ by applying Φ. So $\Phi \circ \Psi = \mathrm{id}$ holds on $\mathrm{img}(\Phi)$ and $\Psi \circ \Phi = \mathrm{id}$ on $\mathrm{img}(\Psi)$ by symmetry.

Proposition.

Let X be compact. Then the closed ideals of $C(X,\mathbb{C})$ are in bijective relationship with the closed subsets of X. To each I from $C(X,\mathbb{C})$ is assigned the closed subset $\Psi(I) := \{x \in X : f(x) = 0 \; \forall f \in I\}$ of X. And conversely, to each subset A of X is assigned the closed ideal $\Phi(A) := \{f \in C(X,\mathbb{C}) : f|_A = 0\}$ of $C(X,\mathbb{C})$.
Furthermore, $C(X,\mathbb{C})/I \cong C(\Psi(I),\mathbb{C})$.

Proof. It only remains to show that the image of Φ consists of the closed ideals of $C(X,\mathbb{C})$ and that of Ψ consists of the closed subsets of X.

Well-definedness. It is obvious that the images consist of closed sets, because $\Psi(I) = \bigcap_{f \in I} f^{-1}(0)$ and $\Phi(A) = \{f : 0 = f(a) = \delta(a)(f) \; \forall a \in A\} = \bigcap_{a \in A} \delta(a)^{-1}(0)$, where $\delta : X \to \mathrm{Alg}(C(X,\mathbb{C}),\mathbb{C})$ is the homeomorphism from 6.42. Since the $\delta(a) : C(X,\mathbb{C}) \to \mathbb{C}$ are algebra homomorphisms, $\Phi(A)$ is an ideal.

Ψ is onto. Let $A \subseteq X$ be closed. We have $A \subseteq \Psi(\Phi(A)) \subseteq X$ by the above. Suppose $A \neq \Psi(\Phi(A))$. According to Urysohn's Lemma, there is an $f \in C(X,[0,1])$ with $f|_A = 0$ and $f|_{\Psi(\Phi(A))} \neq 0$, which means $f \in \Phi(A)$ but $f \notin \Phi(\Psi(\Phi(A))) = \Phi(A)$, a contradiction.

Φ is onto. Conversely, let $I \subseteq C(X, \mathbb{C})$ be a closed ideal. Then $C(X, \mathbb{C})/I$ is a commutative C^*-algebra by 7.27, which is isomorphic to $C(Y, \mathbb{C})$ for some compact space Y by 7.10. The canonical quotient mapping thus induces a $*$-homomorphism $\pi : C(X, \mathbb{C}) \to C(X, \mathbb{C})/I \cong C(Y, \mathbb{C})$. We have $\pi = \alpha^*$ in terms of the continuous mapping $\alpha : Y \to X$ given by

$$\begin{tikzcd} \mathrm{Alg}(C(Y,\mathbb{C}),\mathbb{C}) \arrow[r, "\pi^*"] & \mathrm{Alg}(C(X,\mathbb{C}),\mathbb{C}) \\ Y \arrow[u, "\cong"', "\delta"] \arrow[r, dotted, "\alpha"'] & X \arrow[u, "\cong"', "\delta"] \end{tikzcd}$$

because

$$\begin{aligned} \alpha^*(f)(y) &= (f \circ \alpha)(y) = f(\alpha(y)) = \delta(\alpha(y))(f) \\ &= (\delta \circ \alpha)(y)(f) = (\pi^* \circ \delta)(y)(f) = (\pi^*(\delta(y)))(f) \\ &= (\delta(y) \circ \pi)(f) = \delta(y)(\pi(f)) = \pi(f)(y). \end{aligned}$$

Thus, $I = \ker(\pi) = \ker(\alpha^*) = \{f : 0 = \alpha^*(f) = f \circ \alpha\} = \{f : f|_{\alpha(Y)} = 0\} = \Phi(\alpha(Y))$, i.e. $I \in \mathrm{img}(\Phi)$.

Finally, $\mathrm{incl}^* : C(X, \mathbb{C}) \to C(\Psi(I), \mathbb{C})$ is a continuous and (by Urysohn's Lemma) surjective mapping with $\ker(\mathrm{incl}^*) = \{f \in C(X, \mathbb{C}) : f|_{\Psi(I)} = 0\} = \Phi(\Psi(I)) = I$, i.e. $C(X, \mathbb{C})/I \cong C(\Psi(I), \mathbb{C})$ by 7.28. $\square$

7.30 Proposition.

Let I be a closed ideal in $A := L(H)$ with $I \neq \{0\}$.
Then I contains the ideal $K(H)$ of all compact operators on H.

We will show in 8.26 that this is the only non-trivial closed ideal provided H is separable. The quotient algebra $L(H)/K(H)$ is called CALKIN ALGEBRA. The operators whose cosets are invertible in the Calkin algebra are called FREDHOLM OPERATORS, see [**5**].

Proof. Let $0 \neq a \in I$. Then there is an $x \neq 0$ with $a(x) \neq 0$. Let $e, f \in H$ be arbitrary with $e \neq 0$. Then $b : h \mapsto \frac{\langle h,e\rangle}{\|e\|^2} x$ and $c : h \mapsto \frac{\langle h,a(x)\rangle}{\|a(x)\|^2} f$ are continuous linear operators with $b(e) = x$ and $b|_{e^\perp} = 0$ and $c(a(x)) = f$. So $b\,a\,c \in I$ is given by $h \mapsto \langle h, e\rangle f$, i.e. maps the vector e to f and $e^\perp$ to 0.

It follows easily that all finite-dimensional operators T are in I, because they can be written as $h \mapsto \sum_{j=1}^n \langle h, e_j\rangle f_j$ with certain $e_j, f_j \in H$:
In fact, let $\{f_1, \dots, f_n\}$ be an orthonormal basis for the finite dimensional image of T. Then $T(h)$ can be written as $T(h) = \sum_{i=1}^n T_i(h)\, f_i$, where $T_i(h) = \langle T(h), f_i\rangle = \langle h, T^*(f_i)\rangle$. Let $\{e_1, \dots, e_m\}$ be an (orthonormal) basis of the image of $T^* \circ T$, hence $T^*(f_i) = \sum_j t_{i,j} e_j$ with $t_{i,j} \in \mathbb{C}$. Thus

$$\begin{aligned} T(h) &= \sum_{i=1}^n T_i(h)\, f_i = \sum_i \langle h, T^*(f_i)\rangle\, f_i = \sum_i \Big\langle h, \sum_j t_{i,j}\, e_j \Big\rangle\, f_i \\ &= \sum_j \langle h, e_j\rangle \sum_i t_{i,j}\, f_i. \end{aligned}$$

Since I is closed, it contains all compact operators because they are contained in the closure of the finite-dimensional ones (by [**18**, 6.4.8]). $\square$

Cyclic representations of C^*-algebras

We want to investigate the structure of non-commutative C^*-algebras. For commutative C^*-algebras we have seen in 7.10 that the $*$-homomorphisms into $\mathbb{C}$ fully describe the algebra, and thus we obtained an isometric $*$-homomorphism onto $C(X,\mathbb{C})$ for a suitable compact space X. Our typical example for non-commutative C^*-algebras is $L(H)$ for Hilbert spaces H. It is therefore reasonable to investigate $*$-homomorphisms $A \to L(H)$.

7.31 Definition (Representations and invariant subspaces).

Let A be a C^*-algebra. A REPRESENTATION of A (on a Hilbert space H) is a $*$-homomorphism $\rho : A \to L(H)$.

Two representations $\rho_i : A \to L(H_i)$ with $i \in \{1,2\}$ are called EQUIVALENT if a surjective isometry $U : H_1 \to H_2$ exists that interwines the actions, i.e. $\forall a \in A$: $\rho_2(a) \circ U = U \circ \rho_1(a)$.

The ORTHOGONAL SUM of a family of representations $\{\rho_i : A \to L(H_i)\}_{i\in I}$ is the representation $\rho := \bigoplus_i \rho_i : A \to L(H)$ on the Hilbert space

$$H := \bigoplus_{i\in I} H_i := \Big\{h = (h_i) \in \prod_{i\in I} H_i : \|h\|^2 := \sum_{i\in I} \|h_i\|^2 < \infty\Big\},$$

given by $\rho(a)(h) = (\rho_i(a)(h_i))_{i\in I}$.

A subset $N \subseteq H$ is called INVARIANT SUBSET for the representation, if $\rho(a)(N) \subseteq N$ for all $a \in A$. If N is an invariant closed linear subspace of H, then the representation $\rho : A \to L(H)$ induces a representation $\rho_N : A \to L(N)$, defined by $\rho_N(a) := \rho(a)|_N$. For each $h \in H$ the ORBIT $\rho(A)h$ is an invariant linear subspace.

If N is an invariant linear subspace, obviously the closure $\overline{N}$ and its orthogonal complement $N^\perp$ are also invariant (in fact, $h \in N^\perp \Rightarrow \langle \rho(a)h, k\rangle = \langle h, \rho(a^*)k\rangle = 0$ for all $k \in N$, because for those we have $\rho(a^*)k \in N$).
Furthermore, ρ is equivalent to the orthogonal sum of $\rho|_{\overline{N}}$ and $\rho|_{N^\perp}$.

A representation $\rho : A \to L(H)$ is called IRREDUCIBLE if there are exactly(!) two closed invariant subspaces, namely $\{0\} \neq H$.

It is now suggestive to attempt to decompose the representation space H of a representation ρ into invariant subspaces N so that they can not be further decomposed, i.e. the restriction ρ_N is irreducible, and to write ρ up to equivalence as the orthogonal sum of these irreducible representations, However, this is generally not possible. To decompose every representation into simple representations we need a weaker notion than irreducibility, namely cyclicity:

An $h \in H$ is called CYCLIC VECTOR if the orbit $\rho(A)h$ of h is dense in H.
A representation $\rho : A \to L(H)$ is called CYCLIC if it has a cyclic vector.
Obviously, every vector $h \neq 0$ of an irreducible representation is a cyclic vector, and thus the representation is cyclic.

Main example of a cyclic representation.

For a σ-finite measure space $(X, \mathcal{A}, \mu)$

$$\rho : L^\infty(X) \to L(L^2(X)), \quad \rho(f)(g) := f \cdot g$$

defines a representation, because

$$\langle h, \rho(f^*)(g)\rangle = \langle h, \overline{f}\cdot g\rangle = \int_X h \cdot f \cdot \overline{g}\, d\mu = \langle h\cdot f, g\rangle = \langle \rho(f)(h), g\rangle = \langle h, \rho(f)^*(g)\rangle$$

This representation is cyclic:
If $\mu(X) < \infty$, we may use $h := \chi_X \in L^2(X)$ as the cyclic vector, since by [18, 4.12.5] even the elementary functions $g \in L^\infty(X)$ are dense in L^2, also $\rho(L^\infty) \cdot h = \{gh : g \in L^\infty\} = L^\infty$ is dense in L^2.
If $\mu(X) = \infty$, then we choose a decomposition $X = \bigsqcup_n A_n$ with $\mu(A_n) < \infty$ and put $h := \sum_n \frac{1}{\sqrt{2^n \mu(A_n)}} \chi_{A_n}$. Then $h \in L^2$ is a cyclic vector, because each $f \in L^2$ is approximated by $f \cdot \chi_{\bigcup_{k \leqslant n} A_k} = \sum_{k \leqslant n} f \cdot \chi_{A_k}$ in L^2 by the Lebesgue Theorem [18, 4.11.12] on Dominated Convergence (in fact, $|f|^2 \geqslant |f - f \cdot \chi_{\bigcup_{k\leqslant n} A_k}|^2 \to 0$ ptw.) and these partial sums can be approximated by the first part by $\{g \cdot h : g \in L^\infty(X)\}$.

However, this representation $\rho : L^\infty(X) \to L(L^2(X))$ of an Abelian C^*-algebra is irreducible by 7.42 only if $L^2(X) \cong \mathbb{C}$, i.e. μ is a point measure δ_a for some $a \in X$.

For a positive Borel measure μ on a compact space X, this induces a representation $\rho|_{C(X,\mathbb{C})} : C(X, \mathbb{C}) \to L(L^2(X))$, $\rho(f)(g) := f \cdot g$.

7.32 Theorem.

Each representation of a C^*-algebra is equivalent to an orthogonal sum of cyclic representations.

Proof. Let $\mathcal{M}$ be the set of all subsets $M \subseteq H\backslash\{0\}$ with $\rho(A)h_1 \perp \rho(A)h_2$ for all $h_1, h_2 \in M$ with $h_1 \neq h_2$. By means of Zorn's Lemma we obtain a maximal element $M \in \mathcal{M}$ with respect to the inclusion. Suppose the subspace $\langle\rho(A)M\rangle$ of H generated by $\rho(A)M$ is not dense. Let $k \neq 0$ be an element of its orthogonal complement. Then $\langle\rho(a)k, \rho(b)h\rangle = \langle k, \rho(a^*b)h\rangle = 0$ for all $a, b \in A$ and $h \in M$, i.e. $\rho(A)k \perp \rho(A)h$, a contradiction to maximality.

For $h \in H$, let H_h be the invariant subspace $\overline{\rho(A)h}$ of H and ρ_h the restriction of the representation to this subspace. Obviously, ρ_h is cyclic with cyclic vector h. Furthermore, $U : \bigoplus_{h \in M} H_h \to H$, $x = (x_h) \mapsto \sum_h x_h$, is a surjective (because $\langle\rho(A)M\rangle$ is dense) isometry (by Pythagoras), with respect to which $\bigoplus_{h\in M} \rho_h$ is equivalent to ρ. □

7.33 From cyclic representations to positive functionals.

So we should study cyclic representations more closely. Let $\rho : A \to L(H)$ be a (cyclic) representation with a (cyclic) vector $h \in H$. Then

$$f : A \to \mathbb{C}, \quad f(a) := \langle\rho(a)h, h\rangle,$$

is a bounded linear functional with $\|f\| = \|h\|^2$, because for $\|a\| \leqslant 1$ also $\|\rho(a)\| \leqslant 1$ by 7.28 and therefore $|f(a)| = |\langle\rho(a)h, h\rangle| \leqslant \|\rho(a)h\| \cdot \|h\| \leqslant \|h\|^2$ and $f(1) = \|h\|^2$. This functional will probably carry a great deal of information of the representation.

Each continuous linear functional $f : A \to \mathbb{C}$ on a C^*-algebra A defines a sesqui-linear form $g : A \times A \to \mathbb{C}$ by $g(a, b) := f(b^*a)$. For the above f, this provides a positive (and thus Hermitian) form because

$$g(a, a) = f(a^*a) = \langle\rho(a^*a)h, h\rangle = \langle\rho(a)h, \rho(a)h\rangle = \|\rho(a)h\|^2 \geqslant 0.$$

Consequently, we define:

Definition. Positive functionals and states.

A linear functional $f : A \to \mathbb{C}$ on a C^*-algebra is called POSITIVE if $f(a) \geqslant 0$ for all $a \in A_+$, i.e. the associated sesqui-linear form $g : (a, b) \mapsto f(b^*a)$ is positive. Such an f is MONOTONE, i.e. $a \leqslant b$ implies $f(a) \leqslant f(b)$.

The functional f is called STATE if it is positive and $\|f\| = 1$.

Proposition.

A linear functional $f : A \to \mathbb{C}$ on a C^*-algebra is positive if and only if $\|f\| = f(1)$ (and thus is bounded).

Proof. ($\Rightarrow$) For Hermitian x we have $x \leqslant \|x\|$ (see 7.17) and thus $f(x) \leqslant f(\|x\|) = \|x\|\, f(1)$.

For arbitrary x we obtain by the Cauchy Schwarz inequality 7.6.5 for $g : (x, y) \mapsto f(y^* x)$ the inequality

$$|f(x)|^2 = |g(x,1)|^2 \leqslant g(x,x)\, g(1,1) = f(x^*x)\, f(1) \leqslant \|x^*x\|\, f(1)^2 = (f(1)\, \|x\|)^2,$$

i.e. $\|f\| \leqslant f(1)$. Because of $|f(1)| = f(1) \cdot \|1\|$ equality holds.

($\Leftarrow$) For this we assume, without loss of generality, that $1 = \|f\| = f(1)$. Because of 7.22 we have to show $f(a^*a) \geqslant 0$. We have $\sigma(a^*a) \subseteq [0, \|a^*a\|]$. This interval is the intersection of all discs $\lambda_0 + K_R := \lambda_0 + \{\lambda \in \mathbb{C} : |\lambda| \leqslant R\}$ with $R > 0$, $\lambda_0 \in \mathbb{C}$ containing it. It is therefore sufficient to show $f(a^*a) - \lambda_0 \in K_R$ for these $\lambda_0 \in \mathbb{C}$ and $R > 0$. This is indeed the case, because $|f(a^*a) - \lambda_0| = |f(a^*a - \lambda_0)| \leqslant \|f\|\, \|a^*a - \lambda_0\| = 1 \cdot r(a^*a - \lambda_0) \leqslant R$ by the Corollary in 7.13, since $\sigma(a^*a - \lambda_0) = \sigma(a^*a) - \lambda_0 \subseteq K_R$. □

Example.

The positive linear functionals on $C(X, \mathbb{C})$ are exactly the positive Baire measures, and the states are exactly the probability measures μ, i.e. $\mu(X) = 1$.

7.34 Extension theorem for positive functionals and for states.

Let A be a C^*-algebra and B a C^*-subalgebra of A.
Then any positive functional and any state of B can be extended to one on A.

Proof. Let $f : B \to \mathbb{C}$ be a positive functional, so by 7.33 it is a linear functional with $\|f\| = f(1)$. By Corollary 5.1.5 of the Theorem of Hahn-Banach there exists a linear extension $\tilde f : A \to \mathbb{C}$ with $\|\tilde f\| = \|f\| = f(1) = \tilde f(1)$. Consequently, $\tilde f$ is also a positive functional. □

7.35 Reconstruction of the representation from the positive functional.

Let $\rho : A \to L(H)$ be a cyclic representation of a C^*-algebra A. We want to try to recover this representation from the functional $f : a \mapsto \langle \rho(a)h, h\rangle$, where h should be a cyclic vector.

First we reconstruct the Hilbert space H: Let $U : A \to H$ be the continuous linear mapping $a \mapsto \rho(a)h$. It has dense image because h is cyclic. Furthermore: $\langle U(a), U(b)\rangle = \langle \rho(a)h, \rho(b)h\rangle = \langle \rho(b^*a)h, h\rangle = f(b^*a)$. Thus the kernel of U is the set $I_f := \{a \in A : f(a^*a) = 0\}$ and H is isometrically isomorphic to the completion H_f of $A/_{I_f} \cong \operatorname{img}(U)$ with respect to the norm $\|a + I_f\|^2 := f(a^*a)$.

$$\begin{tikzcd}
\ker U = I_f \arrow[r, hook] & A \arrow[r, two heads, "U"] \arrow[dr, "\pi"'] & \operatorname{img}(U) \arrow[r, hook] & H \\
 & & A/I_f \arrow[u, tail, two heads, "\cong"', "\tilde U"] \arrow[r, hook] & H_f \arrow[u, dotted, "\cong"', "\tilde U"]
\end{tikzcd}$$

Now we reconstruct the representation ρ:
The representation ρ_f induced on H_f by ρ via $\tilde U$ is given by

$$\begin{aligned}\tilde U(\rho_f(a)(b+I_f)) &:= \rho(a)(\tilde U(b+I_f)) = \rho(a)(U(b)) = \rho(a)(\rho(b)h) = \rho(ab)h\\ &= U(ab) = \tilde U(ab+I_f).\end{aligned}$$

Hence $\rho_f(a) : b+I_f \mapsto ab+I_f$ is induced by the left multiplication with a on A. The cyclic vector $h \in H$ obviously corresponds via $\tilde U$ to $h_f := 1+I_f \in H_f$.

Let now $f : A \to \mathbb{C}$ an arbitrary positive functional on some C^*-algebra, which we assume to be commutative for now, i.e. without loss of generality $A = C(X) := C(X,\mathbb{C})$ for some compact space X. According to Riesz's Theorem 5.3.4, $f(g) = \int_X g\,d\mu$ for a positive Baire measure μ and all $g \in C(X)$.

Thus $f(g^*g) = \int_X \overline{g}\,g\,d\mu =: \|g\|_2^2$ and hence

$$I_f = \Big\{g : 0 = f(g^*g)\,g\,d\mu = \|g\|_2^2\Big\} = \Big\{g \in C(X) : g = 0 \ \ \mu\text{-a.e.}\Big\},$$

i.e. the completion H_f of $C(X)/I_f$ is isomorphic to $L^2(X,\mu)$.

The induced representation ρ_f is nothing else but the representation of $C(X)$ on $L^2(X,\mu)$ by multiplication. So we have shown the following:

Proposition.

Up to equivalence, the cyclic representations of the commutative C^*-algebras $C(X)$ are exactly the representations $C(X) \to L(L^2(\mu))$ by multiplication for Baire measures μ on X. □

Now let's generalize this to arbitrary C^*-algebras:

7.36 Theorem (Gelfand-Naimark-Segal).

Let A be a C^*-algebra. Then there exists a bijection between equivalence classes of cyclic representations with distinguished cyclic (normed) vectors and positive linear functionals (states) on A. This assignment is given as follows:

- ($\mapsto$) *To a representation $\rho : A \to L(H)$ with cyclic vector h one associates the positive linear functional $f = f_{\rho,h} : a \mapsto \langle \rho(a)h, h\rangle$ on A.*
- ($\leftarrow\!\shortmid$) *For a positive linear functional $f : A \to \mathbb{C}$ one considers the subspace $I_f := \{a \in A : f(a^*a) = 0\}$ and the completion H_f of A/I_f with respect to the sesqui-linear form $\langle a+I_f, b+I_f\rangle := f(b^*a)$. The associated representation $\rho_f : A \to L(H_f)$ is given by $\rho_f(a)(b+I_f) := ab+I_f$ and $h_f := 1+I_f$ is a distinguished cyclic vector.*

Proof. ($\leftarrow\!\shortmid$) This was shown in 7.33.

($\mapsto$) Let $f : A \to \mathbb{C}$ be a positive linear functional and $g : (a,b) \mapsto f(b^*a)$ be the associated positive sesqui-linear form. Then

$$I_f := \Big\{a : f(a^*a) = g(a,a) = 0\Big\} = \Big\{a : g(a,b) = 0 \text{ for all } b \in A\Big\}$$

is a closed linear subspace, where the equation holds since $|g(a,b)|^2 \leqslant g(a,a)\,g(b,b)$. Consequently, g factors to a positive-definite sesqui-linear form $\tilde g$ on $A/_{I_f}$ given by $\tilde g(a+I_f, b+I_f) := g(a,b) = f(b^*a)$. Let H_f be the Hilbert space obtained by completing $A/_{I_f}$ with respect to $\tilde g$. For $x \in I_f$,

$$g(a\,x, b) = f(b^*ax) = f((a^*b)^*x) = g(x, a^*b) = 0,$$

hence $a\,I_f \subseteq I_f$, and thus

$$\rho_f : A \times (A/_{I_f}) \to A/_{I_f}, \quad (a, b+I_f) \mapsto a\,b+I_f$$

is a well-defined bilinear mapping. We have to show the continuity of $b+I_f \mapsto ab+I_f$ with respect to the norm $\|b + I_f\|^2 := f(b^*b)$:

$$\|ab + I_f\|^2 = f(b^*a^*ab) \leqslant \|a\|^2 f(b^*b) = \|a\|^2 \|b + I_f\|^2,$$

because $a^*a \leqslant \|a^*a\|$ and thus $b^* a^*a\, b \leqslant b^* \|a^*a\|\, b = \|a\|^2 b^*b$ by 7.23.1.
As we easily see, ρ_f induces an algebra-homomorphism $\rho_f : A \to L(H_f)$ by extending it to the completion H_f of $A/_{I_f}$. This mapping $\rho_f : A \to L(H_f)$ is even a $*$-homomorphism, because

$$\begin{aligned}\tilde g\Big(\rho_f(a)(x + I_f), y + I_f\Big) &= \tilde g(ax + I_f, y + I_f) = g(ax, y)\\ &= g(x, a^*y) = \tilde g\Big(x + I_f, \rho_f(a^*)(y + I_f)\Big).\end{aligned}$$

Moreover, $h_f := 1 + I_f$ is a cyclic vector for ρ_f, because its orbit $\rho_f(A)\,(1 + I_f) = \{a + I_f : a \in A\} = A/_{I_f}$ is dense in H_f by construction.

($\rho \curvearrowright f$) Any f coincides with the functional $a \mapsto \tilde g(\rho_f(a)(h_f), h_f) = \tilde g(a + I_f, 1 + I_f) = f(1^*a) = f(a)$ associated to ρ_f and h_f.

($\rho \circlearrowleft f$) Let $\rho : A \to L(H)$ be a representation with cyclic vector h and associated $f = f_{\rho,h} : a \mapsto \langle \rho(a)h, h\rangle$. In 7.35 we have shown that the representation $\rho_h : A \to L(H_f)$ constructed from it is isomorphic via the surjective isometry $\tilde U$ to ρ. $\square$

7.37 Definition. The space of all states.

Let $stat(A)$ be the space of all states $f : A \to \mathbb{C}$ supplied with the topology of pointwise convergence.

Proposition.

Let A be a C^*-algebra. Then the space $stat(A)$ of all states is a compact convex subspace of the unit sphere of A^* and $\|a\| = \max\{f(a) : f \in stat(A)\}$ for all $a \in A_+$.

Proof. The space $\{f \in A^* : \|f\| \leqslant 1 = f(1)\}$ of all states ($|f(1)| \leqslant \|f\|$ is always valid) is obviously a closed convex set in the unit ball of A^* with respect to the topology of pointwise convergence, thus also compact according to 5.4.13.

Let $C^*(a)$ be the commutative C^*-subalgebra of A generated by $a \geqslant 0$. Since $\|a\| = r(a) \in \sigma(a) \subseteq [0, \|a\|]$ the composite $f : C^*(a) \cong C(\sigma(a), \mathbb{C}) \xrightarrow{\mathrm{ev}_{\|a\|}} \mathbb{C}$ is an algebra homomorphism with $f(a) = \|a\|$ and $\|f\| \leqslant 1 = f(1)$. Thus, f is a state on $C^*(a)$ and hence can be extended to a state $f : A \to \mathbb{C}$ by 7.34.

On the other hand, states f clearly satisfy $|f(a)| \leqslant \|f\|\,\|a\| = \|a\|$. $\square$

7.38 Theorem.

Each C^*-algebra A has a faithful *(i.e. injective and thus isometric by 7.28) representation $\rho : A \to L(H)$ on some Hilbert space H.*

If A is separable, the representation can be choosen cyclic, see [5, S.259], [3, S.265].

Proof. Let $H = \bigoplus_{f \in statA} H_f$ and $\rho(a) := \bigoplus_{f \in statA} \rho_f(a)$. Then $\rho : A \to L(H)$ is a representation.

It is faithful: Let $\rho(a) = 0$ and thus $\rho_f(a) = 0$ for all $f \in stat(A)$. Since $a^*a \geqslant 0$ by 7.22, there is a state $f : A \to \mathbb{C}$ with $f(a^*a) = \|a^*a\| = \|a\|^2$ by 7.37. The cyclic vector $h \in H_f$ belonging to the representation ρ_f fulfills $\|h\| = 1$ and $f(b) = \langle \rho_f(b)h, h\rangle_{H_f}$ for all $b \in A$. In particular, $\|a\|^2 = f(a^*a) = \langle \rho_f(a^*a)h, h\rangle = \langle \rho_f(a)h, \rho_f(a)h\rangle = \|\rho_f(a)h\|^2 = 0$, so $a = 0$. $\square$

Irreducible representations of C^*-algebras

So we should study (invariant) closed subspaces of H more closely. Any such subspace can be described as the image of an orthogonal projection. We need the following two lemmas.

7.39 Lemma.

Let H be a Banach space and $P \in L(H)$ be IDEMPOTENT, *i.e. $P^2 = P$ (with other words, P is a* PROJECTION*). Then:*

1. *$1 - P$ is also idempotent;*
2. *$\operatorname{img} P = \ker(1 - P)$ and $\ker P = \operatorname{img}(1 - P)$;*
3. *$H = \operatorname{img} P \oplus \ker P$;*
4. *For $A \in L(H)$: $P \circ A = A \circ P \Leftrightarrow \operatorname{img} P$ and $\ker P$ are A-invariant.*

Proof. ($\boxed{1}$) $(1 - P)^2 = 1 - 2\,P + P^2 = 1 - 2\,P + P = 1 - P$.

($\boxed{2}$) $h \in \operatorname{img} P \Leftrightarrow h = Pk$ with $k \in H \Leftrightarrow Ph = P^2k = Pk = h \Leftrightarrow h \in \ker(1 - P)$. Further, $\operatorname{img}(1 - P) = \ker P$ follows by $\boxed{1}$.

($\boxed{3}$) $\operatorname{img} P \cap \ker P = \{0\}$ because $h \in \operatorname{img} P$ implies $Ph = h$, and $Ph = 0$ for $h \in \ker P$. Each $h \in H$ can be written as $h = Ph + (1 - P)h$, with $Ph \in \operatorname{img} P$ and $(1 - P)h \in \operatorname{img}(1 - P) = \ker P$.

($\boxed{4}$) ($\Rightarrow$) This holds for arbitrary $P \in L(H)$:
We have $A(\operatorname{img} P) = A(P(H)) = P(A(H)) \subseteq P(H) = \operatorname{img} P$, i.e. $\operatorname{img} P$ is A-invariant, and $P(A(\ker P)) = A(P(\ker P)) = 0$, i.e. $\ker P$ is also A-invariant.

($\Leftarrow$) Let now P be a projection with A-invariant kernel and image. For $x \in H$ we have $x = x_0 + x_1$ by ($\boxed{3}$) with $x_0 \in \ker P$ and $x_1 \in \operatorname{img} P$ and thus $Ax_0 \in \ker P$ and $Ax_1 \in \operatorname{img} P$, i.e. $P(Ax_0) = 0 = A(0) = A(Px_0)$ and $P(Ax_1) = Ax_1 = A(Px_1)$, altogether thus $(P \circ A)(x) = (A \circ P)(x)$. □

7.40 Lemma.

For Hilbert spaces H and idempotent $P \in L(H)$ t.f.a.e.:

1. *P is an orthogonal projection, i.e. $\ker P = (\operatorname{img} P)^\perp$;*

$\Leftrightarrow$ 2. *$\ker P \perp \operatorname{img} P$;*

$\Leftrightarrow$ 3. *$\|P\| \leqslant 1$, i.e. P is a contraction;*

$\Leftrightarrow$ 4. *$P \geqslant 0$, i.e. P is positive;*

$\Leftrightarrow$ 5. *$P^* = P$, i.e. P is Hermitian;*

$\Leftrightarrow$ 6. *$P^*P = PP^*$, i.e. P is normal.*

Proof. ($\boxed{1} \Rightarrow \boxed{2}$) is trivial.

($\boxed{2} \Rightarrow \boxed{3}$) $\|h\|^2 = \|Ph\|^2 + \|h - Ph\|^2$ because $\operatorname{img} P \ni Ph \perp h - Ph \in \ker P$. Thus $\|Ph\| \leqslant \|h\|$.

($\boxed{3} \Rightarrow \boxed{4}$) We have $h - Ph = (1 - P)h \in \operatorname{img}(1 - P) = \ker P$. For $h \in \ker P^\perp$, therefore, $0 = \langle h - Ph, h\rangle = \|h\|^2 - \langle Ph, h\rangle$ holds, and thus $\|h\|^2 = \langle Ph, h\rangle \leqslant \|Ph\|\,\|h\| \leqslant \|h\|^2$. Hence $\|Ph\| = \|h\|$ and $\|h - Ph\|^2 = \|h\|^2 - 2\Re e(\langle Ph, h\rangle) + \|Ph\|^2 = 0$ for such h, i.e. $(\ker P)^\perp \subseteq \ker(1 - P) = \operatorname{img} P$.

Let $h = h_0 + h_1$ with $h_0 \in \ker P$ and $h_1 \in (\ker P)^\perp \subseteq \operatorname{img} P$. Consequently, $\langle Ph, h\rangle = \langle Ph_1, h_0 + h_1\rangle = \langle h_1, h_1\rangle \geqslant 0$, i.e. $P \geqslant 0$ by the corollary in $\boxed{7.22}$.

($\boxed{4} \Rightarrow \boxed{5}$) and ($\boxed{5} \Rightarrow \boxed{6}$) are trivial.

($\boxed{6} \Rightarrow \boxed{1}$) Because of $\|Ph\| = \|P^*h\|$ for normal P by 7.7.2, $\ker P = \ker(P^*) = (\operatorname{img} P)^\perp$ by 5.4.3. □

7.41 Theorem.

For each $*$-closed subset $A \subseteq L(H)$ t.f.a.e.:

1. *The set A is irreducible;*

$\Leftrightarrow$ 2. *The commutant A^k consists only of the multiples of the identity;*

$\Leftrightarrow$ 3. $P \in A^k$, $0 \leqslant P \leqslant 1 \Rightarrow \exists \lambda \in [0,1]$: $P = \lambda \cdot \mathrm{id}$;

$\Leftrightarrow$ 4. *The only orthogonal projections in A^k are 0 and 1.*

Proof. ($\boxed{1} \Rightarrow \boxed{2}$) If $b \in A^k$, then $\ker b$ is an invariant subspace by 7.39.4 and thus equal to $\{0\}$ or H, i.e. b is injective or $b = 0$. So the C^*-subalgebra A^k of $L(H)$ has no zero divisors: In fact, let $b_1, b_2 \in A^k$ with $b_1 b_2 = 0$ and $b_1 \neq 0$, hence b_1 is injective and thus $b_2 = 0$. Let $0 \neq b \in A^k$ be Hermitian, then $C(\sigma(b)) \cong C^*(b) \subseteq A^k$ has no zero divisors and thus $\sigma(b)$ is one-pointed, so $C^*(b) = \mathbb{C} \cdot 1$. Since by 7.8.2 each $a \in A^k$ can be written as $\mathfrak{Re}(a) + i\,\mathfrak{Im}(a)$ with Hermitian elements $\mathfrak{Re}(a), \mathfrak{Im}(a) \in A^k$ (because A is $*$-closed), we have $A^k = \mathbb{C} \cdot 1$.

($\boxed{2} \Rightarrow \boxed{3}$) is trivial because it follows from $0 \leqslant P = \lambda \cdot 1 \leqslant 1$ that $0 \leqslant \lambda \leqslant 1$.

($\boxed{3} \Rightarrow \boxed{4}$) For orthogonal projections P we have $0 \leqslant P \leqslant \|P\| \leqslant 1$ by 7.40.4, 7.17 and 7.40.3. Since $P^2 = P$ we get $\lambda^2 = \lambda$, hence $\lambda \in \{0, 1\}$.

($\boxed{1} \Leftarrow \boxed{4}$) Let N be a closed A-invariant subspace of H and let P be the orthogonal projection onto N. Then $\operatorname{img} P = N$ and $\ker P = N^\perp$ are both A-invariant and thus $P \in A^k$ by 7.39.4, i.e. $P = \mathrm{id}$ or $P = 0$ by $\boxed{4}$, hence $N = \{0\}$ or $N = H$. □

7.42 Corollary.

If $A \subseteq L(H)$ is a commutative $*$-closed irreducible subset, then H is 1-dimensional.

Proof. Since A is commutative, $A \subseteq A^k = \mathbb{C}$ by 7.41. Hence every linear subspace is invariant. Since A is irreducible, H has to be 1-dimensional. □

Corollary.

The irreducible representations of commutative C^*-algebras A are given up to equivalence exactly by the algebra homomorphisms $A \to L(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$.

Proof. According to the previous corollary, the representation space H of any irreducible representation of A is necessary isomorphic to $\mathbb{C}$ and thus the representation ρ is given by the algebra-homomorphism $f := \mathrm{ev}_1 \circ \rho : A \to L(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$ by 7.9. □

7.43 Proposition.

Let f be a positive functional on a C^*-algebra A and $\rho : A \to L(H)$ the (by 7.36) associated representation with distinguished cyclic vector h. Then there exists a bijection

$$\Big\{P \in \rho(A)^k \subseteq L(H) : 0 \leqslant P \leqslant 1\Big\} \cong \Big\{g \in A^* : 0 \leqslant g \leqslant f\Big\}$$

which is uniquely determined by the relation

$$g(a) = \langle P(\rho(a)h), h\rangle \text{ for all } a \in A$$

Proof. Let $U : A \to H$ be the continuous linear mapping $a \mapsto \rho(a)h$ with dense image. It satisfies $\langle Ua, Ub\rangle = \langle \rho(a)h, \rho(b)h\rangle = \langle \rho(b^*a)h, h\rangle = f(b^*a)$ by 7.35.

($\mapsto$) Let $P \in \rho(A)^k$ with $0 \leqslant P \leqslant 1$. Then $g_P : a \mapsto \langle P(\rho(a)h), h\rangle$ is a positive linear functional, because

$$\begin{aligned} g_P(a^*a) &= \langle P(\rho(a^*a))h, h\rangle = \langle (P \circ \rho(a^*))(\rho(a)h), h\rangle \\ &= \langle (\rho(a^*) \circ P)(\rho(a)h), h\rangle = \langle \rho(a)^*(P(\rho(a)h)), h\rangle \\ &= \langle P(\rho(a)h), \rho(a)h\rangle = \langle P(Ua), Ua\rangle \geqslant 0, \text{ since } P \geqslant 0. \end{aligned}$$

We have $g_P \leqslant f$, because by the second corollary in 7.22

$$g_P(a^*a) = \langle P(Ua), Ua\rangle \leqslant \langle Ua, Ua\rangle = f(a^*a), \text{ since } P \leqslant 1.$$

($\leftarrow$) Let $g \in A^*$ with $0 \leqslant g \leqslant f$. Then $(a, b) \mapsto g(b^*a)$ is a positive sesqui-linear form on A, which vanishes on $\ker U = \{a \in A : f(a^*a) = 0\}$ by $g \leqslant f$, hence factors over $H/\ker U$ (see 7.35) to a continuous positive sesquilinear form (compare with 7.36). And, since $\operatorname{img} U$ is dense in H it extends to a uniquely determined positive sesqui-linear form $\tilde g : H \times H \to \mathbb{C}$, which corresponds by 7.5 to a positive $P_g \in L(H)$.

We have $P_g \leqslant 1$, because $\langle P_g Ua, Ua\rangle = \tilde g(Ua, Ua) = g(a^*a) \leqslant f(a^*a) = \langle Ua, Ua\rangle$.

Finally, $P_g \in \rho(A)^k$, because $\rho(a)\,U(b) = U(a\,b)$ for $a \in A$:

$$\begin{aligned} \langle (P_g \circ \rho(a))(Ub), Uc\rangle &= \langle P_g(U(ab)), Uc\rangle = g(c^*ab) \\ &= g((a^*c)^*b) = \langle P_g(Ub), U(a^*c)\rangle = \langle P_g(Ub), \rho(a)^*\,(Uc)\rangle \\ &= \langle (\rho(a) \circ P_g)(Ub), (Uc)\rangle. \end{aligned}$$

($g \mapsto P \mapsto g$) For $0 \leqslant g \leqslant f$, let $P := P_g$. Then

$$g_P(a) := \langle P_g(\rho(a)h), h\rangle = \langle P_g(Ua), U1\rangle = g(1^*a) = g(a).$$

($P \mapsto g \mapsto P$) For $0 \leqslant P \leqslant 1$ in $\rho(A)^k$ and $g := g_P$ we have:

$$\langle P_g(Ua), Ub\rangle = g_P(b^*a) = \langle P(\rho(b^*a)h), h\rangle = \langle P(\rho(a)h), \rho(b)h\rangle = \langle P(Ua), Ub\rangle,$$

hence $P_g = P$. □

7.44 Theorem.

For each state $f : A \to \mathbb{C}$ on a C^*-algebra A t.f.a.e.:

1. *The representation associated to f is irreducible;*
2. *For each $0 \leqslant g \leqslant f$ there exists a $0 \leqslant \lambda \leqslant 1$ with $g = \lambda f$;*
3. *The functional f is an extremal point (see 5.5.1) of $stat(A)$.*

Proof. Let $\rho : A \to L(H)$ be the representation associated to f with cyclic vector h.

(1 $\Leftrightarrow$ 2) By 7.41.3, ρ is irreducible if and only if every $P \in \rho(A)^k$ with $0 \leqslant P \leqslant 1$ is a multiple of the identity. By 7.43, these P uniquely correspond to the $g \in A^*$ with $0 \leqslant g \leqslant f$ and $\lambda \cdot \mathrm{id}$ corresponds to $\lambda \cdot f$.

(2 $\Rightarrow$ 3) Let $f = \lambda\, g + (1-\lambda)h$ with states g and h and $0 < \lambda < 1$. Then $0 \leqslant \lambda\, g \leqslant f$ and thus $\lambda\, g = \mu\, f$ for some $0 \leqslant \mu \leqslant 1$ by (2). Because of $f(1) = 1 = g(1)$ we obtain $\lambda = \mu$ and hence $g = f$ and thus also $h = f$, i.e. f is an extremal point.

(3 $\Rightarrow$ 2) Let $0 \leqslant g \leqslant f$ and without loss of generality $g \neq 0$ and $g \neq f$. Then $0 \leqslant f - g \neq 0$, so $0 < \|f - g\| = (f-g)(1) = f(1) - g(1)$ and thus $0 < \lambda := \|g\| =$

$g(1) < f(1) = 1$. The functionals $f_0 := \frac{1}{\lambda} g \geqslant 0$ and $f_1 := \frac{1}{1-\lambda}(f-g) \geqslant 0$ are states because $f_0(1) = \frac{g(1)}{\lambda} = 1$ and $f_1(1) = \frac{f(1)-g(1)}{1-\lambda} = 1$, and clearly $f = \lambda\, f_0 + (1-\lambda)\, f_1$, so $f = f_0 = f_1$ by (3), and thus $g = \lambda\, f_0 = \lambda\, f$. $\square$

7.45 Theorem.

The irreducible representations of any C^*-algebra are point separating.

Proof. Let $a \neq 0$. Then there is an extremal state f with $f(a^*a) > 0$, otherwise the continuous linear mapping $\mathrm{ev}_{a^*a} : A^* \to \mathbb{C}$ would vanish on $\mathrm{Ext}(stat(A))$ and thus also on its closed convex hull which, according to Krein-Millman 5.5.1, coincides with the compact convex (by 7.37) set $stat(A)$. But we have seen in 7.37 that a state $f : A \to \mathbb{C}$ exists with $f(a^*a) = \|a^*a\| = \|a\|^2 \neq 0$, a contradiction. Now let $\rho : A \to L(H)$ be the irreducible representation according to 7.44 with cyclic vector h, which corresponds to the extremal state $f : A \to \mathbb{C}$. Then $0 \neq f(a^*a) = \langle \rho(a^*a)h, h\rangle = \langle \rho(a)h, \rho(a)h\rangle = \|\rho(a)h\|^2$, i.e. $\rho(a) \neq 0$. $\square$

Group Representations

7.46 The group algebra.

Let G be a discrete (or, in particular, a finite group). We want to solve the following universal problem: We are looking for a $\mathbb{K}$-algebra $\mathbb{K}(G)$ and a homomorphism $\delta : G \to \mathbb{K}(G)$ with respect to the multiplication of the algebra, s.t. for each homomorphism $\tau : G \to A$ into an algebra A a unique algebra homomorphism $\tilde\tau : \mathbb{K}(G) \to A$ exists with $\tilde\tau \circ \delta = \tau$, i.e. the following diagram commutes:

$$\begin{tikzcd} G \arrow[rr, "\delta"] \arrow[dr, "\tau"'] & & \mathbb{K}(G) \arrow[dl, dotted, "\exists!"', "\tilde\tau"] \\ & A & \end{tikzcd}$$

In order to achieve this, we first solve the universal problem of finding a $\mathbb{K}$-vector space $\mathbb{K}(G)$ and a mapping $\delta : G \to \mathbb{K}(G)$ for the set G, so that for each mapping $\tau : G \to A$ with values in a $\mathbb{K}$-vector space a unique linear mapping $\tilde\tau : \mathbb{K}(G) \to A$ with $\tilde\tau \circ \delta = \tau$ exists, i.e. the following diagram commutes:

$$\begin{tikzcd} G \arrow[rr, "\delta"] \arrow[dr, "\tau"'] & & \mathbb{K}(G) \arrow[dl, dotted, "\exists!"', "\tilde\tau"] \\ & A & \end{tikzcd}$$

The solution for $\mathbb{K}(G)$ is the free vector space $\coprod_G \mathbb{K} = \bigoplus_G \mathbb{K}$ with the injective mapping $\delta : G \to \coprod_G \mathbb{K}$, $\delta_t := \delta(t) := (\delta_t^s)_{s\in G}$, where $\delta_t^s := 1$ for $t = s$ and 0 else. The elements $f \in \mathbb{K}(G) := \coprod_G \mathbb{K}$ can be written uniquely as finite sum $f = \sum_{t\in G} f(t)\delta_t$, i.e. $\mathbb{K}(G)$ can be identified with the space of all functions $f : G \to \mathbb{K}$ with finite support.
The mapping $\tilde\tau$ is given by

$$\tilde\tau(f) := \tilde\tau\Big(\sum_{t\in G} f(t)\,\delta_t\Big) = \sum_{t\in G} f(t)\,\tilde\tau(\delta_t) = \sum_{t\in G} f(t)\,\tau(t).$$

It is easy to see that this vector space also has the universal property for multi-linear mappings, i.e. every mapping $\tau : G \times \dots \times G \to A$ with values in a $\mathbb{K}$-vector space corresponds to a multi-linear mapping $\tilde\tau : \mathbb{K}(G) \times \ldots \times \mathbb{K}(G) \to A$ with $\tilde\tau \circ (\delta \times \ldots \times \delta) = \tau$ given by $\tilde\tau(f^1, \ldots, f^n) := \sum_{t_1,\ldots,t_n \in G} f^1(t_1) \cdot \cdots \cdot f^n(t_n)\, \tau(t_1, \ldots, t_n)$. If we apply this to the multiplication $G \times G \to G \xrightarrow{\delta} \mathbb{K}(G)$, we obtain a bilinear mapping $\star : \mathbb{K}(G) \times \mathbb{K}(G) \to \mathbb{K}(G)$, which is given by

$$\begin{aligned} f \star g &= \Big(\sum_t f(t)\, \delta_t\Big) \star \Big(\sum_s g(s)\, \delta_s\Big) = \sum_{t,s} f(t)\, g(s)\; \delta_t \star \delta_s \\ &= \sum_{t,s} f(t)\, g(s)\; \delta_{ts} = \sum_r \sum_{ts=r} f(t)\, g(s)\; \delta_r, \end{aligned}$$

i.e. by

$$(f \star g)(r) := \sum_{ts=r} f(t)\, g(s) = \sum_t f(t)\, g(t^{-1}r).$$

Because of the universal property, this multiplication $\star$ is associative (since the multiplication in G is it) and δ_e is a unit, where $e \in G$ is the neutral element of the group. So $\mathbb{K}(G)$ is an associative algebra with unit.

If $\tau : G \to A$ is a group homomorphism, it is easy to see that $\tilde\tau$ becomes an algebra homomorphism, and vice versa.

7.47 Representations of G on $\mathbb{K}(G)$.

The group homomorphism $\delta : G \to \mathbb{K}(G)$ also provides a representation λ of G on the vector space $\mathbb{K}(G)$, i.e. a group homomorphism $\lambda : G \to L(\mathbb{K}(G))$, defined by $\lambda(t)(f) := \lambda_t(f) := \delta_t \star f$. This representation can also be expressed differently:

$$\begin{aligned} \lambda(t)(f) &= \delta_t \star f = \delta_t \star \sum_{s\in G} f(s)\, \delta_s = \sum_{s \in G} f(s)\; \delta_t \star \delta_s \\ &= \sum_{s\in G} f(s)\; \delta_{ts} = \sum_{r \in G} f(t^{-1}r)\; \delta_r = f \circ \ell_{t^{-1}} = (\ell_{t^{-1}})^*(f), \end{aligned}$$

where ℓ_t denotes the so-called LEFT-TRANSLATION on the group G, which is defined by $\ell_t(s) := t\,s$. This ℓ is a group homomorphism of G into the set of all bijections on G.

If $\tilde\tau : \mathbb{K}(G) \to L(H)$ is a representation, and $\tau := \tilde\tau \circ \delta : G \to \mathbb{K}(G) \to L(H)$ is the associated representation of G, then the adjacent diagram is commutative:

$$\begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\ell_t"] & \mathbb{K}(G) \arrow[r, "\tilde\tau"] \arrow[d, "(\ell_{t^{-1}})^* = \lambda_t"] & L(H) \arrow[d, "\tau(t)_*"] \\ G \arrow[r, "\delta"] & \mathbb{K}(G) \arrow[r, "\tilde\tau"'] & L(H), \end{tikzcd}$$

$$\begin{aligned} (\tau(t)_* \circ \tilde\tau)(f) &= \tau(t) \circ \tilde\tau(f) = \tilde\tau(\delta(t)) \circ \tilde\tau(f) \\ &= \tilde\tau(\delta(t) \star f) = \tilde\tau(f \circ \ell_{t^{-1}}) = \tilde\tau(\lambda_t(f)) = (\tilde\tau \circ \lambda_t)(f). \end{aligned}$$

7.48 From $\mathbb{K}(G)$ to $L^1(G)$.

We do not want to remain purely algebraic and instead would like to have a universal property for continuous Banach algebra homomorphisms. For this we have to supply $\mathbb{K}(G)$ with a norm. The p-norms $\|f\|_p := \big(\sum_{t\in G} |f(t)|^p\big)^{1/p}$ satisfy:

$$\|f \star g\|_r \leqslant \|f\|_p \cdot \|g\|_q \text{ if } \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1.$$

In particular, the completion of $\mathbb{K}(G)$ with respect to the 1-norm is a Banach algebra with unit

$$L^1(G) := \Big\{ f : G \to \mathbb{K} : \|f\|_1 := \sum_{t\in G} |f(t)| < \infty \Big\}.$$

Note that these are really the integrable functions with respect to the counting measure $\mu : A \mapsto \sum_{g\in A} 1$.

As we saw in 6.39 and 7.9 together with 7.28, algebra homomorphisms are often automatically continuous and even contractions. The associated algebra homomorphism $\tilde\tau : \mathbb{K}(G) \to A$ with values in a Banach algebra is a contraction (and thus can be extended to $L^1(G)$) if and only if $\|\tau(t)\| \leqslant 1$ for all $t \in G$. However, because of $1 = \|1\| = \|\tau(e)\| = \|\tau(t)\,\tau(t^{-1})\| \leqslant \|\tau(t)\|\,\|\tau(t^{-1})\|$, $\|\tau(t)\| \geqslant \frac{1}{\|\tau(t^{-1})\|} \geqslant 1$ also holds, so τ has values in $U(A) := \{a \in \operatorname{inv}(A) : \|a\| = 1 = \|a^{-1}\|\}$, the set of all invertible elements in the unit sphere of A. If $A = L(H)$ for a Banach space H, then $U(H) := U(L(H))$ is the set of all bijective isometric unitary operators in the case of a Hilbert space H by 7.4, because $\|a\| = 1 = \|a^{-1}\|$ implies

$$\|ax\| \leqslant \|a\|\,\|x\| = \|x\| = \|a^{-1}ax\| \leqslant \|a^{-1}\|\,\|ax\| = \|ax\|.$$

So we have shown the following:

Proposition.

Let G be a discrete group. Then $\delta : G \to L^1(G)$ is a group homomorphism into a Banach algebra which induces a bijection

$$\delta_* : \operatorname{Hom}(L^1(G), L(H)) \cong \operatorname{Hom}(G, U(H))$$

for each Banach space H, where $\operatorname{Hom}(L^1(G), L(H))$ *is the set of contractionary algebra homomorphisms and* $\operatorname{Hom}(G, U(H))$ *is the group of homomorphisms into*

$$U(H) := \{a \in L(H) : a \text{ is an invertible isometry}\}$$

denoted. The elements ρ of the first set are called REPRESENTATIONS *of the Banach algebra $L^1(G)$ on H and the elements τ of the second set are called* UNITARY REPRESENTATIONS *of the group G on H. The bijection is given by*

$$\tau(t) := \rho(\delta_t)$$
$$\rho(f) := \sum_{t\in G} f(t)\tau(t).$$

7.49 The left-regular representations of $L^1(G)$ and the involution.

The representation of $\mathbb{K}(G)$ on the vector space $\mathbb{K}(G)$, given by the convolution, induces well-defined representations (the so-called LEFT-REGULAR REPRESENTATIONS) $\tilde\lambda$ of $L^1(G)$ on the Banach spaces $L^p(G)$, which can be obtained by completing $\mathbb{K}(G)$ with respect to the p-norm. Because the equation $\|f \star g\|_p \leqslant \|f\|_1\,\|g\|_p$ states that the representations are contractions. By composing with $\delta : G \to L^1(G)$ we therefore obtain representations λ of G on the Banach spaces $L^p(G)$.

In case $p = 2$, $H := L^p(G)$ is a Hilbert space and thus $L(H)$ is a C^*-algebra. We now also want to try to make $L^1(G)$ a C^*-algebra so that the left-regular representation $\tilde\lambda : L^1(G) \to L(L^2(G))$ is a $*$-homomophism, i.e.

$$\langle \tilde\lambda(f^*)h_1, h_2\rangle = \langle \tilde\lambda(f)^*h_1, h_2\rangle = \langle h_1, \tilde\lambda(f)h_2\rangle$$

is satisfied for all $f \in L^1(G)$ and $h_1, h_2 \in L^2(G)$. If we choose $h_1 := \delta_e$ and $h_2 := \delta_t$ we obtain

$$\begin{aligned} f^*(t) &= \langle f^* \star \delta_e, \delta_t\rangle = \langle \tilde\lambda(f^*)h_1, h_2\rangle \\ &= \langle h_1, \tilde\lambda(f)h_2\rangle = \langle \delta_1, f \star \delta_t\rangle = \overline{(f \star \delta_t)(1)} = \overline{f(t^{-1})}. \end{aligned}$$

and a corresponding calculation with general h_1 and h_2 shows that $\tilde\lambda : L^1(G) \to L(L^2(G))$ with this definition of f^* is a $*$-homomorphism. Obviously, $(\_)^*$ is an isometric involution (i.e. is conjugated-linear, idempotent, and an anti-homomorphism). However, $L^1(G)$ is not a C^*-algebra, as the following example shows for $G := \mathbb{Z}$.

Example.

For the discrete group $G = \mathbb{Z}$ and $f^*(k) := \overline{f(-k)}$ we have

$$(f^* \star f)(k) = \sum_j f^*(j)\, f(k-j) = \sum_j \overline{f(j)}\, f(k+j).$$

Now let f be real-valued and concentrated on $\{-1,0,1\}$, then $f^* \star f$ is concentrated on $\{-2,-1,0,1,2\}$ and has the following values:

$$f^* \star f : \begin{cases} -2 & \mapsto \overline{f_{+1}}\, f_{-1} \\ -1 & \mapsto \overline{f_0}\, f_{-1} + \overline{f_{+1}}\, f_0 \\ 0 & \mapsto \overline{f_{-1}}\, f_{-1} + \overline{f_0}\, f_0 + \overline{f_{+1}}\, f_{+1} \\ +1 & \mapsto \overline{f_{-1}}\, f_0 + \overline{f_0}\, f_{+1} \\ +2 & \mapsto \overline{f_{-1}}\, f_{+1} \end{cases}$$

Consequently,

$$\|f^* \star f\|_1 = 2\,|f_{+1}\, f_{-1}| + 2\,|f_0|\,|f_{+1} + f_{-1}| + {f_{-1}}^2 + {f_0}^2 + {f_{+1}}^2$$

and

$$\|f\|_1{}^2 = {f_{-1}}^2 + {f_0}^2 + {f_{+1}}^2 + 2\,|f_{+1}\, f_{-1}| + 2\,|f_0\, f_{-1}| + 2\,|f_0\, f_{+1}|.$$

If $f_0 \neq 0$ and $f_{-1} \cdot f_{+1} < 0$ then $\|f^* \star f\|_1 < \|f\|_1^2$.

In summary, we have shown the following:

Proposition.

For each discrete group G, the space $L^1(G)$ is a B^*-ALGEBRA, i.e. a Banach algebra with an involution $*$, which is an isometry but does not necessarily satisfy $\|f^* f\| = \|f\|^2$. The involution on $L^1(G)$ is given by $f^*(t) := \overline{f(t^{-1})}$. □

Lemma.

Let $\rho : B \to A$ be an $*$-homomorphism from a B^*-algebra into a C^*-algebra, then ρ is a contraction.

Proof.

$$\begin{aligned}\|\rho(f)\|^2 &= \|\rho(f)^*\, \rho(f)\| = r(\rho(f)^*\, \rho(f)) = r(\rho(f^*\, f)) \\ &\leqslant r(f^*\, f) \leqslant \|f^*\, f\| \leqslant \|f^*\|\,\|f\| \leqslant \|f\|^2 \quad \square\end{aligned}$$

7.50 Corollary.

The unitary representations of any discrete group G on a Hilbert space H correspond exactly to the $*$-homomorphisms of the B^*-algebra $L^1(G)$ by $L(H)$.

$$\mathrm{Hom}(G, U(H)) \cong \mathrm{Hom}(L^1(G), L(H))$$

Proof. Each $*$-homomorphism $\rho : L^1(G) \to L(H)$ is a contraction according to the lemma above and thus induces a unitary representation $\tau : G \to U(H)$ by 7.48.

Conversely, let $\tau : G \to U(H)$ be a unitary representation and $\rho : L^1(G) \to L(H)$ the algebra homomorphism $\rho : f \mapsto \sum_{t\in G} f(t)\tau(t)$ associated to it by 7.48. Then

$$\begin{aligned}\rho(f^*) &= \sum_{t\in G} \overline{f(t^{-1})}\tau(t) = \sum_{s\in G} \overline{f(s)}\tau(s)^{-1}\\ &= \sum_{s\in G} \overline{f(s)}\tau(s)^* = \Big(\sum_{s\in G} f(s)\tau(s)\Big)^* = \rho(f)^*,\end{aligned}$$

i.e. ρ a $*$-homomorphism. □

7.51 The Haar measure on locally compact groups.

We want to transfer all this as far as possible to LOCALLY COMPACT GROUPS, i.e. groups G, which are additionally locally compact Hausdorff spaces, and for which the multiplication $G\times G \to G$ and the inversion $G \to G$ are continuous. To construct $L^1(G)$ we need a distinguished measure μ on G. We want the left-multiplication ℓ (given by $\ell_t \cdot s = ts$) to induce a representation λ of G on $L^p(G)$ (given by $\lambda_s(f)(t) := (f \circ \ell_{s^{-1}})(t) = f(s^{-1}t)$). So, in particular for $p = 1$ and $f \geqslant 0$, the following should hold:

$$\int_G f(s^{-1}t)\, d\mu(t) = \|\lambda_s(f)\|_1 = \|f\|_1 = \int_G f(t)\, d\mu(t).$$

Thus the measure should be left-invariant, i.e. $\mu(sA) = \mu(A)$ for all measurable A. In fact, it can be shown that such a measure μ (the so-called HAAR MEASURE) always exists on G, and that it is unique up to a constant positive factor, provided one additionally requires that $\mu(U) > 0$ for all open $U \neq \varnothing$. For a proof of this statement, see [**13**, S.185]. For $G = \mathbb{R}$ and $G = S^1$ it is the usual Lebesgue measure and for $G = \mathbb{Z}$ it is the counting measure. We generally write $\int_G f(t)\, dt$ instead of $\int_G f(t)\, d\mu(t)$ for $f \in L^1(G) := L^1(G, \mu)$.

Definition (Convolution).

With $L^p(G) := L^p(G, \mu)$, we denote the Banach space of all equivalence classes of p-integrable functions with respect to the Haar measure μ.

The convolution of two functions is defined analogously to the discrete case by

$$(f \star g)(s) := \int_G f(t)\, g(t^{-1}s)\, dt = \int_G f(st)\, g(t^{-1})\, dt.$$

It provides a bilinear mapping $L^1(G) \times L^p(G) \to L^p(G)$ with $\|f \star g\|_p \leqslant \|f\|_1 \cdot \|g\|_p$ (see [**13**, 20.19]).

The convolution of functions in $L^1(G)$ is associative and thus $L^1(G)$ is a Banach algebra and the convolution induces representations $\tilde\lambda$ of $L^1(G)$ on $L^p(G)$, the so-called LEFT-REGULAR REPRESENTATIONS defined by $\tilde\lambda(f)(g) := f \star g$. To see the associativity, we use the Theorem of Fubini in the following way:

$$\begin{aligned}((f \star g) \star h)(r) &= \int_G (f \star g)(t)\, h(t^{-1}r)\, dt \\ &= \int_G \int_G f(s)\, g(s^{-1}t)\, h(t^{-1}r)\, ds\, dt \\ &= \int_G \int_G f(s)\, g(s^{-1}t)\, h(t^{-1}r)\, dt\, ds \qquad (t = su) \\ &= \int_G \int_G f(s)\, g(u)\, h(u^{-1}s^{-1}r)\, du\, ds \\ &= \int_G f(s)\, (g \star h)(s^{-1}r)\, ds \\ &= (f \star (g \star h))(r).\end{aligned}$$

Since $L^1(G)$ has no unit (see [**18**, 4.7.7]), the group homomorphism $\delta : G \to L^1(G)$ from the discrete case no longer exists.

Nevertheless, we still have a counterpart to the left-regular representation $\tilde{\lambda}$ of $L^1(G)$ on $L^p(G)$, namely the unitary representation $\lambda : G \to L(L^p(G))$, $t \mapsto (f \mapsto f \circ \ell_{t^{-1}})$, which is induced by the left translation ℓ. So there is hope to put representations of $L^1(G)$ in bijective relationship to unitary representations of G. Since G is no longer discrete, we should make continuity assumptions on the representations of G.

7.52 Proposition (Unitary Representations).
Let $\tau : G \to U(H)$ be a group homomorphism into the group of bijective isometries of a Banach space H, then t.f.a.e.:

1. *The mapping $\tau^\wedge : G \times H \to H$ is continuous;*
$\Leftrightarrow$ 2. *The sequence $\tau(t) \to 1$ converges pointwise for $t \to e$;*
$\Leftrightarrow$ 3. *The mapping $\tau : G \to U(H)$ is continuous, with respect to the pointwise convergence on $U(H)$;*
$\Leftrightarrow$ 4. *The mapping $\tau^\wedge : G \times H \to H$ is separately continuous.*

A mapping $\tau : G \to U(H)$ with the above equivalent properties is called UNITARY REPRESENTATION of the group G on the Banach space H.

Proof. ($\boxed{1} \Rightarrow \boxed{2}$) is trivial.

($\boxed{2} \Rightarrow \boxed{3}$) Because $\tau(t) = \tau(t\,t_0^{-1}\,t_0) = \tau(t\,t_0^{-1}) \circ \tau(t_0)$, $\tau(t) \to \tau(t_0)$ converges pointwise for $t\,t_0^{-1} \to e$, i.e. for $t = t\,t_0^{-1}\,t_0 \to e\,t_0 = t_0$.

($\boxed{3} \Rightarrow \boxed{4}$) Assuming that τ has values in $U(H) \subset L(H)$, $\tau^\wedge(t, \_)$ is always continuous. Conversely, $\tau^\wedge(\_, h) = \mathrm{ev}_h \circ \tau$ is continuous for all $h \in H$ if and only if $\tau : G \to U(H)$ is continuous with respect to the pointwise convergence, because this is just the initial topology with respect to $\mathrm{ev}_h : L(H) \to H$ for $h \in H$.

($\boxed{4} \Rightarrow \boxed{1}$) Let $t_0 \in G$, $h_0 \in H$ and $\varepsilon > 0$. Then, because of the continuity of $\tau^\wedge(\_, h_0)$, there is a neighborhood U of t_0 in G, s.t. $\|\tau(t)h_0 - \tau(t_0)h_0\| < \varepsilon$ for all $t \in U$. Consequently,

$$\begin{aligned}\|\tau(t)h - \tau(t_0)h_0\| &\leqslant \|\tau(t)h - \tau(t)h_0\| + \|\tau(t)h_0 - \tau(t_0)h_0\| \\ &\leqslant \|\tau(t)\|\, \|h - h_0\| + \|\tau(t)h_0 - \tau(t_0)h_0\| \\ &\leqslant 1\,\varepsilon + \varepsilon = 2\,\varepsilon\end{aligned}$$

holds for all $\|h - h_0\| < \varepsilon$ and $t \in U$. □

Obviously, a mapping $\tau : G \to L(H)$ that is continuous with respect to the operator-norm on $L(H)$ is also continuous with respect to the coarser topology of pointwise convergence. The fact that the converse implication does not hold is shown by the following

Lemma (Continuity of the left translation).

The mapping

$$\lambda : G \to U(L^1(G)) \subseteq L(L^1(G)), \quad \lambda_s(f) := f \circ \ell_{s^{-1}}$$

induced by the left translation ℓ is a unitary representation of G on $L^1(G)$. It is not continuous with respect to the operator norm on $L(L^1(G))$.

The right translation also induces a group homomorphism $G \to L(L^1(G))$, but which does not have values in $U(L^1(G))$, so it is not a unitary representation.

Proof. Let $t \in G$, $f \in L^1(G)$ and $\varepsilon > 0$. Then there is a $g \in C_c(G)$ with $\|f - g\|_1 < \frac{\varepsilon}{3}$. Since $g \in C_c$ (let $K := \operatorname{Trg} g$), g is uniformly continuous, i.e. there exists a 1-neighborhood U with $|g(s) - g(r)| < \frac{\varepsilon}{6\mu(K)}$ for $rs^{-1} \in U$. Let $s \in V := tU$. Then $s = tu$ for a $u \in U$ and $(t^{-1}r)(s^{-1}r)^{-1} = t^{-1}s = u \in U$ holds and thus

$$\begin{aligned} \|\lambda_s g - \lambda_t g\|_1 &= \int_{\{r : s^{-1}r \in K \text{ or } t^{-1}r \in K\}} |g(s^{-1}r) - g(t^{-1}r)|\, dr \\ &\leqslant \frac{\varepsilon}{6\mu(K)}\, \mu(sK \cup tK) \leqslant \frac{\varepsilon}{3}. \end{aligned}$$

Since the Haar measure is left-invariant and thus $\|\lambda_s f - \lambda_s g\|_1 = \|f - g\|_1 < \frac{\varepsilon}{3}$, we have for $s \in V$:

$$\|\lambda_s f - \lambda_t f\|_1 \leqslant \|\lambda_s(f - g)\|_1 + \|\lambda_s g - \lambda_t g\|_1 + \|\lambda_t(g - f)\|_1 < \varepsilon.$$

The following example shows that mapping $\lambda : G \to U(L^1(G))$ is not continuous with respect to the operator norm: Let $G = \mathbb{R}$. Suppose there were an $\delta > 0$, s.t. $\|\lambda(t) - \lambda(0)\| < 1$ for $|t| \leqslant \delta$. Then, for the characteristic function f of $(0, \delta]$, the supports of $f = \lambda(0)f$ and $\lambda(\delta)f$ would be disjoint and thus $\|\lambda(\delta)f - \lambda(0)f\|_1 = \|\lambda(\delta)f\|_1 + \|\lambda(0)f\|_1 = 2\,\|f\|_1 > \|f\|_1$, a contradiction.

For the right translation, note that

$$f(st) = f((t^{-1}\,s^{-1})^{-1}) = Sf(t^{-1}\,s^{-1}) = S(\lambda_t(Sf))(s),$$

where $Sf(t) := f(t^{-1})$ denotes the reflection and 7.54. □

Lemma.

The representation $G \to L(L^1(G))$, $s \mapsto (f \mapsto f_s(t \mapsto f(ts)))$ by right multiplication is also continuous with respect to the topology of pointwise convergence.

Proof. By the lemma above $\lambda_s f \to f$ converges for $s \to e$ and each $f \in L^1(G)$, hence also $\Delta(s) \cdot \lambda_s f^* \to \Delta(e) \cdot f^* = f^*$, whereby Δ denotes the modulus function to be defined in 7.53 and * the involution which will be defined in 7.55. We have

$$\begin{aligned} \big(\Delta(s) \cdot \lambda_s f^*\big)(t) &= \Delta(s) \cdot f^*(s^{-1}t) = \Delta(s) \cdot \Delta(s^{-1}t) \cdot \overline{f((s^{-1}t)^{-1})} \\ &= \Delta(t) \cdot \overline{f_s(t^{-1})} = (f_s)^*(t). \end{aligned}$$

Thus

$$\|f_s - f\|_1 = \|(f_s)^* - f^*\|_1 = \|\Delta(s) \cdot \lambda_s f^* - f^*\|_1 \to 0 \text{ for } s \to e. \quad \square$$

7.53 The modulus function

The failure of right-invariance of the Haar measure can be described as follows:

Lemma.

Let the modulus Δ be defined by

$$\int_G f(ts)\,d\mu(t) = \Delta(s) \int_G f(t)\,d\mu(t) \text{ for all } f \in L^1(G),\ s \in \mathbb{R}.$$

Then $\Delta : G \to (\mathbb{R}^+, \cdot)$ is a continuous group homomorphism.

Proof. See [13, p196]. Because of the denseness of the subspace generated by the positive continuous functions with compact support, it is sufficient to consider such functions. Let $\mu_s : C_c(G) \to \mathbb{C}$ be defined by $\mu_s(f) := \int_G f(ts)\,d\mu(t)$. Then μ_s is also a left-invariant measure on G. Consequently, there is a positive number $\Delta(s)$ with $\mu_s(f) = \Delta(s)\,\mu(f)$. Furthermore, we have, where f_t denotes the right-translated function $s \mapsto f(st)$:

$$(f_t)_s(r) = f_t(rs) = f((rs)t) = f(r(st)) = f_{st}(r)$$

and thus

$$\Delta(ts)\,\mu(f) = \mu(f_{ts}) = \mu((f_s)_t) = \Delta(t)\,\mu(f_s) = \Delta(t)\,\Delta(s)\,\mu(f).$$

Let U be a relatively compact 1-neighborhood in G, furthermore let $f \neq 0$ and ω be continuous positive functions with compact support on G with $\omega(\operatorname{Trg}(f) \cdot \overline{U}^{-1}) = \{1\}$. Because of the uniform continuity of f, every $\varepsilon > 0$ has a 1-neighborhood $V \subseteq U$ with $|f(st) - f(s)| < \frac{\varepsilon\,\mu(f)}{\mu(\omega)}$ for all $t \in V$ and all $s \in G$. Thus

$$\begin{aligned} |\Delta(t) - 1|\,\mu(f) &= |\mu(f_t) - \mu(f)| \\ &\leqslant \int_{st \in \operatorname{Trg} f \text{ or } s \in \operatorname{Trg} f} |f(st) - f(s)|\,ds \\ &= \int_{s \in \omega^{-1}(1)} |f(st) - f(s)|\,ds \leqslant \varepsilon\,\mu(f), \end{aligned}$$

i.e. $|\Delta(t) - 1| \leqslant \varepsilon$ for all $t \in V$. □

Each discrete, each Abelian, and each compact group G is UNIMODULAR, i.e. $\Delta = 1$, equivalently, the Haar measure is also right-invariant: For discrete G, the counting measure is obviously right-invariant, for Abelian G this is trivial, and for compact G the image under Δ is a compact subgroup of $(\mathbb{R}_+, \cdot)$, which is equal to $\{1\}$.

With respect to the reflection $S : f \mapsto (t \mapsto f(t^{-1}))$, the following holds:

7.54 Lemma.

For $f \in L^1(G)$:

$$\int_G f(t)\,d\mu(t) = \int_G \Delta(t)\,f(t^{-1})\,d\mu(t).$$

Proof. Let $\nu(f) := \int_G \Delta(t)\,f(t^{-1})\,d\mu(t) = \mu(\Delta \cdot Sf)$. Then

$$\begin{aligned} \nu(\lambda_s f) &= \int_G \Delta(t)\,f(s^{-1}t^{-1})\,d\mu(t) = \int_G \Delta(t)\,f((ts)^{-1})\,d\mu(t) \\ &= \int_G \Delta(ts)\,\Delta(s^{-1})\,f((ts)^{-1})\,d\mu(t) = \Delta(s^{-1})\,\mu((\Delta \cdot Sf)_s) \\ &= \Delta(s^{-1})\,\Delta(s)\,\mu(\Delta \cdot Sf) = \nu(f). \end{aligned}$$

So ν is left-invariant and obviously $\nu(U) > 0$ for $U \neq \emptyset$, so $c > 0$ exists with $\nu = c\,\mu$. For $\varepsilon > 0$ we choose a function $g \in C_c(G)$ with $g = Sg$ and $\mathrm{Trg}(g) \subseteq \{t : |\Delta(t) - 1| < \varepsilon\}$. Thus $|g(t) - \Delta(t)g(t)| \leqslant \varepsilon\, g(t)$ and therefore

$$\Big|(1-c)\int_G g\Big| = \Big|\int_G g - \nu(g)\Big| = \Big|\int_G g - \int_G \Delta\, g\Big| \leqslant \varepsilon\Big|\int_G g\Big|,$$

hence $|1-c| \leqslant \varepsilon$, i.e. $c = 1$. So

$$\int_G f(t)\, d\mu(t) = \int_G \Delta(t)\, f(t^{-1})\, d\mu(t). \quad \Box$$

Remark.

One could analogously to the discrete case, define the convolution as

$$\begin{aligned}(f \star_2 g)(r) &:= \int_G f(rs^{-1})\, g(s)\, ds \qquad (s = tr)\\ &= \Delta(r)^{-1} \int_G f(t^{-1})\, g(tr)\, dt \qquad \text{by}\\ &= \Delta(r)^{-1} \int_G \Delta(t)\, f(t)\, g(t^{-1}r)\, dt\\ &= \Delta(r)^{-1}\, ((\Delta\, f) \star g)(r),\end{aligned}$$

i.e. $\Delta \cdot (f \star_2 g) = (\Delta \cdot f) \star g$.

For this second convolution we can not expect associativity, because

$$\begin{aligned}\Delta \cdot ((f \star_2 g) \star_2 h) &= (\Delta \cdot (f \star_2 g)) \star h = ((\Delta \cdot f) \star g) \star h\\ &= (\Delta \cdot f) \star (g \star h) = \Delta \cdot (f \star_2 (g \star h)) \neq \Delta \cdot (f \star_2 (g \star_2 h)).\end{aligned}$$

7.55 The involution on $L^1(G)$.

As in the discrete case, we try to provide $L^1(G)$ with an involution $*$, so that the left-regular representation on $L^2(G)$ is a $*$-representation, i.e. $\langle h_1, f \star h_2\rangle = \langle f^* \star h_1, h_2\rangle$. We have

$$\begin{aligned}\langle h_1, f \star h_2\rangle &= \int_G h_1(r) \int_G \overline{f(t)}\, \overline{h_2(t^{-1}r)}\, dt\, dr\\ &= \int_G \int_G h_1(ts)\, \overline{f(t)}\, \overline{h_2(s)}\, dt\, ds\\ &\overset{\boxed{7.54}}{=\!=} \int_G \int_G \Delta(t)\, h_1(t^{-1}s)\, \overline{f(t^{-1})}\, \overline{h_2(s)}\, dt\, ds\end{aligned}$$

and

$$\langle f^* \star h_1, h_2\rangle = \int_G \int_G f^*(t)\, h_1(t^{-1}s)\, \overline{h_2(s)}\, dt\, ds,$$

consequently we put $f^*(t) := \Delta(t)\, \overline{f(t^{-1})}$, cf. 7.49.

Lemma.

The space $L^1(G)$ is a B^*-algebra (without unit) with involution given by $f^*(t) := \Delta(t)\overline{f(t^{-1})}$.

Proof. Because of 7.54, $\|f^*\|_1 = \|f\|_1$ and

$$(f^*)^*(t) = \Delta(t)\, \overline{f^*(t^{-1})} = \Delta(t)\, \overline{\Delta(t^{-1})}\, \overline{\overline{f((t^{-1})^{-1})}} = f(t).$$

Furthermore:

$$\begin{aligned}(g^* \star f^*)(s) &= \int_G g^*(t)\, f^*(t^{-1}\, s)\, dt = \int_G g^*(st)\, f^*(t^{-1})\, dt \\ &= \int_G \Delta(st)\, \overline{g(t^{-1}\, s^{-1})}\, \Delta(t^{-1})\, \overline{f(t)}\, dt \\ &= \Delta(s)\, \overline{\int_G f(t)\, g(t^{-1}\, s^{-1})\, dt} = (f \star g)^*(s) \qquad \square\end{aligned}$$

As a partial replacement for a unit we have:

7.56 Proposition (Approximating unit).

Let $f \in L^1(G)$ and $\varepsilon > 0$. Then there is a (compact) neighborhood U of e, so that for all $0 \leqslant g \in L^1(G)$ with $\int_G g = 1$ and $g|_{G\setminus U} = 0$ we have

$$\|f \star g - f\|_1 \leqslant \varepsilon.$$

In particular, there is an approximating unit for $L^1(G)$, i.e. a net $i \mapsto u_i$ with $\|u_i\| = 1$ as well as $f \star u_i \to f$ and $u_i \star f \to f$ for all $f \in L^1(G)$.

Proof. Let g be as indicated. Then it is easy to see that $f \star g$ is defined everywhere and lies in $L^1(G)$. Since $\int_G \Delta(t)\, g(t^{-1})\, dt = \int_G g(t)\, dt = 1$ by 7.54,

$$\begin{aligned}(f \star g)(s) - f(s) &= \int_G f(st)\, g(t^{-1})\, dt - f(s) \int_G \Delta(t)\, g(t^{-1})\, dt \\ &= \int_G \underbrace{(f(st) - \Delta(t)\, f(s))\, g(t^{-1})}_{=:F(s,t)}\, dt\end{aligned}$$

holds. We have $F(s,t) = f(st)\,(1 - \Delta(t))\, g(t^{-1}) + (f(st) - f(s))\, \Delta(t)\, g(t^{-1})$, consequently

$$\begin{aligned}k(t) := \int_G |F(s,t)|\, ds &\leqslant \|f_t\|_1\, |1 - \Delta(t)|\, g(t^{-1}) + \|f_t - f\|_1\, \Delta(t)\, g(t^{-1}) \\ &= \Delta(t)\, \|f\|_1\, |1 - \Delta(t)|\, g(t^{-1}) + \|f_t - f\|_1\, \Delta(t)\, g(t^{-1}) \\ &= \Big(\|f\|_1\, |1 - \Delta(t)| + \|f_t - f\|_1\Big)\, \Delta(t)\, g(t^{-1}).\end{aligned}$$

Now let $\varepsilon > 0$. We choose a symmetric neighborhood of e in U

$$\|f\|_1\, |1 - \Delta(t)| \leqslant \frac{\varepsilon}{2} \text{ and } \|f_t - f\|_1 \leqslant \frac{\varepsilon}{2} \text{ for all } t \in U.$$

Now let g be as assumed. Since $g = 0$ outside is $U^{-1} = U$, we obtain $0 \leqslant k \leqslant \varepsilon\, \Delta\, S(g)$. Thus $k \in L^1(G)$ and by Fubini we have

$$\begin{aligned}\|f \star g - f\|_1 &= \int_G \Big|\int_G F(s,t)\, dt\Big|\, ds \leqslant \int_G \int_G |F(s,t)|\, dt\, ds = \int_G \int_G |F(s,t)|\, ds\, dt \\ &= \int_G k(t)\, dt \leqslant \varepsilon \int_G \Delta(t)\, g(t^{-1})\, dt = \varepsilon \int_G g(t)\, dt = \varepsilon.\end{aligned}$$

To obtain an approximating unit, we choose now the index set to be the neighborhood basis of the unit (consisting of compact symmetric neighborhoods) and for each such neighborhood $i := U$ the corresponding weighted characteristic function $\frac{1}{\mu(U)}\, \chi_U$ as u_i. Then, according to above calculation, $f \star u_i \to f$ holds to all $f \in L^1(G)$. Because of $\|u_i^*\| = \|u_i\| = 1$, $\mathrm{Trg}(u_i^*) = \mathrm{Trg}(u_i)^{-1} = U^{-1} = U$ and $u_i^*(t) = \Delta(t)\overline{u_i(t^{-1})} \geqslant 0$ also $g \star u_i^* \to g$ is valid for all $g \in L^1(G)$ and thus $u_i \star f = (f^* \star u_i^*)^* \to (f^*)^* = f$. $\square$

7.57 Theorem.

The left regular representation $\tilde\lambda$ of $L^1(G)$ on $L^2(G)$ is an injective $*$-homomorphism and a contraction.

Proof. We have just choosen $*$ so that $\tilde\lambda : L^1(G) \to L(L^2(G))$ is a $*$-homomorphism. It is injective, because $0 = \tilde\lambda(f)(g) = f \star g$ for all $g \in L^2(G)$ implies $f \star u_i = 0$ and since by $0 = f \star u_i \to f$ we have $f = 0$. In 7.49 we have shown that every $*$-homomorphism from a B*-algebra B (with unit) into a C*-algebra A is a contraction. This even holds for B*-algebras B without unit, because $B_1 := B \oplus \mathbb{C}$ is the associated Banach algebra with unit by 6.4. By virtue of $(x \oplus z)^* := x^* \oplus \overline{z}$ it is a B*-algebra with unit. And every $*$-homomorphism $\rho : B \to A$ extends to a unique, $*$-homomorphism $\rho_1 : B_1 \to A$ by virtue of $\rho_1(x \oplus z) := \rho(x) + z$. So ρ_1 is a contraction and thus also $\rho := \rho_1|_B$. □

7.58 Lemma.

With $A(G)$, we denote the C-algebra generated by the image of the left-regular representation of $L^1(G)$ on $L^2(G)$. Each representation of the C*-algebra $A(G)$ induces a $*$-representation of $L^1(G)$. The commutants of these two representations agree, and thus irreducibility is synonymous for them by 7.41.*

Proof. Note that $A(G)$ is the closure of $\{f \star (\_) + t : f \in L^1(G), t \in \mathbb{C}\}$ in $L(L^2(G))$.

Let $\varphi : A(G) \to L(H)$ be a representation and $\rho := \varphi \circ \tilde\lambda : L^1(G) \to A(G) \to L(H)$ the corresponding representation of $L^1(G)$, then:

$$\begin{aligned} & T \text{ commutes with } \rho(f) = \varphi(f \star (\_)) \text{ for all } f \in L^1(G) \\ \Leftrightarrow\ & T \text{ commutes with } \rho(f) + t = \varphi(f \star (\_) + t) \text{ for all } f \in L^1(G) \text{ and } t \in \mathbb{C} \\ \Leftrightarrow\ & T \text{ commutes with } \varphi(a) \text{ for all } a \in A(G). \qquad \square \end{aligned}$$

7.59 Comparison of the representations of G and of $L^1(G)$

For locally compact groups G we are now trying to relate unitary representations $\tau : G \to U(H)$ and representations $\rho : L^1(G) \to L(H)$ with each other.

$$\begin{tikzcd} G \arrow[d, "\tau"] & L^1(G) \arrow[d, "\rho"'] \\ U(H) \arrow[r, hook] & L(H) \end{tikzcd}$$

($\mapsto$) In the discrete case we had $\rho(f) := \sum_{t\in G} f(t)\tau(t)$. In the general case, we expect $\rho(f) = \int_G f(t)\,\tau(t)\,dt \in L(H)$. Since unitary representations τ need not be continuous with respect to the operator norm by 7.52, the integral in $L(H)$ does not exist, but $\int_G f(t)\,\tau(t)h\,dt \in H$ exists for each $h \in H$, and thus we define

$$\rho(f)h := \int_G f(t)\,\tau(t)h\,dt \in H \text{ for } f \in L^1(G) \text{ and } h \in H.$$

($\leftarrow\!\shortmid$) Conversely, in the discrete case we had $\tau = \rho \circ \delta$, i.e. $\tau(t) = \rho(\delta_t)$. In general, we do not have a unit $\delta_e \in L^1(G)$ but only an approximate unit $u_i \in L^1(G)$, which we can use instead of δ_e. So instead of $\delta_t = \delta_t \star \delta_e = \lambda_t(\delta_e)$ we should use $\lambda_t(u_i)$ and put $\tau(t) := \lim_i \rho(\lambda_t(u_i))$, for which we have to show the existence of the limit.

Another possibility is to use the identity $\tau(t)_* \circ \rho = \rho \circ \lambda_t$ for $t \in G$ of the discrete case, i.e. $\tau(t) \circ \rho(f) = \rho(\lambda_t f)$. This clearly fixes τ on $\rho(L^1(G))H$. If $L^1(G)$ had a unit and ρ preserved it, then $\rho(L^1(G))H = H$ and τ would be fixed. However, since $L^1(G)$ has no unit, representations $\rho : L^1(G) \to L(H)$ may be DEGENERATED, where an algebra homomorphism is called $\rho : A \to L(H)$ NON-DEGENERATED if $\rho(A)H$ generates a dense subspace of H. If ρ is a $*$-homomorphism, this is equivalent to $\rho(A)h = 0 \Rightarrow h = 0$, because

$$\begin{aligned}\langle\rho(A)H\rangle \text{ is dense in } H &\Leftrightarrow \Big(\forall a \in A, \forall k \in H\ \overbrace{\langle\rho(a)k, h\rangle}^{=\langle k,\rho(a^*)h\rangle} = 0\Big) \Rightarrow h = 0\\ &\Leftrightarrow \Big(\rho(A)h = 0 \Rightarrow h = 0\Big).\end{aligned}$$

The space $N := \{h \in H : \rho(A)h = 0\}$ is clearly invariant, hence also $N^\perp$ and $\rho := \rho|_{N^\perp} + 0|_N$, where $\rho|_{N^\perp}$ is not degenerated. So we have no significant restriction when we consider only non-degenerate representations of $L^1(G)$.

Now to the existence of $\lim_i \rho(\lambda_t(u_i))$. For the composition with $\rho(f)$ we obtain:

$$\rho(\lambda_t(u_i)) \circ \rho(f) = \rho(\lambda_t(u_i) \star f) = \rho(\lambda_t(u_i \star f)) \to \rho(\lambda_t(f)),$$

since $u_i \star f \to f$ in $L^1(G)$ and thus $(\rho \circ \lambda_t)(u_i \star f) \to (\rho \circ \lambda_t)(f)$. Since ρ is a contraction, $\|\rho(\lambda_t(u_i))\| \leqslant \|\lambda_t(u_i)\| = \|u_i\| = 1$ holds, and thus $\lim_i \rho(\lambda_t(u_i))$ exists pointwise not only on image of $\rho(f)$ but on all of H. And so $\tau(t) \in L(H)$ is well-defined by

$$\tau(t) := \lim_i \rho(\lambda_t(u_i)) \text{ pointwise on } H$$

and $\|\tau(t)\| \leqslant 1$ and $\tau(t) \circ \rho(f) = \rho(\lambda_t f)$ for all $f \in L^1(G)$. Because of the last equation, we also see that $\tau(t)$ does not depend on the choice of approximating unit u_i.

Theorem.

For locally compact groups G and Hilbert spaces H we have a bijection

$$\mathrm{Hom}(G, U(H)) \cong \mathrm{Hom}(L^1(G), L(H))$$

between the set of unitary representations τ of G on H and those of non-degenerated representations ρ of $L^1(G)$ on H, i.e. the non-degenerated algebra homomorphisms which commute with $*$, or equivalent, are contractions. We have

$$\begin{aligned}\langle\rho(f)h, k\rangle &= \int_G f(t)\,\langle\tau(t)h, k\rangle\,dt \quad \forall h, k \in H, f \in L^1(G),\\ \tau(t) &= \lim_j \rho(\lambda_t u_j) \quad \forall t \in G,\end{aligned}$$

where u_j is an approximating unit of $L^1(G)$.
Furthermore, $\tau(t)$ is uniquely determined by the identity $\tau(t)_* \circ \rho = \rho \circ \lambda_t$.
The irreducible representations also correspond to each other.

Proof. $(\mapsto)$ Let $\tau : G \to L(H)$ be a unitary representation. As mentioned in the introduction we aim to define ρ by

$$\rho(f)h := \int_G f(t)\,\tau(t)h\,dt \in H \text{ for } f \in L^1(G) \text{ and } h \in H.$$

To do so, we consider the sesqui-linear form

$$b_f(h, k) := \int_G f(t)\,\langle\tau(t)h, k\rangle\,dt.$$

Obviously, $\|b_f(h, k)\| \leqslant \|f\|_1\,\|h\|\,\|k\|$ holds. So there is a unique operator $\rho(f) \in L(H)$ with $\langle\rho(f)h, k\rangle = b_f(h, k)$ and $\|\rho(f)\| \leqslant \|f\|_1$. It is easy to see that $\rho : L^1(G) \to L(H)$ is a linear mapping.

Furthermore, ρ is multiplicative because

$$\begin{aligned}\langle\rho(f\star g)h,k\rangle &= \int_G\int_G f(s)\,g(s^{-1}t)\,ds\,\langle\tau(t)h,k\rangle\,dt\\ &= \int_G f(s)\int_G g(s^{-1}t)\,\langle\tau(t)h,k\rangle\,dt\,ds \qquad \text{(Fubini)}\\ &= \int_G f(s)\int_G g(t)\,\langle\tau(st)h,k\rangle\,dt\,ds \qquad (s^{-1}t\mapsto t)\\ &= \int_G f(s)\int_G g(t)\,\langle\tau(t)h,\tau(s)^*k\rangle\,dt\,ds\\ &= \int_G f(s)\langle\rho(g)h,\tau(s)^*k\rangle\,ds = \int_G f(s)\langle\tau(s)\rho(g)h,k\rangle\,ds\\ &= \langle\rho(f)\rho(g)h,k\rangle.\end{aligned}$$

We claim that ρ is a $*$-representation (and thus a contraction):

$$\begin{aligned}\langle\rho(f)^*h,k\rangle &= \langle h,\rho(f)k\rangle = \overline{\langle\rho(f)k,h\rangle} = \overline{b_f(k,h)}\\ &= \int_G \overline{f(t)\,\langle\tau(t)k,h\rangle}\,dt = \int_G \overline{f(t)}\,\langle h,\tau(t)k\rangle\,dt\\ &= \int_G \Delta(t)\overline{f(t^{-1})}\,\langle h,\tau(t^{-1})k\rangle\,dt = \int_G f^*(t)\,\langle\tau(t)h,k\rangle\,dt\\ &= \langle\rho(f^*)h,k\rangle.\end{aligned}$$

The representation ρ is not degenerated: Let $h\in H$ with $\|h\|=1$. Because of $\langle\tau(1)h,h\rangle=\|h\|^2=1$ and because $t\mapsto\tau(t)h$ is continuous, a neighborhood U of the unit exists in G with $|\langle\tau(t)h,h\rangle-1|\leqslant\frac12$ for all $t\in U$. Let $f\in L^1(G)$ with $f\geqslant 0$, $\int_G f=1$ and $\operatorname{Trg}(f)\subseteq U$. Then

$$\langle\rho(f)h,h\rangle-1=\int_G f(t)\,\langle\tau(t)h,h\rangle\,dt-\int_G f(t)\,dt=\int_U f(t)\,\big(\langle\tau(t)h,h\rangle-1\big)\,dt$$

and thus $|\langle\rho(f)h,h\rangle-1|\leqslant\int_U f(t)\,\big|\langle\tau(t)h,h\rangle-1\big|\,dt\leqslant\frac12\int_U f(t)\,dt=\frac12$, i.e. $\langle\rho(f)h,h\rangle\neq 0$.

($\hookleftarrow$) Let $\rho: L^1(G)\to L(H)$ be a non-degenerate contractionary algebra homomorphism. As stated in the introduction, $\tau(t)\in L(H)$ exists as pointwise limit $\lim_i\rho(\lambda_t(u_i))$ and complies with $\|\tau(t)\|\leqslant 1$ and $\tau(t)_*\circ\rho=\rho\circ\lambda_t$. Because of the non-degeneracy of ρ, the last equation immediately implies that $\tau(1)=1$ and $\tau(t_1t_2)=\tau(t_1)\circ\tau(t_2)$ hold. Consequently, $\tau(t^{-1})=\tau(t)^{-1}$ and thus $\tau: G\to U(H)$ is a group homomorphism.

We next show that τ is a unitary representation, i.e. $\tau(t)\to 1$ converges pointwise for $t\to e$. In fact, $\lambda_t f\to f$ and thus $\rho(f)h=\lim_t\rho(\lambda_t f)h=\lim_t(\tau(t)\circ\rho(f))h$. So $\tau(t)(\rho(f)h)\to\rho(f)h$ and, since the vectors $\rho(f)h$ generate a dense linear subspace and $\|\tau(t)\|\leqslant 1$, we obtain $\tau(t)\to 1$ pointwise.

To show that the mappings are inverse to each other, on the one hand, we need to show the equation

$$\langle\rho(f)h,k\rangle=\int_G f(t)\,\langle\tau(t)h,k\rangle\,dt\quad\forall h,k\in H, f\in L^1(G),$$

where τ is the unitary representation associated with ρ. Both sides represent continuous linear functionals with respect to f. It suffice for $\|h\|=1=\|k\|$, $\varepsilon>0$ and characteristic functions $f=\chi_A$ of Baire sets A with finite Haar measure to show that

$$\Big|\langle\rho(f)h,k\rangle-\int_G f(t)\,\langle\tau(t)h,k\rangle\,dt\Big|\leqslant\varepsilon\int_G f(t)\,dt.$$

There is a neighborhood U of $e \in G$ with $\|\rho(g)h - h\| \leqslant \varepsilon$ for all $g \geqslant 0$ with $\|g\| = 1$ and $\operatorname{Trg}(g) \subseteq U$, because one may approximate h by a linear combination of finite many $\rho(f_i)h_i$ with $\|h_i\| \leqslant 1$ and choose U by 7.56, s.t. $\|\rho(g) \circ \rho(f_i) - \rho(f_i)\| \leqslant \|g \star f_i - f_i\|_1 < \frac{\varepsilon}{3}$ for all i.

Let $A^{-1}A \subseteq U$ for the moment. If $\mu(A) = 0$, then nothing is to be shown. Let $\alpha := \mu(A) > 0$ and $g := \frac{1}{\alpha} f$. Then g is bounded and $g \geqslant 0$ and $\int_G g(t)\,dt = 1$. For $t \in A$ the function $\lambda_{t^{-1}} g$ has compact support in U, because for $t' \notin U$ we have $t' \notin A^{-1}A$, i.e. $At' \cap A = \emptyset$, and thus $\lambda_{t^{-1}} g(t') = g(tt') = \frac{1}{\alpha} f(tt') = \frac{1}{\alpha}\chi_A(tt') = 0$. So $\|\tau(t^{-1})\rho(g)h - h\| = \|\rho(\lambda_{t^{-1}} g)h - h\| \leqslant \varepsilon$. Since $\tau(t)$ is unitary, $\|\rho(g)h - \tau(t)h\| = \|\tau(t)\big(\tau(t)^{-1}\rho(g)h - h\big)\| \leqslant \varepsilon$ holds. From $f = \alpha\, g = \chi_A$ it follows that $\langle \rho(f)h, k\rangle - \int_G f(t)\,\langle \tau(t)h, k\rangle\,dt = \int_A \langle (\rho(g) - \tau(t))\,h, k\rangle\,dt$. So the special case is proven.

Let now $f = \chi_A$ with $\mu(A) < \infty$ and let W be a neighborhood of e with $W^{-1}W \subseteq U$. Without loss of generality, W is a Baire set. Let t_n be a sequence in G with $A \subseteq \bigcup_{n\in\mathbb{N}} t_n W$ (cover A with a sequence of compact sets and eacho of them by finitely many translates of W). Let $A_n := A \cap t_n W$. Then $A = \bigcup_{n\in\mathbb{N}} A_n$ and A_n are Baire sets with $A_n^{-1}A_n \subseteq (W^{-1}t_n{}^{-1})(t_n W) = W^{-1}W \subseteq U$. Without loss of generality, these sets are disjoint (replace A_n with $A_n \setminus \bigcup_{j<n} A_j$). Let $f_n := \chi_{A_n}$ and $s_n := \sum_{j\leqslant n} f_j$. For each f_j, the desired equation holds, so

$$\begin{aligned}\Big|\langle \rho(s_n)h, k\rangle - \int_G s_n(t)\,\langle \tau(t)h, k\rangle\,dt\Big| &= \Big|\sum_{j\leqslant n}\Big(\langle \rho(f_j)h, k\rangle - \int_G f_j(t)\,\langle \tau(t)h, k\rangle\,dt\Big)\Big|\\ &\leqslant \sum_{j\leqslant n} \varepsilon \int_G f_j(t)\,dt = \varepsilon \int_G s_n(t)\,dt.\end{aligned}$$

due to linearity. Since $s_j \nearrow f$ pointwise, $\|s_j - f\|_1 \to 0$ holds because of the Theorem [**18**, 4.11.10] of Beppo Levi and thus the desired equation also follows for f.

For the other composition, let ρ be the representation associated to τ. Then

$$\begin{aligned}\langle \rho(\lambda_t f)h, k\rangle &= \int_G \lambda_t f(s)\,\langle \tau(s)h, k\rangle\,ds = \int_G f(t^{-1}s)\,\langle \tau(s)h, k\rangle\,ds\\ &= \int_G f(s)\,\langle \tau(ts)h, k\rangle\,ds \qquad (t^{-1}s \mapsto s)\\ &= \int_G f(s)\,\langle \tau(s)h, \tau(t)^* k\rangle\,ds = \langle \rho(f)h, \tau(t)^*k\rangle = \langle \tau(t)\rho(f)h, k\rangle,\end{aligned}$$

i.e. $\quad \rho \circ \lambda_t = \tau(t)_* \circ \rho.$

Thus τ is the unitary representation associated to ρ.

Finally, $\rho(L^1(G))^k = \tau(G)^k$ holds, from which the statement about irreducibility follows by means of 7.41:

If $T \in L(H)$ commutes with all $\tau(t)$, then

$$\begin{aligned}\langle T\rho(f)h, k\rangle &= \langle \rho(f)h, T^*k\rangle = \int_G f(t)\,\langle \tau(t)h, T^*k\rangle\,dt\\ &= \int_G f(t)\,\langle T\tau(t)h, k\rangle\,dt = \int_G f(t)\,\langle \tau(t)Th, k\rangle\,dt = \langle \rho(f)Th, k\rangle,\end{aligned}$$

i.e. T commutes with $\rho(f)$ for each $f \in L^1(G)$.

Conversely, $T \in L(H)$ converges with $\rho(f)$ for each $f \in L^1(G)$. Let u_i be an approximating unit of $L^1(G)$. Then

$$T\,\tau(t)\,\rho(u_i) = T\,\rho(\lambda_t(u_i)) = \rho(\lambda_t(u_i))T = \tau(t)\,\rho(u_i)\,T$$

and since $\rho(u_i) \to 1$ pointwise, $T\,\tau(t) = \tau(t)\,T$ follows. □

Corollary (Gelfand-Raikov 1955).

The irreducible unitary representations of a locally compact group are point separating, i.e. for each $e \neq s \in G$, such a representation ρ exists on a Hilbert space H with $\rho(s) \neq 1$.

Proof.

$$\begin{tikzcd}
s & h \arrow[r] & a & \\
G \arrow[d, "\tau"] & L^1(G) \arrow[r, tail, "\tilde\lambda"] \arrow[d, "\rho"] & A(G) \arrow[r, hook] \arrow[dl, "\varphi"] & L(L^2(G)) \\
U(H) \arrow[r, hook] & L(H) & &
\end{tikzcd}$$

Let $s \neq e$ in G. Then there is a $f \in C_c(G) \subseteq L^1(G)$ with $f(s^{-1}) \neq f(e)$ and thus $\lambda_s f \neq f$. Let $h := \lambda_s f - f \neq 0 \in L^1(G)$. Because the representation of $L^1(G)$ is injective on $L^2(G)$ by 7.57, we have $0 \neq a := h \star (\_) \in A(G)$. So by 7.45 there is an irreducible representation $\varphi : A(G) \to L(H)$ with $\varphi(a) \neq 0$. The representation $\rho : L^1(G) \to A(G) \to L(H)$ is thus irreducible, i.e. is cyclic and therefore non-degenerated and $\rho(h) \neq 0$. So also the associated representation τ from G on $L(H)$ is irreducible and because of $\rho(\lambda_s f) - \rho(f) = \rho(\lambda_s f - f) = \rho(h) = \varphi(a) \neq 0$, we have $\tau(s) \circ \rho(f) = \rho(\lambda_s f) \neq \rho(f)$, so $\tau(s) \neq 1$. □

7.60 Corollary (Irreducible representations in the Abelian case).

Let G be a locally compact Abelian group. Then the irreducible unitary representations are exactly the CHARACTERS*, i.e. the continuous group homomorphisms $\tau : G \to S^1$. The irreducible non-degenerate $*$-representations of $L^1(G)$ are exactly the $\mathbb{C}$-valued algebra homomorphisms $0 \neq \rho : L^1(G) \to \mathbb{C}$. And the bijection*

$$\operatorname{Hom}(G, S^1) \cong \operatorname{Hom}(L^1(G), \mathbb{C}) \setminus \{0\}$$

of 7.59 is given for $f \in L^1(G)$ by

$$\rho(f) = \int_G f(t)\, \tau(t)\, dt.$$

Proof. If G is Abelian, then the same holds for $L^1(G)$.

According to 7.59, the irreducible unitary representations τ of G correspond exactly to the non-degenerate irreducible representations ρ of $L^1(G)$, and these are 1-dimensional by 7.42, i.e. $H = \mathbb{C}$.

Since the pointwise convergence on $L(\mathbb{C})$ coincides with the norm convergence, the irreducible unitary representations of G are just the continuous group homomorphisms $\tau : G \to U(\mathbb{C}) = S^1$.

The non-degenerate representations of $L^1(G)$ on $\mathbb{C}$ are, by 7.59, just the contractionary algebra homomorphisms $\rho : L^1(G) \to \mathbb{C}$ that are surjective. According to 6.39, every $\mathbb{C}$-valued algebra homomorphism on a Banach algebra with unit has norm 1. Hence every $\mathbb{C}$-valued algebra homomorphism ρ on a Banach algebra A (without unit) is a contraction, because $\rho_1 : A_1 \to \mathbb{C}$ is an algebra homomorphism on $A_1 := A \oplus \mathbb{C}$ by 6.4 and thus is $\|\rho\| = \|\rho_1|_A\| \leqslant \|\rho_1\| = 1$. A scalar-valued linear mapping ρ is surjective if and only if $\rho \neq 0$.

The injection from 7.59 is clearly given by

$$\rho(f) = \int_G f(t)\, \tau(t)\, dt$$

in the case of $H = \mathbb{C}$. □

7.61 The character group.

As in 6.43, one shows that $\operatorname{Hom}(L^1(G), \mathbb{C})$ is a compact space with respect to pointwise convergence (there we used 6.39, but $L^1(G)$ has no unit, however we have assumed $\|f\| \leqslant 1$ for all $f \in \operatorname{Hom}(L^1(G), \mathbb{C})$). Consequently, $\operatorname{Hom}(L^1(G), \mathbb{C})\backslash\{0\}$ is a locally compact space, and the bijection from 7.60 also makes $\operatorname{Hom}(G, S^1)$ a locally compact space. It can be shown that this topology on $\operatorname{Hom}(G, S^1)$ is precisely that of uniform convergence on compact subsets of G. Obviously, $\operatorname{Hom}(G, S^1)$ is a group with respect to pointwise multiplication, and it is easy to see that $\widehat{G} := \operatorname{Hom}(G, S^1)$ is a topological group, the so-called CHARACTER GROUP of G, of all continuous group homomorphisms $G \to S^1$, the so-called CHARACTERS. We will now switch the variables in the homeomorphism

$$\tilde{\mathcal{F}} : \widehat{G} \to \operatorname{Hom}(L^1(G), \mathbb{C})\backslash\{0\} \subseteq \operatorname{Hom}(L^1(G), \mathbb{C}), \quad \tau \mapsto \Big(f \mapsto \int_G f(t)\,\tau(t)\,dt\Big),$$

i.e. consider the associated mapping

$$L^1(G) \to C(\widehat{G}, \mathbb{C}), \quad f \mapsto \Big(\tau \mapsto \int_G f(t)\,\tau(t)\,dt\Big).$$

This is an $*$-homomorphism because $\tilde{\mathcal{F}}(\tau)$ is a $*$-homomorphism for all $\tau \in \widehat{G}$. To get a more familiar from for it, we compose this with the $*$-isomorphism

$$\operatorname{inv}^* : C(\widehat{G}, \mathbb{C}) \cong C(\widehat{G}, \mathbb{C}), \quad g \mapsto \big(\tau \mapsto g(\overline{\tau}) = g(\frac{1}{\tau})\big)$$

and get the following $*$-homomorphism $\mathcal{F}$:

Theorem. Fourier transformation.

Let G be a locally compact Abelian group and $\widehat{G}$ its character group. Then there is a $*$-homomorphism

$$\mathcal{F} : L^1(G) \to C(\widehat{G}, \mathbb{C}), \quad f \mapsto \Big(\tau \mapsto \int_G f(t)\,\overline{\tau(t)}\,dt\Big). \quad \square$$

Theorem of Parseval.

The Fourier transformation of a function $f \in L^1(G)$ thus provides a function $\mathcal{F}(f) : \hat{G} \to \mathbb{C}$. This does not have to be integrable, see [18, 5.4.7]. However, if we restrict the Fourier transform to $L^1(G) \cap L^2(G)$, it has values in $L^1(\widehat{G}) \cap C_0(\widehat{G}) \subseteq L^1(\widehat{G}) \cap L^2(\widehat{G})$, and with proper normalization of the Haar measure on G and $\widehat{G}$, it is an isometry with respect to the 2-norm. Because of the denseness of $L^1(G) \cap L^2(G)$, *it can be extended to a surjective isometry*

$$\mathcal{F} : L^2(G) \xrightarrow{\cong} L^2(\widehat{G}).$$

This is the theorem of Parseval.

7.62 Pontryagin's Duality Theorem.

The mapping $\delta : G \to G^{\wedge\wedge}$, $g \mapsto \operatorname{ev}_g$ is a group homeomorphism.

For a proof, see [13, Vol.2].

7.63 Example.

Let $G := \mathbb{R}$. Then $t \mapsto (s \mapsto e^{its})$ is a group homeomorphism from $\mathbb{R}$ onto the character group $\hat{G} = \operatorname{Hom}(\mathbb{R}, S^1)$. With respect to this isomorphism, the Fourier transform looks like follows

$$\mathcal{F}(f)(s) = \int_{-\infty}^{+\infty} f(t)\, e^{-its}\, dt \text{ for } f \in L^1(\mathbb{R}) \text{ and } s \in \mathbb{R} \cong \widehat{\mathbb{R}}$$

Compare this with the Fourier transform from [**18**, 8.1.2].

Proof. Let $\varphi : \mathbb{R} \to S^1$ be a continuous group homomorphism. Then there is a $\delta > 0$ with $\int_0^\delta \varphi(x)\, dx =: a > 0$ because of $\varphi(0) = 1$. Hence

$$a \cdot \varphi(x) = \varphi(x) \int_0^\delta \varphi(y)\, dy = \int_0^\delta \varphi(x+y)\, dy = \int_x^{x+\delta} \varphi(z)\, dz.$$

Since $a \neq 0$ we have $\varphi(x) = \frac{1}{a}\int_x^{x+\delta} \varphi(y)\, dy$, hence φ is differentiable and

$$\varphi'(x) = \lim_{h\to 0} \frac{\varphi(x+h) - \varphi(x)}{h} = \varphi(x) \lim_{h\to 0} \frac{\varphi(h) - \varphi(0)}{h} = \varphi(x)\, \varphi'(0).$$

So $\varphi(x) = e^{\varphi'(0)x}$ because $\varphi(0) = 1$. Because of $1 = |\varphi(x)| = |e^{\varphi'(0)x}|$ we have $\varphi'(0) \in i\,\mathbb{R}$, i.e. $\varphi(x) = e^{isx}$ for a $s \in \mathbb{R}$. Consequently, $\operatorname{Hom}(\mathbb{R}, S^1) \cong (\mathbb{R}, +)$, and with respect to this isomorphism we have $\mathcal{F}(f)(s) = \int_{\mathbb{R}} f(x)\, e^{-isx}\, dx$. $\square$

Example.

Let $G := S^1$. Then $k \mapsto (z \mapsto z^k)$ is a group homeomorphism from $\mathbb{Z}$ onto the character group $\hat{G} = \operatorname{Hom}(S^1, S^1)$. With respect to this isomorphism and the identification $L^1(S^1) \cong L^1[-\pi, \pi]$, the Fourier transform looks like follows

$$\mathcal{F}(f)(k) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(t)\, e^{-itk}\, dt \text{ for } f \in L^1([-\pi, \pi]) \text{ and } k \in \mathbb{Z} \cong \widehat{S^1}.$$

Compare this to the Fourier coefficients in [**18**, 5.4].

Proof. We have $h : t \mapsto e^{it}$, a continuous surjective group homomorphism on $\mathbb{R} \to S^1$. So $h^* : \operatorname{Hom}(S^1, S^1) \to \operatorname{Hom}(\mathbb{R}, S^1) \cong \mathbb{R}$ defines an injective group homomorphism. Namely, $s \in \mathbb{R}$ is in the image if and only if $x \mapsto e^{isx}$ is 2π-periodic, i.e. $s \in \mathbb{Z}$. Thus $\operatorname{Hom}(S^1, S^1) \cong \mathbb{Z}$ and with respect to this homomorphism and $h^* : L^1(S^1) \cong L^1[-\pi, \pi]$, $\mathcal{F}$ looks like follows:

$$\mathcal{F}(f)(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-itk}\, dt. \quad \square$$

Example.

Let $G := \mathbb{Z}$. Then $a \mapsto (k \mapsto a^k)$ is a group homeomorphism from S^1 onto the character group $\hat{G} = \operatorname{Hom}(\mathbb{Z}, S^1)$. With respect to this isomorphism, the Fourier transform looks like follows

$$\mathcal{F}(f)(a) = \sum_{k=-\infty}^{+\infty} f(k)\, a^{-k} \text{ for } f \in L^1(\mathbb{Z}) \text{ and } a \in S^1 \cong \hat{\mathbb{Z}}.$$

Cf. the Fourier series in [**18**, 5.4].

Proof. Each group homomorphism $\varphi : \mathbb{Z} \to S^1$ is uniquely determined by its value $a := \varphi(1) \in S^1$, because $\varphi(k) = \varphi(\sum_{j=1}^k 1) = \varphi(1)^k$. Consequently, $\widehat{G} \cong S^1$. With respect to this isomorphism, $\mathcal{F}$ now looks like follows:

$$\mathcal{F}(f)(a) := \sum_{k\in\mathbb{Z}} f(k)\, a^{-k}. \quad \square$$

7.64 Theorem of Wiener.

Let $f(t) := \sum_{k\in\mathbb{Z}} f_k\, e^{ikt}$ be an absolutely convergent Fourier series. If f vanishes nowhere, then also $\frac{1}{f}$ can be developed into an absolutely convergent Fourier series.

Proof by Gelfand. We have $A := L^1(\mathbb{Z},\mathbb{C})$, a commutative Banach algebra with unit with respect to the convolution. By 7.60 and the last example in 7.63, the algebra homomorphisms $\rho \in \sigma(A) := \operatorname{Alg}(A,\mathbb{C})$ are described by the $a \in S^1 \cong \operatorname{Hom}(\mathbb{Z},S^1) =: \widehat{\mathbb{Z}}$ via $\rho : f \mapsto \sum_{k\in\mathbb{Z}} f_k\, a^{-k}$. The Gelfand transformation

$$\mathcal{G} : A \to C(\sigma(A),\mathbb{C}), \quad f \mapsto \operatorname{ev}_f(: \rho \mapsto \rho(f))$$

from 6.43 thus maps $f \in L^1(\mathbb{Z},\mathbb{C})$ onto $a \mapsto \sum_{k\in\mathbb{Z}} f_k\, a^{-k}$ up to this isomorphism, so it is $\mathcal{F}$. We have $\mathcal{F}(f) \in C(S^1,\mathbb{C}) \cong C_{2\pi}(\mathbb{R},\mathbb{C})$. As an element of $C_{2\pi}(\mathbb{R},\mathbb{C})$ we have $\mathcal{F}(f)(t) := \sum_{k\in\mathbb{Z}} f_k\, e^{-ikt}$. If $\mathcal{F}(f)$ vanishes nowhere, then $1/\mathcal{F}(f) \in C_{2\pi}(\mathbb{R},\mathbb{C})$ is also in the image of the Gelfand transform (and thus an absolutely convergent Fourier series) because if $\mathcal{G}(f)$ vanishes nowhere, then $\rho(f) = \mathcal{G}(f)(\rho) \neq 0$ for all $\rho \in \operatorname{Alg}(A,\mathbb{C})$ and thus $0 \notin \sigma(\mathcal{G}(f)) = \sigma(f)$, i.e. f is invertible in A and obviously $1 = \mathcal{G}(f^{-1} f) = \mathcal{G}(f^{-1})\,\mathcal{G}(f)$ holds, so $\mathcal{G}(f^{-1}) = \frac{1}{\mathcal{G}(f)}$.

$$\begin{tikzcd}
L^1(\mathbb{Z},\mathbb{C}) \arrow[r, "\mathcal{F}"] \arrow[d, equal] & C(\widehat{\mathbb{Z}},\mathbb{C}) \arrow[r, "7.63", "\cong"'] \arrow[d, "\cong"', "7.60"] & C(S^1,\mathbb{C}) \arrow[r, "\cong"] & C_{2\pi}(\mathbb{R},\mathbb{C}) \\
A \arrow[r, "\mathcal{G}"'] & C(\operatorname{Alg}(A,\mathbb{C}),\mathbb{C}) & &
\end{tikzcd}$$

□

8. Spectral theory for normal operators

Let $N \in L(H)$ be a normal operator, then the C^*-subalgebra $C^*(N)$ generated by N is commutative and thus by 7.10 isomorphic to $C(X, \mathbb{C})$, where $X := \sigma(N) \subseteq \mathbb{C}$ is compact. The inverse of the Gelfand Isomorphism $\mathcal{G}$ thus provides a representation

$$\rho : C(X, \mathbb{C}) \xrightarrow{\cong} C^*(N) \subseteq L(H),$$

the function calculus from 7.14. An in-depth investigation of this representation should provide us also with essential information about normal operators. So we start deepening our study of representations of Abelian C^*-algebras.

Representations of Abelian C^*-algebras and spectral measures

In this section, X is a compact space and H is a Hilbert space.
The irreducible $*$-representations of $C(X, \mathbb{C})$ are 1-dimensional by 7.42, i.e. are algebra homomorphisms $\rho : C(X, \mathbb{C}) \to \mathbb{C}$ by 7.9. By 6.42 these are exactly the point evaluations ev_x with $x \in X$. More generally, according to Riesz's theorem theorem 5.3.4, the continuous linear functionals $C(X, \mathbb{C}) \to \mathbb{C}$ correspond exactly to the regular complex Borel measures on X. The σ-algebra $\mathcal{B}(X)$ of all Borel sets is by definition generated by the compact (equivalent, open or closed sets), see 4.1.3.
A regular complex Borel measure on X is a σ-additive mapping $\mu : \mathcal{B}(X) \to \mathbb{C}$ which satisfies

$$|\mu|(A) = \sup\{|\mu|(K) : K \subseteq A, K \text{ compact}\}.$$

The absolute value $|\mu|$ of a complex measure μ is the positive measure defined by

$$|\mu|(B) := \sup\Big\{\sum_{n=0}^{\infty} |\mu(B_n)| : B_n \in \mathcal{B}, B = \bigsqcup_{n=0}^{\infty} B_n, B_n \text{ pairwise disjoint}\Big\}.$$

The isometric isomorphism

$$C(X, \mathbb{C})^* \cong M(X) := \Big\{\mu : \mu \text{ is a regular complex Borel measure on } X\Big\},$$

is defined by $(f \mapsto \int_X f(x)\, d\mu(x)) \leftarrow\!\shortmid \mu$ and conversely $\mu(B) := \int_X \chi_B(x)\, d\mu(x)$, where we have to extend the functional $C(X, \mathbb{C}) \to \mathbb{C}$ to the measurable and generally not continuous functions χ_B.
The variation norm on $M(X)$ is defined by $\|\mu\| := |\mu|(X)$.

In analogy to the Riesz representation theorem 5.3.4, a general representation $\rho : C(X, \mathbb{C}) \to L(H)$ should be of the form $\rho(f) = \int_X f(x)\, dP(x)$ for some kind of "measure" P with values in $L(H)$ and hence should extend to $\mathrm{Borel}_b(X)$.

8.1 Representations of Borel_b give ortho-projection valued measures.

Let $\rho : \mathrm{Borel}_b(X) \to L(H)$ be a $*$-representation of the algebra $\mathrm{Borel}_b(X)$ of bounded Borel-measurable functions $X \to \mathbb{C}$, furthermore, $\chi : \mathcal{B}(X) \to \mathrm{Borel}_b(X)$

the mapping which assigns to each $B \in \mathcal{B}(X)$ the characteristic function χ_B and $P := \rho \circ \chi : \mathcal{B}(X) \to \mathrm{Borel}_b(X) \to L(H)$. Since $\chi_{B_1 \cap B_2} = \chi_{B_1} \cdot \chi_{B_2}$ we have

$$P(B_1) \circ P(B_2) = P(B_1 \cap B_2) = P(B_2) \circ P(B_1).$$

In particular, $P(B) = P(B \cap B) = P(B)^2$, i.e. $P(B)$ is idempotent, and $P(B)^* = \rho(\chi_B)^* = \rho(\overline{\chi_B}) = \rho(\chi_B) = P(B)$, i.e. $P(B)$ is an ortho-projection.

Orthogonal projections $P \in L(H)$ are in bijective relationship to closed subspaces $E \subseteq H$, via $E = \mathrm{img}\, P = (\ker P)^\perp$, because the unique orthogonal projection $P \in L(H)$ with image E is given by $x \mapsto x_1$, where $x = x_1 + x_2$ is the unique orthogonal decomposition of H in $E \oplus E^\perp$.

We have the partial ordering of "being a subset" for closed subspaces and the one from 7.17 for positive operators and in particular for orthogonal projections. We now relate these two orderings to each other.

8.2 Lemma. Description of the ordering.

For two orthogonal projection P_1 and P_2 t.f.a.e.:

1. $P_1 \leqslant P_2$;
$\Leftrightarrow$ 2. $\|P_1 x\|^2 \leqslant \|P_2 x\|^2$ *for all* x;
$\Leftrightarrow$ 3. $\ker P_1 \supseteq \ker P_2$;
$\Leftrightarrow$ 4. $\mathrm{img}\, P_1 \subseteq \mathrm{img}\, P_2$;
$\Leftrightarrow$ 5. $P_1 = P_1 \circ P_2$;

Proof. (1 $\Leftrightarrow$ 2) By 7.22, $P_1 \leqslant P_2 \Leftrightarrow \langle P_1 x, x\rangle \leqslant \langle P_2 x, x\rangle$ for all x, and $\langle P_j x, x\rangle = \langle P_j^2 x, x\rangle = \langle P_j x, P_j^* x\rangle = \|P_j x\|^2$.

(2 $\Rightarrow$ 3) is obvious.

(3 $\Leftrightarrow$ 4) holds because $\mathrm{img}\, P_j = (\ker P_j)^\perp$.

(3 $\Rightarrow$ 5) We have $x = x_0 + x_1$ with $x_0 \in \ker P_2 \subseteq \ker P_1$ and $x_1 \in (\ker P_2)^\perp = \mathrm{img}\, P_2$. Thus, $(P_1 \circ P_2)x = P_1(P_2(x_0) + P_2(x_1)) = P_1(x_1) = P_1(x_0 + x_1) = P_1(x)$.

(5 $\Rightarrow$ 2) We have $\|P_1 x\| = \|P_1(P_2 x)\| \leqslant \|P_1\|\, \|P_2 x\| \leqslant 1\, \|P_2 x\|$. □

8.3 Lemma. Description of orthogonality.

Let P_1 and P_2 be two orthogonal projections. Then $\mathrm{img}\, P_1 \perp \mathrm{img}\, P_2 \Leftrightarrow P_1 \circ P_2 = 0$.

Proof. $\mathrm{img}\, P_1 \perp \mathrm{img}\, P_2 \Leftrightarrow \mathrm{img}\, P_2 \subseteq (\mathrm{img}\, P_1)^\perp = \ker P_1 \Leftrightarrow P_1 \circ P_2 = 0$. □

Next, let's examine which operations on orthogonal projections correspond to the formation of the intersection and to the orthogonal sum of subspaces.

8.4 Lemma. Description of orthogonal sums.

Let P_i be orthogonal projections with pairwise orthogonal images. Then the orthogonal projection on the closed subspace $\bigoplus_i \mathrm{img}\, P_i$ generated by $\bigcup_i \mathrm{img}\, P_i$ is given by $\sum_i P_i$. This sum converges pointwise, but not with respect to the operator norm.

Proof. Let $E_i := \mathrm{img}\, P_i = (\ker P_i)^\perp$. Then the closed subspace of H generated by $\bigcup_i E_i$ is given by

$$\bigoplus_i E_i := \Big\{\sum_i h_i : h_i \in E_i \text{ and } \sum_i \|h_i\|^2 < \infty\Big\}.$$

In fact, on the one hand $\sum_i h_i$ converges because of the theorem [18, 6.2.3] of Pythagoras ($\|\sum_i h_i\|^2 = \sum_i \|h_i\|^2$) and on the other hand $\bigoplus_i E_i$ is a closed subspace containing all E_i.

Each $h \in H$ can be uniquely written as $h = h_\perp + \sum_i h_i$ with $h_\perp \in (\bigoplus_i E_i)^\perp$ and $\sum_i h_i \in \bigoplus_i E_i$. We have $P_i(h_\perp) = 0$, $P_i(h_i) = h_i$ and $P_i(h_j) = 0$ for $i \neq j$. Consequently, $\big(\sum_{i\in F} P_i\big)h = \sum_{i\in F} h_i \to \sum_i h_i$ holds for the net of the finite partial sums. I.e. the finite sums $\sum_{i\in F} P_i$ converge pointwise towards the orthogonal projection $h = h_\perp + \sum_i h_i \mapsto \sum_i h_i$ with image $\bigoplus_i E_i$.

Since $\|P_i\| = 1$ the sum $\sum_i P_i$ does not converge in the norm. $\square$

For the intersection we have the following pendant.

8.5 Lemma. Description of the intersection.

Let $1 \leqslant i \leqslant n$ be pairwise commuting orthogonal projections P_i. Then the orthogonal projection onto $\bigcap_i \operatorname{img} P_i$ is given by $P_1 \circ P_2 \circ \ldots \circ P_n$.

Proof. It suffices to show this statement for $n = 2$, because the rest follows by induction. Because of the commutativity $(P_1 \circ P_2)^2 = P_1 \circ P_2 \circ P_1 \circ P_2 = (P_1)^2 \circ (P_2)^2 = P_1 \circ P_2$ and $(P_1 \circ P_2)^* = (P_2)^* \circ (P_1)^* = P_2 \circ P_1 = P_1 \circ P_2$, i.e. $P_1 \circ P_2$ is an orthogonal projection with $\operatorname{img}(P_1 \circ P_2) \subseteq \operatorname{img} P_1$. Because of the commutativity $\operatorname{img}(P_1 \circ P_2) = \operatorname{img}(P_2 \circ P_1) \subseteq \operatorname{img} P_2$, hence $\operatorname{img}(P_1 \circ P_2) \subseteq \operatorname{img} P_1 \cap \operatorname{img} P_2$. Let conversely $h \in \operatorname{img} P_1 \cap \operatorname{img} P_2$. Then $(P_1 \circ P_2)h = P_1(P_2 h) = P_1(h) = h$, i.e. $h \in \operatorname{img}(P_1 \circ P_2)$. $\square$

8.6 Example. The representation given by multiplication.

Let μ be a Borel measure on a compact space X and $\rho : f \mapsto M_f$ be the representation of $L^\infty(\mu)$ on $L^2(\mu)$ by multiplication operators $M_f : g \mapsto f \cdot g$.

The mapping $B \mapsto P(B) := \rho(\chi_B)$ is σ-additive in the following sense: $\mathcal{B}_0 \subseteq \mathcal{B}(X)$, countable, pairwise disjoint $\Rightarrow P(\bigsqcup_{B\in\mathcal{B}_0} B) = \sum_{B\in\mathcal{B}_0} P(B)$, where the sum converges pointwise.

$$\begin{tikzcd}
\mathcal{B}(X) \arrow[r, dotted, "P"] \arrow[d, tail, "\chi"] & L(L^2(\mu)) \\
\operatorname{Borel}_b(X,\mathbb{C}) \arrow[r, two heads] & L^\infty(\mu) \arrow[u, "\rho"']
\end{tikzcd}$$

Proof. We have already seen in 8.1 that all $P(B)$ are orthogonal projections and that $P(B_1 \cap B_2) = P(B_1) \circ P(B_2)$. Thus, for disjoint B_1 and B_2, the images of $P(B_1)$ and $P(B_2)$ are normal to each other by 8.3. The image of $P(B)$ is obviously $\{g \in L^2(\mu) : g|_{X\setminus B} = 0\}$. And with $g_B := \chi_B \cdot g$ we obtain:

$$\begin{aligned}
\operatorname{img}\Big(P\Big(\bigsqcup_{B\in\mathcal{B}_0} B\Big)\Big) &= \Big\{g \in L^2(\mu) : g|_{X\setminus\bigcup\mathcal{B}_0} = 0\Big\} = \Big\{\sum_{B\in\mathcal{B}_0} g_B \in L^2(\mu) : g_B|_{X\setminus B} = 0\Big\} \\
&= \bigoplus_{B\in\mathcal{B}_0} \operatorname{img} P(B) \overset{\boxed{8.4}}{=\!=} \operatorname{img}\Big(\sum_{B\in\mathcal{B}_0} P(B)\Big).
\end{aligned}$$

Hence $P(\bigsqcup_{B\in\mathcal{B}_0} B) = \sum_{B\in\mathcal{B}_0} P(B)$. $\square$

8.7 Definition. Spectral-measure.

We call a mapping $P : \mathcal{B}(X) \to L(H)$ defined on the Borel algebra (or any σ-algebra $\mathcal{B}$ of a space X) a spectral measure on X with respect to the Hilbert space H if:

1. The operator $P(B)$ is an orthogonal projection for each $B \in \mathcal{B}$:
2. $P(X) = 1$ and $P(\emptyset) = 0$.

3. $\mathcal{B}_0 \subseteq \mathcal{B}$, countable, pairwise disjoint $\Rightarrow P(\bigsqcup_{B\in\mathcal{B}_0} B) = \sum_{B\in\mathcal{B}_0} P(B)$ pointwise.

Note that by $\boxed{1}$, in the case of $H = \mathbb{C}$, the spectral measures are the $\{0,1\}$-valued measures.

8.8 Lemma. Basics about spectral measures.

For spectral measures P the following statements are valid:

1. *If $B_1 \cap B_2 = \emptyset$, then $\operatorname{img} P(B_1) \perp \operatorname{img} P(B_2)$.*
2. *We have $P(B_1 \cap B_2) = P(B_1) \circ P(B_2)$.*
3. *The spectral measure P is monotone.*
4. *For $h, k \in H$ the function $B \mapsto P_{h,k}(B) := \langle P(B)h, k\rangle$ gives a complex Borel measure on X with total variation $\|P_{h,k}\| \leqslant \|h\|\,\|k\|$. In particular, $P_{h,h}$ is a positive Borel measure.*

Proof. ($\boxed{1}$) Let B_1 and B_2 be disjoint. Suppose the images of $P_1 := P(B_1)$ and $P_2 := P(B_2)$ are not normal to each other, i.e. $P_2 \circ P_1 \neq 0$ by $\boxed{8.3}$. Let $x \in \operatorname{img} P_1$ with $P_2 x \neq 0$. Then

$$\|(P_1 + P_2)x\|^2 = \langle x + P_2 x, x + P_2 x\rangle = \|x\|^2 + 3\|P_2 x\|^2 > \|x\|^2,$$

so $P_1 + P_2 \overset{\boxed{8.7.3}}{=\!=\!=} P(B_1 \sqcup B_2)$ is not an orthogonal projection by $\boxed{7.40.3}$, a contradiction.

($\boxed{2}$) Now let B_1 and B_2 be arbitrary and $P_1 := P(B_1 \backslash B_2)$, $P_2 := P(B_2 \backslash B_1)$ and $P_0 := P(B_1 \cap B_2)$. Then P_0, P_1 and P_2 are by ($\boxed{1}$) pairwise orthogonal projections. Furthermore, by $\boxed{8.7.3}$,

$$\begin{aligned} P(B_1) &= P((B_1\backslash B_2) \sqcup (B_1 \cap B_2)) = P_1 + P_0,\\ P(B_2) &= P((B_2\backslash B_1) \sqcup (B_1 \cap B_2)) = P_2 + P_0. \end{aligned}$$

Folglich ist

$$\begin{aligned} P(B_1) \circ P(B_2) &= (P_1 + P_0) \circ (P_2 + P_0) \\ &= P_1 \circ P_2 + P_0 \circ P_2 + P_1 \circ P_0 + P_0 \circ P_0 \overset{\boxed{7.3}}{=\!=\!=} 0 + 0 + 0 + P_0 \\ &= P(B_1 \cap B_2) \end{aligned}$$

($\boxed{3}$) Let $B_1 \subseteq B_2$, i.e. $B_1 = B_1 \cap B_2$ and thus $P(B_1) = P(B_1 \cap B_2) \overset{\boxed{2}}{=} P(B_1) \circ P(B_2)$, i.e. $P(B_1) \leqslant P(B_2)$ by $\boxed{8.2}$.

($\boxed{4}$) We have that $\mu := P_{h,k}$ is a complex Borel measure, because from $P(\bigsqcup_i B_i)h = \sum_i P(B_i)h$ for pairwise disjoint Borel sets B_n, the σ additivity of μ follows:

$$\mu\Big(\bigsqcup_i B_i\Big) = \Big\langle P\Big(\bigsqcup_i B_i\Big)h, k\Big\rangle = \Big\langle \sum_i P(B_i)h, k\Big\rangle = \sum_i \langle P(B_i)h, k\rangle = \sum_i \mu(B_i).$$

We have $|\mu(B_j)| = \alpha_j\, \mu(B_j)$ with $\alpha_j \in S^1 \subseteq \mathbb{C}$. Hence

$$\sum_j |\mu(B_j)| = \sum_j \alpha_j \langle P(B_j)h, k\rangle = \Big\langle \sum_j \alpha_j P(B_j)h, k\Big\rangle \leqslant \Big\|\sum_j \alpha_j P(B_j)h\Big\|\,\|k\|,$$

and, since the $P(B_j)h$ are pairwise orthogonal,

$$\Big\|\sum_j \alpha_j P(B_j)h\Big\|^2 = \sum_j \|\alpha_j P(B_j)h\|^2 = \Big\|\sum_j P(B_j)h\Big\|^2 = \Big\|P\Big(\bigsqcup_j B_j\Big)h\Big\|^2 \leqslant \|h\|^2.$$

Thus $\sum_j |\mu(B_j)| \leqslant \|h\|\,\|k\|$, i.e. $\|\mu\| := \sup\{\sum_j |\mu(B_j)|\} \leqslant \|h\|\,\|k\|$. □

8.9 Definition. Operator topologies.

We will use the following topologies on $L(H)$:

1. The norm topology, i.e. the topology of uniform convergence on the unit ball (or on bounded sets) of H. A generating norm is the operator norm $\|T\| := \sup\{\|Tx\| : \|x\| \leqslant 1\}$;
2. The strong operator topology (SOT), namely the pointwise convergence on $h \in H$. It has as subbasis the seminorms $T \mapsto \|T(h)\|$ for all $h \in H$;
3. The weak operator topology (WOT), namely the pointwise convergence with respect to the weak topology $\sigma(H, H')$ on H. It has as subbasis the seminorms $T \mapsto |\langle Th, k\rangle|$ for all $h, k \in H$.

Lemma.

The involution $*$ is continuous with respect to the WOT. The composition is separately continuous with respect to the WOT and also with respect to the SOT.

Proof. We have $\langle T^* h, k\rangle = \langle h, Tk\rangle = \overline{\langle Tk, h\rangle}$ and therefore $\langle T_i^* h, k\rangle \to \langle T^* h, k\rangle$ converges provided $\langle T_i k, h\rangle \to \langle Tk, h\rangle$ for all $h, k \in H$.

We have $\langle (T \circ S)h, k\rangle = \langle T(Sh), k\rangle$ and therefore with $T_i \to T$ also $T_i \circ S \to T \circ S$ converges with respect to the WOT.

Finally, $\langle STh, k\rangle = \langle Th, S^* k\rangle$ and thus $\langle ST_i h, k\rangle \to \langle STh, k\rangle$ converges for all $h, k \in H$ if $T_i \to T$ with respect to the WOT.

If $T_i \to T$ in the SOT, then $T_i(Sh) \to T(Sh)$ for $h \in H$, i.e. $T_i \circ S \to T \circ S$ in the SOT and further $T_i h \to Th$ and thus $S(T_i h) \to S(Th)$, i.e. $S \circ T_i \to S \circ T$ in the SOT. □

We aim at constructing a representation ρ of $C(X, \mathbb{C})$ and, more generally, of $\mathrm{Borel}_b(X, \mathbb{C})$ for a given spectral measure P on X by

$$\rho(f) := \int_X f(x)\, dP(x) \text{ for } f \in \mathrm{Borel}_b(X, \mathbb{C}).$$

In order for this to make sense, we have to give a meaning to this integral. We first consider the integral of bounded measurable functions with respect to a complex Borel measure μ on X.

8.10 Proposition. $\mathbb{C}$-Integration.

1. *Denseness of the elementary functions in* $\mathrm{Borel}_b(X, \mathbb{C})$ *with respect to* $\|\_\|_\infty$*: For each bounded Borel measurable function $f : X \to \mathbb{C}$ and $\varepsilon > 0$ there exists a decomposition of X in finitely many Borel-measurable sets B_j, s.t.*
$$\sup\{|f(x) - f(x')| : x, x' \in B_j\} \leqslant \varepsilon \text{ for all } j.$$
2. *Approximation of the integral by a sum: If μ is a $\mathbb{C}$-valued Borel measure on X then any $f \in \mathrm{Borel}_b(X, \mathbb{C})$ is integrable with respect to μ. Moreover, for $\varepsilon > 0$, the B_j choosen as in (1), and $x_j \in B_j$ we have:*
$$\Big|\int_X f\, d\mu - \sum_j f(x_j)\, \mu(B_j)\Big| \leqslant \varepsilon\, \|\mu\|.$$

3. *Embedding of* $\mathrm{Borel}_b(X,\mathbb{C})$ *into* $M(X,\mathbb{C})'$*:*
   The Banach space $\mathrm{Borel}_b(X) := \mathrm{Borel}_b(X,\mathbb{C})$ *of all bounded Borel-measurable functions on* X *considered with the supremum norm embeds by virtue of the mapping* $f \mapsto (\mu \mapsto \int_X f(x)\,d\mu(x))$ *isometrically into* $M(X,\mathbb{C})' \cong C(X,\mathbb{C})''$*. Where* $M(X) := M(X,\mathbb{C})$ *is the Banach space of the regular* $\mathbb{C}$*-valued Borel measures with respect to the variation norm.*
4. *Weak denseness of* $C(X,\mathbb{C})$ *in* $\mathrm{Borel}_b(X,\mathbb{C})$*:*
   For each $f \in \mathrm{Borel}_b(X)$ *there exists a net of continuous functions* $f_i \in C(X)$ *with* $\|f_i\|_\infty \leqslant \|f\|_\infty$ *and* $f_i \to f$ *with respect to* $\sigma(M(X)', M(X))$*, i.e.* $\int_X f_i\,d\mu \to \int_X f\,d\mu$ *for all* $\mu \in M(X)$*.*

Proof. ($\boxed{1}$) Let $f \in \mathrm{Borel}_b(X)$ and $\varepsilon > 0$. We choose a covering of $\{z \in \mathbb{C} : |z| \leqslant \|f\|_\infty\}$ with finite many open balls U_j with radius $\frac{\varepsilon}{2}$ and centers z_j. Let $B_k := f^{-1}(U_k)\setminus\bigcup_{j<k} f^{-1}(U_j)$. Then the B_j form a decomposition of X into measurable sets and for $x, x' \in B_j$ the following holds:

$$|f(x) - f(x')| \leqslant |f(x) - z_j| + |z_j - f(x')| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

For any fixed choosen $x_j \in B_j$ and all $x \in B_i$ we have

$$\Big|\Big(f - \sum_j f(x_j)\chi_{B_j}\Big)(x)\Big| = |f(x) - f(x_i)| \leqslant \varepsilon, \text{ hence } \Big\|f - \sum_j f(x_j)\chi_{B_j}\Big\|_\infty \leqslant \varepsilon.$$

($\boxed{2}$) Now let μ be a $\mathbb{C}$-valued Borel measure and $x_j \in B_j$ arbitrary. Then

$$\begin{aligned}\Big|\int_X \sum_j f(x_j)\chi_{B_j}\,d\mu\Big| &:= \Big|\sum_j f(x_j)\,\mu(B_j)\Big| \\ &\leqslant \sum_j |f(x_j)|\,|\mu(B_j)| \leqslant \|f\|_\infty \sum_j |\mu(B_j)| \leqslant \|f\|_\infty\,\|\mu\|.\end{aligned}$$

Thus, because of $\|f - \sum_j f(x_j)\chi_{B_j}\|_\infty \leqslant \varepsilon$, the function f is integrable and $\int_X f\,d\mu = \lim \int_X \sum_j f(x_j)\chi_{B_j}$ by Lebesgue's Theorem [**18**, 4.11.12] on dominated convergence. In particular,

$$\Big|\int f\,d\mu\Big| \leqslant \|f\|_\infty\,\|\mu\|$$

and

$$\begin{aligned}\Big|\int f\,d\mu - \sum_j f(x_j)\,\mu(B_j)\Big| &= \Big|\int\Big(f - \sum_j f(x_j)\chi_{B_j}\Big)\,d\mu\Big| \\ &\leqslant \Big\|f - \sum_j f(x_j)\chi_{B_j}\Big\|_\infty\,\|\mu\| \leqslant \varepsilon\,\|\mu\|.\end{aligned}$$

($\boxed{3}$) Because of $\Big|\int f\,d\mu\Big| \leqslant \|f\|_\infty\,\|\mu\|$, the mapping $f \mapsto (\mu \mapsto \int f\,d\mu)$ is a contraction $\mathrm{Borel}_b(X) \to M(X)'$. In order to show that this is even an isometry, let $\varepsilon > 0$. Then there is an $x \in X$ with $|f(x)| \geqslant \|f\|_\infty - \varepsilon$. Let μ_x be the point measure of x, i.e. $\mu_x(B) = 1$ if $x \in B$ and 0 otherwise. Then $\|\mu_x\| = 1$ and thus $\|\mu \mapsto \int f\,d\mu\| \geqslant |\int f\,d\mu_x| = |f(x)| \geqslant \|f\|_\infty - \varepsilon$.

($\boxed{4}$) Without loss of generality, let $\|f\| \leqslant 1$. Then this is a consequence of the following lemma for $E := C(X,\mathbb{C})$. □

8.11 Lemma.

Let E *be a normed space.*
Then the 1-ball of E *is dense in the 1-ball of* E'' *with respect to* $\sigma(E'', E')$*.*

Proof. Let B be the $\sigma(E'', E')$-closure of the 1-ball of E in E''. We have to show that the 1-ball of E'' is included in B. Suppose not, then let $x'' \in E''\backslash B$ with $\|x''\| \leqslant 1$. By the separation theorem 5.2.1 there exists an $x' \in (E'', \sigma(E'', E'))' = E'$ and a $\alpha \in \mathbb{R}$ with

$$\mathfrak{Re}(\langle x', x\rangle) < \alpha < \mathfrak{Re}(\langle x', x''\rangle) \text{ for all } x \in B.$$

Without loss of generality $\alpha = 1$, because 0 is in the 1-ball of E we have $0 < \alpha$ and we can divide the inequality by α and replace x' by $\frac{1}{\alpha}x'$.

For $\|x\| \leqslant 1$ we choose $\lambda \in S^1$ so that $\langle x', x\rangle = \lambda\,|\langle x', x\rangle|$. Then $\lambda x \in B$ and thus

$$|\langle x', x\rangle| = \mathfrak{Re}(\bar\lambda\,\langle x', x\rangle) = \mathfrak{Re}(\langle x', \lambda\, x\rangle) < 1,$$

hence $\|x'\| \leqslant 1$ and

$$1 < \mathfrak{Re}(\langle x', x''\rangle) \leqslant |\langle x', x''\rangle| \leqslant \|x'\|\,\|x''\| \leqslant 1$$

yields a contradiction. □

8.12 Corollary. Operator-valued integration.

Let $P : \mathcal{B}(X) \to L(H)$ be a spectral measure.

1. *Operator-valued integral:*
   For each $f \in \mathrm{Borel}_b(X, \mathbb{C})$ there is a unique operator

$$\textit{denoted } \int_X f\, dP = \int_X f(x)\, dP(x) \in L(H)$$
$$\textit{and determined by } \Big\langle \Big(\int_X f\, dP\Big) h, k\Big\rangle = \int_X f\, dP_{h,k} \textit{ for all } h, k \in H.$$

2. *Approximation of the integral by a sum:*
   For $f \in \mathrm{Borel}_b(X, \mathbb{C})$ and $\varepsilon > 0$ let $\{B_1, \dots, B_n\}$ be a decomposition of X as in 8.10.1 and $x_j \in B_j$ be chosen arbitrary. Then the following estimate holds:

$$\Big\|\int_X f\, dP - \sum_{j=1}^n f(x_j)\, P(B_j)\Big\| \leqslant \varepsilon.$$

3. *Representation of $\mathrm{Borel}_b(X, \mathbb{C})$ on H:*

$$\rho : \mathrm{Borel}_b(X, \mathbb{C}) \to L(H), \textit{ given by } f \mapsto \int_X f\, dP,$$

   is a $*$-representation of the Abelian C^*-algebra $\mathrm{Borel}_b(X, \mathbb{C})$ of all bounded measurable functions on X. It is continuous with regard to $\sigma(M(X)', M(X))$ on $\mathrm{Borel}_b(X, \mathbb{C})$ and the WOT on $L(H)$. By restriction, we also get a $*$-representation of $C(X, \mathbb{C})$.

Proof. (1) By 8.8.4 and 8.10.2, $b(h, k) := \int_X f\, dP_{h,k} \in \mathbb{C}$ is well-defined for all $h, k \in H$ and b is a sesquilinear form with $\|b\| \leqslant \|f\|_\infty$ by 8.10.3. So by 7.5 there is a unique bounded operator, which we denote with $\int_X f\, dP$, such that

$$\Big\langle \Big(\int_X f\, dP\Big)\, h, k\Big\rangle = b(h, k) = \int_X f\, dP_{h,k} \text{ for all } h, k \in H.$$

(2) Let now a decomposition $\{B_1, \dots, B_n\}$ of X be given as in 8.10.1. For $x_j \in B_j$ and all $h, k \in H$ we have

$$\begin{aligned}\Big|\Big\langle\Big(\int_X f\,dP\Big)\,h,k\Big\rangle - \sum_{j=1}^n f(x_j)\,\big\langle P(B_j)h,k\big\rangle\Big| &= \Big|\int_X f\,dP_{h,k} - \sum_{j=1}^n f(x_j)\,P_{h,k}(B_j)\Big|\\ &\leqslant \varepsilon\,\|P_{h,k}\| \quad (\text{by } 8.10.2)\\ &\leqslant \varepsilon\|h\|\,\|k\| \quad (\text{by } 8.8.4).\end{aligned}$$

Consequently,

$$\Big\|\int_X f\,dP - \sum_j f(x_j)\,P(B_j)\Big\| \leqslant \varepsilon$$

(3) We only show the multiplicativity in detail because the remaining algebraic properties are easier to show. Let f_1 and f_2 be measurable and $\varepsilon > 0$. We choose a decomposition $\{B_1, \dots, B_n\}$ of X into Borel sets and $x_j \in B_j$, such that $\sup\{|f(x) - f(x')| : x, x' \in B_j\} < \varepsilon$ for all $f \in \{f_1, f_2, f_1 f_2\}$ and all $j \in \{1, \dots, n\}$. By 2 then

$$\Big\|\int_X f\,dP - \sum_j f(x_j)\,P(B_j)\Big\| < \varepsilon \text{ for } f \in \{f_1, f_2, f_1 f_2\}.$$

Since the images of $P(B_j)$ are orthogonal to each other,

$$\begin{aligned}\Big\|\Big(\sum_j f(x_j)\,P(B_j)\Big)h\Big\|^2 &= \sum_j \|f(x_j)\,P(B_j)h\|^2 = \sum_j |f(x_j)|^2\,\|P(B_j)h\|^2\\ &\leqslant \|f\|_\infty^2 \sum_j \|P(B_j)h\|^2 = \|f\|_\infty^2 \Big\|\sum_j P(B_j)h\Big\|^2\\ &= \|f\|_\infty^2 \Big\|P\Big(\bigsqcup_j B_j\Big)h\Big\|^2 = \|f\|_\infty^2\,\|h\|^2\end{aligned}$$

and by 2 thus

$$\Big\|\int f\,dP\Big\| \leqslant \|f\|_\infty.$$

By means of the triangle inequality we obtain:

$$\begin{aligned}&\Big\|\int f_1\,f_2\,dP - \Big(\int f_1\,dP\Big)\,\Big(\int f_2\,dP\Big)\Big\|\\ &\quad\leqslant \Big\|\int_X f_1\,f_2\,dP - \sum_j f_1(x_j)\,f_2(x_j)\,P(B_j)\Big\|\\ &\qquad+ \Big\|\sum_j f_1(x_j)\,f_2(x_j)\,P(B_j) - \Big(\sum_j f_1(x_j)\,P(B_j)\Big)\,\Big(\sum_j f_2(x_j)\,P(B_j)\Big)\Big\|\\ &\qquad+ \Big\|\sum_j f_1(x_j)\,P(B_j)\Big\| \cdot \Big\|\sum_j f_2(x_j)\,P(B_j) - \int f_2\,dP\Big\|\\ &\qquad+ \Big\|\sum_j f_1(x_j)\,P(B_j) - \int f_1\,dP\Big\| \cdot \Big\|\int f_2\,dP\Big\|\end{aligned}$$

Because of $P(B_j) \circ P(B_{j'}) = P(B_j \cap B_{j'}) = P(\emptyset) = 0$ for $j \neq j'$, the second term is 0. And because of $\|\sum_j f(x_j)\,P(B_j)\| \leqslant \|f\|_\infty$ for $f \in \{f_1, f_2\}$ we have finally

$$\Big\|\int f_1\,f_2\,dP - \Big(\int f_1\,dP\Big)\,\Big(\int f_2\,dP\Big)\Big\| \leqslant \varepsilon(1 + \|f_1\|_\infty + \|f_2\|_\infty).$$

Since $\varepsilon > 0$ was arbitrary, $\int f_1\,f_2\,dP = \big(\int f_1\,dP\big)\,\big(\int f_2\,dP\big)$ follows.

The $*$-homomorphism property follows from

$$\int \overline{f}\, dP \approx \sum \overline{f(x_j)}\, P(B_j) = \Big(\sum f(x_j)\, P(B_j)\Big)^* \approx \Big(\int f\, dP\Big)^*.$$

The weak continuity holds, since for $f_j \to f$ in $\sigma(\mathrm{Borel}_b, M(X))$, i.e. $\int f_j\, d\mu \to \int f\, d\mu$ for all $\mu \in M(X)$, and in particular for $\mu := P_{h,k}$ we have

$$\Big\langle \Big(\int f_j\, dP\Big) h, k \Big\rangle = \int f_j\, dP_{h,k} \to \int f\, dP_{h,k} = \Big\langle \Big(\int f\, dP\Big) h, k \Big\rangle,$$

hence $\int f_j\, dP \to \int f\, dP$ with respect to the WOT. $\square$

8.13 Theorem (Counterpart to the representation theorem of Riesz).

Let X be a compact space and H a Hilbert space.
Then the $*$-representations ρ of $C(X,\mathbb{C})$ on H are in bijection to the spectral measures P on X with respect to H via the relation

$$\rho(f) = \int_X f(x)\, dP(x) \text{ for all } f \in C(X,\mathbb{C}).$$

In short:

$$\mathrm{Hom}(C(X,\mathbb{C}), L(H)) \cong M(X, L(H)),$$

where $M(X, L(H))$ denotes the set of all spectral measures on X with respect to H.

Proof. $(\rho \leftarrow P)$ This is 8.12.

$(\rho \mapsto P)$ As for the Riesz representation theorem we extend ρ first to a representation $\tilde\rho$ of $\mathrm{Borel}_b(X,\mathbb{C})$ to get the spectral measure P as $P := \tilde\rho \circ \chi$ afterwards:

$(\rho \mapsto \tilde\rho)$
Since $\mathrm{Borel}_b(X)$ can be considered as a subspace of $C(X)''$ by 8.10, it makes sense to use the bidual mapping

$$\rho^{**} : C(X)'' \to L(H)''.$$

$$\begin{tikzcd}
C(X) \arrow[rr, hook, "\rho"] \arrow[dd, hook, "\delta"'] \arrow[dr, hook] & & L(H) \arrow[r, "\mathrm{id}"] \arrow[dd, hook, "\delta"'] & L(H) \\
& \mathrm{Borel}_b(X) \arrow[dl, hook] & & \\
C(X)'' \arrow[rr, "\rho^{**}"] & & L(H)'' \arrow[uur, dotted, "?\ \tau"'] &
\end{tikzcd}$$

Unfortunately the space $L(H)$ is not reflective and we can only hope to find a retraction (i.e. a left inverse) τ for the canonical embedding $\delta : L(H) \hookrightarrow L(H)''$.
The canonical embedding $\delta : E \hookrightarrow E''$ of a Banach space E into its bidual space has the following property: $\mathrm{ev}_\ell \circ \delta = \ell$ holds for each $\ell \in E'$, because $(\mathrm{ev}_\ell \circ \delta)(x) = \mathrm{ev}_\ell(\delta(x)) = \delta(x)(\ell) = \ell(x)$.

$$\begin{tikzcd}
& E \arrow[dl, "\ell"'] \arrow[dd, hook, "\delta"] \\
\mathbb{C} & \\
& E'' \arrow[ul, "\mathrm{ev}_\ell"]
\end{tikzcd}$$

$$\begin{tikzcd}
L(H) \arrow[rr, "\mathrm{id}"] \arrow[dr, "\delta"] \arrow[ddr, "\ell:=\ell_{h,k}"'] & & L(H) \arrow[ddl, "\ell_{h,k}"] \\
& L(H)'' \arrow[ur, dotted, "\tau"] \arrow[d, "\mathrm{ev}_\ell"] & \\
& \mathbb{C} &
\end{tikzcd}$$

For $h,k \in H$, let the linear functional $\ell_{h,k} : L(H) \to \mathbb{C}$ be defined by $\ell_{h,k}(T) := \langle Th, k\rangle$.
We have $|\ell_{h,k}(T)| = |\langle Th, k\rangle| \leqslant \|T\|\,\|h\|\,\|k\|$. Thus $\ell_{h,k}$ is continuous with $\|\ell_{h,k}\| \leqslant \|h\|\,\|k\|$.
The searched for τ has to fulfill $\ell_{h,k} \circ \tau = \mathrm{ev}_{\ell_{h,k}}$, and is obviously uniquely determined this property because the functionals $\ell_{h,k}$ separate points.

This condition means that the following holds for all $\Psi \in L(H)''$:

$$\langle \tau(\Psi)h, k\rangle = (\ell_{h,k} \circ \tau)(\Psi) = (\mathrm{ev}_{\ell_{h,k}})(\Psi) = \Psi(\ell_{h,k}).$$

In fact, by 7.5, a continuous linear operator $\tau(\Psi)$ is defined by this implicite equation, because $(h,k) \mapsto \Psi(\ell_{h,k})$ is obviously a sesqui-linear form with $|\Psi(\ell_{h,k})| \leqslant \|\Psi\|\,\|\ell_{h,k}\| \leqslant \|\Psi\|\,\|h\|\,\|k\|$. So $\|\tau(\Psi)\| \leqslant \|\Psi\|$, i.e. $\tau : L(H)'' \to L(H)$ is a contraction and clearly linear.

Sideremark: For Banach spaces E and F, one has more generally a $\tau : L(E,F)'' \to L(E,F'')$, which, composed with $\delta : L(E,F) \hookrightarrow L(E,F)''$, yields the inclusion $\delta_* : L(E,F) \to L(E,F'')$. This τ is associated with the 3-linear form

$$L(E,F)'' \times E \times F' \to L(E,F)'' \times L(E,F)' \to \mathbb{C},$$

which is described by the bilinear mapping $E \times F' \to L(E,F)'$, which in turn is associated to $E \times F' \times L(E,F) \to F' \times E \times L(E,F) \to F' \times F \to \mathbb{C}$.

So we obtained the following commutative diagram:

$$\begin{tikzcd}
C(X) \arrow[rrr, "\rho"] \arrow[ddd, hook, "\delta"'] \arrow[dr, hook] \arrow[ddr, dotted, "\mu_{h,k}"] & & & L(H) \arrow[ddl, "\ell_{h,k}"] \\
 & & & \\
 & \mathrm{Borel}_b(X) \arrow[r, phantom] \arrow[dl, hook] \arrow[uurr, dotted, "\tilde\rho"] & \mathbb{C} & \\
C(X)'' \arrow[rrr, "\rho^{**}"'] \arrow[ur, phantom] \arrow[urr, dotted, "\mathrm{ev}_{\mu_{h,k}}"] & & & L(H)'' \arrow[ul, "\mathrm{ev}_{\ell_{h,k}}"'] \arrow[uuu, "\tau"']
\end{tikzcd}$$

Where $\tilde\rho := (\tau \circ \rho^{**})|_{\mathrm{Borel_b(X)}}$ defines a linear extension of ρ that satisfies $\|\tilde\rho\| \leqslant \|\tau \circ \rho^{**}\| \leqslant \|\tau\|\,\|\rho\| \leqslant 1 \cdot 1 = 1$.
Furthermore, $\mu_{h,k} := \ell_{h,k} \circ \rho$ is a continuous linear functional on $C(X)$, and thus can be considered as regular Borel measure. The lower triangle commutes, because for $\ell := \ell_{h,k} \in L(H)'$ the following holds: $(\mathrm{ev}_\ell \circ \rho^{**})(\Phi) = \mathrm{ev}_\ell(\rho^{**}(\Phi)) = \rho^{**}(\Phi)(\ell) = \Phi(\rho^*(\ell)) = \Phi(\ell \circ \rho) = \mathrm{ev}_{\ell\circ\rho}(\Phi)$. Thus the inner parallelogram commutes and hence

$$\langle \tilde\rho(f)h, k\rangle = (\ell_{h,k} \circ \tilde\rho)(f) = \mathrm{ev}_{\mu_{h,k}}(f) \overset{\boxed{8.10.3}}{=\!=\!=} \int_X f(x)\, d\mu_{h,k},$$

Therefore $\tilde\rho$ is also continuous from $\sigma(\mathrm{Borel}_b(X), M(X))$ to $L(H)$ with the WOT. Since $C(X)$ is dense in $C(X)'' = M(X)'$ by 8.11 with respect to the topology $\sigma(M(X)', M(X))$, it is also dense in $\mathrm{Borel}_b(X)$ with respect to the trace topology $\sigma(\mathrm{Borel}_b(X), M(X))$.

Now we use this to show the multiplicativity of $\tilde\rho$:
Let $f \in \mathrm{Borel}_b(X)$. By 8.10.4 there is a net $f_i \in C(X)$, with $\int f_i\, d\mu \to \int f\, d\mu$ for all $\mu \in M(X)$. Since with $g \in \mathrm{Borel}_b(X)$ and $\mu \in M(X)$ also $g\,\mu$ defined by $(g\,\mu)(f) := \int_X f\,g\,d\mu$ lies in $M(X)$ (because $g\,\mu : C(X) \xrightarrow{g\cdot} \mathrm{Borel}_b(X) \hookrightarrow M(X)' \xrightarrow{\mathrm{ev}_\mu} \mathbb{C}$ is continuous by 8.10.3), $f_i\,g \to f\,g$ holds in the weak topology $\sigma(\mathrm{Borel}_b(X), M(X))$ and thus $\tilde\rho(f_i\,g) \to \tilde\rho(f\,g)$ with respect to the WOT. In particular, if $g \in C(X)$, then $\tilde\rho(f_i\,g) = \rho(f_i\,g) = \rho(f_i) \circ \rho(g) \to \tilde\rho(f) \circ \rho(g)$ with respect to the WOT, since the composition is continuous in the first variable with respect to the WOT by 8.9. Consequently, $\tilde\rho(f\,g) = \tilde\rho(f) \circ \rho(g)$. If $g \in \mathrm{Borel}_b(X)$ is now arbitrary, then $\tilde\rho(f_i\,g) = \tilde\rho(g\,f_i) = \tilde\rho(g) \circ \rho(f_i) \to \tilde\rho(g) \circ \tilde\rho(f)$ in the WOT, since the composition is also continuous in the second variable with respect to the WOT by 8.9. So $\tilde\rho(g\,f) = \tilde\rho(g) \circ \tilde\rho(f)$.

In order to show that $\tilde\rho$ is a $*$-representation, it remains to prove the $*$-homomorphy:
Let $f \in \mathrm{Borel}_b(X)$ and $f_i \in C(X)$ be a net as before. For $\mu \in M(X)$, let the measure $\overline{\mu}$ be defined by $\overline{\mu}(B) = \overline{\mu(B)}$. Then, with respect to the WOT, $\rho(f_i) \to \tilde\rho(f)$, and

hence $\rho(f_i)^* \to \tilde\rho(f)^*$ by 8.9. On the other hand: Because of $\int \overline{f_i}\, d\mu = \overline{\int f_i\, d\overline{\mu}} \to \overline{\int f\, d\overline{\mu}} = \int \overline{f}\, d\mu$ for each measure μ, we have $\rho(f_i)^* = \rho(\overline{f_i}) \to \tilde\rho(\overline{f})$, i.e. $\tilde\rho(f)^* = \tilde\rho(\overline{f})$.

($\tilde\rho \mapsto P$) We claim that $B \mapsto P(B) := \tilde\rho(\chi_B)$ defines a spectral measure P:
By 8.1 we know that $P(B)$ is an orthogonal projection, $P(X) = 1$, and we have $P(B_1 \cap B_2) = \tilde\rho(\chi_{B_1} \cdot \chi_{B_2}) = P(B_1) \circ P(B_2)$ and $P(B_1 \sqcup B_2) = \tilde\rho(\chi_{B_1} + \chi_{B_2}) = P(B_1) + P(B_2)$. All that remains to prove is the σ-additivity.
Let B_j be pairwise disjoint Borel sets, $B_{>n} := \bigcup_{j>n} B_j$ and $h \in H$. Then

$$\begin{aligned}\Big\| P\Big(\bigsqcup_{k=1}^{\infty} B_k\Big)h - \sum_{k=1}^{n} P(B_k)h\Big\|^2 &= \Big\| P(B_{>n})h + P\Big(\bigsqcup_{k=1}^{n} B_k\Big)h - P\Big(\bigsqcup_{k=1}^{n} B_k\Big)h\Big\|^2 \\ &= \|P(B_{>n})h\|^2 = \langle P(B_{>n})h, h\rangle \\ &= \langle \tilde\rho(\chi_{B_{>n}})h, h\rangle = \ell_{h,h}(\tilde\rho(\chi_{B>n})) \\ &= \mu_{h,h}(B_{>n}) = \sum_{j>n} \mu_{h,h}(B_j) \to 0,\end{aligned}$$

because $\mu_{h,k}$, as a measure, is obviously σ-additive. So P is a spectral measure.

($\rho \mapsto \tilde\rho \mapsto P \mapsto \rho$) For each representation ρ with associated spectral measure $P := \tilde\rho \circ \chi$ we have to show that $\int f\, dP = \rho(f)$ holds for all $f \in C(X)$:
Let $f \in \mathrm{Borel}_b(X)$ be arbitrary, $\varepsilon > 0$ and $B_j \ni x_j$ as in 8.10.1, hence

$$\Big\| f - \sum_{j=1}^{n} f(x_j)\, \chi_{B_j}\Big\|_\infty < \varepsilon.$$

Because of $\|\tilde\rho\| \leqslant 1$ and 8.12.2, it follows that

$$\Big\|\tilde\rho(f) - \int f\, dP\Big\| \leqslant \Big\|\tilde\rho\Big(f - \sum_{j=1}^{n} f(x_j)\, \chi_{B_j}\Big)\Big\| + \Big\|\sum_{j=1}^{n} f(x_j)\, P(B_j) - \int f\, dP\Big\| \leqslant 2\varepsilon,$$

so $\tilde\rho(f) = \int f\, dP$.

($P \mapsto \rho \mapsto \tilde\rho \mapsto P$) Let $P : \mathcal{B}(X) \to L(H)$ be a spectral measure with representation $\tilde\rho : \mathrm{Borel}_b(X, \mathbb{C}) \to L(H)$ associated by 8.12.3, i.e. $\ell_{h,k}(\tilde\rho(f)) = \int_X f\, d(\ell_{h,k} \circ P)$ for all $f \in \mathrm{Borel}_b(X, \mathbb{C})$ and $h, k \in H$ by 8.12.2. In particular, $\ell_{h,k}((\tilde\rho \circ \chi)(B)) = \int_X \chi_B\, d(\ell_{h,k} \circ P) = \ell_{h,k}(P(B))$, and since the $\ell_{h,k}$ separate operators, $\tilde\rho \circ \chi = P$. Remains show, that $\tilde\rho$ is the unique extension of $\tilde\rho|_{C(X,\mathbb{C})}$, which holds, since $\tilde\rho$ is continuous from $\sigma(\mathrm{Borel}_b(X, \mathbb{C})', M(X))$ into the WOT by 8.12.3 and $C(X, C)$ is dense in $\sigma(\mathrm{Borel}_b(X, \mathbb{C})', M(X))$ by 8.10.4. □

Spectral theory for normal operators

Remark.

Let H be a finite-dimensional Hilbert space. Then the spectral theorem of linear algebra says that every normal operator N can be diagonalized. In particular, there is an orthonormal basis consisting of eigenvectors u_i to eigenvalues λ_i. Thus

$$N(x) = N\Big(\sum_i \langle x, u_i\rangle u_i\Big) = \sum_i \lambda_i\, \langle x, u_i\rangle\, u_i.$$

In the infinite-dimensional case, a corresponding theorem has to look different, since a normal operator does not need to have eigenvalues, such as for example the multiplication operator $N = M_{\mathrm{id}}$ with the identity on $L^2[0,1]$: Let $\lambda\, f(t) = t\, f(t)$

a.e. for some $f \in L^2[0,1]$. Then $(\lambda - t)\, f(t) = 0$ a.e. and thus $f = 0$ a.e., i.e. $f = 0$ in $L^2[0,1]$.

However, one can also rewrite the finite-dimensional theorem as follows. For each eigenvalue $\lambda \in \sigma(N)$, let P_λ be the orthogonal projection onto the eigenspace $\ker(N - \lambda)$. Then

$$\begin{aligned} N(x) &= \sum_i \lambda_i \langle x, u_i\rangle u_i = \sum_\lambda \sum_{i:\lambda_i=\lambda} \lambda_i \langle x, u_i\rangle u_i \\ &= \sum_\lambda \lambda \sum_{i:\lambda_i=\lambda} \langle x, u_i\rangle u_i = \sum_{\lambda\in\sigma(N)} \lambda\, P_\lambda(x) \end{aligned}$$

Let's generalize this to Hilbert spaces and for this we have to simplify $\{N, N^*\}^k$:

8.14 Fugledge-Putnam Theorem.

Let N_1 and N_2 be normal operators on H_1 and H_2. If $T \in L(H_1, H_2)$ intertwines the operator N_1 with N_2 (i.e. $T\, N_1 = N_2\, T$), it also intertwines N_1^* with N_2^*.

Proof. $N_2\, T = T\, N_1 \Rightarrow p(N_2)\, T = T\, p(N_1)$ for each polynomial p and, furthermore, for every entire function $p \in H(\mathbb{C}, \mathbb{C})$. In particular,

$$T = \exp(-i\overline{z}N_2)\, T\, \exp(i\overline{z}N_1).$$

Since $\exp(X + Y) = \exp(X)\exp(Y)$, if X and Y commute with each other, and since the N_j are normal, we have

$$\begin{aligned} f(z) &:= \exp(-izN_2^*)\, T\, \exp(izN_1^*) \\ &= \exp(-izN_2^*)\exp(-i\overline{z}N_2)\, T\, \exp(i\overline{z}N_1)\exp(izN_1^*) \\ &= \exp(-i(zN_2^* + \overline{z}N_2))\, T\, \exp(i(\overline{z}N_1 + zN_1^*)). \end{aligned}$$

For each $z \in \mathbb{C}$, both $zN_2^* + \overline{z}N_2$ and $\overline{z}N_1 + zN_1^*$ are Hermitian operators, so $\exp(-i(zN_2^* + \overline{z}N_2))$ and $\exp(i(\overline{z}N_1 + zN_1^*))$ are unitary (for $(\exp(iA))^* \exp(iA) = \exp(-iA^*)\exp(iA) = \exp(i(A - A)) = 1$) and hence $\|f(z)\| \leqslant \|T\|$. The bounded mapping $f : \mathbb{C} \to L(H_1, H_2)$ is holomorphic, thus according to Liouville's Theorem 6.16 it is constant, and in particular

$$0 = f'(0) = -i\, N_2^* \exp(0)\, T\, \exp(0) + i\, \exp(0)\, T\, N_1^* \exp(0) = i\, (T\, N_1^* - N_2^*\, T). \quad \square$$

8.15 Spectral theorem (for normal bounded operators).

Let N be a normal operator on a Hilbert space H.
Then there is a unique spectral measure P on $\sigma(N)$, such that N has the following spectral decomposition

$$N = \int_{\sigma(N)} z\, dP(z).$$

If $U \neq \emptyset$ is relatively open in $\sigma(N)$, then $P(U) \neq 0$.
Furthermore $\int_{\sigma(N)} f\, dP \in \{N\}^{kk}$ for all $f \in \mathrm{Borel}_b(\sigma(N), \mathbb{C})$, resp.

$$\{N, N^*\}^k = \{N\}^k = \big\{P(B) : B \in \mathcal{B}(\sigma(N))\big\}^k = \Big\{\int_{\sigma(N)} f\, dP : f \in \mathrm{Borel}_b(\sigma(N))\Big\}^k$$

Function calculus: $f \mapsto f(N) := \int_{\sigma(N)} f(z)\, dP(z)$, is the unique representation of the C^*-algebra $\mathrm{Borel}_b(\sigma(N), \mathbb{C})$ on H, which is additionally continuous with respect to the topology $\sigma(\mathrm{Borel}_b(\sigma(N)), M(\sigma(N)))$ on $\mathrm{Borel}_b(\sigma(N))$ and the WOT on $L(H)$, and maps id to N.

Proof. Existence of P:

$$
\begin{aligned}
& N \in L(H), \text{ normal} \\
\overset{\boxed{7.14}}{\Longrightarrow} & \exists! \rho : C(\sigma(N),\mathbb{C}) \xrightarrow{\cong} C^*(N) \subseteq L(H), \text{ a representation with } \rho(\mathrm{id}) = N \\
\overset{\boxed{8.13}}{\Longleftrightarrow} & \exists! P : \mathcal{B}(\sigma(N)) \to L(H), \text{ a spectral measure with } N = \int_{\sigma(N)} z\, dP(z) \\
\overset{\boxed{8.12}}{\Longrightarrow} & \exists! \tilde\rho : \mathrm{Borel}_b(\sigma(N),\mathbb{C}) \to L(H), \text{ a weakly continuous representation.}
\end{aligned}
$$

Here $\int f\, dP = \rho(f)$ for all continuous f by $\boxed{8.13}$, thus in particular $\int z\, dP(z) = \int \mathrm{id}\, dP = \rho(\mathrm{id}) = N$.

Uniqueness of P: Each spectral measure P on $\sigma(N)$ with $N = \int_{\sigma(N)} z\, dP(z)$ corresponds by $\boxed{8.13}$ to a unique $*$-representation $\rho : f \mapsto \int_{\sigma(N)} f\, dP$ of $C(\sigma(N))$ with $\rho(\mathrm{id}) = N$, i.e. the unique function calculus from $\boxed{7.14}$.

Continuity of the function calculus: This follows from $\boxed{8.12.3}$.

Uniqueness of the function calculus: Let ρ be any representation as claimed. Because of the uniqueness of the function calculus $\boxed{6.28}$ and $\boxed{7.14}$, this coincides with $f \mapsto f(N)$ for all $f \in C(\sigma(N))$. Because of the continuity with respect to $\sigma(\mathrm{Borel}_b, M)$ and the denseness of $C(X)$ by $\boxed{8.10.4}$, this coincides with $\int f\, dP$ also for all $f \in \mathrm{Borel}_b$.

Non-degeneracy of P: Let now $U \neq \emptyset$ be open in $\sigma(N)$. Then there is a continuous function $f \neq 0$ on $\sigma(N)$ with $0 \leqslant f \leqslant \chi_U$. Hence, $P(U) = \tilde\rho(\chi_U) \geqslant \rho(f) \neq 0$ by $\boxed{8.8.3}$, $\boxed{8.12.2}$ and $\boxed{7.14}$, so P is not degenerated.

Commutator identities:

$$
\{P(B) : B \in \mathcal{B}\} \hookrightarrow \{f(N) : f \in \mathrm{Borel}_b\} \hookleftarrow \{f(N) : f \in C\} \underset{\boxed{7.16}}{\hookrightarrow} \{N, N^*\}^{kk}
$$

The inclusion in the middle is WOT-dense according to $\boxed{8.10.4}$ and $\boxed{8.12.3}$, and the inclusion on the left is dense in the operator norm according to $\boxed{8.12.2}$. Since the composition is separately continuous with respect to these topologies according to $\boxed{8.9}$, all sets to the left of $\{N, N^*\}^{kk}$ have the same commutant $\{N, N^*\}^k = \{N\}^k$ by $\boxed{7.16}$ and $\boxed{8.14}$. □

Definition. Support of a measure.

Let μ be a regular Borel measure on X and $U \subseteq X$ an open set. One says that μ vanishes on U, if $\int f\, d\mu = 0$ holds for all $f \in C_c(X)$ with $f|_{X\setminus U} = 0$. Equivalently, it is sufficient to request this (as with distributions in [**18**, 4.13.3]) for all $f \in C_c(X)$ with support $\mathrm{supp}(f) \subseteq U$, because if $f|_{X\setminus U} = 0$, then $h_n f \to f$ converges uniformly and $\mathrm{supp}(h_n f) \subseteq U$, where continuous functions $h_n \in C(X, [0,1])$ are choosen by Tietze-Urysohn so that $\mathrm{supp}(h_n) \subseteq U$ and $h_n = 1$ on $\{x : |f(x)| \geqslant \frac{1}{n}\}$.

The union of all open sets U with this property has the same property (i.e. there is a largest set among them), because the (compact) support of f is already covered by finite many such U and thus f can be written as $f = \sum_i h_i f$ by means of a subordinate partition $\{h_i\}_i$ of unity. Since $\int h_i f\, d\mu = 0$, the same holds for f.

The complement of the largest open set U with the above property is called the SUPPORT $\mathrm{supp}(\mu)$ of μ.

Note that for the spectral measure P of a normal $N \in L(H)$,

$$\langle f(N)h, k\rangle = \Big\langle\Big(\int_{\sigma(N)} f\,dP\Big)h, k\Big\rangle = \int_{\sigma(N)} f\,dP_{h,k}$$

for all $h, k \in H$ and $f \in \mathrm{Borel}_b(\sigma(N))$. In particular, $\langle f|_{\sigma(N)}(N)h, k\rangle = \int_{\mathbb{C}} f\,dP_{h,k}$ holds for all $h, k \in H$ and $f \in \mathrm{Borel}_b(\mathbb{C})$, as $P_{h,k}$ is a measure on $\sigma(N)$ and hence can be considered as a measure on $\mathbb{C}$ with support included in $\sigma(N)$.

8.16 Lemma.

Let E be a Banach space and $T \in L(E)$. If $\sigma(T) = K_1 \sqcup K_0$ with disjoint closed K_1 and K_0, then a decomposition $E = E_1 \oplus E_0$ into invariant subspaces E_j of T exists, s.t. $\sigma(T|_{E_j}) = K_j$.

So if $\sigma(T)$ is discrete (and therefore finite), we find a decomposition $E = \bigoplus_{\lambda\in\sigma(T)} E_\lambda$ in invariant subspaces for which $T|_{E_\lambda}$ has spectrum $\{\lambda\}$.

Proof. Let $p \in H(\sigma(T), \mathbb{C})$ be the holomorphic germ with $p = j$ locally at K_j for $j \in \{0, 1\}$ as in 6.33. Then $P := p(T) \in \{T\}^{kk}$ (by 6.32) is idempotent. Thus, $E_1 := \mathrm{img}(P)$ and $E_0 := \mathrm{img}(1 - P) = \ker(P)$ is invariant under $\{T\}^k \supseteq \{T\}$ by 7.39.4.

Let $T_j := T|_{E_j}$. Then $T - \lambda$ is invertible in $L(E)$ if and only if $T_j - \lambda$ is invertible in $L(E_j)$ for $j = 0$ and $j = 1$, and thus $K_1 \sqcup K_0 = \sigma(T) = \sigma(T_1) \cup \sigma(T_0)$: In fact an inverse B to $T - \lambda$ belongs to $\{T\}^k$, hence has to keep the subspaces E_j-invariant by 7.39.4 because $P \in \{T\}^{kk} \subseteq \{B\}^k$.

($\sigma(T_i) \subseteq K_i$) Let $\lambda \notin K_i$ and w.l.o.g. $i = 1$. We define the holomorphic germ f by $f : z \mapsto \frac{1}{\lambda - z}$ locally around K_1 and by $f = 0$ locally around K_0. Then $(\lambda - z)f(z) = p(z)$ and thus $(\lambda - T)\,f(T) = p(T) = P$. Since E_1 is invariant under all occurring operators, the restriction T_1 of T to E_1 satisfies $\lambda \notin \sigma(T_1)$, i.e. $\sigma(T_1) \subseteq K_1$.

Because of $K_1 \sqcup K_0 = \sigma(T_1) \cup \sigma(T_0)$ we obtain $\sigma(T_1) = K_1$ and $\sigma(T_0) = K_0$. □

8.17 Proposition.

Let N be a normal operator on a Hilbert space H with spectral measure P and $\lambda \in \sigma(N)$. Then $\mathrm{img}(P(\{\lambda\})) = \ker(N - \lambda)$. Thus, λ is an eigenvalue of N if and only if $P(\{\lambda\}) \neq 0$ and then $P(\{\lambda\})$ is the orthogonal projection onto the eigenspace of λ.

Proof. ($\subseteq$) We have $(z - \lambda) \cdot \chi_{\{\lambda\}} = 0$ and therefore $(N - \lambda)\,P(\{\lambda\}) = 0$, i.e. $\mathrm{img}(P(\{\lambda\})) \subseteq \ker(N - \lambda)$.

($\supseteq$) For $h \in \ker(N - \lambda)$,

$$\begin{aligned} 0 = \|(N - \lambda)h\|^2 &= \langle (N - \lambda)h, (N - \lambda)h\rangle = \langle (N - \lambda)^*(N - \lambda)h, h\rangle \\ &= \int |z - \lambda|^2\, d\langle P(z)h, h\rangle \end{aligned}$$

and, since $\mu := P_{h,h} = \langle P(\_)h, h\rangle$ is a positive measure by 8.8.4, therefore $\mathrm{supp}(\mu) \subseteq \{z \in \mathbb{C} : |z - \lambda|^2 = 0\} = \{\lambda\}$ (In fact: $\lambda \notin \mathrm{supp}(f) \Rightarrow |f(z)| \leqslant C\,|z - \lambda|^2 \Rightarrow 0 \leqslant |\int f\,d\mu| \leqslant \int |f|\,d\mu \leqslant C \int |z - \lambda|^2 d\mu(z) = 0$) and thus $\|P(\{\lambda\})h\|^2 = \langle P(\{\lambda\})h, h\rangle = \mu(\{\lambda\}) = \mu(\sigma(N)) = \langle (\int_{\sigma(N)} dP)h, h\rangle = \|h\|^2$, i.e. $h \in \mathrm{img}\,P(\{\lambda\})$. □

Spectral theory of compact operators

8.18 Lemma.

Let E and F be Banach spaces. An operator $T \in L(E,F)$ is compact if and only if its adjoint operator $T^* \in L(F^*,E^*)$ is it.

Proof. ($\Rightarrow$) This is [**18**, 6.4.13]

($\Leftarrow$) Let T^* be compact. Then T^{**} is compact by the first part, and thus also its restriction T to $E \subseteq E^{**}$ and $F \subseteq F^{**}$. $\square$

8.19 Lemma.

Let T be a compact operator, $0 \neq \lambda \in \mathbb{C}$. Then λ is an eigenvalue if and only if $\inf\{\|(T-\lambda)h\| : \|h\| = 1\} = 0$.

Proof. ($\Rightarrow$) is clear, because then a $h \neq 0$ exists with $Th = \lambda h$.
($\Leftarrow$) By assumption, there is a sequence $h_n \in E$ with $\|h_n\| = 1$ and $\|(T-\lambda)h_n\| \to 0$. Since T is compact, we may assume that $y := \lim_n Th_n$ exists. Therefore $h_n = \frac{1}{\lambda}\big((\lambda - T)h_n + Th_n\big) \to \frac{1}{\lambda}y$ and consequently $1 = \|\frac{1}{\lambda}y\| = \frac{1}{|\lambda|}\|y\|$, i.e. $y \neq 0$. Due to $Th_n \to T(\frac{1}{\lambda}y) = \frac{1}{\lambda}Ty$, $\frac{1}{\lambda}Ty = y$ holds, i.e. λ is an eigenvalue of T with eigenvector y. $\square$

8.20 Lemma.

Let T be a compact operator on a Banach space E and $0 \neq \lambda \in \sigma(T)$. Then λ is an eigenvalue of T or T^*.

Proof. Indirectly. Suppose λ is neither eigenvalue of $T \in L(E)$ nor of $T^* \in L(E^*)$. By the previous lemma 8.19 there exists a $c > 0$ with $\|(T-\lambda)h\| \geqslant c\|h\|$ for all $h \in E$. So $T - \lambda$ is a homeomorphism onto its image, and thus this is complete and therefore closed. Because λ is not an eigenvalue of the Banach space adjoint T^*,

$$\operatorname{img}(T-\lambda) = \overline{\operatorname{img}(T-\lambda)} \overset{5.4.3}{=\!=\!=} (\ker(T-\lambda)^*)_o \overset{!}{=} (\ker(T^*-\lambda))_o = \{0\}_o = E,$$

holds because $T \mapsto T^*$ is $\mathbb{C}$-linear! Thus, $(T-\lambda) : E \to E$ is bijective and because of $\|(T-\lambda)h\| \geqslant c\|h\|$ (or by the open mapping theorem), the inverse mapping $(T-\lambda)^{-1}$ is also continuous, i.e. $\lambda \notin \sigma(T)$. $\square$

8.21 Lemma.

Let $F \subset E$ be a true closed subspace of a Banach space E and $\varepsilon > 0$. Then there is an $x \in E$ with $\|x\| = 1$ and $\operatorname{dist}(x,F) \geqslant 1 - \varepsilon$.

Proof. Let $d(x) := \operatorname{dist}(x,F) := \inf\{\|x-y\| : y \in F\}$. We choose $x_1 \in E\backslash F$. Then there is a $y_1 \in F$ with $0 < d(x_1) \leqslant \|x_1 - y_1\| \leqslant (1+\varepsilon)\,d(x_1)$. Let $x_2 := x_1 - y_1$, then $d(x_2) = \inf\{\|x_2 - y\| : y \in F\} = \inf\{\|x_1 - y_1 - y\| : y \in F\} = d(x_1)$ and $(1+\varepsilon)\,d(x_2) = (1+\varepsilon)\,d(x_1) \geqslant \|x_1 - y_1\| = \|x_2\| > 0$. Finally let $x := \frac{1}{\|x_2\|}x_2$. Then $\|x\| = 1$ and for $y \in F$ we have

$$\begin{aligned}\|x-y\| &= \left\|\frac{1}{\|x_2\|}x_2 - y\right\| = \frac{1}{\|x_2\|}\left\|x_2 - \|x_2\|\,y\right\| \\ &\geqslant \frac{1}{(1+\varepsilon)\,d(x_2)}\left\|x_2 - \|x_2\|\,y\right\| \geqslant \frac{1}{(1+\varepsilon)\,d(x_2)}\,d(x_2) \geqslant \frac{1}{1+\varepsilon} \\ &> 1-\varepsilon. \quad \square\end{aligned}$$

8.22 Spectral theorem for compact operators on Banach spaces.

Let E be an infinite-dimensional Banach space and $T \in L(E)$ a compact operator. Then $0 \in \sigma(T)$ and all $0 \neq \lambda \in \sigma(T)$ are isolated in $\sigma(T)$ and eigenvalues of T with finite-dimensional eigenspaces $\ker(T-\lambda)$. If there are infinitely many such λ's, then they can be arranged in the form of a 0-sequence.

Proof. Claim: *Each sequence of pairwise distinct eigenvalues $\lambda_n \neq 0$ of T converges towards 0:*
For each n we choose an $h_n \in \ker(T-\lambda_n)\setminus\{0\}$. Let E_n be the linear subspace generated by $\{h_1,\dots,h_n\}$. This space is n-dimensional since the h_n are linear independent: Let $\sum_k \mu_k h_k = 0$ be a linear combination of minimal length, then $0 = (T-\lambda_1)(\sum_k \mu_k h_k) = \sum_{k>1} \mu_k(\lambda_k - \lambda_1)h_k$ is a contradiction to the minimality. By the previous lemma 8.21 there exist $y_n \in E_n$ with $\|y_n\| = 1$ and $d(y_n, E_{n-1}) > \frac{1}{2}$. Let $y_n =: \sum_{k\leqslant n} \mu_k\, h_k$. Then $(T-\lambda_n)y_n = \sum_{k<n} \mu_k\,(\lambda_k - \lambda_n)\, h_k \in E_{n-1}$ and thus for $n > m$:

$$\begin{aligned} T\left(\frac{1}{\lambda_n}y_n\right) - T\left(\frac{1}{\lambda_m}y_m\right) &= \frac{1}{\lambda_n}(T-\lambda_n)y_n - \frac{1}{\lambda_m}(T-\lambda_m)y_m + y_n - y_m \\ &= y_n + \Bigg(\underbrace{\frac{1}{\lambda_n}(T-\lambda_n)y_n}_{\in E_{n-1}} - \underbrace{\frac{1}{\lambda_m}(T-\lambda_m)y_m}_{\in E_{m-1}} - \underbrace{y_m}_{\in E_m}\Bigg) \in y_n + E_{n-1}. \end{aligned}$$

Consequently,

$$\left\| T\left(\frac{1}{\lambda_n}y_n\right) - T\left(\frac{1}{\lambda_m}y_m\right) \right\| \geqslant \operatorname{dist}(y_n, E_{n-1}) > \frac{1}{2}.$$

Thus $(T(\frac{1}{\lambda_n}y_n))_n$ has no convergent subsequence. But since T is compact, and hence the images of bounded sets are relatively compact, $(\frac{1}{\lambda_n}y_n)_n$ can not have a bounded subsequence. So $\|\frac{1}{\lambda_n}y_n\| = \frac{1}{|\lambda_n|} \to \infty$, i.e. $\lambda_n \to 0$.

Claim: *All $0 \neq \lambda \in \sigma(T)$ are isolated points of $\sigma(T)$.*
If $\lambda_n \in \sigma(T)$ with $\lambda_n \neq \lambda$ converges to $\lambda \neq 0$, according to 8.20, λ_n is an eigenvalue of T or T^*. Without loss of generality, we can assume that all λ_n are eigenvalues of T or all of T^*. The previous claim yields – since also T^* is compact by 8.18 – $\lambda_n \to 0$, a contradiction.

Claim: *All $0 \neq \lambda \in \sigma(T)$ are eigenvalues of T.*
Since λ is isolated, there exists by 8.16 a closed invariant subspace E_λ of E, s.t. $T_\lambda := T|_{E_\lambda}$ has as spectrum $\{\lambda\}$. So, T_λ is an invertible ($0 \notin \sigma(T_\lambda)$) compact operator and thus E_λ is finite-dimensional (because the image of the unit ball is then a relatively-compact 0-neighborhood). As a result, $\lambda \in \sigma(T_\lambda)$ is an eigenvalue of T_λ and thus of T.

Claim: *The eigenspace $\ker(T-\lambda)$ is finite-dimensional.*
Since $\ker(T-\lambda)$ is a T-invariant closed subspace and $\lambda\, \mathrm{id}_{\ker(T-\lambda)} = T|_{\ker(T-\lambda)}$ is compact, $\ker(T-\lambda)$ is finite-dimensional. □

8.23 Lemma.

*Let N be a normal operator on a Hilbert space with spectral measure P.
Then N is compact if and only if $P(\{z \in \sigma(N) : |z| > \varepsilon\})$ has finite-dimensional image for all $\varepsilon > 0$.*

Proof. ($\Leftarrow$) Let $\varepsilon > 0$ and $B_\varepsilon := \{z \in \sigma(N) : |z| \leqslant \varepsilon\}$ and $P_\varepsilon := P(\sigma(N)\setminus B_\varepsilon)$. Then for $f : z \mapsto z\,\chi_{B_\varepsilon}(z)$ we have

$$\begin{aligned} N - N\,P_\varepsilon &= N\,(1 - P_\varepsilon) = N\,P(B_\varepsilon) \\ &= \int z\,\chi_{B_\varepsilon}(z)\,dP(z) = f(N). \end{aligned}$$

So $\|N - N\,P_\varepsilon\| \leqslant \|f\|_\infty = \sup\{|z| : z \in B_\varepsilon\} \leqslant \varepsilon$. Since P_ε has finite-dimensional image for each ε, so does $N\,P_\varepsilon$, and hence N is compact by [**18**, 6.4.8].

($\Rightarrow$) Let N be compact and $\varepsilon > 0$. Consider $g : z \mapsto \frac{1}{z}\chi_{\sigma(N)\setminus B_\varepsilon}(z)$ in $\mathrm{Borel}_b(\mathbb{C})$. Since N is compact, the same is true for

$$N\,g(N) = \int z\,\frac{1}{z}\,\chi_{\sigma(N)\setminus B_\varepsilon}(z)\,dP(z) = P_\varepsilon.$$

Since P_ε is a projection, its image has to be finite-dimensional. □

8.24 Spectral theorem for compact normal operator on Hilbert spaces.

Let N be a compact and normal operator on a Hilbert space.
Then the eigenvalues unequal 0 of N form a finite or a convergent sequence λ_j. The eigenspaces $\ker(N - \lambda_j)$ are finite-dimensional and pairwise orthogonal and with respect to the orthogonal projections P_j onto $\ker(N - \lambda_j)$ the following holds:

$$N = \sum_j \lambda_j P_j.$$

Conversely, every operator N is compact and normal, provided it has a representation $N = \sum_j \lambda_j P_j$ with finite-dimensional orthogonal projections $P_j \neq 0$ with pairwise orthogonal images and pairwise different $0 \neq \lambda_j \to 0$. Then the λ_j are the eigenvalues other than 0, and the images of the P_j are the associated eigenspaces.

Proof. ($\Rightarrow$) According to the Spectral Theorem 8.15, a unique spectral measure P exists on $\sigma(N)$ with $N = \int_{\sigma(N)} z\,dP(z)$. By the Spectral Theorem 8.22 $\sigma(N) = \{0, \lambda_1, \lambda_2, \dots\}$ and each λ_k is isolated and an eigenvalue. So by 8.17 $P_k := P(\{\lambda_k\})$ is the orthogonal projection onto the eigenspace $\ker(N - \lambda_k)$. Now let $\varepsilon > 0$, and let n be so large that $|\lambda_k| < \frac{\varepsilon}{2}$ for $k > n$. Then the sets $\{\lambda_1\}, \dots, \{\lambda_n\}, \{0, \lambda_{n+1}, \lambda_{n+2}, \dots\}$ form a decomposition of $\sigma(N)$ into Borel sets with $|z - z'| \leqslant \varepsilon$ for z, z' in the same set. Thus $\|\int_{\sigma(N)} z\,dP(z) - \sum_{j\leqslant n} \lambda_j P_j - 0\,P(\{0, \lambda_{n+1}, \dots\})\| < \varepsilon$, i.e. the sum $\sum_j \lambda_j P_j$ converges towards $N = \int_{\sigma(N)} z\,dP(z)$. Since the λ_j are pairwise distinct, the images of P_j are pairwise orthogonal to 8.8.1.

($\Leftarrow$) Since $\lambda_j \to 0$ and, furthermore, $\|P_j\| \leqslant 1$ for orthogonal projections P_j and the images of P_j are orthogonal, it follows that the sum converges in the operator norm because

$$\begin{aligned} \Big\|\sum_{j\geqslant n} \lambda_j P_j h\Big\|^2 = \sum_{j\geqslant n} |\lambda_j|^2\,\|P_j h\|^2 &\leqslant \max\{|\lambda_j|^2 : j \geqslant n\} \cdot \Big\|\Big(\sum_{j\geqslant n} P_j\Big)h\Big\|^2 \\ &\leqslant \max\{|\lambda_j|^2 : j \geqslant n\} \cdot \|h\|^2. \end{aligned}$$

Its partial sums are assumed to be finite-dimensional operators, so N is compact. We have $N^* = \sum_j \overline{\lambda_j} P_j$, hence $N^*N = NN^* = \sum_j |\lambda_j|^2 P_j$ and thus N is normal.

Let $\lambda \neq 0$ be an eigenvalue of N and h an associated eigenvector. So $0 \neq \lambda\,h = N(h) = \sum_j \lambda_j P_j(h)$, hence at least on $P_k(h) \neq 0$ and by 8.3, using the the orthogonality of the images of the P_j, we get $\lambda\,P_k(h) = \sum_j \lambda_j (P_k \circ P_j)(h) = \lambda_k P_k(h)$. Thus $\lambda = \lambda_k$, i.e. this k is unique and $h = P_k(h)$, i.e. $\ker(N - \lambda_k) \subseteq \operatorname{img} P_k$.

Conversely, $h \in \operatorname{img} P_k \Rightarrow h = P_k h \Rightarrow N(h) = \sum_j \lambda_j P_j(P_k h) = \lambda_k P_k h = \lambda_k h$, i.e. h is an eigenvector with corresponding eigenvalue λ_k. $\square$

8.25 Spectral representation of Hermitian operators.

Let N be a Hermitian operator, P its spectral measure and $p(t) := P(\{s \in \sigma(N) : s < t\})$. Then $p : \mathbb{R} \to L(H)$ is a monotonous, with respect to the SOT left-continuous mapping with $p(t) = 0$ for $t \leqslant -\|N\|$ and $p(t) = 1$ for $t \geqslant \|N\|$. Moreover, $f(N) = \int_{-\infty}^{+\infty} f(t)\, dp(t)$, an operator valued Riemann-Stieltjes integral, for each $f \in C(\sigma(N))$.

Proof. Since $t \mapsto \{s \in \sigma(N) : s < t\}$ is monotonously increasing, $p : t \mapsto P(\{s \in \sigma(N) : s < t\})$ is monotonously increasing by 8.8.3 and because of $\sigma(N) \subseteq \{s \in \mathbb{R} : -\|N\| \leqslant s \leqslant \|N\|\}$, $p(t) = 0$ by 8.8.1 for $t < -\|N\|$ and $p(t) = 1$ for $t \geqslant \|N\|$. Because of the σ additivity of P, p is left-continuous with respect to the SOT: In fact, $t_n \nearrow t_\infty$ implies that $(-\infty, t_\infty) = (-\infty, t_0) \sqcup \bigsqcup_i [t_{i-1}, t_i)$ is a decomposition and thus with respect to the SOT

$$\begin{aligned} p(t_\infty) = P[(-\infty, t_\infty)] &= P[(-\infty, t_0)] + \sum_{i=1}^{\infty} P([t_{i-1}, t_i)) \\ &= p(t_0) + \sum_{i=1}^{\infty} (p(t_i) - p(t_{i-1})) = \lim_{i\to\infty} p(t_i). \end{aligned}$$

Now let $f \in C(\sigma(N))$, so there is a monotonously increasing sequence of $t_j \in \mathbb{R}$ with $|f(x) - f(x')| \leqslant \varepsilon$ for $t_{j-1} \leqslant x, x' \leqslant t_j$. Then

$$\int f(z)\, dP(z) \approx \sum_j f(x_j)\, P([t_{j-1}, t_j)) = \sum_j f(x_j)\, (p(t_j) - p(t_{j-1})),$$

a Riemann-Stieltjes sum for $\int f(z)\, dp(z)$. $\square$

8.26 Corollary.

Let H be a separable Hilbert space. Then the only non-trivial closed ideal is that of all compact operators.

Proof. Because of the Proposition 7.30 every closed ideal $I \neq \{0\}$ contains all compact operators. Suppose it contains also a non-compact operator A. Then $N := A^*A$ is positive and non-compact: Otherwise, $N = \sum_j \lambda_j P_j$ with certain $0 < \lambda_j \to 0$ and orthogonal projections P_j with pairwise orthogonal images by 8.24. Thus $|A| := \sqrt{A^*A} = \sqrt{N} = \sum_j \sqrt{\lambda_j} P_j$ would also be compact by 8.24, and hence $A = U\,|A|$ (by 7.24) would be compact as well, a contradiction.

By 8.23, an $\varepsilon > 0$ exists so that $P_\varepsilon := P(\sigma(N)\backslash B_\varepsilon) = N\,g(N) \in I$ has infinite-dimensional image, where P is the spectral measure for N, $B_\varepsilon := \{z \in \sigma(N) : |z| \leqslant \varepsilon\} = [0, \varepsilon] \cap \sigma(N)$ and $g(z) := \frac{1}{z}\,\chi_{\sigma(N)\backslash B_\varepsilon}$. Since H is separable, there is a surjective isometry $U : H \to \operatorname{img}(P_\varepsilon)$. Then $1 = U^*U = U^* P_\varepsilon U \in I$, i.e. $I = L(H)$. $\square$

Normal operators as multiplication operators

An analogy to a diagonal operator would be a multiplication operator $M_f : g \mapsto f \cdot g$, which we will study now.

8.27 Diagonal operators.

Let (X,Ω,μ) be a σ-finite measure space. Let $f \mapsto M_f$ be the faithful and therefore isometric representation of $L^\infty(\mu)$ on $L^2(\mu)$, which was given in 8.6 by the multiplication operators $M_f : g \mapsto f \cdot g$. Then we have:

1. *The operator M_f is normal and $(M_f)^* = M_{\overline{f}}$.*
2. *We have $\sigma(M_f) = \text{ess-img}(f) := \bigcap\{\overline{f(A)} : A \in \Omega, \mu(X\setminus A) = 0\}$.*
3. *The spectral measure P for M_f on $\sigma(M_f)$ is given by $B \mapsto M_{\chi_{f^{-1}(B)}}$.*

Proof. (1) We have $\langle h, M_f^* k\rangle = \langle M_f h, k\rangle = \int f\, h\, \overline{k}\, d\mu = \int h\, \overline{\overline{f}\, k}\, d\mu = \langle h, M_{\overline{f}} k\rangle$,, i.e. $(M_f)^* = M_{\overline{f}}$, and therefore $M_f \circ (M_f)^* = M_f \circ M_{\overline{f}} = M_{|f|^2} = (M_f)^* \circ M_f$.

(2) ($\subseteq$) Let $\lambda \notin \text{ess-img}(f)$. Then there is an $A \in \Omega$ with $\mu(X\setminus A) = 0$ and $\lambda \notin \overline{f(A)}$, i.e. there is an $\delta > 0$ with $|f(x) - \lambda| \geqslant \delta$ for all $x \in A$. We have $g := \frac{1}{f-\lambda} \in L^\infty(\mu)$ and $M_g = (M_f - \lambda)^{-1}$, hence $\lambda \notin \sigma(M_f)$.
($\supseteq$) Conversely, let $\lambda \in \text{ess-img}(f)$. For $n \in \mathbb{N}$, let $A_n := \{x : |f(x) - \lambda| > \frac{1}{n}\}$. Then $A_n \in \Omega$ with $0 < \mu(X\setminus A_n) \leqslant \infty$ because $\lambda \notin \overline{f(A_n)}$. Since (X,Ω,μ) is σ-finite, there is a measurable $A'_n \subseteq X\setminus A_n$ with $0 < \mu(A'_n) < \infty$. We put $f_n := \frac{1}{\sqrt{\mu(A'_n)}}\chi_{A'_n}$. Then $f_n \in L^2(\mu)$ with $\|f_n\|_2 = 1$ and $\|(M_f - \lambda) f_n\|^2 = \frac{1}{\mu(A'_n)}\int_{A'_n} |f - \lambda|^2\, d\mu \leqslant \frac{1}{n^2}$. Hence $M_f - \lambda$ is not an open mapping and thus $\lambda \in \sigma(M_f)$.

(3) We choose a finite decomposition of the bounded set $\overline{f(X)}$ into Borel sets B_j with $z, z' \in B_j \Rightarrow |z - z'| \leqslant \varepsilon$ and pick $z_j \in B_j$. Then the sets $f^{-1}(B_j)$ form a decomposition of X into measurable sets and for all $x \in f^{-1}(B_j)$ the estimate $|(f - \sum_j z_j \chi_{f^{-1}(B_j)})(x)| = |f(x) - z_j| \leqslant \varepsilon$ holds. Due to $\|M_g\| \leqslant \|g\|_\infty$ for all $g \in L^\infty$, we obtain

$$\Big\| M_f - \sum_j z_j M_{\chi_{f^{-1}(B_j)}} \Big\| \leqslant \Big\| f - \sum_j z_j \chi_{f^{-1}(B_j)} \Big\|_\infty \leqslant \varepsilon.$$

Therefore $\sum_j z_j M_{\chi_{f^{-1}(B_j)}}$ converges towards M_f and also towards $\int z\, dP(z)$, where P is the spectral measure defined by $P(B) := M_{\chi_{f^{-1}(B)}}$. □

8.28 Example.

In particular, if $X = \mathbb{C}$ and $\mu \geqslant 0$ is a regular Borel measure with compact support $K := supp(\mu) \subseteq \mathbb{C}$, then we denote with N_μ the multiplication operator M_{id} on $L^2(\mu)$ with the identity $\text{id} : \mathbb{C} \to \mathbb{C}$. The following holds:

1. *N_μ is normal, and $\sigma(N_\mu) = supp(\mu)$.*
2. *$f(N_\mu)$ is the multiplication operator M_f for each $f \in \text{Borel}_b(\mathbb{C})$.*
3. *The spectral measure P for N_μ is $B \mapsto M_{\chi_B}$.*

Proof. (1) This follows from 8.27.1 and 8.27.2 because $N_\mu = M_{\text{id}}$ and since $\text{ess-img}(f) = f(\text{supp}(\mu))$ for each continuous f (e.g. $f := \text{id}$):
($\subseteq$) We put $K := \text{supp}(\mu)$. Since the characteristic function $\chi_{\mathbb{C}\setminus K}$ of the open set $\mathbb{C}\setminus K$ can be written as pointwise limit of a monotonous sequence of continuous functions $g_n \in C_c(\mathbb{C})$ with $g_n|_K = 0$ (hence $\int g_n\, d\mu = 0$), we obtain $\mu(\mathbb{C}\setminus K) = \int \chi_{\mathbb{C}\setminus K}\, d\mu = \lim_n \int g_n\, d\mu = 0$. Since f is continuous, the image $f(K)$ is compact and thus closed and therefore $\text{ess-img}(f) \subseteq f(K) = f(\text{supp}(\mu))$.
($\supseteq$) Let A be any Borel set with $\mu(\mathbb{C}\setminus A) = 0$. Then for each $0 \leqslant g \in C_c(\mathbb{C})$ with $g|_A = 0$ we have $0 \leqslant \int g\, d\mu \leqslant \|g\|_\infty\, \mu(\mathbb{C}\setminus A) = 0$. Thus the support of μ is contained in $\overline{A}$, hence $f(\text{supp}(\mu)) \subseteq f(\overline{A}) \subseteq \overline{f(A)}$ for each continuous f, i.e. $f(\text{supp}(\mu)) \subseteq \text{ess-img}(f)$.

(2) Because of the Spectral Theorem 8.15, we only have to show that $f \mapsto M_f$ has the characterising continuity properties:
So let $f_j \to 0$ in $\mathrm{Borel}_b(K)$ with respect to the topology $\sigma(\mathrm{Borel}_b(K), M(K))$. We have to show that $M_{f_j} \to 0$ in the WOT. Let $h, k \in L^2(\mu)$. Then, by Cauchy-Schwarz, $h\,\overline{k} \in L^1(\mu)$ and thus $h\,\overline{k}\,\mu \in M(K)$, therefore

$$\langle M_{f_j} h, k \rangle = \int_K f_j\, h\, \overline{k}\, d\mu \to 0.$$

(3) This immediately follows from 8.27.3 or from (2) because $P(B) = \chi_B(N_\mu) = M_{\chi_B}$. □

We now want to show that every normal operator is unitary equivalent to a multiplication operator. Hence the following

8.29 Definition.

We transfer some notions of the representation theory of Abelian C^*-algebras to normal operators $N \in L(H)$ by considering the C^*-subalgebra $C^*(N) \subseteq L(H)$ generated by N and the associated representation $\rho_N : C(\sigma(N)) \cong C^*(N) \subseteq L(H)$, i.e. the function calculus from 7.14.

An $h \in H$ is called CYCLIC VECTOR for N, if it is one for the representation ρ_N, i.e. $\{p(N, N^*)h : p \in \mathbb{C}[z, \overline{z}]\}$ is dense in H.

The normal operator N is called CYCLIC if it has a cyclic vector.

Two normal operators $N_1 \in L(H_1)$ and $N_2 \in L(H_2)$ are called UNITARY EQUIVALENT, if an isometric isomorphism $U : H_1 \to H_2$ exists with $N_2 \circ U = U \circ N_1$, i.e. $N_2 = U \circ N_1 \circ U^{-1}$.

Lemma.

Two normal operators $N_1 \in L(H_1)$ and $N_2 \in L(H_2)$ are unitary equivalent if and only if $\sigma(N_1) = \sigma(N_2)$ and the associated representations ρ_{N_1} and ρ_{N_2} are unitary equivalent:

Proof. ($\Rightarrow$) If $N_1 - \lambda$ is invertible, so is $N_2 - \lambda = U \circ (N_1 - \lambda) \circ U^{-1}$, and vice versa. Hence the two spectra coincide. Furthermore, ρ_{N_2} and $f \mapsto U \circ \rho_{N_1}(f) \circ U^{-1}$ are two $*$-representations of $C(\sigma(N_2))$, which both yield N_2 on the identity. So they agree, hence ρ_{N_1} and ρ_{N_2} are unitary equivalent via U.
($\Leftarrow$) Let $U : H_1 \to H_2$ be a surjective isometry with $\rho_{N_2}(f) \circ U = U \circ \rho_{N_1}(f)$ for all $f \in C(X)$, where $X := \sigma(N_1) = \sigma(N_2)$. Then, in particular, $N_2 \circ U = U \circ N_1$ for $f := \mathrm{id}$. □

8.30 Corollary.

Every normal operator is unitary equivalent to an orthogonal sum of cyclic operators.

Proof. Let N be a normal operator on H. By 7.32, H is an orthogonal sum of closed invariant subspaces H_j of the representation $\rho_N : C(\sigma(N)) \to L(H)$, s.t. the trace representations $\rho_j : f \mapsto \rho_N(f)|_{H_j}$ are cyclic and ρ_N is unitary equivalent to $\bigoplus_j \rho_j$ via the natural isometry $U : \bigoplus_j H_j \to H$. In particular, because of the lemma in 8.29, N is unitary equivalent to $\bigoplus_j N_j$ via U, where the $N_j := N|_{H_j}$ are cyclic operators. □

As for representation theory, we should first study cyclic operators.

8.31 Proposition.

A normal operator N is cyclic if and only if a positive measure μ exists on $\sigma(N)$, s.t. N is unitary equivalent to the multiplication operator N_μ on $L^2(\mu)$ by the identity. The equivalence U is uniquely determined by the condition $U(h_0) = 1$ for a fixed cyclic vector h_0, We have $\mu = P_{h_0,h_0}$, where P is the spectral measure of N.

$$\begin{tikzcd} \mathrm{Borel}_b(\sigma(N),\mathbb{C}) \arrow[d, "\pi"] & C(\sigma(N),\mathbb{C}) \arrow[l, hook'] \arrow[r, "\rho_N"] & L(H) \arrow[d, "\mathrm{konj}_U", "\cong"'] & H \arrow[d, "U"] \\ L^\infty(\mu) \arrow[rr, "M"'] & & L(L^2(\mu)) & L^2(\mu) \end{tikzcd}$$

Proof. By definition, a normal operator $N \in L(H)$ is cyclic if and only if the representation $\rho_N : C(\sigma(N)) \to L(H)$ is cyclic. By 7.35, such a representation $C(\sigma(N))$ is cyclic if and only if it is equivalent to the representation M on $L^2(\mu)$ for some positive Borel measure μ on $\sigma(N)$, where the unitary equivalence $U : L^2(\mu) \to H$ is uniquely determined by $U(1) = h_0$ for the given cyclic vector $h_0 \in H$. By 8.29, this is exactly the cases when N and $N_\mu = M_{\mathrm{id}}$ are unitary equivalent. We have $P_{h_0,h_0} = \mu$, because

$$\begin{aligned} \int f\, dP_{h_0,h_0} &\overset{8.12}{=\!=} \langle \rho_N(f)h_0, h_0\rangle = \langle \rho_N(f)U1, U1\rangle = \langle U^*\rho_N(f)U1, 1\rangle \\ &= \langle U^{-1}\rho_N(f)U1, 1\rangle \overset{7.35}{=\!=} \langle M_f 1, 1\rangle = \int f\, d\mu. \quad \square \end{aligned}$$

8.32 Remark. Unitary equivalent N_μ's.

To determine the unitary equivalence classes of all cyclic operators, we need to decide for which positive Borel measures μ_j on $\mathbb{C}$ with compact supports the operators N_{μ_1} and N_{μ_2} are unitary equivalent.

Suppose there is a surjective isometry $U : L^2(\mu_1) \to L^2(\mu_2)$ with $U\, N_{\mu_1}\, U^{-1} = N_{\mu_2}$: From the equivalence of N_{μ_1} and N_{μ_2} follows by 8.29 that the two spectra $\sigma(N_{\mu_j}) = \mathrm{supp}(\mu_j)$ (by 8.28.1) are the same (say $K := \sigma(N_{\mu_j})$) and that ρ_{N_1} is unitary equivalent to ρ_{N_2} via U. Let $f := U(1) \in L^2(\mu_2)$, i.e. $|f|^2 \in L^1(\mu_2)$. Then $U\, g = U\, M_g\, 1 = M_g\, U\, 1 = g\, f$ for all $g \in C(K)$ and since U is an isometry we have $\int |g|^2\, d\mu_1 = \int |g|^2 |f|^2\, d\mu_2$. Because of the uniqueness of the Riesz representation 5.3.4 we have $\mu_1 = |f|^2\, \mu_2$, where $|f|^2 \in L^1(\mu_2)$.

This raises the question, which measures μ_1 can be writen as $f\, \mu_2$ with $f \in L^1(\mu_2)$.

8.33 Theorem of Radon Nikodym.

Let (X, Ω, μ) be a σ-finite measure space and ν a $\mathbb{C}$-valued measure on (X, Ω). Then t.f.a.e.:

1. $\forall B \in \Omega : (\mu(B) = 0 \Rightarrow \nu(B) = 0)$;

$\Leftrightarrow$ 2. $\exists! f \in L^1(X, \Omega, \mu)$: $\nu(B) = \int_B f\, d\mu$ *for all* $B \in \Omega$.

Under this equivalent assumptions, ν is called ABSOLUTELY CONTINUOUS *with respect to μ, the function f is called the* RADON-NIKODYM DERIVATIVE*, and is also denoted by $\frac{d\nu}{d\mu}$. Furthermore $f\, g \in L^1(\mu)$ for all $g \in L^1(|\nu|)$ and we have:*

$$\int g\, d\nu = \int g\, \frac{d\nu}{d\mu}\, d\mu.$$

For a proof, see [**10**, S505].

As a special case one shows for example in Analysis that – provided the derivative g' of g is Riemann-integrable – one has for Riemann-Stieltjes integrals:

$$\int_a^b f(x)\,dg(x) = \int_a^b f(x)\,g'(x)\,dx.$$

8.34 Proposition.

Two positive measures on $\mathbb{C}$ with compact support are mutually absolutely continuous (we then write $\mu_1 \sim \mu_2$) iff the multiplication operators N_{μ_1} on $L^2(\mu_1)$ and N_{μ_2} on $L^2(\mu_2)$ are unitary equivalent.

Proof. ($\Leftarrow$) We have shown in $\boxed{8.32}$ that the unitary equivalence of N_{μ_1} and N_{μ_2} implies the mutual absolute continuity of measures μ_1 and μ_2.

($\Rightarrow$) Let the measures μ_1 and μ_2 be mutually absolutely continuous and $0 \leqslant f := \frac{d\mu_1}{d\mu_2} \in L^1(\mu_2)$ the Radon-Nikodym derivative. If $g \in L^1(\mu_1)$, then $f\,g \in L^1(\mu_2)$ and $\int f\,g\,d\mu_2 = \int g\,d\mu_1$. So, if $g \in L^2(\mu_1)$, then $|g|^2 \in L^1(\mu_1)$, hence $f\,|g|^2 \in L^1(\mu_2)$ and thus $\sqrt{f}\,|g| \in L^2(\mu_2)$ and $\|\sqrt{f} g\|_2 = \|g\|_2$, i.e. the mapping $U : L^2(\mu_1) \to L^2(\mu_2)$, $g \mapsto \sqrt{f}\,g$ is an isometry. Since obviously $\frac{d\mu_1}{d\mu_2} \cdot \frac{d\mu_2}{d\mu_1} = 1$, the multiplication with $\frac{1}{\sqrt{f}}$ is the inverse to U. For $g \in L^2(\mu_2)$ we have

$$U\,N_{\mu_1}\,U^{-1}\,g = \sqrt{f}\cdot \mathrm{id}\cdot\frac{1}{\sqrt{f}}\cdot g = \mathrm{id}\cdot g = N_{\mu_2} g$$

and hence $U\,N_{\mu_1}\,U^{-1} = N_{\mu_2}$. □

8.35 Theorem. Diagonalization of normal operators.

Let N be a normal operator on H. Then there is a measure space (X, Ω, μ) and a function $f \in L^\infty(X, \Omega, \mu)$, so that N is unitary equivalent to the multiplication operator with f on $L^2(X, \Omega, \mu)$. If H is separable, then the measure μ is σ-finite.

Proof.

$\boxed{8.30} \Rightarrow \exists H_i < H,$ closed, invariant :

$$H \cong \bigoplus_i H_i \text{ and } N \sim \bigoplus_i N_i \text{ with } N_i := N|_{H_i} \text{ cyclic}$$

$\boxed{8.31} \Rightarrow \exists \mu_i$ measure on $X_i := \sigma(N_i) \subseteq \sigma(N) : N_i \sim N_{\mu_i}$.

Let $X := \bigsqcup X_i$, $\mathcal{B} := \{B \subseteq X : B \cap X_i \in \mathcal{B}(X_i)\}$, $\mu(B) := \sum_i \mu_i(B \cap X_i)$

$$U : L^2(X, \mathcal{B}, \mu) \to \bigoplus_i L^2(\mu_i),\ g \mapsto \bigsqcup g|_{X_i} \text{ is an isometric isomorphism.}$$

Let $f := \bigsqcup_i \mathrm{id}_{X_i}$, i.e. $f|_{X_i} := \mathrm{id}$. Then $\bigoplus_i N_{\mu_i} \overset{U}{\sim} M_f$ and

$f^{-1}(W) \cap X_i = W \cap X_i \in \mathcal{B}(X_i)$ for all open $W \subseteq X$, i.e. f is measurable;

$f(X) = \bigcup_i X_i \subseteq \sigma(N)$, hence f is bounded, thus $f \in L^\infty(X, \mathcal{B}, \mu)$ and

$$N \sim \bigoplus_i N_i \sim \bigoplus_i N_{\mu_i} \overset{U}{\sim} M_f.$$

If H is separable, only countable many H_i are non-zero. Thus X is σ-finite because $\mu(X_i) = \mu_i(X_i) = P_{h_i,h_i}(X_i) \leqslant \|h_i\|^2 = 1$ for a normed cyclic vector h_i by 8.8.4 and 8.31. □

8.36 Proposition.

Let $N_j \in L(H_j)$ be normal operators, and $B \in L(H_1, H_2)$ so that $B\,N_1 = N_2\,B$. Then $\overline{\operatorname{img} B}$ is N_2-invariant, $(\ker B)^\perp$ is N_1 invariant, and $N_1|_{(\ker B)^\perp}$ and $N_2|_{\overline{\operatorname{img} B}}$ are unitary equivalent.

Proof. As in 7.39.4, we show the following:

1. For $h_1 \in H_1$ we have $N_2 B h_1 = B N_1 h_1 \in \operatorname{img} B$. Thus also the closure of $\operatorname{img} B$ is N_2-invariant.

2. We have $B N_1 h_1 = N_2 B h_1 = N_2 0 = 0$ for each $h_1 \in \ker B$, thus $\ker B$ is N_1-invariant and also N_1^*-invariant according to the Fugledge-Putnam Theorem 8.14, so $(\ker B)^\perp$ is also N_1-invariant.

$$\begin{tikzcd}
H_1 \arrow[rrr, "B"] \arrow[ddd, "N_1"'] & & & H_2 \arrow[ddd, "N_2"] \\
& (\ker B)^\perp \arrow[ul, hook'] \arrow[r, tail, two heads, "B"] \arrow[d, "N_1|_{(\ker B)^\perp}"'] & \overline{\operatorname{img} B} \arrow[ur, hook] \arrow[d, "N_2|_{\overline{\operatorname{img} B}}"] & \\
& (\ker B)^\perp \arrow[dl, hook] \arrow[r, tail, two heads, "B"'] & \overline{\operatorname{img} B} \arrow[dr, hook'] & \\
H_1 \arrow[rrr, "B"'] & & & H_2
\end{tikzcd}$$

Hence the inner rectangle of this commuting diagram is well-defined.

3. Since $B|_{(\ker B)^\perp}$ is injective and $\operatorname{img} B|_{(\ker B)^\perp} = \operatorname{img} B$, we may assume w.l.o.g. that B is injective with dense image. Let $B = U\,|B|$ be the polar decomposition 7.24 of B with the positive operator $|B| = \sqrt{B^* B}$ and $\operatorname{img} U = \overline{\operatorname{img} B} = H_2$, as well as $(\operatorname{img} |B|)^\perp = \ker |B| = \ker U = \ker B = \{0\}$. Thus $\operatorname{img} |B|$ is dense in H_1 and $U : H_1 \to H_2$ is a surjective isometry. Furthermore:

$$\begin{tikzcd}
H_1 \arrow[rr, tail, two heads, "B"] \arrow[dr, "|B|"] \arrow[ddd, "N_1"'] & & H_2 \arrow[ddd, "N_2"] \\
& H_1 \arrow[ur, "U"'] \arrow[d, "N_1"'] & \\
& H_1 \arrow[dr, "U"] & \\
H_1 \arrow[rr, tail, two heads, "B"'] \arrow[ur, "|B|"] & & H_2
\end{tikzcd}$$

$$N_2\,B = B\,N_1 \Rightarrow B^*\,N_2^* = N_1^*\,B^* \xRightarrow{8.14} B^*\,N_2 = N_1\,B^*$$
$$\Rightarrow N_1\,B^*\,B = B^*\,N_2\,B = B^*\,B\,N_1$$

So $|B|^2 = B^* B \in \{N_1\}^k$ and by 8.15 we have $|B| = \sqrt{|B|^2} \in \{|B|^2\}^{kk} \subseteq (\{N_1\}^k)^{kk} = \{N_1\}^k$. Consequently,

$$N_2\,U\,|B| = N_2\,B = B\,N_1 = U\,|B|\,N_1 = U\,N_1\,|B|,$$

i.e. $N_2\,U = U\,N_1$ on the dense image of $|B|$, hence everywhere. □

8.37 Corollary.

Similar normal operators are unitary equivalent. □

Two operators N_1 and N_2 are called SIMILAR, if $N_2\,B = B\,N_1$ for some invertible bounded linear mapping B.

8.38 Corollary.

Let A be a C^*-subalgebra of $L(H)$ which is additionally closed with respect to the WOT. Then A is the closure with respect to the norm of the subspace generated by the orthogonal projections in A.

Proof. We have to show that *every $a \in A$ can be approximated in the operator norm by linear combinations of orthogonal projections $P \in A$*: Since A is a C^*-algebra also $\Re e(a) = \frac{1}{2}(a + a^*)$ and $\Im m(a) = \frac{1}{2i}(a - a^*)$ are in A. Thus w.l.o.g. $a \in A$ is Hermitian. By 8.25, the Riemann-Stieltjes sums $\sum_j t_{j-1}(p_{t_j} - p_{t_{j-1}})$ converge to a, where $p_t := P((-\infty, t))$. So we only have to show that the orthogonal projections p_t are in A. The characteristic function $\chi_{(-\infty,t)}$ is a pointwise limit of a monotonously increasing sequence of continuous functions $f_n \in C(\mathbb{R})$. Therefore $f_n \to \chi_{(-\infty,t)}$ in the weak topology by the Theorem on Dominated Convergence and hence $A \supseteq C^*(a) \ni f_n(a) \to \chi_{(-\infty,t)}(a) = p_t$ converges in the WOT. □

Commutants and von Neumann algebras

Our goal is to determine for normal operators $N \in L(H)$ on Hilbert spaces H with spectral measure P, the kernel and the image of the function calculus

$$\rho_N : \mathrm{Borel}_b(\sigma(N), \mathbb{C}) \to \{N\}^{kk} \subseteq L(H), \quad f \mapsto f(N) := \int_{\sigma(N)} f \, dP$$

in order to obtain a faithful representation (a functional calculus) by factoring out the kernel. Since the functional calculus is also continuous with respect to the WOT, we should examine this topology more closely.

8.39 Lemma. Functionals being continuous with respect to operator topologies.

Let $\ell : L(H) \to \mathbb{C}$ be a linear functional. T.f.a.e.:

1. *The functional ℓ is SOT-continuous;*

$\Leftrightarrow$ 2. *The functional ℓ is WOT-continuous;*

$\Leftrightarrow$ 3. *There are finite many h_j and k_j in H with $\ell(T) = \sum_j \langle Th_j, k_j\rangle$.*

Proof. (1 $\Leftarrow$ 2 $\Leftarrow$ 3) is trivial.

(1 $\Rightarrow$ 3) Let ℓ be continuous with respect to the SOT. Then there are finite many h_j with $|\ell(T)| \leqslant \sum_{j=1}^n \|Th_j\|$ for all $T \in L(H)$. Because of the Cauchy-Schwarz inequality [**18**, 6.2.1], we have

$$\sum_{j=1}^n \|Th_j\| = \sum_{j=1}^n 1 \cdot \|Th_j\| \leqslant \sqrt{n} \cdot \Big(\sum_{j=1}^n \|Th_j\|^2\Big)^{1/2} = \Big(\sum_{j=1}^n \|T(\sqrt{n}\, h_j)\|^2\Big)^{1/2}.$$

If one replaces h_j by $\sqrt{n}\, h_j$, then for the seminorm

$$p : T \mapsto \Big(\sum_{j=1}^n \|Th_j\|^2\Big)^{1/2}$$

we have $|\ell(T)| \leqslant p(T)$.

$$\begin{tikzcd}
L(H) \arrow[r, two heads, "\pi"] \arrow[dr, "\ell"'] & H_0 \arrow[r, hook] \arrow[d, dotted, "\ell_0"] & \bigoplus^n H \arrow[dl, dotted, "\tilde{\ell}_0"] \\
& \mathbb{C} &
\end{tikzcd}$$

Let the linear mapping $\pi : L(H) \to \bigoplus^n H$ be given by $\pi(T) := \bigoplus_j Th_j$ and H_0 be its image, then $p(T) = \|\pi(T)\|$. Due to the implications $\pi(T) = 0 \Rightarrow 0 = p(T) \geqslant |\ell(T)| \Rightarrow \ell(T) = 0$, ℓ factors over π to a linear functional $\ell_0 : H_0 \to \mathbb{C}$ and $|\ell_0(\pi(T))| = |\ell(T)| \leqslant p(T) = \|\pi(T)\|$ holds, so ℓ_0 is extendable by 5.1.5 to a continuous linear functional on $\bigoplus^n H$ and there is a vector $\oplus_j k_j$ in the Hilbert space $\bigoplus^n H$ with

$$\ell(T) = \ell_0(\pi(T)) = \tilde{\ell_0}(\pi(T)) = \langle \oplus_j T h_j, \oplus_j k_j \rangle = \sum_j \langle T h_j, k_j \rangle. \quad \square$$

8.40 Corollary. The closure with respect to operator topologies.

Let A be a convex subset of $L(H)$, then the WOT-closure coincides with the SOT-closure of A.

Proof. 8.39 and 5.4.8. $\square$

8.41 Definition.

For $n \in \mathbb{N}$ we define a C^*-algebra homomorphism $\Delta^n : L(H) \to L(\bigoplus^n H)$ by

$$\Delta^n(T) := \bigoplus^n T : \oplus_{j=1}^n h_j := (h_j)_{j=1}^n \mapsto (T h_j)_{j=1}^n.$$

Lemma.

Let A be a subalgebra of $L(H)$ with unit.
Then the SOT-closure of A is given by all those $T \in L(H)$, s.t. for each finite n each closed $\Delta^n(A)$-invariant subspace of $\bigoplus_{j=1}^n H$ is also $\Delta^n T$-invariant.

Proof. ($\subseteq$) Let $T \in L(H)$ be an operator in the SOT-closure of A. Then there is a net $T_i \in A$ which converges pointwise towards T. Now let E be a closed $\Delta^n(A)$-invariant subspace of $\bigoplus_{j=1}^n H$. This is then in particular $\Delta^n T_i$-invariant and thus also $\Delta^n T$-invariant.

($\supseteq$) $T \in L(H)$ satisfies the condition on the invariant subspaces. Let $h_j \in H$ and $\varepsilon > 0$. We have to show the existence of an $S \in A$, with $\|(T - S)h_j\| < \varepsilon$ for all $j \in \{1, \dots, n\}$. Let E be the closure of the linear subspace $\Delta^n(A)(\oplus_j h_j) \subseteq \bigoplus^n H$. Since A is an algebra, E is a $\Delta^n(A)$-invariant subspace, so is also $\Delta^n T$-invariant by assumption. Since $1 \in A$, we have $\oplus_j h_j \in E$ and thus $\oplus_j T h_j = (\Delta^n T)(\oplus_j h_j) \in E$ and, since $\Delta^n(A)(\oplus_j h_j)$ dense is in E, there is an $S \in A$ with $\sum_j \|(T - S)h_j\|^2 < \varepsilon^2$. $\square$

8.42 Remark.

For $A \subseteq L(H)$, the commutant A^k is SOT-closed because of the lemma in 8.9, see 6.31.

If A is closed with respect to $*$, then A^k is a C^*-algebra:
We only have to prove the $*$-closedness of A^k. Let $b \in A^k$ and $a \in A$. Since $a^* \in A$, we have $b^* a = (a^* b)^* = (b a^*)^* = a b^*$, so $b^* \in A^k$.

Furthermore, a $*$-closed subset A is a maximal Abelian subset (or even C^*-algebra) if and only if $A = A^k$ holds:
($\Leftarrow$) Let $A \subseteq B$ with Abelian B. Then $B \subseteq B^k \subseteq A^k = A$, so A is maximal Abelian.
($\Rightarrow$) Let A be Abelian, i.e. $A \subseteq A^k$. Since A is $*$-closed, A^k is a C^*-algebra and it suffices to show that $\mathfrak{Re}(A^k) \subseteq A$. Let $x \in A^k$ be Hermitian and A_x be the C^*-algebra generated by A and x. Because of $x \in A^k$ it is Abelian, and because of the maximality we have $x \in A_x = A$.

8.43 Lemma.

Let $A \subseteq L(H)$. Then

$$A^{kk} = (\Delta^n)^{-1}((\Delta^n A)^{kk})$$

holds

Proof. The following holds:

$$t = (t_{i,j})_{i,j} \in (\Delta^n A)^k \Leftrightarrow \forall a \in A \forall i,j : t_{i,j}\, a = a\, t_{i,j} \Leftrightarrow \forall i,j : t_{i,j} \in A^k,$$

because

$$\begin{pmatrix} t_{1,1} & \dots & t_{1,n} \\ \vdots & \ddots & \vdots \\ t_{n,1} & \dots & t_{n,n} \end{pmatrix} \cdot \begin{pmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{pmatrix} = \begin{pmatrix} t_{1,1}\, a & \dots & t_{1,n}\, a \\ \vdots & \ddots & \vdots \\ t_{n,1}\, a & \dots & t_{n,n}\, a \end{pmatrix}$$

$$\begin{pmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{pmatrix} \cdot \begin{pmatrix} t_{1,1} & \dots & t_{1,n} \\ \vdots & \ddots & \vdots \\ t_{n,1} & \dots & t_{n,n} \end{pmatrix} = \begin{pmatrix} a\, t_{1,1} & \dots & a\, t_{1,n} \\ \vdots & \ddots & \vdots \\ a\, t_{n,1} & \dots & a\, t_{n,n} \end{pmatrix}.$$

Consequently,

$$\begin{aligned} \Delta^n a \in (\Delta^n A)^{kk} &\Leftrightarrow \forall t = (t_{i,j})_{i,j} \in (\Delta^n A)^k : t\, \Delta^n(a) = \Delta^n(a)\, t \\ &\Leftrightarrow \forall t_{i,j} \in A^k : t_{i,j}\, a = a\, t_{i,j} \Leftrightarrow a \in A^{kk}. \quad \square \end{aligned}$$

8.44 Double Commutant Theorem, by Neumann 1929.

Let A be a C^*-subalgebra of $L(H)$, then A^{kk} is the closure of A with respect to the SOT or the WOT, i.e.

$$A^{kk} = \overline{A}^{SOT} = \overline{A}^{WOT}.$$

Proof.

($\subseteq$)

$$\begin{aligned} T \in A^{kk} &\overset{\boxed{8.43}}{\Longleftrightarrow} \Delta^n T \in (\Delta^n A)^{kk} \\ &:\Leftrightarrow \Delta^n T\, P = P\, \Delta^n T \text{ for all } P \in (\Delta^n A)^k \\ &\Rightarrow \Delta^n T\, P = P\, \Delta^n T \text{ for all ortho-projections } P \in (\Delta^n A)^k \\ &\xRightarrow{\boxed{7.39.4}\text{, cf. }\boxed{7.41}} \text{Each closed } \Delta^n A\text{-invariant subspace is } \Delta^n T\text{-invariant} \\ &\overset{\boxed{8.41}}{\Longleftrightarrow} T \in \overline{A}^{SOT} \overset{\boxed{8.40}}{=\!=} \overline{A}^{WOT} \end{aligned}$$

($\supseteq$) Being a commutant $A^{kk} \supseteq A$ is closed with respect to SOT and by $\boxed{8.40}$ also with respect to WOT, so $\overline{A}^{WOT} = \overline{A}^{SOT} \subseteq A^{kk}$. $\square$

8.45 Definition.

A VON NEUMANN ALGEBRA A in $L(H)$ is a C^*-subalgebra, with $A^{kk} = A$, i.e. it is closed with respect to the SOT (or WOT).

Therefore $\{N\}^{kk}$ is the smallest (Abelian) von Neumann algebra containing the normal operator N. By $\boxed{8.44}$ this is the WOT-closure of $C^*(N)$ or also of $\{p(N, N^*) : p \in \mathbb{C}[z, \overline{z}]\}$, because this lies dense in $C^*(N)$.

8.46 Proposition.

Let (X, Ω, μ) be a σ-finite measure space and

$$A_\mu := \{M_f : f \in L^\infty(\mu)\} \subseteq L(L^2(\mu)),$$

be the subalgebra generated by the multiplication operators. Then $A_\mu = {A_\mu}^k$, hence is an Abelian von Neumann algebra in $L(L^2(\mu))$.

If μ is a finite measure, then the representation $f \mapsto M_f$, $L^\infty(\mu) \to A_\mu$, from 8.27 is a homeomorphism with respect to the weak topology $\sigma(L^\infty(\mu), L^1(\mu))$ and the WOT on A_μ.

Let μ be a positive Borel measure on $\mathbb{C}$ with compact support. Then $\{N_\mu\}^k = A_\mu{}^k$ and thus $\{N_\mu\}^{kk} = A_\mu$, i.e.

$$L^\infty(\mu) \xrightarrow[\cong]{M} A_\mu = \{N_\mu\}^{kk} \hookrightarrow L(L^2(\mu))$$

Proof. ($A_\mu = A_\mu^k$) Since A_μ is Abelian, $A_\mu \subseteq A_\mu^k$. Conversely, for $a \in A_\mu^k$ we have to show that $a = M_f$ for some $f \in L^\infty(\mu)$. W.l.o.g. $a \neq 0$.

Let first $\mu(X) < \infty$. Then $1 \in L^2(\mu)$. For $f := a(1) \in L^2(\mu)$ we have $a(g) = a(M_g 1) = M_g(a1) = M_g f = g\,f$ holds for $g \in L^\infty(\mu) \subseteq L^2(\mu)$. So $\|f\,g\|_2 = \|a(g)\|_2 \leqslant \|a\|\,\|g\|_2$. In particular, for $g := \chi_{X_0}$ with $X_0 := \{x \in X : |f(x)| \geqslant 2\|a\|\}$. we obtain

$$\|a\|^2\mu(X_0) = \|a\|^2\,\|g\|^2 \geqslant \|a(g)\|^2 = \|f\,g\|^2 = \int_{X_0} |f|^2\,d\mu \geqslant 4\,\|a\|^2\,\mu(X_0).$$

So $\mu(X_0) = 0$, i.e. $[f] \in L^\infty(\mu)$. Since $a = M_f$ holds on the dense subspace $L^\infty(\mu)$ of $L^2(\mu)$, it holds on all of $L^2(\mu)$.

Let now $X = \bigsqcup_n X_n$ with $\mu(X_n) < \infty$. For B with $\mu(B) < \infty$, $L^2(\mu|_B) \cong \{f \in L^2(\mu) : f = 0 \text{ outside of } B\}$ is a-invariant because $a(f) = a(\chi_B \cdot f) = \chi_B \cdot a(f) \in L^2(\mu|_B)$ for $f \in L^2(\mu|_B)$ since $a \in A_\mu^k$. Let a_B be the restriction of a to $L^2(\mu|_B)$. By the first part there is an $f_B \in L^\infty(\mu|_B)$ with $a_B = M_{f_B}$. We write f_n for f_{X_n} and define $f := \bigsqcup_n f_n$, i.e. $f|_{X_n} := f_{X_n}$. Then f is a well-defined measurable function on X and $\|f_n\|_\infty = \|M_{f_n}\| = \|a_{X_n}\| \leqslant \|a\|$. So $\|f\|_\infty \leqslant \|a\|$ and obviously $a = M_f$.

Let μ again be a finite measure.

(Injectivity) We have $f \mapsto M_f$ is injective since $1 \in L^2(\mu)$.

(Homeomorphy) Let $f_i \in L^\infty(\mu)$ be a net. Then this converges to 0 in the weak topology $\sigma(L^\infty, L^1)$ if and only if for all $g \in L^1(\mu)$ the following holds: $\int f_i\,g\,d\mu \to 0$. These g are exactly the products $h_1 \cdot \overline{h_2}$ with $h_1, h_2 \in L^2(\mu)$, because by Hölder's inequality $h_1 \cdot \overline{h_2} \in L^1(\mu)$, and vice versa, both $h_2 := \sqrt{|g|}$ and $h_1 := \operatorname{sign}(g)\,h_2$ are in $L^2(\mu)$. So the convergence statement is equivalent to $\langle M_{f_i} h_1, h_2\rangle = \int f_i\,h_1\,\overline{h_2}\,d\mu \to 0$, i.e. to $M_{f_i} \to 0$ in the WOT on $L(L^2(\mu))$.

($\{N_\mu\}^k = A_\mu{}^k$) Let μ be a positive Borel measure on $\mathbb{C}$ with compact support X. By 8.14, $\{N_\mu\}^k = \{N_\mu, N_\mu^*\}^k = \{M_p : p \in \mathbb{C}[z, \overline{z}]\}^k = \{M_f : f \in C(X)\}^k$, since the set of polynomials $p \in \mathbb{C}[z, \overline{z}]$ is dense in $C(X)$. We may consider $L^1(\mu)$ as subspace of $C(X)'$ via the isometry embedding $L^1(\mu) \hookrightarrow C(X)'$, $f \mapsto f\,d\mu$: In fact, $|\int g\,f\,d\mu| \leqslant \|g\|_\infty\,\|f\|_1$ and $\|f\,d\mu\| := |f\,d\mu|(X) = \int 1\,|f|\,d\mu = \|f\|_1$. Thus, for each $f \in L^\infty = (L^1)'$ there exists by Hahn-Banach a $\tilde{f} \in C(X)''$ with $\tilde{f}|_{L^1} = f$. By 8.11, $\delta : C(X) \to \sigma(C(X)'', C(X)')$ has dense image and thus for given $f_1, \dots, f_n \in L^1$ and $\varepsilon > 0$ there is a $g \in C(X)$ with $\varepsilon > |(\tilde{f} - \delta(g))(f_i\,d\mu)| = |\int f\,f_i\,d\mu - \int g\,f_i\,d\mu|$, hence $C(X)$ is dense in $\sigma(L^\infty(\mu), L^1(\mu))$. And since $f \mapsto M_f$ is a homeomorphism $\sigma(L^\infty, L^1) \cong (A_\mu, \text{WOT})$, we have $A_\mu^k = \{M_f : f \in L^\infty(\mu)\}^k = \{M_f : f \in C(X)\}^k = \{N_\mu\}^k$. □

Remark.

We aim at modifying the function calculus

$$\rho : \operatorname{Borel}_b(\sigma(N)) \to L(H), \quad f \mapsto \int_{\sigma(N)} f\,dP$$

from 8.15 so that it becomes a bijection. In order to achieve this, we first try to find a Borel measure μ on $\sigma(N)$, so that ρ factores over the quotient map π : $\mathrm{Borel}_b(\sigma(N),\mathbb{C}) \to L^\infty(\mu)$ to an injective mapping

$$\begin{tikzcd} \mathrm{Borel}_b(\sigma(N)) \arrow[rr, two heads, "\pi"] \arrow[dr, "\rho"'] & & L^\infty(\mu) \arrow[dl, dotted, tail] \\ & L(H) & \end{tikzcd}$$

i.e. we should have $\ker\rho = \ker\pi = \{f : f = 0\ \mu\text{-a.e.}\}$ and, because of $P = \rho \circ \chi : \mathcal{B}(\sigma(N)) \to \mathrm{Borel}_b(\sigma(N)) \to L(H)$, at least

$$\begin{aligned} \{B \in \mathcal{B}(\sigma(N)) : P(B) = 0\} &= \ker(P) = \ker(\rho\circ\chi) = \chi^{-1}(\ker(\rho)) \\ &= \chi^{-1}(\{f : f = 0\ \mu\text{ -a.e.}\}) \\ &= \{B \in \mathcal{B}(\sigma(N)) : \mu(B) = 0\}. \end{aligned}$$

We therefore define:

8.47 Definition.

A SCALAR-VALUED SPECTRAL MEASURE for a normal operator N is a measure $\mu \geqslant 0$ on $\sigma(N)$, which vanishes on exactly those Borel sets where the spectral measure of N does.

A possibility to find such a measure is to take a vector $h \in H$ and consider $\mu_h := P_{h,h}$. For these

$$\mu_h(B) := P_{h,h}(B) = \langle P(B)h, h\rangle = \|P(B)h\|^2.$$

holds. Thus, μ_h is scalar-valued spectral measure if and only if

$$\forall B \in \mathcal{B}(\sigma(N)) : P(B)h = 0 \Rightarrow P(B) = 0.$$

This leads to the definition:

Let $A \subseteq L(H)$. Then an $h \in H$ is called SEPARATING VECTOR for A, if

$$\forall a \in A : ah = 0 \Rightarrow a = 0.$$

An $h \in H$ is a SEPARATING VECTOR for the normal operator $N \in L(H)$ if h is separating for the von Neumann algebra $\{N\}^{kk}$ generated by N.

8.48 Lemma.

Let $h \in H$ be a separating vector for a normal operator N and P its spectral measure. Then the measure $\mu_h := P_{h,h}$ is a scalar-valued spectral measure for N.

Proof. h separating for $N :\Leftrightarrow h$ separating for $\{N\}^{kk} \supseteq \{P(B) : B\}$ (because of 8.15), so $\forall B \in \mathcal{B}(\sigma(N))$: $(\mu_h(B) = \|P(B)h\|^2 = 0 \Rightarrow P(B) = 0)$, i.e. μ_h is a scalar-valued spectral measure for N. □

Cyclic versus separating vectors.

Let $\dim H > 1$.

1. If $A = L(H)$, then all $h \neq 0$ are cyclic vectors, but no $h \in H$ is separating.
2. If $A = \mathbb{C}$, then A has no cyclic vectors, but each $h \neq 0$ is separating.

Our next task is to prove the existence of separating vectors.

8.49 Lemma.

Let h be a cyclic vector for A. Then h is a separating vector for A^k.

Proof. $b \in A^k \Rightarrow \ker b$ is A-invariant (in fact: $ba(\ker b) = ab(\ker b) = \{0\}$); Let $bh = 0$, i.e. $h \in \ker b \Rightarrow Ah \subseteq \ker b \Rightarrow \ker b = H$, because Ah is dense $\Rightarrow b = 0$, i.e. h is separating for A^k. □

8.50 Corollary.

Let $A \subseteq L(H)$ be Abelian. Then every cyclic vector of A is also separating.

Proof. Since A is Abelian, $A \subseteq A^k$ is valid and because h is separating for A^k by 8.49, it is also for the subset A. □

8.51 Corollary.

Let H be separable. Then each Abelian C^*-subalgebra of $L(H)$ has a separating vector.

Proof. According to Zorn's Lemma, A is contained in a maximal Abelian C^*-algebra. Since a separating vector is also separating for each subset, we may assume without loss of generality that A is maximal Abelian and thus $A = A^k$ by 8.42.

By 7.32, an orthogonal decomposition $H = \bigoplus_n H_n$ exists into A-invariant subspaces H_n with cyclic, and by 8.50, separating unit vectors $h_n \in H_n$. Since H is separable, the index set is countable (i.e. without loss of generality $\mathbb{N}$). Let $h_\infty := \sum_{n=1}^\infty \frac{1}{\sqrt{2^n}} h_n$. Then $\|h_\infty\|^2 = \sum_{n=1}^\infty \frac{1}{2^n} = 1$, hence $h_\infty \in H$. Suppose $ah_\infty = 0$ for some $a \in A$. Let P_n be the orthogonal projection on H_n. Since each H_n is A-invariant, $P_n \in A^k = A$ by 7.39.4 and thus $0 = P_n a h_\infty = a P_n h_\infty = \frac{1}{\sqrt{2^n}} a h_n$, hence $a = 0$, i.e. h_∞ is separating. □

8.52 Corollary.

Let $N \in L(H)$ be normal and H be separable, then there is a separating vector for N.

Proof. Since the set $\{N\}^{kk}$ is Abelian by 6.31, it has a separating vector h by 8.51. □

This corollary is the reason we will from now on assume that:
all occurring Hilbert spaces are separable.

8.53 Localization of the function calculus.

Let H be separable and $N \in L(H)$ normal.

For $h \in H$, let $\mu_h := P_{h,h}$ and H_h be the closure of $\{N\}^{kk}h$ in H. This is obviously $\{N\}^{kk}$-invariant hence also N-invariant and thus the restriction of N is an operator $N_h := N|_{H_h} \in L(H_h)$.

$$\begin{tikzcd}
\{N\}^{kk}h \arrow[r, hook, two heads] \arrow[d, dotted] & H_h \arrow[r, hook] \arrow[d, dotted, "N_h"] & H \arrow[d, "N"] \\
\{N\}^{kk}h \arrow[r, hook, two heads] & H_h \arrow[r, hook] & H
\end{tikzcd}$$

Lemma.

We have the following commutative diagram consisting of $*$-homomorphisms:

$$\begin{tikzcd}
\mathrm{Borel}_b(\sigma(N)) \arrow[r, "\rho_N"] \arrow[d, "\mathrm{inkl}^*"'] & \{N\}^{kk} \arrow[d, dotted, "\rho_h"] & \\
\mathrm{Borel}_b(\sigma(N_h)) \arrow[r, "\rho_{N_h}"'] & \{N_h\}^{kk} \arrow[r, hook] & L(H_h)
\end{tikzcd}$$

where $\rho_h : a \mapsto a|_{H_h}$ is WOT-continuous.

Proof. ($\rho_h : \{N\}^{kk} \to L(H_h)$, $a \mapsto a|_{H_h}$, is well-defined) This is obvious since $H_h = \overline{\{N\}^{kk}h}$ is obviously $\{N\}^{kk}$-invariant.

(ρ_h is WOT-continuous) If $a_i \to a_\infty$ in $\{N\}^{kk}$ with respect to the WOT, then $\langle a_i v, w\rangle \to \langle a_\infty v, w\rangle$ holds for all $v, w \in H$, in particular, for those in $H_h \subseteq H$, i.e. $\rho_h(a_i) = a_i|_{H_h} \to a_\infty|_{H_h} = \rho_h(a_\infty)$ in $L(H_h)$ with respect to the WOT.

($\rho_h(\{N\}^{kk}) \subseteq \{N_h\}^{kk}$) By 8.45, $\{N\}^{kk} = \overline{\{p(N, N^*) : p \in \mathbb{C}[z, \overline{z}]\}}^{\mathrm{WOT}}$, i.e. for $a \in \{N\}^{kk}$ there exists a net of such polynomials p_i with $p_i(N, N^*) \to a$ with respect to the WOT. By the previous point, $\{N_h\}^{kk} \ni p_i(N_h, N_h^*) = \rho_h(p_i(N, N^*)) \to \rho_h(a)$ in the WOT, so $\rho_h(a) \in \{N_h\}^{kk}$ by the Double Commutant Theorem 8.44.

(The diagram commutes) Let $f \in \mathrm{Borel}_b(\sigma(N))$. By 8.11 (compare with the proof of 8.13) there exists a net of polynomials $p_i \in \mathbb{C}[z, \overline{z}]$ with $\int p_i\, d\mu \to \int f\, d\mu$ for all $\mu \in M(\sigma(N))$. Since $\sigma(N_h) \subseteq \sigma(N)$, this also holds for all $\mu \in M(\sigma(N_h))$. By 8.15 both $p_i(N, N^*) \to f(N)$ and $p_i(N_h, N_h^*) \to f(N_h)$ converge with respect to the WOT. Because of $p_i(N_h, N_h^*) = \rho_h(p_i(N, N^*)) \to \rho_h(f(N))$ with respect to the WOT, we obtain $\rho_h(f(N)) = f(N_h)$. □

8.54 Lemma.

We have the following commutative diagram of $*$-homomorphisms:

$$\begin{tikzcd}
\mathrm{Borel}_b(\sigma(N_h)) \arrow[r, two heads, "\rho_{N_h}"] \arrow[d, "\pi_h"] & \{N_h\}^{kk} \arrow[r, hook] \arrow[d, tail, two heads, "\mathrm{conj}_{U_h}"', "\cong"] & L(H_h) \arrow[d, tail, two heads, "\mathrm{conj}_{U_h}", "\text{8.31},\cong"'] \\
L^\infty(\mu_h) \arrow[r, tail, two heads, "M", "\cong,\ \text{8.46}"'] & \{N_{\mu_h}\}^{kk} \arrow[r, hook] & L(L^2(\mu_h))
\end{tikzcd}$$

Where $U_h : H_h \to L^2(\mu_h)$ is the unique bijective isometry from 8.31 that interchanges N_h and N_{μ_h} and maps h to 1. Furthermore, $\mathrm{conj}_{U_h} : a \mapsto U_h\, a\, U_h^{-1}$. The mappings denoted by $\twoheadrightarrow$ are surjective and continuous and those with $\cong$ are even homeomorphisms with respect to $\sigma(\mathrm{Borel}_b, M)$, $\sigma(L^\infty, L^1)$ and the WOT's.

Proof. (h is a cyclic vector for N_h) Since $\{N\}^{kk}$ is the closure of $C^*(N)$ in the SOT by 8.44, we have $\{N\}^{kk}h = \mathrm{ev}_h(\overline{C^*(N)}) \subseteq \overline{\mathrm{ev}_h(C^*(N))} = \overline{C^*(N)h}$ and $C^*(N)h \subseteq \overline{C^*(N_h)h}$, because for $a \in C^*(N)$ there are polynomials $p_i \in \mathbb{C}[z, \overline{z}]$ with $p_i(N, N^*) \to a$ and thus $ah = \lim_i p_i(N, N^*)h = \lim p_i(N_h, N_h^*)h \in \overline{C^*(N_h)h}$. Thus $C^*(N_h)h$ is dense in $H_h = \overline{\{N\}^{kk}h}$, i.e. h is a cyclic vector of the restriction N_h.

(The right arrow is a homeomorphism) By 8.31, $\mu_h := P_{h,h}$ is a measure on $\sigma(N_h)$ so that N_h is unitary equivalent to N_{μ_h} on $L^2(\mu_h)$ with respect to a bijective isometry $U = U_h : H_h \to L^2(\mu_h)$ being uniquely determined by $U_h(h) := 1$. Conjugation $a \mapsto U \circ a \circ U^{-1}$ provides a $*$-isomorphism $L(H_h) \to L(L^2(\mu_h))$, which maps N_h to N_{μ_h} and thus $\{N_h\}^{kk}$ to $\{N_{\mu_h}\}^{kk}$. This is obviously also a homeomorphism with respect to the WOT's.

(The lower arrow is a homeomorphism) According to 8.46, $f \mapsto M_f$ is a surjective isometry $L^\infty \to A_\mu = \{N_{\mu_h}\}^{kk}$ and a homeomorphism with respect to $\sigma(L^\infty, L^1)$ and the WOT.

(Commutativity) The surjective C^*-homomorphism $\pi_h : f \mapsto [f]$ is continuous with respect to $\sigma(\mathrm{Borel}_b, M(\sigma(N_h)))$ and $\sigma(L^\infty, L^1)$, because each $g \in L^1(\mu_h)$ defines a measure $g\, d\mu_h$. Thus, ρ_{N_h} and $\mathrm{conj}_{U_h}^{-1} \circ M \circ \pi_h$ both have the characteristic properties of the function calculus 8.15, hence they coincide. □

8.55 Lemma.

Let $e \in H$ be so that μ_e is a scalar-valued spectral measure for N. The measures μ being absolutely continuous with respect to μ_e are exactly the μ_h for $h \in H$.

Proof. ($\Leftarrow$) Since μ_e is a scalar-valued spectral measure, $\mu_e(B) = 0$ implies $P(B) = 0$ and thus $\mu_h(B) = \langle P(B)h, h\rangle = \|P(B)h\|^2 = 0$.

($\Rightarrow$) By the Theorem 8.33 of Radon-Nikodym $f := \sqrt{\frac{d\mu}{d\mu_e}} \in L^2(\mu_e)$ exists. Let $h := U_e^{-1} f \in H_e$ where $U_e : H_e \to L^2(\mu_e)$ is the isometric isomorphism from 8.31. For every Borel set B we have:

$$\begin{aligned}\mu(B) = \int \chi_B\, d\mu &\overset{\boxed{8.33}}{=\!=} \int \chi_B\, f^2\, d\mu_e = \langle M_{\chi_B} f, f\rangle_{L^2(\mu_e)} \overset{\boxed{8.31}}{=\!=} \langle U_e^{-1} M_{\chi_B} f, U_e^{-1} f\rangle_{H_e} \\ &\overset{\boxed{8.54}}{=\!=} \langle \rho_{N_e}(\chi_B) U_e^{-1} f, U_e^{-1} f\rangle \overset{\boxed{8.53}}{=\!=} \langle \rho_N(\chi_B) U_e^{-1} f, U_e^{-1} f\rangle \\ &= \langle P(B)h, h\rangle = \mu_h(B). \quad \square\end{aligned}$$

8.56 Lemma.

All mappings denoted by $\twoheadrightarrow$ in the following diagram from 8.53 are surjective.

$$\begin{tikzcd} \mathrm{Borel}_b(\sigma(N)) \arrow[r, two heads, "\rho_N"] \arrow[d, "\mathrm{inkl}^*"'] & \{N\}^{kk} \arrow[d, dotted, two heads, "\rho_h"] \\ \mathrm{Borel}_b(\sigma(N_h)) \arrow[r, two heads, "\rho_{N_h}"', "\boxed{8.54}"] & \{N_h\}^{kk} \end{tikzcd}$$

Proof. (Surjectivity below) This holds by 8.54.

(Surjectivity on the right) Because of the commutativity of the diagram and because the path over the left lower vertex is surjective, ρ_h is also surjective.

(Surjectivity above) Let $A := \{f(N) : f \in \mathrm{Borel}_b(\sigma(N))\}$ be the image. Then A is a C^*-algebra by 8.15 and 7.28 with $C^*(N) \subseteq A \subseteq \{N\}^{kk}$. Because of 8.45 it suffices to show that A is WOT-closed:
So let $f_i \in \mathrm{Borel}_b(\sigma(N))$ be a net with $f_i(N) \to a$ in the WOT. Then $a \in \{N\}^{kk}$ by 8.45. Let $h \in H$ be arbitrary. Since ρ_{N_h} is onto, there exists an $f_h \in \mathrm{Borel}_b(\mathbb{C})$ with $a|_{H_h} = f_h(N_h)$. Since $\rho_h : \{N\}^{kk} \to \{N_h\}^{kk}$ is continuous, $f_i(N_h) = \rho_h(f_i(N)) \to \rho_h(a) = a|_{H_h} = f_h(N_h) \in \{N_h\}^{kk}$ in the WOT and thus $f_i \to f_h$ in $\sigma(L^\infty(\mu_h), L^1(\mu_h))$ by 8.46 for each h. Due to Corollary 8.52, there is a separating vector e for $\{N\}^{kk}$ and μ_e is a scalar-valued spectral measure for N by 8.48 with μ_h being absolutely continuous with respect to μ_e by 8.55, i.e. $\exists \frac{d\mu_h}{d\mu_e} \in L^1(\mu_e)$.

Thus $\int_B f_i\, d\mu_h = \int_B f_i \frac{d\mu_h}{d\mu_e}\, d\mu_e \to \int_B f_e \frac{d\mu_h}{d\mu_e}\, d\mu_e = \int_B f_e\, d\mu_h$ for each Borel set B. On the other hand $\int_B f_i\, d\mu_h \to \int_B f_h\, d\mu_h$. Thus $0 = \int_B (f_e - f_h)\, d\mu_h$, i.e. $f_e = f_h$ μ_h-a.e.. Since for each $g \in H_h$ the measure μ_g is absolutely continuous with respect to μ_h by $\boxed{8.55}$, we have $f_e = f_h$ μ_g-a.e. and hence $\langle f_h(N_h)g, g\rangle = \langle f_h(N)g, g\rangle = \int f_h\, d\mu_g = \int f_e\, d\mu_g = \langle f_e(N_h)g, g\rangle$, i.e. $f_h(N_h) = f_e(N_h)$ by $\boxed{7.6.3}$ and in particular $ah = a|_{H_h}h = f_h(N_h)h = f_e(N_h)h = \rho_{N_h}(f_e)(h) = \rho_N(f_e)(h) = f_e(N)h$. Since $h \in H$ was arbitrary, $a = f_e(N)$ holds. $\square$

8.57 Lemma.

We have $\rho_N(f) \in \ker(\rho_h) \Leftrightarrow f = 0$ μ_h-a.e., i.e. $\rho_N|_{\ker(\pi)} : \ker(\pi) \to \ker(\rho_h)$ is well-defined and surjective, where $\pi := \pi_h \circ \mathrm{incl}^*$.

Proof. Let $a \in \{N\}^{kk}$, i.e. $a = f(N) = \rho_N(f)$ for a $f \in \mathrm{Borel}_b(\sigma(N))$ by $\boxed{8.56}$. Then:

$$\begin{aligned} & a = \rho_N(f) \in \ker(\rho_h) \\ \Leftrightarrow\ & 0 = \rho_h(\rho_N(f)) \overset{\boxed{8.56}}{=\!=} \rho_{N_h}(f|_{\sigma(N_h)}) \\ \overset{\boxed{8.54}}{\Longleftrightarrow}\ & f|_{\sigma(N_h)} = 0\ \mu_h\text{-a.e.} \\ \overset{\boxed{8.28}}{\Longleftrightarrow}\ & f = 0\ \mu_h\text{-a.e., because } \mathrm{supp}(\mu_h) = \sigma(N_h). \end{aligned}$$

$$\begin{tikzcd} \ker(\pi) \arrow[r, dotted, two heads] \arrow[d, hook] \arrow[dr, phantom, "\text{pullback}"] & \ker(\rho_h) \arrow[d, hook] \\ \mathrm{Borel}_b(\sigma(N)) \arrow[r, two heads, "\rho_N", "\boxed{8.56}"'] \arrow[d, "\pi=\pi_h\circ\mathrm{inkl}^*"'] & \{N\}^{kk} \arrow[d, "\rho_h"] \\ L^\infty(\mu_h) \arrow[r, tail, two heads, "\cong", "\boxed{8.54}"'] & \{N_h\}^{kk} \end{tikzcd}$$

$\square$

8.58 Proposition.

Let N be normal and $e \in H$. Then t.f.a.e.:

1. *The mapping $\rho_e : \{N\}^{kk} \to \{N_e\}^{kk}$ is a $*$-isomorphism (or at least injective);*

$\Leftrightarrow$ 2. *$\forall f \in \mathrm{Borel}_b(\sigma(N))$: $f(N) = 0 \Leftrightarrow f = 0$ μ_e-a.e..*

$\Leftrightarrow$ 3. *e is separating for $\{f(N) : f \in \mathrm{Borel}_b(\sigma(N))\} = \{N\}^{kk}$;*

$\Leftrightarrow$ 4. *$\mu_e := P_{e,e}$ is a scalar-valued spectral measure for N;*

Proof. ($\boxed{1} \Rightarrow \boxed{2}$) $f(N) = 0 \overset{(\boxed{1})}{\Longleftrightarrow} \rho_e(f(N)) = 0 \overset{\boxed{8.57}}{\Longleftrightarrow} f = 0$ μ_e-a.e..

($\boxed{2} \Rightarrow \boxed{3}$) Let $a \in \{N\}^{kk}$ with $ae = 0$. By $\boxed{8.56}$, $f \in \mathrm{Borel}_b(\sigma(N))$ exists with $f(N) = a$. So $0 = \|ae\|^2 = \langle a^*ae, e\rangle = \langle \rho_N(|f|^2)\, e, e\rangle = \langle \int |f|^2 dP\, e, e\rangle = \int |f|^2\, d\mu_e$. And thus $f = 0$ μ_e-a.e.. Consequently $0 = f(N) = a$ by ($\boxed{2}$).

($\boxed{3} \Rightarrow \boxed{4}$) is $\boxed{8.48}$.

($\boxed{4} \Rightarrow \boxed{1}$) By $\boxed{8.56}$, ρ_e is a surjective $*$-morphism. By $\boxed{8.57}$, $\ker \rho_e = \{f(N) : f = 0\ \mu_e\text{-a.e.}\}$, so ρ_e is also injective, because if $f = 0$ outside a Borel set B with $\mu_e(B) = 0$, so $P(B) = 0$ by ($\boxed{4}$), then $f(N) = \int_B f\, dP = 0$. $\square$

Summary.

$$
\begin{tikzcd}
\ker(\pi) \arrow[r, two heads, "\boxed{8.57}"'] \arrow[d, hook] & \ker(\rho_e) \arrow[r, equal, "\boxed{8.58}"] \arrow[d, hook] & \{0\} \\
\mathrm{Borel}_b(\sigma(N)) \arrow[r, two heads, "\rho_N", "\boxed{8.56}"'] \arrow[d, "\mathrm{inkl}^*"'] \arrow[dd, bend right, "\pi"'] & \{N\}^{kk} \arrow[r, hook] \arrow[d, tail, two heads, dotted, "\rho_e"', "{\cong, \boxed{8.58}}"] & L(H) \\
\mathrm{Borel}_b(\sigma(N_e)) \arrow[r, two heads, "\rho_{N_e}", "\boxed{8.54}"'] \arrow[d, "\pi_e"'] & \{N_e\}^{kk} \arrow[r, hook] \arrow[d, tail, two heads, "\cong"', "\mathrm{conj}_{U_e}"] & L(H_e) \arrow[d, tail, two heads, "{\boxed{8.31}, \cong}"', "\mathrm{conj}_{U_e}"] \\
L^\infty(\mu_e) \arrow[r, tail, two heads, "M", "{\cong, \boxed{8.46}}"'] & \{N_{\mu_e}\}^{kk} \arrow[r, hook] & L(L^2(\mu_e))
\end{tikzcd}
$$

8.59 Theorem. Function calculus.

Let N be a normal operator on a separable Hilbert space H. Then there is an up to equivalence unique scalar-valued spectral measure μ for N.

The function calculus ρ_N from $\boxed{8.15}$ factorizes via $\pi : \mathrm{Borel}_b(\sigma(N)) \twoheadrightarrow L^\infty(\mu)$ to a well-defined (isometric) $*$-isomorphism $\rho : L^\infty(\mu) \to \{N\}^{kk}$, which is also a homeomorphism from the topology $\sigma(L^\infty, L^1)$ to the WOT.

$$
\begin{tikzcd}
\mathrm{Borel}_b(\sigma(N)) \arrow[r, "\rho_N"'] \arrow[d, "\pi"'] & L(H) \\
L^\infty(\mu) \arrow[r, tail, two heads, dotted, "\rho", "\cong"'] & \{N\}^{kk} \arrow[u, hook]
\end{tikzcd}
$$

Proof. Obviously, all scalar-valued spectral measures are equivalent, i.e. mutually absolutely continuous, because they have the same 0-sets by definition.

The functional calculus $\mathrm{Borel}_b(\sigma(N)) \to \{N\}^{kk}$ can be writen as composition because of ($\boxed{1} \Leftarrow \boxed{3}$) in $\boxed{8.58}$

$$\mathrm{Borel}_b(\sigma(N)) \xrightarrow{\pi} L^\infty(\mu_e) \cong \{N_{\mu_e}\}^{kk} \cong \{N_e\}^{kk} \cong \{N\}^{kk} \subseteq L(H),$$

where the mapping ρ is defined as the composition $L^\infty(\mu_e) \cong \{N_{\mu_e}\}^{kk} \cong \{N_e\}^{kk} \cong \{N\}^{kk}$. Thus it is a bijective $*$-homomorphism and a homeomorphism with respect to $\sigma(L^\infty, L^1)$ and the WOT by $\boxed{8.46}$ and $\boxed{8.56}$, because $f_i \to 0$ with respect to $\sigma(L^\infty, L^1)$ implies conversely that for $h \in H$

$$\langle f_i(N)h, h\rangle = \langle f_i(N_h)h, h\rangle = \int f_i \, d\mu_h = \int f_i \frac{d\mu_h}{d\mu_e} d\mu_e \to 0,$$

since by $\boxed{8.55}$ for $h \in H$ the measure μ_h is absolutely continuous with respect to μ_e, and thus by $\boxed{8.33}$ the Radon-Nikodym derivative $\frac{d\mu_h}{d\mu_e} \in L^1(\mu_e)$ exists.

□

8.60 Spectral Mapping Theorem.

Let H be a separable Hilbert space, $N \in L(H)$ a normal operator, P its spectral measure, μ a scalar-valued spectral measure for N, and finally $f \in L^\infty(\mu)$.
Then the spectrum $\sigma(f(N))$ of $f(N)$ is the μ-essential image of $f \in L^\infty(\mu)$.
Furthermore, $P \circ f^{-1}$ is the spectral measure and $\mu \circ f^{-1}$ a scalar-valued for $f(N)$.

Proof. First the statement about the spectrum:

$$\begin{aligned}\sigma_{L(H)}(f(N)) &\overset{\boxed{7.13}}{=} \sigma_{\{N\}^{kk}}(f(N)) \overset{\boxed{8.59}}{=} \sigma_{\{N_\mu\}^{kk}}(M_f)\\ &\overset{\boxed{7.13}}{=} \sigma_{L(L^2(\mu))}(M_f) \overset{\boxed{8.27}}{=} \mu\text{-ess-image}(f).\end{aligned}$$

Since f is measurable, $f^*P := P \circ f^{-1}$ is a spectral measure on $X := \{z \in \mathbb{C} : |z| \leqslant \|f\|_\infty\} \supseteq f(\sigma(N)) \supseteq \sigma(f(N))$. For $\varepsilon > 0$ we choose a partition of X and thus of $\sigma(f(N))$ into Borel sets B_j with $|z - z'| < \varepsilon$ for $z, z' \in B_j$. For $f^{-1}(B_j) \neq \emptyset$ let $x_j \in f^{-1}(B_j)$ be choosen fixed and $y_j := f(x_j)$. Then the $f^{-1}(B_j) \neq \emptyset$ form a partition of $\sigma(N)$ and thus by $\boxed{8.12.2}$

$$\begin{aligned}\Big\|f(N) - \int_X z\, d\, f^*P\,(z)\Big\| &= \Big\|\int_{\sigma(N)} f(z)\, dP(z) - \int_X z\, d\, f^*P\,(z)\Big\|\\ &\leqslant \Big\|\int_{\sigma(N)} f(z)\, dP(z) - \sum_j f(x_j)\, P(f^{-1}(B_j))\Big\|\\ &\quad + \Big\|\sum_j y_j\, f^*P\,(B_j) - \int_X z\, d\, f^*P\,(z)\Big\|\\ &\leqslant 2\varepsilon,\end{aligned}$$

hence equality holds and thus f^*P is the spectral measure for $f(N)$ by $\boxed{8.15}$.

We have that $f^*\mu := \mu \circ f^{-1}$ is a scalar-valued spectral measure of $f(N)$, because $0 = P_{f(N)}(B) = f^*P\,(B) = P(f^{-1}(B))$ if and only if $0 = \mu(f^{-1}(B)) = f^*\mu\,(B)$ holds. □

Multiplicity Theory for Normal Operators

8.61 Theorem (Hellinger 1907).

Let N be a normal operator on a separable Hilbert space. Then there is a sequence of measures μ_n on $\mathbb{C}$ with compact supports and μ_{n+1} absolutely continuous with respect to μ_n and

$$N \cong N_{\mu_1} \oplus N_{\mu_2} \oplus \dots.$$

Up to unitary equivalence, N is uniquely determined by the equivalence classes of these measures.

Remark.

The measure μ_1 has to be a scalar-valued spectral measure for N: Because $\bigoplus_j N_{\mu_j} - \lambda$ is invertible if and only if all $N_{\mu_j} - \lambda$ are, i.e. $\sigma(N) = \bigcup_j \sigma(N_{\mu_j}) = \bigcup_j \operatorname{supp}(\mu_j)$. Furthermore, $P(B) = 0$ exactly when $P_j(B) = 0$ for all j, i.e. B is an μ_j zero set. However, since μ_{j+1} is absolutely continuous with respect to μ_j, this is exactly the case when $\mu_1(B) = 0$.

Before turning to the proof, let us deduce a few variants. For the first we need the following

8.62 Lemma.

Let ν be an absolutely continuous measure with respect to another μ measure. Then there is a measurable set B, so that $\mu|_B$ and ν are equivalent (i.e. are mutually absolutely continuous).

Proof. Let $0 \leqslant f := \frac{d\nu}{d\mu} \in L^1(\mu)$ be the Radon-Nikodym derivative. Furthermore, let $B := \{x : f(x) \neq 0\}$. This measurable set is uniquely determined except for a zero set μ. For all measurable A we have: $0 = \nu(A) = \int \chi_A \, d\nu = \int \chi_A f \, d\mu = \int_B \chi_A f \, d\mu \Leftrightarrow \mu|_B(A) = 0$, i.e. ν and $\mu|_B$ are equivalent. □

8.63 Corollary.

Let N be a normal operator on a separable Hilbert space and μ a scalar spectral measure for N. Then there is decreasing (a respect to the inclusion) sequence of Borel sets $B_n \subseteq \sigma(N)$ with $B_1 := \sigma(N)$ and

$$N \cong N_\mu \oplus N_{\mu|_{B_2}} \oplus \dots .$$

Up to unitary equivalence, N is uniquely determined by the equivalence class of μ and the Borel sets up to μ-zero-sets. □

Remark.

If H is finite-dimensional, then $\sigma(N) = \{\lambda_1, \dots, \lambda_n\}$ is finite. By 8.61 we have $N = \bigoplus_k N_k$, where the $N_k \cong N_{\mu|_{B_k}}$ are cyclic diagonal operators on invariant subspaces $H_k \subseteq H$. The entries on the diagonal of N_k must therefore be pairwise distinct, i.e. all eigenvalues of N_k have multiplicity 1. Since μ_1 is a scalar spectral measure for N, the support of μ_1 must be the entire spectrum, i.e. the first summand $\sigma(N_1) = \sigma(N)$. The absolute continuity means that the respective spectrum becomes smaller, i.e. the diagonal elements of N_{k+1} must be a subset of those of N_k. So the N_k are the diagonal operators with pairwise distinct entries and exactly the eigenvalues of N with multiplicity at least k.

Remark.

However, there is another representation. Let Λ_k be the set of eigenvalues with multiplicity k, i.e. $\dim\ker(N-\lambda) = k$ for $\lambda \in \Lambda_k$. Let N_k be the diagonal operator which has Λ_k as diagonal elements, each with multiplicity k. Then $N_k \cong {A_k}^{(k)} := \bigoplus^k A_k$, where A_k is a diagonal operator with Λ_k as diagonal elements with multiplicity 1, i.e. $\sigma(A_k) = \Lambda_k$. Thus,

$$N \cong A_1 \oplus {A_2}^{(2)} \oplus {A_3}^{(3)} \dots$$

with $\sigma(A_j) \cap \sigma(A_k) = \emptyset$ for $j \neq k$. The following theorem provides an infinite-dimensional generalization.

8.64 Theorem.

Let N be a normal operator on a separable Hilbert space H. Then there are pairwise singular measures $\mu_\infty, \mu_1, \dots$ and an isomorphism

$$U : H \to L^2(\mu_\infty)^{(\infty)} \oplus L^2(\mu_1) \oplus L^2(\mu_2)^{(2)} \oplus \dots$$

which translates N into the sum of multiplication operators with z. Two operators are unitary equivalent if and only if the corresponding measures are.

Two measures μ_1 and μ_2 are called mutually SINGULAR, in case a decomposition $X = B_1 \sqcup B_2$ exists with $\mu_1(B_1) = 0$ and $\mu_2(B_2) = 0$.

Proof. Let μ be a spectral measure for N and B_n the Borel subsets of $\sigma(N)$ obtained by 8.63. Let $\Delta_\infty := \bigcap_{n=1}^\infty B_n$ and $\Delta_n := B_n \setminus B_{n+1}$ for $1 \leqslant n < \infty$. Let $\mu_n := \mu|_{\Delta_n}$ and $\nu_n := \mu|_{B_n}$ for $1 \leqslant n < \infty$. Since $B_n = \bigcap_{k=1}^\infty B_k \sqcup (B_n \setminus B_{n+1}) \sqcup (B_{n+1} \setminus B_{n+2}) \sqcup \dots = \Delta_\infty \sqcup \Delta_n \sqcup \Delta_{n+1} \sqcup \dots,$, hence $\nu_n = \mu_\infty + \mu_n + \mu_{n+1} + \dots$ and the measures

μ_∞, μ_n, $\mu_{n+1}, \dots$ are pairwise singular. So $N_{\nu_n} \cong N_{\mu_\infty} \oplus N_{\mu_n} \oplus N_{\mu_{n+1}} \oplus \dots$. Using 8.63 thus follows

$$\begin{aligned} N &\cong N_{\nu_1} \oplus N_{\nu_2} \oplus \dots \\ &\cong (N_{\mu_\infty} \oplus N_{\mu_1} \oplus N_{\mu_2} \oplus \dots) \oplus (N_{\mu_\infty} \oplus N_{\mu_2} \oplus N_{\mu_3} \oplus \dots) \oplus \dots \\ &\cong N_{\mu_\infty}^{(\infty)} \oplus N_{\mu_1}^{(1)} \oplus N_{\mu_2}^{(2)} \oplus \dots . \end{aligned}$$

The uniqueness is up to the reader. □

Proof of the existence statement of Theorem 8.61. The idea of proof of 8.61 is to construct a decomposition of H into the orthogonal sum $\bigoplus H_{h_n}$ of the cyclic subspaces generated by h_n by selecting a sequence from h_n. This can not be done by Zorn's Lemma, since the absolutely continuity of the associated measures can not be enforced. Inductive one could proceed as follows: Let $e_1 \in H$ be a separating vector for $\{N\}^{kk}$ as well as H_1 the closure of $\{N\}^{kk}e_1$ and $\mu_1(B) := \|P(B)e_1\|^2$. In the next step we consider $N_2 := N|_{H_1^\perp}$. Again by 8.52 there is a separating vector $e_2 \in H_1^\perp \subseteq H$ for $\{N_2\}^{kk}$. Let H_2 be the closure of $\{N_2\}^{kk}e_2$. Then $\mu_2 := P_{e_2,e_2}$ is absolutely continuous with respect to μ_1 by 8.55. If we proceed by induction, we can not guarantee that $\bigoplus H_k$ will fill all of H.

To ensure the termination after countable many steps, we choose an orthonormal basis $\{f_j\}$ from H with $f_1 = e_1$: We would like to choose the separating vector e_2 for $\{N_2\}^{kk}$ so that the orthogonal projection f_2' of f_2 onto $H_1^\perp$ lies in the closure of H_2in $\{N_2\}^{kk}e_2$. Then we would have $f_2 \in H_1 \oplus \{f_2'\} \subseteq H_1 \oplus H_2$. And inductively we would get $f_n \in H_1 \oplus \dots \oplus H_n$, so $H = \bigoplus_n H_n$ woulf hold. To justify this particular choice, we need the following lemma. □

8.65 Lemma.

Let N be a normal operator and $e \in H$. Then there is a separating vector e_0 for $\{N\}^{kk}$ with e in the $\{N\}^{kk}e_0$ closure.

Proof. Let f_0 be a separating vector for $\{N\}^{kk}$ and let P be the spectral measure of N. We define $\mu(B) := \|P(B)f_0\|^2$ and denote the closure of $\{N\}^{kk}f_0$ by H_0. We have $e =: h_0 + h_1$ with $h_0 \in H_0$ and $h_1 \in (H_0)^\perp$. Let $\eta(B) := \|P(B)h_1\|^2$ and H_1 be the closure of $\{N\}^{kk}h_1$. Then both H_0 and H_1 are invariant with respect to N. Furthermore, $H_0 \perp H_1$ and $N|_{H_0} \cong N_\mu$ and $N|_{H_1} \cong N_\eta$. Since η is absolutely continuous with respect to μ by 8.55, it follows that a Borel set B exists so that η and $\nu := \mu|_B$ are mutually absolutely continuous by 8.62. So $N|_{H_1} \cong N_\nu$ by 8.34. Let $U : H_0 \oplus H_1 \to L^2(\mu) \oplus L^2(\nu)$ be the canonical isomorphism with $U(N|_{H_0} \oplus N|_{H_1})U^{-1} = N_\mu \oplus N_\nu$. Because of $e = h_0 \oplus h_1 \in H_0 \oplus H_1$ we have $Ue = e_0 \oplus e_1$. Since h_1 is a cyclic vector for $N|_{H_1}$, e_1 is also one of N_ν and therefore $e_1 \neq 0$ ν-a.e..

We now want to show that an $f \in L^2(\mu)$ exists, so that $f \oplus e_1$ is a separating vector of $\{N_\mu \oplus N_\nu\}^{kk}$ and $e_0 \oplus e_1$ is in the closure of $\{N_\mu \oplus N_\nu\}^{kk}(f \oplus e_1)$:
We define $f(z) := e_0(z)$ for $z \in B$ and $f(z) := 1$ otherwise. Let H be the closure of $\{N_\mu \oplus N_\nu\}^{kk}(f \oplus e_1) = \{g\,(f \oplus e_1) : g \in L^\infty(\mu)\}$ (where the equality holds by 8.59 because μ is a scalar-valued spectral measure for $N_\mu \oplus N_\nu$). Let B^c be the complement of B, then: $g\,\chi_{B^c} \oplus 0 = g\,\chi_{B^c}(f \oplus e_1)$ for all $g \in L^\infty(\mu)$. So $L^2(\mu|_{B^c}) \oplus 0 \subseteq H$ and thus $(1-e_0)\,\chi_{B^c} \oplus 0 \in H$ and finally $e_0 \oplus e_1 = f \oplus e_1 - (1-e_0)\,\chi_{B^c} \oplus 0 \in H$.

On the other hand, it follows from $g \in L^\infty(\mu)$ and $0 = g\,(f \oplus e_1)$ that $g\,f = g\,e_1 = 0$ is μ-a.e.. Since $e_1 \neq 0$ is ν-a.e., $g = 0$ is μ-a.e. on B. Since $f = 1$ on B^c it follows that also $g = 0$ is μ-a.e. on B^c. So $f \oplus e_1$ is a separating vector of $\{N_\mu \oplus N_\nu\}^{kk}$. □

Proof of the uniqueness statement of the theorem 8.61. Since $\nu \sim \mu$ implies that $N_\nu \cong N_\mu$ we only need to show the converse implication. Let $N \cong M$, more precisely: Let U be a surjective isometry with $UNU^{-1} = M$. If e_1 is a separating vector for $\{N\}^{kk}$, then $f_1 := U(e_1)$ is one for $\{M\}^{kk}$. Since μ_1 and ν_1 are scalar-valued spectral measures for N and M, respectively, $\nu_1 \sim \mu_1$ follows and thus $N_{\mu_1} \cong N_{\nu_1}$, i.e. if $H = \oplus_n H_n$ and $K = \oplus_n K_n$ with $N|_{H_n} = N_{\mu_n}$ and $M|_{K_n} = N_{\nu_n}$, then $N|_{H_1} \cong M|_{K_1}$. However, this isomorphism does not have to be a restriction of U, i.e. we do not know wether $U(H_1) \subseteq K_1$. So we have to show that $N|_{H_1^\perp} \cong M|_{K_1^\perp}$. This is done in the following Proposition 8.66. The result then follows by means of induction. □

8.66 Proposition.

Let N, A and B be normal operators, N cyclic and $N \oplus A \cong N \oplus B$. Then $A \cong B$.

Proof. Let $N \in L(H)$, $A \in L(H_A)$ and $B \in L(H_B)$. And let $U : H \oplus H_A \to H \oplus H_B$ be an isomorphism with $U(N \oplus A)U^{-1} = N \oplus B$. We write U as matrix

$$U = \begin{pmatrix} U_{1,1} & U_{1,2} \\ U_{2,1} & U_{2,2} \end{pmatrix}$$

with $U_{1,1} \in L(H, H)$, $U_{1,2} \in L(H_A, H)$, $U_{2,1} \in L(H, H_B)$ and $U_{2,2} \in L(H_A, H_B)$. Then

$$U^* = \begin{pmatrix} U_{1,1}^* & U_{2,1}^* \\ U_{1,2}^* & U_{2,2}^* \end{pmatrix}$$

and furthermore

$$N \oplus A = \begin{pmatrix} N & 0 \\ 0 & A \end{pmatrix} \text{ and } N \oplus B = \begin{pmatrix} N & 0 \\ 0 & B \end{pmatrix}.$$

The equation $U\,(N \oplus A) = (N \oplus B)\,U$ reads:

$$\begin{pmatrix} U_{1,1}N & U_{1,2}A \\ U_{2,1}N & U_{2,2}A \end{pmatrix} = \begin{pmatrix} NU_{1,1} & NU_{1,2} \\ BU_{2,1} & BU_{2,2} \end{pmatrix}.$$

and $U\,(N \oplus A)^* = (N \oplus B)^*\,U$ reads:

$$\begin{pmatrix} U_{1,1}N^* & U_{1,2}A^* \\ U_{2,1}N^* & U_{2,2}A^* \end{pmatrix} = \begin{pmatrix} N^*U_{1,1} & N^*U_{1,2} \\ B^*U_{2,1} & B^*U_{2,2} \end{pmatrix}.$$

The equations $U^*U = 1$ and $UU^* = 1$ are:

$$\begin{pmatrix} U_{1,1}^*U_{1,1} + U_{2,1}^*U_{2,1} & U_{1,1}^*U_{1,2} + U_{2,1}^*U_{2,2} \\ U_{1,2}^*U_{1,1} + U_{2,2}^*U_{2,1} & U_{1,2}^*U_{1,2} + U_{2,2}^*U_{2,2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} U_{1,1}U_{1,1}^* + U_{1,2}U_{1,2}^* & U_{1,1}U_{2,1}^* + U_{1,2}U_{2,2}^* \\ U_{2,1}U_{1,1}^* + U_{2,2}U_{1,2}^* & U_{2,1}U_{2,1}^* + U_{2,2}U_{2,2}^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

From equation $(2,2)$ for N and for N^* and 8.36 it follows that $(\ker U_{2,2})^\perp$ is A-invariant, $(\ker U_{2,2}^*)^\perp$ B-invariant, and $A|_{(\ker U_{2,2})^\perp} \cong B|_{(\ker U_{2,2}^*)^\perp}$. It suffices to show that $A|_{\ker U_{2,2}} \cong B|_{\ker U_{2,2}^*}$, because then $A \cong B$. If $h \in \ker U_{2,2} \subseteq H_A$, then

$$\begin{pmatrix} U_{1,1} & U_{1,2} \\ U_{2,1} & U_{2,2} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ h \end{pmatrix} = \begin{pmatrix} U_{1,2}h \\ 0 \end{pmatrix}.$$

Since U is an isometry it follows that $U_{1,2}$ maps the kernel of $U_{2,2}$ isometrically to a closed subspace E of H. From the equations $(1,2)$ for N and $(1,2)$ for N^* and the fact that $\ker U_{2,2}$ is A-invariant, it follows that E is N-invariant. Thus, the restriction of $U_{1,2}$ to $\ker U_{2,2}$ is an equivalence for $A|_{\ker U_{2,2}} \cong N|_E$.

Similarly, we obtain that $U_{2,1}^*$ maps the kernel of $U_{2,2}^*$ isometrically to a closed subspace E_* of H, which is N-invariant, and provides an equivalence $B|_{\ker U_{2,2}^*} \cong N|_{E_*}$.

It remains to show $E = E_*$. If $h \in \ker U_{2,2}$, then $U_{1,1}^* U_{1,2} h = -U_{2,1}^* U_{2,2} h = 0$ by the equation $(1,2)$ for U^*U and thus $E = U_{1,2}(\ker U_{2,2}) \subseteq \ker U_{1,1}^*$. On the other hand, because of $(1,1)$ for UU^* for $f \in \ker U_{1,1}^*$, the equation $f = (U_{1,1}U_{1,1}^* + U_{1,2}U_{1,2}^*)f = U_{1,2}U_{1,2}^* f$ is valid. Because of $(2,1)$ for UU^* we have $U_{2,2}U_{1,2}^* f = -U_{2,1}U_{1,1}^* f = 0$, and hence $U_{1,2}^* f \in \ker U_{2,2}$. Consequently, $f \in U_{1,2}(\ker U_{2,2})$ and thus $E = \ker U_{1,1}^*$. Analogously we obtain $E_* := \ker U_{1,1}$. From equation $(1,1)$ for N it follows that $U_{1,1} \in \{N\}^k$, and since N is cyclic, it follows from 8.46 that $U_{1,1}$ is normal (because $\{N_\mu\}^k = A_\mu$) and hence $E = \ker U_{1,1}^* = \ker U_{1,1} = E_*$. □

9. Spectral theory for unbounded operators

Unbounded Operators

Quantum Mechanics.

In Quantum Mechanics one wants to represent physical quantities as self adjoint operators on a separable Hilbert space. For the position operator Q and the impulse operator P, the following version of the Heisenberg uncertainity principle $[P, Q] := PQ - QP = \frac{\hbar}{i}$ has to hold, where $\hbar \neq 0$ denotes the Plank quantum.

So let P and Q be elements of a Banach algebra ($A = L(H)$) satisfying this commutation relation. Induction immediately shows that $P^k\, Q = Q\, P^k + k\, \frac{\hbar}{i}\, P^{k-1}$ holds:

$$\begin{aligned} P^{k+1}\, Q = P\, P^k\, Q = P\, \left(Q\, P^k + k\, \frac{\hbar}{i}\, P^{k-1}\right) \\ = \left(Q\, P + \frac{\hbar}{i}\right)\, P^k + k\, \frac{\hbar}{i}\, P^k = Q\, P^{k+1} + (k+1)\, \frac{\hbar}{i}\, P^k. \end{aligned}$$

For $t \in \mathbb{C}$, we obtain

$$\begin{aligned} e^{i\,t\,P}\, Q &= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} P^k\, Q = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \left(Q\, P^k + k\, \frac{\hbar}{i}\, P^{k-1}\right) \\ &= Q \sum_{k=0}^{\infty} \frac{(it)^k}{k!}\, P^k + \frac{\hbar}{i} \sum_{k=1}^{\infty} \frac{(it)^k}{(k-1)!}\, P^{k-1} = (Q + t\,\hbar)\, e^{i\,t\,P}. \end{aligned}$$

Since $e^{i\,t\,P}$ is invertible, with inverse mapping $e^{-i\,t\,P}$, we have that Q and $Q + t\,\hbar$ similar and thus they have the same spectrum. However, since the spectrum of $Q + t\,\hbar$ is that of Q shifted by $t\,\hbar$, the spectrum of Q would have to be all of $\mathbb{C}$, and thus Q can not be an element of a Banach algebra by 6.24, and hence, in particular, not a bounded linear operator. A similar calculation shows that also P can not be a bounded operator.

If we define the impulse operator P by $(Pf)(x) := \frac{\hbar}{i}\frac{d}{dx} f(x)$ and the position operator Q by $(Qf)(x) := x\, f(x)$, then

$$[P, Q]f(x) = \frac{\hbar}{i}\frac{d}{dx}(x\, f(x)) - x\frac{\hbar}{i}\frac{d}{dx}\, f(x) = \frac{\hbar}{i}\Big(f(x) + x\, f'(x) - x\, f'(x)\Big) = \frac{\hbar}{i} f(x).$$

These operators are not defined for all f in the Hilbert space $L^2(\mathbb{R})$, so we need an extension of the notion "bounded linear operator" on Hilbert spaces.

9.1 Definition.

A linear operator $T : H_1 \rightsquigarrow H_2$ between Hilbert spaces H_1 and H_2 is a linear mapping T defined on a linear subspace $\operatorname{dom} T$ of H_1, the domain of T. Particularly important is the case where $\operatorname{dom} T$ is dense in H_1, which we may assume without loss of generality by replacing H_1 with the Hilbert space $\overline{\operatorname{dom} T}$. The sum $T_1 + T_2$ of two such operators T_1 and T_2 is defined on $\operatorname{dom} T_1 \cap \operatorname{dom} T_2$ and the COMPOSITION $T \circ S$ on $S^{-1}(\operatorname{dom} T)$.

An operator $\tilde{T} : H_1 \rightsquigarrow H_2$ is called EXTENSION of $T : H_1 \rightsquigarrow H_2$ if $\tilde{T} \supseteq T$, i.e. $\operatorname{dom} \tilde{T} \supseteq \operatorname{dom} T$ and $\tilde{T}|_{\operatorname{dom} T} = T$.

If T is bounded, then there is a bounded linear extension to the closure of $\operatorname{dom} T$, and if we put $T = 0$ on the orthogonal complement of $\operatorname{dom} T$, we obtain a bounded linear extension to H_1. The interesting non-globally defined operators are therefore all unbounded.

However, the operators should have some continuity property, since otherwise we would do only linear algebra. Therefore, we call an operator $T : H_1 \rightsquigarrow H_2$ CLOSED OPERATOR if its GRAPH $\operatorname{graph}(T) := \{(x, Tx) : x \in \operatorname{dom} T\}$ is closed in $H_1 \oplus H_2$. An operator is called CLOSEABLE if it has a closed extension.

9.2 Proposition.

Let $T : H_1 \rightsquigarrow H_2$ be a linear operator. Then t.f.a.e.:

1. *It is closeable;*

$\Leftrightarrow$ 2. *The closure of its graph is the graph of a mapping;*

$\Leftrightarrow$ 3. $(0, h) \in \overline{\operatorname{graph} T}$ *implies* $h = 0$.

In this situation, the operator with the closure of $\operatorname{graph} T$ as graph is called the CLOSURE of T.

Not every operator is closeable. Let e.g. $T : \ell^2 \rightsquigarrow \mathbb{C}$ defined by $T((x_n)_n) := \sum_n n\, x_n$ on $\operatorname{dom} T := \{(x_n)_n : \sum_n n\, |x_n| < \infty\}$. Then also $(0, 1) = \lim_n(\frac{1}{n} e_n, 1) \in \overline{\operatorname{graph} T}$, so this can not be a graph of a function.

Proof. ($\boxed{1} \Rightarrow \boxed{3}$) Let $\tilde{T} \supseteq T$ be a closed operator. Therefore, the closure $\overline{\operatorname{graph} T}$ of the graph of T is a subset of $\operatorname{graph} \tilde{T}$. Let $(0, h) \in \overline{\operatorname{graph} T} \subseteq \operatorname{graph} \tilde{T}$, then $h = \tilde{T}(0) = 0$.

($\boxed{2} \Leftarrow \boxed{3}$) Let $H_0 := \operatorname{pr}_1(\overline{\operatorname{graph} T}) = \{h \in H_1 : \exists k \in H_2 \text{ mit } (h, k) \in \overline{\operatorname{graph} T}\}$. Then we have to show that for each $h \in H_0$ exactly one $k \in H_2$ exists with $(h, k) \in \overline{\operatorname{graph} T}$. Let k_1 and k_2 be two such k. Then $(0, k_1 - k_2) = (h, k_1) - (h, k_2) \in \overline{\operatorname{graph} T}$ and thus $k_1 - k_2 = 0$, i.e. $k_1 = k_2$.

($\boxed{1} \Leftarrow \boxed{2}$) Let $\overline{\operatorname{graph} T}$ be the graph of a mapping $\tilde{T}$. This mapping $\tilde{T}$ has to be linear because the closure of the linear subspace $\operatorname{graph} T$ is itself a linear subspace. Furthermore, $\tilde{T}$ is by construction closed and $T \subseteq \tilde{T}$. $\square$

Adjoint operator

9.3 Definition of the adjoint operator.

In order to define uniquely a vector T^*k by the equation $\langle Th, k\rangle = \langle h, T^*k\rangle$ we need on the one hand that this holds for h in a dense subset, thus $\operatorname{dom} T$ has to be dense, and on the other hand $h \mapsto \langle Th, k\rangle$ has to be a bounded linear functional (on $\operatorname{dom} T$). So we define:
For a densely defined operator $T : H_1 \rightsquigarrow H_2$, the ADJOINT OPERATOR $T^* : H_2 \rightsquigarrow H_1$ is the operator with domain

$$\operatorname{dom}(T^*) := \Big\{k \in H_2 : \langle T(\_), k\rangle \text{ is bounded linear on } \operatorname{dom} T\Big\},$$

which is defined by $\langle Th, k\rangle = \langle h, T^*k\rangle$ for all $h \in \operatorname{dom} T$.

9.4 Multiplication operator as an example.

Let (X, Ω, μ) be a σ-finite measure space and $\lambda : X \to \mathbb{C}$ be an Ω-measurable function. Let $D := \{g \in L^2(\mu) : \lambda\, g \in L^2(\mu)\}$ and $T(g) := \lambda\, g$ for all $g \in D$. Then $T = M_\lambda$ is a closed densely defined operator. Its adjoint has the same domain D and is given by $T^* := M_{\overline{\lambda}}$:

Let $\Delta_n \subseteq \{x : |\lambda(x)| \leqslant n\}$ with $\mu(\Delta_n) < \infty$ and $\bigcup_n \Delta_n = X$. Then $L^2(\Delta_n) \subseteq D$, because for $\lambda \in L^\infty$ and $g \in L^2$ also $\lambda\, g \in L^2$ by the Hölder inequality . Thus D is dense.

Let now $g_k \to g$ and $Tg_k \to h$ in L^2. Then $\lambda\, g_k \to \lambda\, g$ converges on Δ_n and on the other hand also towards h, so $\lambda\, g = h$ a.e. and thus $g \in D$ and $(g, h) = (g, Tg) \in \operatorname{graph} T$, i.e. the graph of T is closed.

We have that $g \mapsto \langle \lambda g, h\rangle = \int g\, \lambda\, \overline{h}$ is bounded by the Theorem [**18**, 6.2.9] of Riesz if and only if $\lambda\, \overline{h} \in L^2$, i.e. $h \in D$. So $\operatorname{dom} T^* = D$ and

$$\langle \lambda g, h\rangle = \int \lambda\, g\, \overline{h} = \int g\, \overline{\overline{\lambda}\, h} = \langle g, \overline{\lambda}\, h\rangle,$$

i.e. $T^* h = \overline{\lambda} h$.

Diagonal operator.

Let, in particular, μ be the counting measure on $X = \mathbb{N}$. Then $L^2(X) = \ell^2$ and $\lambda : X \to \mathbb{C}$ is a sequence $(\lambda_n)_n$. The multiplication operator T has $D := \{h \in \ell^2 : \sum_k |\lambda_k\, h_k|^2 = \sum_k |\lambda_k \langle h, e_k\rangle|^2 < \infty\}$ as domain and is given by $Th := (\lambda_k h_k)_k = \sum_k \lambda_k \langle h, e_k\rangle e_k$ for all $h \in D$.

Position operator.

Let, in particular, μ be the Lebesgue measure on $X := \mathbb{R}$ and $\lambda := \mathrm{id}_{\mathbb{R}}$. Then T is the position operator of (1-dimensional) Quantum Mechanics.

We now show that T is the closure of $T|_{C_c^\infty}$:
Since T is closed, we have to find for each $f \in \operatorname{dom} T = \{f \in L^2 : \lambda\, f \in L^2\}$ a sequence $f_n \in C_c^\infty$, with $(f_n, Tf_n) \to (f, Tf)$:
Let $\rho \in C_c^\infty$ with $\rho = 1$ on a neighborhood U_0 of 0. Since C_c^∞ is dense in L^2, there exist $g_n, h_n \in C_c^\infty$ with $h_n \to f$ and $g_n \to Tf$. Therefore $\rho\, h_n \to \rho\, f$ and both sides vanish outside $\operatorname{supp}(\rho)$, so $T(\rho\, h_n) = \lambda\, \rho\, h_n \to T(\rho\, f) = \rho\, Tf$. Moreover, $(1-\rho) g_n \to (1-\rho) Tf$ and both sides vanish on U_0, so the functions $\frac{1-\rho}{\lambda} g_n$ from a sequence of C_c^∞-functions converging in L^2 towards $\frac{1-\rho}{\lambda} Tf = (1-\rho)\, f$. Finally, $f_n := \frac{1-\rho}{\lambda}\, g_n + \rho\, h_n \in C_c^\infty$ converges to $(1-\rho)\, f + \rho \cdot f = f$ in L^2 and $Tf_n = (1-\rho)\, g_n + \rho\, Th_n \to (1-\rho)\, Tf + \rho\, Tf = Tf$.

9.5 Differentiation operator as an example.

Let

$$D_0 := \Big\{ f : [-1, 1] \to \mathbb{C} : f \textit{ is absolutely continuous, } f' \in L^2 \textit{ and } f(-1) = 0 = f(1) \Big\}.$$

and let T_0 be defined by $T_0(f) := i\, f'$ for all $f \in D_0$. Note that the absolutely continuous functions f are just the antiderivatives of the L^1-functions.

Since the polynomials p with $p(-1) = 0 = p(1)$ are in D_0, we have that D_0 *is dense in* $L^2[-1, 1]$.

The operator T_0 *is closed*: Let $f_n \in D_0$ with $(f_n, if_n') \to (f,g)$ in $L^2 \oplus L^2$. Let $h(x) := -i\int_{-1}^x g(t)\,dt$. Because of the Cauchy-Schwarz inequality ($\|\_\|_1 \leqslant \|1\|_2\,\|\_\|_2 = \sqrt{2}\,\|\_\|_2$), h is absolutely continuous and

$$|f_n(x) - h(x)| = \left|\int_{-1}^x (f_n'(t) + i\,g(t))\,dt\right| \leqslant \sqrt{2}\|f_n' + i\,g\|_2 = \sqrt{2}\|if_n' - g\|_2 \to 0.$$

So $f_n \to h$ uniformly on $[-1,1]$. Since $f_n \to f$ in $L^2[-1,1]$, we have that $f = h$ a.e.. We can thus assume that $f(x) = h(x)$ for all x, and thus f is absolutely continuous and $f_n \to f$ uniformly on $[-1,1]$. In particular, $f(-1) = \lim_n f_n(-1) = \lim_n 0 = 0$ and analogously we have $f(1) = 0$. Furthermore, $f' = h' = -i\,g \in L^2[-1,1]$. Hence $f \in D_0$ and $(f,g) = (f, if') \in \operatorname{graph} T_0$.

Let $\operatorname{img} T_0 = \{f' : f \in D_0\} = \left\{h \in L^2[-1,1] : 0 = \int_{-1}^1 h(x)\,dx = \langle h, 1\rangle\right\} = \{1\}^\perp$.

Finally we have:

$$\operatorname{dom} T_0^* = D := \{g : g \textit{ is absolutely continuous on } [-1,1], \textit{ and } g' \in L^2[-1,1]\}$$

and $T_0^* g = i\,g'$, *i.e.* $T_0 \subset T_0^*$*:*
($\subseteq$) Let $g \in \operatorname{dom} T_0^*$ and $h := T_0^* g$. We put $H(x) := \int_{-1}^x h(t)\,dt$. By means of partial integration, we obtain the following for each $f \in D_0$ because of $f(-1) = 0 = f(1)$:

$$\begin{aligned}\langle T_0 f, g\rangle = \langle f, T_0^* g\rangle = \langle f, h\rangle &= \int_{-1}^1 f\,\overline{h} = \int_{-1}^1 f(x)\,\overline{H}'(x)\,dx \\ &= f(x)\,\overline{H(x)}\Big|_{x=-1}^1 - \int_{-1}^1 f'(x)\overline{H(x)}\,dx = -\int_{-1}^1 if'(x)\overline{iH(x)}\,dx \\ &= -\langle T_0 f, i\,H\rangle.\end{aligned}$$

So $\langle T_0 f, g + i\,H\rangle = 0$ for all $f \in \operatorname{dom} T$. Hence $g + i\,H \in (\operatorname{img} T_0)^\perp = \{1\}^{\perp\perp} = \mathbb{C}$, i.e. $c := g + i\,H$ is constant and thus $g = c - i\,H$ is absolutely continuous, $g' = -i\,H' = -i\,h \in L^2$ and $T_0^* g = h = i\,g'$.
($\supseteq$) Let g be absolutely continuous with $g' \in L^2$. By means of partial integration it follows for all $h \in D_0$ because of $h(-1) = 0 = h(1)$ that $\langle i\,h', g\rangle = -i\int h\,\bar{g}'$ and thus is continuous with respect to h, i.e. $g \in \operatorname{dom} T^*$.

Note that the factor i was necessary in order to get the same formula for T_0^* as for T_0.

Example of an extension.

We now extend the domain D_0.

$$D_1 := \left\{f : [-1,1] \to \mathbb{C} : f \text{ is absolutely continuous, } f' \in L^2[-1,1] \text{ and } f(-1) = f(1)\right\}$$

Let T_1 be given by the same formula as before, namely $T_1(f) = i\,f'$ for all $f \in D_1$.

Of course, T_1 is also densely defined, because $D_0 \subseteq D_1$. As before, one shows that T_1 is closed (this also follows from 9.8) and that $\operatorname{img} T_1 = \{1\}^\perp$.

This time, however, $\operatorname{dom} T_1^* = D_1 = \operatorname{dom} T_1$ and $T_1^* g = i\,g'$, i.e. $T_1 = T_1^*$:
($\subseteq$) Let again $g \in \operatorname{dom} T_1^*$ and $h := T_1^* g$ and $H(x) := \int_{-1}^x h(t)\,dt$. Then $H(-1) = 0$. And, because of $1 \in D_1$, we now have $H(1) = \int_{-1}^1 h = \langle T_1^* g, 1\rangle = \langle g, T_1 1\rangle = 0$. By partial integration and $H(-1) = 0 = H(1)$ we obtain again $\langle T_1 f, g + i\,H\rangle = 0$ for all $f \in D_1$. So, as before, $g = c - i\,H$ is absolutely continuous, $g' = -i\,H' = -i\,h \in L^2$ and $T_1^* g = h = i\,g'$.
($\supseteq$) For $g \in D_1$ it follows by means of partial integration (because $h(-1) = h(1)$ and $g(-1) = g(1)$) that $\langle i\,h', g\rangle = -i\int h\,\bar{g}'$ and thus it is continuous in $h \in D_1$, i.e. $g \in \operatorname{dom} T^*$.

The impulse operator on $L^2(\mathbb{R})$.

Let now

$$\begin{aligned} D &:= \Big\{f \in L^2(\mathbb{R}) : f \text{ is locally absolutely continuous and } f' \in L^2\Big\},\\ T(f) &:= i\, f' \text{ for all } f \in D. \end{aligned}$$

This operator is also densely defined, because for each interval $[a,b]$ and $n \in \mathbb{N}$ we consider the trapezoid function, which is 1 on $[a,b]$ and vanishes outside of an $1/n$-neighborhood. These functions are in D and their linear span is dense in L^2.

Claim: $T = T^*$ and hence T is closed by 9.8.
$(\subseteq)$ Let $g \in \operatorname{dom} T^*$ and $T^* g = h$. Then $\int_{\mathbb{R}} i\, f'\, \overline{g} = \langle T\, f, g\rangle = \langle f, h\rangle = \int_{\mathbb{R}} f\, \overline{h}$ for all $f \in D$. If we specifically choose a trapezoid function f_n for f as above, then

$$n \int_{a-\frac{1}{n}}^{a} i\, \overline{g} - n \int_{b}^{b+\frac{1}{n}} i\, \overline{g} = \int_{\mathbb{R}} f_n\, \overline{h}.$$

Multiplication with i, conjugating, and passing to the limit for $n \to \infty$, yields $g(b) - g(a) = -i \int_a^b h$ for almost all a and b, as the antiderivative $t \mapsto G(t) = \int_0^t g(s)\, ds$ of a $L^2[0,b] \subseteq L^1[0,b]$ function is almost everywhere differentiable and has g as derivative and thus $\lim_{n\to\infty} n \int_t^{t\pm\frac{1}{n}} g = \lim_{n\to\infty} \frac{G(t\pm\frac{1}{n})-G(t)}{\pm\frac{1}{n}} = G'(t) = g(t)$. Because $L^2 \subseteq L^1_{loc}$, we have that g is locally absolutely continuous and $g' = -i\, h$ almost everywhere. So g is in D and $T^* g = h = i\, g'$.

$(\supseteq)$ Let $g \in D$. Partial integration yields $\int_a^b i\, f' \overline{g} = i\, f\, \overline{g}|_a^b + \int_a^b f\, \overline{i\, g'}$, and since $f\, \overline{g}$ is integrable, we have $\liminf_{a\to-\infty, b\to\infty} |(f\,\overline{g})(b) - (f\,\overline{g})(a)| = 0$, hence $\int_{-\infty}^{+\infty} i\, f' \overline{g} = \int_{-\infty}^{+\infty} f\, \overline{i\, g'}$, I.e. $g \in \operatorname{dom} T^*$ and $T^* g = i\, g'$.

Claim: *T is the closure of $T|_{C_c^\infty}$.* For this we have to show that for each $f \in D$ functions $f_n \in C_c^\infty$ exist with $f_n \to f$ and $T f_n \to T f$ in L^2.
We first show that we find $f_n \in C^\infty \cap L^2$. For this we choose a $\rho \in C_c^\infty$ with $\rho \geqslant 0$ and $\int_{\mathbb{R}} \rho = 1$ and put $\rho_n : x \mapsto n\, \rho(n x)$ and $f_n := \rho_n \star f$. As in [**18**, 4.13.9], one shows that $\|f_n - f\|_2 = \|\rho_n \star f - f\|_2 \to 0$ (see also [**2**, 55]) and $\rho_n \star f \in C^\infty \cap L^2$, since $\rho_n \in C^\infty \cap L^1$ and $f \in L^2$. Furthermore, $(\rho_n \star f)' = \rho_n \star f'$. Since $f' \in L^2$, we have $T f_n = f_n' \in L^2$ and $\|T f_n - T f\|_2 = \|\rho_n \star f' - f'\|_2 \to 0$.
Let now $f \in C^\infty \cap L^2$ and choose $\rho \in C_c^\infty$ with $\rho(x) = 1$ for $|x| \leqslant 1$ and $\rho_n(x) := \frac{1}{n}\rho(\frac{x}{n})$. Let $f_n := \rho_n \cdot f$. Then $f_n \in C_c^\infty$ and $f_n(x) = f(x)$ for $|x| \leqslant n$. So $f_n \to f$ pointwise and since $|f_n(x)| \leqslant |f(x)|$, because of the theorem about dominated convergence, the convergence is also with respect to the 2-norm. Furthermore, $\|T f_n - T f\|_2 \leqslant \|\rho_n' \cdot f\|_2 + \|\rho_n \cdot f' - f'\|_2 \leqslant \|\rho_n'\|_\infty \cdot \|f\|_2 + \|\rho_n \cdot f' - f'\|_2 \leqslant \frac{1}{n}\|\rho'\|_\infty \cdot \|f\|_2 + \|\rho_n \cdot f' - f'\|_2 \to 0$.
We will give a second proof of this fact in 9.46.

9.6 Remark.

Let $T := \sum_{|\alpha|\leqslant m} a_\alpha\, \partial^\alpha$ be a linear partial differential operator of degree $\leqslant m$ on $\mathbb{R}^n$, i.e.

$$(T u)(x) := \sum_{|\alpha|\leqslant m} a_\alpha(x)\, \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} u(x).$$

with C^m-functions a_α. The transposed operator is given by

$$T^t : v \mapsto \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} \partial^\alpha (a_\alpha \cdot v).$$

For $u, v \in C^m$ we have:

$$T(u) \cdot v - u \cdot T^t(v) = \operatorname{div} J(u,v),$$

where $J = (J_1, \dots, J_n)$ is an n-tuple of bilinear partial differential operators J_n of degree $< m$.

Proof. We the prove this for $m = 2$ only (the general case is analogous). Let

$$T := \sum_{j,k} a_{j,k}\, \partial_j \partial_k + \sum_j b_j\, \partial_j + c.$$

We want to move the partial derivatives in the product $T(u) \cdot v$ from u to v. Let's start first with a term of 1-st degree

$$b_j\, \partial_j u \cdot v = \partial_j(b_j\, u\, v) - u \cdot \partial_j(b_j\, v).$$

For the terms of 2-nd degree we obtain

$$\begin{aligned} a_{j,k}\, \partial_j \partial_k u \cdot v &= \partial_j(a_{j,k}\, \partial_k u \cdot v) - \partial_j(a_{j,k}\, v) \cdot \partial_k u \\ &= \partial_j(a_{j,k}\, \partial_k u \cdot v) - \partial_k\left(\partial_j(a_{j,k}\, v) \cdot u\right) + \partial_k \partial_j(a_{j,k}\, v) \cdot u. \end{aligned}$$

So,

$$\begin{aligned} T(u) \cdot v = u \cdot &\Big(\sum_{j,k} \partial_k \partial_j(a_{j,k}\, v) - \sum_j \partial_j(b_j\, v) + c\, v\Big) \\ &+ \sum_j \partial_j \underbrace{\Big(\sum_k a_{j,k}\, \partial_k u \cdot v - \sum_k u \cdot \partial_k(a_{k,j}\, v) + b_j\, u \cdot v\Big)}_{=:J_j(u,v)} \\ &= u \cdot T^t(v) + \operatorname{Div} J(u,v), \end{aligned}$$

where $J := (J_1, \dots, J_n)$ and J_j is the following bilinear partial differential operator of degree 1:

$$\begin{aligned} J_j(u,v) &= \sum_k a_{j,k}\, \partial_k u \cdot v - \sum_k u \cdot \partial_k(a_{k,j}\, v) + b_j\, u \cdot v \\ &= \sum_k \Big(a_{j,k}\, \partial_k u \cdot v - a_{k,j} u \cdot \partial_k v\Big) - \left(\sum_k \partial_k(a_{k,j}) - b_j\right) u \cdot v. \end{aligned}$$

The application of the divergence theorem thus provides

$$\int_B T(u) \cdot v - u \cdot T^t(v) = \int_B \operatorname{div} J(u,v) = \int_{\partial B} \langle J(u,v), n_{\partial B} \rangle \operatorname{vol}_{\partial \mathrm{B}},$$

where $n_{\partial B} = (n_j)_j$ denotes the outward facing unit normal to the surface ∂B and $\operatorname{vol}_{\partial B}$ the surface area element.

In particular, $T(u) := \sum_{j,k} \partial_j(a_{j,k}\, \partial_k) + c\, u$ with $\mathbb{R}$-valued C^2-functions $a_{j,k} = a_{k,j}$ and c. Then $a_{j,k}$ is exactly the coefficient in the general formula at the beginning of the proof and $b_k = \sum_j \partial_j(a_{j,k})$. The transposed operator in this situation is $T^t = T$, because

$$\begin{aligned}
T^t(v) &:= \sum_{j,k} \partial_j \partial_k (v\, a_{j,k}) - \sum_j \partial_j \Big(v \sum_k \partial_k a_{k,j} \Big) + c\, v \\
&= \sum_{j,k} \Bigg(\partial_j \partial_k v\, a_{j,k} + \partial_k v\, \partial_j a_{j,k} + \partial_j v\, \partial_k a_{j,k} + \partial_j \partial_k a_{j,k}\, v \Bigg) \\
&\quad - \sum_{j,k} (\partial_j v\, \partial_k a_{k,j} + v\, \partial_j \partial_k a_{k,j}) + c\, v \\
&= \sum_{j,k} (a_{j,k}\, \partial_j \partial_k v + \partial_j v\, \partial_k a_{j,k}) + c\, v \\
&= T(v).
\end{aligned}$$

Let the derivative $\frac{\partial}{\partial n}$ in the "normal" direction be defined by

$$\frac{\partial}{\partial n} := \sum_{j,k} a_{j,k}\, n_j\, \partial_k.$$

Then

$$\begin{aligned}
\int_B T(u) \cdot v - u \cdot T^t(v) = \int_B \operatorname{div} J(u,v) &= \int_{\partial B} \langle J(u,v), n_{\partial B} \rangle \operatorname{vol}_{\partial B} \\
&= \int_{\partial B} \sum_j \Bigg(\sum_k \Big(a_{j,k}\, \partial_k u \cdot v - a_{k,j} u \cdot \partial_k v \Big) \\
&\quad - \Big(\sum_k \partial_k (a_{k,j}) - \sum_k \partial_k (a_{k,j}) \Big)\, u \cdot v \Bigg)\, n_j \operatorname{vol}_{\partial B} \\
&= \int_{\partial B} \Big(\frac{\partial u}{\partial n} \cdot v - u \cdot \frac{\partial v}{\partial n} \Big) \operatorname{vol}_{\partial B},
\end{aligned}$$

This integral vanishes if and only if the normal part of $J(u,v)|_{\partial B}$ vanishes, and, in particular, if $u|_{\partial B} = 0$ and either $v|_{\partial B} = 0$ or $\frac{\partial u}{\partial n}|_{\partial B} = 0$. □

We need the following description (of the graph) of T^*:

9.7 Proposition.

Let $T : H_1 \rightsquigarrow H_2$ be densely defined and $J : H_1 \oplus H_2 \to H_2 \oplus H_1$ be given by $J(f,g) = (-g, f)$. Then J is a bijective isometry and

$$\operatorname{graph} T^* = (J(\operatorname{graph} T))^\perp.$$

Proof. Obviously, J is a bijective isometry.

($\subseteq$) Let $g \in \operatorname{dom} T^*$ and $f \in \operatorname{dom} T$, then

$$\Big\langle (g, T^* g), J(f, T f) \Big\rangle = -\langle g, T f \rangle + \langle T^* g, f \rangle = 0.$$

($\supseteq$) Let $(g,h) \in (J(\operatorname{graph} T))^\perp$. For all $f \in \operatorname{dom} T$ we have $0 = \langle (g,h), (-Tf, f) \rangle = -\langle g, Tf \rangle + \langle h, f \rangle$. Thus $g \in \operatorname{dom} T^*$ and $h = T^* g$. □

9.8 Proposition.

Let $T : H_1 \rightsquigarrow H_2$ be a densely defined operator. Then:

1. *T^* is a closed operator.*
2. *T^* is densely defined if and only if T is closeable.*
3. *If T is closeable then its closure is T^{**}.*

Proof. (1) By 9.7, graph T^* is an orthogonal complement hence closed, i.e. T^* is a closed operator.

For the rest, note that the mapping J is a bijective isometry with inverse $J^{-1} : H_2 \oplus H_1 \to H_1 \oplus H_2$, $(g, f) \mapsto (f, -g)$.

(2) ($\Leftarrow$) We have to show that $(\operatorname{dom} T^*)^\perp = \{0\}$: For $k \in (\operatorname{dom} T^*)^\perp$ we have $(k, 0) \in (\operatorname{graph} T^*)^\perp \overset{9.7}{=\!=} (J(\operatorname{graph} T))^{\perp\perp} = \overline{J(\operatorname{graph} T)} = J(\overline{\operatorname{graph} T})$, i.e. $(0, -k) = J^{-1}(k, 0) \in J^{-1}J(\overline{\operatorname{graph} T}) = \overline{\operatorname{graph} T}$. Since T is closeable, $k = 0$ by 9.2.

($\Rightarrow$) Let $\operatorname{dom} T^*$ be dense. Then $T^{**} = (T^*)^*$ is well-defined and is by (1) a closed operator. We have $T \subseteq T^{**}$ (so T^{**} is a closed extension), because for all $f \in \operatorname{dom} T$ $g \mapsto \langle g, Tf \rangle = \langle T^* g, f \rangle$ is a well-defined bounded functional on $\operatorname{dom} T^*$, i.e. $f \in \operatorname{dom} T^{**}$ and $T^{**} f = Tf$.

(3) By 9.7 applied to T^*, we have $\operatorname{graph} T^{**} = (J' \operatorname{graph} T^*)^\perp$ where $J' : H_2 \oplus H_1 \to H_1 \oplus H_2$ is given by $J'(g, f) := (-f, g) = -(f, -g) = -J^{-1}(g, f)$. So

$$\begin{aligned}\operatorname{graph} T^{**} &= (-J^{-1} \operatorname{graph} T^*)^\perp \overset{9.7}{=\!=} (-J^{-1}(J \operatorname{graph} T)^\perp)^\perp \\ &\overset{J^{-1}\text{ Isometr.}}{=\!=\!=\!=} (-J^{-1} J \operatorname{graph} T)^{\perp\perp} = \overline{-\operatorname{graph} T} = \overline{\operatorname{graph} T}. \quad \square\end{aligned}$$

9.9 Corollary.

Let T be closed and densely defined. Then also T^* is closed and densely defined and $T^{**} = T$. $\square$

9.10 Proposition.

Let $T : H_1 \rightsquigarrow H_2$ be densely defined. Then

$$(\operatorname{img} T)^\perp = \ker T^*.$$

If T is additionally closed, then

$$(\operatorname{img} T^*)^\perp = \ker T.$$

Proof. ($\subseteq$) If $g \perp \operatorname{img} T$, then $\langle Tf, g \rangle = 0 = \langle f, 0 \rangle$ holds for all $f \in \operatorname{dom} T$. So $g \in \operatorname{dom} T^*$ and $T^* g = 0$.
($\supseteq$) Let $g \in \ker T^*$. Then, for all $f \in \operatorname{dom} T$, $\langle Tf, g \rangle = \langle f, T^* g \rangle = \langle f, 0 \rangle = 0$ holds.

By Corollary 9.9, we have $T^{**} = T$ for closed, densely defined T, and thus the second equation follows from the first one. $\square$

9.11 Theorem on closed image.

Let $T : H_1 \rightsquigarrow H_2$ be a densely defined, closed operator.
Then $\operatorname{img} T$ is closed if and only if $\operatorname{img} T^*$ is it.

Proof. We first show that we may replace T by a bounded operator S in the proof. Let $S : H_1 \times H_2 \supseteq \operatorname{graph} T \to H_2$ be the projection onto the 2-nd factor. We have the following commutative diagram:

$$\begin{tikzcd}
H_1 & \supseteq & \operatorname{dom}T \arrow[rr, two heads, "T"] & & \operatorname{img}T & \subseteq & H_2 \\
& & & \operatorname{graph}T \arrow[ul, "\mathrm{pr}_1"'] \arrow[ur, two heads, "S"] \arrow[d, hook, "\iota"', "|\cap"] & & & \\
& & & H_1\oplus H_2 \arrow[uulll, "\mathrm{pr}_1"] \arrow[uurrr, "\mathrm{pr}_2"'] & & &
\end{tikzcd}$$

We now show that the following holds for the image of the adjoint operator

$$S^*: H_2^* \to (\operatorname{graph} T)^* = (H_1^* \oplus H_2^*)/(\operatorname{graph} T)^o$$

$$(\iota^*)^{-1}(\operatorname{img} S^*) = \operatorname{img} T^* \oplus H_2^* \subseteq H_1^* \oplus H_2^*.$$

$$\begin{aligned}
&(f^*, g^*) \in (\iota^*)^{-1}(\operatorname{img} S^*)\\
\Leftrightarrow\ &(f^*, g^*)|_{\operatorname{graph} T} =: \iota^*(f^*, g^*) \in \operatorname{img} S^*\\
\Leftrightarrow\ &\exists h^* \in H_2^* : (f^*, g^*)|_{\operatorname{graph} T} = S^*(h^*)\\
\Leftrightarrow\ &\exists h^* \in H_2^* \forall f \in \operatorname{dom} T : \underbrace{(f^*, g^*)(f, Tf)}_{f^*(f)+g^*(Tf)} = \underbrace{S^*(h^*)(f, Tf)}_{h^*(Tf)}\\
\Leftrightarrow\ &\exists h^* \in H_2^* \forall f \in \operatorname{dom} T : f^*(f) = (h^* - g^*)(Tf)\\
&\quad \text{i.e. } h^* - g^* \in \operatorname{dom} T^*,\ T^*(h^* - g^*) = f^*\\
\Leftrightarrow\ &\exists h^* \in g^* + \operatorname{dom} T^* : T^*(h^* - g^*) = f^*\\
\Leftrightarrow\ &f^*, g^* \in \operatorname{img} T^* \oplus H_2^*,
\end{aligned}$$

Where the last ($\Leftarrow$) follows by $\exists k^* \in \operatorname{dom} T^* : f^* = T^* k^*$, now choose $h^* = g^* + k^*$.

Because ι is a closed embedding, ι^* is a quotient map by 5.2.4, and thus $\operatorname{img} S^*$ is closed if and only if $(\iota^*)^{-1}(\operatorname{img} S^*) = \operatorname{img} T^* \oplus H_2^*$, or equivalent $\operatorname{img} T^*$, is it. Because of $\operatorname{img} T = \operatorname{img} S$ it suffices to show the theorem for the bounded operator S.

($\Rightarrow$) So let $T : H_1 \to H_2$ be a bounded linear operator with closed image. Since the adjoint of the inclusion $\operatorname{img} T \to H_2$ is surjective by Hahn-Banach, we may assume without loss of generality that T is surjective. By the open mapping theorem, there is a $\delta > 0$ with $\{g : \|g\| \leqslant \delta\} \subseteq \{Tf : \|f\| \leqslant 1\}$. So there is a $f \in T^{-1}(g)$ for $g \in H_2$ with $\|f\| \leqslant \frac{\|g\|}{\delta}$. For all $g^* \in H_2^*$ we obtain

$$|g^*(g)| = |g^*(Tf)| = |T^* g^*(f)| \leqslant \|f\| \, \|T^* g^*\| \leqslant \frac{\|g\|}{\delta} \, \|T^* g^*\|.$$

Consequently, $\|g^*\| =\leqslant \frac{1}{\delta} \|T^* g^*\|$. So $T^* : H_2^* \to H_1^*$ is injective and is a homeomorphism onto its image, so $\operatorname{img} T^*$ is closed.

Since $T^{**} = T$ by 9.9, the converse implication also holds. □

This theorem also holds for Banach spaces.

Proof for Banach spaces. ($\Rightarrow$) In the above proof we have used nowhere that the spaces are Hilbert spaces.

($\Leftarrow$) So let $T : H_1 \to H_2$ be a bounded linear operator and let $T^* : H_2^* \to H_1^*$ have closed image. We replace T by the operator $T_1 : H_1 \to \overline{\operatorname{img} T}$. Since $T = \iota \circ T_1$, where ι denotes the closed inclusion of $\overline{\operatorname{img} T}$ into H_2, we have $T^* = T_1^* \circ \iota^*$ and ι^* is surjective. So T_1^* has the same closed image as T^* and we just have to show that T_1 is surjective. So let $T = T_1$ without loss of generality, which means T has denses image.

We have that $T^* : H_2^* \to H_1^*$ is injective, because $T^* g^* = 0$ implies $\langle Tf, g^* \rangle = \langle f, T^* g^* \rangle = 0$. Since the image of T is dense in H_2, we have $g^* = 0$. By the open mapping theorem, $T^* : H^* \to \operatorname{img} T^*$ is a homeomorphism onto its closed image. In order to show that T is surjective, we apply the Closed Graph Theorem to the inverse $S := \tilde{T}^{-1}$ of the injective mapping $\tilde{T} : H_1/_{\mathrm{kernelT}} \to H_2$ as in the proof of the theorem of open mappings.

$$\begin{tikzcd} H_1 \arrow[r, two heads, "T"] \arrow[dr, "\pi"'] & \operatorname{img} T \arrow[r, hook] & H_2 \\ & H_1/\ker T \arrow[u, tail, two heads, "\tilde{T}"'] & \end{tikzcd}$$

In the proof of the Closed Graph Theorem, we have used the non-meagerness of $G := \operatorname{img} T$ only for showing that S is almost continuous, i.e. that
the closure of $S^{-1}(\{z : \|z\| \leqslant \delta\}) = T(\{x : \|x\| \leqslant \delta\})$ contains a zero-neighborhood for all $\delta > 0$. Hence it is sufficient to show this.

Suppose there is a $\delta > 0$ so that the closure of the image of the ball $\{Tx : \|x\| \leqslant \delta\}$ does not contain a 0-neighborhood, i.e. $\exists y_n \notin \overline{\{Tx : \|x\| \leqslant \delta\}}$ with $y_n \to 0$. Since this closure is absolutely convex, by Mazur's lemma 5.2.4 there exists a continuous linear functional f_n with $f_n(y_n) > \sup_{\|x\| \leqslant \delta} |f_n(Tx)| = \sup_{\|x\| \leqslant \delta} |T^*(f_n)(x)|$. Hence $\|T^* f_n\| < \frac{1}{\delta} \|f_n\| \, \|y_n\|$ and because of $y_n \to 0$ it follows that T^* can not be a homeomorphism onto its image, a contradiction. □

Invertibility and spectrum

9.12 Definition .

Let $T : H_1 \rightsquigarrow H_2$ be a linear operator. Then T is called BOUNDED INVERTIBLE if a **bounded** linear operator $S : H_2 \to H_1$ exists with $T S = 1$ and $S T \subseteq 1$, i.e. $S T = 1$ on $\operatorname{dom} T$ (because $\operatorname{dom}(ST) = T^{-1}(\operatorname{dom}(S)) = \operatorname{dom} T$). Warning: This definition is quite asymmetrical!

9.13 Proposition .

Let $T : H_1 \rightsquigarrow H_2$ be a linear operator. Then T is bounded invertible if and only if T is closed and $T : \operatorname{dom} T \to H_2$ is bijective. Under these assumptions, its inverse is unique.

We will denote the uniquely determined inverse of a bounded invertible operator T by T^{-1}.

Proof. ($\Rightarrow$) Let S be a bounded inverse. Since $S T \subseteq 1$, we have $\ker T = \{0\}$. Because $T S = 1$, we have $\operatorname{img} T = H_2$, i.e. $T : \operatorname{dom} T \to H_2$ is bijective and $S : H_2 \to \operatorname{dom} T$ is its inverse, because $T S = 1$ and $S T = 1$ on $\operatorname{dom} T$. So S is unique. Finally, $\operatorname{graph} T = \{(h, Th) : h \in \operatorname{dom} T\} = \{(Sk, k) : k \in H_2\}$. Since S is bounded, this graph is closed.

($\Leftarrow$) If T has the given properties, the inverse $S : \operatorname{img} T = H_2 \to \operatorname{dom} T$ is a well-defined linear mapping with $\operatorname{graph} S = \{(k, Sk) : k \in H_2\} = \{(Th, h) : h \in \operatorname{dom} T\}$. So this graph is closed and according to the Closed Graph Theorem S is bounded. □

Lemma.

Let $T : H_1 \rightsquigarrow H_2$ be a densely defined, closed operator. Then T is bounded invertible if and only if T^* is it. Under this condition we have $(T^{-1})^* = (T^*)^{-1}$.

Proof. ($\Rightarrow$) Let T be bounded invertible and $S : H_2 \to \operatorname{dom} T \subseteq H_1$ the bounded inverse. Then $S^* \in L(H_1^*, H_2^*)$ is well-defined.

($S^* T^* \subseteq 1$) Let $k \in \operatorname{dom}(S^* T^*) = \operatorname{dom}(T^*)$. For $g \in \operatorname{dom} S = H_2$ we have

$$\langle g, S^* T^* k\rangle = \langle S g, T^* k\rangle \xlongequal{S g \in \operatorname{dom} T} \langle T S g, k\rangle = \langle g, k\rangle.$$

($T^* S^* = 1$) Let $h \in H_1$. Then $S^* h \in \operatorname{dom} T^*$, because $f \mapsto \langle T f, S^* h\rangle = \langle S T f, h\rangle = \langle f, h\rangle$ is bounded. Moreover, $\langle f, T^* S^* h\rangle = \langle T f, S^* h\rangle = \langle S T f, h\rangle = \langle f, h\rangle$ holds for all $f \in \operatorname{dom} T$, thus $T^* S^* = 1$.

($\Leftarrow$) With T^* also T^{**} is bounded invertible because of ($\Rightarrow$), and $T^{**} = T$ by 9.9. $\square$

9.14 Definition.

Let $T : H \rightsquigarrow H$ be a linear operator. The resolvent set $\rho(T)$ is the set

$$\rho(T) := \Big\{\lambda \in \mathbb{C} : T - \lambda \text{ is bounded invertible}\Big\}.$$

The spectrum of T is the set $\sigma(T) = \mathbb{C}\backslash\rho(T)$. The resolvent set $\rho(T)$ is now defined as a subset of $\mathbb{C}$ and not of $\mathbb{C}_\infty$, since we will show in 9.15 that every closed subset of $\mathbb{C}$ appears as spectrum of some operator and if it is not bounded then it is not closed in $\mathbb{C}_\infty$.

9.15 Proposition.

Let $T : H \rightsquigarrow H$ be a linear operator. Then $\sigma(T)$ is closed in $\mathbb{C}$ and the resolvent function $\rho(T) \to L(H)$, $z \mapsto (z - T)^{-1}$, is holomorphic.

Proof.

Let $\lambda_0 \in \rho(T)$ and $(\lambda_0 - T)^{-1}$ the bounded inverse. We use the Ansatz

$$\begin{aligned}
(\lambda - T)^{-1} &:= \frac{1}{(\lambda_0 - T) - (\lambda_0 - \lambda)} := (\lambda_0 - T)^{-1} \frac{1}{1 - (\lambda_0 - \lambda)(\lambda_0 - T)^{-1}} \\
&:= (\lambda_0 - T)^{-1} \sum_{k \geqslant 0} (\lambda_0 - \lambda)^k \Big((\lambda_0 - T)^{-1}\Big)^k \\
&= \sum_{k \geqslant 0} (\lambda_0 - \lambda)^k \Big((\lambda_0 - T)^{-1}\Big)^{k+1}.
\end{aligned}$$

This series converges absolutely for $|\lambda_0 - \lambda| < \frac{1}{\|(\lambda_0 - T)^{-1}\|}$ and $(\lambda - T)^{-1}$ has values in $\operatorname{img}(\lambda_0 - T)^{-1} = \operatorname{dom}(\lambda_0 - T) = \operatorname{dom} T$. We have

$$\begin{aligned}
&(\lambda_0 - T)^{-1} \sum_{k \geqslant 0} (\lambda_0 - \lambda)^k \Big((\lambda_0 - T)^{-1}\Big)^k (\lambda - \lambda_0 + \lambda_0 - T) \\
&\quad = -\sum_{k \geqslant 0} (\lambda_0 - \lambda)^{k+1} \Big((\lambda_0 - T)^{-1}\Big)^{k+1} + \sum_{k \geqslant 0} (\lambda_0 - \lambda)^k \Big((\lambda_0 - T)^{-1}\Big)^k = 1
\end{aligned}$$

on $\operatorname{dom} T$. Analogously, it can be shown that on all of H the reverse composition yields 1. So $\rho(T)$ is open and the resolvent function can be developed locally into a power series with coefficients in $L(H)$. $\square$

Remark.

If $T : H \rightsquigarrow H$ is a linear operator and $\lambda \in \mathbb{C}$, then $\operatorname{graph} T$ is closed if and only if the set $\operatorname{graph}(T-\lambda)$ obtained by shearing with $(x,y) \mapsto (x, y-\lambda x)$ is closed. So for non-closed operators the spectrum is all of $\mathbb{C}$.

If T is defined as in example 9.4, then $\sigma(T) = \overline{\{\lambda_n : n \in \mathbb{N}\}}$, because every λ_n is an eigenvalue and by 9.15 $\sigma(T)$ is closed. Conversely, for μ with $\delta := d(\mu, \{\lambda_n : n \in \mathbb{N}\}) > 0$ the mapping $T-\mu$, $(x_n)_n \mapsto ((\lambda_n - \mu)x_n)_n$, is obviously injective and closed. But it is also surjective, because each $(y_n)_n \in \ell^2$ has an inverse image given by $x_n := \frac{1}{\lambda_n - \mu_n} y_n$ since $(\frac{1}{\lambda_n-\mu_n})_n \in \ell^\infty$.

Thus, any closed set $A \neq \varnothing$ occurs as spectrum of some closed densely defined linear operator T: One may choose decompositions of $\mathbb{C}$ into squares with side length $\frac{1}{2^n}$ and for each square which meets A an intersection point. So one obtains a countable subset $\{\lambda_n : n \in \mathbb{N}\}$ being dense in A and we can choose the corresponding multiplication operator as T.

It may also occur that $\sigma(T) = \varnothing$. To see this let an $S \in L(H)$ with dense image and $\sigma(S) = \{0\}$ be given (see example 9.16). We put $\operatorname{dom} T := \operatorname{img} S$ and $T := S^{-1} : \operatorname{img} S \twoheadrightarrow H$. Then T is closed, densely defined and bounded invertible with $T^{-1} = S$. We now show that all $\lambda \neq 0$ are also in $\rho(T)$. For this we use the Ansatz

$$(\lambda - T)^{-1} := -T^{-1} \sum_{k=0}^{\infty} (\lambda T^{-1})^k = -S \sum_{k=0}^{\infty} \lambda^k S^k.$$

as in 9.15 This series converges absolutely in $L(H)$ for all λ by the root test, because $\sqrt[k]{\|\lambda^k S^k\|} = |\lambda|\, \|S^k\|^{1/k} \to \|\lambda\|\, r(S) = 0$ by 6.25. That it is an inverse to $\lambda - T$ follows as in 9.15.

9.16 Example.

Let $T \in L(\ell^2(\mathbb{Z}))$ be given by $(Tx)_n := e^{-n^2} x_{n-1}$, i.e. as composition of the shift operator with the multiplication operator with $n \mapsto e^{-n^2}$.

Since all $e_n \in \operatorname{img} T$, we have that $\operatorname{img} T$ is dense in ℓ^2.

We now show $\sigma(T) = \{0\}$, i.e. $0 = r(T) = \lim_k \|T^k\|^{1/k}$ by 6.25. Obviously,

$$(T^k x)_n = e^{-n^2} e^{-(n-1)^2} \cdots e^{-(n-k+1)^2} x_{n-k}$$

and thus

$$\begin{aligned} \|T^k x\|_2^2 &= \sum_n |(T^k x)_n|^2 = \sum_n \Big| e^{-n^2} e^{-(n-1)^2} \cdots e^{-(n-k+1)^2} x_{n-k} \Big|^2 \qquad (m := n-k) \\ &= \sum_m e^{-2((m+k)^2 + \cdots + (m+1)^2)} |x_m|^2 \leqslant e^{-(k-1)^2} |x_m|^2 \qquad \text{for } k \geqslant 2, \end{aligned}$$

because $(m+k)^2 + \cdots + (m+1)^2 \geqslant (m+k)^2 + (m+1)^2 = 2(m+k)(m+1) + (k-1)^2 \geqslant (k-1)^2$. So $\|T^k\| \leqslant e^{-(k-1)^2}$ and $r(T) = \lim_{k\to\infty} \|T^k\|^{1/k} = \lim_{k\to\infty} e^{-\frac{(k-1)^2}{k}} = 0$.

9.17 Proposition.

Let $T : H \rightsquigarrow H$ be a closed, densely defined linear operator. Then:

1. *$\lambda \in \rho(T)$ if and only if $(T-\lambda) : \operatorname{dom} T \to H$ is bijective.*
2. *We have $\sigma(T^*) = \{\overline{\lambda} : \lambda \in \sigma(T)\}$ and $(T^* - \overline{\lambda})^{-1} = ((T-\lambda)^{-1})^*$ for $\lambda \in \rho(T)$.*

Proof. By 9.13, $T - \lambda$ is bounded invertible if and only if $T - \lambda$ is bijective from $\operatorname{dom}(T - \lambda) = \operatorname{dom} T$ to H and the graph is closed. This shows (1).

(2) The following holds:

$$\begin{aligned}\lambda \notin \sigma(T) &\Leftrightarrow T - \lambda \text{ is bounded invertible}\\ &\xLeftrightarrow{9.13} T^* - \overline{\lambda} = (T - \lambda)^* \text{ is bounded invertible}\\ &\Leftrightarrow \overline{\lambda} \notin \sigma(T^*)\end{aligned}$$

and for such λ we have $(T^* - \overline{\lambda})^{-1} = ((T - \lambda)^*)^{-1} \overset{9.13}{=\!=} ((T - \lambda)^{-1})^*$. □

Symmetric and self adjoint operators

9.18 Definition.

An operator $T : H \rightsquigarrow H$ is called SYMMETRIC if it is densely defined and satisfies $\langle Th, k\rangle = \langle h, Tk\rangle$ for all $h, k \in \operatorname{dom} T$.

Lemma.

Let

$$T(u)(x) := \sum_{j,k} \frac{\partial}{\partial x^j} \left(a_{j,k}(x) \frac{\partial}{\partial x_k} u(x) \right) + c(x)\, u(x)$$

be a 2-nd order partial differential operator with real C^2-functions c and $a_{j,k} = a_{k,j}$ as coefficients. Then T is symmetric as operator with $\operatorname{dom} T := C_c^\infty(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$ or, if $G \subseteq \mathbb{R}^n$ is a bounded domain with smooth boundary ∂G, also as operator T with $\operatorname{dom} T := \{f \in C^\infty(\overline{G}) : f|_{\partial G} = 0\} \subseteq L^2(\overline{G})$.

Proof. By 9.6, the transposed operator is $T^t = T$ and satisfies

$$\int_G T(u) \cdot v = \int_G u \cdot T^t(v),$$

so for $v = \overline{w}$ also

$$\langle T(u), w\rangle = \int_G T(u) \cdot v = \int_G u \cdot Tv = \langle u, \overline{T(v)}\rangle = \langle u, T(w)\rangle,$$

because T has real coefficients. Thus T is symmetrical. We have

$$\frac{\partial}{\partial x_j} \left(a_{j,k}(x) \frac{\partial}{\partial x_k} u(x) \right) = \frac{\partial}{\partial x_j} a_{j,k}(x) \cdot \frac{\partial}{\partial x_k} u(x) + a_{j,k}(x) \frac{\partial^2}{\partial x_j \partial x_k} u(x)$$

and thus the formally adjoint differential operator T^* on $v \in C^2$ is given by:

$$\begin{aligned}
T^*(v)(x) &:= \sum_{j,k} \frac{\partial^2}{\partial x_k \partial x_j}\Big(v(x)\, a_{j,k}(x)\Big) \\
&\quad - \sum_j \frac{\partial}{\partial x_j}\Big(v(x) \sum_k \frac{\partial}{\partial x_k} a_{k,j}(x)\Big) + c(x)\, v(x) \\
&= \sum_{j,k}\Bigg(\frac{\partial^2 v(x)}{\partial x_k \partial x_j} a_{j,k}(x) + \frac{\partial v(x)}{\partial x_k}\frac{\partial a_{j,k}(x)}{\partial x_j} \\
&\qquad + \frac{\partial v(x)}{\partial x_j}\frac{\partial a_{j,k}(x)}{\partial x_k} + \frac{\partial^2 a_{j,k}(x)}{\partial x_k \partial x_j} v(x)\Bigg) \\
&\quad - \sum_{j,k}\left(\frac{\partial v(x)}{\partial x_j}\frac{\partial a_{k,j}(x)}{\partial x_k} + v(x)\frac{\partial^2 a_{k,j}(x)}{\partial x_j \partial x_k}\right) + c(x)\, v(x) \\
&= \sum_{j,k}\left(a_{j,k}(x)\frac{\partial^2 v(x)}{\partial x_j \partial x_k} + \frac{\partial v(x)}{\partial x_j}\frac{\partial a_{j,k}(x)}{\partial x_k}\right) + c(x)\, v(x) \\
&= T(v)(x). \qquad \square
\end{aligned}$$

9.19 Lemma.

Let $T : H \rightsquigarrow H$ be densely defined. Then t.f.a.e.:

1. *T is symmetrical;*

$\Leftrightarrow$ 2. *$T \subseteq T^*$.*

$\Leftrightarrow$ 3. *$\langle Th, h\rangle \in \mathbb{R}$ for all $h \in \operatorname{dom} T$;*

Proof. ($\boxed{1} \Leftrightarrow \boxed{2}$), because

$$\begin{aligned}
(\boxed{2}) &\Leftrightarrow \forall g \in \operatorname{dom} T : g \in \operatorname{dom} T^* \text{ and } T^* g = Tg \\
&\Leftrightarrow \forall g \in \operatorname{dom} T : f \mapsto \langle Tf, g\rangle \text{ is bounded on } \operatorname{dom} T \\
&\qquad\qquad \text{and } \forall f \in \operatorname{dom} T : \langle Tf, g\rangle = \langle f, Tg\rangle \\
&\Leftrightarrow (\boxed{1}),
\end{aligned}$$

because the second condition of the penultimate row obviously implies the first one.

($\boxed{1} \Leftrightarrow \boxed{3}$)

$$\begin{aligned}
(\boxed{1}) &\Leftrightarrow \forall f, g \in \operatorname{dom} T : p(f,g) := \langle Tf, g\rangle - \langle f, Tg\rangle = 0 \\
&\Leftrightarrow \forall f \in \operatorname{dom} T : 0 = p(f,f) = \langle Tf, f\rangle - \overline{\langle Tf, f\rangle} \\
&\Leftrightarrow \forall f \in \operatorname{dom} T : \langle Tf, f\rangle \in \mathbb{R},
\end{aligned}$$

because of the polarization-equation $\boxed{7.6}$ for the sesqui-linear form $p : \operatorname{dom} T \times \operatorname{dom} T \to \mathbb{C}$. $\square$

9.20 Definition.

For a symmetric operator T, $\operatorname{dom} T = \operatorname{dom} T^*$ might fail, see example $\boxed{9.5}$. So we call an operator $T : H \rightsquigarrow H$ SELF ADJOINT if it is defined and satisfies $T = T^*$. In particular, every self adjoint operator is symmetric. Corollary $\boxed{9.8.1}$ shows that every self adjoint operator is closed.

Also, the adjoint of a symmetric operator does not have to be symmetric: In example $\boxed{9.5}$ we saw that $T_0^* \supset T_1^* = T_1 \supset T_0 = T_0^{**}$ by $\boxed{9.9}$. So we call a densely defined operator $T : H \rightsquigarrow H$ ESSENTIALLY SELF ADJOINT if T and T^* are symmetric.

Lemma.

Let $T : H \rightsquigarrow H$ be a densely defined operator. Then:

1. *The operator T is essentially self adjoint if and only if T^* is self adjoint.*
2. *If T is symmetric, then T is closeable and its closure T^{**} is also symmetric.*

Proof. (1) ($\Rightarrow$) Since T is symmetric, $T \subseteq T^*$ holds. It easily follows $T^* \supseteq (T^*)^* = T^{**}$. Since T^* is symmetric, the converse inclusion also holds.
($\Leftarrow$) If T^* is self adjoint, then it is dense and thus $T^* = T^{**}$ is the closure of T by 9.8.2 and 9.8.3, so T is also symmetric as restriction of T^*.

(2) Since T is densely defined, T^* makes sense. And because T is symmetric, $\operatorname{dom} T \subseteq \operatorname{dom} T^*$ holds. So also T^* is dense defined and thus T^{**} is the closure of T again by 9.8.2 and 9.8.3.

Since $\operatorname{dom} T \subseteq \operatorname{dom} T^{**}$, also T^{**} is densely defined. Thus T^{***} makes sense. From $T \subseteq T^*$ follows $T^* \supseteq T^{**}$ and finally $T^{**} \subseteq T^{***}$, so T^{**} is symmetrical. $\square$

9.21 Proposition.

Let $T : H \rightsquigarrow H$ be a symmetric operator.

1. *If $\operatorname{img} T$ is dense then T is injective.*
2. *If T is self adjoint and injective, then $\operatorname{img} T$ is dense and T^{-1} is also self adjoint.*
3. *If $\operatorname{dom} T = H$, then T is self adjoint and T is bounded.*
4. *If $\operatorname{img} T = H$, T is self adjoint and T^{-1} is bounded.*

Proof. (1) Let $Th = 0$, then $0 = \langle Th, k\rangle = \langle h, Tk\rangle$ for all $k \in \operatorname{dom} T$ and because $\operatorname{img} T = T(\operatorname{dom} T)$ is dense, we have $h = 0$.

(2) Because of 9.10 we have $(\operatorname{img} T)^\perp = \ker T^* = \ker T = \{0\}$, i.e. $\operatorname{img} T$ is dense. An operator S is self adjoint if and only if $\operatorname{graph} S = \operatorname{graph} S^* = (J \operatorname{graph} S)^\perp$ by 9.7. Furthermore,

$$\begin{aligned}\operatorname{graph}(T^{-1}) &= \{(g, T^{-1}g) : g \in \operatorname{dom}(T^{-1}) = \operatorname{img} T\} = \{(Tf, f) : f \in \operatorname{dom} T\}\\ &= J \operatorname{graph}(-T).\end{aligned}$$

Because of $(-T)^* = -T^* = -T$ it finally follows

$$\begin{aligned}(J \operatorname{graph} T^{-1})^\perp &= (J\, J \operatorname{graph}(-T))^\perp\\ &= J\,((J \operatorname{graph}(-T))^\perp)\\ &= J(\operatorname{graph}(-T))\\ &= \operatorname{graph}(T^{-1}),\end{aligned}$$

and thus T^{-1} is self adjoint.

(3) By 9.19, $T \subseteq T^*$ and, if $\operatorname{dom} T = H$, then $T = T^*$ and therefore closed by 9.8. By the Closed Graph Theorem T is bounded.

(4) If $\operatorname{img} T = H$, then T is injective by (1). Let $S := T^{-1}$ with $\operatorname{dom} S = \operatorname{img} T = H$. We have that S is symmetric, because for $f, g \in \operatorname{dom} S$, i.e. $f = Th$ and $g = Tk$ with $h, k \in \operatorname{dom} T$, we have $\langle Sf, g\rangle = \langle h, Tk\rangle = \langle Th, k\rangle = \langle f, Sg\rangle$. By (3) S is a bounded self adjoint injective operator and by (2) $T = S^{-1}$ is self adjoint. $\square$

Spectrum of symmetric operators

We need to determine $\rho(T)$ for symmetric T. By 9.17, $\lambda \in \rho(T)$ is for closed T equivalent to the bijectivity of $T-\lambda : \operatorname{dom} T \to H$, so we should determine $\ker(T-\lambda)$ and $\operatorname{img}(T-\lambda)$.

9.22 Proposition.

Let T be symmetric and $\lambda = \alpha + i\,\beta$ with $\alpha, \beta \in \mathbb{R}$. Then:

1. $\|(T-\lambda)f\|^2 = \|(T-\alpha)f\|^2 + \beta^2\|f\|^2$ *for all* $f \in \operatorname{dom} T$.
2. *For* $\beta \neq 0$ *we have* $\ker(T-\lambda) = \{0\}$.
3. *If T is closed and $\beta \neq 0$, then* $\operatorname{img}(T-\lambda)$ *is closed.*

Proof. (1) The following holds:

$$\begin{aligned}\|(T-\lambda)f\|^2 &= \|(T-\alpha)f - i\,\beta\, f\|^2\\ &= \|(T-\alpha)f\|^2 + 2\,\Re e(\langle (T-\alpha)f, i\,\beta f\rangle) + \|\beta\, f\|^2\\ &= \|(T-\alpha)f\|^2 + 2\beta\,\Im m(\langle (T-\alpha)f, f\rangle) + \beta^2\|f\|^2.\end{aligned}$$

Because of $\langle (T-\alpha)f, f\rangle = \langle Tf, f\rangle - \alpha\|f\|^2 \in \mathbb{R}$ we have (1).

(2) follows directly from (1).

(3) We have $\|(T-\lambda)f\|^2 \geqslant \beta^2\|f\|^2$. Let now $f_n \in \operatorname{dom} T$ with $(T-\lambda)f_n \to g$. Because of the inequality, f_n is a Cauchy sequence. Let $f := \lim_n f_n$. Since $(f_n, (T-\lambda)f_n) \in \operatorname{graph}(T-\lambda)$ and $(f_n, (T-\lambda)f_n) \to (f, g)$, we conclude that $(f, g) \in \operatorname{graph}(T-\lambda)$ because the graph of $(T-\lambda)$ is closed, so $g = (T-\lambda)f \in \operatorname{img}(T-\lambda)$. □

9.23 Proposition.

Let T be a closed symmetric operator.
Then $\lambda \mapsto \dim\ker(T^*-\lambda)$ is locally constant on $\mathbb{C}\backslash\mathbb{R}$.

Here dim denotes the VECTOR SPACE DIMENSION, i.e. the cardinality of a HAMEL BASIS. Note that by 9.10 we have $\ker(T^*-\lambda) = (\operatorname{img}(T-\bar\lambda))^\perp$ and thus $T-\lambda$ is onto if and only if $\dim\ker(T^*-\bar\lambda) = 0$ is it.

Sublemma.

Let H_1 and H_2 be closed subspaces of H with $H_1 \cap H_2^\perp = \{0\}$. Then $\dim H_1 \leqslant \dim H_2$.

Proof. Let P be the orthonormal projection from H onto H_2. Because of $H_1 \cap H_2^\perp = \{0\}$, the restriction $P|_{H_1} : H_1 \to H_2$ is injective. Consequently, $\dim H_2 \geqslant \dim P(H_1) = \dim H_1$. □

Proof of 9.23. Let $\lambda = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$.
We claim that $\ker(T^*-\mu) \cap \ker(T^*-\lambda)^\perp = \{0\}$ for $|\lambda-\mu| < |\beta|$:
Supose this were not true. Then there is an $f \in \ker(T^*-\mu) \cap (\ker(T^*-\lambda))^\perp$ with $\|f\| = 1$. By 9.10, $f \in (\ker(T^*-\lambda))^\perp = \overline{\operatorname{img}(T-\overline{\lambda})}$ and, by 9.22.3, $\operatorname{img}(T-\overline{\lambda})$ is closed. So there is a $g \in \operatorname{dom} T$ with $f = (T-\overline{\lambda})g$. Since $f \in \ker(T^*-\mu)$ we have

$$\begin{aligned}0 &= \langle (T^*-\mu)f, g\rangle = \langle f, (T-\overline{\mu})g\rangle = \langle f, (T-\overline{\lambda}+\overline{\lambda}-\overline{\mu})g\rangle\\ &= \|f\|^2 + (\lambda-\mu)\langle f, g\rangle.\end{aligned}$$

So $1 = \|f\|^2 = |\lambda - \mu|\,|\langle f, g\rangle| \leqslant |\lambda - \mu|\,\|g\|$. From 9.22.1 it follows that $1 = \|f\| = \|(T - \overline{\lambda})g\| \geqslant |\beta|\,\|g\| > |\lambda - \mu|\,\|g\| \geqslant 1$, a contradiction.

From the claim follows by means of the sublema that $\dim\ker(T^* - \mu) \leqslant \dim\ker(T^* - \lambda)$ if $|\lambda - \mu| < |\beta| = |\mathfrak{Im}(\lambda)|$. If $|\lambda - \mu| < \frac{1}{2}|\beta|$, then $|\mathfrak{Im}(\lambda) - \mathfrak{Im}(\mu)| \leqslant |\lambda - \mu| < \frac{1}{2}|\beta| = \frac{1}{2}|\mathfrak{Im}(\lambda)|$, i.e. $|\mathfrak{Im}(\mu)| \geqslant \frac{1}{2}|\mathfrak{Im}(\lambda)|$, and thus also the other inequality holds because of $|\mu - \lambda| < \frac{1}{2}|\mathfrak{Im}(\lambda)| \leqslant |\mathfrak{Im}(\mu)|$. This shows that $\lambda \mapsto \dim\ker(T^* - \lambda)$ is locally constant on $\mathbb{C}\backslash\mathbb{R}$. □

9.24 Theorem.

Let $T : H \rightsquigarrow H$ be a closed symmetric operator, then exactly one of the following things happens:

1. $\sigma(T) \subseteq \mathbb{R}$;
2. $\sigma(T) = \{\lambda \in \mathbb{C} : \mathfrak{Im}(\lambda) \geqslant 0\}$;
3. $\sigma(T) = \{\lambda \in \mathbb{C} : \mathfrak{Im}(\lambda) \leqslant 0\}$;
4. $\sigma(T) = \mathbb{C}$.

Proof. Let $\mathbb{C}_\pm := \{\lambda \in \mathbb{C} : \pm\mathfrak{Im}(\lambda) > 0\}$ be the upper and lower open half-plane. By 9.22.2, $T - \lambda$ is injective and has closed image for all $\lambda \in \mathbb{C}_\pm$ by 9.22.3. Thus, by 9.17.1, $\lambda \in \rho(T)$ if and only if $T - \lambda$ is surjective. Because $(\mathrm{img}(T - \lambda))^\perp = \ker(T^* - \overline{\lambda})$ by 9.10, according to the previous theorem 9.23, either $\mathbb{C}_\pm \cap \sigma(T) = \varnothing$ or $\mathbb{C}_\pm \subseteq \sigma(T)$ (and hence $\overline{\mathbb{C}_\pm} \subseteq \sigma(T)$, since $\sigma(T)$ is closed). So either (1), i.e. $\sigma(T) \cap (\mathbb{C}_+ \cup \mathbb{C}_-) = \varnothing$, or one of the other 3 cases, namely $\sigma(T) \in \{\overline{\mathbb{C}_\pm}, \mathbb{C}\}$. □

9.25 Corollary.

Let $T : H \rightsquigarrow H$ be a closed symmetric operator, then t.f.a.e.:

1. *T is self adjoint;*

$\Leftrightarrow$ 2. $\sigma(T) \subseteq \mathbb{R}$;

$\Leftrightarrow$ 3. $\ker(T^* - i) = \{0\} = \ker(T^* + i)$.

Proof. (1 $\Rightarrow$ 2) From $T = T^*$ and $\mathfrak{Im}(\lambda) \neq 0$, follows $\mathrm{img}(T - \lambda)^\perp = \ker(T^* - \overline{\lambda}) = \ker(T - \overline{\lambda}) = \{0\}$ by 9.22.2. Since $\mathrm{img}(T - \lambda)$ is closed by 9.22.3, $T - \lambda : \mathrm{dom}\, T \to H$ is bijective and thus $\lambda \in \rho(T)$ by 9.17.1. So $\sigma(T) \subseteq \mathbb{R}$.

(2 $\Rightarrow$ 3) If $\sigma(T) \subseteq \mathbb{R}$, then $\pm i \in \rho(T)$, i.e. $\mathrm{img}(T \pm i) = H$ and thus $\ker(T^* \pm i) = \mathrm{img}(T \mp i)^\perp = \{0\}$.

(3 $\Rightarrow$ 1) By 9.22.2, $T \pm i$ is injective, and because $\mathrm{img}(T \pm i)^\perp = \ker(T^* \mp i) = \{0\}$ by (3) and $\mathrm{img}(T - \lambda)$ is closed by 9.22.3, $T \pm i$ is also surjective. Because of 9.13, $T \pm i$ is bounded invertible and according to the lemma in 9.13 also $T^* \mp i$. Let $h \in \mathrm{dom}\, T^*$. Since $T + i$ is invertible, $f \in \mathrm{dom}\, T$ exists with $(T + i)f = (T^* + i)h$. But $T^* + i \supseteq T + i$ and thus $(T^* + i)f = (T + i)f = (T^* + i)h$. Because $T^* + i$ is injective, we have $h = f \in \mathrm{dom}\, T$ and hence $T = T^*$. □

9.26 Corollary.

Let $T : H \rightsquigarrow H$ be a closed symmetric operator.
If $\sigma(T)$ does not contain $\mathbb{R}$, then T is self adjoint.

Proof. None of the cases 2 – 4 in 9.24 can occur, so $\sigma(T) \subseteq \mathbb{R}$ and T is self adjoint by 9.25. □

Symmetrical extensions

A symmetric operator T is not self adjoint if its domain strictly smaller than that of T^*. So we should examine symmetric extensions $\tilde{T}$ of T. In particular we are interested in the question of whether a self adjoint extension exists. For each symmetric extension $\tilde{T}$ of T we have $T \subseteq \tilde{T}$ and thus $\tilde{T}^* \subseteq T^*$, i.e. $T \subseteq \tilde{T} \subseteq \tilde{T}^* \subseteq T^*$. Each symmetric extension of T is therefore a restriction of T^*.

Corollary 9.25 suggests to study the eigenspaces of T^* to eigenvalue $\pm i$ for symmetric operators T. Hence the following

9.27 Definition.

Let $T : H \rightsquigarrow H$ be a closed symmetric operator. The DEFICIENCY-SUBSPACES of T are the eigenspaces of T^* with eigenvalue $\pm i$:

$$D_+ := (\operatorname{img}(T+i))^\perp = \ker(T^* - i) = \{f \in \operatorname{dom} T^* : T^*(f) = +i\,f\},$$
$$D_- := (\operatorname{img}(T-i))^\perp = \ker(T^* + i) = \{f \in \operatorname{dom} T^* : T^*(f) = -i\,f\}.$$

Furthermore, $G_\pm$ are the following closed subspaces of $H \oplus H$:

$$G_+ := \{(f, +i\,f) : f \in D_+\} = \operatorname{graph}(+i) \cap \operatorname{graph}(T^*)$$
$$G_- := \{(g, -i\,g) : g \in D_-\} = \operatorname{graph}(-i) \cap \operatorname{graph}(T^*).$$

The deficiency spaces are therefore also closed, because $\mathrm{pr}_1 : G_\pm \to D_\pm$ is a linear isomorphism with inverse $f \mapsto (f, \pm i\,f)$. The DIMENSIONS of $D_\pm$ as Hilbert space, i.e. the cardinality of a complete orthonormal basis, are denoted as DEFICIENCY INDICES $d_\pm$.

Now for a symmetric operator T we want to determine the part of T^* that extends beyond T.

9.28 Lemma.

Let T be a closed symmetric operator, then

$$\operatorname{graph} T^* = \operatorname{graph} T \oplus G_+ \oplus G_- = \operatorname{graph}\Big(T \oplus (+i)|_{D_+} \oplus (-i)|_{D_-}\Big).$$

In particular, $\operatorname{dom} T^* = \operatorname{dom} T \oplus D_+ \oplus D_-$ *is a direct-sum decomposition in not necessarily orthogonal subspaces.*

Proof. We have $G_\pm \perp \operatorname{graph} T$, because for $f \in D_\pm$ and $h \in \operatorname{dom} T$:

$$\langle h \oplus Th, f \oplus (\pm i\,f)\rangle = \langle h, f\rangle \mp i\langle Th, f\rangle = \mp i\langle (T \pm i)h, f\rangle = 0,$$

because $D_\pm = \operatorname{img}(T \pm i)^\perp$.

We also have $G_+ \perp G_-$, because $\langle f \oplus i\,f, g \oplus (-i\,g)\rangle = \langle f, g\rangle - \langle i\,f, i\,g\rangle = 0$ for $f \in D_+$ and $g \in D_-$.

Since $\operatorname{graph} T \oplus G_+ \oplus G_- \subseteq \operatorname{graph} T^*$ is obviously closed, it suffices to show that this sum has a trivial orthogonal complement in $\operatorname{graph} T^*$: Let $h \in \operatorname{dom} T^*$ with $h \oplus T^*h \perp \operatorname{graph} T \oplus G_+ \oplus G_-$. Because $h \oplus T^*h \perp \operatorname{graph} T$, we have $0 = \langle h \oplus T^*h, f \oplus Tf\rangle = \langle h, f\rangle + \langle T^*h, Tf\rangle$ for all $f \in \operatorname{dom} T$. Consequently, $T^*h \in \operatorname{dom} T^*$ and $(T^*)^2 h = -h$. So $(T^* - i)(T^* + i)h = ((T^*)^2 + 1)h = 0$, and hence $g := (T^* + i)h \in D_+ = \ker(T^* - i)$. Consequently, $0 = \langle h \oplus T^*h, g \oplus ig\rangle = \langle h, g\rangle - i\langle T^*h, g\rangle = -i\langle (T^* + i)h, g\rangle = -i\|(T^* + i)h\|^2$, hence $(T^* + i)h = 0$, i.e. $h \in D_-$. For symmetry reasons $h \in D_+$ also holds. So $h \in D_+ \cap D_- = \{0\}$.

Since $\mathrm{pr}_1 : \operatorname{graph} T^* \to \operatorname{dom} T^*$ is a linear bijection, the direct-sums decomposition of $\operatorname{dom} T^*$ immediately follows from that of $\operatorname{graph} T^*$. $\square$

9.29 Lemma.

Each symmetric operator T has a maximal symmetric extensions. Every such extension $\tilde T$ is closed. Each self adjoint operator is a maximal symmetric operator.

Proof. The fact that each self adjoint operator T is maximally symmetric follows immediately from the fact that every symmetric extension of T is a restriction of $T^* = T$.

The existence of maximal symmetric extensions follows directly from Zorn's Lemma.

Now let $\tilde T$ be a maximal symmetric operator. Since, according to the lemma in 9.20, the operator $\tilde T^{**}$ is a closed symmetric extension of $\tilde T$, we have $\tilde T = \tilde T^{**}$ and thus $\tilde T$ is also closed. □

9.30 Lemma.

Let $T : H \rightsquigarrow H$ be a closed symmetric operator. Then there is a bijection

$$\left\{\tilde T \supseteq T : \tilde T \text{ closed, symm.}\right\} \cong \left\{F < D_+ \oplus D_- : T^*|_F \text{ closed, symm.}\right\},$$

i.e. the closed symmetric extensions $\tilde T$ of T are in bijective correspondance with the subspaces F of $D_+ \oplus D_-$ for which $T^*|_F$ is a closed symmetric operator. This relation between $\tilde T$ and F is given by:

$$\operatorname{graph} \tilde T = \operatorname{graph} T \oplus \operatorname{graph}(T^*|_F).$$

Proof. ($\leftarrow$) Let F be such a subspace. We put $D := \operatorname{dom} T \oplus F \subseteq \operatorname{dom} T^*$ and $\tilde T := T^*|_D \supseteq T$. Then $\tilde T$ is symmetric, because for $f = f_0 + f_1$ and $g = g_0 + g_1$ with $f_0, g_0 \in \operatorname{dom} T$ and $f_1, g_1 \in F$ we have

$$\begin{aligned}\langle \tilde T f, g\rangle &= \langle T^* f_0 + T^* f_1, g_0 + g_1\rangle \\ &= \langle T f_0, g_0\rangle + \langle T f_0, g_1\rangle + \langle T^* f_1, g_0\rangle + \langle T^* f_1, g_1\rangle\end{aligned}$$

(By the symmetry of T and of $T^*|_F$ and the adjointness of T^* zu T)

$$\begin{aligned}&= \langle f_0, T g_0\rangle + \langle f_0, T^* g_1\rangle + \langle f_1, T g_0\rangle + \langle f_1, T^* g_1\rangle \\ &= \langle f, \tilde T g\rangle.\end{aligned}$$

By 9.28, $\operatorname{graph} \tilde T = \operatorname{graph} T \oplus \operatorname{graph}(T^*|_F)$ is an orthogonal decomposition, and since both summands are closed, T is closed.

($\to$) Let $\tilde T \supseteq T$ be closed and symmetrical. Then $T \subseteq \tilde T \subseteq T^*$ and thus $\operatorname{graph} T \subseteq \operatorname{graph} \tilde T \subseteq \operatorname{graph} T^* = \operatorname{graph} T \oplus G_+ \oplus G_-$. Let $G := \operatorname{graph} \tilde T \cap (G_+ \oplus G_-)$ and $F := \operatorname{pr}_1(G) \subseteq (D_+ \oplus D_-) \cap \operatorname{dom} \tilde T$. Then $T^*|_F = \tilde T|_F$ is also symmetric and because $\operatorname{graph}(T^*|_F) = G$ we deduce that $T^*|_F$ is also closed.

For $h \oplus \tilde T h \in \operatorname{graph} \tilde T \subseteq \operatorname{graph} T^*$, we have $h \oplus \tilde T h = (f \oplus T f) + k$ with $f \in \operatorname{dom} T$ and $k \in G_+ \oplus G_-$ by 9.28. And because $T \subseteq \tilde T$ we have $k \in \operatorname{graph} \tilde T$ and thus $k \in G$, thus $\operatorname{graph} \tilde T = \operatorname{graph} T \oplus \operatorname{graph}(T^*|_F)$.

The two assignments are inverse to each other, because if $F := \operatorname{pr}_1(\operatorname{graph} \tilde T \cap (G_+ \oplus G_1)) < D_+ \oplus D_-$ is the subspace associated with extension $\tilde T$, then obviously $\tilde T = T \cup T^*|_F = T^*|_{\operatorname{dom} T \oplus F}$ because of the last equation. And on the other hand, if $\tilde T = T^*|_{\operatorname{dom} T \oplus F}$ is the extension belonging to the subspace F, then $G := \operatorname{graph} \tilde T \cap (G_+ \oplus G_-) = \operatorname{graph}(T^*|_F)$ and thus $F = \operatorname{pr}_1(G)$. □

9.31 Theorem.

Let $T : H \rightsquigarrow H$ be a closed symmetric operator. Then there is a bijection

$$\Big\{\tilde T \supseteq T : \tilde T \text{ closed, symm.}\Big\} \cong$$
$$\cong \Big\{U : U \text{ is part. iso. with initial space } I_+ \subseteq D_+ \text{ and final space } I_- \subseteq D_-\Big\},$$

i.e. the closed symmetric extensions $\tilde T$ of T are in bijection with the partial isometries U with initial space $I_+ \subseteq D_+$ and final-space $I_- \subseteq D_-$. This relation between $\tilde T$ and U is given by:

$$\operatorname{dom}\tilde T = \{h + k + Uk : h \in \operatorname{dom} T, k \in I_+\}$$
$$\tilde T(h + k + Uk) = Th + i\,k - i\,Uk.$$

For the deficiency indices we have $d_\pm(\tilde T) + \dim I_\pm = d_\pm(T)$.

Proof. Because of 9.30 it suffices to describe a bijection between subspaces F of $D_+ \oplus D_-$ with $T^*|_F$ symmetric and closed and the specified partial isometrics U.

($\to$) Let F be a subspace of $D_+ \oplus D_-$ with $T^*|_F$ closed and symmetric. We want to show that F is the graph of a (unique) isometry $U : I_+ \to I_-$ with $I_\pm \subseteq D_\pm$. For $f \in F$ let $f = f^+ \oplus f^-$ be the direct sum decomposition with $f^\pm \in D_\pm$. Furthermore, let $I_\pm := \{f^\pm : f \in F\}$. Since $T^*|_F$ is symmetric, $0 = \langle T^* f, f\rangle - \langle f, T^* f\rangle = \langle i f^+ - i f^-, f^+ + f^-\rangle - \langle f^+ + f^-, i f^+ - i f^-\rangle = 2i\langle f^+, f^+\rangle - 2i\langle f^-, f^-\rangle$ holds, so $\|f^+\| = \|f^-\|$. If $f^+ \oplus f^{1-}$ and $f^+ \oplus f^{2-}$ are two vectors from $F \subseteq D_+ \oplus D_-$, then $0 \oplus (f^{1-} - f^{2-}) \in F$ and thus $\|f^{1-} - f^{2-}\| = \|0\| = 0$ by what has just been shown, i.e. $f^{1-} = f^{2-}$. So F is the graph of the bijective isometry $U : I_+ \to I_-$ defined by $U(f^+) := f^-$.

We have that I_+ is closed: Let $f_n \in F$ with $f_n^+ \to g^+$. Since $\|f_n^+ - f_m^+\| = \|f_n^- - f_m^-\|$, there exists an g^- with $f_n^- \to g^-$. Obviously, $f_n = f_n^+ + f_n^-$ converges towards $g^+ + g^- =: g$. Furthermore, $T^* f_n^\pm = \pm i\, f_n^\pm \to \pm i g^\pm$ holds. And it follows $(g^+ + g^-) \oplus (i g^+ - i g^-) \in \overline{\operatorname{graph}(T^*|_F)} = \operatorname{graph}(T^*|_F)$, i.e. $g^+ \in I^+$.

($\leftarrow$) Let U be a partial isometry with initial space $I_+ \subseteq D_+$ and final-space $I_- \subseteq D_-$. We define $F := \operatorname{graph} U|_{\operatorname{ini} U} := \{g \oplus Ug : g \in I_+\} \subseteq I_+ \oplus I_- \subseteq D_+ \oplus D_-$.

Then $T^*|_F$ *is symmetrical*, because $Ug, Uh \in I_- \subseteq D_- = \ker(T^* + i)$ for $g, h \in I_+ \subseteq D_+ = \ker(T^* - i)$ and thus

$$\begin{aligned}\langle T^*(g + Ug), h + Uh\rangle &= \langle T^* g, h\rangle + \langle T^* g, Uh\rangle + \langle T^* Ug, h\rangle + \langle T^* Ug, Uh\rangle\\ &= i\langle g, h\rangle + i\langle g, Uh\rangle - i\langle Ug, h\rangle - i\langle Ug, Uh\rangle\\ &= i\langle g, Uh\rangle - i\langle Ug, h\rangle.\end{aligned}$$

And similary one shows $\langle g + Ug, T^*(h + Uh)\rangle = i\langle g, Uh\rangle - i\langle Ug, h\rangle$.

Furthermore, $T^*|_F$ *is closed*: For $g_n \in I_+$ with $(g_n + Ug_n) \oplus (i g_n - i\,U\,g_n) \to f \oplus h$, we have that $2i g_n = i(g_n + Ug_n) + (i g_n - i\,U\,g_n) \to i f + h$ and $2i\,U\,g_n = i(g_n + Ug_n) - (i g_n - i\,U\,g_n) \to i f - h$ hold. Thus, $U(i\,f + h) = i\,f - h$ and for $g := \frac{1}{2i}(i f + h)$ we have that $f = g + Ug$ and $h = i\,g - i\,U\,g$ hold.

Obviously, the two assignments $U \leftrightarrow \operatorname{graph} U|_{\operatorname{ini} U} = F$ are inverse to each other.

By 9.30 we obtain the desired bijection with

$$\begin{aligned}\operatorname{dom}\tilde T &:= \operatorname{dom} T \oplus F = \operatorname{dom} T \oplus \operatorname{graph} U|_{\operatorname{ini} U}\\ &= \{h \oplus k \oplus U(k) : h \in \operatorname{dom} T, k \in \operatorname{ini} U\}\\ \tilde T &:= T^*|_{\operatorname{dom}\tilde T} = (h \oplus k \oplus Uk \mapsto Th + i\,k - i\,Uk).\end{aligned}$$

We finally show $d_+(\tilde T) + \dim I_+ = d_+(T)$: Let $f \in \operatorname{dom} T$ and $g \in I_+$. Then

$$(\tilde T + i)(f + g + Ug) = (T + i)f + ig - i\,U\,g + ig + i\,U\,g = (T + i)f + 2ig.$$

So we have the orthogonal decomposition $\operatorname{img}(\tilde T + i) = \operatorname{img}(T + i) \oplus I_+$ and thus $\operatorname{img}(T + i)^\perp = \operatorname{img}(\tilde T + i)^\perp \oplus I_+$. So $d_+(T) = \dim(\operatorname{img}(T + i)^\perp) = \dim(\operatorname{img}(\tilde T + i)^\perp) + \dim(I_+) = d_+(\tilde T) + \dim I_+$. Similarly one shows $d_-(\tilde T) = d_-(T) - \dim I_-$. □

9.32 Theorem.

Let $T : H \rightsquigarrow H$ be a closed symmetric operator with deficiency indices $d_\pm < \infty$. Then:

1. *T is a maximal symmetric operator if and only if $d_+ = 0$ or $d_- = 0$.*
2. *T is self-adjoint if and only if $d_+ = 0 = d_-$.*
3. *T has a self adjoint extension if and only if $d_+ = d_-$. In this case, the self adjoint extensions are in bijective correspondance with the isometries from D_+ onto D_-.*

Proof. (1) is a direct corollary to 9.31, because only the trivial partial isometry $U = 0$ exists, provided D_+ or D_- is equal to $\{0\}$.

(2) is a reformulation of 9.25.

(3) If T has a self adjoint extension $\tilde T$, then $d_\pm(\tilde T) = d_\pm(T) - \dim(I_\pm)$, where $U : I_+ \to I_-$ is the associated bijective isometry. So $\dim(I_+) = \dim(I_-)$ as well as $d_+(\tilde T) = d_-(\tilde T)$ by (2), and thus $d_+(T) = d_-(T)$. Conversely, it follows from $d_+ = d_-$ that a bijective isometry $U : D_+ \to U_-$ exists, and the associated extension $\tilde T$ thus satisfies $d_+(\tilde T) = d_+(T) - \dim(I_+) = d_-(T) - \dim(I_-) = d_-(\tilde T)$, i.e. is self adjoint by (2). □

9.33 Example.

Let $T_0 : f \mapsto if'$ be the symmetric operator from Example 9.5. In order to determine all closed symmetric extensions of T_0 we have to specify D_+ and D_-. We have $f \in D_\pm$ if and only if $f \in \operatorname{dom} T_0^*$ and $\pm if = T_0^* f = if'$. So $D_\pm = \{x \mapsto \alpha e^{\pm x} : \alpha \in \mathbb{C}\}$ and $d_\pm = 1$. All partial isometries $U \neq 0$ from D_+ to D_- are of form $U_\lambda(x \mapsto e^x)(x) = \lambda e^{-x}$ with $|\lambda| = 1$. Let

$$D_\lambda := \Big\{x \mapsto f(x) + \alpha e^x + \lambda\alpha e^{-x} : \alpha \in \mathbb{C}, f \in \operatorname{dom} T_0\Big\}$$

$$T_\lambda\Big(x \mapsto f(x) + \alpha e^x + \lambda\alpha e^{-x}\Big)(x) := if'(x) + \alpha i e^x - i\,\lambda\alpha e^{-x},$$

for $f \in \operatorname{dom} T_1$ and $\alpha \in \mathbb{C}$. By 9.31 these are all true symmetric closed (self adjoint) extensions of T_0. In particular, the domain

$$\begin{aligned} D_1 &= \big\{f + 2\,\alpha\, \cosh : f \in \operatorname{dom} T_0, \alpha \in \mathbb{C}\big\} \\ &= \Big\{g \in L^2 : g \text{ is absolutely continuous}, g' \in L^2, g(-1) = g(1)\Big\} \\ T_1(g) &= \tilde T_1(f + 2\,\alpha, \cos h) = i\,f' + i\,\alpha\,2\,\sinh = i\,g', \end{aligned}$$

is exactly the self adjoint extension of T_0 in Example 9.5.

Let T be a linear differential operator with real coefficients functions. Then $\operatorname{dom} T$ is invariant under conjugation and $\overline{Tf} = T\overline{f}$. We now want to show that symmetric operators with such a property possess self adjoint extensions.

9.34 Corollary.

Let $T : H \rightsquigarrow H$ be a symmetric operator and $J : H \to H$ a conjugated linear bounded operator (such as the conjugation for example) with $J^2 = 1$ and $T \circ J \subseteq J \circ T$. Then T has a self adjoint extension.

Proof. From $TJ \subseteq JT$ follows $JT = JTJ^2 \subseteq J^2TJ = TJ$ and thus $TJ = JT$. Consequently, $\operatorname{dom} T = \operatorname{dom}(J \circ T) = \operatorname{dom}(T \circ J) = J^{-1}(\operatorname{dom} T) = J(\mathrm{DomT})$. Since J is not linear, we need to define the adjoint J^*: For $h \in H$, the mapping $f \mapsto \langle h, Jf\rangle$ is a bounded linear functional, so a unique $J^*h \in H$ exists with $\langle h, Jf\rangle = \langle f, J^*h\rangle$. Obviously, J^* is additive and conjugated linear since $\langle f, J^*(\lambda h)\rangle = \langle \lambda h, Jf\rangle = \lambda\langle h, Jf\rangle = \lambda\langle f, J^*h\rangle = \langle f, \overline{\lambda}J^*h\rangle$. Because of $J^2 = 1$ also $(J^*)^2 = 1$.

We next claim that $J^*T^* = T^*J^*$.
Let $h^* \in \operatorname{dom} T^*$ and $h \in \operatorname{dom} T$. Then $\langle TJh, h^*\rangle = \langle Jh, T^*h^*\rangle = \langle J^*T^*h^*, h\rangle$ and thus $\langle TJh, h^*\rangle = \langle JTh, h^*\rangle = \langle J^*h^*, Th\rangle$. Consequently, $\langle J^*T^*h^*, h\rangle = \langle J^*h^*, Th\rangle$, i.e. $J^*h^* \in \operatorname{dom} T^*$ and $T^*J^*h^* = J^*T^*h^*$, and thus $T^*J^* \subseteq J^*T^*$. Because of $(J^*)^2 = 1$, equality follows as before.

Let now $h^* \in \ker(T^* \pm i)$. Then $T^*J^*h^* = J^*T^*h^* = J^*(\mp i\, h^*) = \pm i J^*h^*$. So $J^*(\ker(T^* \pm i)) \subseteq \ker(T^* \mp i)$. Because of $(J^*)^2 = 1$, the other inclusion also holds, so the two deficiency-spaces are via J^* isomorphic as real lcs's and thus also as complex Hilbert spaces (Choose orthonormal basis and extend the bijection to a linear isometry) and thus T has a self adjoint extension by 9.31, cf. 9.32. □

Cayley Transformation

For the Möbius transformation $\mu : z \mapsto \frac{z-i}{z+i}$ we have: $0 \mapsto -1$, $1 \mapsto -i$, $\infty \mapsto 1$, $i \mapsto 0$. Since Möbius transformations map straight lines to straight lines or circles, μ maps $\mathbb{R} \cup \{\infty\}$ to $\partial\mathbb{D}$ and thus the upper half-plane to the unit disk $\mathbb{D}$. The inverse mapping is given by $w \mapsto i\frac{1+w}{1-w}$, because $\frac{z-i}{z+i} = w$ implies $z(1-w) = i\,(1+w)$. Since the spectrum of self adjoint operators is included in $\mathbb{R}$ and that of unitary operator in $\mu(\mathbb{R}) = \partial\mathbb{D}$, this μ should yield a correspondance between these classes of operators. In fact, we have

9.35 Theorem (Cayley Transformation).

The closed symmetric operators $T : H \rightsquigarrow H$ are in bijective correspondance to the partial isometries U, for which $(1-U)\operatorname{ini} U$ lies dense, i.e.

$$\Big\{T : H \rightsquigarrow H,\ \textit{closed, symm.}\Big\} \cong \Big\{U \in L(H) : U \textit{ part. iso., } (1-U)\operatorname{ini} U \textit{ dense}\Big\},$$

with respect to the relations:

$$\begin{aligned} U &= (T-i)\,(T+i)^{-1} \\ T &= i\,(1+U)\,(1-U)^{-1} \\ D_+(T) &= \operatorname{ini} U^{\perp} \\ D_-(T) &= \operatorname{fin} U^{\perp}. \end{aligned}$$

This assignment is called the CAYLEY TRANSFORMATION, and the U belonging to T is called the CAYLEY TRANSFORM of T.

Proof.

($\to$) Let T be a closed symmetric operator. By 9.22.3, $\operatorname{img}(T \pm i)$ is closed, so $D_\pm^\perp = \operatorname{img}(T \pm i)$. By 9.22.2, $\ker(T+i) = \{0\}$, so $(T+i)^{-1}$ is well-defined on $D_+^\perp$

and $(T+i)^{-1}D_+^\perp = \operatorname{dom} T = \operatorname{dom}(T-i)$ and thus the described U is a well-defined operator.

$$\begin{tikzcd}
& & \operatorname{dom} T \arrow[dl, "T+i"'] \arrow[dr, "T-i"] & & \\
D_+^\perp \arrow[r, equal] & \operatorname{img}(T+i) \arrow[ur, dotted, bend right, "\frac{1}{2i}(1-U)"'] \arrow[rr, dotted, "U", "\cong"'] & & \operatorname{img}(T-i) \arrow[r, equal] & D_-^\perp
\end{tikzcd}$$

If $h \in D_+^\perp$, then $h = (T+i)f$ with a unique $f \in \operatorname{dom} T$. So $\|Uh\|^2 = \|(T-i)f\|^2 = \|Tf\|^2 + \|f\|^2 = \|(T+i)f\|^2 = \|h\|^2$ by 9.22.1. Hence U can be uniquely extended to a partial isometry with $\operatorname{ini} U := (\ker U)^\perp = D_+^\perp$ and $\operatorname{fin} U := \operatorname{img} U = D_-^\perp$.

We have $(T+i)^{-1} = \frac{1}{2i}(1-U) : D_+^\perp \to \operatorname{dom} T$, because $(1-U)h = h - (T-i)f = (T+i)f - (T-i)f = 2if$ for $f \in \operatorname{dom} T$ and $h = (T+i)f$.
Consequently, $(1-U)\operatorname{ini} U = \operatorname{dom} T$ and thus is dense.

Furthermore, $(1+U)(T+i) = 2T$, because $(1+U)(T+i)f = (T+i)f + Uh = (T+i)f + (T-i)f = 2Tf$, and consequently $i(1+U)(1-U)^{-1} = i(1+U)\frac{1}{2i}(T+i) = \frac{1}{2}\,2T = T$.

($\Leftarrow$) Let now U be a partial isometry as stated. Then $\ker(1-U) = \{0\}$, because $Uf = f$ is valid for $f \in \ker(1-U)$ and thus $\|f\| = \|Uf\|$, i.e. $f \in \operatorname{ini} U$. Since U^*U is the orthogonal projection on $\operatorname{ini} U$ (see 7.24), $f = U^*Uf = U^*f$, so $f \in \ker(1-U^*) = \operatorname{img}(1-U)^\perp = \{0\}$, i.e. $f = 0$, because $\operatorname{img}(1-U) \supseteq (1-U)\operatorname{ini} U$ is dense.

Let $D := (1-U)\operatorname{ini} U$. Then $(1-U)^{-1} : D \to \operatorname{ini} U$ is well-defined. So $T := i(1+U)(1-U)^{-1}$ is a well-defined operator with domain D.

$$\begin{tikzcd}
& \operatorname{ini} U \arrow[dl, "1-U"'] \arrow[dr, "i(1+U)"] & \\
D = (1-U)\operatorname{ini} U \arrow[rr, dotted, "T"] & & (1+U)\operatorname{ini} U
\end{tikzcd}$$

Again $(1-U)^{-1} = \frac{1}{2i}(T+i) : D \to \operatorname{ini} U$, because for $h \in \operatorname{ini} U$ and $f = (1-U)h$ we have $(T+i)f = Tf + if = i(1+U)h + i(1-U)h = 2ih$.
Consequently, $\operatorname{ini} U = \operatorname{img}(T+i) = D_+(T)^\perp$.

Furthermore, $(T-i)(1-U) = 2iU$, since $(T-i)(1-U)h = i(1+U)h - i(1-U)h = 2iU$, and thus $(T-i)(T+i)^{-1} = (T-i)\frac{1}{2i}(1-U) = \frac{1}{2i}2iU = U$ and $\operatorname{fin} U = \operatorname{img}(T-i) = D_-(T)^\perp$.

We have that T is closed: Let $f_n \in (1-U)\operatorname{ini} U$ with $f_n \to f$ and $Tf_n \to g$ and let $h_n \in \operatorname{ini} U$ be so that $(1-U)h_n = f_n$. Then $Tf_n = i(1+U)h_n$ and thus $2\,i\,h_n = i(1-U)h_n + i(1+U)h_n = if_n + Tf_n \to if + g =: 2ih \in \operatorname{ini} U$. So $f_n = (1-U)h_n \to (1-U)h$ and $Tf_n = i(1+U)h_n \to i(1+U)h$, and thus $g = i(1+U)h = T(1-U)h = Tf$.
Furthermore, T is symmetrical: For $f, g \in D$, let $f = (1-U)h$ and $g = (1-U)k$ with $h, k \in \operatorname{ini} U$. Then

$$\langle Tf, g\rangle = i\langle (1+U)h, (1-U)k\rangle = i(\langle h,k\rangle + \langle Uh,k\rangle - \langle h,Uk\rangle - \langle Uh,Uk\rangle).$$

Since $h, k \in \operatorname{ini} U$, we have $\langle Uh, Uk\rangle = \langle h, k\rangle$, so $\langle Tf, g\rangle = i(\langle Uh,k\rangle - \langle h,Uk\rangle)$ and analogously one shows $\langle f, Tg\rangle = -i\langle (1-U)h, (1+U)k\rangle = -i(\langle h,Uk\rangle - \langle Uh,k\rangle) = \langle Tf, g\rangle$. □

9.36 Corollary.

The self adjoint operators are, via the Cayley transformation, in bijective correspondance to the unitary operators, for which 1 is not an eigenvalue.

Proof. A symmetric closed operator is self adjoint by 9.32 if and only if $\{0\} = D_\pm$, i.e. by 9.35, if and only if for the associated partial isometry $I_\pm = H$ holds, i.e. it is unitary. Finally, we have seen in the proof of 9.35 that the denseness of $\operatorname{img}(1-U)$ implies the equation $\ker(1-U) = \{0\}$ – i.e. 1 is not an eigenvalue of U. Conversely, 1 is not an eigenvalue of U and $f \perp \operatorname{img}(1-U)$, i.e. $f \in \operatorname{img}(1-U)^\perp = \ker(1-U^*)$. So $U^*f = f$ and thus $Uf = UU^*f = f$, i.e. $f \in \ker(1-U) = \{0\}$, so $\operatorname{img}(1-U) = (1-U)(\operatorname{ini} U)$ is dense. □

One can use the Cayley transformation to deduce from the spectral decomposition for bounded unitary operators also one for unbounded self adjoint operators. However, in the next section we will develop more generally the spectral theory of normal unbounded operators.

Unbounded normal operators

9.37 Definition.

A linear operator $T : H \rightsquigarrow H$ is called NORMAL if it is densly defined, closed and satisfies $T^*T = TT^*$. Obviously, any self adjoint operator is normal. The multiplication operator T in example 9.4 is normal, but note that $\operatorname{dom} T^*T \subset \operatorname{dom} T$ holds.

9.38 Lemma.

For densely defined closed T, the following holds:

1. *The graph of $T|_{\operatorname{dom}(T^*T)}$ is dense in the graph of T.*
2. *T^*T is self adjoint (and, in particular, densely defined).*
3. *$1 + T^*T$ is bounded invertible, and for the inverse $0 \leqslant (1+T^*T)^{-1} \leqslant 1$.*
4. *The operator $T(1+T^*T)^{-1}$ is a global contraction.*

Proof. (3) *$1 + T^*T$ is surjective:* Let $J : H \oplus H \to H \oplus H$ be again defined by $J(h,k) = (-k,h)$. By 9.7 we have $H \oplus H = J \operatorname{graph} T + \operatorname{graph} T^*$. For $h \in H$, therefore, $f \in \operatorname{dom} T$ and $g \in \operatorname{dom} T^*$ exist with $(0,h) = J(f,Tf) + (g,T^*g) = (-Tf,f) + (g,T^*g)$, i.e. $0 = -Tf + g$ and $h = f + T^*g = f + T^*Tf = (1+T^*T)f$. So $\operatorname{img}(1+T^*T) = H$.
$1 + T^*T$ is injective: For $f \in \operatorname{dom} T^*T$ we have $Tf \in \operatorname{dom} T^*$ and $\|f + T^*Tf\|^2 = \|f\|^2 + 2\|Tf\|^2 + \|T^*Tf\|^2 \geqslant \|f\|^2$. Hence $\ker(1+T^*T) = \{0\}$.
We have $0 \leqslant S := (1+T^*T)^{-1} \leqslant 1$: From $\|(1+T^*T)f\| \geqslant \|f\|$ for all $f \in \operatorname{dom} T^*T$ we deduce the inequality $\|Sh\| \leqslant \|h\|$ for $h = (1+T^*T)f$ and $S := (1+T^*T)^{-1}$, i.e. $\|S\| \leqslant 1$. Furthermore, $\langle Sh, h\rangle = \langle f, (1+T^*T)f\rangle = \|f\|^2 + \|Tf\|^2 \geqslant 0$, i.e. $S \geqslant 0$.

(1) Since T is closed, it suffices to show that for no vector $g \neq 0$ the vector $(g,Tg) \in \operatorname{graph} T$ is orthogonal to $\{(h,Th) : h \in \operatorname{dom} T^*T\}$. Let $h \in \operatorname{dom} T^*T$. Then

$$0 = \langle (g,Tg),(h,Th)\rangle = \langle g,h\rangle + \langle Tg,Th\rangle = \langle g,h\rangle + \langle g,T^*Th\rangle = \langle g,(1+T^*T)h\rangle.$$

So $g \perp \operatorname{img}(1+T^*T) \overset{(3)}{=\!=} H$ and thus $g = 0$.

(2) It follows from (1) that $\operatorname{dom} T^*T$ is dense in $\operatorname{dom} T$ and hence in H. Let $f, g \in \operatorname{dom} T^*T$, i.e. $f, g \in \operatorname{dom} T$ and $Tf, Tg \in \operatorname{dom} T^*$. Consequently, $\langle T^*Tf, g\rangle =$

$\langle Tf, Tg\rangle = \langle f, T^*Tg\rangle$ holds. So T^*T is symmetrical. Furthermore, $1 + T^*T$ has a bounded inverse by (3), so $-1 \notin \sigma(T^*T)$ and $1 + T^*T$ is closed by 9.13 and therefore also T^*T. Because of 9.26, T^*T is self adjoint.

(4) We put $R := T(1+T^*T)^{-1} = TS : H \to \operatorname{dom}(T^*T) \subseteq \operatorname{dom} T \to H$. If $h = (1+T^*T)f$ with $f \in \operatorname{dom} T^*T \subseteq \operatorname{dom} T$, then $\|Rh\|^2 = \|Tf\|^2 \leqslant \|(1 + T^*T)f\|^2 = \|h\|^2$ by the proof of (3). So $\|R\| \leqslant 1$. □

9.39 Corollary.

For each normal operator $T : H \rightsquigarrow H$ *we have* $\operatorname{dom} T = \operatorname{dom} T^*$ *and* $\|Th\| = \|T^*h\|$ *for all* $h \in \operatorname{dom} T$. *Normal operators do not have non-trivial normal extensions.*

Proof. If $h \in \operatorname{dom} T^*T = \operatorname{dom} TT^*$, then $Th \in \operatorname{dom} T^*$ and $T^*h \in \operatorname{dom} T$. So $\|Th\|^2 = \langle T^*Th, h\rangle = \langle TT^*h, h\rangle = \|T^*h\|^2$.

If $f \in \operatorname{dom} T$, it follows from 9.38.1 that a sequence $h_n \in \operatorname{dom} T^*T$ exists with $(h_n, Th_n) \to (f, Tf)$, so $\|Th_n - Tf\| \to 0$. By the first part $\|T^*h_n - T^*h_m\| = \|Th_n - Th_m\|$ holds and thus there is an $g \in H$ with $T^*h_n \to g$. So $(h_n, T^*h_n) \to (f, g)$ holds. Because T^* is closed by 9.8.1, $f \in \operatorname{dom} T^*$ and $g = T^*f$. So $\operatorname{dom} T \subseteq \operatorname{dom} T^*$ and $\|Tf\| = \lim_n \|Th_n\| = \lim_n \|T^*h_n\| = \|g\| = \|T^*f\|$.

By 9.9, $T^{**} = T$ and, by 9.8.1 and 9.8.2, also T^* is normal, i.e. by the previous part $\operatorname{dom} T^* \subseteq \operatorname{dom}(T^*)^* = \operatorname{dom} T \subseteq \operatorname{dom} T^*$, i.e. $\operatorname{dom} T = \operatorname{dom} T^*$.

Let now $\tilde T \supseteq T$ be a normal extension. Then $\tilde T^* \subseteq T^*$ and hence $\operatorname{dom} T \subseteq \operatorname{dom} \tilde T = \operatorname{dom} \tilde T^* \subseteq \operatorname{dom} T^* = \operatorname{dom} T$. So $T = \tilde T$. □

9.40 Remark.

Let $S, S_1, S_2 : H_1 \rightsquigarrow H_2$ and $T, T_1, T_2 : H_2 \rightsquigarrow H_3$, then

$$\begin{aligned} &T_1 \circ S + T_2 \circ S = (T_1 + T_2) \circ S;\\ &T \circ S_1 + T \circ S_2 \subseteq T \circ (S_1 + S_2);\\ &T \circ S_1 + T \circ S_2 = T \circ (S_1 + S_2) \text{ if } T \text{ is globally defined.}\end{aligned}$$

The first row follows from

$$\begin{aligned}\operatorname{dom}((T_1 + T_2) \circ S) &= S^{-1}(\operatorname{dom}(T_1 + T_2)) = S^{-1}(\operatorname{dom}(T_1) \cap \operatorname{dom}(T_2))\\ &= S^{-1}(\operatorname{dom}(T_1)) \cap S^{-1}(\operatorname{dom}(T_2))\\ &= \operatorname{dom}(T_1 \circ S) \cap \operatorname{dom}(T_2 \circ S) = \operatorname{dom}(T_1 \circ S + T_2 \circ S).\end{aligned}$$

The second row follows from

$$\begin{aligned}\operatorname{dom}(T \circ S_1 + T \circ S_2) &= \operatorname{dom}(T \circ S_1) \cap \operatorname{dom}(T \circ S_2)\\ &= S_1^{-1}(\operatorname{dom} T) \cap S_2^{-1}(\operatorname{dom} T)\\ &\subseteq (S_1 + S_2)^{-1}(\operatorname{dom} T) = \operatorname{dom}(T \circ (S_1 + S_2)).\end{aligned}$$

If T is globally defined then equality holds, because then $S^{-1}(\operatorname{dom} T) = \operatorname{dom} S$ for $S \in \{S_1, S_2, S_1 + S_2\}$. Otherwise, the inclusion might be strict, as the example $S_1 = \mathrm{id} = -S_2$ shows, because then $T \circ (S_1 + S_2) = 0$ is globally defined and $\operatorname{dom}(T \circ S_1 + T \circ S_2) = \operatorname{dom}(T \circ S_1) \cap \operatorname{dom}(T \circ S_2) = \operatorname{dom} T$.

9.41 Lemma.

Let H_n be Hilbert spaces and $T_n \in L(H_n)$. Let $H := \bigoplus_n H_n$ and $\bigoplus_n T_n : H \rightsquigarrow H$ be defined on $D := \{(h_n) \in \bigoplus_n H_n : \sum_n \|T_n h_n\|^2 < \infty\}$ by $(h_n)_n \mapsto (T_n h_n)_n$.
Then $\bigoplus_n T_n$ is a closed densely defined operator. Its adjoint is $(\bigoplus_n T_n)^* = \bigoplus_n T_n^*$ and $\bigoplus_n T_n$ is normal if and only if all T_n are so.

For any second sequence of operators $S_n \in L(H_n)$ we have: $(\bigoplus_n T_n) \circ (\bigoplus_n S_n) \subseteq \bigoplus_n (T_n \circ S_n)$. If additionally $(\|S_n\|)_n$ is bounded, then equality holds.

Proof. Obviously, D is a linear subspace and $T := \bigoplus_n T_n$ is linear on D. Since $H_n \subseteq D$ for all n, D is dense in H.

Claim: *T is closed.*
Let $h^{(j)}$ be a sequence in $\operatorname{dom} T$ with $(h^{(j)}, Th^{(j)}) \to (h, g)$ in $H \oplus H$. Then for the components we have $(h_n^{(j)}, T_n h_n^{(j)}) \to (h_n, g_n)$. Since T_n is bounded, we have $T_n h_n = g_n$ and thus $\sum_n \|T_n h_n\|^2 = \sum_n \|g_n\|^2 = \|g\|^2 < \infty$, i.e. $h \in \operatorname{dom} T$, and obviously $Th = g$, so T is closed.

Claim: $T^*((k_n)_n) = (T_n^* k_n)_n$ for $(k_n)_n \in \operatorname{dom} T^* = \{(k_n) : \sum_n \|T_n^* k_n\|^2 < \infty\}$.
($\supseteq$) We have $k \in \operatorname{dom} T^*$ if and only if

$$h \mapsto \langle h, T^* k\rangle := \langle Th, k\rangle = \sum_n \langle T_n h_n, k_n\rangle = \sum_n \langle h_n, T_n^* k_n\rangle$$

is a bounded linear functional on $\operatorname{dom} T$. Because of the Cauchy-Schwarz inequality this is the case for k with $\sum_n \|T_n^* k_n\|^2 < \infty$. That $T^* k$ is given for such k by $T^* k = (T_n^* k_n)_n$ is obvious.
($\subseteq$) For $k \in \operatorname{dom} T^*$ there is an $C > 0$ with $|\langle Th, k\rangle| \leqslant C\,\|h\|$ and thus, with $h_n := T_n^* k_n$ for each finite partial sum $\sum \|T_n^* k_n\|^2 = \sum \langle h_n, T_n^* k_n\rangle \leqslant C\,\sqrt{\sum \|h_n\|^2} = C\,\sqrt{\sum \|T_n^* k_n\|^2}$, we have $\sum \|T_n^* k_n\|^2 \leqslant C^2$. Hence $\sum_{n=1}^\infty \|T_n^* k_n\|^2 \leqslant C^2$.

Now let $S_n \in L(H_n)$ be a second sequence of operators, and let $T := \bigoplus_n T_n$ and $S := \bigoplus_n S_n$. For

$$h \in \operatorname{dom}(T \circ S) = \left\{ h = (h_n)_n : \begin{array}{c} \sum_n \|h_n\|^2 < \infty, \\ \sum_n \|S_n h_n\|^2 < \infty, \\ \sum_n \|T_n(S_n h_n)\|^2 < \infty \end{array} \right\}$$

obviously $h \in \operatorname{dom}(\bigoplus_n (T_n \circ S_n))$ and we have

$$\begin{aligned} \Big(\bigoplus_n (T_n \circ S_n)\Big)(h) &= \Big((T_n \circ S_n)(h_n)\Big)_n = \Big(\bigoplus_n T_n\Big)\Big(S_n(h_n)\Big)_n \\ &= \Big(\bigoplus_n T_n\Big)\Big(\Big(\bigoplus_n S_n\Big) h\Big) = \Big(\Big(\bigoplus_n T_n\Big) \circ \Big(\bigoplus_n S_n\Big)\Big) h, \end{aligned}$$

i.e. $(\bigoplus_n T_n) \circ (\bigoplus_n S_n) \subseteq \bigoplus_n (T_n \circ S_n)$.

If $\|S_n\|$ is bounded, then because of the Cauchy-Schwarz inequality, the domain of $S = \bigoplus_n S_n$ is all of H and $\|S\| = \sup_n \|S_n\|$. For $h = (h_n)_n \in \operatorname{dom}(\bigoplus_n (T_n \circ S_n))$, $\sum_n \|h_n\|^2 < \infty$ implies the estimate $\sum_n \|S_n h_n\|^2 \leqslant \|S\|^2 \sum_n \|h_n\|^2 < \infty$, so $h \in \operatorname{dom}(T \circ S)$ and hence we have equality.

If $\bigoplus_n T_n$ is normal, obviously also the restrictions T_n are normal.
Conversely, by 9.39,

$$\begin{aligned} \operatorname{dom}(T^* \circ T) &= \{h \in \operatorname{dom} T : Th \in \operatorname{dom} T^*\} \\ &= \left\{ h = (h_n)_n : \begin{array}{c} \sum_n \|h_n\|^2 < \infty, \\ \sum_n \|T_n^* h_n\|^2 = \sum_n \|T_n h_n\|^2 < \infty, \\ \sum_n \|T_n T_n^* h_n\|^2 = \sum_n \|T_n^* T_n h_n\|^2 < \infty \end{array} \right\} \\ &= \{h \in \operatorname{dom} T^* : T^* h \in \operatorname{dom} T\} \\ &= \operatorname{dom}(T \circ T^*), \end{aligned}$$

and both $T^* \circ T$ and $T \circ T^*$ are restrictions of $\bigoplus T_n^* \circ T_n = \bigoplus T_n \circ T_n^*$. So T is normal. □

9.42 Theorem .

Let $P : \mathcal{B}(X) \to L(H)$ be a spectral measure as in 8.7. For a measurable function $f : X \to \mathbb{C}$, consider a partition of X in measurable sets Δ_n on which f is bounded (e.g., $\Delta_n := \{x \in X : n-1 \leqslant |f(x)| < n\}$). We also use $H_n := P(\Delta_n)H$ and let $P_n : \mathcal{B}(\Delta_n) \to L(H_n)$ be the spectral measure $P_n(\Delta) := P(\Delta)|_{H_n}$.

Then $H = \bigoplus_{n=1}^\infty H_n$ and with respect to this decomposition

$$\int_X f\, dP := \bigoplus_{n=1}^\infty \int_{\Delta_n} f|_{\Delta_n}\, dP_n,$$

is the normal operator

$$\int_X f\, dP : h = (h_n)_n \mapsto \bigoplus_{n=1}^\infty \Big(\int_{\Delta_n} f\, dP_n\Big)\, h_n$$

with domain of definition

$$D_f := \Big\{h \in H : \sum_{n=1}^\infty \Big\|\Big(\int_{\Delta_n} f\, dP_n\Big)\, h_n\Big\|^2 < \infty\Big\} = \Big\{h : \int_X |f|^2\, dP_{h,h} < \infty\Big\}$$

and for $h \in D_f$ and $k \in H$ we have $f \in L^1(|P_{h,k}|)$ with

$$\int_X |f|\, d|P_{h,k}| \leqslant \Big(\int_X |f|^2\, dP_{h,h}\Big)^{1/2} \|k\| \quad \text{and} \quad \Big\langle \Big(\int_X f\, dP\Big)\, h, k\Big\rangle = \int_X f\, dP_{h,k}.$$

In particular, the operator $\int_X f\, dP$ and its domain do not depend on the selection of the Δ_n.

Proof. Since $P(\Lambda) \circ P(\Delta_n) = P(\Lambda \cap \Delta_n) = P(\Delta_n) \circ P(\Lambda)$, we have that $H_n := P(\Delta_n)H$ is an $P(\Lambda)$-invariant subspace, and thus P_n is a well-defined spectral measure for H_n. Because of $1 = P(X) = P(\bigsqcup_n \Delta_n) = \sum_n P(\Delta_n)$, we have $H = \bigoplus_n H_n$ and the orthogonal projection onto H_n is given by $h \mapsto h_n := P(\Delta_n)h$.

Since $f|_{\Delta_n}$ is bounded, $\int_{\Delta_n} f\, dP_n$ is a well-defined bounded normal operator on H_n by 8.12. Thus, by 9.41, $\int_X f\, dP := \bigoplus_n \int_{\Delta_n} f\, dP_n$ is a normal unbounded operator with domain D_f.

Next we show the claimed equation for D_f:
According to the spectral theory 8.12 for bounded operators we have:

$$\begin{aligned}\Big\|\Big(\int f_{\Delta_n}\, dP_n\Big)\, h_n\Big\|^2 &= \Big\langle\Big(\int_{\Delta_n} f\, dP_n\Big)^*\Big(\int_{\Delta_n} f\, dP_n\Big) h_n, h_n\Big\rangle \\ &= \Big\langle\Big(\int_{\Delta_n} \overline{f}\, f\, dP_n\Big)\, h_n, h_n\Big\rangle = \Big\langle\Big(\int_{\Delta_n} |f|^2\, dP_n\Big) h_n, h_n\Big\rangle \\ &= \int_{\Delta_n} |f|^2\, d(P_n)_{h_n,h_n} = \int_{\Delta_n} |f|^2\, dP_{h,h},\end{aligned}$$

since for $\Lambda \subseteq \Delta_n$ we have:

$$\begin{aligned}P_{h,h}(\Lambda) &= \langle P(\Lambda)h, h\rangle \\ &= \langle P(\Delta_n \cap \Lambda \cap \Delta_n)h, h\rangle = \langle P(\Delta_n)P(\Lambda)P(\Delta_n)\, h, h\rangle \\ &= \langle P(\Lambda)\, P(\Delta_n)h, P(\Delta_n)h\rangle = \langle P(\Lambda)\, h_n, h_n\rangle \\ &= \langle P_n(\Lambda)\, h_n, h_n\rangle = (P_n)_{h_n,h_n}(\Lambda).\end{aligned}$$

From this follows the asserted equation on D_f. And thus the domain of $\int_X f\,dP$ is independent of the choice of the partition into sets Δ_n.

Let now $h \in D_f$ and $k \in H$. According to the Radon-Nikodym Theorem 8.33, there is a measurable function u with $|u| = 1$ and $|P_{h,k}| = u\,P_{h,k}$, where $|P_{h,k}|$ is the variation of $P_{h,k}$. Let $f_{\leqslant n} := f|_{\bigsqcup_{k\leqslant n}\Delta_k} = \sum_{k=1}^n \chi_{\Delta_k} f$. We have both $f_{\leqslant n}$ and $u\,f_{\leqslant n}$ bounded and therefore:

$$\begin{aligned}\int |f_{\leqslant n}|\,d|P_{h,k}| &= \int |f_{\leqslant n}|\,u\,dP_{h,k} = \Big\langle\Big(\int |f_{\leqslant n}|\,u\,dP\Big)h, k\Big\rangle \\ &\leqslant \Big\|\Big(\int |f_{\leqslant n}|\,u\,dP\Big)h\Big\| \cdot \|k\|.\end{aligned}$$

and further

$$\begin{aligned}\Big\|\Big(\int |f_{\leqslant n}|\,u\,dP\Big)h\Big\|^2 &= \Big\langle\Big(\int |f_{\leqslant n}|\,u\,dP\Big)h, \Big(\int |f_{\leqslant n}|\,u\,dP\Big)h\Big\rangle \\ &= \Big\langle\Big(\int |f_{\leqslant n}|^2\,dP\Big)h, h\Big\rangle = \int |f_{\leqslant n}|^2\,dP_{h,h} \leqslant \int |f|^2\,dP_{h,h}.\end{aligned}$$

So $\int |f_{\leqslant n}|\,d|P_{h,k}| \leqslant \Big(\int |f|^2\,dP_{h,h}\Big)^{1/2}\|k\|$ for all n. Since $|f_{\leqslant n}|$ monotonously converges pointwise towards $|f|$, it follows by means of the theorem of Beppo Levi on monotone convergence that $f \in L^1(|P_{h,k}|)$ and the desired inequality

$$\int |f|\,d|P_{h,k}| \leqslant \Big(\int_X |f|^2\,dP_{h,h}\Big)^{1/2}\|k\|.$$

Since $f_{\leqslant n}$ is bounded, also

$$\Big\langle\Big(\int f_{\leqslant n}\,dP\Big)h, k\Big\rangle = \int f_{\leqslant n}\,dP_{h,k}$$

holds by 8.12.1. If $h \in D_f$ and $k \in H$, it follows by the theorem on dominated convergence that

$$\int f_{\leqslant n}\,dP_{h,k} \to \int f\,dP_{h,k} \text{ for } n \to \infty.$$

On the other hand:

$$\begin{aligned}\Big(\int f_{\leqslant n}\,dP\Big)\,h &= \Big(\bigoplus_{j\leqslant n}\int_{\Delta_j} f|_{\Delta_j}\,dP_j\Big)\,(\dots,h_n,0,\dots) \\ &= \Big(\int f\,dP\Big)\,P\Big(\bigcup_{j=1}^n \Delta_j\Big)\,h \xlongequal{\boxed{9.41}} P\Big(\bigcup_{j=1}^n \Delta_j\Big)\,\Big(\int f\,dP\Big)\,h,\end{aligned}$$

and since $P\Big(\bigcup_{j=1}^n \Delta_j\Big) \to P(X) = 1$ in the SOT, finally follows

$$\Big\langle\Big(\int f_{\leqslant n}\,dP\Big)\,h, k\Big\rangle \to \Big\langle\Big(\int f\,dP\Big)\,h, k\Big\rangle.$$

So

$$\Big\langle\Big(\int_X f\,dP\Big)\,h, k\Big\rangle = \int_X f\,dP_{h,k}.$$

This also shows that the operator $\int_X f\,dP$ is independent on the selection of the partition in sets Δ_n. □

9.43 Proposition.

Let $P : \mathcal{B}(X) \to L(H)$ be a spectral measure. For each measurable function $f : X \to \mathbb{C}$ a linear operator $\rho(f) : H \rightsquigarrow H$ is defined by $\rho(f) := \int_X f\,dP$. Then for measurable functions $f, g : X \to \mathbb{C}$ holds:

1. $\rho(f)^* = \rho(\overline{f})$.
2. $\rho(f\,g) \supseteq \rho(f)\,\rho(g)$ *and* $\mathrm{dom}(\rho(f)\,\rho(g)) = D_g \cap D_{f\,g}$.
3. *If g is bounded, so is* $\rho(f)\,\rho(g) = \rho(f\,g)$.
4. $\rho(f)^*\,\rho(f) = \rho(|f|^2)$.

Proof. For given measurable functions $f, g : X \to \mathbb{C}$ we choose a partition of X into measurable sets Δ_n and define a spectral measure P_n on Δ_n for $H_n := P(\Delta_n)H$ as in 9.42. Let ρ_n be the associated C^*-representation of the bounded functions on Δ_n on H_n. Then $\rho(h) := \bigoplus_n \rho_n(h)$ for $h \in \{f, \overline{f}, g, f \cdot g\}$. For the C^*-representation ρ_n of course (1) - (4) holds with equality everywhere. Using 9.41 we now obtain:

(1) because

$$\rho(f)^* = \Big(\bigoplus_n \rho_n(f)\Big)^* = \bigoplus_n \rho_n(f)^* = \bigoplus_n \rho_n(\overline{f}) = \rho(\overline{f}).$$

(2) The inclusion is valid because

$$\begin{aligned}\rho(f) \circ \rho(g) &= \Big(\bigoplus_n \rho_n(f)\Big) \circ \Big(\bigoplus_n \rho_n(g)\Big) \subseteq \bigoplus_n (\rho_n(f) \circ \rho_n(g)) = \bigoplus_n (\rho_n(f\,g)) \\ &= \rho(f\,g).\end{aligned}$$

Furthermore, $h \in \mathrm{dom}(\rho(f) \circ \rho(g))$ holds exactly when $h \in \mathrm{dom}(\rho(g)) =: D_g$ and $\rho(g)\,h \in \mathrm{dom}(\rho(f)) =: D_f$. The latter means that $\infty > \sum_n \|\rho_n(f)(\rho_n(g)h)\|^2 = \sum_n \|\rho_n(f\,g)h\|^2$, i.e. $h \in D_{f\,g}$.

(3) If g is bounded, then $D_g = H$ and thus $\mathrm{dom}(\rho(f)\rho(g)) = H \cap \mathrm{dom}(\rho(fg)) = \mathrm{dom}(\rho(fg))$.

Note that under this assumption, $\rho(g\,f) = \rho(g)\,\rho(f)$ does not hold, in contrast to what is stated in [**5**, X.4.10]. Namely, let e.g. $g = 0$, then $g\,f = 0$ and $D_{g\,f} = H$ but $\mathrm{dom}(\rho(g)\,\rho(f)) = D_f \cap D_{gf} = \mathrm{dom}(\rho(f)) \subset H$.

(4) By (1) and (2), we have $\rho(f)^* \circ \rho(f) = \rho(\overline{f}) \circ \rho(f) \subseteq \rho(|f|^2)$ and $\mathrm{dom}(\rho(f)^* \circ \rho(f)) = \mathrm{dom}(\rho(\overline{f}) \circ \rho(f)) = D_f \cap D_{|f|^2}$. So it only remains to show $D_{|f|^2} \subseteq D_f$. Let $h = (h_n)_n \in D_{|f|^2}$, i.e. $\sum_n \|\rho_n(|f|^2)h_n\|^2 < \infty$. Two-fold application of Cauchy-Schwarz's inequality shows

$$\begin{aligned}\sum \|\rho_n(f)h_n\|^2 &= \sum \langle \rho_n(f)^*\rho_n(f)h_n, h_n\rangle \leqslant \sum \|\rho_n(f)^*\rho_n(f)h_n\|\,\|h_n\| \\ &\leqslant \Big(\sum \|\rho_n(|f|^2)h_n\|^2\Big)^{1/2}\ \|h\| < \infty,\end{aligned}$$

i.e. $h \in D_f$. □

9.44 Theorem.

Let $N : H \rightsquigarrow H$ be a normal operator on H.
Then there is a unique spectral measure P defined on the Borel sets of $\mathbb{C}$, s.t.

1. $N = \int_{\mathbb{C}} z\,dP(z)$.
2. $P(\Lambda) = 0$ *if* $\Lambda \cap \sigma(N) = \varnothing$.
3. *If $U \subseteq \mathbb{C}$ is open and $U \cap \sigma(N) \neq \varnothing$ then $P(U) \neq 0$.*
4. *If $A \in L(H)$ with $AN \subseteq NA$ and $AN^* \subseteq N^*A$,*
   then $A\left(\int_{\mathbb{C}} f\,dP\right) \subseteq \left(\int_{\mathbb{C}} f\,dP\right) A$ for all Borel functions f on $\mathbb{C}$.

The Fugledge-Putnam theorem is also valid for unbounded normal operators, and thus the hypothesis $AN^* \subseteq N^*A$ in (4) can be dropped.

About the idea of the proof: If $N := \int z\, dP(z)$, we could split $\mathbb{C}$ into annuli Δ_n. Then $H_n := P(\Delta_n)H$ would be invariant subspaces with $H = \bigoplus_n H_n$ and we could compare N with the unbounded sum $\bigoplus_n N|_{H_n}$.

Conversely, we should therefore find a decomposition $H = \bigoplus_n H_n$ into $\{N, N^*\}$-invariant subspaces H_n so that $N_n := N|_{H_n}$ is a bounded normal operator. By the spectral theorem for bounded operators the spectral measures P_n with $N_n = \int z\, dP_n$ exist. We want to sum this up to get a spectral measure P for N.
The function $f : z \mapsto \frac{1}{1+|z|^2} = (1 + \bar{z}z)^{-1}$ maps $\mathbb{C}$ to the interval $(0, 1]$. The annuli correspond to subintervals. So in order to find the spaces H_n without using the not yet available spectral measure P of N, we consider the contraction $S := (1+N^*N)^{-1} \geqslant 0$ from 9.38 and the images of its spectral projectors (which would be $P \circ f^{-1}$ by 8.59 for bounded N) on subintervals of $(0, 1] \subset [0, 1] \supseteq \sigma(S)$.

Sublemma.

Let $N : H \rightsquigarrow H$ be normal, $S := (1 + N^*N)^{-1}$ and $S = \int_0^1 t\, dP(t)$ the spectral representation.
Then $S\,N \subseteq N\,S$ and $S\,N\,S = N\,S\,S$.
If Δ is a Borel subset in $[\delta, 1]$ with $0 < \delta < 1$, then $H_\Delta := P(\Delta)H$ is an $\{S, N, N^*\}$-invariant subset of $\operatorname{dom} N$, furthermore $S|_{H_\Delta}$ is invertible and $N|_{H_\Delta}$ is a bounded normal operator with $\|N|_{H_\Delta}\| \leqslant \sqrt{\frac{1}{\delta} - 1}$.

Proof. By 9.38.3 and 9.38.4, S and $N\,S$ are global contractions.

$S\,N \subseteq N\,S$:
Let $f \in \operatorname{dom} SN$. Then $g := Sf \in \operatorname{img} S = \operatorname{dom} N^*N \subseteq \operatorname{dom} N$, i.e. $f = (1+N^*N)g$ and thus $N^*Ng = f - g \in \operatorname{dom} SN - \operatorname{dom} N^*N \subseteq \operatorname{dom} N$. Hence $Ng \in \operatorname{dom} NN^*$ and consequently $Nf = N(1 + N^*N)g = Ng + NN^*Ng = (1 + NN^*)Ng = (1 + N^*N)Ng$, due to the normality of N. Finally $SNf = S(1+N^*N)Ng = Ng = NSf$, i.e. $S\,N \subseteq N\,S$.

Also $S\,N\,S \subseteq N\,S\,S$ follows and, since $\operatorname{dom} NS = H$ by 9.38.4 and thus also $\operatorname{dom} SNS = H$, thus $S\,N\,S = N\,S\,S$.

Let now $\Delta \subseteq [\delta, 1]$ be a Borel set.

Claim: $S : H_\Delta \to H_\Delta$ is an isomorphism.

Since S commutes with its spectral projectors $P(\Delta)$, we have the nearby commutative diagram. Consequently, $S|_{H_\Delta}$ has dense image in H_Δ since $S(H_\Delta) = S(P(\Delta)H) = P(\Delta)(SH)$ is dense in $P(\Delta)H = H_\Delta$ because $SH = \operatorname{dom} N^*N$ is dense in H by 9.38.2.

$$\begin{tikzcd}
H \arrow[r, tail, two heads, "S"] \arrow[d, "P(\Delta)"'] & \operatorname{dom} N^*N \arrow[r, hook] & H \arrow[d, "P(\Delta)"] \\
H_\Delta \arrow[rr, tail, dotted, "S|_{H_\Delta}"] \arrow[d, hook] & & H_\Delta \arrow[d, hook] \\
H \arrow[rr, tail, "S"] & & H
\end{tikzcd}$$

For $h \in H_\Delta$, we have $h = P(\Delta)h$ and hence

$$\begin{aligned}
\|Sh\|^2 &= \langle S^2 P(\Delta)h, h\rangle = \Big\langle \Big(\int_0^1 t^2\, \chi_\Delta\, dP\Big)\, h, h \Big\rangle = \int_\Delta t^2\, dP_{h,h} \\
&\geqslant \delta^2 P_{h,h}(\Delta) = \delta^2 \langle P(\Delta)h, h\rangle = \delta^2\, \|h\|^2.
\end{aligned}$$

So $S|_{H_\Delta}$ has closed image in H_Δ and since this is dense, $S|_{H_\Delta}$ is an isomorphism.

We have $H_\Delta \subseteq \operatorname{dom} N$, because $H_\Delta = S(H_\Delta) \subseteq \operatorname{img} S = \operatorname{dom}(N^*N) \subseteq \operatorname{dom} N$.

Claim: H_Δ is N-invariant.
Let $h \in H_\Delta$ and $g \in H_\Delta$ with $h = S\,g$. Let $R := N\,S \in L(H)$. Then $S\,R = S\,N\,S = N\,S\,S = R\,S$ by the above and thus $P(\Delta)\,R = R\,P(\Delta)$ by 8.15, so H_Δ is R-invariant. Consequently, $N\,h = N\,S\,g = R\,g \in H_\Delta$.

Claim: H_Δ is N^*-invariant.
If $N_1 := N^*$ and $S_1 := (1 + N_1^* N_1)^{-1} = (1 + NN^*)^{-1} = (1 + N^*N)^{-1} = S$. From the previous claim follows that $N^* H_\Delta = N_1 H_\Delta \subseteq H_\Delta$.

It follows that the restriction $N|_{H_\Delta}$ is also normal.

Finally let $h \in H_\Delta$. Then, similar we obtain

$$\|Nh\|^2 = \langle N^*Nh, h\rangle = \langle (S^{-1} - 1)h, h\rangle = \int_\delta^1 (\tfrac{1}{t} - 1)\, dP_{h,h}(t) \leqslant \|h\|^2\, (\tfrac{1}{\delta} - 1).$$

So $\|N|_{H_\Delta}\| \leqslant \sqrt{\frac{1}{\delta} - 1}$. □

Proof of 9.44. As in the sublemma, let $S := (1 + N^*N)^{-1}$ and $R := NS$. Furthermore, $S = \int_0^1 t\, dP(t)$ is the spectral representation, and let $P_n := P(\frac{1}{n+1}, \frac{1}{n}]$ and $H_n := P_n H$ for $n \geqslant 1$. So $1 = P(\sigma(S)) = P(\{0\}) + \sum_{n=1}^\infty P_n$. Since $\ker S = \{0\}$, $\lambda = 0$ is not an eigenvalue of S and thus $P(\{0\}) = 0$ by 8.17, hence $1 = \sum_{n=1}^\infty P_n$ and thus $H = \bigoplus_n H_n$. By the sublemma, H_n is an $\{N, N^*\}$-invariant subspace of $\operatorname{dom} N$ and $N_n := N|_{H_n}$ is a bounded normal operator with $\|N_n\| \leqslant \sqrt{n}$.

So if $\lambda \in \sigma(N_n)$, then

$$\begin{aligned}
\frac{1}{1 + |\lambda|^2} \in \sigma((1 + N_n^* N_n)^{-1}) &= \sigma(S|_{H_n}) = \sigma((S \circ P_n)|_{H_n}) \\
&= \sigma\Bigg(\Big(\int t \cdot \chi_{(\frac{1}{n+1}, \frac{1}{n}]}(t)\, dP(t)\Big)\Big|_{H_n}\Bigg) \\
&\subseteq \sigma\Big(\int t\, \chi_{(\frac{1}{n+1}, \frac{1}{n}]}(t)\, dP(t)\Big) \\
&= \text{ess-image}(\{t\, \chi_{(\frac{1}{n+1}, \frac{1}{n}]}(t) : t \in \sigma(S)\}) \quad \text{by } 8.60 \\
&\subseteq \overline{\{t\, \chi_{(\frac{1}{n+1}, \frac{1}{n}]}(t) : t \in (0, 1])\}} \subseteq \{0\} \cup [\tfrac{1}{n+1}, \tfrac{1}{n}],
\end{aligned}$$

i.e. $\sigma(N_n) \subseteq \{\lambda \in \mathbb{C} : \frac{1}{n+1} \leqslant \frac{1}{1+|\lambda|^2} \leqslant \frac{1}{n}\} = \{\lambda \in \mathbb{C} : \sqrt{n} \geqslant |\lambda| \geqslant \sqrt{n-1}\} =: \Delta_n$.

Now let $P_n : \mathcal{B}(\Delta_n) \to L(H_n)$ be the spectral measure of N_n and let P be defined on each Borel set $\Lambda \subseteq \mathbb{C}$ by

$$P(\Lambda) := \bigoplus_{n=1}^\infty P_n(\Lambda \cap \Delta_n).$$

In order to show that P is a spectral measure, we first note that clearly $P(X) = 1$. We have that $P_n(\Lambda \cap \Delta_n)$ is an orthogonal projection with image in H_n and thus $P(\Lambda)$ is an orthogonal projection in $L(H)$. Since the H_n are pairwise orthogonal, we have for Borel sets Λ_i:

$$\begin{aligned}
P(\Lambda_1)\, P(\Lambda_2) &= \Big(\bigoplus_{n=1}^\infty P_n(\Lambda_1 \cap \Delta_n)\Big) \Big(\bigoplus_{n=1}^\infty P_n(\Lambda_2 \cap \Delta_n)\Big) \\
&= \bigoplus_{n=1}^\infty P_n(\Lambda_1 \cap \Delta_n)\, P_n(\Lambda_2 \cap \Delta_n) = \bigoplus_{n=1}^\infty P_n(\Lambda_1 \cap \Lambda_2 \cap \Delta_n) \\
&= P(\Lambda_1 \cap \Lambda_2).
\end{aligned}$$

For $h \in H$ we obtain $P(\Lambda)h = \big(P_n(\Lambda \cap \Delta_n)h_n\big)_n$. If Λ_n are pairwise disjoint Borel sets, then $\sum_j P(\Lambda_j)$ converges pointwise to $\boxed{8.4}$, and thus:

$$\Big\langle P\Big(\bigsqcup_{j=1}^{\infty} \Lambda_j\Big)h, h\Big\rangle = \sum_{n=1}^{\infty}\Big\langle P_n\Big(\bigsqcup_{j=1}^{\infty} \Lambda_j \cap \Delta_n\Big)h_n, h_n\Big\rangle = \sum_{n=1}^{\infty}\Big\langle \sum_{j=1}^{\infty} P_n(\Lambda_j \cap \Delta_n)h_n, h_n\Big\rangle$$
$$= \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \underbrace{\langle P_n(\Lambda_j \cap \Delta_n)h_n, h_n\rangle}_{\geqslant 0} = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\langle P_n(\Lambda_j \cap \Delta_n)h_n, h_n\rangle$$
$$= \sum_{j=1}^{\infty}\langle P(\Lambda_j)h, h\rangle = \Big\langle \sum_{j=1}^{\infty} P(\Lambda_j)h, h\Big\rangle.$$

Hence, P is σ-additive.

($\boxed{1}$) For $h = (h_n)_n \in \operatorname{dom}(\bigoplus_n N_n)$, we have $\big((h_1, \dots, h_n, 0, \dots), (N_1 h_1, \dots, N_n h_n, 0, \dots)\big) \in \operatorname{graph} N$ (because of $N_n := N|_{H_n}$) and this expression converges to $\big(h, (\bigoplus_n N_n)h\big)$. Since N is closed, $h \in \operatorname{dom} N$ and $Nh = (\bigoplus_n N_n)h$. However, since both N and $\bigoplus_n N_n$ are normal, $N = \bigoplus_n N_n = \bigoplus_n \int z\, dP_n(z) =: \int z\, dP(z)$ by $\boxed{9.39}$.

Claim: $\sigma(N) = \overline{\bigcup_{n=1}^{\infty} \sigma(N_n)}$.
Obviously, $\sigma(N) \supseteq \bigcup_{n=1}^{\infty} \sigma(N_n)$ and, since $\sigma(N)$ is closed, this shows ($\supseteq$). Conversely, let $\lambda \notin \overline{\bigcup_{n=1}^{\infty} \sigma(N_n)}$. Then there is a $\delta > 0$ with $|\lambda - z| \geqslant \delta$ for all $z \in \bigcup_{n=1}^{\infty} \sigma(N_n)$. So $(N_n - \lambda)^{-1}$ and $\|(N_n - \lambda)^{-1}\| = \|z \mapsto (z-\lambda)^{-1}\|_\infty \leqslant \frac{1}{\delta}$ exists for each n. Consequently, $\bigoplus_{n=1}^{\infty}(N_n - \lambda)^{-1}$ is a bounded operator and equal to $(N-\lambda)^{-1}$, i.e. $\lambda \notin \sigma(N)$.

($\boxed{2}$) The following holds: $\Lambda \cap \sigma(N) = \emptyset \Rightarrow \forall n : \Lambda \cap \sigma(N_n) = \emptyset \Rightarrow \forall n : P_n(\Lambda) = 0 \Rightarrow P(\Lambda) = 0$.

($\boxed{3}$) If U is open and $U \cap \sigma(N) \neq \emptyset$, then the above claim implies that $U \cap \sigma(N_n) \neq \emptyset$ for an n. Since then $P_n(U) \neq 0$ by $\boxed{8.15}$, we also have $P(U) \neq 0$.

($\boxed{4}$) Now let $A \in L(H)$ with $AN \subseteq NA$ and $AN^* \subseteq N^*A$. Then $A(1+N^*N) \subseteq (1+N^*N)A$ by $\boxed{9.40}$. So $SA \subseteq AS$, and since both sides are globally defined, $SA = AS$ holds. Thus, according to $\boxed{8.15}$, A commutes with the spectral projections of S and, in particular, H_n is invariant with respect to A. Thus, $A_n := A|_{H_n} \in L(H_n)$ and $A_n N_n = N_n A_n$. So $A_n\, f(N_n) = f(N_n)\, A_n$ holds for any bounded Borel function f. By $\boxed{9.41}$, $A\big(\int_X f\, dP\big) = (\bigoplus_n A_n) \circ (\bigoplus_n f(N_n)) \subseteq \bigoplus(A_n \circ f(N_n)) = \bigoplus(f(N_n) \circ A_n) = (\bigoplus_n f(N_n)) \circ (\bigoplus_n A_n) = \big(\int_X f\, dP\big) A$ now follows, since $\bigoplus_n A_n$ is a bounded operator. $\square$

9.45 Theorem.

Let $N : H \rightsquigarrow H$ be a normal operator on a separable Hilbert space H. Then there is a σ-finite measure space (X, Ω, μ) and a Ω-measurable function $f : X \to \mathbb{C}$, so that N is unitary equivalent to M_f on $L^2(\mu)$.

Proof. We decompose N into the unbounded sum of bounded normal operators N_n as in the proof of $\boxed{9.44}$. According to theorem $\boxed{8.35}$, there are σ-finite measure spaces (X_n, Ω_n, μ_n) and bounded Ω_n-measurable function f_n, so that N_n is unitary-equivalent to M_{f_n}. Let X be the disjoint union of X_n and $\Omega := \{\Delta \subseteq X : \Delta \cap X_n \in \Omega_n$ for all $n\}$. If $\Delta \in \Omega$ then let $\mu(\Delta) := \sum_{n=1}^{\infty} \mu_n(\Delta \cap X_n)$. Furthermore, $f : X \to \mathbb{C}$ is defined by $f|_{X_n} := f_n$. Then f is Ω-measurable and $N = \bigoplus_n N_n \sim \bigoplus_n M_{f_n} = M_f$ on $L^2(X, \Omega, \mu)$. $\square$

9.46 Example.

We now want to find a unitary operator U, which transforms the impulse operator $P : f \mapsto i\, f'$ into a multiplication operator. For this purpose we recall the Fourier transform $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ from chapter [18, 8]. It was defined by

$$\mathcal{F}f(y) := \int_{\mathbb{R}} f(x)\, e^{-ixy}\, dx$$

and satisfied the Parseval equation

$$\langle \mathcal{F}f, \mathcal{F}g\rangle = \frac{1}{2\pi}\langle f, g\rangle.$$

To make it truly unitary, we modify it with a factor $\frac{1}{\sqrt{2\pi}}$, i.e. redefine

$$\mathcal{F}f(y) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\, e^{-ixy}\, dx.$$

Since $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is a surjective isometry with inverse $\mathcal{F}^{-1}f = S(\mathcal{F}f)$ (where S denotes the reflection) and $\mathcal{S}$ is dense in L^2, it can be extended to a unique unitary operator $\mathcal{F} : L^2 \to L^2$.

For $f \in \mathcal{S}$, as we have seen in [18, 8.1.5], we have:

$$\begin{aligned}(P \circ \mathcal{F})f(y) &= i\frac{d}{dy}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)\, e^{-ixy}\, dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)\,(-i^2)\, x\, e^{-ixy}\, dx\\ &= (\mathcal{F} \circ Q)f(y),\end{aligned}$$

where Q denotes the location operator. So we have $P|_{\mathcal{S}} = \mathcal{F} \circ Q|_{\mathcal{S}} \circ \mathcal{F}^{-1}$, and since P is the closure of $P|_{C_c^\infty}$ by $\boxed{9.6}$ and thus also of $P|_{\mathcal{S}}$, and analogously Q is that of $Q|_{\mathcal{S}}$ by $\boxed{9.4}$, we have $P = \overline{P|_{\mathcal{S}}} = \overline{\mathcal{F} \circ Q|_{\mathcal{S}} \circ \mathcal{F}^{-1}} = \mathcal{F} \circ \overline{Q|_{\mathcal{S}}} \circ \mathcal{F}^{-1} = \mathcal{F} \circ Q \circ \mathcal{F}^{-1}$. In fact, it is sufficient to show that Q is the closure of $Q|_{\mathcal{S}}$, because obviously P contains the closure of $P|_{\mathcal{S}}$, i.e. the self adjoint operator $\overline{P|_{\mathcal{S}}} = \overline{\mathcal{F} \circ Q|_{\mathcal{S}} \circ \mathcal{F}^{-1}} = \mathcal{F} \circ \overline{Q|_{\mathcal{S}}} \circ \mathcal{F}^{-1} = \mathcal{F} \circ Q \circ \mathcal{F}^{-1}$. Since self adjoint operators are maximally symmetric, this has to be P.

Because $\mathcal{F}^{-1} = S \circ \mathcal{F}$, we have conversely $Q = \mathcal{F}^{-1} \circ P \circ \mathcal{F} = S \circ \mathcal{F} \circ P \circ S^{-1} \circ \mathcal{F}^{-1} = -\mathcal{F} \circ S \circ S^{-1} \circ P \circ \mathcal{F}^{-1} = -\mathcal{F} \circ P \circ \mathcal{F}^{-1}$, since

$$\begin{aligned}(S \circ \mathcal{F})f(y) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)\, e^{-ix(-y)}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)\, e^{-i(-x)y}\, dx\\ &= -\frac{1}{\sqrt{2\pi}}\int_{+\infty}^{-\infty} f(-x)\, e^{-ixy}\, dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} Sf(x)\, e^{-ixy}\, dx = (\mathcal{F} \circ S)f(y)\end{aligned}$$

and

$$(S \circ P)f(y) = P(f)(-y) = i\, f'(-y) = -i\frac{d}{dy}(f'(-y)) = -(P \circ S)f(y).$$

1-parameter groups and infinitesimal generators

Motivation.

In classical mechanics, the equation of motion is given by Newton's law

$$F(x) = m \cdot \ddot{x} \qquad \text{(Force = mass} \times \text{acceleration)}\ .$$

With the Ansatz $q := x$ and $p := m\dot{x}$ (impulse = mass × velocity), this ordinary 2-nd order differential equation is converted into the following first order differential equation:

$$\dot{q} = \frac{1}{m} p$$
$$\dot{p} = F(q)$$

If the field of force is a gradient field, i.e. $F = -\operatorname{Grad} U$, and the energy E (=Hamilton function H) is defined as sum of the KINETIC ENERGY $\frac{m\|\dot{x}\|^2}{2} = \frac{\|p\|^2}{2m}$ and the POTENTIAL ENERGY $U(q)$, one obtains

$$E(q, p) := \frac{|p|^2}{2m} + U(q)$$

and $\frac{\partial E}{\partial q} = \operatorname{Grad} U = -F$ and $\frac{\partial E}{\partial p} = \frac{1}{m} p$. So the energy is a MOTION INVARIANT, i.e. $\frac{d}{dt} E(p, q) = \frac{\partial E}{\partial p}\,\dot{p} + \frac{\partial E}{\partial q}\,\dot{q} = \frac{p}{m}\,F(q) - F(q)\,\frac{p}{m} = 0$, and the equation of motion is equivalent to

$$\dot{q} = \frac{\partial E}{\partial p}$$
$$\dot{p} = -\frac{\partial E}{\partial q}.$$

If we translate this into quantum mechanics, p becomes the differential operator $P = \frac{\hbar}{i}\frac{d}{dx} : f \mapsto f'$ and q the multiplication operator $Q = x$ with the identity. The energy function then becomes the SCHRÖDINGER OPERATOR: $S := -\frac{\hbar^2}{2m}\left(\frac{d}{dx}\right)^2 + U(x)$, or in several variables

$$S = -\frac{\hbar^2}{2m}\Delta + U(x).$$

The corresponding equation of motion is the SCHRÖDINGER EQUATION

$$i\hbar \frac{d}{dt} u = S\,u.$$

Of quite similar form is the heat conduction equation

$$\frac{d}{dt} u = \Delta u.$$

The wave equation $\frac{d^2 u}{dt^2} = \Delta u$ can also be transformed into the form

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix}$$

by means of the Ansatz $v = \frac{d}{dt} u$.

So we have to solve equations of the form $\dot{u} = A\,u$, a linear first order ordinary differential equation. For bounded operators on Banach spaces the solution to [18, 3.5.1] is given by $u(t) = u(0)\,e^{t\,A}$. The operators occurring in the above situations, however, are partial differential operators of second order, i.e. not continuous operators on Banach spaces. For Fréchet spaces like $C^\infty(\mathbb{R}, \mathbb{R})$, however, the series $e^{t\,A} = \sum_n \frac{t^n}{n!} A^n$ does not have to converge. So we should take A as linear (unbounded) operators on L^2, and define $e^{t\,A}$ for them.

Note that the Laplace operator is self adjoint. According to a result of [15], the Schrödinger operator $S = -\frac{\hbar^2}{2m}\Delta + U(x)$ is essentially self adjoint under suitable growth conditions on the potential U, see also [37, 253].

Let $t \mapsto u_x(t)$ be the solution curve for the initial value $u(0) = x$ of an ordinary differential equation $\dot u = A(u)$. Then the mapping $U : (t, x) \mapsto u_x(t)$ obviously has the following properties where it is defined :

$$U(0, x) = x$$
$$U(t + s, x) = U(t, U(s, x)).$$

It is also called the FLOW of the differential equation. If A is linear, then clearly $x \mapsto U(t, x)$ is also linear, and thus $\check U : \mathbb{R} \to L(H)$ is a curve with $\check U(0) = 1$ and $\check U(t + s) = \check U(t) \circ \check U(s)$. So we have a group homomorphism $\check U : \mathbb{R} \to L(H)$. And for all $x \in H$, $\frac{d}{dt}\check U(t)(x) = \frac{\partial}{\partial t} u_x(t) = A(u_x(t)) = (A \circ \check U(t))(x)$ holds. In particular, the pointwise derivative of the curve $\check U$ at 0 is precisely A. We now want to transfer this correspondance between operators and 1-parameter subgroups to unbounded self adjoint operators.

9.47 Stone's Theorem

Let $S : H \rightsquigarrow H$ be self adjoint and $S = \int_{-\infty}^{+\infty} t\, dP(t)$ its spectral representation. Since for $t \in \mathbb{R}$ the mapping $s \mapsto e^{its}$ is bounded on $\mathbb{R}$, $U(t) := e^{itS} := \int_{-\infty}^{+\infty} e^{its}\, dP(s) \in L(H)$ exists. Furthermore, $U(t)^* = e^{-itS}$ and thus $U(t) \circ U(t)^* = e^{itS} \circ e^{-itS} = e^0 = 1$ and $U(t)^* \circ U(t) = e^{-itS} \circ e^{itS} = 1$, i.e. $U(t)$ is unitary.

Because of $e^z \cdot e^w = e^{z+w}$ we have $U(t) \circ U(s) = U(t + s)$. Furthermore U is SOT-continuous, because $\|U(t)h - U(s)h\| = \|U(t - s + s)h - U(s)h\| = \|U(s)\,(U(t - s)h - h)\| = \|U(t - s)h - h\|$. So it suffice to show that $\|U(t)h - h\|^2 = \int_{\mathbb{R}} |e^{i\,t\,s} - 1|^2\, dP_{h,h}(s) \to 0$ for $t \to 0$. We have that $P_{h,h}$ is a finite measure on $\mathbb{R}$, and for every $s \in \mathbb{R}$ $|e^{i\,t\,s} - 1|^2 \to 0$ holds for $t \to 0$ and $|e^{i\,t\,s} - 1|^2 \leqslant 4$. So the theorem on dominated convergence implies that $U(t)h \to h$ for $t \to 0$.

Theorem.

We have a bijection

$$\Big\{S : H \rightsquigarrow H,\ \textit{self adjoint}\Big\} \cong \Big\{U : \mathbb{R} \to L(H),\ \textit{unitary representation}\ \Big\}$$

via

$$U(t) := \int_{-\infty}^{+\infty} e^{-it}\, dP(t) \textit{ for } S = \int_{-\infty}^{+\infty} t\, dP(t)$$
$$i\,S := \frac{d}{dt}\Big|_{t=0} U(t)h \textit{ for } h \in \operatorname{dom} S := \Big\{h : \exists \frac{d}{dt}|_{t=0} U(t)h\Big\}.$$

Proof. We have just shown that U is a unitary representation.
We have $\frac{1}{t}(U(t) - 1) - i\,S = f_t(S)$, where $f_t(s) := \frac{1}{t}(e^{i\,t\,s} - 1) - i\,s$. So

$$\left\|\frac{1}{t}(U(t)h - h) - i\,S\,h\right\|^2 = \|f_t(S)h\|^2 = \int_{\mathbb{R}} \left|\frac{e^{i\,t\,s} - 1}{t} - i\,s\right|^2 dP_{h,h}(s).$$

for $h \in \operatorname{dom} S$. For $t \to 0$, we have $\frac{1}{t}(e^{i\,t\,s} - 1) - i\,s \to 0$ for all $s \in \mathbb{R}$ because by the Mean Value Theorem $|e^{i\,s} - 1| \leqslant |s|$. Thus $|f_t(s)| \leqslant \frac{1}{t}|e^{i\,t\,s} - 1| + |s| \leqslant 2|s|$. Since $\mathrm{id} \in L^2(P_{h,h})$ by 9.42, we obtain $\lim_{t\to 0} \frac{1}{t}(U(t) - 1)h = i\,S\,h$ by the theorem of dominant convergence.

Let $D := \Big\{h \in H : \frac{d}{dt}|_{t=0} U(t)\, h$ existiert in $H\Big\}$. For $h \in D$, $\tilde S h$ is defined by

$$\tilde S h := -i\, \tfrac{d}{dt}|_{t=0} U(t)\, h.$$

One sees immediately that $\tilde S$ is a linear operator. According to the above, $\tilde S$ is an extension of S and thus also $\tilde S$ is densely defined. For $h, g \in D$ we have:

$$\langle \tilde S h, g\rangle = -i \lim_{t\to 0}\Big\langle \frac{U(t)h-h}{t}, g\Big\rangle = \lim_{t\to 0}\Big\langle h, -i\,\frac{U(-t)g-g}{-t}\Big\rangle = \langle h, \tilde S g\rangle,$$

because $U(t)^* = U(t)^{-1} = U(-t)$. So $\tilde S$ is a symmetric extension of S and, since by 9.29 the self adjoint operator S is maximally symmetric, $\tilde S = S$ and $D = \operatorname{dom} S$ holds.

Let conversely $U : \mathbb{R} \to U(H)$ be a unitary representation, $D := \{h \in H : \exists \frac{d}{dt}|_{t=0} U(t)\,h\}$ and $Sh := -i\frac{d}{dt}|_{t=0}U(t)h$ for $h \in D$.

Claim: D is dense in H.
In order to see this we define operators R_n by

$$R_n h := \int_0^\infty e^{-t}\, U(\tfrac{t}{n}) h\, dt.$$

Since $\|U(t)h\| = \|h\|$ and $(t \mapsto e^{-t}) \in L^1(\mathbb{R}^+)$, this integral is well-defined and $\|R_n h\| \leqslant \int_0^\infty e^{-t}\,\|h\|\,dt = -e^{-t}\,\|h\|\,|_{t=0}^\infty = \|h\|$ holds. Obviously, $R_n : H \to H$ is a bounded linear operator with $\|R_n\| \leqslant 1$.
We now want to show that the image of R_n is completely contained in D. Let $h \in H$, then

$$\begin{aligned}
-\frac{i}{t}(U(t)-1)\,R_n h &= -\frac{i}{t}\int_0^\infty e^{-s}\,U(t+\tfrac{s}{n})\,h\,ds + \frac{i}{t}\int_0^\infty e^{-s}\,U(\tfrac{s}{n})\,h\,ds\\
&= -\frac{i}{t}\int_{nt}^\infty e^{-(r-nt)}\,U(\tfrac{r}{n})\,h\,dr + \frac{i}{t}\int_0^\infty e^{-s}\,U(\tfrac{s}{n})\,h\,ds\\
&= -in\,\frac{e^{nt}-1}{nt}\int_0^\infty e^{-s}\,U(\tfrac{s}{n})\,h\,ds + in\,\frac{1}{nt}\int_0^{nt} e^{-r+nt}\,U(\tfrac{r}{n})\,h\,dr\\
&= -i\,n\,\frac{e^{nt}-1}{nt}\,R_n h + i\,n\,e^{nt}\,\tfrac{1}{nt}\int_0^{nt} e^{-r}\,U(\tfrac{r}{n})\,h\,dr.
\end{aligned}$$

For $t \to 0$ we have

$$\frac{e^{nt}-1}{nt} \to 1, \quad e^{nt} \to 1 \quad \text{and } \frac{1}{nt}\int_0^{nt} e^{-r}\,U(\tfrac{r}{n})\,h\,dr \to e^0\,U(0)\,h = h.$$

So $R_n h \in D$ and $S\,R_n\,h = -i\,n\,(R_n - 1)h$.

For the denseness of D, it suffices to show that $R_n h \to h$ for arbitrary $n \to \infty$ and $h \in H$. We have

$$\begin{aligned}
R_n h - h &= \int_0^\infty e^{-t}\,U(\tfrac{t}{n})\,h\,dt - \int_0^\infty e^{-t}\,h\,dt\\
&= \int_0^\infty e^{-t}\,(U(\tfrac{t}{n})h - h)\,dt.
\end{aligned}$$

For $\varepsilon > 0$, let $\delta > 0$ is be choosen so that $\|U(t)h - h\| < \varepsilon$ for all $|t| \leqslant \delta$. Then

$$\begin{aligned}
\|R_n h - h\| &\leqslant \int_0^\infty e^{-t}\,\|U(\tfrac{t}{n})h - h\|\,dt\\
&\leqslant \int_0^{n\delta} e^{-t}\,\varepsilon\,dt + \int_{n\delta}^\infty e^{-t}\,(\|U(\tfrac{t}{n})h\| + \|h\|)\,dt\\
&\leqslant \varepsilon + \int_{n\delta}^\infty e^{-s}\,2\,ds\\
&\leqslant 2\varepsilon,
\end{aligned}$$

if $n = n(\varepsilon,\delta)$ was choosen so large that $\int_{n\delta}^\infty e^{-s}\,ds \leqslant \frac{\varepsilon}{2}$.

Claim: S is symmetric because

$$\begin{aligned}0 &= -i\tfrac{d}{dt}|_{t=0}\langle h,k\rangle\\ &= -i\tfrac{d}{dt}|_{t=0}\langle U(t)h,U(t)k\rangle\\ &= \langle -i\tfrac{d}{dt}|_{t=0}U(t)h,U(0)k\rangle - \langle U(0)h,-i\tfrac{d}{dt}|_{t=0}U(t)k\rangle\\ &= \langle Sh,k\rangle - \langle h,Sk\rangle.\end{aligned}$$

for $h,k\in D$

By 9.19, S is closeable and we denote the closure of S again with S. By 9.25, we only have to show for self-adjointness that $\ker(S^*\pm i)=\{0\}$, or, equivalently, that $\operatorname{img}(S\pm i)$ is dense. For this we calculate

$$(S+i)\circ(-i\,R_1) = i^2(R_1-1) - i^2R_1 = 1,$$

hence $S+i$ is surjective.

If we define analogously to R_n an operator T_n by $T_nh := \int_0^\infty e^{-t}\,U(-\frac{t}{n})h\,dt$, and show $S\circ T_n = i\,n\,(T_n-1)\,h$, we obtain

$$(S-i)\circ(i\,T_1) = i^2(T_1-1) - i^2T_1 = 1,$$

hence $S-i$ is also surjective.

Let $h\in D$. Then $\frac{U(t+s)h-U(t)h}{s} = U(t)\frac{U(s)-1}{s}h$ and since $\frac{d}{ds}|_{s=0}U(s)h = \lim_{s\to0}\frac{U(s)-1}{s}h$ exists, this also holds for

$$\begin{aligned}\frac{d}{dt}U(t)h &= \lim_{s\to0}\frac{U(t+s)h-U(t)h}{s}\\ &= \lim_{s\to0}U(t)\frac{U(s)-1}{s}h = U(t)\lim_{s\to0}\frac{U(s)-1}{s}h\\ &= U(t)(i\,S\,h).\end{aligned}$$

On the other hand, $\frac{U(t+s)h-U(t)h}{s} = \frac{U(s)-1}{s}U(t)\,h$, thus $U(t)h\in D$ and

$$\frac{d}{dt}U(t)\,h = \lim_{s\to0}\frac{U(t+s)h-U(t)h}{s} = \lim_{s\to0}\frac{U(s)-1}{s}U(t)\,h = i\,S\,U(t)\,h.$$

The previous calculation showed that for $h\in D=\operatorname{dom}S=\operatorname{dom}(U(t)\,S)$ the equation $U(t)\,S\,h = i\frac{d}{dt}U(t)\,h = S\,U(t)\,h$ holds, i.e. $U(t)\,S\subseteq S\,U(t)$. This follows also directly from 9.43.

Let $V(t) := \exp(i\,S\,t)$. We have to show $U=V$. Let $h\in D$. By the above we have $V(t)h\in D$ and

$$\frac{d}{dt}V(t)\,h = i\,S\,V(t)\,h.$$

Similarly:

$$\frac{d}{dt}U(t)\,h = i\,S\,U(t)\,h.$$

Therefore, $t\mapsto h(t) := U(t)\,h - V(t)\,h$ is differentiable and

$$h'(t) = i\,S\,U(t)h - i\,S\,V(t)h = i\,S\,h(t).$$

We have

$$\begin{aligned}\tfrac{d}{dt}\|h(t)\|^2 &= \tfrac{d}{dt}\langle h(t),h(t)\rangle\\ &= \langle\tfrac{d}{dt}h(t),h(t)\rangle + \langle h(t),\tfrac{d}{dt}h(t)\rangle\\ &= \langle i\,S\,h,h\rangle + \langle h,i\,S\,h\rangle\\ &= i\langle Sh,h\rangle - i\langle h,Sh\rangle = 0.\end{aligned}$$

Thus, h is constant, and thus $h(t)=h(0)=0$ for all t, i.e. $U(t)h=V(t)h$ for all $h\in D$ and all $t\in\mathbb{R}$. Since D is dense, $U=V$. $\square$

9.48 Proposition.

The infinitesimal generator is bounded if and only if $\lim_{t\to 0}\|U(t)-1\| = 0$, *i.e. U is norm continuous.*

Proof. ($\Rightarrow$) This holds since $\|U(t)-1\| = \|\exp i\,t\,T - 1\| = \|\int_{\sigma(T)}(e^{its}-1)\,dP(s)\| = \|s\mapsto e^{its}-1\|_\infty = sup\{|e^{i\,t\,s}-1| : s\in\sigma(T)\}\to 0$ for $t\to 0$ because $\sigma(T)$ is bounded.

($\Leftarrow$) Suppose $\|U(t)-1\|\to 0$ for $t\to 0$. Let $0<\varepsilon<\frac{\pi}{4}$. Then there is a $t_0>0$ with $\|U(t)-1\|<\varepsilon$ for $|t|\leqslant t_0$. Because $U(t)-1 = \int_{\sigma(T)}(e^{i\,t\,s}-1)\,dP(s)$, we have $\sup\{|e^{i\,t\,s}-1| : s\in\sigma(T)\} = \|U(t)-1\|<\varepsilon$ for these t. For δ depending on ε, therefore $t\,s\in\bigcup_{n\in\mathbb{Z}}]2\pi n-\delta, 2\pi n+\delta[=: G$ for all $s\in\sigma(T)$ and $|t|\leqslant t_0$. Since these intervals are disjoint components of G and the interval $\{ts : 0\leqslant t\leqslant t_0\}$ is contained in G for $s\in\sigma(T)$, we have $|t\,s|<\delta$ for all $|t|\leqslant t_0$. In particular, $t_0\,\sigma(T)\subseteq[-\delta,\delta]$. And thus $\sigma(T)$ is bounded and hence T is bounded, because $T=\int_{\sigma(T)} z\,dP(z)\in L(H)$, since $(z\mapsto z)$ is bounded on $\sigma(T)$. □

9.49 Theorem.

Let H be separable and $U:\mathbb{R}\to L(H)$ be a unitary representation. If for all $h,k\in H$ the mapping $t\mapsto\langle U(t)h,k\rangle$ is Lebesgue-measurable, then U is SOT-continuous.

Proof. Let $0<a<\infty$ and $h,g\in H$. Then $t\mapsto\langle U(t)h,g\rangle$ is a bounded measurable function on $[0,a]$, so

$$\int_0^a |\langle U(t)h,g\rangle|\,dt \leqslant a\,\|h\|\,\|g\|.$$

Therefore, $h\mapsto\int_0^a\langle U(t)h,g\rangle\,dt$ is a bounded linear functional on H. So there is a $g_a\in H$ with $\langle h,g_a\rangle = \int_0^a\langle U(t)h,g\rangle\,dt$ for all $h\in H$ and $\|g_a\|\leqslant a\,\|g\|$.

We now claim that the linear span of $\{g_a : g\in H, a>0\}$ is dense in H.
In fact, if $h\in H$ is assumed to be orthogonal to all g_a, then $0=\langle h,g_a\rangle = \int_0^a\langle U(t)h,g\rangle\,dt$ for all $a>0$ and $g\in H$. So $\langle U(\_)h,g\rangle = 0$ is almost everywhere on $\mathbb{R}$. Since H is separable, there exists a subset $\Delta\subset\mathbb{R}$ of measure 0, s.t. $\langle U(t)h,g\rangle = 0$ for all $t\notin\Delta$ and g in a fixed countable dense subset of H. So $\|h\| = \|U(t)h\| = 0$ for $t\notin\Delta$.

For $s\in\mathbb{R}$, now the following holds:

$$\begin{aligned}\langle h,U(s)g_a\rangle &= \langle U(-s)h,g_a\rangle\\ &= \int_0^a\langle U(t)\,U(-s)\,h,g\rangle\,dt\\ &= \int_0^a\langle U(t-s)h,g\rangle\,dt\\ &= \int_{-s}^{a-s}\langle U(t)h,g\rangle\,dt\\ &\to \int_0^a\langle U(t)h,g\rangle\,dt = \langle h,g_a\rangle\end{aligned}$$

So $\langle h,U(s)g_a\rangle\to\langle h,g_a\rangle$ for $s\to 0$. Because $\{g_a : a>0, g\in H\}$ is dense and because of the uniform boundedness, $U:\mathbb{R}\to B(H)$ is continuous at 0 with respect to the WOT. Because of the group property, U is continuous with respect to the WOT everywhere. So U is also SOT-continuous. We have:

$$\begin{aligned}\|U(t)h-h\|^2 &= \langle (U(t)-1)h,(U(t)-1)h\rangle \\ &= \langle (U(t)-1)^*(U(t)-1)h,h\rangle \\ &= \langle (U(-t)-1)(U(t)-1)h,h\rangle \\ &= \langle (U(0)-U(-t)-U(t)+1)h,h\rangle \\ &= -\langle U(t)h-h,h\rangle - \langle U(-t)h-h,h\rangle \\ &\to 0+0=0. \quad \square\end{aligned}$$

Since self adjoint operators on separable Hilbert spaces can be represented as multiplication operators, one only needs to determine the 1-parameter subgroups of these operators:

9.50 Proposition.

Let (X,Ω,μ) be a σ-finite measure space and f a real-valued Ω-measurable function on X. Let $S := M_f$ on $L^2(\mu)$. Then $\exp(i\,t\,S) = M_{e_t}$, where $e_t(x) := \exp(i\,t\,f(x))$.

Proof. We have $\operatorname{dom} M_f = \{h \in L^2 : f\,h \in L^2\}$. So we just have to show that $\frac{d}{dt}|_{t=0} e^{itf}h = ifh$ for all $h \in \operatorname{dom} M_f$. Pointwise, we have obviously

$$\frac{d}{dt}|_{t=0} e^{itf(x)}h(x) = if(x)e^0h(x) = if(x)h(x).$$

To apply the theorem on dominated convergence, we need an upper bound for $|\frac{e^{itf(x)}-1}{t}h(x) - if(x)h(x)|^2$ which we obtain as in the proof of 9.47 with $s = f(x)$:

$$\begin{aligned}\Big|\frac{e^{itf(x)}-1}{t}h(x) - if(x)h(x)\Big|^2 &= \Big|\frac{e^{its}-1}{t}h(x) - ish(x)\Big|^2 \\ &= \Big|\Big(\frac{e^{its}-1}{t}h(x) - is\Big)\,h(x)\Big|^2 \\ &= |f_t(s)h(x)|^2 \\ &\leqslant |2\,s\,h(x)|^2 = 4|f(x)h(x)|^2,\end{aligned}$$

and since $f\,h \in L^2$ the proof is complete. $\square$

9.51 Theorem.

Let $P : f \mapsto if'$ be defined on

$$D := \Big\{f \in L^2(\mathbb{R}) : f \text{ is locally absolutely continuous and } f' \in L^2(\mathbb{R})\Big\}.$$

Then P is self adjoint and the associated 1-parameter subgroup U is given by $U(t)f : x \mapsto f(x-t)$.

Proof. We have seen, that the Fourier transform $\mathcal{F} : L^2 \to L^2$ is a unitary operator which transforms P to Q, i.e. $P = \mathcal{F}Q\mathcal{F}^{-1}$. By 9.50, the unitary 1-parameter group U_Q, associated to Q by $U_Q(t)$, is the multiplication with $x \mapsto e^{itx}$. The unitary 1-parameter group U_P for P is thus given by $U_P(t) = \mathcal{F}U_Q(t)\mathcal{F}^{-1}$. We saw in [**18**, 8.1.5] that the following holds for $g \in \mathcal{S}$

$$\begin{aligned}\mathcal{F}(U_Q(t)g)(y) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{itx}\,g(x)\,e^{-ixy}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} g(x)\,e^{-ix(y-t)}\,dx \\ &= \mathcal{F}(g)(y-t) \\ &= (T_t\mathcal{F}g)(y),\end{aligned}$$

where T_t denotes the translation operator. Consequently, we have

$$U_P(t)(f) = (\mathcal{F} U_Q(t) \mathcal{F}^{-1}) f = \mathcal{F}(U_Q(t)(\mathcal{F}^{-1} f)) = T_t(\mathcal{F}\mathcal{F}^{-1} f) = T_t f. \quad \square$$

Literaturverzeichnis

[1] An elementary example of a Banach space not isomorphic to its complex conjugate.

[2] Hans Wilhelm Alt. *Lineare Funktionalanalysis*. Springer, 1985. Hochschultext.

[3] Sterling K. Berberian. *Lectures in Functional Analysis and Operator Theory*, volume 15. Springer, 1974. Graduate Texts in Mathematics.

[4] H.F. Bohnenblust and S. Karlin. *On a theorem of Ville*, volume 1 of *Theory of Games*, pages 155–160. Princeton, 1950.

[5] John B. Conway. *A Course in Functional Analysis*, volume 96. Springer, 1985. Graduate Texts in Maths.

[6] M. De Wilde. Réseaux dans les espaces linéaires à semi-normes. *Mém. Soc. Roy. Sci. Liege 18*, 1969.

[7] Ky Fan. Fixed-point and minimax theorems in locally convex topological linear spaces. *Proc. NAS USA*, 38:121–126, 1952.

[8] H. G. Garnir. Solovay's axiom and functional analysis. In *Functional analysis and its applications (Internat. Conf., Eleventh Anniversary of Matscience, Madras, 1973; dedicated to Alladi Ramakrishnan)*, pages 189–204. Lecture Notes in Math., Vol. 399. Springer, Berlin, 1974.

[9] I.L. Glicksberg. A further generalization of the Kakutani fixed point theorem, with applications to Nash equilibrium points. *Proc. AMS*, 3:170–174, 1952.

[10] Ernst Henze. *Einführung in die Maßtheorie*. Bibliographisches Institut, Mannheim, 1971. Hochschultaschenbücher Band 505.

[11] Harro Heuser. *Lehrbuch der Analysis*. Teubner, 1980. Mathematische Leitfäden.

[12] E. Hewitt. Rings of continuous functions, I. *Trans. Am. Math. Soc.*, 64:45–99, 1948.

[13] Edwin Hewitt and Kenneth A. Ross. *Abstract Harmonic Analysis*. Springer, 1963. Grundlehren.

[14] Hans Jarchow. *Locally convex spaces*. Teubner, Stuttgart, 1981.

[15] Tosio Kato. Fundamental Properties of Hamilton Operators of Schrödinger Type. *Trans. Am. Math. Soc.*, 70:195–211, 1950.

[16] H. J. Keisler and A. Tarski. From accessible to inaccessible cardinals. *Fund. Math.*, 53:225–308, 1964.

[17] Andreas Kriegl. Algebraische Topologie. Skriptum, 2006.

[18] Andreas Kriegl. Funktionalanalysis 1. Skriptum, 2006.

[19] Andreas Kriegl. Komplexe Analysis. Universität Wien, SS 2011.

[20] Andreas Kriegl. Analysis 1. Skriptum, 2005.

[21] Andreas Kriegl. Proseminar zur Analysis 1. Skriptum, 2005.

[22] Andreas Kriegl. Analysis 2. Skriptum, 2005.

[23] Andreas Kriegl. Proseminar zur Analysis 2. Skriptum, 2005.

[24] Andreas Kriegl. Analysis 3. Skriptum, 2005.

[25] Andreas Kriegl. Proseminar zur Analysis 3. Skriptum, 2005.

[26] Andreas Kriegl. Topologie 1. Skriptum, 2000.

[27] Andreas Kriegl and Peter W. Michor. *The convenient setting of global analysis*. Am. Math. Soc., 1997.

[28] K. Kuratowski and A. Mostokwski. *Set theory, 2nd edition*. Amsterdam, 1976.

[29] G. W. Mackey. Equivalence of a problem im measure theory to a problem in the theory of vector lattices. *Bull. Am. Math. Soc.*, 50:719–722, 1944.

[30] L. Nachbin. Topological vector spaces of continuous functions. *Proc. Nat. Acad. Sci. USA*, 40:471–474, 1954.

[31] Kakutani S. A generalization of Brouwer's fixed point theorem. *Duke Math. J.*, 8:457–459, 1941.

[32] Jean Schmets. Bornological and ultrabornological $C(X, E)$ spaces. *Manuscripta Math.*, 21:117–133, 1977.

[33] Jean Schmets. An example of the barrelled space associated with $C(X; E)$. *Advances in Functional Analysis, Holomorphy, and Approximation Theory, Springer Lecture Notes*, 843:561–571, 1981.
[34] T. Shirota. On locally convex vector spaces of continuous functions. *Proc. Japan Acad.*, 30:294–298, 1954.
[35] Robert M. Solovay. A model of set-theory in which every set of reals is Lebesgue measurable. *Ann. of Math. (2)*, 92:1–56, 1970.
[36] S. Ulam. Zur Maßtheorie in der allgemeinen Mengenlehre. *Fund. Math.*, 16:140–150, 1930.
[37] N. Ya. Vilenkin et al. *Functional Analysis.* Walters Noordhoff, 1972.

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