THE DYNAMICS OF ALGEBRAIC $\mathbb{Z}^d$-ACTIONS

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1. Algebraic $\mathbb{Z}^d$-actions and their dual modules

An algebraic $\mathbb{Z}^d$-action is an action $\alpha: \mathbb{N} \to \mathbb{Z}^d$ of $\mathbb{Z}^d$, $d \geq 1$, by continuous automorphisms of a compact abelian group $X$ with Borel field $\mathcal{B}_X$ and normalized Haar measure $\lambda_X$. Two algebraic $\mathbb{Z}^d$-actions $\alpha$ and $\beta$ on compact abelian groups $X$ and $Y$ are algebraically conjugate if there exists a continuous group isomorphism $\phi: X \to Y$ with $\phi \cdot \alpha_n = \beta_n \cdot \phi$ \hspace{1cm} (1.1)

for every $n \in \mathbb{Z}^d$. If the map $\phi$ in (1.1) is a homeomorphism then $\alpha$ and $\beta$ are topologically conjugate. Finally we call $\alpha$ and $\beta$ measurably conjugate if there exists a measure space isomorphism $\phi: (X, \mathcal{B}_X, \lambda_X) \to (Y, \mathcal{B}_Y, \lambda_Y)$ satisfying (1.1) $\lambda_X$-a.e. for every $n \in \mathbb{Z}^d$.

In [4] and [13], Pontryagin duality was shown to imply a one-to-one correspondence between algebraic $\mathbb{Z}^d$-actions (up to algebraic conjugacy) and modules over the ring of Laurent polynomials $R_d = \mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$ with integral coefficients in the commuting variables $u_1, \ldots, u_d$ (up to module isomorphism).

In order to explain this correspondence we write a typical element $f \in R_d$ as

$$f = \sum_{m \in \mathbb{Z}^d} c_f(m) u^m$$ \hspace{1cm} (1.2)

with $u^m = u_1^{m_1} \cdots u_d^{m_d}$ and $c_f(m) \in \mathbb{Z}$ for every $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$, where $c_f(m) = 0$ for all but finitely many $m$. If $\alpha$ is an algebraic $\mathbb{Z}^d$-action on a compact abelian group $X$, then the additively-written dual group $M = \hat{X}$ is a module over the ring $R_d$ with operation

$$f \cdot a = \sum_{m \in \mathbb{Z}^d} c_f(m) \hat{\alpha}^m(a)$$ \hspace{1cm} (1.3)

for $f \in R_d$ and $a \in M$, where $\hat{\alpha}^m$ is the automorphism of $M = \hat{X}$ dual to $\alpha^m$. In particular,

$$u^m \cdot a = \hat{\alpha}^m(a)$$ \hspace{1cm} (1.4)

for $m \in \mathbb{Z}^d$ and $a \in M$. Conversely, any $R_d$-module $M$ determines an algebraic $\mathbb{Z}^d$-action $\alpha_M$ on the compact abelian group $X_M = \hat{M}$ with $\alpha_M^m$ dual to multiplication by $u^m$ on $M$ for every $m \in \mathbb{Z}^d$ (cf. (1.4)). Note that $X_M$ is metrizable if and only if its dual module $M$ is countable.

**Examples 1.1.** (1) Let $M = R_d$. Since $R_d$ is isomorphic to the direct sum $\sum_{Z^d} Z$ of copies of $Z$, indexed by $Z^d$, the dual group $X = \hat{R}_d$ is isomorphic to the Cartesian product $T^{\mathbb{Z}^d}$ of copies of $T = \mathbb{R}/\mathbb{Z}$. We write a typical element $x \in T^{\mathbb{Z}^d}$ as $x = (x_n)$
with $x_n \in T$ for every $n \in \mathbb{Z}^d$ and choose the following identification of $X_{R_d} = \widehat{R}_d$ and $T^\mathbb{Z}^d$: for every $x \in T^\mathbb{Z}^d$ and $f \in R_d$,

$$\langle x, f \rangle = e^{2\pi i \sum_{n \in \mathbb{Z}^d} c_f(n)x_n},$$

where $f$ is given by (1.2). Under this identification the $\mathbb{Z}^d$-action $\alpha_{R_d}$ on $X_{R_d} = T^\mathbb{Z}^d$ becomes the shift-action

$$(\alpha^n_{R_d}x)_n = x_{m+n}.$$  

(2) Let $I \subset R_d$ be an ideal and $M = R_d/I$. Since $M$ is a quotient of the additive group $R_d$ by an $\alpha_{R_d}$-invariant subgroup (i.e. by a submodule), the dual group $X_M = \hat{M}$ is the closed $\alpha_{R_d}$-invariant subgroup

$$X_{R_d/I} = \{ x \in X_{R_d} = T^\mathbb{Z}^d : \langle x, f \rangle = 1 \text{ for every } f \in I \}
= \left\{ x \in T^\mathbb{Z}^d : \sum_{n \in \mathbb{Z}^d} c_f(n)x_{m+n} = 0 \pmod{1} \text{ for every } f \in I \text{ and } m \in \mathbb{Z}^d \right\},$$

and $\alpha_{R_d/I}$ is the restriction of the shift-action $\alpha_{R_d}$ in (1.6) to the shift-invariant subgroup $X_{R_d/I} \subset T^\mathbb{Z}^d$.

Conversely, let $X \subset T^\mathbb{Z}^d = \widehat{R}_d$ be a closed subgroup, and let

$$X^\perp = \{ f \in R_d : \langle x, f \rangle = 1 \text{ for every } x \in X \}$$
be the annihilator of $X$ in $\widehat{R}_d$. Then $X$ is shift-invariant if and only if $X^\perp$ is an ideal in $R_d$.

The correspondence between algebraic $\mathbb{Z}^d$-actions $\alpha = \alpha_M$ and $R_d$-modules $M$ yields a correspondence (or ‘dictionary’) between dynamical properties of $\alpha_M$ and algebraic properties of the module $M$ (cf. [16]). It turns out that some of the principal dynamical properties of $\alpha_M$ can be expressed entirely in terms of the prime ideals associated with the module $M$, where a prime ideal $p \subset R_d$ is associated with $M$ if

$$p = \{ f \in R_d : f \cdot a = 0_M \}$$

for some $a \in M$. The set of all prime ideals associated with $M$ is denoted by $\text{asc}(M)$; if $M$ is Noetherian, then $\text{asc}(M)$ is finite.

Figure 1 on the facing page provides a small illustration of this correspondence; all the relevant results can be found in [16]. In the third column we assume that the $R_d$-module $M = \widehat{X}$ defining $\alpha$ is of the form $R_d/p$, where $p \subset R_d$ is a prime ideal, and describe the algebraic condition on $p$ equivalent to the dynamical condition on $\alpha = \alpha_{R_d/p}$ appearing in the second column. In the fourth column we consider a countable $R_d$-module $M$ and state the algebraic property of $M$ corresponding to the property of $\alpha = \alpha_M$ in the second column.

The notation in Figure 1 is as follows. In (1),

$$V_C(p) = \{ c \in (C \setminus \{0\})^d : f(c) = 0 \text{ for every } f \in p \}$$

is the variety of $p$, and $S = \{ c \in C : |c| = 1 \}$. From (2)–(4) it is clear that $\alpha$ is ergodic if and only if $\alpha^n$ is ergodic for some $n \in \mathbb{Z}^d$, and that $\alpha$ is mixing if and
only if $\alpha^n$ is ergodic for every nonzero $n \in \mathbb{Z}^d$. In (5), $\alpha$ is mixing of order $r \geq 2$ if
\[
\lim_{\|n_1, \ldots, n_r\| \to \infty} \lambda_X \left( \bigcap_{1 \leq i < j \leq r} \alpha^{-n_i} B_i \right) = \prod_{i=1}^r \lambda_X (B_i)
\]
for all Borel sets $B_i \subset X$, $i = 1, \ldots, r$. In (6)–(8), $h(\alpha)$ stands for the topological entropy of $\alpha$ (which coincides with the metric entropy $h_{\lambda_X}(\alpha)$). In [8] and [16] there is an explicit entropy formula for algebraic $\mathbb{Z}^d$-actions. In the special case where $\alpha = \alpha_{R_d/p}$ for some prime ideal $p \subset R_d$ this formula reduces to
\[
h(\alpha) = \begin{cases} 
|\log M(f)| & \text{if } p = (f) = fR_d \text{ is principal,} \\
0 & \text{otherwise,}
\end{cases}
\]
where
\[
M(f) = \begin{cases} 
\exp \left( \int_{S^d} \log |f(s)| \, ds \right) & \text{if } f \neq 0, \\
0 & \text{if } f = 0,
\end{cases}
\]
is the Mahler measure of the polynomial $f$. Here $ds$ denotes integration with respect to the normalized Haar measure on the multiplicative subgroup $S^d \subset \mathbb{C}^d$.

For background, details and proofs of these and further results we refer to [16] and the original articles cited there. The remainder of this note is devoted to two particular problems: the higher order mixing behaviour and the conjugacy problem for algebraic $\mathbb{Z}^d$-actions.

2. Higher order mixing properties of algebraic $\mathbb{Z}^d$-actions

In this section we describe the connection between higher order mixing properties of algebraic $\mathbb{Z}^d$-actions and certain diophantine results on additive relations in fields due to Mahler ([9]), Masser ([10], [5]) and Schlickewei, W. Schmidt and van der Poorten ([1], [17]). In the discussion below we shall use the following elementary consequence of Pontryagin duality:

<table>
<thead>
<tr>
<th>Property of $\alpha$</th>
<th>$\alpha = \alpha_{R_d/p}$</th>
<th>$\alpha = \alpha_M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) $\alpha$ is expansive</td>
<td>$\forall {p} \cap \mathbb{Z}^d = \emptyset$</td>
<td>$M$ is Noetherian and $\alpha_{R_d/p}$ is expansive for every $p \in \text{asc}(M)$</td>
</tr>
<tr>
<td>(2) $\alpha^n$ is ergodic for some $n \in \mathbb{Z}^d$</td>
<td>$a^{kn} - 1 \notin p$ for every $k \geq 1$</td>
<td>$\alpha_{R_d/p}$ is ergodic for every $p \in \text{asc}(M)$</td>
</tr>
<tr>
<td>(3) $\alpha$ is ergodic</td>
<td>${a^{kn} - 1 : n \in \mathbb{Z}^d} \notin p$ for every $k \geq 1$</td>
<td>$\alpha_{R_d/p}$ is ergodic for every $p \in \text{asc}(M)$</td>
</tr>
<tr>
<td>(4) $\alpha$ is mixing</td>
<td>$a^{n} - 1 \notin p$ for every non-zero $n \in \mathbb{Z}^d$</td>
<td>$\alpha_{R_d/p}$ is mixing for every $p \in \text{asc}(M)$</td>
</tr>
<tr>
<td>(5) $\alpha$ is mixing of every order</td>
<td>Either $p$ is equal to $pR_d$ for some rational prime $p$, or $p \cap \mathbb{Z} = {0}$ and $\alpha_{R_d/p}$ is mixing for every $p \in \text{asc}(M)$</td>
<td>$\alpha_{R_d/p}$ is mixing of every order</td>
</tr>
<tr>
<td>(6) $h(\alpha) &gt; 0$</td>
<td>$p$ is principal and $\alpha_{R_d/p}$ is mixing</td>
<td>$h(\alpha_{R_d/p}) &gt; 0$ for at least one $p \in \text{asc}(M)$</td>
</tr>
<tr>
<td>(7) $h(\alpha) &lt; \infty$</td>
<td>$p \neq {0}$</td>
<td>If $M$ is Noetherian: $p \neq {0}$ for every $p \in \text{asc}(M)$</td>
</tr>
<tr>
<td>(8) $\alpha$ has completely positive entropy (or is Bernoulli)</td>
<td>$h(\alpha_{R_d/p}) &gt; 0$</td>
<td>$h(\alpha_{R_d/p}) &gt; 0$ for every $p \in \text{asc}(M)$</td>
</tr>
</tbody>
</table>

Figure 1: A Pocket Dictionary
Lemma 2.1. Let $\alpha$ be an algebraic $\mathbb{Z}^d$-action on a compact abelian group $X$ with dual module $M$. Then $X$ is connected if and only if no prime ideal $\mathfrak{p} \in \text{asc}(M)$ contains a nonzero constant, and $X$ is zero-dimensional if and only if every $\mathfrak{p} \in \text{asc}(M)$ contains a nonzero constant.

Let $\mathfrak{p} \subset R_d$ be a prime ideal, and let $\alpha = \alpha_{R_d/\mathfrak{p}}$ be the algebraic $\mathbb{Z}^d$-action with dual module $M = R_d/\mathfrak{p} = \hat{X}$. If $\alpha$ is not mixing, then there exist Borel sets $B_1, B_2 \subset X$ and a sequence $(n_k, k \geq 1)$ in $\mathbb{Z}^d$ with $\lim_{k \to \infty} n_k = \infty$ and

$$\lim_{k \to \infty} \lambda_X(B_1 \cap \alpha^{-n_k}B_2) = c$$

for some $c \neq \lambda_X(B_1)\lambda_X(B_2)$. Fourier expansion implies that the latter condition is equivalent to the existence of nonzero elements $a_1, a_2 \in M$ such that

$$a_1 + u_n a_2 = 0$$

for infinitely many $k \geq 1$. In particular,

$$(u^m - 1) \cdot a_2 = 0$$

for some nonzero $u \in \mathbb{Z}^d$ (cf. Figure 1 (4)). A very similar argument shows that $\alpha$ is not mixing of order $r \geq 2$ if and only if there exist elements $a_1, \ldots, a_r$ in $M$, not all equal to zero, and a sequence $((n_k^{(1)}, \ldots, n_k^{(r)}), k \geq 1)$ in $(\mathbb{Z}^d)^r$ such that $\lim_{k \to \infty} \|n_k^{(i)} - n_k^{(j)}\| = \infty$ for all $i, j$ with $1 \leq i < j \leq r$, and with

$$u^{n_k^{(1)}} a_1 + \cdots + u^{n_k^{(r)}} a_r = 0$$

for every $k \geq 1$.

Below we shall see that higher order mixing of an algebraic $\mathbb{Z}^d$-action $\alpha$ on a compact abelian group $X$ can break down in a particularly regular way (cf. Examples 2.7 and 2.10). We call a nonempty finite subset $S \subset \mathbb{Z}^d$ mixing under $\alpha$ if

$$\lim_{k \to \infty} \lambda_X\left(\bigcap_{n \in S} \alpha^{-kn}B_n\right) = \prod_{n \in S} \lambda_X(B_n)$$

(2.3)

for all Borel sets $B_n \subset X$, $n \in S$, and nonmixing otherwise. If $\alpha$ is $r$-mixing, then every set $S \subset \mathbb{Z}^d$ with cardinality $|S| = r$ is obviously mixing. The validity of the reverse implication for algebraic $\mathbb{Z}^d$-actions is an open problem (cf. Problem 2.11 and Conjecture 2.12)

As in (2.3) one sees that a nonempty finite set $S \subset \mathbb{Z}^d$ is nonmixing if and only if there exist elements $a_n \in M$, $n \in S$, not all equal to zero, such that

$$\sum_{n \in S} u^{kn} a_n = 0$$

(2.4)

for infinitely many $k \geq 1$.

The higher order mixing behaviour of an algebraic $\mathbb{Z}^d$-action $\alpha$ with dual module $M$ is again completely determined by that of the actions $\alpha_{R_d/\mathfrak{p}}$ with $\mathfrak{p} \in \text{asc}(\mathbb{R})$.

Theorem 2.2. Let $\alpha$ be an algebraic $\mathbb{Z}^d$-action on a compact abelian group $X$ with dual module $M = \hat{X}$.

(1) For every $r \geq 2$, the following conditions are equivalent:
   (a) $\alpha$ is $r$-mixing,
   (b) $\alpha_{R_d/\mathfrak{p}}$ is $r$-mixing for every $\mathfrak{p} \in \text{asc}(M)$.

(2) For every nonempty finite set $S \subset \mathbb{Z}^d$, the following conditions are equivalent:
(a) $S$ is $\alpha$-mixing,
(b) $S$ is $\alpha_{R_d/p}$-mixing for every $p \in \text{asc}(M)$.

In order to exhibit the connection between mixing properties and additive relations in fields we begin with a theorem by Mahler.

**Theorem 2.3** ([9]). Let $K$ be a field of characteristic $0$, $r \geq 2$, and let $x_1, \ldots, x_r$ be nonzero elements of $K$. If we can find nonzero elements $c_1, \ldots, c_r$ such that the equation

$$\sum_{i=1}^{r} c_i x_i^k = 0$$

has infinitely many solutions $k \geq 0$, then there exist integers $s \geq 1$ and $i, j$ with $1 \leq i < j \leq r$ such that $x_i^s = x_j^s$.

We denote by $K$ the field of fractions of the integral domain $R_d/p$, choose a finite set $S = \{n_1, \ldots, n_r\} \subset \mathbb{Z}^d$ with $r \geq 2$, and set $x_i = u^{n_i}$ for $i = 1, \ldots, r$. In view of Figure 1 (4)–(5), Lemma 2.1, (2.1), (2.4) and Theorem 2.2, Theorem 2.3 implies (and is, in fact, equivalent to) the following statement:

**Theorem 2.4** ([14]). Let $\alpha$ be a mixing algebraic $\mathbb{Z}^d$-action on a compact connected abelian group $X$. Then every nonempty finite subset $S \subset \mathbb{Z}^d$ is mixing.

If an algebraic $\mathbb{Z}^d$-action $\alpha$ is not mixing of every order, then there exists a smallest integer $r \geq 2$ such that $\alpha$ is not $r$-mixing. As a consequence of Lemma 2.1 and (2.2) one obtains the equivalence of the Theorems 2.5 and 2.6 below.

**Theorem 2.5** ([1], [17]). Let $K$ be a field of characteristic $0$ and $G$ a finitely generated multiplicative subgroup of $K^\times = K \setminus \{0\}$. If $r \geq 2$ and $(c_1, \ldots, c_r) \in (K^\times)^r$, then the equation

$$\sum_{i=1}^{r} c_i x_i = 0$$

has only finitely many solutions $(x_1, \ldots, x_r) \in G^r$ such that no sub-sum of (2.5) vanishes.

**Theorem 2.6** ([15]). Let $\alpha$ be a mixing algebraic $\mathbb{Z}^d$-action on a compact connected abelian group $X$. Then $\alpha$ is mixing of every order.

The ‘absolute’ version of the $S$-unit theorem in [1] contains a bound on the number of solutions of (2.5) without vanishing subsums which is expressed purely in terms of the integer $r$ and the rank of the group $G$ (in our setting: the order of mixing and the rank of the group $\mathbb{Z}^d$). This bound could be used, for example, to obtain quite remarkable uniform statements on the speed of multiple mixing for all irreducible and mixing algebraic $\mathbb{Z}^d$-actions (cf. Definition 3.1).

For algebraic $\mathbb{Z}^d$-actions on disconnected groups the situation is considerably more complicated due to the possible presence of nonmixing sets (cf. (2.3)).

**Example 2.7** ([7]). Let $p = (2, 1 + u_1 + u_2) = 2R_2 + (1 + u_1 + u_2)R_2$, $M = R_2/p$, and let $\alpha = \alpha_M$ be the algebraic $\mathbb{Z}^2$-action on $X = X_M = \hat{M}$ defined in Example 1.1 (2). Then $\alpha$ is mixing by Figure 1 (4), but not three-mixing.

Indeed, $(1 + u_1 + u_2)^{2n} \cdot a = 0$ for every $n \geq 0$ and $a \in M$. For $a = 1 + (2, 1 + u_1 + u_2) \in M$ our identification of $M$ with $\hat{X}$ in Example 1.1 (2) implies that
and hence that

$$B \cap \alpha^{-\langle 2^n,0 \rangle}(B) \cap \alpha^{-\langle 0,2^n \rangle}(B) = B \cap \alpha^{-\langle 2^n,0 \rangle}(B),$$

and hence that

$$\lambda_X(B \cap \alpha^{-\langle 2^n,0 \rangle}(B) \cap \alpha^{-\langle 0,2^n \rangle}(B)) = \lambda_X(B \cap \alpha^{-\langle 2^n,0 \rangle}(B)) = 1/4$$

for every $n \geq 0$. If $\alpha$ were three-mixing, we would have that

$$\lim_{n \to \infty} \lambda_X(B \cap \alpha^{-\langle 2^n,0 \rangle}(B) \cap \alpha^{-\langle 0,2^n \rangle}(B)) = \lambda_X(B)^3 = 1/8.$$  

By comparing this with (2.3) we see that the set $S = \{(0,0),(1,0),(0,1)\} \subset \mathbb{Z}^2$ is nonmixing.

A mixing algebraic $\mathbb{Z}^d$-action $\alpha$ on a disconnected compact abelian group $X$ has nonmixing sets if and only if it is not Bernoulli (cf. Figure 1 (8), [5] and [16, Section 27]). In particular, if $\alpha$ is an ergodic algebraic $\mathbb{Z}^d$-action on a compact zero-dimensional abelian group $X$ with zero entropy, then $\alpha$ has nonmixing sets. The description of the nonmixing sets of such an action $\alpha$ is facilitated by a Theorem of Masser ([5], [10]), which should be seen as an analogue of Theorem 2.3 in positive characteristic.

**Theorem 2.8.** Let $K$ be an algebraically closed field of characteristic $p > 0$, $r \geq 2$, and let $(x_1, \ldots, x_r) \in (K^\times)^r$. The following conditions are equivalent:

1. There exists an element $(c_1, \ldots, c_r) \in (K^\times)^r$ such that

$$\sum_{i=1}^r c_ix_i^k = 0$$

for infinitely many $k \geq 0$;

2. There exists a rational number $s > 0$ such that the set $\{x_1, \ldots, x_r\}$ is linearly dependent over the algebraic closure $\bar{F}_p \subset K$ of the prime field $F_p = \mathbb{Z}/p\mathbb{Z}$.

**Corollary 2.9.** Let $p \subset R_d$ be a prime ideal containing a rational prime $p > 1$, and let $\alpha = \alpha_{R_d/p}$ be the algebraic $\mathbb{Z}^d$-action on $X = X_{R_d/p}$ defined in Example 1.1 (2). We denote by $K = Q(R_2/p) \supseteq R_2/p$ the quotient field of $R_d/p$, write $\bar{K}$ for its algebraic closure, and set $x_n = n^a + p \in R_d/p \subset K \subset \bar{K}$ for every $n \in \mathbb{Z}^d$. If $S \subset \mathbb{Z}^d$ is a nonempty finite set, then the following conditions are equivalent:

1. $S$ is not $\alpha$-mixing;

2. There exists a rational number $s > 0$ such that the set $\{x_1^s, \ldots, x^s_r\} \subset \bar{K}$ is linearly dependent over $\bar{F}_p \subset \bar{K}$.

**Examples 2.10 ([5]).** (1) In the notation of Examples 2.7 and 1.1 (2) we set $f = 1 + u_1 + u_2 + u_1^2 + u_1u_2 + u_2^2 \in R_2$ and put $p = (2, f) \subset R_2$, $M = R_2/p$, $x = x_{M}=M$. We claim that the set $S = \{(0,0),(1,0),(0,1)\}$ is nonmixing.

In order to verify this we define $\{x_n : n \in \mathbb{Z}^d\} \subset K = Q(R_2/p)$ as in Corollary 2.9 and choose $\omega \in \bar{F}_2 \subset \bar{K}$ with $1 + \omega + \omega^2 = 0$. Since

$$f = (1 + \omega u_1 + \omega^2 u_2)(1 + \omega u_1 + \omega u_2),$$

we obtain that $x_{(0,0)} + \omega x_{(1,0)} + \omega^2 x_{(0,1)} = 0$, so that $S$ is nonmixing by Corollary 2.9.
Since the element $\omega' = \frac{1 + u_1}{u_1 + u_2} + p \in K$ satisfies that $1 + \omega' + \omega'^2 = 0$, we can recover (2.4) from the fact that

$$(u_1 + u_2) + (1 + u_2)u_1^{3k} + (1 + u_1)u_2^{3k} \in p$$

for every $k \geq 0$.

(2) Let $g = 1 + u_1 + u_2 + u_1^2 + u_1u_2 + u_2^2 + u_1^3u_2 + u_1u_2^2 + u_2^3$ and $q = (2, g) \subset R_2$, $M = R_2/q$, $\alpha = \alpha_M$ and $X = X_M = \tilde{M}$. We claim that the set $S = \{(0, 0), (1, 0), (0, 1)\}$ is again nonmixing.

In Example (1) above we used the fact that $f$ is irreducible over $F_2$, but not over $\tilde{F}_2$. Here the polynomial $g$ is irreducible over $\tilde{F}_2$; however, the polynomial $g(u_1^2, u_2^2)$ turns out to be divisible by $1 + u_1 + u_2$, which can be translated into the statement that the set $\{x_{(0,0)}^{1/3}, x_{(1,0)}^{1/3}, x_{(0,1)}^{1/3}\}$ is linearly dependent over $\tilde{F}_2$.

The main open question concerning higher order mixing is the following:

**Problem 2.11.** Let $\alpha$ be an algebraic $\mathbb{Z}^d$-action on a compact abelian group $X$, and let $r \geq 2$. If every subset $S \subset \mathbb{Z}^d$ of cardinality $r$ is mixing, is $\alpha$ $r$-mixing?

A positive answer to Problem 2.11 would be equivalent to the following analogue of Theorem 2.5 in characteristic $p > 0$:

**Conjecture 2.12.** Let $K$ be an algebraically closed field of characteristic $p > 0$, $G \subset K^\times = K \setminus \{0\}$ a finitely generated multiplicative group, $r \geq 2$, and $(c_1, \ldots, c_r) \in (K^\times)^r$. Let us call a solution $(x_1, \ldots, x_r) \in G^r$ of the equation

$$\sum_{i=1}^{r} c_ix_i = 0$$

regular if there exists a rational number $s > 0$ such that $\{x_1^s, \ldots, x_r^s\}$ is linearly dependent over $\tilde{F}_p \subset K$, and irregular otherwise.

Then the equation (2.6) has only finitely many irregular solutions.

3. **Conjugacy of algebraic $\mathbb{Z}^d$-actions**

Every algebraic $\mathbb{Z}^d$-action $\alpha$ with completely positive entropy is measurably conjugate to a Bernoulli shift (cf. Figure 1 (8)). Since entropy is a complete invariant for measurable conjugacy of Bernoulli shifts by [11], $\alpha$ is measurable conjugate to the $\mathbb{Z}^d$-action

$$\alpha^A: n \mapsto \alpha^{An}$$

for every $A \in \text{GL}(d, \mathbb{Z})$, since the entropies of all these actions coincide. In general, however, $\alpha$ and $\alpha^A$ are not topologically conjugate.

Every algebraic $\mathbb{Z}^d$-action $\alpha$ with positive entropy has Bernoulli factors by [8] and [12], and two such actions may again be measurably conjugate without being algebraically or topologically conjugate. For zero entropy actions, however, there is some evidence for a very strong form of isomorphism rigidity. Let us begin with a special case.

**Definition 3.1.** An algebraic $\mathbb{Z}^d$-action $\alpha$ on a compact abelian group $X$ is irreducible if every closed, $\alpha$-invariant subgroup $Y \subseteq X$ is finite.
Irreducibility is an extremely strong hypothesis: if \( \alpha \) is mixing it implies that \( \alpha^n \) is Bernoulli with finite entropy for every nonzero \( n \in \mathbb{Z}^d \). If \( \beta \) is a second irreducible and mixing algebraic \( \mathbb{Z}^d \)-action on a compact abelian group \( Y \) such that \( h(\alpha^n) = h(\beta^n) \) for every \( n \in \mathbb{Z}^d \), then \( \alpha^n \) is measurably conjugate to \( \beta^n \) for every \( n \in \mathbb{Z}^d \). However, if \( d > 1 \), then the actions \( \alpha \) and \( \beta \) are generally nonconjugate.

**Theorem 3.2** ([2], [6]). Let \( d > 1 \), and let \( \alpha \) and \( \beta \) be irreducible and mixing algebraic \( \mathbb{Z}^d \)-actions on compact abelian groups \( X \) and \( Y \), respectively. If \( \phi : X \to Y \) is a measurable conjugacy of \( \alpha \) and \( \beta \), then \( \phi \) is \( \lambda_X \)-a.e. equal to an affine map (a map \( \psi : X \to Y \) affine if it is of the form \( \phi(x) = \psi(x) + y \) for every \( x \in X \), where \( \psi : X \to Y \) is a continuous group isomorphism and \( y \in Y \)). In particular, measurable conjugacy implies algebraic conjugacy.

If the irreducible actions \( \alpha \) and \( \beta \) in Theorem 3.2 are of the form \( \alpha = \alpha_{R,d/p} \) and \( \beta = \alpha_{R,d/q} \) for some prime ideals \( p, q \subset R_d \), then measurable conjugacy implies that \( p = q \). This allows the construction of algebraic \( \mathbb{Z}^d \)-actions with very similar properties which are nevertheless measurably nonconjugate.

**Example 3.3.** Consider the algebraic \( \mathbb{Z}^2 \)-actions \( \alpha, \alpha', \alpha'' \) on \( X = T^3 \) generated by the matrices

\[
A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 8 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 1 \\ -1 & 8 & 4 \end{pmatrix},
\]

\[
A' = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ -6 & 9 & 2 \end{pmatrix} \quad \text{and} \quad B' = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 1 \\ -6 & 9 & 4 \end{pmatrix},
\]

\[
A'' = \begin{pmatrix} -3 & 4 & 0 \\ -3 & 4 & 0 \\ -10 & 11 & 2 \end{pmatrix} \quad \text{and} \quad B'' = \begin{pmatrix} -1 & 4 & 0 \\ -1 & 4 & 0 \\ -10 & 11 & 4 \end{pmatrix},
\]

respectively. In [2] it was shown that these actions are not measurably conjugate, although it appears difficult to distinguish them with the usual invariants of measurable conjugacy.

**Example 3.4** (Nonconjugacy of \( \mathbb{Z}^2 \)-actions with positive entropy). Let

\[
f_1 = 1 + u_1 + u_1^2 + u_1 u_2 + u_2^2, \\
_f_2 = 1 + u_1^2 + u_2 + u_1 u_2 + u_2^2, \\
_f_3 = 1 + u_1 + u_1^2 + u_2 + u_2^2, \\
_f_4 = 1 + u_1 + u_1^2 + u_2 + u_1 u_2 + u_2^2,
\]

in \( R_2 \), put \( p_i = (2, f_i) \subset R_2, J_i = (4, f_i) \subset R_2, M_i = R_2/J_i \), and define the algebraic \( \mathbb{Z}^2 \)-actions \( \alpha_i = \alpha_{R_2/J_i} \) on \( X_i = X_{R_2/J_i} \) as in Example 1.1 (2). Then \( h(\alpha_{R_2/q}) = \log 2 \) and \( h(\alpha_{R_2/p_i}) = 0 \), and [8, Theorem 6.5] implies that the Pinsker algebra \( \pi(\alpha_i) \) of \( \alpha_i \) is the sigma-algebra \( \mathcal{B}_{X_i/Y_i} \) of \( Y_i \)-invariant Borel sets in \( X_i \), where \( Y_i = X_i^{1/2} \) and

\[
N_i = \{ a \in M_i : p_i \cdot a = 0 \} = 2M_i \cong R_2/p_i.
\]

In other words, the \( \mathbb{Z}^2 \)-action \( \beta_i \) induced by \( \alpha_i \) on the Pinsker algebra \( \pi(\alpha_i) \) is measurably conjugate to \( \alpha_{R_2/p_i} \).

Since any measurable conjugacy of \( \alpha_i \) and \( \alpha_j \) would map \( \pi(\alpha_i) \) to \( \pi(\alpha_j) \) and induce a conjugacy of \( \beta_i \) and \( \beta_j \), Theorem 3.2 implies that \( \alpha_i \) and \( \alpha_j \) are measurably nonconjugate for \( 1 \leq i < j \leq 4 \).
The basic idea of the proof of Theorem 3.2 in [2] and [5] was suggested by Thouvenot: if $\phi: X \to Y$ is a measurable conjugacy of $\alpha$ and $\beta$, then there exists a unique probability measure $\nu$ on the graph $\Gamma(\phi) = \{(x, \phi(x)) : x \in X\} \subset X \times Y$ which projects to $\lambda_X$ and $\lambda_Y$, respectively, and which is invariant under the product-action $\alpha \times \beta: \mathbb{N} \to \alpha^n \times \beta^n$ of $\mathbb{Z}^d$ on $X \times Y$. Since $\alpha \times \beta$, acting on $(X \times Y, \nu)$, is measurably conjugate both to $\alpha$ and to $\beta$, the measure $\nu$ is mixing and has positive entropy under $\alpha^n \times \beta^n$ for every nonzero $n \in \mathbb{Z}^d$. The proof of Theorem 3.2 consists of showing that $\nu$ is a translate of the Haar measure of some closed $(\alpha \times \beta)$-invariant subgroup of $X \times Y$ (this obviously implies that $\phi$ is affine). If $X$ and $Y$ are connected, the relevant property of $\nu$ follows from [3], and if $X$ and $Y$ are zero-dimensional, the nonmixing sets of $\nu$ provide the necessary tool in [6].

Since there are considerable difficulties in extending either of these techniques to general algebraic $\mathbb{Z}^d$-actions with zero entropy, the following conjecture may seem a little premature, but I would still like to risk stating it:

**Conjecture 3.5.** Let $d > 1$, and let $\alpha$ and $\beta$ be mixing algebraic $\mathbb{Z}^d$-actions on compact abelian groups $X$ and $Y$, respectively. If $h(\alpha) = 0$, and if $\phi: X \to Y$ is a measurable conjugacy of $\alpha$ and $\beta$, then $\phi$ is $\lambda_X$-a.e. equal to an affine map. In particular, measurable conjugacy implies algebraic conjugacy.

**References**


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