HOMOCLINIC POINTS AND ISOMORPHISM RIGIDITY OF ALGEBRAIC $\mathbb{Z}^d$-ACTIONS ON ZERO-DIMENSIONAL COMPACT ABELIAN GROUPS

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ABSTRACT. Let $d > 1$, and let α and β be mixing $\mathbb{Z}^d$-actions by automorphisms of zero-dimensional compact abelian groups X and Y, respectively. By analyzing the homoclinic groups of certain sub-actions of α and β we prove that, if the restriction of α to some subgroup $\Gamma \subset \mathbb{Z}^d$ of infinite index is expansive and has completely positive entropy, then every measurable factor map $\phi\colon (X,\alpha) \longrightarrow (Y,\beta)$ is almost everywhere equal to an affine map. The hypotheses of this result are automatically satisfied if the action α contains an expansive automorphism $\alpha^{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^d$, or if α is arises from a nonzero prime ideal in the ring of Laurent polynomials in d variables with coefficients in a finite prime field. Both these corollaries generalize the main theorem in [9]. In several examples we show that this kind of isomorphism rigidity breaks down if our hypotheses are weakened.

1. INTRODUCTION

Throughout this paper the term *compact abelian group* will denote an infinite compact metrizable abelian group.

Let X be an additive compact abelian group with identity element 0_X, normalized Haar measure λ_X and additive dual group $\widehat{X}$. For every $x \in X$ and $a \in \widehat{X}$ we denote by $\langle a, x\rangle \in \mathbb{S} = \{z \in \mathbb{C}\colon |z| = 1\}$ the value of the character $a \in \widehat{X}$ at the point $x \in X$. An *algebraic action* α of a countable group Γ on X is a homomorphism $\alpha\colon \gamma \mapsto \alpha^\gamma$ from Γ into the group $\mathrm{Aut}(X)$ of continuous automorphisms of X. An algebraic Γ-action α on a compact abelian group X is *expansive* if there exists an open set $\mathcal{O} \subset X$ with

$$\bigcap_{\gamma\in\Gamma} \alpha^\gamma(\mathcal{O}) = \{0_X\},$$

and *mixing* if there exists, for all nonempty open subsets $\mathcal{O}_1, \mathcal{O}_2 \subset X$, a finite set $F \subset \Gamma$ with

$$\mathcal{O}_1 \cap \alpha^\gamma(\mathcal{O}_2) \neq \varnothing$$

for every $\gamma \in \Gamma \smallsetminus F$.

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Let α and β be algebraic Γ-actions on compact abelian groups X and Y, respectively. A Borel map $\phi\colon X \longrightarrow Y$ is *equivariant* if

$$\phi \circ \alpha^\gamma = \beta^\gamma \circ \phi \;\; \lambda_X\text{-a.e., for every } \gamma \in \Gamma. \tag{1.1}$$

A surjective equivariant Borel map $\phi\colon X \longrightarrow Y$ in (1.1) with $\lambda_Y = \lambda_X \phi^{-1}$ is called a *measurable factor map*

$$\phi\colon (X, \alpha) \longrightarrow (Y, \beta). \tag{1.2}$$

If there exists a measurable (or continuous) factor map $\phi\colon (X, \alpha) \longrightarrow (Y, \beta)$ then (Y, β) is a *measurable* (or *topological*) *factor* of (X, α). If the factor map ϕ in (1.2) is a continuous surjective group homomorphism then (Y, β) is an *algebraic factor* of (X, α) and ϕ is an *algebraic factor map*. The actions α and β are *measurably, topologically* or *algebraically conjugate* if the map ϕ in (1.2) can be chosen to be a Borel isomorphism, a homeomorphism or a continuous group isomorphism (in which case ϕ is called a *measurable, topological* or *algebraic conjugacy* of (X, α) and (Y, β)).

A map $\psi\colon X \longrightarrow Y$ is *affine* if there exist a continuous group homomorphism $\psi'\colon X \longrightarrow Y$ and an element $y \in Y$ with

$$\psi(x) = \psi'(x) + y$$

for every $x \in X$. If there exists an affine factor map $\psi\colon (X, \alpha) \longrightarrow (Y, \beta)$ then (Y, β) is obviously an algebraic factor of (X, α).

For $d = 1$, any algebraic $\mathbb{Z}$-action is determined by the powers of a single group automorphism α. If α is ergodic, then it is Bernoulli (cf. e.g. [1]), which implies that two such actions with equal entropy are measurably conjugate even if they are algebraically non-conjugate.

If $d > 1$ and α_1, α_2 are algebraic $\mathbb{Z}^d$-actions with completely positive entropy with respect to Haar measure, then they are Bernoulli by [11] and can thus again be measurably conjugate without being algebraically conjugate. However, if these actions are mixing with zero entropy, then measurable conjugacy implies — under certain additional conditions — not only algebraic conjugacy, but also that every measurable conjugacy between such actions is (almost everywhere equal to) an affine map. For irreducible[1] and mixing algebraic $\mathbb{Z}^d$-actions with $d > 1$ this kind of strong isomorphism rigidity was proved in [8]–[9], and in [13] the (cautious) conjecture was formulated that *every* measurably conjugate pair of expansive and mixing zero-entropy algebraic $\mathbb{Z}^d$-actions with $d > 1$ is algebraically conjugate, and that every measurable conjugacy between such actions is affine.

In [2], the first author presented a counterexample to this conjecture: there exist two measurably conjugate expansive and mixing zero-entropy algebraic $\mathbb{Z}^8$-actions α_1 and α_2 on non-isomorphic zero-dimensional compact abelian groups X_1 and X_2, respectively. On the positive side it was shown in [3] that, for $d > 1$, every measurable conjugacy between expansive and mixing zero-entropy algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups is (almost everywhere equal to) a continuous map with certain additional algebraic properties.

[1]An algebraic $\mathbb{Z}^d$-action α on a compact abelian group X is *irreducible* if every closed α-invariant subgroup $Y \subsetneq X$ is finite.

In this paper we present further counterexamples to the rigidity conjecture in [13], including two measurably conjugate, but algebraically non-conjugate, expansive and mixing zero-entropy $\mathbb{Z}^3$-actions on zero-dimensional compact abelian groups. However, if $d > 1$, and if α_1 and α_2 are mixing algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups X_1 and X_2 such that the restriction of α_1 to some subgroup $\Gamma \subset \mathbb{Z}^d$ of infinite index is expansive and has completely positive entropy, then every measurable factor map between α_1 and α_2 is affine (Theorem 4.1). Since this condition is automatically satisfied if α_1 is an expansive $\mathbb{Z}^2$-action with zero entropy (or, more generally, if α_1 contains an expansive element $\alpha_1^{\mathbf{n}}$), all expansive and mixing zero-entropy algebraic $\mathbb{Z}^2$-actions (or all mixing algebraic $\mathbb{Z}^d$-actions containing an expansive element) on zero-dimensional compact abelian groups exhibit strong isomorphism rigidity (Corollary 4.2). In a second corollary (Corollary 4.3) we show that any measurable conjugacy between two mixing algebraic $\mathbb{Z}^d$-actions α_1, α_2 arising from nonzero prime ideals in the ring $R_d^{(p)}$ of Laurent polynomials in d variables with coefficients in a finite prime field F_p via the construction (2.10)–(2.11) is affine.

The key tools for the proof of Theorem 4.1 are the continuity of measurable equivariant maps proved in [3] and a detailed investigation of the homoclinic groups of certain sub-actions of the $\mathbb{Z}^d$-actions α_1 and α_2 in Proposition 3.5.

In [5] Manfred Einsiedler has recently given a proof of Theorem 4.1 by a different method based on relative entropy considerations in the sense of [7].

2. Algebraic $\mathbb{Z}^d$-actions on zero-dimensional groups

Let α be an algebraic Γ-action on a compact abelian group X. For every subgroup $\Gamma' \subset \Gamma$ we denote by $\alpha^{\Gamma'}$ the restriction of α to Γ'. If $Z \subset X$ is a closed α-invariant subgroup we write α_Z and $\alpha_{X/Z}$ for the algebraic $\mathbb{Z}^d$-actions induced by α on Z and X/Z, respectively.

We denote by $R_d = \mathbb{Z}[u_1^{\pm1}, \dots, u_d^{\pm1}]$ the ring of Laurent polynomials with integral coefficients in the variables $u_1, \dots, u_d$ and write the elements $f \in R_d$ as

$$f = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} u^{\mathbf{n}} \tag{2.1}$$

with $u^{\mathbf{n}} = u_1^{n_1} \cdots u_d^{n_d}$ and $f_{\mathbf{n}} \in \mathbb{Z}$ for all $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$, where $f_{\mathbf{n}} = 0$ for all but finitely many $\mathbf{n} \in \mathbb{Z}^d$.

If α is an algebraic $\mathbb{Z}^d$-action on a compact abelian group X, then the additively-written dual group $M = \widehat{X}$ is a module over the ring R_d with respect to the operation

$$f \cdot a = f(\widehat{\alpha})(a) = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} \widehat{\alpha^{\mathbf{n}}}(a) \tag{2.2}$$

for $f \in R_d$ and $a \in M$, where $\widehat{\alpha^{\mathbf{n}}}$ denotes the automorphism of $\widehat{X}$ dual to $\alpha^{\mathbf{n}}$. The module $M = \widehat{X}$ is called the *dual module* of α.

Conversely, if M is a module over R_d, then we obtain an algebraic $\mathbb{Z}^d$-action α_M on $X_M = \widehat{M}$ by setting

$$\widehat{\alpha_M^{\mathbf{n}}}(a) = u^{\mathbf{n}} \cdot a \tag{2.3}$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $a \in M$. Clearly, M is the dual module of α_M.

Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X with dual module $M = \widehat{X}$. For every $f = \sum_{\mathbf{n}\in\mathbb{Z}^d} f_{\mathbf{n}} u^{\mathbf{n}} \in R_d$ we define a continuous group homomorphism $f(\alpha)\colon X \longrightarrow X$ by setting, for every $x \in X$,

$$f(\alpha)(x) = \sum_{\mathbf{n}\in\mathbb{Z}^d} f_{\mathbf{n}} \alpha^{\mathbf{n}} x. \tag{2.4}$$

Note that $f(\alpha)$ is dual to multiplication by f on $M = \widehat{X}$ (or, equivalently, that $\widehat{f(\alpha)} = f(\widehat{\alpha})$ in (2.2)). Hence $f(\alpha)$ is surjective if and only if f does not lie in any prime ideal associated[2] with M. For details we refer to [12].

In this paper we restrict our attention to algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups. We recall the following results (cf. [12, Propositions 6.6 and 6.9]).

Lemma 2.1. *Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X. Then the group X is zero-dimensional if and only if every prime ideal $\mathfrak{p}$ associated with the dual module $M = \widehat{X}$ of α contains a rational prime constant $p(\mathfrak{p}) > 1$.*

Lemma 2.2. *Let α be an algebraic $\mathbb{Z}^d$-action on a zero-dimensional compact abelian group X with dual module $M = \widehat{X}$.*

1. *The following conditions are equivalent.*
   - (a) *α is expansive;*
   - (b) *The module M is Noetherian.*
2. *The following conditions are equivalent.*
   - (a) *α_M is mixing;*
   - (b) *$\alpha_{R_d/\mathfrak{p}}$ is mixing for every $\mathfrak{p} \in \mathrm{Asc}(M)$;*
   - (c) *$\mathfrak{p} \cap \{u^{\mathbf{n}} - 1\colon 0 \neq \mathbf{n} \in \mathbb{Z}^d\} = \varnothing$ for every $\mathfrak{p} \in \mathrm{Asc}(M)$.*
3. *The following conditions are equivalent.*
   - (a) *α_M has positive entropy (with respect to the normalized Haar measure λ_X of X);*
   - (b) *$\alpha_{R_d/\mathfrak{p}}$ has positive entropy for some $\mathfrak{p} \in \mathrm{Asc}(M)$;*
   - (c) *Some $\mathfrak{p} \in \mathrm{Asc}(M)$ is principal (and hence of the form $\mathfrak{p} = (p) = pR_d$ for some rational prime constant $p > 1$).*
4. *The following conditions are equivalent.*
   - (a) *α_M has completely positive entropy (with respect to λ_X);*
   - (b) *$\alpha_{R_d/\mathfrak{p}}$ has positive entropy for every $\mathfrak{p} \in \mathrm{Asc}(M)$;*
   - (c) *Every $\mathfrak{p} \in \mathrm{Asc}(M)$ of the form $\mathfrak{p} = (p) = pR_d$ for some rational prime constant $p = p(\mathfrak{p}) > 1$.*

Lemma 2.3. *Let α be an expansive algebraic $\mathbb{Z}^d$-action on a zero-dimensional compact abelian group X with dual module $M = \widehat{X}$. If $\mathrm{Asc}(M) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$, then there exist Noetherian R_d-modules $N \supseteq M \supseteq N'$ with the following properties.*

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[2] A prime ideal $\mathfrak{p} \subset R_d$ is *associated with an R_d-module M* if $\mathfrak{p} = \mathrm{ann}(a) = \{f \in R_d\colon f \cdot a = 0_M\}$ for some $a \in M$, and the module M is *associated with a prime ideal* $\mathfrak{p} \subset R_d$ if $\mathfrak{p}$ is the only prime ideal associated with M. The set of prime ideals associated with a Noetherian R_d-module M is finite and denoted by $\mathrm{Asc}(M)$.

(1) $N = N^{(1)} \oplus \cdots \oplus N^{(m)}$, *where each of the modules* $N^{(j)}$ *has a finite sequence of submodules* $N^{(j)} = N^{(j)}_{s_j} \supset \cdots \supset N^{(j)}_0 = \{0\}$ *with* $N^{(j)}_k / N^{(j)}_{k-1} \cong R_d/\mathfrak{p}_j$ *for* $k = 1, \dots, s_j$;

(2) N *and* N' *are isomorphic as* R_d*-modules.*

In view of the Lemmas 2.1–2.3 it is useful to have an explicit realization of $\mathbb{Z}^d$-actions of the form $\alpha_{R_d/\mathfrak{p}}$, where $\mathfrak{p} \subset R_d$ is a prime ideal containing a rational prime constant $p > 1$.

Denote by $R_d^{(p)} = F_p[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ the ring of Laurent polynomials in the variables $u_1, \dots, u_d$ with coefficients in the prime field $F_p = \mathbb{Z}/p\mathbb{Z}$ and define a ring homomorphism $f \mapsto f_{/p}$ from R_d to $R_d^{(p)}$ by reducing each coefficient of f modulo p. As in (2.1) we write every $h \in R_d^{(p)}$ as $h = \sum_{\mathbf{n} \in \mathbb{Z}^d} h_{\mathbf{n}} u^{\mathbf{n}}$ with $h_{\mathbf{n}} \in F_p$ for every $\mathbf{n} \in \mathbb{Z}^d$. The set

$$\mathcal{S}(h) = \{\mathbf{n} \in \mathbb{Z}^d \colon c_h(\mathbf{n}) \neq 0\} \tag{2.5}$$

is called the *support* of $h \in R_d^{(p)}$.

If $\mathfrak{p} \subset R_d$ is a prime ideal containing the constant p, then

$$\overline{\mathfrak{p}} = \{f_{/p} \colon f \in \mathfrak{p}\} \subset R_d^{(p)} \tag{2.6}$$

is again a prime ideal, and the map $f \mapsto f_{/p}$ induces an R_d-module isomorphism

$$R_p/\mathfrak{p} \cong R_d^{(p)}/\overline{\mathfrak{p}}. \tag{2.7}$$

Let $\Omega = F_p^{\mathbb{Z}^d}$, furnished with the product topology and component-wise addition. We write every $\omega \in \Omega$ as $\omega = (\omega_{\mathbf{n}})$ with $\omega_{\mathbf{n}} \in F_p$ for every $\mathbf{n} \in \mathbb{Z}^d$ and define the shift-action σ of $\mathbb{Z}^d$ on Ω by

$$(\sigma^{\mathbf{m}} \omega)_{\mathbf{n}} = \omega_{\mathbf{m}+\mathbf{n}} \tag{2.8}$$

for every $\mathbf{m} \in \mathbb{Z}^d$ and $\omega = (\omega_{\mathbf{n}}) \in \Omega$. For every $h = \sum_{\mathbf{n} \in \mathbb{Z}^d} h_{\mathbf{n}} u^{\mathbf{n}} \in R_d^{(p)}$ we define a continuous group homomorphism $h(\sigma) \colon \Omega \longrightarrow \Omega$ as in (2.4) by

$$h(\sigma) = \sum_{\mathbf{n} \in \mathbb{Z}^d} h_{\mathbf{n}} \sigma^{\mathbf{n}}.$$

The additive group $R_d^{(p)}$ can be identified with the dual group $\widehat{\Omega}$ of Ω by setting

$$\langle h, \omega \rangle = e^{2\pi i \left(\sum_{\mathbf{n} \in \mathbb{Z}^d} h_{\mathbf{n}} \omega_{\mathbf{n}}\right)/p} \tag{2.9}$$

for every $h \in R_d^{(p)}$ and $\omega \in \Omega$. With this identification the shift $\sigma^{\mathbf{m}} \colon \Omega \longrightarrow \Omega$ is dual to multiplication by $u^{\mathbf{m}}$ on $\widehat{\Omega} = R_d^{(p)}$, and $h(\sigma)$ is dual to multiplication by h on $R_d^{(p)}$ for every $h \in R_d^{(p)}$.

If $\mathfrak{q} \subset R_d^{(p)}$ is an ideal with generators $\{h^{(1)}, \dots, h^{(k)}\}$, then

$$\begin{aligned} \mathfrak{q}^{\perp} = \widehat{R_d^{(p)}/\mathfrak{q}} = X_{R_d^{(p)}/\mathfrak{q}} &= \{\omega \in \Omega \colon \langle h, \omega \rangle = 1 \text{ for every } h \in \mathfrak{q}\} \\ &= \bigcap_{h \in \mathfrak{q}} \ker(h(\sigma)) = \bigcap_{i=1}^{k} \ker(h^{(i)}(\sigma)). \end{aligned} \tag{2.10}$$

is a closed, shift-invariant subgroup of Ω, and

$$\alpha_{R_d^{(p)}/\mathfrak{q}} = \sigma_{X_{R_d^{(p)}/\mathfrak{q}}} \tag{2.11}$$

is the restriction of the shift-action σ to $X_{R_d^{(p)}/\mathfrak{q}} \subset \Omega$.

We will use the following result from [3] on measurable equivariant maps between algebraic $\mathbb{Z}^d$-actions on zero-dimensional groups (cf. [3, Corollary 1.2]).

Lemma 2.4. *Let $d > 1$, and let α and β be mixing zero-entropy algebraic $\mathbb{Z}^d$-actions on compact abelian groups X and Y, respectively. Then there exists, for every measurable $\mathbb{Z}^d$-equivariant map $\phi\colon (X,\alpha) \longrightarrow (Y,\beta)$, a continuous $\mathbb{Z}^d$-equivariant map $\phi'\colon (X,\alpha) \longrightarrow (Y,\beta)$ such that $\phi = \phi'$ λ_X-a.e.*

3. Homoclinic points

Definition 3.1. Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X, and let $\Gamma \subset \mathbb{Z}^d$ be a subgroup. An element $x \in X$ is (α,Γ)-*homoclinic* (to the identity element 0_X of X), if

$$\lim_{\substack{\mathbf{n}\to\infty \\ \mathbf{n}\in\Gamma}} \alpha^{\mathbf{n}} x = 0_X.$$

The α-invariant subgroup $\Delta_{(\alpha,\Gamma)}(X) \subset X$ of all (α,Γ)-homoclinic points is an R_d-module under the operation

$$f \cdot x = f(\alpha)(x)$$

for every $f \in R_d$ and $x \in \Delta_{(\alpha,\Gamma)}(X)$ (cf. (2.4)), and is called the Γ-*homoclinic module of* α (cf. [10]).

Proposition 3.2. *Let α be an expansive algebraic $\mathbb{Z}^d$-action on a compact abelian group X, and let $\Gamma \subset \mathbb{Z}^d$ be a subgroup. Then $\Delta_{(\alpha,\Gamma)} \neq \{0_X\}$ if and only if the entropy $h(\alpha^\Gamma)$ of the algebraic Γ-action α^Γ on X is positive, and $\Delta_{(\alpha,\Gamma)}$ is dense in X if and only if α^Γ has completely positive entropy (where entropy is always taken with respect to Haar measure).*

Proof. This is [10, Theorems 4.1 and 4.2]. □

If an expansive and mixing algebraic $\mathbb{Z}^d$-action α on a compact abelian group X has zero entropy, then the homoclinic group $\Delta_\alpha(X)$ of this $\mathbb{Z}^d$-action is trivial by Proposition 3.2, but $\Delta_{(\alpha,\Gamma)}$ will be dense in X for appropriate subgroups $\Gamma \subset \mathbb{Z}^d$. We investigate this phenomenon in the special case where $p > 1$ is a rational prime, $f \in R_d^{(p)}$ an irreducible Laurent polynomial such that the convex hull $\mathcal{C}(f) \subset \mathbb{R}^d$ of the support $\mathcal{S}(f) \subset \mathbb{Z}^d$ of f contains an interior point (cf. (2.5)), and where $\alpha = \alpha_{R_d^{(p)}/(f)}$ is the shift-action of $\mathbb{Z}^d$ on the compact abelian group $X = X_{R_d^{(p)}/(f)} \subset F_p^{\mathbb{Z}^d}$ defined in (2.10)–(2.11).

We write $[\,.\,,.\,]$ and $\|\cdot\|$ for the Euclidean inner product and norm on $\mathbb{R}^d$ and set, for every nonzero element $\mathbf{m} \in \mathbb{Z}^d$,

$$\Gamma_{\mathbf{m}} = \{\mathbf{n} \in \mathbb{Z}^d \colon [\mathbf{m},\mathbf{n}] = 0\}. \tag{3.1}$$

Let

$$\mathsf{S}_{d-1} = \{\mathbf{v} \in \mathbb{R}^d \colon \|\mathbf{v}\| = 1\}$$

be the unit sphere in $\mathbb{R}^d$ and put, for every $\mathbf{v} \in \mathsf{S}_{d-1}$,

$$H_{\mathbf{v}} = \{\mathbf{w} \in \mathbb{Z}^d \colon [\mathbf{v}, \mathbf{w}] \leq 0\},$$
$$X_{\mathbf{v}} = \{x \in X \colon x_{\mathbf{n}} = 0 \text{ for every } \mathbf{n} \in H_{\mathbf{v}}\}.$$

Following [6] we observe that the set

$$\mathsf{N}(\alpha) = \{\mathbf{v} \in \mathsf{S}_{d-1} \colon X_{\mathbf{v}} \neq \{0_X\}\}$$

consists of all $\mathbf{v} \in \mathsf{S}_{d-1}$ such that

$$\{\mathbf{w} \in \mathcal{C}(f) \colon [\mathbf{w}, \mathbf{v}] = \max_{\mathbf{w}' \in \mathcal{C}(f)} [\mathbf{w}', \mathbf{v}]\}$$

contains a (one-dimensional) edge of $\mathcal{C}(f)$ (recall that $\mathcal{C}(f) \subset \mathbb{R}^d$ is the convex hull of the support of f in (2.5)). The complement

$$\mathsf{E}(\alpha) = \mathsf{S}_{d-1} \smallsetminus \mathsf{N}(\alpha) \tag{3.2}$$

of $\mathsf{N}(\alpha)$ is dense, open, and consists of finitely many connected components. Hence the set

$$\mathsf{E}^*(\alpha) = \mathsf{E}(\alpha) \cap (-\mathsf{E}(\alpha)) = \mathsf{S}_{d-1} \smallsetminus (\mathsf{N}(\alpha) \cup (-\mathsf{N}(\alpha))) \tag{3.3}$$

is again dense, open, and has finitely many connected components, called the *Weyl chambers* of α. For every nonzero $\mathbf{m} \in \mathbb{Z}^d$ with

$$\mathbf{m}^* = \tfrac{\mathbf{m}}{\|m\|} \in \mathsf{E}^*(\alpha) \tag{3.4}$$

we denote by $W(\mathbf{m})$ the connected component of $\mathsf{E}(\alpha)$ containing $\mathbf{m}^*$ and write $W^*(\mathbf{m}) = W(\mathbf{m}) \cap W(-\mathbf{m})$ for the Weyl chamber of $\mathsf{E}^*(\alpha)$ containing $\mathbf{m}^*$. In this notation we have the following lemma.

Lemma 3.3. *Let $f \in R_d^{(p)}$ be an irreducible Laurent polynomial such that the convex hull $\mathcal{C}(f) \subset \mathbb{R}^d$ of the support $\mathcal{S}(f) \subset \mathbb{Z}^d$ of f contains an interior point, and let $\alpha = \alpha_{R_d^{(p)}/(f)}$ be the shift-action of $\mathbb{Z}^d$ on the compact abelian group $X = X_{R_d^{(p)}/(f)} \subset F_p^{\mathbb{Z}^d}$ defined in* (2.10)*–*(2.11)*.*

(1) *For every nonzero element $\mathbf{m} \in \mathbb{Z}^d$, the action $\alpha^{\Gamma_{\mathbf{m}}}$ is expansive if and only if $\mathbf{m}$ satisfies* (3.4)*;*
(2) *If $\mathbf{m}$ satisfies* (3.4) *then $\Delta_{(\alpha,\Gamma_{\mathbf{m}})}$ is dense in X and there exists a* fundamental homoclinic point *$x^{\Delta} \in \Delta_{(\alpha,\Gamma_{\mathbf{m}})}$ such that*
$$\{h(\alpha)(x^{\Delta}) \colon h \in R_d^{(p)}\} = \Delta_{(\alpha,\Gamma_{\mathbf{m}})} \tag{3.5}$$
and
$$h(\alpha)(x^{\Delta}) = 0_X \text{ if and only if } h \in (f). \tag{3.6}$$
(3) *If $\mathbf{n} \in \mathbb{Z}^d$ is a second nonzero element satisfying* (3.4)*, then*
$$\Delta_{(\alpha,\Gamma_{\mathbf{m}})} = \Delta_{(\alpha,\Gamma_{\mathbf{n}})}$$
whenever $W^*(\mathbf{m}) = W^*(\mathbf{n})$.

Proof. The assertion (1) follows from [4], [6] or an elementary direct argument. In order to prove the existence of a fundamental homoclinic point x^{Δ} in (2) we choose an element $\mathbf{m}' \in \mathbb{Z}^d$ with $\mathbb{Z}^d = \Gamma_{\mathbf{m}} + \{k\mathbf{m}' \colon k \in \mathbb{Z}\}$ and write f as $f = \sum_{k=k_1}^{k_2} u^{k\mathbf{m}'} g^{(k)}$ for appropriate integers $k_1 < k_2$, where $\mathcal{S}(g^{(k)}) \subset \Gamma_{\mathbf{m}}$ for every $k = k_1, \dots, k_2$, and where $g^{(k_1)}$ and $g^{(k_2)}$ each have a single nonzero entry. As $X = \ker(f(\sigma))$ by (2.10), every $x \in X$ is determined completely by its coordinates in the subset $S = \Gamma_{\mathbf{m}} + \{k_1\mathbf{m}', \dots, (k_2-1)\mathbf{m}'\} \subset \mathbb{Z}^d$; furthermore, the projection $\pi_S \colon X \longrightarrow F_p^S$ onto the coordinates in S is bijective and

$$\Delta_{(\alpha,\Gamma_{\mathbf{m}})} = \{x = (x_{\mathbf{k}}) \in X \colon x_{\mathbf{k}} \neq 0 \text{ for only finitely many } \mathbf{k} \in S\}.$$

The point $x^\Delta \in X$ with $x^\Delta_{k_1\mathbf{m}'} = 1$ and $x^\Delta_{\mathbf{k}} = 0$ for every $\mathbf{k} \in S \smallsetminus \{k_1\mathbf{m}'\}$ will satisfy (3.5)–(3.6). Note that we have proved in passing that $\alpha^{\Gamma_{\mathbf{m}}}$ is the shift-action of $\Gamma_{\mathbf{m}}$ on $A^{\Gamma_{\mathbf{m}}}$ for some finite abelian group A, and that $\Delta_{(\alpha,\Gamma_{\mathbf{m}})}$ is dense in X.

For (3) we consider the convex cone

$$C'(\mathbf{m}) = \{\mathbf{v} \in \mathbb{R}^d \smallsetminus \{\mathbf{0}\} \colon \mathbf{v}^* \in W(\mathbf{m})\}$$

with dual cone

$$C(\mathbf{m}) = \{\mathbf{w} \in \mathbb{R}^d \colon [\mathbf{w}, \mathbf{v}] \leq 0 \text{ for every } \mathbf{v} \in C'(\mathbf{m})\}. \tag{3.7}$$

If $\mathbf{l} \in \mathcal{C}(f)$ is the unique vertex with

$$[\mathbf{l}, \mathbf{m}] = \max \{[\mathbf{k}, \mathbf{m}] \colon \mathbf{k} \in \mathcal{S}(f)\},$$

then $C(\mathbf{m})$ is the smallest cone in $\mathbb{R}^d$ containing $\mathcal{S}(f) - \mathbf{l} = \mathcal{S}(u^{-\mathbf{l}}f)$. Furthermore, if $\mathbf{n} \in \mathbb{Z}^d \smallsetminus \{\mathbf{0}\}$ and $\mathbf{n}^* \in \mathsf{E}^*(\alpha)$, then

$$C(\mathbf{m}) = C(\mathbf{n}) \text{ if and only if } W(\mathbf{m}) = W(\mathbf{n}) \tag{3.8}$$

(cf. (3.7)), but the interiors of $C(\mathbf{m})$ and $C(\mathbf{n})$ may obviously have nonempty intersection even if $W(\mathbf{m}) \neq W(\mathbf{n})$.

For every homoclinic point $x \in \Delta_{(\alpha,\Gamma_{\mathbf{m}})}(X)$ we set

$$\mathcal{S}(x) = \{\mathbf{n} \in \mathbb{Z}^d \colon x_{\mathbf{n}} \neq 0\}$$

and note that there exist elements $\mathbf{k}^\pm \in \mathbb{Z}^d$ with

$$\mathcal{S}(x) \subset (\mathbf{k}^+ - C(\mathbf{m})) \cup (\mathbf{k}^- - C(-\mathbf{m})). \tag{3.9}$$

This hows that x is homoclinic for every $\alpha^{\Gamma_{\mathbf{n}}}$ with $\mathbf{n}^* \in W^*(\mathbf{m})$. Since $x \in \Delta_{(\alpha,\Gamma_{\mathbf{m}})}(X)$ was arbitrary, and since the situation is symmetric in $\mathbf{m}$ and $\mathbf{n}$, this proves (3). ☐

Lemma 3.4. *Let $d > 1$, $p > 1$ a rational prime, and let $f \in R_d^{(p)}$ be an irreducible Laurent polynomial such that the convex hull $\mathcal{C}(f) \subset \mathbb{R}^d$ of the support $\mathcal{S}(f) \subset \mathbb{Z}^d$ contains an interior point. Let $\alpha = \alpha_{R_d^{(p)}/(f)}$ be the shift-action of $\mathbb{Z}^d$ on the compact abelian group $X = X_{R_d^{(p)}/(f)} \subset F_p^{\mathbb{Z}^d}$ defined in* (2.10)–(2.11), *and let $z \in X$ be a point with the following property: there exist an integer $k \geq 1$ and elements $\mathbf{n}_i$, $i = 1, \dots k$, in $\mathbb{Z}^d \smallsetminus \{\mathbf{0}\}$ such that*

$$\mathcal{S}(z) = \{\mathbf{n} \in \mathbb{Z}^d \colon z_{\mathbf{n}} \neq 0\} \subset \left(\bigcup_{i=1}^k \Gamma_{\mathbf{n}_i}\right) + Q(N) \tag{3.10}$$

for some integer $N \geq 0$, where

$$Q(M) = \{-M, \dots, M\}^d \subset \mathbb{Z}^d \tag{3.11}$$

for every $M \geq 0$. Then there exists a Laurent polynomial $g \in R_d^{(p)} \smallsetminus (f)$ with $g(\alpha)(z) = 0_X$.

Proof. We write f in the form (2.1), assume without loss in generality (by multiplying f by a monomial $u^{\mathbf{k}}$, if necessary) that

$$S = \mathcal{S}(f) \cap \Gamma_{\mathbf{n}_k} \neq \varnothing,$$

and set

$$h_k = \sum_{\mathbf{n} \in \Gamma_{\mathbf{n}_k}} f_{\mathbf{n}} u^{\mathbf{n}}.$$

Since the convex hull of the support of h_k has no interior point, $h_k \notin (f)$.

Choose $M \geq 1$ with $\mathcal{S}(f) \subset Q(M)$ (cf. (3.11)), and let $r \geq 1$ be an integer with $p^r > 2dN$. For every $\mathbf{k} \in \mathbb{Z}^d$ with

$$\mathbf{k} \notin \left(\bigcup_{i=1}^{k-1} \Gamma_{\mathbf{n}_i} \right) + Q(p^r M + N),$$

the support of the Laurent polynomial $u^{\mathbf{k}} f^{p^r}$ does not intersect

$$\left(\bigcup_{i=1}^{k-1} \Gamma_{\mathbf{n}_i} \right) + Q(N).$$

Furthermore, if

$$\mathcal{S}(u^{\mathbf{k}} h_k^{p^r}) \cap (\Gamma_{\mathbf{n}_k} + Q(N)) = \mathcal{S}(u^{\mathbf{k}} h_k^{p^r}) \cap \left[\left(\bigcup_{i=1}^{k} \Gamma_{\mathbf{n}_i} \right) + Q(N) \right] \neq \varnothing,$$

then

$$\begin{aligned} \mathcal{S}(u^{\mathbf{k}} f^{p^r}) \cap (\Gamma_{\mathbf{n}_k} + Q(N)) &= \mathcal{S}(u^{\mathbf{k}} h_k^{p^r}) \cap \left[\left(\bigcup_{i=1}^{k} \Gamma_{\mathbf{n}_i} \right) + Q(N) \right] \\ &= \mathcal{S}(u^{\mathbf{k}} h_k^{p^r}) \cap (\Gamma_{\mathbf{n}_k} + Q(N)). \end{aligned} \tag{3.12}$$

According to the definition of X in (2.11), $f^{pM}(\alpha)(z) = 0_X$, and hence

$$\begin{aligned} 0 = f^{p^r}(\alpha)(z)_{-\mathbf{k}} = (u^{\mathbf{k}} f^{p^r})(\alpha)(z)_{\mathbf{0}} &= \sum_{\mathbf{n} \in \mathcal{S}(f)} f_{\mathbf{n}} z_{\mathbf{k}+p^r \mathbf{n}} \\ \stackrel{*}{=} \sum_{\mathbf{n} \in \mathcal{S}(h_k)} f_{\mathbf{n}} z_{\mathbf{k}+p^r \mathbf{n}} &= (u^{\mathbf{k}} h_k^{p^r})(\alpha)(z)_{\mathbf{0}} = h_k^{p^r}(\alpha)(z)_{-\mathbf{k}}, \end{aligned}$$

where the identity marked $\stackrel{*}{=}$ follows from (3.12). The Laurent polynomial $h_k' = h_k^{p^r} \notin (f)$ thus has the property that

$$\mathcal{S}(h_k'(\alpha)(z)) \subset \left(\bigcup_{i=1}^{k-1} \Gamma_{\mathbf{n}_i} \right) + Q(N')$$

for some integer $N' \geq 1$.

We repeat the argument with k, z and N replaced by $k-1, h_k'(\alpha)(z)$ and N', respectively. After k steps we obtain Laurent polynomials $h_1', \ldots, h_k'$ in $R_d^{(p)}$ such that $g = \prod_{i=1}^{k} h_i' \notin (f)$ and $\mathcal{S}(g(\alpha)(z))$ is finite. In other words, the point $g(\alpha)(z)$ is homoclinic and hence, since α has entropy zero, equal to 0_X by Proposition 3.2. □

Now we can state the main results of this section.

Proposition 3.5. *Let $f \in R_d^{(p)}$ be an irreducible Laurent polynomial such that the convex hull $\mathcal{C}(f) \subset \mathbb{R}^d$ of the support $\mathcal{S}(f) \subset \mathbb{Z}^d$ of f contains an interior point, and let $\alpha = \alpha_{R_d^{(p)}/(f)}$ be the shift-action of $\mathbb{Z}^d$ on the compact abelian group $X = X_{R_d^{(p)}/(f)} \subset F_p^{\mathbb{Z}^d}$ defined in (2.10)–(2.11). Then there exists, for every Weyl chamber W_1^* of α, a Weyl chamber W_2^* of α such that the following properties are satisfied for all nonzero $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$ with $\mathbf{m}^* \in W_1^*$ and $\mathbf{n}^* \in W_2^*$.*

(1) *The homoclinic groups $\Delta_{(\alpha,\Gamma_{\mathbf{m}})}(X)$ and $\Delta_{(\alpha,\Gamma_{\mathbf{n}})}(X)$ are dense in X;*
(2) *$\Delta_{(\alpha,\Gamma_{\mathbf{m}})}(X) \cap \Delta_{(\alpha,\Gamma_{\mathbf{n}})}(X) = \{0_X\}$.*

Proof. We fix a nonzero element $\mathbf{m} \in \mathbb{Z}^d$ with $\mathbf{m}^* \in W_1^*$. Then the homoclinic group $\Delta_{(\alpha,\Gamma_{\mathbf{m}})}$ is dense in X and isomorphic to $R_d^{(p)}/(f)$ by Lemma 3.3.

Suppose that $\Delta_{(\alpha,\Gamma_{\mathbf{m}})} \cap \Delta_{(\alpha,\Gamma_{\mathbf{n}})} \neq \{0_X\}$ for every nonzero $\mathbf{n} \in \mathbb{Z}^d$ satisfying (3.4) (with $\mathbf{n}$ replacing $\mathbf{m}$). Under this hypothesis we shall prove the existence of a Laurent polynomial $g \in R_d^{(p)} \smallsetminus (f)$ such that $g(\alpha)(X) = \{0_X\}$. By duality, $(g) = gR_d^{(p)} \subset (f)$, and this contradiction will prove the proposition.

In order to construct such a Laurent polynomial g we choose an enumeration $W_1^*, \dots, W_k^*$ of the Weyl chambers of α, set $\mathbf{n}_1 = \mathbf{m}$, and choose elements $\mathbf{n}_i \in \mathbb{Z}^d \smallsetminus \{\mathbf{0}\}$ such that $\mathbf{n}_i^* \in W_i^*$ for $i = 2, \dots, k$. By hypothesis, $\Delta_{(\alpha,\Gamma_{\mathbf{m}})} \cap \Delta_{(\alpha,\Gamma_{\mathbf{n}_i})} \neq \{0_X\}$ for $i = 2, \dots, k$, and (3.5)–(3.6) allows us to find Laurent polynomials $h^{(i)} \in R_d^{(p)} \smallsetminus (f)$ with $h^{(i)}(\alpha)(x^\Delta) \in \Delta_{(\alpha,\Gamma_{\mathbf{n}_i})} \smallsetminus \{0_X\}$ for $i = 2, \dots, k$. The Laurent polynomial $h = \prod_{i=2}^k h^{(i)} \in R_d^{(p)} \smallsetminus (f)$ has the property that

$$0_X \neq y^\Delta = h(\alpha)(x^\Delta) \in \Delta_{(\alpha,\Gamma_{\mathbf{n}_i})} \tag{3.13}$$

for $i = 1, \dots, m$. It follows that $y^\Delta \in \Delta_{(\alpha,\Gamma_{\mathbf{n}})}$ and hence that

$$\lim_{\substack{\mathbf{k}\to\infty \\ \mathbf{k}\in\Gamma_{\mathbf{n}}}} \alpha^{\mathbf{k}} y^\Delta = 0_X \tag{3.14}$$

for every nonzero $\mathbf{n} \in \mathbb{Z}^d$ for which $\alpha^{\Gamma_{\mathbf{n}}}$ is expansive.

From (3.9) we conclude that there exist elements $\mathbf{k}_i^\pm \in \mathbb{Z}^d$, $i = 1, \dots, k$, with

$$\mathcal{S}(y^\Delta) \subset \bigcap_{i=1}^{k} \bigl((\mathbf{k}_i^+ - C(\mathbf{n}_i)) \cup (\mathbf{k}_i^- - C(-\mathbf{n}_i)\bigr). \tag{3.15}$$

We write $\mathcal{F}(f)$ for the set of ($(d-1)$-dimensional) faces of the convex polyhedron $\mathcal{C}(f)$, choose, for every face $F \in \mathcal{F}(f)$, an element $\mathbf{v}_F \in \mathsf{N}(\alpha)$ orthogonal to F, and set

$$\Gamma(F) = \Gamma_{\mathbf{v}_F}$$

as in (3.1). From (3.15) and the definition of $X = X_{R_d^{(p)}/(f)}$ in (2.10) we conclude that that there exists an integer $N \geq 0$ with

$$\mathcal{S}(y^\Delta) \subset \biggl(\bigcup_{F\in\mathcal{F}(f)} \Gamma(F)\biggr) + Q(N). \tag{3.16}$$

Lemma 3.4 implies the existence of a Laurent polynomial $g \in R_d^{(p)} \smallsetminus (f)$ with

$$g(\alpha)(y^\Delta) = (gh)(\alpha)(x^\Delta) = 0_X.$$

As explained above, this completes the proof of the proposition. $\square$

Proposition 3.6. *Let $d > 1$, $p > 1$ a rational prime, $f \in R_d^{(p)}$ an irreducible Laurent polynomial such that the shift-action $\alpha = \alpha_{R_d^{(p)}/(f)}$ of $\mathbb{Z}^d$ on the compact abelian group $X = X_{R_d^{(p)}/(f)} \subset F_p^{\mathbb{Z}^d}$ in (2.10)–(2.11) is mixing, and let $\mathbf{m} \subset \mathbb{Z}^d$ be a nonzero element such that the restriction $\alpha^{\Gamma_\mathbf{m}}$ of α to the subgroup $\Gamma_\mathbf{m}$ in (3.1) is expansive. Then the homoclinic group $\Delta_{(\alpha,\Gamma_\mathbf{m})}(X)$ is dense in X. Furthermore there exists an open subset $W \subset \mathsf{S}_{d-1}$ such that every nonzero element $\mathbf{n} \in \mathbb{Z}^d$ with $\mathbf{n}^* \in \mathsf{S}_{d-1}$ has the following properties.*

1. *$\Delta_{(\alpha,\Gamma_\mathbf{n})}(X)$ is dense in X;*
2. *$\Delta_{(\alpha,\Gamma_\mathbf{m})}(X) \cap \Delta_{(\alpha,\Gamma_\mathbf{n})}(X) = \{0_X\}$.*

Proof. If the convex hull $\mathcal{C}(f) \subset \mathbb{R}^d$ of the support $\mathcal{S}(f) \subset \mathbb{Z}^d$ of f contains an interior point then Proposition 3.6 is essentially a re-statement of Proposition 3.5.

If $\mathcal{C}(f)$ does not contain an interior point, then we may assume without loss in generality (after multiplying f by a monomial $u^\mathbf{m}$, if necessary) that $\mathcal{S}(f)$ is contained in some subspace $V \subset \mathbb{R}^d$ of dimension $d' < d$, where we assume that d' is minimal (i.e. that there does not exist a $\mathbf{n} \in \mathbb{Z}^d$ such that $\mathcal{S}(u^\mathbf{n} h)$ is contained in a subspace of lower dimension). Since α is mixing, Lemma 2.2 (2) implies that $d' \geq 2$.

Put $\Gamma = V \cap \mathbb{Z}^d \cong \mathbb{Z}^{d'}$ and choose a subgroup $\Gamma' \subset \mathbb{Z}^d$ with $\Gamma \cap \Gamma' = \{\mathbf{0}\}$ and $\Gamma + \Gamma' = \mathbb{Z}^d$. We identify Γ with $\mathbb{Z}^{d'}$, regard f as an element of $R_{d'}^{(p)}$, and apply Proposition 3.5 to the $\mathbb{Z}^{d'}$-action $\alpha_{R_{d'}^{(p)}/(f)}$ on $X_{R_{d'}^{(p)}/(f)}$ to find, for every $\mathbf{m} \in \Gamma$ such that the restriction of $\alpha_{R_{d'}^{(p)}/(f)}$ to the group

$$\Gamma_\mathbf{m} = \{\mathbf{n} \in \Gamma \colon [\mathbf{n}, \mathbf{m}] = 0\}$$

is expansive, a Weyl chamber W_2 of the $\mathbb{Z}^{d'}$-action $\alpha_{R_{d'}^{(p)}/(f)}$ such that, for every nonzero $\mathbf{n} \in \mathbb{Z}^{d'}$ with $\mathbf{n}^* \in W_2$, the restriction of $\alpha_{R_{d'}^{(p)}/(f)}$ to $\Gamma_\mathbf{n}$ is again expansive and the homoclinic groups

$$\Delta_{(\alpha_{R_{d'}^{(p)}/(f)},\Gamma_\mathbf{m})}(X_{R_{d'}^{(p)}/(f)}),\ \ \Delta_{(\alpha_{R_{d'}^{(p)}/(f)},\Gamma_\mathbf{n})}(X_{R_{d'}^{(p)}/(f)})$$

have trivial intersection.

Since the restriction α^Γ of α to Γ is algebraically conjugate to the product action of Γ on $X \cong (X_{R_{d'}^{(p)}/(f)})^{\Gamma'}$, we obtain that the restrictions of α to the groups $\Gamma_\mathbf{m} + \Gamma'$ and $\Gamma_\mathbf{n} + \Gamma'$ are expansive, and that the homoclinic groups $\Delta_{(\alpha,\Gamma_\mathbf{m}+\Gamma')}(X)$ and $\Delta_{(\alpha,\Gamma_\mathbf{n}+\Gamma')}(X)$ have trivial intersection. It is easy to see that this implies the statement of the proposition in the case where $\mathcal{C}(f)$ does not have an interior point (in fact, the open set $W \subset \mathsf{S}_{d-1}$ can again be interpreted as a *Weyl chamber* of α). $\square$

4. Isomorphism rigidity

In this section we prove the following rigidity result for measurable factor maps between algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups.

Theorem 4.1. *Let $d > 1$, and let α and β be mixing algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups X and Y, respectively. Suppose that there exists a subgroup $\Gamma \subset \mathbb{Z}^d$ of of infinite index such that the restriction α^Γ of α to Γ is expansive and has completely positive entropy. Then every measurable factor map $\phi\colon (X,\alpha) \longrightarrow (Y,\beta)$ is λ_X-a.e. equal to an affine map.*

Before turning to the proof of this result we mention a couple of corollaries which generalize the main result in [9] in different directions.

Corollary 4.2. *Let $d > 1$, and let α and β be mixing algebraic $\mathbb{Z}^d$-actions on zero-dimensional compact abelian groups X and Y, respectively. Suppose that there exists a nonzero element $\mathbf{n} \in \mathbb{Z}^d$ such that the automorphism $\alpha^{\mathbf{n}}$ is expansive. Then every measurable factor map $\phi\colon (X,\alpha) \longrightarrow (Y,\beta)$ is λ_X-a.e. equal to an affine map.*

Proof. Since every mixing (= ergodic) group automorphism has completely positive entropy, this is Theorem 4.1 with Γ of rank one. □

Corollary 4.3. *Let $d > 1$, p a rational prime, and $\mathfrak{p},\mathfrak{q} \subset R_d^{(p)}$ nonzero prime ideals such that the $\mathbb{Z}^d$-actions $\alpha = \alpha_{R_d^{(p)}/\mathfrak{p}}$ and $\beta = \alpha_{R_d^{(p)}/\mathfrak{q}}$ on the compact zero dimensional groups $X = X_{R_d^{(p)}/\mathfrak{p}}$ and $Y = X_{R_d^{(p)}/\mathfrak{q}}$ in (2.10)–(2.11) are mixing. Then α and β are measurably conjugate if and only if they are algebraically conjugate, and hence if and only if $\mathfrak{p} = \mathfrak{q}$. Furthermore, every measurable conjugacy $\phi\colon (X,\alpha) \longrightarrow (Y,\beta)$ is λ_X-a.e. equal to an affine map.*

Proof. The existence of a subgroup $\Gamma \subset \mathbb{Z}^d$ of infinite index with the properties required by Theorem 4.1 is proved in [6] (the rank of Γ is the maximal number of algebraically independent elements in the set $\{u^{\mathbf{n}} + \mathfrak{p} \colon \mathbf{n} \in \mathbb{Z}^d\} \subset R_d^{(p)}/\mathfrak{p}$). Let $\phi\colon (X,\alpha) \longrightarrow (Y,\beta)$ be a measurable conjugacy. By Theorem 4.1, there exist $y \in Y$ and a continuous homomorphism $\theta\colon X \longrightarrow Y$ such that $\phi(x) = y + \theta(x)$ for λ_X-a.e. $x \in X$. It is easy to verify that θ is an algebraic conjugacy of (X,α) and (Y,β).

In order to see that algebraic conjugacy implies that $\mathfrak{p} = \mathfrak{q}$ we note that, for every $f \in R_d^{(p)}$, the maps $f(\alpha)$ and $f(\beta)$ in (2.4) are surjective if and only if $f \notin \mathfrak{p}$ (resp. $f \notin \mathfrak{q}$). □

We begin the proof of Theorem 4.1 with a lemma.

Lemma 4.4. *For $i = 1,2,3$, let α_i be a mixing algebraic $\mathbb{Z}^d$-action on a compact abelian group X_i, and let $\phi\colon (X_1 \times X_2, \alpha_1 \times \alpha_2) \longrightarrow (X_3,\alpha_3)$ be a continuous factor map such that $\phi(x_1,x_2) = 0_{X_3}$ whenever $x_1 = 0_{X_1}$ or $x_2 = 0_{X_2}$. Suppose furthermore that there exist subgroups Γ_1,Γ_2 in $\mathbb{Z}^d$ such that the homoclinic groups $\Delta_{(\alpha_i,\Gamma_i)}(X_i)$ are dense in X_i for $i = 1,2$, and that $\Delta_{(\alpha_3,\Gamma_1)}(X_3) \cap \Delta_{(\alpha_3,\Gamma_2)}(X_3) = \{0_{X_3}\}$. Then $\phi(X_1 \times X_2) = \{0_{X_3}\}$.*

Proof. Since ϕ is a continuous factor map,

$$\begin{aligned}\lim_{\substack{\mathbf{m}\to\infty\\ \mathbf{m}\in\Gamma_1}} \alpha_3^{\mathbf{m}}\phi(x_1,x_2) &= \lim_{\substack{\mathbf{m}\to\infty\\ \mathbf{m}\in\Gamma_1}} \phi(\alpha_1^{\mathbf{m}}x_1,\alpha_2^{\mathbf{m}}x_2) = 0_{X_3}\\ &= \lim_{\substack{\mathbf{n}\to\infty\\ \mathbf{n}\in\Gamma_2}} \alpha_3^{\mathbf{n}}\phi(x_1,x_2) = \lim_{\substack{\mathbf{n}\to\infty\\ \mathbf{n}\in\Gamma_2}} \phi(\alpha_1^{\mathbf{n}}x_1,\alpha_2^{\mathbf{n}}x_2)\end{aligned}$$

for every $x_i \in \Delta_{(\alpha_i,\Gamma_i)}(X_i)$, $i=1,2$. Hence

$$\phi(x_1,x_2) \in \Delta_{(\alpha_3,\Gamma_1)}(X_3) \cap \Delta_{(\alpha_3,\Gamma_2)}(X_3) = \{0_{X_3}\}.$$

As $\Delta_{(\alpha_i,\Gamma_i)}(X_i) \subset X_i$ is dense for $i=1,2$ and ϕ is continuous this implies our assertion. □

Proof of Theorem 4.1. We assume without loss in generality that the group $\mathbb{Z}^d/\Gamma$ is torsion-free and that $\Gamma \cong \mathbb{Z}^{d'}$ with $d' < d$. Choose a primitive[3] element $\mathbf{n} \in \mathbb{Z}^d \smallsetminus \Gamma$ and set $\Gamma' = \Gamma + \{k\mathbf{n} \colon k \in \mathbb{Z}\} \cong \mathbb{Z}^{d'+1}$. Since α is mixing, the same is true for $\alpha' = \alpha^{\Gamma'}$, and the expansiveness of α^Γ implies that of $\alpha^{\Gamma'}$. Furthermore, since α^Γ is expansive, the Γ-action α^Γ has finite entropy and hence $\alpha^{\Gamma'}$ has zero entropy. We restrict α and β to Γ' and assume that $d = d'+1$, that α is an expansive and mixing $\mathbb{Z}^d$-action, and that $\Gamma \subset \mathbb{Z}^d$ is a subgroup of rank $d-1$ such that α^Γ is expansive and has completely positive entropy. Since the restriction to subgroups $\Gamma'' \subset \Gamma$ of finite index changes neither expansiveness nor completely positive entropy we shall assume for simplicity that

$$\Gamma = \{\mathbf{n} = (n_1,\dots,n_d) \in \mathbb{Z}^d \colon n_d = 0\} = \mathbb{Z}^{d-1}.$$

As the $\mathbb{Z}^{d-1}$-action α^Γ has finite and completely positive entropy, the same is true for β^Γ, and Lemma 2.2 shows that every prime ideal $\mathfrak{q} \subset R_{d-1}$ associated with the dual module $N' = \widehat{Y}$ of the $\mathbb{Z}^{d-1}$-action β^Γ is of the form $\mathfrak{q} = p(\mathfrak{q})$ for some rational prime $p(\mathfrak{q}) > 1$. The existence of the filtrations described in Lemma 2.3 guarantees that N' is Noetherian as a module over R_{d-1} and hence that β^Γ is expansive. It follows that β is expansive, that the dual module $N = \widehat{Y}$ of the $\mathbb{Z}^d$-action β is Noetherian, and that every prime ideal $\mathfrak{p} \subset R_d$ associated with N is of the form $\mathfrak{p} = (p,f) = pR_d + fR_d$ for some rational prime $p \geq 2$ and some Laurent polynomial $f \in R_d$ whose reduction $f_{/p}$ modulo p is nonzero (otherwise β would have positive entropy by Lemma 2.2).

We apply Lemma 2.3 and choose isomorphic R_d-modules $L \supseteq N \supseteq L'$ with the properties mentioned there. As L and L' are isomorphic, the restrictions to Γ of the $\mathbb{Z}^d$-actions α_L, β, $\beta' = \alpha_{L'}$ all have the same entropy. The inclusion $L' \subset N$ induces a dual algebraic factor map $\psi \colon (Y,\beta) \longrightarrow (X_{L'},\beta')$, and the filtration of $L' \cong L$ described in Lemma 2.3 induces a filtration $Y_k = X_{L'} \supset \dots Y_0 = \{0\}$ of Y by β'-invariant subgroups such that each $(Y_j/Y_{j-1}, \beta_{Y_j/Y_{j-1}})$ is algebraically conjugate to $(X_{R_d^{(p)}/(f)}, \alpha_{R_d^{(p)}/(f)})$ for some rational prime $p \geq 2$ and some nonzero element $f \in R_d^{(p)}$ such that $\alpha_{R_d^{(p)}/(f)}$ is mixing. For every $j = 0,\dots,k$ we denote by $\pi_j \colon Y_k \longrightarrow Y_k/Y_j$ the quotient map.

[3]A nonzero element $\mathbf{n} = (n_1,\dots,n_d) \in \mathbb{Z}^d$ is *primitive* if $\gcd(n_1,\dots,n_d) = 1$.

Suppose that $\phi\colon (X,\alpha) \longrightarrow (Y,\beta)$ is a measurable factor map. Lemma 2.4 allows us to assume that ϕ is continuous, and we set $\phi_j = \pi_j \circ \psi \circ \phi\colon X \longrightarrow Y_k/Y_j$ for $j = 0,\dots,k-1$.

We set $j = k-1$, $Y'' = Y_k/Y_{k-1}$, and write $\beta'' = \beta'_{Y''}$ for the $\mathbb{Z}^d$-action induced by β' on Y''. Then the restriction β''^{Γ} of β'' to Γ is expansive, and Proposition 3.6 and Lemma 3.3 (1) allow us to find a nonzero element $\mathbf{n} \in \mathbb{Z}^d$ such that the restrictions $\alpha^{\Gamma_{\mathbf{n}}}$ and $\beta''^{\Gamma_{\mathbf{n}}}$ of α and β'' to $\Gamma_{\mathbf{n}}$ are expansive, the homoclinic group $\Delta_{(\alpha,\Gamma_{\mathbf{n}})}(X)$ is dense[4] in X, and the homoclinic groups $\Delta_{(\beta'',\Gamma)}(Y'')$ and $\Delta_{(\beta'',\Gamma_{\mathbf{n}})}(Y'')$ have trivial intersections. We write $\Phi\colon X \times X \longrightarrow Y''$ for the map

$$\Phi(x_1,x_2) = \phi_{k-1}(x_1+x_2) - \phi_{k-1}(x_1) - \phi_{k-1}(x_2) + \phi_{k-1}(0_X)$$

and obtain from Lemma 4.4 that $\Phi \equiv 0_{Y''}$ or, equivalently, that

$$\psi\circ\phi(x_1+x_2) - \psi\circ\phi(x_1) - \psi\circ\phi(x_2) + \psi\circ\phi(0_X) \in Y_{k-1}$$

for every $x_1,x_2 \in X$. By repeating this argument we obtain inductively that

$$\psi\circ\phi(x_1+x_2) - \psi\circ\phi(x_1) - \psi\circ\phi(x_2) + \psi\circ\phi(0_X) \in Y_j$$

for every $j = k-1,\dots,0$, which implies that

$$\phi(x_1+x_2) - \phi(x_1) - \phi(x_2) + \phi(0_X) \in \ker(\psi)$$

for every $x_1,x_2 \in X$. From Lemma 2.3 we know that the Γ-action induced by β on $Y_k = X_{L'}$ has the same entropy as β^{Γ}, and hence that the restriction $\beta^{\Gamma}_{\ker(\psi)}$ of β^{Γ} to $\ker(\psi)$ has zero entropy. Since the map

$$(x_1,x_2) \mapsto \phi(x_1+x_2) - \phi(x_1) - \phi(x_2) + \phi(0_X)$$

is a measurable factor map from $(X \times X, \alpha^{\Gamma} \times \alpha^{\Gamma})$ to $(\ker(\psi), \beta^{\Gamma}_{\ker(\psi)})$, and since the first of these Γ-actions has completely positive entropy by assumption and the second one zero entropy, it follows that

$$\phi(x_1+x_2) - \phi(x_1) - \phi(x_2) + \phi(0_X) = 0_Y$$

for every $x_1,x_2 \in X$, i.e. that ϕ is affine. □

The following examples show that Theorem 4.1 and Corollary 4.3 do not hold if any of the assumptions are dropped. Our first example implies that the surjectivity of ϕ is necessary in Corollary 4.3 (and hence in Theorem 4.1).

Example 4.5. Let $d = 3$, $p = 2$, and consider the polynomials $f_1, f_2 \in R_3^{(2)}$ defined by $f_1 = 1 + u_1 + u_2, f_2 = 1 + u_1 + u_2 + u_1^2 + u_1u_2 + u_2^2 + u_3$. Let $\mathfrak{p} = (f_1,f_2) \subset R_3^{(2)}$ denote the ideal generated by f_1 and f_2, and let $\mathfrak{q} = (f_2) \subset R_3^{(2)}$ be the principal ideal generated by f_2. It is easy to see that $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals. We define the shift-actions $\alpha_1 = \alpha_{R_3^{(2)}/\mathfrak{p}}$ and $\alpha_2 = \alpha_{R_3^{(2)}/\mathfrak{q}}$ on $X_1 = X_{R_3^{(2)}/\mathfrak{p}} \subset F_2^{\mathbb{Z}^3}$ and $X_2 = X_{R_3^{(2)}/\mathfrak{q}} \subset F_2^{\mathbb{Z}^3}$, respectively, by (2.10)–(2.11). From Lemma 2.2 it is clear that α_1 and α_2 are mixing and have zero entropy.

[4]The density of the homoclinic group $\Delta_{(\alpha,\Gamma)}(X)$ in X is clear from Proposition 3.2, since α^{Γ} is expansive and has completely positive entropy.

We write $\star$ for the component-wise multiplication $(z \star z')_{\mathbf{n}} = z_{\mathbf{n}} z'_{\mathbf{n}}$ in $F_2^{\mathbb{Z}^3}$ and observe that

$$\sigma^{\mathbf{n}}(z \star z') = (\sigma^{\mathbf{n}} z) \star (\sigma^{\mathbf{n}} z')$$

for every $z, z' \in F_2^{\mathbb{Z}^3}$ and $\mathbf{n} \in \mathbb{Z}^3$ (cf. (2.8)). We claim that

$$x \star x' \in X_2 \text{ for every } x, x' \in X_1. \tag{4.1}$$

In order to verify this we define subsets $S_i \subset \mathbb{Z}^3$, $i = 0, \dots, 3$, by

$$\begin{aligned} S_0 &= \mathcal{S}(f_2), \quad S_1 = \mathcal{S}(f_1), \\ S_2 &= \{(1,0,0),(1,1,0),(2,1,0)\} = \mathcal{S}(u_1 f_1), \\ S_3 &= \{(0,1,0),(0,2,0),(1,1,0)\} = \mathcal{S}(u_2 f_1), \end{aligned}$$

and consider the set Z of all $z \in F_2^{S_0}$ with $\sum_{\mathbf{n} \in S_i} z_{\mathbf{n}} = 0$ for $i = 0, \dots, 3$. A calculation shows that, for every $z, z' \in Z$, the component-wise product $w = z \star z' \in F_2^{S_0}$ satisfies that $\sum_{\mathbf{n} \in S_0} w_{\mathbf{n}} = 0$. This implies (4.1).

Take a non-zero $\mathbf{m} \in \mathbb{Z}^3$ such that $\alpha_1^{\mathbf{m}} z = z$ for some non-zero $z \in X_1$ and define $\phi \colon X_1 \longrightarrow X_2$ by $\phi(x) = x \star \alpha_1^{\mathbf{m}} x$. Clearly ϕ is a $\mathbb{Z}^3$-equivariant map from (X_1, α_1) to (X_2, α_2). We choose $y \in X_1$ such that $z \star (\alpha_1^{\mathbf{m}} y - y) \neq 0_{X_2}$. Since $\phi(0_{X_1}) = 0_{X_2}$ and $\phi(z+y) - \phi(z) - \phi(y) = z \star (\alpha_1^{\mathbf{m}} y - y) \neq 0_{X_2}$, the map ϕ is not affine.

In the next example we construct a non-affine factor map $\psi \colon (X, \alpha) \longrightarrow (X', \alpha')$ between expansive and mixing zero-entropy algebraic $\mathbb{Z}^3$-actions, where α' has an expansive $\mathbb{Z}^2$-sub-action with completely positive entropy.

Example 4.6. We use the same notation as in the previous example. Let $\mathfrak{r} = \mathfrak{p}\mathfrak{q} = (f_1 f_2, f_2^2) \subset R_3^{(2)}$ be the ideal generated by $f_1 f_2$ and f_2^2 and let β denote the algebraic $\mathbb{Z}^3$-action $\alpha_{R_3^{(2)}/\mathfrak{r}}$ on $Y = X_{R_3^{(2)}/\mathfrak{r}} \subset F_2^{\mathbb{Z}^3}$. From Lemma 2.2 it follows that the action (Y, β) is mixing and has zero entropy. We define continuous group homomorphisms $\theta_1 \colon Y \longrightarrow X_1$ and $\theta_2 \colon Y \longrightarrow X_2$ by

$$\theta_1(y) = f_2(\sigma)(y), \qquad \theta_2(y) = f_1(\sigma)(y).$$

It is easy to verify that for $i = 1, 2$, $\theta_i \colon (Y, \beta) \longrightarrow (X_i, \alpha_i)$ is an algebraic factor map. Let $\psi \colon (Y, \beta) \longrightarrow (X_2, \alpha_2)$ be the $\mathbb{Z}^3$-equivariant continuous map defined by

$$\psi(x) = \theta_2(x) + \phi \circ \theta_1(x),$$

where $\phi \colon X_1 \longrightarrow X_2$ is as in the previous example. Since θ_1 is a surjective homomorphism and ϕ is non-affine, it follows that $\phi \circ \theta_1$ is non-affine, i.e. that ψ is a non-affine map. It is easy to see that the restriction of θ_2 to X_2 is a surjective map from X_2 to itself. Since $\theta_1(x) = 0$ for all $x \in X_2 \subset Y$, this shows that ψ is a non-affine factor map from (Y, β) to (X_2, α_2) (in fact, it can be shown that $\tau \circ \psi$ is non-affine for every surjective α_2-equivariant group homomorphism $\tau \colon X_2 \longrightarrow X_2$).

Our final example shows that there exist measurably conjugate expansive and mixing zero-entropy algebraic $\mathbb{Z}^3$-actions on non-isomorphic compact zero-dimensional abelian groups.

Example 4.7. Let (X_1, α_1) and (X_2, α_2) be as in Example 4.5, and let (X, α) denote the product action $(X_1, \alpha_1) \times (X_2, \alpha_2)$. Following [2] we define a zero-dimensional compact abelian group Y and an algebraic $\mathbb{Z}^3$-action β on Y by setting $Y = X_1 \times X_2$ with composition

$$(x, y) \odot (x', y') = (x + x', x \star x' + y + y')$$

for every $(x, x'), (y, y') \in Y$, and by letting

$$\beta^{\mathbf{n}}(x, y) = (\alpha_1^{\mathbf{n}} x, \alpha_2^{\mathbf{n}} y)$$

for every $(x, y) \in Y$ and $\mathbf{n} \in \mathbb{Z}^3$. The 'identity' map $\phi\colon X \longrightarrow Y$, defined by

$$\phi(x, y) = (x, y)$$

for every $(x, y) \in X$, is obviously a topological conjugacy of (X, α) and (Y, β) with $\lambda_X \phi^{-1} = \lambda_Y$ (by Fubini's theorem). However, ϕ is not a group isomorphism. In fact, the groups X and Y are not isomorphic: since X is a subgroup $(F_2 \oplus F_2)^{\mathbb{Z}^3}$, every element in X has order 2, whereas $(x, 0_{X_2}) \in Y$ and $(x, 0_{X_2}) \odot (x, 0_{X_2}) = (0_{X_2}, x) \neq 0_Y$ for every nonzero $x \in X_1$.

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