REMARKS ON LIVŠIC' THEORY FOR NONABELIAN COCYCLES

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Abstract. Let (X, ϕ) be a hyperbolic dynamical system and (G, δ) a Polish group. Motivated by [4] and [5] we study conditions under which two Hölder maps $f, g \colon X \longrightarrow G$ are Hölder cohomologous.

In the context of [4] we show that, if f and g are measurably cohomologous and the distortion of the metric δ by the cocycles defined by f and g is *bounded* in an appropriate sense, then f and g are Hölder cohomologous.

Two further results extend the main theorems in [5]. Under the hypothesis of bounded distortion we show that, if f and g give *equal* weight to all periodic points of ϕ, then f and g are Hölder cohomologous. If the metric δ is bi-invariant, and if the skew-product ϕ_f defined by f is topologically transitive, then *conjugacy* of weights implies that g is Hölder conjugate to $\alpha \cdot f$ for some isometric automorphism α of G. The weaker condition that g-weights of periodic points are close to the identity whenever their f-weights are close to the identity implies that g is continuously cohomologous to a homomorphic image of f.

1. Introduction

This note is a companion to the paper [5] on the characterisation of cohomology of Hölder cocycles of hyperbolic dynamical systems in terms of weights of periodic orbits and arose out of a series of discussions with W. Parry. Although all results in this paper hold for general hyperbolic dynamical systems we restrict our discussion to shifts of finite type for most of the paper and explain in the last section how to deal with arbitrary hyperbolic systems.

Let A be a finite set, $X \subset A^{\mathbb{Z}}$ a mixing shift of finite type and (G, δ) a complete metric group with identity element 1_G. Two Hölder maps $f, g \colon X \longrightarrow G$ are *continuously* or *Hölder cohomologous* if there exists a continuous or Hölder map $b \colon X \longrightarrow G$ satisfying the equation

$$f(x) = b(\sigma x)^{-1} g(x) b(x) \tag{1.1}$$

for every $x \in X$, where σ is the shift on X (cf. Section 2). The map f is a *coboundary* if it is cohomologous to the constant function $g = 1_G$, i.e. if there exists a (continuous or Hölder) map $b \colon X \longrightarrow G$ with

$$f(x) = b(\sigma x)^{-1} b(x) \tag{1.2}$$

for every $x \in X$.

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In many problems in dynamics it is important to decide when a given Hölder map $f\colon X \longrightarrow G$ is a coboundary, or when two such maps $f,g\colon X \longrightarrow G$ are cohomologous. The first — and still most fundamental — results in this direction are due to A. Livšic: if the metric δ is *bi-invariant* (i.e. invariant both under left and right translations), then the following statements are true (cf. [2]–[3]).

(1) f is a coboundary if and only if the f-*weight*

$$f(\sigma^{p-1}x)\cdots f(x)$$

of every periodic point $x \in X$ with period p is equal to 1_G;

(2) If b is a measurable map satisfying (1.2) *a.e.* with respect to a suitable shift-invariant measure μ, then b is μ-*a.e.* equal to a Hölder map.

If G is abelian these results also gives a necessary and sufficient condition for cohomology of two Hölder maps $f,g\colon X \longrightarrow G$:

(3) f and g are cohomologous if and only if their weights of periodic points coincide;

(4) Any measurable solution b of (1.1) must be Hölder.

In the absence of a bi-invariant metric M. Nicol and M. Pollicott have proved (2) for bounded Borel maps $b\colon X \longrightarrow GL(2,\mathbb{R})$ satisfying (1.2) μ-*a.e.* for some appropriate shift-invariant probability measure μ on X (cf. [4]). In Theorem 3.1 we extend their result by dropping the requirement that $g = 1_G$: if (G,δ) is a Lipschitz metric group (a notion related to the condition of 'goodness' in [3]) and $f,g\colon X \longrightarrow G$ are Hölder maps defining cocycles with bounded distortion (cf. Section 2), then any measurable solution b of (1.1) must be Hölder. For Hölder maps with values in a compact group G (where bounded distortion holds automatically) this result already appears in [6].

Concerning (3) and (4) W. Parry proved that, if G is compact and $f,g\colon X \longrightarrow G$ are Hölder maps which give equal weights to all periodic points, then f and g are Hölder cohomologous (Theorem 7.1 in [5]). If the f- and g-weights of periodic points are not equal, but only conjugate, then there exists an isometric automorphism α of G such that $\alpha \cdot f$ and g are cohomologous (Theorem 6.5).

Theorem 4.1 in this paper yields the first of these results under the weaker hypotheses that (G,δ) is a Lipschitz metric group and the cocycles defined by f and g have bounded distortion. If δ is bi-invariant and the skew-product extension σ_f of σ is topologically transitive, then Theorem 5.1 (2) yields the second result without the hypothesis that G is compact.

Perhaps surprisingly, a much weaker hypothesis than conjugacy of the f- and g-weights of periodic points still implies that g is cohomologous to a homomorphic image of f: it suffices to assume that the g-weights of periodic points are close to 1_G whenever the same is true for their f-weights (Theorem 5.1 (1)).

The final section describes the changes required for proving the same results for general hyperbolic dynamical systems (Theorems 6.2 and 6.3).

The ideas involved in the proofs of these results are closely related to those in [5], but the approach is somewhat different: in [5] the author initially assumes the existence of a fixed point x of σ with $f(x) = g(x) = 1_G$ and constructs a kind of *loop group* from the set of weights of points $y \in X$ which are homoclinic to x. The conjugacy of the f- and g-weights of periodic points leads to a homomorphism of these loop groups, from which the automorphism α is constructed. Although this construction is interesting, its hypotheses about the fixed point x are too restrictive, and the author includes a recipe for their removal: replace the fixed point by a suitable periodic point x' whose f- and g-weights tend to 1_G along some sequence of periods (this is one of the places where the compactness of G comes in), and then use an asymptotic version of the same construction by moving along this sequence of periods.

Here we avoid dependence on any particular fixed or periodic point by working on the entire homoclinic equivalence relation of the hyperbolic map.

As pointed out in Remark 2.2, the hypothesis of bounded distortion of a map f taking values in a Lipschitz metric group (G,δ) is extremely strong and may conceivably imply measurable cohomology of f to a map f' taking values in a subgroup $H\subset G$ which has a bi-invariant metric. If one could prove that f is actually *Hölder* cohomologous to such an f' then the Theorems 3.1, 4.1, 6.1 and 6.2 would become elementary consequences of known results. However, the Theorems 5.1 and 6.3 appear to be new even in the case where G is abelian or compact (in which case every G-valued map automatically has bounded distortion).

2. Cocycles with bounded distortion

Let G be a Polish group with identity element 1_G. We call a metric δ on G *Lipschitz* and (G,δ) a *Lipschitz metric group* if (G,δ) is complete and if there exists, for every $g\in G$, a constant a_g with

$$\begin{aligned} a_g^{-1}\delta(h,h') &\le \min\left(\delta(gh,gh'),\delta(hg,h'g)\right)\\ &\le \max\left(\delta(gh,gh'),\delta(hg,h'g)\right)\le a_g\delta(h,h') \end{aligned} \tag{2.1}$$

for all $h,h'\in G$. A Polish group G is *Lipschitz* if there exists a Lipschitz metric δ on G.

If (G,δ) is a Lipschitz metric group we denote by

$$D_\delta(g)=\inf\{a_g\ge 1 : a_g \text{ satisfies (2.1) for every } h,h'\in G\} \tag{2.2}$$

the *distortion* of δ by an element $g\in G$.

Any Polish group which is abelian, discrete or compact has a bi-invariant complete metric δ and is thus Lipschitz. For every $n\ge 2$, the group $G=GL(n,\mathbb{C})$ has the Lipschitz metric

$$\delta(A,B)=\|A-B\|+\|A^{-1}-B^{-1}\|, \tag{2.3}$$

where $\|A\|$ stands for the operator norm of $A\in G$, regarded as a linear map on the complex Hilbert space $\mathbb{C}^n$ with its usual Euclidean inner product. Since any closed subgroup of a Lipschitz group is Lipschitz it follows very easily that every finite-dimensional Lie group is Lipschitz.

Let T be an ergodic measure-preserving automorphism of a probability space $(X,\mathcal{S},\mu)$, G a Polish group and $f\colon X\longrightarrow G$ a Borel map. We set

$$\mathbf{f}(n,x)=\begin{cases} f(T^{n-1}x)\cdots f(x) & \text{if } n>1,\\ 1_G & \text{if } n=0,\\ f(-n,T^nx)^{-1} & \text{if } n<0. \end{cases} \tag{2.4}$$

and note that the map $\mathbf{f}\colon \mathbb{Z}\times X \longrightarrow G$ satisfies the *cocycle equation*

$$\mathbf{f}(m, T^n x)\mathbf{f}(n, x) = \mathbf{f}(m+n, x) \tag{2.5}$$

for every $m, n \in \mathbb{Z}$ and $x \in X$. Two Borel maps $f, g\colon X \longrightarrow G$ are μ-*cohomologous* if there exists a Borel map $b\colon X \longrightarrow G$ such that

$$f(x) = b(Tx)^{-1} g(x) b(x) \tag{2.6}$$

for μ-a.e. $x \in X$. The map b in (2.6) is called the *transfer function* of (f, g). A Borel map $f\colon X \longrightarrow G$ is a *coboundary* if it is cohomologous to the constant function $g = 1_G$.

If (G, δ) is Lipschitz we say that the cocycle $\mathbf{f}$ has *bounded distortion* if there exists a constant $c \geq 1$ with $D_\delta(\mathbf{f}(n, x)) \leq c$ for μ-a.e. $x \in X$ and every $n \in \mathbb{Z}$.

If a cocycle $\mathbf{f}$ has bounded distortion then it might be tempting to ask whether there exist a subgroup $H \subset G$ and a metric δ' on G such that f is μ-cohomologous to a Borel map $g\colon X \longrightarrow H$ and δ' is bi-invariant under H. The following proposition shows that this answer to this question is positive for $G = GL(n, \mathbb{C})$.

Proposition 2.1. *Let $f\colon X \longrightarrow G = GL(n, \mathbb{C})$ be a Borel map such that $\mathbf{f}\colon \mathbb{Z}\times X \longrightarrow G$ has bounded distortion with respect to the metric* (2.3). *Then f is μ-cohomologous to a Borel map $g\colon X \longrightarrow \mathrm{U}(n)$.*

Proof. The metric δ in (2.3) has the property that the set

$$C(a) = \{g \in G : D_\delta(g) \leq a\}$$

is compact for every $a \geq 0$, where $D_\delta(g)$ is defined in (2.2). In particular, if $\mathbf{f}$ has bounded distortion, then $\mathbf{f}$ is bounded in the sense of Definition 4.6 in [8], and Theorem 4.7 in [8] yields that f is μ-cohomologous to a map taking values in a compact subgroup $K \subset G$. Since $\mathrm{U}(n)$ is a maximal compact subgroup of G we can conjugate K into $\mathrm{U}(n)$, which implies that f is μ-cohomologous to a Borel map taking values in $\mathrm{U}(n)$. □

Remark 2.2. Suppose that X is a mixing (i.e. irreducible and aperiodic) shift of finite type, $T = \sigma$ the shift map on X, and μ a fully supported shift-invariant and ergodic measure on X (for the necessary definitions we refer to Section 3). If $f\colon X \longrightarrow G = GL(n, \mathbb{C})$ is Hölder continuous (cf. Section 3) and $\mathbf{f}\colon \mathbb{Z}\times X \longrightarrow G$ has bounded distortion, can one choose the map $b\colon X \longrightarrow G$ in Proposition 2.1 to be Hölder? If this is possible then some of the results in this paper become redundant for Hölder functions $f\colon X \longrightarrow GL(n, \mathbb{C})$.

3. Hölder cocycles with bounded distortion

Let A be a finite set and $X \subset A^{\mathbb{Z}}$ a mixing shift of finite type. We write $\sigma\colon X \longrightarrow X$ for the shift on X, defined by

$$(\sigma x)_n = x_{n+1} \tag{3.1}$$

for every $x = (x_n) \in X$, and denote by

$$\Delta_X = \{(x, y) \in X \times X : x_n \neq y_n \text{ for only finitely many } n \in \mathbb{Z}\} \tag{3.2}$$

the *homoclinic* or *Gibbs equivalence relation* of X. The equivalence class of a point $x \in X$ is given by

$$\Delta_X(x) = \{y \in X : (x,y) \in \Delta_X\}. \tag{3.3}$$

Sometimes it will be convenient to choose a countable group Γ of homeomorphisms of X with the following properties:

(a) For every $V \in \Gamma$ there exists an integer $M \geq 0$ such that $x_m = (Vx)_m$ whenever $x = (x_n) \in X$ and $|m| \geq M$,
(b) $\Delta_X = \{(x, Vx) : x \in X, V \in \Gamma\}$.

If (G, δ) is a complete separable metric group a map $f \colon X \longrightarrow G$ is *Hölder* if there exist constants $c > 0, 0 < \gamma < 1$, with

$$\omega_n = \sup_{\{(x,y)\in X\times X \colon x_k = y_k \ \text{ for every } \ k=-n,\dots,n\}} \delta(f(x), f(y)) \leq c\gamma^n \tag{3.4}$$

for every $n \geq 0$. The infimum of the possible constants γ in (3.4) is denoted by γ_f and is called the *Hölder constant* of f.

If $\theta \colon X \longrightarrow \mathbb{R}$ is a Hölder function (with respect to the usual metric on $\mathbb{R}$), then there exists a unique probability measure μ_θ on the sigma-algebra $\mathcal{B}_X$ of Borel sets in X which is quasi-invariant under Γ and satisfies that, for every $V \in \Gamma$,

$$\log \frac{d\mu_\theta V}{d\mu_\theta}(x) = \sum_{k=-\infty}^{\infty} (\theta(\sigma^k Vx) - \theta(\sigma^k x)) \ \text{ for } \ \mu_\theta\text{-a.e. } x \in X. \tag{3.5}$$

The measure μ_θ is shift-invariant and is called the *Gibbs measure* of θ. We are ready to state the main result of this section.

Theorem 3.1. *Let A be a finite set, $X \subset A^{\mathbb{Z}}$ a mixing shift of finite type, $\theta \colon X \longrightarrow \mathbb{R}$ a Hölder function and μ_θ the Gibbs measure of θ.*

Suppose furthermore that (G, δ) is a Lipschitz metric group and $f, g \colon X \longrightarrow G$ are Hölder maps such that the corresponding cocycles $\mathbf{f}, \mathbf{g} \colon \mathbb{Z} \times X \longrightarrow G$ in (2.4) *have bounded distortion. If f and g are μ_θ-cohomologous then they are Hölder cohomologous.*

The remainder of this section is devoted to the proof of Theorem 3.1. For every Hölder map $f \colon X \longrightarrow G$ with $\mathbf{f}$ of bounded distortion we define maps $a_f^{\pm} \colon \Delta_X \longrightarrow G$ by setting, for every $(x,y) \in \Delta_X$,

$$\begin{aligned} a_f^+(x,y) &= \lim_{n\to\infty} \mathbf{f}(n,x)^{-1}\mathbf{f}(n,y), \\ a_f^-(x,y) &= \lim_{n\to\infty} \mathbf{f}(-n,x)^{-1}\mathbf{f}(-n,y), \end{aligned} \tag{3.6}$$

for every $(x,y) \in \Delta_X$. The following lemma justifies these definitions.

Lemma 3.2. *The maps $a_f^{\pm} \colon \Delta_X \longrightarrow G$ in* (3.6) *are well-defined and satisfy the equations*

$$\begin{aligned} a_f^+(x,y)a_f^+(y,z) &= a_f^+(x,z), \\ a_f^-(x,y)a_f^-(y,z) &= a_f^-(x,z) \end{aligned} \tag{3.7}$$

for all $(x,y), (x,z) \in \Delta_X$. Furthermore, if γ_f is the Hölder constant of f and $\gamma_f < \gamma < 1$, then there exists a constant $c' > 0$ with the following properties.

(1) *If* $(x,y)\in\Delta_X$ *satisfies that* $x_n=y_n$ *for all* $n\ge N\ge 0$, *then*

$$\delta(a_f^+(x,y),\mathbf{f}(M+N,x)^{-1}\mathbf{f}(M+N,y))\le c'\gamma^M$$

for every $M\ge 0$;

(2) *If* $(x,y),(x',y')\in\Delta_X$ *and* $M,N\ge 0$ *satisfy that* $x_n=y_n$, $x'_n=y'_n$ *for all* $n\ge N$ *and* $x_n=x'_n$, $y_n=y'_n$ *for* $n=-M,\dots,2M+N$, *then*

$$\delta(a_f^+(x,y),a_f^+(x',y'))\le c'\gamma^M;$$

(3) *If* $(x,y)\in\Delta_X$ *satisfy that* $x_n=y_n$ *for all* $n\le -N-1<0$, *then*

$$\delta(a_f^-(x,y),\mathbf{f}(-M-N,x)^{-1}\mathbf{f}(-M-N,y))\le c'\gamma^M$$

for every $M\ge 0$;

(4) *If* $(x,y),(x',y')\in\Delta_X$ *and* $N\ge 0$ *satisfy that* $x_n=y_n$, $x'_n=y'_n$ *for all* $n\le -N-1$, *and* $x_n=x'_n$, $y_n=y'_n$ *for* $n=-2M-N-1,\dots,M-1$, *then*

$$\delta(a_f^-(x,y),a_f^-(x',y'))\le c'\gamma^M.$$

Proof. Let $a\ge 1$ be chosen so that

$$a^{-1}\delta(h,h')\le\delta(\mathbf{f}(m,x)h\mathbf{f}(n,y),\mathbf{f}(m,x)h'\mathbf{f}(n,y))\le a\delta(h,h')$$

for every $m,n\in\mathbb{Z}$, $x,y\in X$ and $h,h'\in G$.

Suppose that $(x,y)\in\Delta_X$ and $N\ge 0$ satisfy that $x_n=y_n$ whenever $n\ge N$. If $N<l<l'<\infty$ then

$$\begin{aligned}
&\delta(\mathbf{f}(l,x)^{-1}\mathbf{f}(l,y),\mathbf{f}(l',x)^{-1}\mathbf{f}(l',y))\\
&\quad=\delta(\mathbf{f}(l,x)^{-1}\mathbf{f}(l,y),\mathbf{f}(l,x)^{-1}\mathbf{f}(l'-l,\sigma^l x)^{-1}\mathbf{f}(l'-l,\sigma^l y)\mathbf{f}(l,y))\\
&\quad\le a\delta(1_G,\mathbf{f}(l'-l,\sigma^l x)^{-1}\mathbf{f}(l'-l,\sigma^l y))\\
&\quad\le a\sum_{j=l}^{l'-1}\delta(f(\sigma^l x)^{-1}\cdots f(\sigma^{l'-1}(x))^{-1}\cdot\\
&\qquad\qquad\cdot f(\sigma^{l'-1}(x))\cdots f(\sigma^j x)f(\sigma^{j-1}y)\cdots f(\sigma^l y),\\
&\qquad\quad f(\sigma^l x)^{-1}\cdots f(\sigma^{l'-1}(x))^{-1}\cdot\\
&\qquad\qquad\cdot f(\sigma^{l'-1}(x))\cdots f(\sigma^{j+1}x)f(\sigma^j y)\cdots f(\sigma^l y))\\
&\quad\le a^2\sum_{j=l}^{l'-1}\delta(f(\sigma^j x),f(\sigma^j y))\le ca^2\sum_{j=l}^{l'-1}\gamma^{j-N}\\
&\quad\le ca^2|\log\gamma|^{-1}\gamma^{l-N},
\end{aligned}\tag{3.8}$$

where c is chosen as in (3.4). For every $(x,y)\in\Delta_X$, $(\mathbf{f}(n,x)^{-1}\mathbf{f}(n,y),\ n\ge 1)$ is thus a Cauchy sequence, which implies that a_f^+ is well-defined. Similarly one proves that a_f^- is well defined. The equations (3.7) are immediate consequences of (3.6).

The inequality (1) follows from (3.8) with $c'=ca^2|\log\gamma|^{-1}$. In order to prove (2) we assume that $(x,y),(x',y')\in\Delta_X$ satisfy the hypotheses stated there and obtain that

$$
\begin{aligned}
&\delta(\mathbf{f}(M+N,y),\mathbf{f}(M+N,y'))\\
&\quad\le \sum_{j=0}^{M+N-1} \delta(f(\sigma^{M+N-1}y)\cdots f(\sigma^j y)f(\sigma^{j-1}y')\cdots f(y'),\\
&\qquad\qquad\qquad f(\sigma^{M+N-1}y)\cdots f(\sigma^{j+1}y)f(\sigma^j y')\cdots f(y'))\\
&\quad\le a\sum_{j=0}^{M+N-1} \delta(f(\sigma^j y), f(\sigma^j y'))\\
&\quad\le 2ca\sum_{j=0}^{M+N-1} \gamma^{M+j} \le 2ca|\log\gamma|^{-1}\gamma^M.
\end{aligned}
\tag{3.9}
$$

Similarly we obtain that

$$\delta(\mathbf{f}(M+N,x)^{-1},\mathbf{f}(M+N,x')^{-1}) \le 2ca|\log\gamma|^{-1}\gamma^M.$$

By combining this with (1) we obtain that

$$\delta(a_f^+(x,y),a_f^+(x',y')) \le 6ca^2|\log\gamma|^{-1}\gamma^M$$

for every $M \ge 0$. The inequalities (3) and (4) are proved similarly. □

Lemma 3.3. *Suppose that $f,g\colon X \longrightarrow G$ are cohomologous Hölder maps such that $\mathbf{f}$ and $\mathbf{g}$ have bounded distortion, where $\mathbf{g}\colon \mathbb{Z}\times X \longrightarrow G$ is defined by* (2.4) *with g replacing f. If $b\colon X \longrightarrow G$ is a Borel map satisfying* (1.1) *for μ_θ-a.e. $x \in X$, then there exists a Borel set $X' \subset X$ with $\mu_\theta(X') = 1$ such that*

$$
\begin{aligned}
a_f^+(x,y) &= b(x)^{-1}a_g^+(x,y)b(y),\\
a_f^-(x,y) &= b(x)^{-1}a_g^-(x,y)b(y)
\end{aligned}
\tag{3.10}
$$

for every $(x,y) \in \Delta_X \cap (X'\times X')$.

Proof. Let $N \subset X$ be a μ_θ-null set such that (1.1) holds for every $x \in X' = X \smallsetminus N$. We set

$$a = \sup_{n\in\mathbb{Z},\, x\in X} \max\,(D_\delta(\mathbf{f}(n,x))^2, D_\delta(\mathbf{g}(n,x))^2),$$

fix $\varepsilon > 0$ and find a countable Borel partition $(B_k,\ k \ge 1)$ of X such that $\delta(b(x),b(y)) < \varepsilon/a$ whenever $k \ge 1$ and $x,y \in B_k$.

Let Γ be the countable group of homeomorphisms of X satisfying the conditions (a)–(b) above. For every $V \in \Gamma$ and every cylinder set $A \subset X$, $\sigma^n A = V\sigma^n A$ for all $n \in \mathbb{Z}$ with $|n|$ sufficiently large. By approximating an arbitrary Borel set by finite unions of cylinder sets we obtain that

$$\lim_{|n|\to\infty} \mu_\theta(\sigma^n B \triangle V\sigma^n B) = 0$$

for every Borel set $B \subset X$. In particular, if $k \ge 1$ and $V \in \Gamma$, then the sets

$$\{n \ge 1 : \sigma^n x \in B_k,\ \sigma^n Vx \in B_k\},\ \ \{n \ge 1 : \sigma^{-n} x \in B_k,\ \sigma^{-n} Vx \in B_k\}$$

are infinite for μ_θ-a.e. $x \in B_k$. Since

$$\mathbf{f}(n,Vx)^{-1}\mathbf{f}(n,x) = b(Vx)^{-1}\mathbf{g}(n,Vx)^{-1}b(\sigma^n Vx)b(\sigma^n x)^{-1}\mathbf{g}(n,x)b(x)$$

whenever $n \ge 1$ and $x, Vx \in X'$, we conclude that, for μ_θ-a.e. $x \in B_k$,

$$\delta(b(Vx)^{-1}\mathbf{g}(n,Vx)^{-1}\mathbf{g}(n,x)b(x),\mathbf{f}(n,Vx)^{-1}\mathbf{f}(n,x))$$

$$\leq a\delta(\mathbf{f}(n,Vx)b(Vx)^{-1}\mathbf{g}(n,Vx)^{-1}\mathbf{g}(n,x)b(x)\mathbf{f}(n,x)^{-1},1_G)$$
$$= a\delta(b(\sigma^n Vx)^{-1}b(\sigma^n x),1_G) \leq \varepsilon$$

for infinitely many $n > 0$. By letting k vary we obtain that

$$\delta(a_f^+(Vx,x), b(Vx)^{-1}a_g^+(Vx,x)b(x)) \leq \varepsilon$$

for μ_θ-a.e. $x \in X$. As ε is arbitrary, this proves that, for every $V \in \Gamma$, $a_f^+(Vx,x) = b(Vx)^{-1}a_g^+(Vx,x)b(x)$ for μ_θ-a.e. $x \in X$, and by letting V vary we have verified the first equation in (3.10) for every $(x,y) \in \Delta_X$. The proof of the second equation is analogous. □

Lemma 3.4. *Suppose that $f, g\colon X \longrightarrow G$ are cohomologous Hölder maps such that $\mathbf{f}$ and $\mathbf{g}$ have bounded distortion, and that $b\colon X \longrightarrow G$ is a Hölder map satisfying* (3.10) *for every $(x,y) \in \Delta_X \cap (X' \times X')$, where $X' \subset X$ is a Borel set with $\mu_\theta(X') = 1$. Then there exists a Hölder map $b'\colon X \longrightarrow G$ with $b = b'$ μ_θ-a.e. such that*

$$a_f^\pm(x,y) = b'(x)^{-1}a_g^\pm(x,y)b'(y)$$

for every $(x,y) \in \Delta_X$.

Proof. As a consequence of bounded distortion there exists a constant $a \geq 1$ with

$$D_\delta(\mathbf{f}(n,x)) \leq a, \qquad D_\delta(\mathbf{g}(n,x)) \leq a \tag{3.11}$$

for every $n \in \mathbb{Z}$ and $x \in X$. From the definition of $a_f^\pm, a_g^\pm$ it is clear that

$$D_\delta(a_f^\pm(x,y)) \leq a^2, \qquad D_\delta(a_g^\pm(x,y)) \leq a^2 \tag{3.12}$$

whenever $(x,y) \in \Delta_X$.

Since the measure μ_θ is quasi-invariant under the group Γ chosen at the beginning of this section we can decrease X', if necessary, and assume in addition that X' is invariant both under σ and Γ.

For every $(w,w') \in \Delta'_X = \Delta_X \cap (X' \times X')$ and $h, h' \in G$ we have that

$$\delta(hb(w'), h'b(w')) \leq D_\delta(a_g^\pm(w',w))\delta(ha_f^\pm(w',w)b(w), h'a_f^\pm(w',w)b(w))$$
$$\leq D_\delta(b(w))D_\delta(a_g^\pm(w',w))D_\delta(a_f^\pm(w',w))\delta(h,h'),$$

and, similarly,

$$\delta(b(w')h, b(w')h') \leq D_\delta(b(w))D_\delta(a_f^\pm(w,w'))D_\delta(a_g^\pm(w',w))\delta(h,h').$$

This implies that

$$D_\delta(b(w')) \leq D_\delta(b(w))a^4 \tag{3.13}$$

whenever $(w,w') \in \Delta'_X$ and hence, by ergodicity, for $(\mu_\theta \times \mu_\theta)$-a.e. $(w,w') \in X' \times X'$. After reducing X' by a Γ- and σ-invariant null-set we may assume that (3.13) holds for every $(w,w') \in X' \times X'$.

Let $c' > 0$ and $0 < \gamma < 1$ be constants satisfying Lemma 3.2 (1)–(4) both for f and g. For every $M \geq 0$ and $(x,y) \in \Delta_X$ with $x_n = y_n$ for $n \geq -M$, Lemma 3.2 (2) (with $x' = y' = x$) implies that

$$\max\left(\delta(1_G, a_f^+(x,y)), \delta(1_G, a_g^+(x,y))\right) \leq 2c'\gamma^M. \tag{3.14}$$

Similarly, if $(x,y) \in \Delta_X$ satisfies that $x_n = y_n$ for $n \leq M$, then

$$\max\left(\delta(1_G, a_f^-(x,y)), \delta(1_G, a_g^-(x,y))\right) \leq 2c'\gamma^M. \tag{3.15}$$

We fix an element $w \in X'$ and claim that

$$\delta(b(x), b(y)) \leq D_\delta(b(w))(1 + D_\delta(b(w))a^4)4c'\gamma^N. \tag{3.16}$$

for every $N \geq 0$ and $x, y \in \Delta_X(w) \subset X'$ with $x_k = y_k$ for $|k| \leq N$ (cf. (3.3)).

In order to verify (3.16) we fix $N \geq 0$ and assume that $x, y \in \Delta_X(w)$ satisfy that $x_k = y_k$ for $|k| \leq N$. The local product structure of X guarantees the existence of a point $z \in \Delta_X(w)$ with $z_k = x_k$ for $k \geq 0$ and $z_k = y_k$ for $k \leq 0$, and (3.14)–(3.15) yields that

$$\max\left(\delta(1_G, a_f^+(x,z)), \delta(1_G, a_g^+(x,z))\right) \leq 2c'\gamma^N,$$
$$\max\left(\delta(1_G, a_f^-(z,y)), \delta(1_G, a_g^+(z,y))\right) \leq 2c'\gamma^N.$$

From (3.7) and (3.13) we obtain that

$$\begin{aligned}
\delta(b(x), b(z)) &\leq \delta(b(x), a_f^+(x,z)b(z)) + \delta(b(z), a_f^+(x,z)b(z)) \\
&\leq a^2\delta(a_f^+(w,x)b(x), a_f^+(w,x)a_f^+(x,z)b(z)) \\
&\qquad\qquad + D_\delta(b(z))\delta(1_G, a_f^+(x,z)) \\
&\leq a^2\delta(a_f^+(w,x)b(x), a_f^+(w,z)b(z)) + D_\delta(b(w))a^4 2c'\gamma^N \\
&\leq D_\delta(b(w))^2 a^2 \delta(b(w)^{-1}a_f^+(w,x)b(x), b(w)^{-1}a_f^+(w,z)b(z)) \\
&\qquad\qquad + D_\delta(b(w))a^4 2c'\gamma^N \\
&\leq D_\delta(b(w))^2 a^2 \delta(a_g^+(w,x), a_g^+(w,z)) + D_\delta(b(w))a^4 2c'\gamma^N \\
&\leq D_\delta(b(w))^2 a^4 \delta(1_G, a_g^+(x,z)) + D_\delta(b(w))a^4 2c'\gamma^N \\
&\leq D_\delta(b(w))(1 + D_\delta(b(w))a^4)2c'\gamma^N.
\end{aligned}$$

Similarly we see that

$$\delta(b(y), b(z)) \leq D_\delta(b(w))(1 + D_\delta(b(w))a^4)2c'\gamma^N,$$

which implies (3.16).

As the map $w \mapsto D_\delta(b(w))$ is measurable and everywhere finite on X' there exists a Borel set $B \subset X'$ with $\mu_\theta(B) > 0$ and $D_\delta(b(w)) < C < \infty$ for every $w \in B$. The ergodicity of the relation Δ_X (cf. [7]) implies that not only $\mu_\theta(\Delta_X(B)) = 1$, where

$$\Delta_X(B) = \bigcup_{w \in B} \Delta_X(w) = \bigcup_{V \in \Gamma} V(B),$$

but also that there exists a Borel set $X'' \subset X$ with $\mu_\theta(X'') = 1$ such that

$$\delta(b(x), b(y)) \leq C(1 + D_\delta(b(w))a^4)4c'\gamma^N$$

for every $N \geq 0$ and every $(x, y) \in X \times X$ with $x_k = y_k$ for $k = -N, \dots, N$. This shows that b coincides μ_θ-a.e. with a Hölder function.

Finally, if $V \in \Gamma$ is fixed, then (3.10) implies that

$$a_f^\pm(Vy, y) = b'(Vy)^{-1}a_g^\pm(Vy, y)b'(y) \tag{3.17}$$

for μ_θ-a.e. $y \in X$. Lemma 3.2 (2) and (4) guarantee that the maps $y \mapsto a_f^\pm(Vy, y)$, $y \mapsto a_g^\pm(Vy, y)$ are continuous, so that both sides of (3.17) are continuous and coincide on a dense subset of X. Hence (3.17) holds for every $y \in X$, and by varying $V \in \Gamma$ we obtain that

$$a_f^{\pm}(x,y) = b'(x)^{-1} a_g^{\pm}(x,y) b'(y)$$

for every $(x,y) \in \Delta_X$ □

Proof of Theorem 3.1. Apply the Lemmas 3.3 and 3.4. □

4. Cocycles with equal weights

Let A be a finite set, $X \subset A^{\mathbb{Z}}$ a mixing shift of finite type, σ the shift (3.1) on X, (G,δ) a complete separable metric group, and $f \colon X \longrightarrow G$ a Hölder map. For every periodic point $x \in X$ we write

$$\pi(x) = \min\{k \geq 1 : \sigma^k(x) = x\} \tag{4.1}$$

for the minimal period of x and denote by

$$w_f(x) = f(\sigma^{\pi(x)-1}x) \cdots f(x) = \mathbf{f}(\pi(x), x) \tag{4.2}$$

the f-weight of x, where $\mathbf{f} \colon \mathbb{Z} \times X \longrightarrow G$ is the cocycle (2.4).

Theorem 4.1. *Let A be a finite set, $X \subset A^{\mathbb{Z}}$ a mixing shift of finite type, (G,δ) a Lipschitz metric group, and $f, g \colon X \longrightarrow G$ Hölder maps such that the corresponding cocycles $\mathbf{f}, \mathbf{g} \colon \mathbb{Z} \times X \longrightarrow G$ have bounded distortion. If $w_f(x) = w_g(x)$ for every periodic point $x \in X$ (cf. (4.2)), then f and g are Hölder cohomologous.*

We begin the proof of Theorem 4.1 with some definitions. The notation is that of Section 3.

A point $x \in X$ is *doubly transitive* if the sets $\{\sigma^k x : k \geq N\}$ and $\{\sigma^{-k} x : k \geq N\}$ are dense in X for every $N \geq 0$. The set $X^* \subset X$ of doubly transitive points is a dense G_δ with $\mu_\theta(X^*) = 1$ for every Hölder map $\theta \colon X \longrightarrow \mathbb{R}$ (cf. 3.5). Furthermore, X^* is invariant under σ and *saturated* with respect to Δ_X in the sense that

$$X^* = \bigcup_{x \in X^*} \Delta_X(x).$$

We define an equivalence relation

$$\Delta_X^* = \{(\sigma^k x, y) : k \in \mathbb{Z},\ (x,y) \in \Delta_X \cap (X^* \times X^*)\} \tag{4.3}$$

and set

$$a_f^*(\sigma^k x, y) = \mathbf{f}(k,x) a_f^+(x,y) \tag{4.4}$$

for every $(x,y) \in \Delta_X \cap (X^* \times X^*)$ and $k \in \mathbb{Z}$. The resulting map $a_f^* \colon \Delta_X^* \longrightarrow G$ is well-defined, since X^* contains no periodic points, and satisfies the cocycle equation

$$a_f^*(x,y) a_f^*(y,z) = a_f^*(x,z) \tag{4.5}$$

for all $(x,y), (x,z) \in \Delta_X^*$. As before we write $\Delta_X^*(x)$ for the Δ_X^*-equivalence class of a point $x \in X^*$.

Lemma 4.2. *Let $f, g \colon X \longrightarrow G$ be Hölder maps. If there exists a Borel map $b \colon X^* \longrightarrow G$ with*

$$a_f^*(x,y) = b(x)^{-1} a_g^*(x,y) b(y)$$

for every $(x,y) \in \Delta_X^*$, then f and g are Hölder cohomologous.

Proof. For every $x \in X^*$, $(\sigma x, x) \in \Delta_X^*$, and

$$f(x) = a_f^*(\sigma x, x) = b(\sigma x)^{-1} a_g^*(\sigma x, x) b(x) = b(\sigma x)^{-1} g(x) b(x).$$

As $\mu_\theta(X^*) = 1$ for every Hölder map $\theta\colon X \longrightarrow \mathbb{R}$, Lemma 3.4 implies that b is μ_θ-a.e. equal to a Hölder map $b'\colon X \longrightarrow G$. By continuity, $f(x) = b'(\sigma x)^{-1} g(x) b'(x)$ for every $x \in X$. □

Lemma 4.3. *Let $a_f^*, a_g^*\colon \Delta_X^* \longrightarrow G$ be the cocycles defined by* (4.4) *for f and g, respectively. Then there exists, for every $\varepsilon > 0$, an $N \geq 0$ with*

$$\delta(a_f^*(x, y), a_g^*(x, y)) < \varepsilon$$

whenever $(x, y) \in \Delta_X^*$ and $x_n = y_n$ for all $|n| \leq N$.

Proof. In the notation of Lemma 3.2 we assume that $\max\{\gamma_f, \gamma_g\} < \gamma < 1$ and choose a constant $c' > 0$ obeying the conditions (1)–(4) of that lemma for both f and g.

Suppose that $(x, y) \in \Delta_X^*$ satisfies that $x_n = y_n$ whenever $|n| \leq N$, and $x_{n+k} = y_n$ whenever $n \geq M$, where $M > N$ are positive integers and $k \in \mathbb{Z}$. Then $(\sigma^k x, y) \in \Delta_X$, and the definitions of $a_f^*(x, y)$, $a_g^*(x, y)$ show that

$$a_f^*(x, y) = \lim_{n\to\infty} \mathbf{f}(k, x)^{-1} \mathbf{f}(n, \sigma^k x)^{-1} \mathbf{f}(n, y) = \lim_{n\to\infty} \mathbf{f}(n + k, x)^{-1} \mathbf{f}(n, y),$$
$$a_g^*(x, y) = \lim_{n\to\infty} \mathbf{g}(k, x)^{-1} \mathbf{g}(n, \sigma^k x)^{-1} \mathbf{g}(n, x) = \lim_{n\to\infty} \mathbf{g}(n + k, x)^{-1} \mathbf{g}(n, y).$$

We choose periodic points x', y' in X such that y' has period $p > M + 3N$, x' has period $p + k$ and

$$x'_n = x_n \ \text{ for } \ n = -N, \dots, M + 2N + k,$$
$$y'_n = \begin{cases} y_n & \text{for } \ n = -N, \dots, M + 2N, \\ x'_{n+k} & \text{for } \ n = M + 2N + 1, \dots, p - N - 1. \end{cases}$$

From Lemma 3.2 (1)–(4) we know that

$$\delta(a_f^*(x, y), \mathbf{f}(p + k, x)^{-1} \mathbf{f}(p, y)) \leq ac'\gamma^N,$$
$$\delta(\mathbf{f}(p + k, x)^{-1} \mathbf{f}(p, y), \mathbf{f}(p + k, x')^{-1} \mathbf{f}(p, y')) \leq 2ac'\gamma^N,$$
$$\delta(a_g^+(x, y), \mathbf{g}(p + k, x)^{-1} \mathbf{g}(p, y)) \leq ac'\gamma^N,$$
$$\delta(\mathbf{g}(p + k, x)^{-1} \mathbf{g}(p, y), \mathbf{g}(p + k, x')^{-1} \mathbf{g}(p, y')) \leq 2ac'\gamma^N.$$

Our condition on weights implies that

$$\mathbf{f}(p + k, x') = w_f(x')^r = w_g(x')^r = \mathbf{g}(p + k, x'),$$
$$\mathbf{f}(p, y') = w_f(y')^{r'} = w_g(y')^{r'} = \mathbf{g}(p, y'),$$

where $p + k = r\pi(x')$ and $p = r'\pi(y')$ (cf. (4.1)). Hence

$$\delta(a_f^*(x, y), a_g^*(x, y)) \leq 6ac'\gamma^N.$$

This implies the assertion of the lemma. □

Proof of Theorem 4.1. Fix $y \in X^*$ and let, for every $x \in \Delta_X^*(y)$, $u(x) = a_g^*(x, y) a_f^*(y, x)$. From (4.5) we obtain that

$$u(x') = a_g^*(x', x) u(x) a_f^*(x, x')$$

for every $x' \in \Delta_X^*(y)$, and Lemma 4.3 and (3.11)–(3.12) together imply that u is the restriction of a Hölder map $b\colon X \longrightarrow G$ to the dense set $\Delta_X^*(y) \subset X$. By using (4.5) again we obtain that

$$a_g^*(x,x') = b(x)a_f^*(x,x')b(x')^{-1} \tag{4.6}$$

for every $x, x' \in \Delta_X^*(y)$. In order to extend (4.6) to Δ_X^* we consider an arbitrary $z \in X^*$ and elements $v, w \in \Delta_X^*(z)$ with $(\sigma^k v, w) \in \Delta_X$, say. We fix an integer $D \geq 0$ with $v_{k+n} = w_n$ whenever $|n| \geq D$ and choose, for every $l \geq 1$, elements $v^l, w^l \in \Delta_X^*(y)$ with $v_{k+n}^l = w_n^l$ for every $n \geq D$ and $v_{k+n}^l = v_{k+n}$, $w_n^l = w_n$ for $-l - D \leq n \leq l + D$. According to Lemma 3.2, (4.6) and (3.11)–(3.12),

$$\begin{aligned} a_g^*(v,w) &= \lim_{l\to\infty} a_g^*(v^l,w^l) = \lim_{l\to\infty} b(v^l)a_f^*(v^l,w^l)b(w^l) \\ &= b(v)a_f^*(v,w)b(w)^{-1}, \end{aligned}$$

so that (4.6) holds on all of Δ_X^*. An application of Lemma 4.2 completes the proof. ☐

5. Cocycles with conjugate weights

Let G be a Polish group with a bi-invariant metric δ, A a finite set, $X \subset A^{\mathbb{Z}}$ a mixing shift of finite type, and $f\colon X \longrightarrow G$ a continuous map. We set $Y = X \times G$, denote by $\sigma_f\colon Y \longrightarrow Y$ the skew-product transformation given by

$$\sigma_f(x,h) = (\sigma x, f(x)h) \tag{5.1}$$

for every $y = (x,h) \in Y$, and call σ_f topologically transitive if it has a dense orbit.

Theorem 5.1. *Let A be a finite set, $X \subset A^{\mathbb{Z}}$ a mixing shift of finite type, G, G' Polish groups with bi-invariant metrics δ, δ', respectively, and $f\colon X \longrightarrow G$, $g\colon X \longrightarrow G'$ Hölder maps such that σ_f is topologically transitive on $X \times G$.*

(1) *Suppose that there exists, for every $\varepsilon > 0$, a $\xi > 0$ such that*

$$\delta'(\mathbf{g}(p,x), 1_G) < \varepsilon$$

for every periodic point $x \in X$ with

$$\sigma^p x = x \text{ and } \delta(\mathbf{f}(p,x), 1_G) < \xi.$$

Then there exists a continuous group homomorphism $\alpha\colon G \longrightarrow G'$ such that g is continuously cohomologous to $\alpha \cdot f$;

(2) *If $(G,\delta) = (G',\delta')$, and if $w_f(x)$ is conjugate to $w_g(x)$ for every periodic point $x \in X$ (cf. (4.2)), then there exists an isometric group isomorphism $\alpha\colon G \longrightarrow G$ such that $\alpha \cdot f$ and g are Hölder cohomologous. Moreover, if G is compact, then α preserves conjugacy classes.*

Remarks 5.2. (1) If G is compact, and if both σ_f and σ_g are topologically transitive, then the group homomorphism $\alpha\colon G \longrightarrow G'$ in Theorem 5.1 (1) is surjective.

(2) It will be clear from the proof of Theorem 5.1 (1) that we do not really require the bi-invariance of δ and δ': we only need that (G,δ) and

(G', δ') are Lipschitz groups and $\mathbf{f}, \mathbf{g}$ have bounded distortion. However, the topological transitivity of σ_f implies that

$$\sup_{h\in G} D_\delta(h) < \infty$$

(cf. (2.2)), so that δ is Lipschitz-equivalent to a bi-invariant metric.

(3) If we drop the hypothesis that σ_f is topologically transitive and assume instead that there exist a Hölder map $\beta\colon X \longrightarrow G$ and a closed subgroup $\tilde{G} \subset G$ such that $f'(x) = \beta(\sigma x)^{-1} f(x)\beta(x) \in \tilde{G}$ for every $x \in X$ and $\sigma_{f'}$ is topologically transitive on $X \times \tilde{G}$, then Theorem 5.1 (1) guarantees the existence of a continuous group homomorphism $\alpha\colon \tilde{G} \longrightarrow G'$ with $\alpha \cdot f'$ continuously cohomologous to g. Note, however, that it may not be possible to extend α to a continuous group homomorphism $\bar{\alpha}\colon G \longrightarrow G'$, so that we cannot guarantee that g is cohomologous to a homomorphic image of f (cf. Theorems 6.1 and 6.5 in [5]).

For the proof of Theorem 5.1 we define the equivalence relation Δ_X^* and the cocycles $a_f^*\colon \Delta_X^* \longrightarrow G$, $a_g^*\colon \Delta_X^* \longrightarrow G'$ by (4.3) and (4.4). Consider the equivalence relation Δ_f on $Y^* = X^* \times G \subset X \times G = Y$ given by

$$\Delta_f = \{((x, a_f^*(x, x')h), (x', h)) : (x, x') \in \Delta_X^*,\ h \in G\}. \tag{5.2}$$

As usual we denote by

$$\Delta_f(y) = \{y' \in Y^* : (y, y') \in \Delta_f\}$$

the equivalence class of a point $y \in Y^*$.

Lemma 5.3. *Suppose that $f\colon X \longrightarrow G$, $g\colon X \longrightarrow G'$ are Hölder maps with the following property: for every $\varepsilon > 0$ there exists a $\xi = \xi(\varepsilon) > 0$ with $\delta'(\mathbf{g}(p, x), 1_G) < \varepsilon$ whenever $x \in X$ is a periodic point with period p and $\delta(\mathbf{f}(p, x), 1_G) < \xi$.*

Then there exist, for every $\varepsilon > 0$, constants $\xi' = \xi'(\varepsilon) > 0$, $M = M(\varepsilon) \geq 0$, with

$$\delta'(a_g^+(x, y), 1_G) < \varepsilon$$

whenever $(x, y) \in \Delta_X^*$, $x_n = y_n$ for $n = -M, \dots, M$ and $\delta(a_f^+(x, y), 1_G) < \xi'$.

Proof. Fix $\varepsilon > 0$ and $\xi = \xi(\varepsilon/3) < \varepsilon/3$ as in the statement of the lemma. We choose constants $c' > 0, 0 < \gamma < 1$ satisfying Lemma 3.2 (1)–(4) both for f and g, let $M \geq 0$ with $c'\gamma^M < \xi/10$, and assume that $(x, y) \in \Delta_X^*$ with $x_n = y_n$ for all $n = -M, \dots, M$. Choose integers $k \in \mathbb{Z}$ and $D \geq 0$ with $x_{k+n} = y_n$ whenever $|n| \geq D$ and find a periodic point $v' \in X$ with (not necessarily minimal) period $p' > D + 3M + k$ such that $v'_n = x_n$ whenever $-M \leq n \leq D + k + 2M$. Since the cylinder set

$$C = \{v \in X : v_n = v'_n \text{ for } n = -M, \dots, p' + 2M\}$$

is nonempty, we can use the topological transitivity of σ_f to find a point $v'' \in C$ and an integer $p > p'$ with $\sigma^p v'' \in C$ and $\delta(1_G, \mathbf{f}(p, v'')) < \xi/3$.

Let $v \in X$ be the periodic point with period p and $v_n = v''_n$ for $n = 0, \dots, p-1$. From (3.9) we see that

$$\delta(1_G, \mathbf{f}(p, v)) \leq \delta(\mathbf{f}(p, v''), \mathbf{f}(p, v)) + \delta(1_G, \mathbf{f}(p, v'')) < c'\gamma^M + \xi/3 < \xi/2.$$

We define a second periodic point $w \in X$ with period $p-k$ by

$$v'_n = \begin{cases} y_n & \text{for } \; -M \le n \le D, \\ v_{n+k} & \text{for } \; D+1 < n \le p-k-M-1. \end{cases}$$

According to Lemma 3.2,

$$\begin{gathered} \delta(a_f^*(x,y), \mathbf{f}(p',x)^{-1}\mathbf{f}(p'-k,y)) < c'\gamma^M, \\ \delta(\mathbf{f}(p',x)^{-1}\mathbf{f}(p'-k,y), \mathbf{f}(p',v)^{-1}\mathbf{f}(p'-k,w)) < c'\gamma^M, \\ \delta(\mathbf{f}(p',v)^{-1}\mathbf{f}(p'-k,w), \mathbf{f}(p,v)^{-1}\mathbf{f}(p-k,w)) < c'\gamma^M, \\ \delta'(a_g^*(x,y), \mathbf{g}(p',x)^{-1}\mathbf{g}(p'-k,y)) < c'\gamma^M, \\ \delta'(\mathbf{g}(p',x)^{-1}\mathbf{g}(p'-k,y), \mathbf{g}(p',v)^{-1}\mathbf{g}(p'-k,w)) < c'\gamma^M, \\ \delta'(\mathbf{g}(p',v)^{-1}\mathbf{g}(p'-k,w), \mathbf{g}(p,v)^{-1}\mathbf{g}(p-k,w)) < c'\gamma^M. \end{gathered}$$

From these inequalities it is clear that

$$\begin{gathered} \delta(a_f^*(x,y), \mathbf{f}(p,v)^{-1}\mathbf{f}(p-k,w)) < 3c'\gamma^M, \\ \delta'(a_g^*(x,y), \mathbf{g}(p,v)^{-1}\mathbf{g}(p-k,w)) < 3c'\gamma^M, \end{gathered}$$

and the bi-invariance of δ allows us to conclude that

$$\begin{aligned} \delta(a_f^*(x,y), \mathbf{f}(p-k,w)) &\le \delta(a_f^*(x,y), \mathbf{f}(p,v)^{-1}\mathbf{f}(p-k,w)) \\ &\quad + \delta(\mathbf{f}(p,v)^{-1}\mathbf{f}(p-k,w), \mathbf{f}(p-k,w)) \\ &< 3c'\gamma^M + \xi/2 < 4\xi/5. \end{aligned} \tag{5.3}$$

Similarly we see that

$$\delta'(a_g^*(x,y), \mathbf{g}(p-k,w)) < 3c'\gamma^M + \varepsilon/3, \tag{5.4}$$

where we are using that $\delta'(\mathbf{g}(p,w), 1_{G'}) < \varepsilon/3$ whenever $\delta(\mathbf{f}(p,v), 1_G) < \xi$.

Now suppose that, in addition, $a_f^*(x,y) < \xi/5$. Then (5.3) implies that

$$\delta(\mathbf{f}(p-k,w), 1_G) < \xi$$

and hence that

$$\delta'(\mathbf{g}(p-k,w), 1_{G'}) < \varepsilon/3.$$

It follows that

$$\begin{aligned} \delta'(a_g^*(x,y), 1_{G'}) &\le \delta'(\mathbf{g}(p-k,w), 1_{G'}) + \delta'(a_g^*(x,y), \mathbf{g}(p-k,w)) \\ &< 3c'\gamma^M + \varepsilon/3 + \varepsilon/3 < \varepsilon. \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, this proves the lemma. □

Lemma 5.4. *Suppose that $f\colon X \longrightarrow G$, $g\colon X \longrightarrow G'$ are Hölder maps with the following properties.*

(1) *σ_f is topologically transitive;*
(2) *For every $\varepsilon > 0$ there exists a $\xi > 0$ with $\delta'(\mathbf{g}(p,x), 1_{G'}) < \varepsilon$ for every periodic point $x \in X$ with period p and $\delta(\mathbf{f}(p,x), 1_G) < \xi$.*

Then there exists a continuous group homomorphism $\alpha\colon G \longrightarrow G'$ such that g is continuously cohomologous to $\alpha \cdot f$.

Proof. For every $h \in G$ we denote by $R_h \colon Y \longrightarrow Y$ the right translation

$$R_h(x,h') = (x,h'h),\ (x,h') \in Y = X \times G.$$

A standard argument shows that the set of doubly transitive points

$$Y^{**} = \big\{y \in Y : \text{the sets } \{\sigma_f^n y : n \geq N\}, \{\sigma_f^{-n} y : n \geq N\} \\ \text{are dense in } Y \text{ for every } N \geq 0\big\}$$

of σ_f is a dense G_δ in Y. Hence

$$Z = Y^{**} \cap Y^*$$

is again a dense G_δ in Y, and $\Delta_f(y)$ is dense in Y for every $y \in Z$ (cf. (5.2)). Since $R_h Z = Z$ for every $h \in G$, Z is of the form $Z = X^{**} \times G$ for some dense G_δ set $X^{**} \subset X^*$, and X^{**} is Δ_X^*-saturated (i.e. a union of Δ_X^*-equivalence classes).

We fix a point $z = (\bar{x}, 1_G) \in Z$, define $b' \colon \Delta_f(z) \longrightarrow G'$ by

$$b'(x,h) = a_g^*(x,\bar{x})$$

for every $(x,h) = (x, a_f^*(x,\bar{x})) \in \Delta_f(z)$, and conclude from Lemma 5.3 and the cocycle equation (4.5) that there exists a continuous map $\bar{b} \colon Y \longrightarrow G'$ with

$$\bar{b}(y) = b'(y)$$

for every $y \in Z$. If $a((x,h),(x',h')) = a_g^*(x,x')$ for all $(x,h),(x',h') \in \Delta_f(z)$, then $a \colon \Delta_f \longrightarrow G'$ satisfies the cocycle equation (4.5), and

$$a(y,y') = \bar{b}(y)\bar{b}(y')^{-1}$$

for every $y,y' \in \Delta_f(z)$. By using the same continuity argument as in the proof of Theorem 3.1 we can extend this equation to all of Δ_f.

Having found a continuous map $\bar{b} \colon Y \longrightarrow G'$ with

$$a_g^*(x,x') = \bar{b}(x, a_f^*(x,x')h')\bar{b}(x',h')^{-1} \tag{5.5}$$

for every $(x,x') \in \Delta_X^*$ and $h' \in G$, we use the transitivity of Δ_f (i.e. the fact that $\Delta_f(y)$ is dense in Y for every $y \in Z$) to choose, for any given $x \in X^{**}$ and $h \in G$, a sequence $(x^k,\ k \geq 1)$ in $\Delta_X^*(x)$ such that $\lim_{k\to\infty} x^k = x$ and $\lim_{k\to\infty} a_f^*(x^k,x) = h$. As $k \to \infty$, the right hand side of (5.5) converges to $\bar{b}(x,hh')\bar{b}(x,h')^{-1}$, and the left hand side of (5.5) guarantees that $\bar{b}(x,hh')\bar{b}(x,h')^{-1}$ is constant in h' for every $x \in X^{**}$ and $h \in G$. By continuity, $\bar{b}(x,hh')\bar{b}(x,h')^{-1}$ is constant in h' for every $x \in X$ and $h \in G$. Hence there exists, for every $x \in X$, a continuous group homomorphism $\alpha_x \colon G \longrightarrow G'$ with $\bar{b}(x,hh') = \alpha_x(h)\bar{b}(x,h')$ for every $h,h' \in G$. By setting $h' = 1_G$ and $b(x) = \bar{b}(x,1_G)$, (5.5) changes into

$$a_g^*(x,x') = \alpha_x(a_f^*(x,x'))b(x)b(x')^{-1} \tag{5.6}$$

for every $(x,x') \in \Delta_X^*$, and the cocycle equation (4.5) yields that

$$\begin{aligned} \alpha_x(a_f^*(x,x''))b(x)b(x'')^{-1} &= \alpha_x(a_f^*(x,x'))\alpha_x(a_f^*(x',x''))b(x)b(x'') \\ &= \alpha_x(x,x')b(x)b(x')^{-1}\alpha_{x'}(a_f^*(x',x''))b(x')b(x'')^{-1} \end{aligned} \tag{5.7}$$

for every $(x,x'),(x,x'') \in \Delta_X^*$. Hence $b(x)b(x')^{-1}$ commutes with $\alpha_{x'}(a_f^*(x', x''))$ for every $x',x'' \in \Delta_X^*(x)$. The transitivity of Δ_f and continuity imply that $b(x)b(x')^{-1}$ commutes with $\alpha_{x'}(G)$ for every $(x,x') \in X \times X$.

Fix an arbitrary element $\bar{x} \in X^{**}$ and replace $\bar{b}$ with

$$\bar{b}'(x,h) = \bar{b}(x,h)\bar{b}(\bar{x},1_G)^{-1}.$$

Then $\bar{b}'$ again satisfies (5.5) and $b'(x) = \bar{b}'(x,1_G) = b(x)b(\bar{x})^{-1}$ commutes with $\alpha_x(G)$ for every $x \in X$. From (5.6) we see that

$$a_g^*(x,x') = b'(x)\alpha_x(a_f^*(x,x'))b'(x')^{-1} \tag{5.8}$$

for every $(x,x') \in \Delta_X^*$, and (5.7) implies that $\alpha_x = \alpha_{x'}$ for every $(x,x') \in \Delta_X^*$. By transitivity, the map $x \mapsto \alpha_x$ is constant and everywhere equal to a continuous group homomorphism $\alpha\colon G \longrightarrow G'$, and (5.8) shows that a_g^* is cohomologous to $\alpha \cdot a_f^*$, with continuous transfer function b'. Lemma 4.2 implies that $\alpha \cdot f$ and g are cohomologous with transfer function b'. □

Proof of Theorem 5.1. Lemma 5.4 proves (1). For the proof of (2) we note that conjugacy of weights implies that

$$\delta(\mathbf{f}(p,x),1_G) = \delta(\mathbf{g}(p,x),1_G)$$

for every $x \in X$ with $\sigma^p x = x$, since δ is bi-invariant. According to Lemma 5.4 there exist a continuous group homomorphism $\alpha\colon G \longrightarrow G$ and a continuous map $b'\colon X \longrightarrow G$ such that

$$\alpha \cdot f(x) = b'(\sigma x)^{-1} g(x) b'(x)$$

for every $x \in X$. We claim that α is an isometry.

We write $[h] = \{h'^{-1}hh' : h' \in G\}$ for the conjugacy class of an element $h \in G$ and obtain that

$$\alpha(w_f(x)^n) \in [w_g(x)^n]$$

for every periodic point $x \in X$ and every $n \geq 1$. As σ_f is topologically transitive, the set

$$W = \{w_f(x)^n : x \in X \text{ is periodic and } n \geq 1\}$$

is dense in G, and

$$\delta(1_G,h) = \delta(1_G,\alpha(h)) \tag{5.9}$$

for every $h \in W$. By continuity, (5.9) holds for every $h \in G$. The bi-invariance of δ yields that

$$\delta(h,h') = \delta(1_G,h^{-1}h') = \delta(1_G,\alpha(h^{-1}h')) = \delta(\alpha(h),\alpha(h'))$$

for every $h,h' \in G$, so that α is an isometric group automorphism and $\alpha \cdot f$ is again Hölder. Theorem 3.1 implies that b' is Hölder.

If G is compact we write $\pi_1\colon G \times G \longrightarrow G$ for the first coordinate projection and obtain that the dense set

$$\{h \in G : \alpha(h) \in [h]\} = \pi_1(\{(h,h'^{-1}hh') : h,h' \in G\} \cap \{(h,\alpha(h)) : h \in G\})$$

is actually closed and thus equal to G. This shows that α preserves conjugacy classes. □

6. Hölder cocycles of hyperbolic diffeomorphisms

Although we have have stated and proved the Theorems 3.1, 4.1 and 5.1 for shifts of finite type, the same statements hold for hyperbolic diffeomorphisms. Let $\phi\colon M \longrightarrow M$ be a C^∞-diffeomorphism of a compact Riemannian manifold M with Riemannian metric d. A closed ϕ-invariant set $X \subset M$ has *hyperbolic structure* if the tangent bundle $T_X(M)$ over X can be split continuously into two $D\phi$-invariant sub-bundles $T_X(M) = E^u \oplus E^s$ such that, for suitable constants $\xi > 0$, $0 < \lambda < 1$, and for every $n \geq 0$,

$$\|D\phi^n v\| \leq \xi\lambda^n\|v\| \qquad \text{if } v \in E^s,$$
$$\|D\phi^{-n} v\| \leq \xi\lambda^n\|v\| \qquad \text{if } v \in E^u.$$

We call the restriction of ϕ to X *hyperbolic* if

(i) X has hyperbolic structure,
(ii) X is locally maximal (or basic), i.e. there exists an open set $\mathcal{U} \supset X$ with closure $\overline{\mathcal{U}}$ such that $\bigcap_{n=\in\mathbb{Z}} \phi^n\overline{\mathcal{U}} = X$,
(iii) the periodic points of X are dense in X,
(iv) ϕ is topologically mixing, i.e. for all nonempty open subsets $\mathcal{U}, \mathcal{V} \subset X$, $\phi^n\mathcal{U} \cap \mathcal{V} \neq \varnothing$ whenever $|n|$ is sufficiently large.

Then X has local product structure (Proposition 6.4.21 in [1]), the periodic points of ϕ are dense (Corollary 6.4.20 in [1]), and ϕ has specification (Theorem 18.3.9 in [1]) and obeys the shadowing lemma (Theorem 18.1.2 in [1]). Two points $x, y \in X$ are called *homoclinic* if $\lim_{|n|\to\infty} d(\phi^n x, \phi^n y) = 0$, and the *homoclinic equivalence relation* is the set $\Delta_X \subset X \times X$ of all homoclinic pairs. From the specification property it is easy to see that Δ_X is *minimal* in the sense that the equivalence class $\Delta_X(x)$ of every $x \in X$ is dense in X. As X may be connected, it may not be possible to choose a group of homeomorphisms Γ whose orbits generate Δ_X (cf. the conditions (a)–(b) in Section 3), but we can choose a countable collection Γ of *local* diffeomorphisms satisfying the analogue of (b).

We fix a hyperbolic system (X, ϕ) with metric d and assume that (G, δ) is a Lipschitz metric group. A map $f\colon X \longrightarrow G$ is *Hölder* if there exist constants $c, \gamma > 0$ such that

$$\delta(f(x), f(y)) \leq c d(x, y)^\gamma \tag{6.1}$$

for every $x, y \in X$.

For every Hölder map $\theta\colon X \longrightarrow \mathbb{R}$ there exists a unique probability measure μ_θ on X satisfying (3.5) for every $V \in \Gamma$ and μ_θ-a.e. x in the domain of V. The measure μ_θ is called the *Gibbs measure* or *equilibrium state* of θ (cf. Corollary 20.3.8 in [1]). As in [7] one can verify easily that Δ_X is μ_θ-ergodic in the sense that every Δ_X-saturated Borel set (i.e. every Borel set which is a union of Δ_X-equivalence classes) has μ_θ-measure equal to zero or one.

For every Hölder map $f\colon X \longrightarrow G$ we define the cocycle $\mathbf{f}\colon \mathbb{Z} \times X \longrightarrow G$ by (2.5) and the notions of cohomology and bounded distortion exactly as in Section 2 with ϕ replacing T. Then the following results are true.

Theorem 6.1. *Let (X, ϕ) by a hyperbolic system, $\theta\colon X \longrightarrow \mathbb{R}$ a Hölder function, and let μ_θ be the Gibbs measure of θ.*

Suppose furthermore that (G,δ) *is a Lipschitz metric group and* $f,g\colon X \longrightarrow G$ *are Hölder maps such that the corresponding cocycles* $\mathbf{f},\mathbf{g}\colon \mathbb{Z}\times X \longrightarrow G$ *have bounded distortion. If* f *and* g *are* μ_θ*-cohomologous then they are Hölder cohomologous.*

Theorem 6.2. *Let* (X,ϕ) *by a hyperbolic system,* (G,δ) *a Lipschitz metric group, and* $f,g\colon X \longrightarrow G$ *Hölder maps such that the corresponding cocycles* $\mathbf{f},\mathbf{g}\colon \mathbb{Z}\times X \longrightarrow G$ *have bounded distortion. If* $w_f(x) = w_g(x)$ *for every periodic point* $x \in X$*, where* $w_f(x), w_g(x)$ *are defined as in* (4.2)*, then* f *and* g *are Hölder cohomologous.*

Theorem 6.3. *Let* (X,ϕ) *be a hyperbolic system,* G, G' *Polish groups with bi-invariant metrics* δ,δ'*, respectively, and* $f\colon X \longrightarrow G$*,* $g\colon X \longrightarrow G'$ *Hölder maps such that such that* ϕ_f *is topologically transitive on* $X\times G$*, where* ϕ_f *is the skew-product map defined as in* (5.1)*.*

(1) *Suppose that there exists, for every* $\varepsilon > 0$*, a* $\xi > 0$ *such that*

$$\delta'(\mathbf{g}(p,x), 1_G) < \varepsilon$$

for every periodic point $x \in X$ *with*

$$\phi^p x = x \text{ and } \delta(\mathbf{f}(p,x), 1_G) < \xi.$$

Then there exists a continuous group homomorphism $\alpha\colon G \longrightarrow G'$ *such that* g *is continuously cohomologous to* $\alpha\cdot f$*;*

(2) *If* $(G,\delta) = (G',\delta')$*, and if* $w_f(x)$ *is conjugate to* $w_g(x)$ *for every periodic point* $x \in X$ *(cf.* (4.2)*), then there exists an isometric group isomorphism* $\alpha\colon G \longrightarrow G$ *such that* $\alpha\cdot f$ *and* g *are Hölder cohomologous. Moreover, if* G *is compact, then* α *preserves conjugacy classes.*

In order to verify these theorems one can imitate directly the proofs of the corresponding results for shifts of finite type, where we have only used the properties of Δ_X (minimality and μ_θ-ergodicity) and the local product structure.

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