MULTI-DIMENSIONAL SYMBOLIC DYNAMICAL SYSTEMS

KLAUS SCHMIDT*

Abstract. The purpose of this note is to point out some of the phenomena which arise in the transition from classical shifts of finite type $X \subset A^{\mathbb{Z}}$ to *multi-dimensional* shifts of finite type $X \subset A^{\mathbb{Z}^d}$, $d \geq 2$, where A is a finite alphabet. We discuss rigidity properties of certain multi-dimensional shifts, such as the appearance of an unexpected intrinsic algebraic structure or the scarcity of isomorphisms and invariant measures. The final section concentrates on group shifts with finite or uncountable alphabets, and with the symbolic representation of such shifts in the latter case.

AMS(MOS) subject classifications. Primary: 37B15, 37B50; Secondary: 37A15, 37A60.

Key words. Multi-dimensional symbolic dynamics, Tiling systems, symbolic representation of $\mathbb{Z}^d$-actions.

1. Shifts of finite type.

Let $d \geq 1$, A a finite set (the *alphabet*), and let $A^{\mathbb{Z}^d}$ be the set of all maps $x\colon \mathbb{Z}^d \longrightarrow A$. For every nonempty subset $F \subset \mathbb{Z}^d$, the map

$$\pi_F\colon A^{\mathbb{Z}^d} \longrightarrow A^F$$

is the projection which restricts each $x \in A^{\mathbb{Z}^d}$ to F. For every $\mathbf{n} \in \mathbb{Z}^d$ we define a homeomorphism $\sigma^{\mathbf{n}}$ of the compact space $A^{\mathbb{Z}^d}$ by

$$(\sigma^{\mathbf{n}}x)_{\mathbf{m}} = x_{\mathbf{n}+\mathbf{m}} \tag{1.1}$$

for every $x = (x_{\mathbf{m}}) \in A^{\mathbb{Z}^d}$. The map $\sigma\colon \mathbf{n} \mapsto \sigma^{\mathbf{n}}$ is the *shift-action* of $\mathbb{Z}^d$ on $A^{\mathbb{Z}^d}$, and a subset $X \subset A^{\mathbb{Z}^d}$ is *shift-invariant* if $\sigma^{\mathbf{n}}(X) = X$ for all $\mathbf{n} \in \mathbb{Z}^d$. A closed, shift-invariant set $X \subset A^{\mathbb{Z}^d}$ is a *shift of finite type* (*SFT*) if there exist a finite set $F \subset \mathbb{Z}^d$ and a subset $P \subset A^F$ such that

$$X = X(F,P) = \{x \in A^{\mathbb{Z}^d} : \pi_F \circ \sigma^{\mathbf{n}}(x) \in P \ \text{ for every } \mathbf{n} \in \mathbb{Z}^d\}. \tag{1.2}$$

A closed shift-invariant subset $X \subset A^{\mathbb{Z}^d}$ is a *SFT* if and only if there exists a finite set $F \subset \mathbb{Z}^d$ such that

$$X = \{x \in A^{\mathbb{Z}^d} : \pi_F \circ \sigma^{\mathbf{n}}(x) \in \pi_F(X) \ \text{ for every } \ \mathbf{n} \in \mathbb{Z}^d\}. \tag{1.3}$$

An immediate consequence of this characterization of *SFT*'s is that the notion *SFT* is an invariant of topological conjugacy. For background and details we refer to [21]–[25].

If $X \subset A^{\mathbb{Z}^d}$ is a *SFT* we may change the alphabet A and assume that

$$F = \{0,1\}^d \ \text{ or } \ F = \{\mathbf{0}\} \cup \bigcup_{i=1}^{d} \{\mathbf{e}^{(i)}\},$$

where $\mathbf{e}^{(\mathbf{i})}$ is the i-th basis vector in $\mathbb{Z}^d$.

*Mathematics Institute, University of Vienna, and Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, A-1090 Vienna, Austria.
Email: `klaus.schmidt@univie.ac.at`

Let $X \subset A^{\mathbb{Z}^d}$ be a *SFT*. A point $x \in X$ is *periodic* if its orbit under σ is finite. In contrast to the case where $d = 1$, a higher-dimensional *SFT* X may not contain any periodic points (we give an example below). This potential absence of periodic points is associated with certain **undecidability problems** (cf. e.g. [1], [9], [19] and [32]):

(1) It is algorithmically undecidable if $X(F, P) \neq \varnothing$ for given (F, P);
(2) It is algorithmically undecidable whether an allowed[1] partial configuration can be extended to a point $x \in X(F, P)$.

In dealing with concrete *SFT*'s undecidability is not really a problem, but it indicates the difficulty of making general statements about higher-dimensional *SFT*'s. There have been several attempts to define more restrictive classes of *SFT*'s with the hope of a systematic approach within such a class (cf. e.g. [16]–[17], the algebraic systems considered in [9], or certain specification properties — such as in [13] — which guarantee 'sufficient similarity' to full shifts).

2. Some examples.

EXAMPLE 1 (Chessboards). *Let $n \geq 2$ and $A = \{0, \dots, n-1\}$. We interpret A as a set of colours and consider the* SFT $X = X^{(n)} \subset A^{\mathbb{Z}^2}$ *consisting of all configurations in which adjacent lattice points must have different colours.*

For $n = 2$, $X^{(2)}$ consists of two points. For $n \geq 3$, $X^{(n)}$ is uncountable.

There is a big difference between $n = 3$ and $n \geq 4$: for $n = 3$ there exist frozen configurations *in $X^{(3)}$, which cannot be altered in only finitely many places. These points are the periodic extensions of*

$$\begin{array}{cccccc} 0&1&2&0&1&2\\ 2&0&1&2&0&1\\ 1&2&0&1&2&0\\ 0&1&2&0&1&2\\ 2&0&1&2&0&1\\ 1&2&0&1&2&0 \end{array} \qquad\qquad \begin{array}{cccccc} 0&2&1&0&2&1\\ 1&0&2&1&0&2\\ 2&1&0&2&1&0\\ 0&2&1&0&2&1\\ 1&0&2&1&0&2\\ 2&1&0&2&1&0 \end{array}$$

EXAMPLE 2 (Wang tilings). *Let T be a finite nonempty set of distinct, closed 1×1 squares (tiles) with coloured edges such that no horizontal edge has the same colour as a vertical edge: such a set T is called a collection of Wang tiles. For each $\tau \in T$ we denote by $\mathsf{r}(\tau), \mathsf{t}(\tau), \mathsf{l}(\tau), \mathsf{b}(\tau)$ the colours of the right, top, left and bottom edges of τ, and we write $\mathcal{C}(T) = \{\mathsf{r}(\tau), \mathsf{t}(\tau), \mathsf{l}(\tau), \mathsf{b}(\tau) : \tau \in T\}$ for the set of colours occurring on the tiles in T. A Wang tiling w by T is a covering of $\mathbb{R}^2$ by translates of copies of elements of T such that*

(i) every corner of every tile in w lies in $\mathbb{Z}^2 \subset \mathbb{R}^2$,
(ii) two tiles of w are only allowed to touch along edges of the same colour, i.e. $\mathsf{r}(\tau) = \mathsf{l}(\tau')$ whenever τ, τ' are horizontally adjacent tiles with τ to the left of τ', and $\mathsf{t}(\tau) = \mathsf{b}(\tau')$ if τ, τ' are vertically adjacent with τ' above τ.

We identify each such tiling w with the point

$$w = (w_{\mathbf{n}}) \in T^{\mathbb{Z}^2},$$

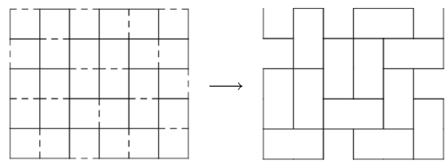
where $w_{\mathbf{n}}$ is the unique element of T whose translate covers the square $\mathbf{n} + [0,1]^2 \subset \mathbb{R}^2$, $\mathbf{n} \in \mathbb{Z}^2$. The set $W_T \subset T^{\mathbb{Z}^2}$ of all Wang tilings by T is obviously a SFT, *and is called the Wang shift of T.*

Here is an explicit example of a two-dimensional Wang shift: let T_D be the set of Wang tiles

with the colours $\mathsf{H}, \mathsf{h}, \mathsf{V}, \mathsf{v}$ on the solid horizontal, broken horizontal, solid vertical and broken vertical edges. The following picture shows a partial Wang tiling of $\mathbb{R}^2$ by T_D

[1]If $X = X(F, P)$ is a *SFT* and $\varnothing \neq E \subset \mathbb{Z}^d$, then an element $x \in A^E$ is an *allowed partial configuration* if $\pi_{(F+\mathbf{n}) \cap E}(x)$ coincides (in the obvious sense) with an element of $\pi_{F \cap (E-\mathbf{n})}(P)$ whenever $F \cap (E - \mathbf{n}) \neq \varnothing$.

and explains the name 'domino tiling' for such a tiling: two tiles meeting along an edge coloured h *or* v *form a single vertical or horizontal 'domino'.*

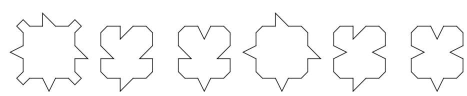


The Wang shift $W_D \subset T_D^{\mathbb{Z}^2}$ of T_D is called the domino (or dimer) shift*, and is one of the few higher dimensional* SFT*'s for which the dynamics is understood to some extent (cf. e.g. [2], [3], [7]). The shift-action σ_{W_D} of $\mathbb{Z}^2$ on W_D is topologically mixing, and its topological entropy $h(\sigma_{W_D})$ was computed by Kastelleyn in [7]:*

$$h(\sigma_{W_D}) = \frac{1}{4}\int_0^1\int_0^1 (4 - 2\cos 2\pi s - 2\cos 2\pi t)\, ds\, dt.$$

The domino-tilings again have frozen configurations which look like 'brick walls'.

Example 3 (A shift of finite type without periodic points). *Consider the following set T' of six polygonal tiles, introduced by Robinson in [19], each of which which should be thought of as a 1×1 square with various bumps and dents.*



We denote by T the set of all tiles which are obtained by allowing horizontal and vertical reflections as well as rotations of elements in T' by multiples of $\frac{\pi}{2}$. Again we consider the set $W_T \subset T^{\mathbb{Z}^2}$ consisting of all tilings of $\mathbb{R}^2$ by translates of elements of T aligned to the integer lattice (as much as their bumps and dents allow). The set W_T is obviously a SFT*, and W_T is uncountable and has no periodic points. If we allow each (or even only one) of these tiles to occur in two different colours with no restriction on adjacency of colours then we obtain a* SFT *with positive entropy, but still without periodic points.*

The paper [19] also contains an explicit set T of Wang tiles for which the extension problem is undecidable.

3. Wang tiles and shifts of finite type.

Theorem 3.1. *Every* SFT *can be represented (in many different ways) as a Wang tiling.*

Proof. Assume that $F = \{0,1\}^2 \subset \mathbb{Z}^2$. We set $T = \pi_F(X(F,P))$ and consider each

$$\tau = \boxed{\begin{smallmatrix} x_{(0,1)} & x_{(1,1)} \\ x_{(0,0)} & x_{(1,0)} \end{smallmatrix}} \in T$$

as a unit square with the 'colours' $\left[\begin{smallmatrix} x_{(0,0)} & x_{(1,0)} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} x_{(0,1)} & x_{(1,1)} \end{smallmatrix}\right]$ along its bottom and top horizontal edges, and $\left[\begin{smallmatrix} x_{(0,1)} \\ x_{(0,0)} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} x_{(1,1)} \\ x_{(1,0)} \end{smallmatrix}\right]$ along its left and right vertical edges. With this interpretation we obtain a one-to-one correspondence between the points $x = (x_{\mathbf{n}}) \in X$ and the Wang tilings $w = (w_{\mathbf{n}}) = (\pi_F \circ \sigma^{\mathbf{n}}(x)) \in T^{\mathbb{Z}^2}$. □

This correspondence allows us to regard each *SFT* as a Wang shift and vice versa. However, the correspondence is a bijection only up to topological conjugacy: if we start with a *SFT* $X \subset A^{\mathbb{Z}^2}$ with $F = \{0,1\}^2$, view it as the Wang shift $W_T \subset T^{\mathbb{Z}^2}$ with $T = \pi_F(X)$, and then interpret W_T as a *SFT* as above, we do not end up with X, but with the 2-block representation of X.

Definition 3.1. *Let A be a finite set and $X \subset A^{\mathbb{Z}^2}$ a* SFT*, T a set of Wang tiles and W_T the associated Wang shift. We say that W_T* represents *X if W_T is topologically conjugate to X. Two Wang shifts W_T and $W_{T'}$ are* equivalent *if they are topologically conjugate as* SFT*'s.*

Since any given infinite *SFT* X has many different representations by Wang shifts one may ask whether these different representations of X have anything in common. The answer to this question turns out to be related to a measure of the 'complexity' of the *SFT* X. For this we need to introduce the *tiling group* associated with a Wang shift.

Let T be a collection of Wang tiles and $W_T \subset T^{\mathbb{Z}^2}$ the Wang shift of T. Following Conway, Lagarias and Thurston ([4], [29]) we write

$$\Gamma(T) = \langle \mathcal{C}(T) | \mathsf{t}(\tau)\mathsf{l}(\tau) = \mathsf{r}(\tau)\mathsf{b}(\tau),\ \tau \in T \rangle$$

for the free group generated by the colours occurring on the edges of elements in T, together with the relations $\mathsf{t}(\tau)\mathsf{l}(\tau) = \mathsf{r}(\tau)\mathsf{b}(\tau)$, $\tau \in T$. The countable, discrete group $\Gamma(T)$ is called the *tiling group* of T (or of the Wang shift W_T). From the definition of $\Gamma(T)$ it is clear that the map $\theta \colon \Gamma(T) \to \mathbb{Z}^2$, given by

$$\begin{aligned} \theta(\mathsf{b}(\tau)) &= \theta(\mathsf{t}(\tau)) = (1,0), \\ \theta(\mathsf{l}(\tau)) &= \theta(\mathsf{r}(\tau)) = (0,1), \end{aligned}$$

for every $\tau \in T$, is a group homomorphism whose kernel is denoted by

$$\Gamma_0(T) = \ker(\theta).$$

Suppose that $E \subset \mathbb{R}^2$ is a bounded set, and that $w \in T^{\mathbb{R}^2 \smallsetminus E}$ is a Wang-tiling of $\mathbb{R}^2 \smallsetminus E$. When can we complete w to a Wang-tiling of $\mathbb{R}^2$ (possibly after enlarging E by a finite amount)? After a finite enlargement we may assume that E is the empty rectangle in the left picture of Figure 1 (the tiles covering the rest of $\mathbb{R}^2 \smallsetminus E$ are not shown).

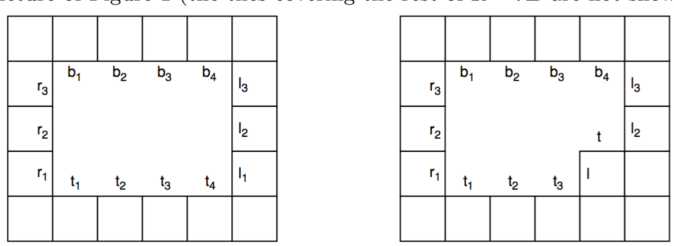


FIGURE 1

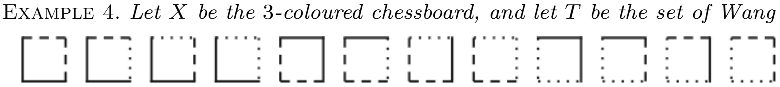
If we add a tile legally (as in the right picture), then the words in $\Gamma(T)$ obtained by reading off the colours along the edges of the two holes coincide because of the tiling relations:

$$\begin{aligned} & r_1^{-1}r_2^{-1}r_3^{-1}b_1^{-1}b_2^{-1}b_3^{-1}b_4^{-1}l_3l_2l_1t_4t_3t_2t_1 \\ &= r_1^{-1}r_2^{-1}r_3^{-1}b_1^{-1}b_2^{-1}b_3^{-1}b_4^{-1}l_3l_2tlt_3t_2t_1 \end{aligned} \tag{3.1}$$

In particular, if the hole can be closed, then the word must be the identity.

If $X \subset A^{\mathbb{Z}^2}$ is a *SFT* and W_T a Wang representation of X then the tiling group $\Gamma(T)$ gives an obstruction to the *weak closing* of bounded holes (i.e. the closing of holes *after finite enlargement*) for points $x \in A^{\mathbb{Z}^2} \smallsetminus E$, where $E \subset \mathbb{Z}^2$ is a finite set. However, different Wang-representations of X may give different answers.

EXAMPLE 4. *Let X be the 3-coloured chessboard, and let T be the set of Wang tiles*



with the colours

$$\mathsf{h}_0 = \quad , \quad \mathsf{h}_1 = \quad , \quad \mathsf{h}_2 = \quad ,$$

$$\mathsf{v}_0 = \quad , \quad \mathsf{v}_1 = \quad , \quad \mathsf{v}_2 = \quad$$

on the horizontal and vertical edges. Then W_T represents X. The tiling group $\Gamma(T)$ is of the form

$$\Gamma(T'_C) = \{\mathsf{h}_i, \mathsf{v}_i,\ i = 0,1,2 | \mathsf{v}_1\mathsf{h}_0 = \mathsf{v}_2\mathsf{h}_0 = \mathsf{h}_1\mathsf{v}_0 = \mathsf{h}_2\mathsf{v}_0,$$

$$\mathsf{v}_2\mathsf{h}_1 = \mathsf{v}_0\mathsf{h}_1 = \mathsf{h}_2\mathsf{v}_1 = \mathsf{h}_0\mathsf{v}_1, \;\; \mathsf{v}_0\mathsf{h}_2 = \mathsf{v}_1\mathsf{h}_2 = \mathsf{h}_0\mathsf{v}_2 = \mathsf{h}_1\mathsf{v}_2\}.$$

Since $\mathsf{h}_0 = \mathsf{h}_1 = \mathsf{h}_2$, $\mathsf{v}_0 = \mathsf{v}_1 = \mathsf{v}_2$ *and* $\mathsf{h}_0\mathsf{v}_0 = \mathsf{v}_0\mathsf{h}_0$, $\Gamma(T) \cong \mathbb{Z}^2$, *and every hole appears closable.*

With a different representation of X *as a Wang shift we obtain more information. Let* T' *be the set of Wang tiles*

$$\boxed{\begin{smallmatrix}1&0\\0&1\end{smallmatrix}}\ \boxed{\begin{smallmatrix}1&2\\0&1\end{smallmatrix}}\ \boxed{\begin{smallmatrix}2&0\\0&1\end{smallmatrix}}\ \boxed{\begin{smallmatrix}1&0\\0&2\end{smallmatrix}}\ \boxed{\begin{smallmatrix}2&0\\0&2\end{smallmatrix}}\ \boxed{\begin{smallmatrix}2&1\\0&2\end{smallmatrix}}\ \boxed{\begin{smallmatrix}0&1\\1&0\end{smallmatrix}}\ \boxed{\begin{smallmatrix}0&2\\1&0\end{smallmatrix}}\ \boxed{\begin{smallmatrix}2&1\\1&0\end{smallmatrix}}$$
$$\boxed{\begin{smallmatrix}0&1\\1&2\end{smallmatrix}}\ \boxed{\begin{smallmatrix}2&0\\1&2\end{smallmatrix}}\ \boxed{\begin{smallmatrix}2&1\\1&2\end{smallmatrix}}\ \boxed{\begin{smallmatrix}0&1\\2&0\end{smallmatrix}}\ \boxed{\begin{smallmatrix}0&2\\2&0\end{smallmatrix}}\ \boxed{\begin{smallmatrix}1&2\\2&0\end{smallmatrix}}\ \boxed{\begin{smallmatrix}0&2\\2&1\end{smallmatrix}}\ \boxed{\begin{smallmatrix}1&0\\2&1\end{smallmatrix}}\ \boxed{\begin{smallmatrix}1&2\\2&1\end{smallmatrix}}$$

with the colours $\mathsf{h}_{ij} = [\,i\ j\,]$ *on the horizontal and* $\mathsf{v}_i^j = \left[\begin{smallmatrix}j\\i\end{smallmatrix}\right]$ *on the vertical edges, where* $i, j \in \{0,1,2\}$ *and* $i \neq j$. *Then* $W_{T'}$ *represents* X.

There exists a group homomorphism $\phi\colon \Gamma(T') \longrightarrow \mathbb{Z}$ *with*

$$\begin{aligned}\phi(\mathsf{h}_{01}) &= \phi(\mathsf{h}_{12}) = \phi(\mathsf{h}_{20}) = \phi(\mathsf{v}_0^1) = \phi(\mathsf{v}_1^2) = \phi(\mathsf{v}_2^0) = 1,\\ \phi(\mathsf{h}_{10}) &= \phi(\mathsf{h}_{21}) = \phi(\mathsf{h}_{02}) = \phi(\mathsf{v}_1^0) = \phi(\mathsf{v}_2^1) = \phi(\mathsf{v}_0^2) = -1.\end{aligned}$$

This homomorphism detects that the hole with the edge

$$\begin{matrix}1&2&1\\2&?&0\\0&1&2\end{matrix}$$

cannot be closed, no matter how it is extended on the outside, and how much it is enlarged initially.

This example raises the alarming possibility that more and more complicated Wang-representations of a *SFT* X will give more and more combinatorial information about X. Remarkably, this is not the case.

Theorem 3.2. *For many topologically mixing* $\mathbb{Z}^2$-SFT*'s there exists a Wang-representation* W_T *of* X *which contains all the combinatorial information obtainable from* all possible *Wang-representations of* X.

For examples we refer to [25] and [5]. In order to make this statement comprehensible one has to express it in terms of the *continuous cohomology of* X.

4. Wang tiles and cohomology.

Let $X \subset A^{\mathbb{Z}^2}$ be a *SFT* and G a discrete group with identity element 1_G. A map $c\colon \mathbb{Z}^2 \times X \longrightarrow G$ is a *cocycle* for the shift-action σ of $\mathbb{Z}^2$ on X if $c(\mathbf{n}, \cdot)\colon X \longrightarrow G$ is continuous for every $\mathbf{n} \in \mathbb{Z}^2$ and

$$c(\mathbf{m}+\mathbf{n}, x) = c(\mathbf{m}, \sigma^{\mathbf{n}}x)c(\mathbf{n}, x)$$

for all $x \in X$ and $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^2$. One can interpret this equation as *path-independence.*

A cocycle $c\colon \mathbb{Z}^2 \times X \longrightarrow G$ is a *homomorphism* if $c(\mathbf{n}, \cdot)$ is constant for every $\mathbf{n} \in \mathbb{Z}^2$, and c is a *coboundary* if there exists a continuous map $b\colon X \longrightarrow G$ such that

$$c(\mathbf{n}, x) = b(\sigma^{\mathbf{n}}x)^{-1}b(x)$$

for all $x \in X$ and $\mathbf{n} \in \mathbb{Z}^2$. Two cocycles $c, c'\colon \mathbb{Z}^2 \times X \longrightarrow G$ are *cohomologous* with continuous *transfer function* $b\colon X \longrightarrow G$, if

$$c(\mathbf{n}, x) = b(\sigma^{\mathbf{n}}x)^{-1}c'(\mathbf{n}, x)b(x)$$

for all $\mathbf{n} \in \mathbb{Z}^2$ and $x \in X$.

For every Wang representation W_T of X we define a *tiling cocycle* $c_T\colon \mathbb{Z}^2 \times W_T \longrightarrow \Gamma(T)$ (and hence a cocycle $c_T'\colon \mathbb{Z}^2 \times X \longrightarrow \Gamma(T)$) by setting

$$c_T((1,0), w) = \mathsf{b}(w_{\mathbf{0}}), \;\; c_T((0,1), w) = \mathsf{l}(w_{\mathbf{0}})$$

for every Wang tiling $w \in W_T \subset T^{\mathbb{Z}^2}$, and by using the cocycle equation to extend c_T to a map $\mathbb{Z}^2 \times W_T \longrightarrow \Gamma(T)$ (the relations $\mathsf{t}(\tau)\mathsf{l}(\tau) = \mathsf{r}(\tau)\mathsf{b}(\tau)$, $\tau \in T$, in the tiling group are precisely what is needed to allow such an extension). Conversely, if G is a discrete group and $c\colon \mathbb{Z}^2 \times X \longrightarrow G$ a cocycle, then Theorem 4.2 in [25] shows that there exists a Wang representation w_T of X and a group homomorphism $\eta\colon \Gamma(T) \longrightarrow G$ such that

$$c = \eta \circ c_T. \tag{4.1}$$

In order to establish a link between cocycles and the 'closing of holes' discussed in the last section we return for a moment to the Wang tiles in Figure 1 and assume that the partial configuration $w_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^2 \smallsetminus E$, shown there extends to an element $w \in W_T$ with $\mathsf{l}(w_{\mathbf{0}}) = r_1$ and $\mathsf{b}(w_{\mathbf{0}}) = t_1$ (i.e. the tile $w_{\mathbf{0}}$ occupies the bottom left hand corner of the 'hole' E in Figure 1). Then

$$c_T((4,3),w) = l_3l_2l_1t_4t_3t_2t_3 = l_3l_2tlt_3t_2t_1 = b_4b_3b_2b_1r_3r_2r_1,$$

depending on the route chosen from $\mathbf{0}$ to $(4,3)$, which is equivalent to (3.1).

If $W_{T'}$ is another Wang representation of X, then there exists a topological conjugacy $\phi\colon W_T \longrightarrow W_{T'}$, and the coordinates $w_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^2 \smallsetminus E$, determine the coordinates $\phi(w)_{\mathbf{m}}$, $\mathbf{m} \in \mathbb{Z}^2 \smallsetminus E'$, for some finite set $E' \subset \mathbb{Z}^2$ which we may again assume to be a rectangle. If the tiling cocycle $c_{T'}$ of $W_{T'}$ is a homomorphic image of w_T in the sense of (4.1), then $w_{T'}$ cannot lead to any new obstructions (other than those already exhibited by c_T). A slightly more refined version of the same argument shows that $w_{T'}$ will not lead to any new obstructions even if it is only cohomologous to a homomorphic image of c_T. This observation is the motivation for the following definition.

Definition 4.1. *A cocycle $c^*\colon \mathbb{Z}^2 \times X \longrightarrow G^*$ with values in a discrete group G^* is* fundamental *if the following is true: for every discrete group G and every cocycle $c\colon \mathbb{Z}^2 \times X \longrightarrow G$ there exists a group homomorphism $\theta\colon G^* \longrightarrow G$ such that c is cohomologous to the cocycle $\theta \circ c^*\colon \mathbb{Z}^2 \times X \longrightarrow G$.*

In this terminology we can state a more precise (but still rather vague) form of Theorem 3.2 (cf. [25]).

Theorem 4.1. *In certain examples of topologically mixing $\mathbb{Z}^2$-SFT's there exists an explicitly computable Wang representation W_T of X whose tiling cocycle $c_T'\colon \mathbb{Z}^2 \times X \longrightarrow \Gamma(T)$ is fundamental.*

For a list of examples (which includes the chessboards in Example 4 and the domino-tilings in Example 2) we refer to [5] and [24]–[25].

Although one can make analogous definitions for classical (one-dimensional) *SFT*'s, they never have fundamental cocycles. The existence of fundamental cocycles is a *rigidity phenomenon* specific to multi-dimensional *SFT*'s.

5. Group shifts and their symbolic representations.

In this section we leave the general setting of multi-dimensional shifts of finite type with all its inherent problems and restrict our attention to *SFT*'s with a group structure. This class of *SFT*'s is of interest in coding theory and allows much more detailed statements about conjugacy and dynamical properties than arbitrary *SFT*'s.

Let $d \geq 1$, and let X be a compact abelian group with normalized Haar measure λ_X. A $\mathbb{Z}^d$-action $\alpha\colon \mathbf{n} \mapsto \alpha^{\mathbf{n}}$ by continuous automorphisms of X is called an *algebraic* $\mathbb{Z}^d$-action on X. An algebraic $\mathbb{Z}^d$-action α on X is *expansive* if there exists an open neighbourhood $\mathcal{U}$ of the identity 0_X in X with $\bigcap_{\mathbf{n}\in\mathbb{Z}^d} \alpha^{-\mathbf{n}}(\mathcal{U}) = \{0_X\}$.

Suppose that α is an algebraic $\mathbb{Z}^d$-action on a compact abelian group X. An α-invariant probability measure μ on the Borel field $\mathcal{B}_X$ of X is *ergodic* if

$$\mu\Bigl(\bigcup_{\mathbf{n}\in\mathbb{Z}^d} \alpha^{-\mathbf{n}}(B)\Bigr) \in \{0,1\}$$

for every $B \in \mathcal{B}_X$, and *mixing* if

$$\lim_{\mathbf{n}\to\infty} \mu(B \cap \alpha^{-\mathbf{n}}(B')) = \mu(B)\mu(B')$$

for all $B, B' \in \mathcal{B}_X$. The action α is *ergodic* or *mixing* if λ_X is ergodic or mixing.

Let α_1, α_2 be algebraic $\mathbb{Z}^d$-actions on compact abelian groups X_1 and X_2, respectively. A Borel bijection $\phi\colon X_1 \longrightarrow X_2$ is a *measurable conjugacy* of α_1 and α_2 if

$$\lambda_{X_1}\phi^{-1} = \lambda_{X_2}$$

and

$$\phi \circ \alpha_1^{\mathbf{n}}(x) = \alpha_2^{\mathbf{n}} \circ \phi(x) \tag{5.1}$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and λ_{X_1}-a.e. $x \in X_1$.

A continuous group isomorphism $\phi\colon X_1 \longrightarrow X_2$ is an *algebraic conjugacy* of α_1 and α_2 if it satisfies (5.1) for every $\mathbf{n} \in \mathbb{Z}^d$ and $x \in X_1$.

The actions α_1, α_2 are *measurably* (resp. *algebraically*) *conjugate* if there exists a measurable (resp. algebraic) conjugacy between them.

Finally we call a map $\phi\colon X_1 \longrightarrow X_2$ *affine* if there exist a continuous group isomorphism $\psi\colon X_1 \longrightarrow X_2$ and an element $x' \in X_2$ such that

$$\phi(x) = \psi(x) + x'$$

for every $x \in X_1$.

Here we are interested in algebraic $\mathbb{Z}^d$-actions of a particularly simple form. Let A be a compact abelian group, and let $\Omega_A^{(d)} = A^{\mathbb{Z}^d}$ be the compact abelian group consisting of all maps $\omega\colon \mathbb{Z}^d \longrightarrow A$, furnished with the product topology and coordinate-wise addition. We write every $\omega \in \Omega_A^{(d)}$ as $\omega = (\omega_{\mathbf{n}})$ with $\omega_{\mathbf{n}} \in A$ for every $\mathbf{n} \in \mathbb{Z}^d$ and define the *shift-action* σ of $\mathbb{Z}^d$ on $\Omega_A^{(d)}$ by (1.1). Clearly, σ is an algebraic $\mathbb{Z}^d$-action on $\Omega_A^{(d)}$. A *group shift* is the restriction of the shift-action σ to a closed, shift-invariant subgroup $X \subset \Omega_A^{(d)}$.

Throughout the following discussion we shall assume that the 'alphabet' A is either finite or $A = \mathbb{T}$. In the former case every group shift $X \subset \Omega_A^{(d)}$ is automatically a d-dimensional shift of finite type (cf. [9]–[10] and [23]). In our earlier discussion of SFT's we were interested in topological conjugacy invariants. Here we are interested in the connection between measurable and algebraic conjugacy.

Example 5. *The shift automorphisms*

$$(\sigma x)_n = x_{n+1}$$

on the compact abelian groups

$$X = (\mathbb{Z}/4\mathbb{Z})^{\mathbb{Z}},$$
$$Y = ((\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}))^{\mathbb{Z}}.$$

are measurably (even topologically) conjugate, but the groups X and Y are not algebraically isomorphic.

Example 6. *For every nonempty finite set $E \subset \mathbb{Z}^d$ we denote by $X_E \subset \bar{X} = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^d}$ the closed shift-invariant subgroup consisting of all $x \in \bar{X}$ whose coordinates sum to 0 in every translate of E in $\mathbb{Z}^d$. If E has at least two points then X_E is uncountable and the restriction σ_E of σ to X_E is an expansive algebraic $\mathbb{Z}^d$-action.*

For $d = 2$ and the subset

$$E = \{(0,0),(1,0),(0,1)\} \subset \mathbb{Z}^2,$$

the $\mathbb{Z}^2$-action σ_E on X_E is called Ledrappier's example*: σ_E is mixing and expansive, but not mixing of order* 3 *(for every $n \geq 0$, $x_{(0,0)} + x_{(2^n,0)} + x_{(0,2^n)} = 0$).*

In this example, 3-mixing breaks down in a particularly regular way: if we call a finite subset $S \subset \mathbb{Z}^d$ mixing *for a group shift X if*

$$\lim_{k\to\infty} \lambda_X\Bigl(\bigcap_{\mathbf{m}\in S} \sigma^{-k\mathbf{m}} B_{\mathbf{m}}\Bigr) = \prod_{\mathbf{m}\in S} \lambda_X(B_{\mathbf{m}}) \tag{5.2}$$

for all Borel sets $B_{\mathbf{m}}$, $\mathbf{m} \in S$, and nonmixing *otherwise, then the last paragraph shows that $S = \{(0,0),(1,0),(0,1)\}$ is nonmixing for Ledrappier's example.*

We also consider the subsets

$$\begin{aligned} E_1 =&\{(0,0),(1,0),(2,0),(1,1),(0,2)\},\\ E_2 =&\{(0,0),(2,0),(0,1),(1,1),(0,2)\},\\ E_3 =&\{(0,0),(1,0),(2,0),(0,1),(1,1),(0,2)\}. \end{aligned}$$

of $\mathbb{Z}^2$*. The shift-actions* $\sigma_i = \sigma_{E_i}$ *of* $\mathbb{Z}^2$ *on* $X_i = X_{E_i}$ *are again mixing, but the set* $S = \{(0,0),(1,0),(0,1)\}$ *is nonmixing for each of these actions (cf. [11]).*

For every $\mathbf{n} \in \mathbb{Z}^2$*, the automorphisms* $\sigma_i^{\mathbf{n}}$ *are measurably conjugate. However, as was shown in [12], these three* $\mathbb{Z}^2$*-actions are not even measurably conjugate.*

In general, if $X \subset A^{\mathbb{Z}^2}$ *is a group shift with finite alphabet* A*, then* X *has nonmixing sets if and only if it does not have completely positive entropy or, equivalently, if and only if it is not measurably conjugate to a full shift* $Y = B^{\mathbb{Z}^2}$*, where* B *is a finite set (cf. [15], [11] and [20]). However, even if* X *has completely positive entropy, it need not be topologically conjugate to a full shift.*

For $d = 1$, algebraic conjugacy of group shifts $X \subset A^{\mathbb{Z}^d}$ to full shifts $Y = B^{\mathbb{Z}^d}$, where A and B are finite abelian groups, is a matter of considerable interest in coding theory (cf. e.g. [18]), and results for $d > 1$ are just beginning to emerge.

Example 6 is based on a special case of another rigidity phenomenon specific to $\mathbb{Z}^d$-actions with $d > 1$. We call an algebraic $\mathbb{Z}^d$-action α on a compact abelian group *irreducible* if every closed, α-invariant subgroup $Y \subsetneq X$ is finite. The following statement is proved in [8] and [12].

THEOREM 5.1. *Let* $d > 1$*, and let* α_1 *and* α_2 *be mixing algebraic* $\mathbb{Z}^d$*-actions on compact abelian groups* X_1 *and* X_2*, respectively. If* α_1 *is irreducible, and if* $\phi\colon X_1 \longrightarrow X_2$ *is a measurable conjugacy of* α_1 *and* α_2*, then* α_2 *is irreducible and* ϕ *is* λ_{X_1}-a.e. *equal to an affine map. Hence measurable conjugacy of* α_1 *and* α_2 *implies algebraic conjugacy.*

Irreducibility of algebraic $\mathbb{Z}^d$-actions with $d > 1$ implies that these actions have zero entropy (as $\mathbb{Z}^d$-actions). For actions with positive entropy one cannot expect this kind of isomorphism rigidity, since positive entropy implies the existence of nontrivial Bernoulli factors (cf. [23]). However, it is sometimes still be possible to apply Theorem 5.1 to prove measurable nonconjugacy of actions with positive entropy.

EXAMPLE 7 (Conjugacy of $\mathbb{Z}^2$-actions with positive entropy). *We modify Example 6 by setting* $\bar{Y} = (\mathbb{Z}/4\mathbb{Z})^{\mathbb{Z}^d}$ *and consider, for every nonempty finite set* $E \subset \mathbb{Z}^d$ *the closed shift-invariant subgroup* $Y_E \subset \bar{Y}$ *consisting of all* $y \in \bar{Y}$ *whose coordinates sum to* $0 \pmod 2$ *in every translate of* E *in* $\mathbb{Z}^d$*. The group* Y_E *is always uncountable, and the restriction* τ_E *of the shift-action* σ *to* Y_E *is an expansive algebraic* $\mathbb{Z}^d$*-action with entropy* $\log 2$*. As in Example 6 we set* d = 2 *and consider the the subsets* $E, E_1, E_2, E_3 \subset \mathbb{Z}^2$ *defined there. Theorem 6.5 in [15] implies that the Pinsker algebra* $\pi(\tau_{E_i})$ *of* τ_{E_i} *is the sigma-algebra* $\mathcal{B}_{Y_{E_i}/Z_{E_i}}$ *of* Z_{E_i}*-invariant Borel sets in* Y_{E_i}*, where*

$$Z_{E_i} = \{x = (x_{\mathbf{n}}) \in Y_{E_i} : x_{\mathbf{n}} = 0 \pmod 2 \text{ for every } \mathbf{n} \in \mathbb{Z}^2\}.$$

Then the $\mathbb{Z}^2$*-action* τ'_{E_i} *induced by* τ_{E_i} *on* Y_{E_i}/Z_{E_i} *is algebraically conjugate to the shift-action* σ_{E_i} *on the group* X_{E_i} *in Example 6.*

Since any measurable conjugacy of τ_{E_i} *and* τ_{E_j} *would map* $\pi(\tau_{E_i})$ *to* $\pi(\tau_{E_j})$ *and induce a conjugacy of* τ'_{E_i} *and* τ'_{E_j} *and hence of* σ_{E_i} *and* σ_{E_j}*, Example 6 implies that* τ_i *and* τ_j *are measurably nonconjugate for* $1 \le i < j \le 3$*.*

EXAMPLE 8 (Group shifts with uncountable alphabet). *We write* $\mathfrak{R}_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ *for the ring of Laurent polynomials with integral coefficients in the commuting variables* $u_1, \dots, u_d$*, and represent every* $f \in \mathfrak{R}_d$ *as* $f = \sum_{\mathbf{m} \in \mathbb{Z}^d} f_{\mathbf{m}} u^{\mathbf{m}}$ *with* $u^{\mathbf{m}} = u_1^{m_1} \cdots u_d^{m_d}$ *and* $f_{\mathbf{m}} \in \mathbb{Z}$ *for every* $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$*.*

Let σ *be the shift-action* (1.1) *of* $\mathbb{Z}^d$ *on* $\Omega^{(d)} = \mathbb{T}^{\mathbb{Z}^d}$*. For every nonzero* $f \in \mathfrak{R}_d$ *and* $x \in X$ *we set*

$$f(\sigma)(x) = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} \sigma^{\mathbf{n}} x \tag{5.3}$$

and note that $f(\sigma)\colon \Omega^{(d)} \longrightarrow \Omega^{(d)}$ *is a continuous surjective group homomorphism. For every ideal* $I \subset \mathfrak{R}_d$ *we set*

$$X_I = \bigcap_{f \in I} \ker(f(\sigma)) \tag{5.4}$$

and denote by σ_I *the restriction of* σ *to* X_I. *If* $\{g_1, \dots, g_L\}$ *is a set of generators of* I *(such a finite set of generators always exists, since* $\mathfrak{R}_d$ *is Noetherian), then*

$$X_I = \bigcap_{j=1}^{m} \ker(g_j(\sigma)).$$

The dynamical properties of group shifts of the form X_I *are described in [22], [15] and [23]. In the special case where the ideal* I *is principal, i.e. where* $I = (f) = f\mathfrak{R}_d$ *for some* $f \in \mathfrak{R}_d$, *the entropy of* $\sigma_{(f)}$ *is given by*

$$h(\sigma_{(f)}) = \begin{cases} \int_0^1 \cdots \int_0^1 \log |f(e^{2\pi i t_1}, \dots, e^{2\pi i t_d})|\, dt_1 \cdots dt_d & \text{if } f \neq 0, \\ \infty & \text{otherwise.} \end{cases}$$

Furthermore, $\sigma_{(f)}$ *is expansive if and only if*

$$f(\mathbf{z}) \neq 0 \ \text{ for all } \ \mathbf{z} = (z_1, \dots, z_d) \in \mathbb{C}^d \ \text{ with } \ |z_1| = \cdots = |z_d| = 1. \tag{5.5}$$

If $\sigma_{(f)}$ *is expansive then it is automatically mixing and Bernoulli (in particular, it has finite and positive entropy).*

Although the group shifts $\sigma_{(f)}$, $f \in \mathfrak{R}_d$, in Example 8 are *of finite type* in the sense that they are determined by restrictions in a finite 'window' of coordinates (consisting of those $\mathbf{n} \in \mathbb{Z}^d$ with $f_{\mathbf{n}} \neq 0$), their uncountable alphabets put them outside the customary framework of symbolic dynamics. In view of this (and for a variety of other reasons) it seems desirable to find 'symbolic' representations of such systems, analogous to the representation of hyperbolic toral automorphisms as *SFT*'s by means of Markov partitions.

Following [6] we consider the Banach space $\ell^\infty(\mathbb{Z}^d, \mathbb{R})$ and write

$$\ell^\infty(\mathbb{Z}^d, \mathbb{Z}) \subset \ell^\infty(\mathbb{Z}^d, \mathbb{R})$$

for the subgroup of bounded integer-valued functions. Consider the surjective map $\eta\colon \ell^\infty(\mathbb{Z}^d, \mathbb{R}) \longrightarrow \mathbb{T}^{\mathbb{Z}^d}$ given by

$$\eta(v)_{\mathbf{n}} = v_{\mathbf{n}} \pmod 1$$

for every $v = (v_{\mathbf{n}}) \in \ell^\infty(\mathbb{Z}^d, \mathbb{R})$ and $\mathbf{n} \in \mathbb{Z}^d$. Let $\bar{\sigma}$ be the shift-action of $\mathbb{Z}^d$ on $\ell^\infty(\mathbb{Z}^d, \mathbb{R})$, defined as in (1.1), and set, for every $h = \sum_{\mathbf{n} \in \mathbb{Z}^d} h_{\mathbf{n}} u^{\mathbf{n}} \in \mathfrak{R}_d$ and $v \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$,

$$h(\bar{\sigma})(v) = \sum_{\mathbf{n} \in \mathbb{Z}^d} h_{\mathbf{n}} \bar{\sigma}^{\mathbf{n}} v.$$

The expansiveness of $\sigma_{(f)}$ can be expressed in terms of the kernel of $f(\bar{\sigma})$: $\sigma_{(f)}$ is expansive if and only if $\ker(f(\bar{\sigma})) = \{0\} \subset \ell^\infty(\mathbb{Z}^d, \mathbb{R})$.

According to Lemma 4.5 in [14] there exists a unique element $w^\Delta \in \ell^\infty(\mathbb{Z}^d, \mathbb{R})$ with the property that

$$f(\bar{\sigma})(w^\Delta)_{\mathbf{n}} = \begin{cases} 1 & \text{if } \mathbf{n} = \mathbf{0}, \\ 0 & \text{otherwise.} \end{cases}$$

The point w^Δ also has the property that there exist constants $c_1 > 0, 0 < c_2 < 1$ with

$$|w^\Delta_{\mathbf{n}}| \leq c_1 c_2^{\|\mathbf{n}\|}$$

for every $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$, where $\|\mathbf{n}\| = \max_{i=1,\dots,d} |n_i|$. From the properties of w^Δ it is clear that

$$\bar{\xi}(v) = \sum_{\mathbf{n} \in \mathbb{Z}^d} v_{\mathbf{n}} \bar{\sigma}^{-\mathbf{n}} w^\Delta$$

is a well-defined element of $\ell^\infty(\mathbb{Z}^d, \mathbb{R})$ for every $v \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$, and we set

$$\xi = \eta \circ \bar{\xi} \colon \ell^\infty(\mathbb{Z}^d, \mathbb{Z}) \longrightarrow X_{(f)}.$$

The map $\xi \colon \ell^\infty(\mathbb{Z}^d, \mathbb{Z}) \longrightarrow X_{(f)}$ is a surjective group homomorphism, and

$$\xi \circ \bar{\sigma}^{\mathbf{n}} = \sigma_{(f)}^{\mathbf{n}} \circ \xi \ \text{ for every } \ \mathbf{n} \in \mathbb{Z}^d,$$
$$\ker(\xi) = f(\bar{\sigma})(\ell^\infty(\mathbb{Z}^d, \mathbb{Z})).$$

The point $x^\Delta = \xi(w^\Delta)$ is *homoclinic*:

$$\lim_{\mathbf{n} \to \infty} \sigma_{(f)}^{\mathbf{n}}(x^\Delta) = 0. \tag{5.6}$$

Furthermore, x^Δ is a *fundamental homoclinic point* in the sense that every homoclinic point of $\sigma_{(f)}$ (i.e. every $x \in X_{(f)}$ satisfying (5.6) with x replacing x^Δ) lies in the countable subgroup of $X_{(f)}$ generated by $\{\sigma_{(f)}^{\mathbf{n}} x^\Delta : \mathbf{n} \in \mathbb{Z}^d\}$. It can be shown that an expansive algebraic $\mathbb{Z}^d$-action α on a compact abelian group X has a fundamental homoclinic point if and only if it is of the form $\alpha = \sigma_{(f)}$, $X = X_{(f)}$, for some $f \in \mathfrak{R}_d$ satisfying (5.5) (cf. [26]).

From the definition of ξ it is clear that its restriction to every bounded subset of $\ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ is continuous in the weak*-topology. One can easily find bounded (and thus weak*-compact) subset $\mathcal{V} \subset \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ with $\xi(\mathcal{V}) = X_{(f)}$:

PROPOSITION 5.1. *For every $h = \sum_{\mathbf{n} \in \mathbb{Z}^d} h_{\mathbf{n}} u^{\mathbf{n}} \in \mathfrak{R}_d$ we set*

$$h^+ = \sum_{\mathbf{n} \in \mathbb{Z}^d} \max(0, h_{\mathbf{n}}) u^{\mathbf{n}}, \qquad h^- = -\sum_{\mathbf{n} \in \mathbb{Z}^d} \min(0, h_{\mathbf{n}}) u^{\mathbf{n}},$$
$$\|h^+\|_1' = \max(\|h^+\|_1 - 1, 0), \qquad \|h^-\|_1' = \max(\|h^-\|_1 - 1, 0),$$
$$\|h\|_1^* = \|h^+\|_1' + \|h^-\|_1'.$$

Then the set

$$\mathcal{V} = \{v \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z}) : 0 \le v_{\mathbf{n}} \le \|f\|_1^* \ \text{ for every } \ \mathbf{n} \in \mathbb{Z}^d\}$$

satisfies that $\xi(\mathcal{V}) = X_{(f)}$.

The restriction of the homomorphism ξ to $\mathcal{V}$ is surjective, but generally not injective, and the key problem in constructing symbolic representations of the $\mathbb{Z}^d$-action $\sigma_{(f)}$ is to find closed, shift-invariant subsets $\mathcal{W} \subset \mathcal{V}$ with the following properties:

(a) $\mathcal{W}$ is a *SFT* or at least *sofic*, i.e. a topological factor of a *SFT*,

(b) $\xi(\mathcal{W}) = X_{(f)}$, and the restriction of ξ to a dense G_δ-set in $\mathcal{W}$ is injective.

Examples of such choices of $\mathcal{W} \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ for appropriate polynomials $f \in \mathfrak{R}_1$ can be found in [26], [27], [28], [30] and [31]. Examples in higher dimensions (with $d \ge 2$) are much more difficult to find, and there are many unresolved problems in this area. We end this section with one of the few successful examples.

EXAMPLE 9 ([6]). *Let $d = 2$ and $f = 3 - u_1 - u_2 \in \mathfrak{R}_2$. Then $\|f\|_1^* = 3$, but the set*

$$\mathcal{W} = \{v \in \ell^\infty(\mathbb{Z}^2, \mathbb{Z}) : 0 \le v_{\mathbf{n}} \le 2\} \subsetneq \mathcal{V} \tag{5.7}$$

also satisfies that $\xi(\mathcal{W}) = X_{(f)}$. Furthermore, the restriction of ξ to $\mathcal{W}$ is almost injective in the sense of Condition (b).

REFERENCES

[1] R. Berger, *The undecidability of the Domino Problem*, Mem. Amer. Math. Soc. **66** (1966).

[2] R. Burton and R. Pemantle, *Local characteristics, entropy and limit theorems for spanning trees and domino tilings via transfer-impedances*, Ann. Probab. **21** (1993), 1329–1371.

[3] C. Cohn, N. Elkies and J. Propp, *Local statistics for random domino tilings of the Aztec diamond*, Duke Math. J. **85** (1996), 117–166.

[4] J.H. Conway and J.C. Lagarias, *Tilings with polyominoes and combinatorial group theory*, J. Combin. Theory Ser. A **53** (1990), 183–208.

[5] M. Einsiedler, *Fundamental cocycles of tiling spaces*, Ergod. Th. & Dynam. Sys. (to appear).

[6] M. Einsiedler and K. Schmidt, *Markov partitions and homoclinic points of algebraic $\mathbb{Z}^d$-actions*, Proc. Steklov Inst. Math. **216** (1997), 259–279.

[7] P.W. Kasteleyn, *The statistics of dimers on a lattice. I*, Phys. D **27** (1961), 1209–1225.

[8] A. Katok, S. Katok and K. Schmidt, *Rigidity of measurable structure for algebraic actions of higher-rank abelian groups*, in preparation.

[9] B. Kitchens and K. Schmidt, *Periodic points, decidability and Markov subgroups*, in: Dynamical Systems, Proceeding of the Special Year, Lecture Notes in Mathematics, vol. 1342, Springer Verlag, Berlin-Heidelberg-New York, 1988, 440–454.

[10] B. Kitchens and K. Schmidt, *Automorphisms of compact groups*, Ergod. Th. & Dynam. Sys. **9** (1989), 691–735.

[11] B. Kitchens and K. Schmidt, *Mixing sets and relative entropies for higher dimensional Markov shifts*, Ergod. Th. & Dynam. Sys. **13** (1993), 705–735.

[12] B. Kitchens and K. Schmidt, *Isomorphism rigidity of irreducible algebraic $\mathbb{Z}^d$-actions*, Preprint (1999).

[13] S. Lightwood, *An aperiodic embedding theorem for square filling subshifts of finite type*, Preprint (1999).

[14] D. Lind and K. Schmidt, *Homoclinic points of algebraic $\mathbb{Z}^d$-actions*, J. Amer. Math. Soc. **12** (1999), 953–980.

[15] D. Lind, K. Schmidt and T. Ward, *Mahler measure and entropy for commuting automorphisms of compact groups*, Invent. Math. **101** (1990), 593–629.

[16] N.G. Markley and M.E. Paul, *Matrix subshifts for $\mathbb{Z}^n$ symbolic dynamics*, Proc. London Math. Soc. **43** (1981), 251–272.

[17] ———, *Maximal measures and entropy for $\mathbb{Z}^n$ subshifts of finite type*, Preprint (1979).

[18] R.J. McEliece, *The algebraic theory of convolutional codes*, in: Handbook of Coding Theory (2 vols.), North Holland, Amsterdam, 1998, 1065–1138.

[19] R.M. Robinson, *Undecidability and nonperiodicity for tilings of the plane*, Invent. Math. **12** (1971), 177–209.

[20] D.J. Rudolph and K. Schmidt, *Almost block independence and Bernoullicity of $\mathbb{Z}^d$-actions by automorphisms of compact groups*, Invent. Math. **120** (1995), 455–488.

[21] K. Schmidt, *Algebraic ideas in ergodic theory*, in: CBMS Lecture Notes, vol. 76, American Mathematical Society, Providence, R.I., 1990.

[22] K. Schmidt, *Automorphisms of compact abelian groups and affine varieties*, Proc. London Math. Soc. **61** (1990), 480–496.

[23] K. Schmidt, *Dynamical systems of algebraic origin*, Birkhäuser Verlag, Basel-Berlin-Boston, 1995.

[24] K. Schmidt, *The cohomology of higher-dimensional shifts of finite type*, Pacific J. Math. **170** (1995), 237–270.

[25] K. Schmidt, *Tilings, fundamental cocycles and fundamental groups of symbolic $\mathbb{Z}^d$-actions*, Ergod. Th. & Dynam. Sys. **18** (1998), 1473–1525.

[26] K. Schmidt, *Algebraic coding of expansive group automorphisms and two-sided beta-shifts*, Monatsh. Math. (to appear).

[27] N.A. Sidorov and A.M. Vershik, *Ergodic properties of Erdös measure, the entropy of the goldenshift, and related problems*, Monatsh. Math. **126** (1998), 215–261.

[28] N. Sidorov and A. Vershik, *Bijective arithmetic codings of the* 2*-torus, and binary quadratic forms*, to appear.

[29] W. Thurston, *Conway's tiling groups*, Amer. Math. Monthly **97** (1990), 757–773.

[30] A. Vershik, *The fibadic expansion of real numbers and adic transformations*, Preprint, Mittag-Leffler Institute, 1991/92.

[31] A.M. Vershik, *Arithmetic isomorphism of hyperbolic toral automorphisms and sofic shifts*, Funktsional. Anal. i Prilozhen. **26** (1992), 22–27.

[32] H. Wang, *Proving theorems by pattern recognition II*, AT&T Bell Labs. Tech. J. **40** (1961), 1–41.