IRREDUCIBILITY, HOMOCLINIC POINTS AND ADJOINT ACTIONS OF ALGEBRAIC $\mathbb{Z}^d$-ACTIONS OF RANK ONE

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Abstract. In this paper we consider $\mathbb{Z}^d$-actions, $d \geq 1$, by automorphisms of compact connected abelian groups which contain at least one expansive automorphism (such actions are called *algebraic $\mathbb{Z}^d$-actions of expansive rank one*). If α is such a $\mathbb{Z}^d$-action on an infinite compact connected abelian group X, then every expansive element $\alpha^{\mathbf{n}}$ of this action has a dense group $\Delta_{\alpha^{\mathbf{n}}}(X)$ of homoclinic points. For different expansive elements $\alpha^{\mathbf{m}}, \alpha^{\mathbf{n}}$ these groups are generally different and may have zero intersection. By obtaining an appropriate structure formula we prove that these groups are canonically isomorphic for different $\mathbf{m}, \mathbf{n}$, and that the restriction of α to any of these groups defines by duality another algebraic $\mathbb{Z}^d$-action α^* of expansive rank one on a compact connected abelian group X^*, called the *adjoint action* of α. The *second adjoint* $\alpha^{**} = (\alpha^*)^*$ obtained by repeating this construction is algebraically conjugate to α.

A class of examples of algebraic $\mathbb{Z}^d$-actions of expansive rank one is obtained by fixing a d-tuple $\mathbf{c}$ of algebraic numbers consisting not entirely of roots of unity and by associating with it a finite set $S_{\mathbf{c}}$ of places of the algebraic number field $K = K(\mathbf{c})$ generated by the entries of $\mathbf{c}$. For every $S_{\mathbf{c}}$-integral ideal $\mathcal{I}$ in K we define an algebraic $\mathbb{Z}^d$-action α of expansive rank one on a compact connected abelian group. In this case the adjoint action α^* arises from the inverse ideal class $\mathcal{I}^{-1}$ of $\mathcal{I}$. If $\mathcal{I}, \mathcal{J}$ are two $S_{\mathbf{c}}$-integral ideals, then the algebraic $\mathbb{Z}^d$-actions arising from them are algebraically conjugate if and only if the ideals lie in the same ideal class.

Earlier work in this direction can be found in [4], [6] and [11].

1. Introduction

An *algebraic $\mathbb{Z}^d$-action* α on a compact abelian group X is a homomorphism $\mathbf{n} \mapsto \alpha^{\mathbf{n}}$ from $\mathbb{Z}^d$ into the group $\mathrm{Aut}(X)$ of continuous group automorphisms of X. An algebraic $\mathbb{Z}^d$-action α on a compact abelian group X is *expansive* if there exists an open neighbourhood $\mathcal{O} \subset X$ of the identity element $0_X \in X$ with

$$\bigcap_{\mathbf{n} \in \mathbb{Z}^d} \alpha^{-\mathbf{n}}(\mathcal{O}) = \{0_X\}.$$

If α is an algebraic $\mathbb{Z}^d$-action on X, then its topological entropy $h(\alpha)$ coincides with the metric entropy $h_{\lambda_X}(\alpha)$ with respect to the normalized Haar measure λ_X on X.

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Let α and β be algebraic $\mathbb{Z}^d$-actions on compact abelian groups X and Y, respectively. The action β is an *algebraic factor* of α if there exists a continuous surjective group homomorphism $\chi\colon X \longrightarrow Y$ with

$$\chi \circ \alpha^{\mathbf{n}} = \beta^{\mathbf{n}} \circ \chi \ \text{ for every } \ \mathbf{n} \in \mathbb{Z}^d. \tag{1.1}$$

The map χ in (1.1) is an *algebraic factor map* from α to β. The actions α and β are *algebraically conjugate* if the factor map $\chi\colon X \longrightarrow Y$ in (1.1) can be chosen to be a continuous group isomorphism. The actions α and β are *finitely equivalent* if each of them is a finite-to-one algebraic factor of the other.

In this paper we restrict ourselves to algebraic $\mathbb{Z}^d$-actions where individual elements have finite entropy.

Definition 1.1. Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X. An algebraic $\mathbb{Z}^d$-action α on X has *expansive rank one* if there exists $\mathbf{n} \in \mathbb{Z}^d$ such that $\alpha^{\mathbf{n}}$ is expansive, has *entropy rank one* if $h(\alpha^{\mathbf{n}}) < \infty$ for every $\mathbf{n} \in \mathbb{Z}^d$, and is *irreducible* if every closed, α-invariant subgroup $Y \subsetneq X$ is finite.

The relationship between these rank one conditions and irreducibility will be studied in Section 4. We refer to [5] for a more general discussion of expansive rank and entropy rank (with a slightly different definition).

Lemma 1.2. *Let α be an irreducible algebraic $\mathbb{Z}^d$-action on a compact connected abelian group X and let (Y, β) be a non-trivial algebraic factor of (X, α). Then the factor map is finite-to-one, and β is irreducible and finitely equivalent to α.*

Proof. Let $\phi : X \longrightarrow Y$ be the factor map between the actions α on X and β on Y. The kernel $K = \ker \phi$ is an α-invariant closed subgroup of X. As $Y \neq \{0\}$ is assumed to be non-trivial, the kernel K is a proper subgroup and must be finite by irreducibility.

Let Z be a proper closed β-invariant subgroup of Y. The subgroup $\phi^{-1}(Z) \subset X$ is again finite by irreducibility. This shows that $Z = \phi(\phi^{-1}(Z))$ is finite and the action β is irreducible.

As K is finite, there exists an integer $m > 0$ with $m \cdot K = \{0\}$. As X is connected and $M = \widehat{X}$ is therefore torsion-free as a group, the multiplication $\widehat{\chi_m} : a \mapsto ma$ by m on M is injective and the dual map $\chi_m : x \mapsto mx$ on X is surjective. It follows that the map $y \mapsto \psi(y) = \chi_m(\phi^{-1}(y))$ is a well-defined factor map from (Y, β) to (X, α). By the first paragraph, ψ is finite-to-one and α and β are finitely equivalent. $\square$

If $d > 1$, and if α and β are two irreducible and mixing algebraic $\mathbb{Z}^d$-actions on compact abelian groups X and Y, respectively, then α and β are algebraically conjugate if and only if they are measurably conjugate (cf. [6] and [7]). In view of this rigidity property the question whether *finite equivalence* of α and β can be replaced by *algebraic conjugacy* acquires special significance: since it is easy to find finitely equivalent irreducible $\mathbb{Z}^d$-actions which are not algebraically conjugate one obtains examples of measurably non-conjugate $\mathbb{Z}^d$-actions which are difficult to distinguish by using classical invariants of ergodic theory.

If α is an algebraic $\mathbb{Z}^d$-action on an infinite compact connected abelian group X, then a point $x \in X$ is *α-homoclinic* (or simply *homoclinic*) if

$$\lim_{\mathbf{n}\to\infty} \alpha^{\mathbf{n}} x = 0.$$

If α is expansive, the set $\Delta_\alpha(X)$ of all α-homoclinic points in X is countable, and $\Delta_\alpha(X)$ is nonzero if and only if α has positive topological entropy (cf. [9]). In [4] the module of homoclinic points $\Delta_\alpha(X)$ was used to define, for any algebraic $\mathbb{Z}^d$-action α on a compact abelian group X, the adjoint action $\alpha^* = \alpha_{\Delta_\alpha(X)}$ on $X^* = \widehat{\Delta_\alpha(X)}$ (where $\Delta_\alpha(X)$ is considered as a discrete module). If the action α is expansive and has completely positive entropy, then α^* is closely related to α and is again expansive with completely positive entropy. Iteration of this construction leads to further algebraic $\mathbb{Z}^d$-actions $\alpha^{**}, \alpha^{***}, \dots, \alpha^{*^k}, \dots$, where $\alpha^{*^{k+2}}$ can be shown to be algebraically conjugate to α^{*^k} for $k \geq 1$ (if not earlier).

For $d \geq 2$, algebraic $\mathbb{Z}^d$-actions with expansive rank one have zero entropy and therefore cannot have nonzero homoclinic points. Individual expansive elements of the action will, however, have nonzero homoclinic points; in fact, the homoclinic group $\Delta_{\alpha^{\mathbf{n}}}(X)$ of every expansive automorphism $\alpha^{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^d$, is dense in X by [9, Theorem 4.2].

For different expansive automorphisms $\alpha^{\mathbf{m}}, \alpha^{\mathbf{n}}$ the homoclinic groups $\Delta_{\alpha^{\mathbf{m}}}(X)$ and $\Delta_{\alpha^{\mathbf{n}}}(X)$ can be quite different and may have zero intersection (cf. [11]). The main purpose of this paper is to investigate the connection between these groups for different expansive elements of such an action.

If we restrict the $\mathbb{Z}^d$-action α to the homoclinic group $\Delta_{\alpha^{\mathbf{n}}}(X)$ of some expansive automorphism $\alpha^{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^d$, then we obtain a $\mathbb{Z}^d$-action by automorphisms of the countable abelian group $\Delta_{\alpha^{\mathbf{n}}}(X)$ which determines, by duality, an algebraic $\mathbb{Z}^d$-action α^* of the compact abelian group $X^* = \widehat{\Delta_{\alpha^{\mathbf{n}}}(X)}$. A priori this construction depends on the expansive automorphism $\alpha^{\mathbf{n}}$ we started off with. In the Theorems 5.1 and 5.5 we show, however, that different expansive elements lead to algebraically conjugate actions α^*. This allows us to suppress the dependence of α^* on $\mathbf{n}$ and to refer to α^* as the *adjoint action* of α (cf. Definition 5.2 and [4]). If α is an expansive $\mathbb{Z}^d$-action by commuting automorphisms of a finite-dimensional torus $X = \mathbb{T}^m$, $m \geq 2$, then α^* can easily be checked to be the *transpose action* $\mathbf{n} \mapsto (\alpha^{\mathbf{n}})^\top$ of $\mathbb{Z}^d$ on X.

The adjoint action α^* again has expansive rank one, and the group X^* is infinite, compact and connected (in fact, α^* is easily seen to be finitely equivalent to α, but α and α^* are usually not algebraically — and hence, by our earlier remark, not measurably – conjugate). One may thus repeat this construction and define a *second adjoint* action $\alpha^{**} = (\alpha^*)^*$ of α which is algebraically conjugate to α (cf. the Corollaries 5.3 and 5.6). This situation is in marked contrast to the case where α has completely positive entropy: there α and α^{**} may be algebraically nonconjugate, although α^* and $\alpha^{***} = (\alpha^{**})^*$ are always conjugate (cf. [4]). Furthermore, since algebraic $\mathbb{Z}^d$-actions with completely positive entropy are Bernoulli by [13], α and α^* are always measurably conjugate in this case.

The main tool for our investigation is a careful analysis of the structure of algebraic $\mathbb{Z}^d$-actions of expansive rank one in the Theorems 3.4 and 4.5.

Although this description is not new in principle (cf. e.g. [1]–[2] and [14]), the precise form required for the proofs of the Theorems 5.1 and 5.5 and of Corollary 5.6 does not appear to be available in the earlier literature.

For irreducible algebraic $\mathbb{Z}^d$-actions on compact connected abelian groups the Theorems 3.4 and 3.8 and Corollary 5.4 elaborate the close connections with algebraic number theory (and, in particular, with ideal classes) already apparent in [6].

2. Algebraic $\mathbb{Z}^d$-actions

For the description of algebraic $\mathbb{Z}^d$-action we need some algebra. Denote by $R_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ the ring of Laurent polynomials with integral coefficients in the variables $u_1, \dots, u_d$ and write every $f \in R_d$ as

$$f = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} u^{\mathbf{n}} \tag{2.1}$$

with $u^{\mathbf{n}} = u_1^{n_1} \cdots u_d^{n_d}$ and $f_{\mathbf{n}} \in \mathbb{Z}$ for all $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$, where $f_{\mathbf{n}} = 0$ for all but finitely many $\mathbf{n} \in \mathbb{Z}^d$. A prime ideal $\mathfrak{p} \subset R_d$ is *associated with an R_d-module M* if $\mathfrak{p} = \{f \in R_d : f \cdot a = 0\}$ for some $a \in M$, and the module M is *associated with a prime ideal* $\mathfrak{p} \subset R_d$ if $\mathfrak{p}$ is the only prime ideal associated with M. The set of prime ideals associated with a Noetherian R_d-module M is finite and denoted by $\operatorname{Asc}(M)$. If M is a Noetherian R_d-module there exists a chain of submodules

$$M_0 = \{0\} \subset M_1 \subset \cdots \subset M_k = M \tag{2.2}$$

such that $M_i/M_{i-1} \cong R_d/\mathfrak{q}_i$ for $i = 1, \dots, k$, where each $\mathfrak{q}_i \subset R_d$ is a prime ideal containing one of the associated primes of M.

If α is an algebraic $\mathbb{Z}^d$-action on a compact abelian group X then the additively-written dual group $M = \widehat{X}$ is a module over the ring R_d with operation

$$f \cdot a = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} \widehat{\alpha^{\mathbf{n}}}(a)$$

for $f \in R_d$ and $a \in M$, where $\widehat{\alpha^{\mathbf{n}}}$ denotes the automorphism of $\widehat{X}$ dual to $\alpha^{\mathbf{n}}$. The module $M = \widehat{X}$ is called the *dual module* of α. Conversely, if M is a module over R_d, then we can define an algebraic $\mathbb{Z}^d$-action α_M on $X_M = \widehat{M}$ by setting

$$\widehat{\alpha_M^{\mathbf{n}}} a = u^{\mathbf{n}} \cdot a \tag{2.3}$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $a \in M$. Clearly, M is the dual module of α_M.

By using duality one can express many topological and dynamical properties of X and α in terms of the dual module $M = \widehat{X}$. For example, X is connected if and only if M is torsion-free as a group, and the dual module M is Noetherian whenever α is expansive (cf. [15]). We also recall the following result from [15].

Lemma 2.1. *Let M be a Noetherian R_d-module with associated prime ideals $\operatorname{Asc}(M) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$. The following conditions are equivalent.*

(1) *α_M is expansive;*
(2) *$\alpha_{R_d/\mathfrak{p}_j}$ is expansive for every $j = 1, \dots, m$;*

(3) $V_{\mathbb{C}}(\mathfrak{p}_j) \cap \mathbb{S}^d = \varnothing$ *for every* $j = 1, \dots, m$, *where*

$$V_{\mathbb{C}}(\mathfrak{p}_j) = \{\mathbf{z} \in (\mathbb{C}^\times)^d : f(\mathbf{z}) = 0 \text{ for every } f \in \mathfrak{p}_j\}, \qquad (2.4)$$
$$\mathbb{C}^\times = \mathbb{C} \smallsetminus \{0\} \text{ and } \mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}.$$

3. Irreducible actions

In this section we describe — up to algebraic conjugacy — all irreducible algebraic $\mathbb{Z}^d$-actions on an infinite compact abelian connected group X. Background and details of the following discussion can be found in [15, Section 7] and [16].

Let K be an algebraic number field, i.e. a finite extension of $\mathbb{Q}$. A *valuation* of K is a map $\phi\colon K \longrightarrow \mathbb{R}_+$ with the property that $\phi(a) = 0$ if and only if $a = 0$, $\phi(ab) = \phi(a)\phi(b)$, and $\phi(a+b) \le c \cdot \max\{\phi(a), \phi(b)\}$ for some $c \ge 1$ in $\mathbb{R}$ and all $a, b \in K$. The valuation ϕ is *non-trivial* if $\phi(K) \supsetneq \{0, 1\}$. A non-trivial valuation ϕ is *non-archimedean* if $\phi(a+b) \le \max\{\phi(a), \phi(b)\}$ for all $a, b \in K$, and *archimedean* otherwise. Two valuations ϕ, ψ of K are *equivalent* if there exists an $s > 0$ with $\phi(a) = \psi(a)^s$ for all $a \in K$. An equivalence class v of non-trivial valuations of K is called a *place* of K; such a place v is *finite* if it consists of non-archimedean valuations, and *infinite* otherwise.

If v is a place of K, then a sequence $(a_n, n \ge 1)$ in K is *v-Cauchy* if $\lim_{k,l\to\infty} \phi(a_k - a_l) = 0$ for some (and hence for every) valuation $\phi \in v$. With this notion of a Cauchy sequence one can define the *completion* K_v of K at the place v.

Ostrowski's Theorem ([3, Theorem 2.2.1]) states that every non-trivial valuation ϕ of $\mathbb{Q}$ is either equivalent to the absolute value $\frac{m}{n} \mapsto |\frac{m}{n}| = |\frac{m}{n}|_\infty$, or to the p-adic valuation $|\frac{m}{n}|_p = p^{(n'-m')}$ for some rational prime p, where $m = p^{m'}m''$, $n = p^{n'}n''$, and neither m'' nor n'' are divisible by p. The completions $\mathbb{Q}_\infty$ and $\mathbb{Q}_p$ of $\mathbb{Q}$ are equal to $\mathbb{R}$ and the field of p-adic rationals, respectively.

For every valuation ϕ of K, the restriction of ϕ to $\mathbb{Q} \subset K$ is a valuation of $\mathbb{Q}$ and is thus equivalent either to $|\cdot|_\infty$ or to $|\cdot|_p$ for some rational prime p. In the first case the place $v \ni \phi$ is infinite (or *lies above* ∞), and in the second case v *lies above* p (or p *lies below* v).

We denote by w the place of $\mathbb{Q}$ below v and observe that the field K_v is a finite-dimensional vector space over the locally compact, metrizable field $\mathbb{Q}_w$ and hence locally compact and metrizable. Choose a Haar measure λ_v on K_v (with respect to addition) and denote by $\mathrm{mod}_{K_v}\colon K_v \longrightarrow \mathbb{R}$ the map satisfying

$$\lambda_v(aB) = \mathrm{mod}_{K_v}(a)\lambda_v(B) \qquad (3.1)$$

for every $a \in K_v$ and every Borel set $B \subset K_v$. The restriction of mod_{K_v} to K is a valuation in v, denoted by $|\cdot|_v$.

Above every place v of $\mathbb{Q}$ there are at least one and at most finitely many places of K. We write $P^{(K)}$, $P_f^{(K)}$, and $P_\infty^{(K)}$, for the sets of places, finite places and infinite places of K. An infinite place v of K is either *real* (if $K_v = \mathbb{R}$) or *complex* (if $K_v = \mathbb{C}$). The field K is *totally real* if $K_v = \mathbb{R}$ for every $v \in P_\infty^{(K)}$, and *totally complex* if $K_v = \mathbb{C}$ for every $v \in P_\infty^{(K)}$.

For every $v \in P^{(K)}$, the sets

$$\mathcal{R}_v = \{a \in K_v : |a|_v \le 1\}, \qquad \mathcal{R}_v^\times = \{a \in K_v : |a|_v = 1\} \tag{3.2}$$

are compact. If $v \in P_f^{(K)}$, then $\mathcal{R}_v$ is the unique maximal compact subring of K_v and is also open, and the ideal

$$\mathcal{P}_v = \{a \in K_v : |a|_v < 1\} \subset \mathcal{R}_v \tag{3.3}$$

is open, closed and maximal. The set

$$\mathfrak{o}_K = \bigcap_{v \in P_f^{(K)}} \{a \in K : |a|_v \le 1\} \tag{3.4}$$

is the ring of integral elements in K.

Now suppose that $d \ge 1$ and $\mathbf{c} = (c_1, \dots, c_d) \in (\bar{\mathbb{Q}}^\times)^d$, where $\bar{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ and $\bar{\mathbb{Q}}^\times = \bar{\mathbb{Q}} \smallsetminus \{0\}$. We set $K = K_{\mathbf{c}} = \mathbb{Q}(c_1, \dots, c_d) = \mathbb{Q}[c_1^{\pm1}, \dots, c_d^{\pm1}]$ and

$$S_{\mathbf{c}} = P_\infty^{(K)} \cup \{v \in P_f^{(K)} : |c_i|_v \ne 1 \text{ for some } i = 1, \dots, d\}. \tag{3.5}$$

The set $S_{\mathbf{c}}$ is finite by [16, Theorem III.3]. We denote by

$$\iota_{\mathbf{c}} \colon K \longrightarrow V_{\mathbf{c}} = \prod_{v \in S_{\mathbf{c}}} K_v \tag{3.6}$$

the diagonal embedding $a \mapsto (a, \dots, a)$, $a \in K$, and put

$$\mathcal{R}_{\mathbf{c}} = \{a \in K : |a|_v \le 1 \text{ for every } v \in P^{(K)} \smallsetminus S_{\mathbf{c}}\} \supset \mathfrak{o}_K. \tag{3.7}$$

The set $V_{\mathbf{c}}$ is a locally compact algebra over K with respect to coordinate-wise addition, multiplication and scalar multiplication, and $\iota_{\mathbf{c}}(\mathcal{R}_{\mathbf{c}})$ is a discrete, co-compact, additive subgroup of $V_{\mathbf{c}}$. Put

$$Y_{\mathbf{c}} = V_{\mathbf{c}} \,/\, \iota_{\mathbf{c}}(\mathcal{R}_{\mathbf{c}}) \tag{3.8}$$

and write

$$\pi_{\mathbf{c}} \colon V_{\mathbf{c}} \longrightarrow Y_{\mathbf{c}} \tag{3.9}$$

for the quotient map. According to [15, (7.6)] we may identify $Y_{\mathbf{c}}$ with the dual group of $\mathcal{R}_{\mathbf{c}}$, i.e.

$$Y_{\mathbf{c}} = \widehat{\mathcal{R}_{\mathbf{c}}}. \tag{3.10}$$

If every c_i, $i = 1, \dots, d$, is a unit in $\mathfrak{o}_K$ then $S_{\mathbf{c}} = P_\infty^{(K)}$ and

$$V_{\mathbf{c}} \cong \mathbb{R}^{r(K)}, \qquad Y_{\mathbf{c}} \cong \mathbb{T}^{r(K)}, \tag{3.11}$$

where

$$r(K) = [K : \mathbb{Q}] = |\{v \in P_\infty^{(K)} : K_v = \mathbb{R}\}| + 2|\{v \in P_\infty^{(K)} : K_v = \mathbb{C}\}|. \tag{3.12}$$

In general,

$$c_i \in \mathcal{R}_{\mathbf{c}}^\times = \{a \in \mathcal{R}_{\mathbf{c}} : a^{-1} \in \mathcal{R}_{\mathbf{c}}\} \tag{3.13}$$

is a unit in $\mathcal{R}_{\mathbf{c}}$ for every $1 \le i \le d$. We put, for every $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$,

$$\mathbf{c}^{\mathbf{n}} = c_1^{n_1} \cdots c_d^{n_d}, \tag{3.14}$$

write every $a \in V_{\mathbf{c}}$ as $a = (a_v) = (a_v,\, v \in S)$ with $a_v \in K_v$ for every $v \in S$, and define a $\mathbb{Z}^d$-action $\bar{\beta}_{\mathbf{c}}$ on $V_{\mathbf{c}}$ by setting

$$\bar{\beta}_{\mathbf{c}}^{\mathbf{n}} a = \iota_{\mathbf{c}}(\mathbf{c}^{\mathbf{n}}) a = (\mathbf{c}^{\mathbf{n}} a_v) \tag{3.15}$$

for every $a = (a_v) \in V_{\mathbf{c}}$ and $\mathbf{n} \in \mathbb{Z}^d$. As $\bar{\beta}_{\mathbf{c}}^{\mathbf{n}}(\iota_{\mathbf{c}}(\mathcal{R}_{\mathbf{c}})) = \iota_{\mathbf{c}}(\mathcal{R}_{\mathbf{c}})$ for every $\mathbf{n} \in \mathbb{Z}^d$, $\bar{\beta}_{\mathbf{c}}$ induces an algebraic $\mathbb{Z}^d$-action $\beta_{\mathbf{c}}$ on the compact abelian group $Y_{\mathbf{c}}$ in (3.8) by

$$\beta_{\mathbf{c}}^{\mathbf{n}}(a + \iota_{\mathbf{c}}(\mathcal{R}_{\mathbf{c}})) = \bar{\beta}_{\mathbf{c}}^{\mathbf{n}} a + \iota_{\mathbf{c}}(\mathcal{R}_{\mathbf{c}}) \tag{3.16}$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $a \in V_{\mathbf{c}}$, whose dual action $\hat{\beta}_{\mathbf{c}} \colon \mathbf{n} \mapsto \hat{\beta}_{\mathbf{c}}^{\mathbf{n}}$ is given by

$$\hat{\beta}_{\mathbf{c}}^{\mathbf{n}} b = \mathbf{c}^{\mathbf{n}} b \tag{3.17}$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $b \in \mathcal{R}_{\mathbf{c}} = \widehat{Y_{\mathbf{c}}}$ (cf. (3.10)).

We denote by $\eta_{\mathbf{c}} \colon f \mapsto f(\mathbf{c})$ the evaluation map and define the ideal $P_{\mathbf{c}} = \ker \eta_{\mathbf{c}}$. Then

$$R_d/P_{\mathbf{c}} \cong \eta_{\mathbf{c}}(R_d) = \mathbb{Z}[\mathbf{c}^{\pm 1}] = \mathbb{Z}[c_1^{\pm 1}, \dots, c_d^{\pm 1}] \subset \mathcal{R}_{\mathbf{c}}, \tag{3.18}$$

and $\mathcal{R}_{\mathbf{c}}$ is a module over the integral domain $\mathbb{Z}[\mathbf{c}^{\pm 1}]$. We need the following basic lemma from algebraic number theory.

Lemma 3.1. *The $\mathbb{Z}[\mathbf{c}^{\pm 1}]$-module $\mathcal{R}_{\mathbf{c}}$ is equal to $\mathfrak{o}_K[\mathbf{c}^{\pm 1}]$ and is thus finitely generated.*

Proof. The inclusion $\mathfrak{o}_K[\mathbf{c}^{\pm 1}] \subset \mathcal{R}_{\mathbf{c}}$ is obvious. For the reverse inclusion let $x \in \mathcal{R}_{\mathbf{c}}$ and put

$$E_x = \{v \in S_{\mathbf{c}} \cap P_f^{(K)} : |x|_v > 1\}.$$

If $E_x = \varnothing$ then $x \in \mathfrak{o}_K$ and we are done. Now assume that $k \geq 1$ and that $y \in \mathfrak{o}_K[\mathbf{c}^{\pm 1}]$ for every $y \in \mathcal{R}_{\mathbf{c}}$ with $|E_y| < k$. If $|E_x| = k$ and $v \in E_x$, then we can find an $\mathbf{n} \in \mathbb{Z}^d$ with $|\mathbf{c}^{\mathbf{n}}|_v > |x|_v$. By the Chinese remainder theorem there exists $a \in \mathfrak{o}_K$ such that

$$|a|_v = 1 \ \text{ and } \ |a\mathbf{c}^{\mathbf{n}}|_w < 1 \ \text{ for every } \ w \in S_{\mathbf{c}} \smallsetminus \{v\}.$$

Then $x(a\mathbf{c}^{\mathbf{n}})^{-1} \in \mathcal{R}_v$ and, since $\mathfrak{o}_K$ is dense in $\mathcal{R}_v$, we can find $b \in \mathfrak{o}_K$ and $d \in \mathcal{R}_v$ such that

$$x(a\mathbf{c}^{\mathbf{n}})^{-1} = b + d \ \text{ with } \ |d|_v \leq |\mathbf{c}^{\mathbf{n}}|_v^{-1}.$$

This shows that

$$\begin{aligned} &|x - ab\mathbf{c}^{\mathbf{n}}|_v = |ad\mathbf{c}^{\mathbf{n}}|_v \leq 1, \\ &|x - ab\mathbf{c}^{\mathbf{n}}|_w = |x|_w && \text{for } \ w \in E_x \smallsetminus \{v\}, \\ &|x - ab\mathbf{c}^{\mathbf{n}}|_w \leq 1 && \text{for } \ w \in P_f^{(K)} \smallsetminus E_x. \end{aligned}$$

Our induction hypothesis implies that $x - ab\mathbf{c}^{\mathbf{n}} \in \mathfrak{o}_K[\mathbf{c}^{\pm 1}]$ and hence that $x \in \mathfrak{o}_K[\mathbf{c}^{\pm 1}]$. By induction, $\mathcal{R}_{\mathbf{c}} \subset \mathfrak{o}_K[\mathbf{c}^{\pm 1}]$, as promised.

Since $\mathfrak{o}_K$ is a finitely generated $\mathbb{Z}$-module, $\mathcal{R}_{\mathbf{c}}$ is finitely generated over $\mathbb{Z}[\mathbf{c}^{\pm 1}]$. □

Let $\mathcal{L} \subset K$ be a finitely generated $\mathbb{Z}[\mathbf{c}^{\pm 1}]$-submodule. We denote by $\hat{\alpha}_{(\mathbf{c},\mathcal{L})}$ the $\mathbb{Z}^d$-action on $\mathcal{L}$ defined by

$$\hat{\alpha}_{(\mathbf{c},\mathcal{L})}^{\mathbf{n}} a = \mathbf{c}^{\mathbf{n}} a \tag{3.19}$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $a \in \mathcal{L}$ and write $\alpha_{(\mathbf{c},\mathcal{L})}$ for the dual algebraic $\mathbb{Z}^d$-action on

$$X_{\mathcal{L}} = \widehat{\mathcal{L}}. \tag{3.20}$$

Since $K=\mathbb{Q}[\mathbf{c}]$ we can write every $a\in K$ as $a=b/n$ for some $b\in\mathcal{R}_{\mathbf{c}}$ and $n\in\mathbb{Z}$. As $\mathcal{L}$ is assumed to be finitely generated, we can find a common integer $N>0$ such that $N\mathcal{L}\subset\mathcal{R}_{\mathbf{c}}$. If

$$\hat{\theta}_{\mathcal{L}}\colon \widehat{X_{\mathcal{L}}}=\mathcal{L}\longrightarrow\mathcal{R}_{\mathbf{c}}=\widehat{Y_{\mathbf{c}}} \tag{3.21}$$

is the injective map defined by multiplication with N, then we obtain a dual algebraic factor map

$$\theta_{\mathcal{L}}\colon Y_{\mathbf{c}}\longrightarrow X_{\mathcal{L}} \tag{3.22}$$

between the algebraic $\mathbb{Z}^d$-actions $\beta_{\mathbf{c}}$ and $\alpha_{(\mathbf{c},\mathcal{L})}$.

For the particular choices $\mathcal{L}=\mathcal{R}_{\mathbf{c}}$ and $\mathcal{L}=\mathbb{Z}[\mathbf{c}^{\pm1}]$ we obtain the actions

$$\begin{aligned}\beta_{\mathbf{c}}&=\alpha_{(\mathbf{c},\mathcal{R}_{\mathbf{c}})} &&\text{on}\ \ Y_{\mathbf{c}}=\widehat{\mathcal{R}_{\mathbf{c}}},\\ \alpha_{\mathbf{c}}&=\alpha_{(\mathbf{c},\mathbb{Z}[\mathbf{c}^{\pm1}])} &&\text{on}\ \ X_{\mathbf{c}}=\widehat{\mathbb{Z}[\mathbf{c}^{\pm1}]}.\end{aligned} \tag{3.23}$$

Proposition 3.2. *For any two nonzero finitely generated $\mathbb{Z}[\mathbf{c}^{\pm1}]$-modules $\mathcal{L}\subset\mathcal{L}'$ the module $\mathcal{L}'/\mathcal{L}$ is finite.*

Furthermore the $\mathbb{Z}^d$-action $\beta_{\mathbf{c}}$ on $Y_{\mathbf{c}}$ is irreducible, the factor map $\theta_{\mathcal{L}}\colon Y_{\mathbf{c}}\longrightarrow X_{\mathcal{L}}$ in (3.21)–(3.22) is finite-to-one, and the action $\alpha_{(\mathbf{c},\mathcal{L})}$ on $X_{\mathcal{L}}$ is irreducible and finitely equivalent to $\beta_{\mathbf{c}}$.

For the proof of Proposition 3.2 we need another lemma.

Lemma 3.3. *Let $\mathfrak{o}\subset K$ be a finitely generated subring with identity of the algebraic number field K. Then every nonzero ideal $\mathfrak{J}\subset\mathfrak{o}$ has finite index.*

Proof. Assume that we have already shown that some finitely generated subring $\mathfrak{o}\subset K$ containing 1 has the property that $|\mathfrak{o}/\mathfrak{J}|<\infty$ for every nonzero ideal $\mathfrak{J}\subset\mathfrak{o}$. By assumption $\mathbb{Z}\subset\mathfrak{o}$.

Let $a\in K$ be an algebraic number with primitive minimal polynomial $f(x)\in\mathbb{Z}[x]$, and let $\mathfrak{J}\subset\mathfrak{o}[a]$ be a nonzero ideal. We set $S=\mathfrak{o}\smallsetminus\{0\}$ and consider the number fields $S^{-1}\mathfrak{o}=L$ and $S^{-1}\mathfrak{o}[a]=L[a]=L'$. As $\{0\}\subsetneq S^{-1}\mathfrak{J}\subset L'$, it follows that $S^{-1}\mathfrak{J}=L'$ and $\mathfrak{J}\cap S\neq\{0\}$

By our hypothesis on $\mathfrak{o}$, the nonzero ideal $\mathfrak{J}\cap\mathfrak{o}$ has finite index in $\mathfrak{o}$. We claim that

$$\text{there exists a monic polynomial } h\in\mathbb{Z}[x] \text{ with } h(a)\in\mathfrak{J}. \tag{3.24}$$

Indeed, since $\mathbb{Z}\subset\mathfrak{o}$ and $\mathfrak{J}\cap\mathfrak{o}$ has finite index in $\mathfrak{o}$, there exists a positive integer $n\in\mathfrak{J}$. We denote by $\mathfrak{I}=\langle n,f\rangle\subset\mathbb{Z}[x]$ the ideal generated by the elements $n,f\in\mathbb{Z}[x]$ and assert that

$$\mathfrak{I} \text{ contains a monic polynomial } h. \tag{3.25}$$

By evaluating the generators of the ideal $\mathfrak{I}$ at a we conclude that $h(a)\in\mathfrak{J}$, which shows that (3.24) is a consequence of (3.25).

In order to prove (3.25) we first assume that $n=p^e$ is a prime power. We write f as a sum $f=f_1-pf_2$ with $f_1,f_2\in\mathbb{Z}[x]$, where the leading coefficient of f_1 is co-prime to p. Multiplication with $a=f_1^{e-1}+f_1^{e-2}pf_2+\cdots+(pf_2)^{e-1}\in\mathbb{Z}[x]$ gives that $f_1^e-p^ef_2^e\in\mathfrak{I}$. We have thus found polynomials $a,b\in\mathbb{Z}[x]$ such that $h_p'=f_1^e=af+bp^e\in\mathfrak{I}$ has a leading coefficient which is co-prime to p and hence to $n=p^e$. If m is the degree of h_p' we can apply Euclid's algorithm to find integers k,k' such that the leading coefficient of

$$h_p=kh_p'+k'nx^m\in\mathfrak{I} \tag{3.26}$$

is one.

If n contains a product of at least two distinct primes we write $n = p_1^{e_1} \cdots p_k^{e_k}$ for the prime power decomposition of n and use the isomorphism $\mathbb{Z}/n\mathbb{Z} \cong \prod_{j=1}^k \mathbb{Z}/p_j^{e_j}\mathbb{Z}$ to obtain an isomorphism

$$\theta \colon (\mathbb{Z}/n\mathbb{Z})[x] \longrightarrow \prod_{j=1}^{k} (\mathbb{Z}/p_j^{e_j}\mathbb{Z})[x]$$

of the polynomial rings. Denote by $\bar f \in R = (\mathbb{Z}/n\mathbb{Z})[x]$ the polynomial obtained by reducing each coefficient of f modulo n and put $\theta(\bar f) = (\bar f_1, \dots, \bar f_k)$ with $\bar f_j \in R_j = (\mathbb{Z}/p_j^{e_j}\mathbb{Z})[x]$ for every $j = 1, \dots, k$. The preceding paragraph shows that the principal ideal generated by $\bar f_j$ in R_j contains a polynomial $\bar h_j$ with leading coefficient 1, i.e. that there exists a $\bar g_j \in R_j$ with $\bar h_j = \bar f_j \bar g_j$. The polynomial $g \in R$ with $\theta(g) = (\bar g_1, \dots, \bar g_k)$ satisfies that $\bar f \bar g \in R$ has leading coefficient 1. This shows that there exists a $g' \in \mathbb{Z}[x]$ such that the polynomial $h = fg + ng' \in \mathfrak{I}$ has leading coefficient 1 and proves (3.25) and hence (3.24).

If m is the degree of the polynomial h found in (3.24), then

$$\big|\mathfrak{o}[a]/\mathfrak{J}\big| = \big|\mathfrak{o} + a\mathfrak{o} + \dots + a^{m-1}\mathfrak{o}/\mathfrak{J}\big| \le \big|\mathfrak{o}/\mathfrak{o} \cap \mathfrak{J}\big|^m < \infty.$$

This shows that the ring $\mathfrak{o}[a]$ again has the property that $|\mathfrak{o}[a]/\mathfrak{J}| < \infty$ for every nonzero ideal $\mathfrak{J} \subset \mathfrak{o}[a]$.

The proof of the lemma is completed by induction on the number of generators of the subring $\mathfrak{o}$. □

Proof of Proposition 3.2. This is a slight extension of [15, Theorem 7.1].

Let $\mathfrak{L} \subset \mathfrak{L}' \subset K$ be two nonzero finitely generated $\mathbb{Z}[\mathbf{c}^{\pm 1}]$-submodules. Since $\mathfrak{L} \subset K$ is nonzero, $\mathbb{Z}[\mathbf{c}^{\pm 1}]a \subset \mathfrak{L}$ for some nonzero $a \in \mathfrak{L}$. Since $\mathfrak{L}'$ is finitely generated as a $\mathbb{Z}[\mathbf{c}^{\pm 1}]$-module and $\mathfrak{L}' \subset K = \mathbb{Q}[\mathbf{c}]$, we can find $M \in \mathbb{Z}$ such that

$$\mathbb{Z}[\mathbf{c}^{\pm 1}]a \subset \mathfrak{L} \subset \mathfrak{L}' \subset \frac{1}{M}\mathbb{Z}[\mathbf{c}^{\pm 1}].$$

Lemma 3.3 shows that $\mathbb{Z}[\mathbf{c}^{\pm 1}]a$ has finite index in $\frac{1}{M}\mathbb{Z}[\mathbf{c}^{\pm 1}]$, which completes the proof of the first statement of the proposition.

For the second statement we consider the action $\beta_{\mathbf{c}}$ on $Y_{\mathbf{c}}$. If $Z \subset Y_{\mathbf{c}}$ is a proper invariant closed subgroup, then the annihilator $\mathfrak{L} = Z^{\perp} \subset \mathfrak{R}_{\mathbf{c}}$ is a nonzero $\mathbb{Z}[\mathbf{c}^{\pm 1}]$-submodule. Therefore has $\mathfrak{L}$ finite index in $\mathfrak{R}_{\mathbf{c}}$ and Z is finite. This shows that $\beta_{\mathbf{c}}$ is irreducible. Lemma 1.2 implies the remaining statements. □

Our next theorem shows that every irreducible algebraic $\mathbb{Z}^d$-action α on a compact abelian connected group is of the form $\alpha_{(\mathbf{c},\mathfrak{L})}$ described in (3.19)–(3.20).

Theorem 3.4. *Suppose that $d \ge 1$, and that α is an algebraic $\mathbb{Z}^d$-action on an infinite compact connected abelian group X. Then α is irreducible if and only if it is finitely equivalent to each of the irreducible algebraic $\mathbb{Z}^d$-actions $\alpha_{\mathbf{c}}$ on $X_{\mathbf{c}}$ and $\beta_{\mathbf{c}}$ on $Y_{\mathbf{c}}$ for some $\mathbf{c} = (c_1, \dots, c_d) \in (\bar{\mathbb{Q}}^{\times})^d$. Furthermore there exists a finitely generated $\mathbb{Z}[\mathbf{c}^{\pm 1}]$-submodule $\mathfrak{L} \subset K$ such that α is algebraically conjugate to the $\mathbb{Z}^d$-action $\alpha_{(\mathbf{c},\mathfrak{L})}$ on $X_{\mathfrak{L}}$ defined in*

(3.19)–(3.20). *Without loss of generality one may assume in addition that* $\mathcal{L} \subset \mathcal{R}_{\mathbf{c}}$.

Proof. Let α be an irreducible algebraic action on the compact connected abelian group X with dual module $M = \widehat{X}$ and let $\mathfrak{p}$ be an associated prime ideal for M. There exists $a \in M$ such that

$$\{f \in R_d : f \cdot a = 0\} = \mathfrak{p}.$$

This shows that the map $f + \mathfrak{p} \mapsto \hat{\theta}(f) = fa$ from $R_d/\mathfrak{p}$ to M is an injective module homomorphism. By duality, $\theta\colon X \longrightarrow X_{R_d/\mathfrak{p}}$ a factor map. From Lemma 1.2 we see that θ is finite-to-one and the action $\alpha_{R_d/\mathfrak{p}}$ on $X_{R_d/\mathfrak{p}}$ is irreducible.

As X is connected, the dual module is torsion-free as an abelian group and $\mathfrak{p}$ does not contain a constant. Hilbert's Nullstellensatz shows that there exists a point

$$\mathbf{c} \in V(\mathfrak{p}) = \{\mathbf{c}' \in \overline{\mathbb{Q}}^{\times} : f(\mathbf{c}') = 0 \text{ for every } f \in \mathfrak{p}\}.$$

Let $\pi\colon R_d/\mathfrak{p} \longrightarrow R_d/\mathfrak{p}_{\mathbf{c}}$ be the canonical projection map, where

$$\mathfrak{p}_{\mathbf{c}} = \{f \in R_d : f(\mathbf{c} = 0)\}. \tag{3.27}$$

Then $\hat{\pi}\colon X_{\mathbf{c}} \longrightarrow X$ is injective. As $\alpha_{R_d/\mathfrak{p}}$ is irreducible by the previous paragraph, every non-trivial closed α-invariant subgroup must be finite and $\widehat{\pi}$ must be surjective. By duality, π is injective and $\mathfrak{p} = \mathfrak{p}_{\mathbf{c}}$.

Lemma 1.2 also shows that the actions $\alpha_{R_d/\mathfrak{p}_{\mathbf{c}}}$ and α are finitely equivalent. Let $\phi : X_{\mathbf{c}} \longrightarrow X$ be a factor map. The dual homomorphism $\hat{\phi}\colon M \longrightarrow R_d/\mathfrak{p}_{\mathbf{c}} \cong \mathbb{Z}[\mathbf{c}^{\pm 1}]$ of ϕ is injective. Hence $\mathcal{L} = \hat{\phi}(M) \subset \mathbb{Z}[\mathbf{c}^{\pm 1}] \subset K$ is a nonzero finitely generated $\mathbb{Z}[\mathbf{c}^{\pm 1}]$-submodule and the $\mathbb{Z}^d$-actions α and $\alpha_{(\mathbf{c},\mathcal{L})}$ are algebraically conjugate. □

By using the locally compact group $V_{\mathbf{c}}$ in (3.6) we can give another description of all irreducible algebraic $\mathbb{Z}^d$-actions.

Corollary 3.5. *Let $d \geq 1$, and let α be an irreducible $\mathbb{Z}^d$-action on an infinite compact connected abelian group X. We denote by $\mathbf{c} \in (\bar{\mathbb{Q}}^{\times})^d$ the point described in Theorem* 3.4 *and define the ring $\mathcal{R}_{\mathbf{c}} \subset K$, the set $S_{\mathbf{c}} \subset P^{(K)}$, the algebra $V_{\mathbf{c}} = \prod_{v \in S} K_v$ and the embedding $\iota_{\mathbf{c}}\colon K \longrightarrow V_{\mathbf{c}}$ as in* (3.5)–(3.6). *Then there exists a finitely generated $\mathbb{Z}[\mathbf{c}^{\pm 1}]$-submodule $\mathcal{K} \subset K$ such that α is algebraically conjugate to the $\mathbb{Z}^d$-action $\alpha'_{(\mathbf{c},\mathcal{K})}$ on the compact abelian group*

$$X'_{\mathcal{K}} = V_{\mathbf{c}}/\iota_S(\mathcal{K}), \tag{3.28}$$

defined as in (3.16) *by*

$$\alpha'^{\mathbf{n}}_{(\mathbf{c},\mathcal{K})}(a + \iota_S(\mathcal{K})) = \bar{\beta}^{\mathbf{n}}_{(\mathbf{c},S)} a + \iota_S(\mathcal{K}) \tag{3.29}$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $a \in V_{\mathbf{c}}$. Furthermore one can always assume that $\mathcal{K} \subset \mathcal{R}_{\mathbf{c}}$.

Conversely, if $\mathcal{K} \subset K$ is a nonzero finitely generated $\mathbb{Z}[\mathbf{c}^{\pm 1}]$-submodule, then the $\mathbb{Z}^d$-action $\alpha'_{(\mathbf{c},\mathcal{K})}$ on $X'_{\mathcal{K}}$ in (3.28)–(3.29) *is irreducible and finitely equivalent to $\alpha_{\mathbf{c}}$ and $\beta_{\mathbf{c}}$.*

Proof. According to Theorem 3.4 there exists a finite-to-one factor map $\phi\colon Y_{\mathbf{c}} \longrightarrow X$. The map ϕ induces a continuous surjective group homomorphism $\psi\colon V_{\mathbf{c}} \longrightarrow X$ with $\psi \circ \bar{\beta}_{\mathbf{c}}^{\mathbf{n}} = \alpha^{\mathbf{n}} \circ \psi$ for every $\mathbf{n} \in \mathbb{Z}^d$, whose kernel $\mathcal{K}' = \ker(\psi)$ is invariant under the $\mathbb{Z}^d$-action $\bar{\beta}_{\mathbf{c}}$ in (3.15) and contains $\iota_S(\mathcal{R}_{\mathbf{c}})$ as a subgroup of finite index.

Choose an integer $N \geq 1$ with $\mathcal{K}'' = N\mathcal{K}' \subset \iota_S(\mathcal{R}_{\mathbf{c}})$ and denote by $\mathcal{K} \subset \mathcal{R}_{\mathbf{c}}$ the $\eta_{\mathbf{c}}(R_d)$-submodule satisfying $\iota_S(\mathcal{K}) = \mathcal{K}''$. If $m_N\colon V_{\mathbf{c}} \longrightarrow V_{\mathbf{c}}$ denotes multiplication by N, then

$$X \cong V_{\mathbf{c}}/\mathcal{K}' \cong m_N(V_{\mathbf{c}})/m_N(\mathcal{K}'') \cong V_{\mathbf{c}}/\mathcal{K}'' \cong V_{\mathbf{c}}/\iota_S(\mathcal{K}) = X'_{\mathcal{K}},$$

and the isomorphism of X and $X'_{\mathcal{K}}$ carries the $\mathbb{Z}^d$-action α to $\alpha'_{(\mathbf{c},\mathcal{K})}$.

The other statements are clear from Proposition 3.2, since $\mathcal{K}$ has finite index in $\mathcal{R}_{\mathbf{c}}$. □

Theorem 3.4 and Corollary 3.5 give a variety of representations of irreducible algebraic $\mathbb{Z}^d$-action on infinite compact connected abelian groups. For a fixed $\mathbf{c}$ all these representations are finitely equivalent. Theorem 3.8 will show that these representations are sometimes, but not always, algebraically conjugate.

We can give an easy characterization of those actions which are algebraically conjugate to $\alpha_{\mathbf{c}}$.

Definition 3.6. Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X. The dual group $\widehat{X}$ of X is *cyclic* under the dual action $\hat{\alpha}$ of α (or α has *cyclic dual*) if there exists a character $a \in \widehat{X}$ such that $\widehat{X}$ is generated by the set $\{\hat{\alpha}^{\mathbf{n}}a : \mathbf{n} \in \mathbb{Z}\}$.

Proposition 3.7. *Let $d \geq 1$, and let α be an irreducible algebraic $\mathbb{Z}^d$-action on an infinite compact connected abelian group X. If $\mathbf{c} \in \bar{\mathbb{Q}}^{\times}$ is the point appearing in Theorem* 3.4*, then α is algebraically conjugate to $\alpha_{\mathbf{c}}$ if and only if α has cyclic dual.*

Proof. The action $\alpha_{\mathbf{c}}$ has cyclic dual, since the element $1 \in \mathbb{Z}[\mathbf{c}^{\pm 1}] = \widehat{X_{\mathbf{c}}}$ is cyclic under $\hat{\alpha}_{\mathbf{c}}$.

If α and $\alpha_{\mathbf{c}}$ are algebraically conjugate, there exists a continuous group isomorphism $\phi\colon X \longrightarrow X_{R_d/\mathfrak{p}_c}$ with $\phi \circ \alpha^{\mathbf{n}} = \alpha_{\mathbf{c}}^{\mathbf{n}} \circ \phi$ for every $\mathbf{n} \in \mathbb{Z}^d$, and the dual isomorphism $\hat{\phi}\colon R_d/\mathfrak{p}_c \longrightarrow \widehat{X}$ sends $1 \in \mathbb{Z}[\mathbf{c}^{\pm 1}]$ to a cyclic element $a \in \widehat{X}$ for $\hat{\alpha}$.

Conversely, if $a \in \widehat{X}$ is a cyclic element of $\hat{\alpha}$, then the map

$$h(\mathbf{c}) \mapsto h(\hat{\alpha})(a) = \sum_{\mathbf{n}\in\mathbb{Z}^d} h_{\mathbf{n}} \widehat{\alpha^{\mathbf{n}}} a$$

for $h = \sum_{\mathbf{n}\in\mathbb{Z}^d} h_{\mathbf{n}} u^{\mathbf{n}} \in R_d$ induces a module-isomorphism $\hat{\psi}\colon \mathbb{Z}[\mathbf{c}^{\pm 1}] \longrightarrow \widehat{X}$ whose dual $\psi\colon X \longrightarrow X_{R_d/P} = \widehat{X_{R_d/P}}$ is an algebraic conjugacy of α and $\alpha_{\mathbf{c}}$. □

We end this section with a connection between irreducible algebraic $\mathbb{Z}^d$-actions and ideal classes in algebraic number fields. This will give us a collection of nonconjugate but finitely equivalent algebraic actions.

Every nonzero ideal $\mathcal{I} \subset \mathcal{R}_S$ is called an *S-integral ideal* of K and has finite index in $\mathcal{R}_S$ by Lemma 3.3. Two S-integral ideals $\mathcal{I}, \mathcal{J}$ of K lie in the same *ideal class* if there exists an element $a \in K$ with $a\mathcal{I} = \mathcal{J}$.

Theorem 3.8. *Suppose that K is an algebraic number field, $\mathbf{c} \in (K^\times)^d$ a vector of nonzero algebraic numbers with $K = \mathbb{Q}(\mathbf{c})$, and let $S_\mathbf{c} \subset P^{(K)}$ be the set of places defined by (3.5). Then the $\mathbb{Z}^d$-action $\alpha_{(\mathbf{c},\mathcal{I})}$ on $X_\mathcal{I} = \hat{\mathcal{I}}$ is irreducible for every nonzero ideal $\mathcal{I} \subset \mathcal{R}_\mathbf{c}$. Furthermore, if $\mathcal{I}, \mathcal{J}$ are nonzero ideals in $\mathcal{R}_\mathbf{c}$, then $\alpha_{(\mathbf{c},\mathcal{I})}$ and $\alpha_{(\mathbf{c},\mathcal{J})}$ are finitely equivalent, and $\alpha_{(\mathbf{c},\mathcal{I})}$ and $\alpha_{(\mathbf{c},\mathcal{J})}$ are algebraically conjugate if and only if $\mathcal{I}$ and $\mathcal{J}$ lie in the same ideal class.*

Proof. Theorem 3.4 shows that the action $\alpha_{(\mathbf{c},\mathcal{I})}$ is irreducible.

If $\mathcal{I}, \mathcal{J}$ are nonzero ideals in $\mathcal{R}_\mathbf{c}$, then $\alpha_{(\mathbf{c},\mathcal{I})}$ and $\alpha_{(\mathbf{c},\mathcal{J})}$ are obviously algebraically conjugate whenever $\mathcal{I}$ and $\mathcal{J}$ lie in the same ideal class.

Conversely, if $\phi \colon X_\mathcal{I} \longrightarrow X_\mathcal{J}$ is an algebraic conjugacy of $\alpha_{(\mathbf{c},\mathcal{I})}$ and $\alpha_{(\mathbf{c},\mathcal{J})}$, then the dual map $\hat{\phi} \colon \mathcal{J} \longrightarrow \mathcal{I}$ is an $\eta_\mathbf{c}(R_d)$-module isomorphism (cf. (3.18)), i.e. $\hat{\phi}(f(c)a) = f(c)\hat{\phi}(a)$ for every $f \in R_d$ and $a \in \mathcal{I}$. Since K is the field of fractions of $\mathbb{Z}[\mathbf{c}^{\pm1}]$ we can extend $\hat{\phi}$ to a K-linear map $\hat{\psi} \colon K \longrightarrow K$ by fixing a nonzero element $a \in \mathcal{I}$ and setting $\hat{\psi}\bigl(\frac{f(c)}{g(c)}a\bigr) = \frac{f(c)}{g(c)}\hat{\phi}(a)$ for every $f, g \in R_d$ with $g(c) \neq 0$. An elementary calculation shows that $\hat{\psi}\bigl(\frac{f(c)}{g(c)}\bigr) = \hat{\phi}\bigl(\frac{f(c)}{g(c)}\bigr)$ whenever $\frac{f(c)}{g(c)} \in \mathcal{I}$. If $b = \hat{\psi}(1)$, then the K-linearity of $\hat{\psi}$ guarantees that $\hat{\psi}(a) = ba$ for every $a \in K$, and hence that $\mathcal{J} = \hat{\phi}(\mathcal{I}) = \hat{\psi}(\mathcal{I}) = b\mathcal{I}$. This shows that $\mathcal{I}$ and $\mathcal{J}$ lie in the same ideal class. □

Theorem 3.8 describes the algebraic $\mathbb{Z}^d$-actions arising from a fixed vector $\mathbf{c} = (c_1, \dots, c_d)$ of algebraic numbers. In the remainder of the section we start with an integral (or an S-integral) ideal in an algebraic number field and consider algebraic $\mathbb{Z}^d$-actions arising from units systems in the number field.

Let $K' \supset \mathbb{Q}$ be an algebraic number field, and let $S' \subset P^{(K')}$ be a finite number of places containing $P_\infty^{(K')}$. We define the ring of S'-integers $\mathcal{R}'_{S'} \subset K'$ as in (3.7) with S' replacing $S_\mathbf{c}$. By [12, Theorem 3.5], the group of units ${\mathcal{R}'}_{S'}^{\times}$ of $\mathcal{R}'_{S'}$ in (3.13) is isomorphic to the cartesian product $F \times \mathbb{Z}^d$, where F is a finite cyclic group consisting of all roots of unity in K' and $d = |S| - 1$.

Definition 3.9. A d-tuple $\mathbf{c} = (c_1, \dots, c_d)$ in ${\mathcal{R}'}_{S'}^{\times}$ is a *free S'-unit system* if it generates a free abelian group, i.e. if the equation $\mathbf{c}^\mathbf{n} = c_1^{n_1} \cdots c_d^{n_d} = 1$ with $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$ implies that $\mathbf{n} = \mathbf{0}$.

A free S'-unit system $c = (c_1, \dots, c_d)$ is a *fundamental S'-unit system* of K' if every $a \in {\mathcal{R}'}_{S'}^{\times}$ can be written uniquely as $a = uc_1^{k_1} \cdots c_d^{k_d}$ with $u \in F$ and $k_1, \dots, k_d \in \mathbb{Z}$.

We fix a free S'-unit system $\mathbf{c} = (c_1, \dots, c_d)$ of K' and set $K = \mathbb{Q}[\mathbf{c}]$. Every S'-integral ideal $\mathcal{I} \subset \mathcal{R}'_{S'}$ of K' is obviously a module over the ring $\eta_\mathbf{c}(R_d)$, and we obtain an algebraic $\mathbb{Z}^d$-action $\alpha_{(\mathbf{c},\mathcal{I})}$ on $X_\mathcal{I} = \hat{\mathcal{I}}$ as in (3.19)–(3.20). In order to simplify terminology we count $\mathcal{R}'_{S'}$ itself among the ideals of $\mathcal{R}'_{S'}$.

Theorem 3.10. *Let K be an algebraic number field, $S \subset P^{(K)}$ a finite number of places containing $P_\infty^{(K)}$ with $|S| \geq 2$, and $\mathbf{c} = (c_1, \ldots, c_d)$ a fundamental S-unit system with $d = |S| - 1$.*

Suppose that K has at least one real place. Then the $\mathbb{Z}^d$-action $\alpha_{(\mathbf{c},\mathfrak{I})}$ is irreducible for every nonzero ideal $\mathfrak{I} \subset \mathfrak{R}_\mathbf{c}$. Furthermore, if $\mathfrak{I}, \mathfrak{J}$ are nonzero ideals in $\mathfrak{R}_\mathbf{c}$, then the $\mathbb{Z}^d$-actions $\alpha_{(\mathbf{c},\mathfrak{I})}$ and $\alpha_{(\mathbf{c},\mathfrak{J})}$ are finitely equivalent, and they are algebraically conjugate if and only if $\mathfrak{I}$ and $\mathfrak{J}$ lie in the same ideal class.

For the proof of Theorem 3.8 we need two lemmas.

Lemma 3.11. *Let K' be an algebraic number field and let $S' \subset P^{(K')}$ be a finite subset containing $P_\infty^{(K')}$ with $d = |S'| - 1 \geq 1$. We fix a fundamental S-unit system $\mathbf{c} = (c_1, \ldots, c_d)$ in K' and write $K = \mathbb{Q}[\mathbf{c}] \subset K'$ for the field of fractions of the integral domain $\mathbb{Z}[\mathbf{c}^{\pm 1}]$.*

If $K \neq K'$ then K' is totally complex, K is totally real and $[K' : K] = 2$.

Proof. This is essentially [12, Proposition 3.5]. For every place v' of K' we denote by v the restriction of v' to $K \subset K'$, and we write $S = \{v : v' \in S'\}$ for the resulting places of K.

Since the group of units $\mathfrak{R}_\mathbf{c}^\times$ of $\mathfrak{R}_\mathbf{c} = \mathfrak{R}'_{S'} \cap K$ contains the free abelian group generated by $\{c_1, \ldots, c_d\}$, [12, Theorem 3.5] implies that $|S'| = |S|$, i.e. that restrictions to K of distinct places of K' yield distinct places of K. However, there is *a priori* no guarantee that the completions $K'_{v'}$ and K_v coincide for every $v' \in S'$.

Assume that there exists a nonempty subset $D \subsetneq S$ such that $K_v \neq K'_{v'}$ for $v \in D$ and $K'_{v'} = K_v$ for $v \in S \smallsetminus D$. According to [12, Assertion (iii) on p.105] there exists a unit $a = c_1^{k_1} \cdots c_d^{k_d} \in \mathfrak{R}_\mathbf{c}^\times \subset \mathfrak{R}'^\times_{S'}$ with $(k_1, \ldots, k_d) \in \mathbb{Z}^d$, $|a|_v > 1$ for $v \in D$, and $|a|_v \leq 1$ for $v \in S \smallsetminus D$. The product formula [3, Theorem 10.2.1], applied to $a \in K'$ and $a \in K$, implies that

$$\prod_{v \in S} |a|_v = \prod_{v \in D} |a|_v \cdot \prod_{v \in S \smallsetminus D} |a|_v = 1,$$

$$\prod_{v \in S} |a|_{v'} = \prod_{v \in D} |a|_{v'} \cdot \prod_{v \in S \smallsetminus D} |a|_{v'} = \prod_{v \in D} |a|_{v'} \cdot \prod_{v \in S \smallsetminus D} |a|_v = 1,$$

which is impossible due to the fact that $|a|_{v'} \geq |a|_v^2$ for every $v \in D$. This contradiction shows that $D = \varnothing$ or $D = S$.

In the former case $K_v = K'_{v'}$ for every $v \in P_\infty^{(K)}$ and $[K : \mathbb{Q}] = r + 2s = [K' : \mathbb{Q}]$, where r and s are the numbers of real and complex places of K, and hence $K = K'$.

In the latter case $K'_{v'} = \mathbb{C}$ and $K_v = \mathbb{R}$ for every $v \in P_\infty^{(K)}$, $s = 0$, $[K' : \mathbb{Q}] = 2r = 2[K : \mathbb{Q}]$, and $[K : K'] = 2$. □

Lemma 3.12. *Let $\mathbf{c} = (c_1, \ldots, c_d)$ be a free S'-unit system of K', and let $K = \mathbb{Q}[\mathbf{c}] \subset K'$.*

1. *$K' = K$ if and only if the $\mathbb{Z}^d$-action $\alpha_{(\mathbf{c},\mathfrak{I})}$ on $X_\mathfrak{I} = \hat{\mathfrak{I}}$ is irreducible for every nonzero ideal $\mathfrak{I} \subset \mathfrak{R}'_{S'}$;*
2. *$K' \neq K$ if and only if the $\mathbb{Z}^d$-action $\alpha_{(\mathbf{c},\mathfrak{I})}$ on $X_\mathfrak{I} = \hat{\mathfrak{I}}$ is reducible for every nonzero ideal $\mathfrak{I} \subset \mathfrak{R}'_{S'}$.*

Proof. Suppose that $K = K'$, and that $\mathcal{J} \subset \mathcal{R}_{\mathbf{c}} = \mathcal{R}'_{S'}$ is a nonzero ideal. As the construction after (3.5) applies, Proposition 3.2 shows that the action $\alpha_{(\mathbf{c},\mathcal{J})}$ is irreducible.

If $K' \neq K$, Lemma 3.11 shows that the ring $\mathcal{R}_{\mathbf{c}} = \mathcal{R}'_{S'} \cap K$ is an infinite $\mathbb{Z}[\mathbf{c}^{\pm 1}]$-submodule of infinite index in $\mathcal{R}_{\mathbf{c}}$. Hence $Y = \mathcal{R}_{\mathbf{c}}^{\perp} = \widehat{\mathcal{R}'_{S'}/\mathcal{R}_{\mathbf{c}}}$ is an infinite, closed, proper $\alpha_{(\mathbf{c},\mathcal{R}'_{S'})}$-invariant subgroup of $X_{\mathcal{R}'_{S'}}$. This implies that $\alpha_{(\mathbf{c},\mathcal{R}'_{S'})}$ is reducible.

Let $\mathcal{J} \subset \mathcal{R}'_{S'}$ be a nonzero ideal and $a \in \mathcal{J} \smallsetminus \{0\}$. By the previous paragraph we see that $a\mathcal{R}_{\mathbf{c}}$ has infinite index in $a\mathcal{R}'_{S'}$ and therefore also in $\mathcal{J}$. Since $Y = (a\mathcal{R}_{\mathbf{c}})^{\perp}$ is an infinite, closed, proper $\alpha_{(\mathbf{c},\mathcal{J})}$-invariant subgroup, the action $\alpha_{(\mathbf{c},\mathcal{J})}$ is reducible. □

Proof of Theorem 3.10. Lemma 3.11 shows that $K = K'$, and the assertions follow from the Lemmas 3.12 and Theorem 3.8. □

Remarks 3.13. (1) Even if $K' \neq K$ in Lemma 3.11 for some choice of fundamental S-units $\{c_1, \dots, c_d\}$, a different choice $\{c'_1, \dots, c'_d\}$ of fundamental S-units may lead to equality of K' and K.

(2) The most interesting special cases of the Theorems 3.8 and 3.10 occur when $S = P_{\infty}^{(K)}$, in which case $\mathcal{R}_{\mathbf{c}} = \mathfrak{o}_K$ is the ring of integers of K, $X_{\mathcal{R}_{\mathbf{c}}} = \mathbb{T}^{r+2s}$ and $\{c_1, \dots, c_d\}$ is a fundamental unit system, where r, s are the numbers of real and complex places of K and $d = r + s - 1$. For information on ideal classes of algebraic number fields we refer to [12].

4. Algebraic actions of expansive rank one

Throughout this section α is an algebraic $\mathbb{Z}^d$-action on an infinite compact connected abelian group X. We discuss the relationship between irreducibility and the rank one conditions introduced in Definition 1.1. In the rank one case we give a description of the action α similar to Theorem 3.4.

In some of the following results we assume the irreducible $\mathbb{Z}^d$-actions to be expansive. By applying Lemma 2.1 to the case of an irreducible action α we see that α is expansive if and only if no Galois conjugate of $\mathbf{c}$ consists entirely of elements of absolute value one. This is the same as saying there exists no valuation $v \in P_{\infty}^{(K)}$ with $|c_i|_v = 1$ for all i.

If one asks if a single automorphism $\alpha^{\mathbf{n}}$ is expansive, the answer is slightly different (Lemma 2.1 might not be applicable directly as $M = \widehat{X}$ might not be Noetherian over $\mathbb{Z}[u^{\pm\mathbf{n}}]$). However, by using the product structure of $V_{\mathbf{c}}$ in (3.6) it is easy to see that $\alpha^{\mathbf{n}}$ is expansive if and only if $|\mathbf{c}^{\mathbf{n}}|_v \neq 1$ for all $v \in S$.

Proposition 4.1. *Let α be an irreducible algebraic $\mathbb{Z}^d$-action on a compact connected abelian group X. Then α has entropy rank one.*

If α is in addition expansive, then α has expansive rank one. In fact, there exists a finite union U of hyperplanes in $\mathbb{R}^d$ such that $\alpha^{\mathbf{n}}$ is expansive for every $\mathbf{n} \in \mathbb{Z}^d \smallsetminus U$.

Proof. Let $\mathbf{c}$ and $S_{\mathbf{c}}$ be as in Theorem 3.4 and (3.5), respectively. The second proof of [15, Proposition 17.2] shows that

$$h_{\lambda_X}(\alpha^{\mathbf{n}}) = \sum_{\{v\in S_{\mathbf{c}}:|c^{\mathbf{n}}|_v>1\}} \log |c^{\mathbf{n}}|_v < \infty$$

for every $\mathbf{n}\in\mathbb{Z}^d$ (cf. (3.14)).

For every $v \in S_{\mathbf{c}}$ we see that $\log|\mathbf{c}^{\mathbf{n}}|_v = \sum_{i=1}^d n_i \log|c_i|_v$. Let U be the finite union of the hyperplanes defined by the linear functions $\mathbf{n} \mapsto \log|\mathbf{c}^{\mathbf{n}}|_v$, $v\in S_{\mathbf{c}}$. For $\mathbf{n}\in\mathbb{Z}^d\smallsetminus U$ we have that $|\mathbf{c}^{\mathbf{n}}|_v\neq 1$ for every $v\in S$ and hence that $\alpha^{\mathbf{n}}$ is expansive. □

Proposition 4.1 has a partial converse.

Proposition 4.2. *Let $\mathfrak{p}\subset R_d$ be a prime ideal and $\alpha_{R_d/\mathfrak{p}}$ the algebraic $\mathbb{Z}^d$-action on the compact connected group $X_{R_d/\mathfrak{p}} = \widehat{R_d/\mathfrak{p}}$ defined in* (2.3)*. If $\alpha_{R_d/\mathfrak{p}}$ has expansive rank one or entropy rank one, then $\alpha_{R_d/\mathfrak{p}}$ is irreducible.*

Proof. Suppose that $\alpha_{R_d/\mathfrak{p}}$ has expansive rank one and that $\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}}$ is expansive for some $\mathbf{n}\in\mathbb{Z}^d$. It is easy to see that the $\mathbb{Z}[u^{\pm\mathbf{n}}]$-module $R_d/\mathfrak{p}$ is associated to the prime ideal $\mathfrak{p}\cap\mathbb{Z}[u^{\pm\mathbf{n}}]$. By Lemma 2.1, the variety $V_{\mathbb{C}}(\mathfrak{p})$ of $\mathfrak{p}\cap\mathbb{Z}[u^{\pm\mathbf{n}}]$ cannot meet the circle $\{z\in\mathbb{C} : |z| = 1\}$. This shows that $f \in \mathfrak{p}\cap\mathbb{Z}[u^{\pm\mathbf{n}}]$ for some nonzero polynomial f. Again by expansiveness, $R_d/\mathfrak{p}$ must be a Noetherian module over $\mathbb{Z}[u^{\pm\mathbf{n}}]$. Hence every $u_i+\mathfrak{p}\in R_d/\mathfrak{p}$ satisfies a monic relation over $\mathbb{Z}[u^{\mathbf{n}}]$ and the variety (2.4) of $\mathfrak{p}$ is finite. Let $\mathbf{c}\in\bar{\mathbb{Q}}^{\times}$ be a point of this variety. Then $\mathfrak{p}\subset\mathfrak{p}_{\mathbf{c}}$ are two prime ideals with finite varieties (cf. (3.27)). If $\mathfrak{p}\neq\mathfrak{p}_{\mathbf{c}}$ it is easy to find polynomials f resp. g which vanish at $V(\mathfrak{p})\smallsetminus V(\mathfrak{p}_{\mathbf{c}})$ resp. $V(\mathfrak{p}_{\mathbf{c}})$. As $f,g\notin\mathfrak{p}$ but fg vanishes on $V(\mathfrak{p})$ and therefore belongs to $\mathfrak{p}$, we get a contradiction to the fact that $\mathfrak{p}$ is a prime ideal. This shows that $\alpha_{R_d/\mathfrak{p}} = \alpha_{R_d/\mathfrak{p}_{\mathbf{c}}}$ is irreducible.

Now assume $\alpha_{R_d/\mathfrak{p}}$ has entropy rank one. Let $\mathbf{n}=\mathbf{e}_i$ be one of the standard basis vectors of $\mathbb{Z}^d$. As $h(\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}})$ is finite, $\mathfrak{p}\cap\mathbb{Z}[u^{\pm\mathbf{n}}]$ cannot be trivial. For if $\mathfrak{p}\cap\mathbb{Z}[u^{\pm\mathbf{n}}]=\{0\}$, the shift σ on $\mathbb{T}^{\mathbb{Z}}$ would be a factor of $\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}}$ and hence $h(\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}})=\infty$. By applying this with $i=1,\dots,d$, we obtain that $V(\mathfrak{p})$ is finite and conclude as above that $\alpha_{R_d/\mathfrak{p}}=\alpha_{R_d/\mathfrak{p}_{\mathbf{c}}}$ is irreducible. □

Definition 4.3. ([15, Definition 3.1]) Let α be an algebraic $\mathbb{Z}^d$-action on the compact abelian group X. The action α satisfies the descending chain condition if there exists, for every non-increasing sequence $X\supset X_1\supset\cdots\supset X_k\supset\cdots$ of closed α-invariant subgroups of X, an integer $K\geq 1$ such that $X_k=X_K$ for all $k\geq K$. Every expansive $\mathbb{Z}^d$-action on a compact abelian group satisfies the descending chain condition by [15, (4.10)].

Theorem 4.4. *Let α be an algebraic $\mathbb{Z}^d$-action on an infinite compact connected abelian group X.*

(1) *If α has expansive rank one it has entropy rank one.*
(2) *If α has entropy rank one then the action $\alpha_{R_d/\mathfrak{p}}$ is irreducible for every associated prime ideal $\mathfrak{p}$ of $M=\widehat{X}$.*
(3) *If every associated prime ideal $\mathfrak{p}$ of $M=\widehat{X}$ satisfies that $\alpha_{R_d/\mathfrak{p}}$ is irreducible and α is expansive, then α has also expansive rank one.*

(4) *Assume that α satisfies the descending chain condition. If $\alpha_{R_d/\mathfrak{p}}$ is irreducible for every associated prime ideal $\mathfrak{p}$ of $M=\widehat{X}$ then α has entropy rank one.*

Proof. Assume α has expansive rank one and $\mathfrak{p}$ is an associated prime ideal of $M=\widehat{X}$. Then $\alpha_{R_d/\mathfrak{p}}$ has also expansive rank one and $\alpha_{R_d/\mathfrak{p}}$ is irreducible by Proposition 4.2. From Proposition 4.1 we know that $\alpha_{R_d/\mathfrak{p}}$ has entropy rank one. This shows that $h(\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}})<\infty$ for all $\mathbf{n}\in\mathbb{Z}^d$ and every associated prime ideal $\mathfrak{p}$. As M has a finite prime filtration (2.2) we get that $h(\alpha^{\mathbf{n}})<\infty$ by Yuzvinskii's addition formula (cf. [10] or [15, Theorem 14.1]) and hence that α has entropy rank one.

If α has entropy rank one and $\mathfrak{p}$ is an associated prime ideal of $M=\widehat{X}$, then $\alpha_{R_d/\mathfrak{p}}$ also has entropy rank one and Proposition 4.2 shows that $\alpha_{R_d/\mathfrak{p}}$ is irreducible.

If the action $\alpha_{R_d/\mathfrak{p}}$ is irreducible and expansive for every associated prime ideal, Proposition 4.1 shows that there exists, for every $\mathfrak{p}\in\operatorname{Asc}(M)$, a hyperplane $U_{\mathfrak{p}}\subset\mathbb{R}^d$ such that $\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}$ is expansive for $\mathbf{n}\notin U_{\mathfrak{p}}\cap\mathbb{Z}^d$. As expansiveness implies that M is a Noetherian R_d-module, there are only finitely many associated prime ideals $\mathfrak{p}\in\operatorname{Asc}(M)$. Therefore there exists $\mathbf{n}$ such that $\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}$ is expansive for every associated prime ideal $\mathfrak{p}$. From the finite filtration (2.2) it follows that $\alpha^{\mathbf{n}}$ is expansive (for prime ideals $\mathfrak{q}\supsetneq\mathfrak{p}\in\operatorname{Asc}(M)$ the space $X_{R_d/\mathfrak{q}}$ is finite by irreducibility and expansiveness is therefore trivial).

Finally we assume that $\alpha_{R_d/\mathfrak{p}}$ is irreducible for every associated prime ideal $\mathfrak{p}$ of $M=\widehat{X}$ and that α satisfies the descending chain condition. By duality the module M is Noetherian. Proposition 4.1 shows that $\alpha_{R_d/\mathfrak{p}}$ has entropy rank one and we conclude as before from the filtration (2.2) and Yuzvinskii's addition formula that α has entropy rank one. □

The remainder of this section will be devoted to a structure formula — analogous to Theorem 3.4 — for algebraic $\mathbb{Z}^d$-actions of expansive rank one. This structure formula will be needed in Section 5 in order to prove that any two expansive elements of an algebraic $\mathbb{Z}^d$-action of expansive rank one have isomorphic homoclinic modules.

For every rational prime p we write $\mathbb{Z}_p=\{x\in\mathbb{Q}_p:|x|_p\le 1\}\subset\mathbb{Q}_p$ for the ring of p-adic integers. For notational consistency we set $\mathbb{Q}_\infty=\mathbb{R}$.

Theorem 4.5. *Let α be an algebraic $\mathbb{Z}^d$-action on an infinite compact connected abelian group X with dual module $M=\widehat{X}$. Assume that α either has expansive rank one, or that α has entropy rank one and satisfies the descending chain condition. Then there exists a finite set of primes $S\subset\mathbb{N}$ with the following properties.*

(1) *For every $p\in S\cup\{\infty\}$ there exists a finite-dimensional vector space M_p over $\mathbb{Q}_p$ and a $\mathbb{Q}_p$-linear $\mathbb{Z}^d$-action $\bar\beta_p$ on M_p.*
(2) *Let $\bar\beta=\prod_{p\in S\cup\{\infty\}}\bar\beta_p$ be the product action on $\bar M=\prod_{p\in S\cup\{\infty\}}M_p$. Then we can find a discrete, co-compact, $\bar\beta$-invariant subgroup $\mathcal{N}\subset\bar M$ such that α is algebraically conjugate to the $\mathbb{Z}^d$-action β induced by $\bar\beta$ on $Y=\bar M/\mathcal{N}$.*

Theorem 4.5 turns out to be a consequence of the following proposition.

Proposition 4.6. *Let M be a Noetherian module over R_d which is torsion-free over $\mathbb{Z}$. Assume furthermore that there exists a finite set of primes $S \subset \mathbb{N}$ such that $M \otimes \mathbb{Z}[\frac{1}{p} : p \in S]$ is a free module of finite rank over the ring $\mathbb{Z}[\frac{1}{p} : p \in S] = \mathbb{Z}\big[1/\prod_{p \in S} p\big]$. Then there exists, for every $p \in S \cup \{\infty\}$, a finite-dimensional vector space M_p over $\mathbb{Q}_p$ which is a $\mathbb{Q}_p$-linear R_d-module such that the following holds.*

(1) *There is an injective module homomorphism*

$$\phi\colon M \longrightarrow V = \prod_{p \in S \cup \{\infty\}} M_p. \tag{4.1}$$

(2) *The image $\phi(M) \subset V$ is a discrete and co-compact subgroup.*

Proof of Theorem 4.5*, given Proposition* 4.6. Since X is connected, the R_d-module $M = \widehat{X}$ is a torsion-free additive group. Furthermore, since the action α_X is expansive or satisfies the descending chain condition, M is Noetherian.

In order to choose the set S we first consider the case where $M = R_d/\mathfrak{p}$ for some prime ideal $\mathfrak{p} = \mathfrak{p}_{\mathbf{c}} \subset R_d$ of the form (3.27) with $\mathbf{c} = (c_1, \dots, c_d) \in (\bar{\mathbb{Q}}^\times)^d$. Let $f_i \in \mathbb{Z}[u]$ be the primitive minimal polynomial of c_i, and let S be the set of primes dividing the leading or trailing coefficients of any of the f_i, $i = 1, \dots, d$. If k_i is the degree of f_i, then any $\mathbf{c}^{\mathbf{m}}$, $\mathbf{m} \in \mathbb{Z}^d$, belongs to the additive group generated by the set

$$\begin{aligned} \{a\mathbf{c}^{\mathbf{n}} : a \in \mathbb{Z}[\frac{1}{p} : p \in S], \\ \mathbf{n} = (n_1, \dots, n_d) \text{ with } 0 \le n_i < k_i \text{ for } i = 1, \dots, d\} \end{aligned}$$

This shows that $R_d/\mathfrak{p} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p} : p \in S]$ is a free module of finite rank over the ring $\mathbb{Z}[\frac{1}{p} : p \in S]$.

In the case of a general Noetherian R_d-module M we take the union of the sets $S = S_{\mathfrak{p}}$ of the preceding paragraph for every associated prime ideal $\mathfrak{p}$ of M. By using the finite filtration (2.2) and enlarging S, if necessary, to kill all finite factors $R_d/\mathfrak{q}$ arising from prime ideals $\mathfrak{q} \subset R_d$ which properly contain one of the associated primes of M, we have found a set S satisfying the hypotheses of Proposition 4.6.

Let M_p, $p \in S$, and ϕ be chosen as in Proposition 4.6. As the dual group $\widehat{\mathbb{Q}_p}$ is isomorphic to $\mathbb{Q}_p$, we can identify M_p and $\bar{M}$ with their duals $\widehat{M_p}$ resp. $\widehat{\bar{M}}$. For every $\mathbf{n} \in \mathbb{Z}^d$ and $p \in S$ we define the $\mathbb{Q}_p$-linear map $\bar{\beta}_p^{\mathbf{n}}\colon M_p \longrightarrow M_p$ as the dual of multiplication with $u^{\mathbf{n}}$ on $M_p = \widehat{M_p}$.

Let $\mathcal{N} = \phi(M)^\perp \subset V = \bar{M} = \prod_{p \in S \cup \{\infty\}} M_p$ be the annihilator of $\phi(M)$. As $\phi(M)$ is a discrete, co-compact submodule of V, $\mathcal{N}$ is invariant under the action $\bar{\beta}$ and is again discrete and co-compact. Furthermore,

$$\widehat{M} \cong \widehat{\phi(M)} \cong V/\phi(M)^\perp = V/\mathcal{N},$$

and these isomorphisms conjugate the $\mathbb{Z}^d$-action α and β. □

Proof of Proposition 4.6. We set $M_\infty = M \otimes_{\mathbb{Z}} \mathbb{R}$. Clearly, M_∞ is a finite-dimensional vector space over $\mathbb{R}$ and a module over R_d, and multiplication with $u_i \in R_d$ is $\mathbb{R}$-linear for $i = 1, \dots, d$. Furthermore the homomorphism $\phi_\infty \colon M \longrightarrow M_\infty$, defined by $\phi_\infty(m) = m \otimes 1_{\mathbb{R}}$ for every $m \in M$, is injective.

For every prime $p \in S$, $V_p = M \otimes_{\mathbb{Z}} \mathbb{Q}_p$ is a finite-dimensional vector space over $\mathbb{Q}_p$ and a module over $\mathbb{Q}_p[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$. We are going to construct an invariant subspace $M_p \subset V_p$ which will be spanned by all expanding and contracting subspaces of the $\mathbb{Q}_p$-linear maps consisting of multiplication by u_i, $i = 1, \dots, d$.

Let A_j be the $\mathbb{Q}_p$-linear map on V_p defined by multiplication with u_j. Since A_j restricts also to the $\mathbb{Q}$-vector space $M \otimes_{\mathbb{Z}} \mathbb{Q}$, the minimal polynomial χ_j of A_j can be chosen such that $\chi_j \in \mathbb{Z}[t]$ and χ_j is primitive; in particular, p does not divide χ_j. Hensel's Lemma allows us to write χ_j as a product $\chi_j(t) = f_j(t) g_j(t)$ with $f_j, g_j \in \mathbb{Z}_p[t]$, where f_j has its leading and trailing coefficients in $\mathbb{Z}_p^\times$ and $g_j \equiv t^{n_j} \pmod{p}$ for some n_j. Obviously $f_j, g_j \in \mathbb{Z}_p[t]$ are co-prime, hence

$$a_j f_j + b_j g_j = 1 \tag{4.2}$$

for some $a_j, b_j \in \mathbb{Q}_p[t]$. We can thus form the polynomial

$$h(u_1, \dots, u_d) = u^{-\mathbf{n}} \prod_{j=1}^{d} g_j(u_j) \in \mathbb{Z}_p[u_1^{\pm 1}, \dots, u_d^{\pm 1}], \tag{4.3}$$

where $\mathbf{n} \in \mathbb{Z}^d$ is chosen such that

$$h \equiv 1 \pmod{p}. \tag{4.4}$$

By taking the product of the Equations (4.2) for $j = 1, \dots, d$, we obtain polynomials $a_j^*, b^* \in \mathbb{Q}_p[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ with

$$1 = \sum_{j=1}^{d} a_j^* f_j' + b^* h, \tag{4.5}$$

where $f_j' = f_j(u_j) \in \mathbb{Z}_p[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$.

Since multiplication by each of the variables u_i defines a $\mathbb{Q}_p$-linear map on V_p, the polynomials h and f_j' induce $\mathbb{Q}_p$-linear maps on V_p. By construction, the linear maps corresponding to the polynomials $\chi_j(u_j)$ and $h f_j'$ are equal to zero, and (4.5) implies that the subspaces

$$M_p = \ker h, \qquad N_p = \bigcap_{j=1}^{d} \ker f_j'$$

satisfy that $V_p = M_p \oplus N_p$; in fact, the projections

$$\pi_1^{(p)} \colon V_p \longrightarrow M_p, \qquad \pi_2^{(p)} \colon V_p \longrightarrow N_p$$

are given by

$$\pi_1^{(p)}(x) = \sum_{j=1}^{d} a_j^* f_j' x \in M_p, \qquad \pi_2^{(p)}(x) = b^* h x \in N_p.$$

Clearly, M_p, N_p are $\mathbb{Q}_p[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$-submodules (and hence R_d-submodules) of V_p, and $\pi_1^{(p)}, \pi_2^{(p)}$ are module homomorphisms.

We set

$$V = \prod_{p\in S\cup\{\infty\}} M_p,$$

write every $v \in V$ as $v = (v_p) = (v_p,\, p \in S \cup \{\infty\})$, define $\phi_p\colon M \longrightarrow M_p$ by $\phi_p(m) = \pi_1^{(p)}(m \otimes 1_{\mathbb{Q}_p})$ for every $p \in S$ and $m \in M_p$, and consider the R_d-module homomorphism $\phi\colon M \longrightarrow V$ given by

$$\phi(m) = (\phi_p(m),\, p \in S \cup \{\infty\})$$

for every $m \in M$ (recall that $\phi_\infty\colon M \longrightarrow M_\infty$ was already defined at the beginning of this proof). To conclude the proof we have to show that $\mathcal{N} = \phi(M)$ is a discrete and co-compact subgroup of V.

From the discussion concerning (3.8) we know that $\mathbb{Z}[\frac{1}{p} : p \in S]$ is a discrete subgroup of $\prod_{p\in S\cup\{\infty\}} \mathbb{Q}_p$. As $M \otimes \mathbb{Z}[\frac{1}{p} : p \in S]$ is a free module over the ring $\mathbb{Z}[\frac{1}{p} : p \in S]$ by hypothesis, the subgroup

$$\bigl\{(m \otimes 1_{\mathbb{Q}_p},\, p \in S \cup \{\infty\}) : m \in M\bigr\} \subset \prod_{p\in S\cup\{\infty\}} M \otimes \mathbb{Q}_p$$

is discrete. We claim that, for fixed $p \in S$, the set $\pi_2^{(p)}(M \otimes 1_{\mathbb{Q}_p})$ is contained in a compact subgroup $C_p \subset N_p$. If we can prove this, then $M \otimes 1_{\mathbb{Q}_p} \subset M_p \times C_p$ of every $p \in S$, and since $\mathcal{N} = \phi(M)$ is obtained by projection along the compact kernel $\prod_{p\in S} C_p$, $\mathcal{N}$ must be a discrete subgroup of V.

To verify this claim we choose $L \geq 1$ with $\deg f_j \leq L$ for $j = 1, \dots, d$, and select a set $\{m_1', \dots, m_k'\}$ of generators of the module M over R_d. Put

$$B = \Bigl\{u^{\mathbf{n}}\pi_2^{(p)}(m_i' \otimes 1_{\mathbb{Q}_p}) : 1 \leq i \leq k, 0 \leq n_j < L \text{ for all } 1 \leq j \leq d\Bigr\}$$

and let $C_p \subset M \otimes \mathbb{Q}_p$ be the compact $\mathbb{Z}_p$-module generated by B. Since each f_j has leading and trailing coefficients in $\mathbb{Z}_p^\times$ and annihilates N_p it follows that $\pi_2^{(p)}(M \otimes 1_{\mathbb{Q}_p})$ is contained in the $\mathbb{Z}_p$-module C_p generated by B.

We proceed with the proof that the quotient $V/\mathcal{N}$ is compact. Let $m_1, \dots, m_r \in M$ be a basis of the $\mathbb{Q}$-vector space $M \otimes_{\mathbb{Z}} \mathbb{Q}$. We assert that every element of $V/\mathcal{N}$ can be represented by an element of the subgroup

$$G = M_\infty \times \prod_{p\in S} \left(\sum_{i=1}^{r} \mathbb{Z}_p \phi_p(m_i)\right).$$

For this it is enough to show that any element $v = (v_q,\, q \in S \cup \{\infty\}) \in V$ of the form

$$v_q = \begin{cases} 0 & \text{for } q \neq p \\ \pi_1^{(p)}(m \otimes a) & \text{for } q = p \end{cases} \tag{4.6}$$

with $p \in S$, $m \in M$ and $a \in \mathbb{Q}_p$, can be represented modulo $\mathcal{N}$ by an element in G.

Multiplying m and dividing a in (4.6) by the same integer does not change v. This allows us to assume without loss of generality that m lies in the subgroup of M generated by $\{m_1, \dots, m_r\}$.

Let $j \geq 0$ be such that $p^j a \in \mathbb{Z}_p$. We denote by h the polynomial in (4.3) and assume without loss in generality (by multiplying m with an integer co-prime to p) that

$$u^{\mathbf{n}}m \in \sum_{i=1}^{r} \mathbb{Z}[\frac{1}{p}]m_i \tag{4.7}$$

for every $\mathbf{n} \in \mathbb{Z}^d$ for which $u^{\mathbf{n}}$ has a nonzero coefficient in $(1-h)^j$ (i.e. with $(1-h)^j_{\mathbf{n}} \neq 0$ in the notation of (2.1)). Since $v \in G$ for $a \in \mathbb{Z}_p$ we may restrict ourselves to the case where $a = \frac{1}{p^j}$ for some $j \geq 1$.

We fix $k \geq 1$ for the moment and choose $h(k) \in R_d$ with $h(k)_{\mathbf{n}} = h_{\mathbf{n}}$ (mod p^k) for every $\mathbf{n} \in \mathbb{Z}^d$.

According to (4.4), $\frac{1-h(k)}{p} \in R_d$. Since h annihilates M_p,

$$\begin{aligned} \phi_p(am) = a\phi_p(m) = \frac{1}{p^j}\phi_p(m) = \frac{(1-h)^j}{p^j}\phi_p(m) \\ = \phi_p\left(\left(\frac{1-h(k)}{p}\right)^j m\right) + \frac{(1-h)^j - (1-h(k))^j}{p^j}\phi_p(m). \end{aligned}$$

For $k \geq j$ large enough we get cancellation with the denominators in (4.7) and therefore

$$y_p = \frac{(1-h)^j - (1-h(k))^j}{p^j}\phi_p(m) \in \sum_{i=1}^{r} \mathbb{Z}_p\phi_p(m_i).$$

For $q \neq p$ (4.7) shows that

$$y_q = -\phi_q\left(\left(\frac{1-h(k)}{p}\right)^j m\right) \in \sum_{i=1}^{r} \mathbb{Z}_q\phi_q(m_i),$$

and hence that $v = \phi\left(\left(\frac{1-h(k)}{p}\right)^j m\right) + y$ with $y \in G$.

We have shown that

$$V/\mathcal{N} \cong G/(G \cap \mathcal{N}).$$

Although G is not compact, the quotient G/H is compact, where $H \subset G \cap \mathcal{N}$ is the subgroup generated by $\{\phi(m_i) : i = 1, \dots, k\}$. Hence $V/\mathcal{N}$ is compact. □

5. Homoclinic points for actions of expansive rank one

In this section we study the groups of homoclinic points for different expansive elements of an algebraic $\mathbb{Z}^d$-action of expansive rank one. The results in this section generalize the results for commuting toral automorphism in [11].

Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X. We define the groups $\Delta_\alpha(X)$ and $\Delta_{\alpha^{\mathbf{n}}}(X)$ of homoclinic points of α and $\alpha^{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^{\mathbf{d}}$, as in the Introduction. Both $\Delta_\alpha(X)$ and $\Delta_{\alpha^{\mathbf{n}}}(X)$ are R_d-modules with respect to the operation

$$(f, x) \mapsto f \cdot x = f(\alpha)(x) = \sum_{\mathbf{m} \in \mathbb{Z}^d} f_{\mathbf{m}}\alpha^{\mathbf{m}}x$$

for $f = \sum_{\mathbf{m} \in \mathbb{Z}^d} f_{\mathbf{m}}u^{\mathbf{m}} \in R_d$ and $x \in \Delta_\alpha(X)$ or $x \in \Delta_{\alpha^{\mathbf{n}}}(X)$.

If an algebraic $\mathbb{Z}^d$-action α with $d' \geq 2$ has expansive rank one, then $h(\alpha) = 0$ and α^* is trivial. Following [4, Section 6] one can, however, use an individual expansive element $\alpha^{\mathbf{n}}$ of the action α to obtain the R_d-module

$\Delta_{\alpha^{\mathbf{n}}}(X)$ and again define an associated $\mathbb{Z}^d$-action $\beta = \alpha_{\Delta_{\alpha^{\mathbf{n}}}(X)}$ on $Y = \widehat{\Delta_{\alpha^{\mathbf{n}}}(X)}$. In this section we investigate how this construction depends on the choice of $\mathbf{n}$, under the hypothesis that the group X is connected.

We begin our discussion in the case where α is irreducible and mixing.

Theorem 5.1. *Let $d \geq 1$, and let α be an expansive, irreducible and mixing algebraic $\mathbb{Z}^d$-action on an infinite compact connected abelian group X. Let $\alpha'_{(\mathbf{c},\mathcal{K})}$ be a realization of α as in Corollary 3.5, where $\mathbf{c} = (c_1, \dots, c_d) \in (\bar{\mathbb{Q}}^\times)^d$, $K = \mathbb{Q}[\mathbf{c}] = \mathbb{Q}[c_1, \dots, c_d]$, $S_{\mathbf{c}}$ is defined in (3.5), and $\mathcal{K} \subset \mathcal{R}_{\mathbf{c}}$ is a finitely generated $\mathbb{Z}[\mathbf{c}^{\pm 1}]$-submodule.*

For every $v \in S_{\mathbf{c}}$ we define the group homomorphism $\omega_v \colon \mathbf{n} \mapsto \log |\mathbf{c}^{\mathbf{n}}|_v$ from $\mathbb{Z}^d$ to $\mathbb{R}$ and set $H_v = \{\mathbf{n} \in \mathbb{Z}^d : \omega_v(\mathbf{n}) = 0\}$. Then

$$E_\alpha = \{\mathbf{m} \in \mathbb{Z}^d : \alpha^{\mathbf{m}} \text{ is expansive}\} = \mathbb{Z}^d \smallsetminus \bigcup_{v \in S_{\mathbf{c}}} H_v. \tag{5.1}$$

Furthermore the following is true.

1. *If $\mathbf{m} \in \mathbb{Z}^d \smallsetminus E_\alpha$, then $\Delta_{\alpha^{\mathbf{m}}}(X) = \{0\}$;*
2. *If $\mathbf{m} \in E_\alpha$, then $\Delta_{\alpha^{\mathbf{m}}}(X)$ is dense in X and isomorphic to $\mathcal{K}$ as a module over R_d;*
3. *If $\mathbf{m}, \mathbf{n} \in E_\alpha$, then*

$$\Delta_{\alpha^{\mathbf{m}}}(X) \cap \Delta_{\alpha^{\mathbf{n}}}(X) = \begin{cases} \Delta_{\alpha^{\mathbf{m}}}(X) & \text{if } \omega_v(\mathbf{m})\omega_v(\mathbf{n}) > 0 \text{ for every } v \in S_{\mathbf{c}}, \\ & \text{or if } \omega_v(\mathbf{m})\omega_v(\mathbf{n}) < 0 \text{ for every } v \in S_{\mathbf{c}}. \\ \{0\} & \text{otherwise,} \end{cases}$$

4. *Let $\mathbf{m} \in E_\alpha$, and let $\alpha^* = \alpha_{\Delta_{\alpha^{\mathbf{m}}}(X)}$ be the algebraic $\mathbb{Z}^d$-action on $X^* = \widehat{\Delta_{\alpha^{\mathbf{m}}}(X)}$ defined by (2.3). Then $E_\alpha = E_{\alpha^*}$ and the homoclinic module $\Delta_{(\alpha^*)^{\mathbf{n}}(X)}$ of any $(\alpha^*)^{\mathbf{n}}$ with $\mathbf{n} \in E_\alpha = E_{\alpha^*}$ is isomorphic to $\widehat{X}$.*

Proof. Define $V_{\mathbf{c}}$ by (3.6) and (3.5) and choose a $\mathbb{Z}[\mathbf{c}^{\pm 1}]$-submodule $\mathcal{K} \subset \mathcal{R}_{\mathbf{c}}$ according to Corollary 3.5. Since α is algebraically conjugate to the $\mathbb{Z}^d$-action $\alpha'_{(\mathbf{c},\mathcal{K})}$ on $X'_{\mathcal{K}} = V_{\mathbf{c}}/\iota_{\mathbf{c}}(\mathcal{K})$ defined by (3.28)–(3.29) we assume for simplicity that $X = X'_{\mathcal{K}}$ and $\alpha = \alpha'_{(\mathbf{c},\mathcal{K})}$.

For every $\mathbf{n} \in \mathbb{Z}^d$ we denote by

$$\begin{aligned} W^u_{\mathbf{n}} &= \{(a_v, v \in S_{\mathbf{c}}) \in V_{\mathbf{c}} : a_v = 0 \text{ for every } v \in S_{\mathbf{c}} \text{ with } |\mathbf{c}^{\mathbf{n}}|_v > 1\}, \\ W^s_{\mathbf{n}} &= \{(a_v, v \in S_{\mathbf{c}}) \in V_{\mathbf{c}} : a_v = 0 \text{ for every } v \in S_{\mathbf{c}} \text{ with } |\mathbf{c}^{\mathbf{n}}|_v < 1\} \end{aligned}$$

the unstable and stable subspaces of $\bar{\beta}^{\mathbf{n}}_{\mathbf{c}}$ in $V_{\mathbf{c}}$ (cf. (3.15)).

If $\mathbf{n} \neq \mathbf{0}$, then $\mathbf{c}^{\mathbf{n}}$ is not a root of unity due to our assumption that α is mixing, and Kronecker's theorem (cf. [8]) guarantees that $|\mathbf{c}^{\mathbf{n}}|_v \neq 1$ for some $v \in S_{\mathbf{c}}$. The product formula [3, Theorem 10.2.1] shows that $W^u_{\mathbf{n}} \neq \{0\}$ and $W^s_{\mathbf{n}} \neq \{0\}$, since $\prod_{v \in S_{\mathbf{c}}} |\mathbf{c}^{\mathbf{n}}|_v = 1$. As $\iota_{\mathbf{c}}(\mathcal{K})$ is a discrete subgroup of $V_{\mathbf{c}}$ and thus has no contracting automorphisms, $\iota_{\mathbf{c}}(\mathcal{K}) \cap W^u_{\mathbf{n}} = \iota_{\mathbf{c}}(\mathcal{K}) \cap W^s_{\mathbf{n}} = \{0\}$. Hence the quotient map $\pi \colon V_{\mathbf{c}} \longrightarrow X = V_{\mathbf{c}}/\iota_{\mathbf{c}}(\mathcal{K})$ is injective on the subspaces $W^u_{\mathbf{n}}$ and $W^s_{\mathbf{n}}$ of $V_{\mathbf{c}}$.

First we assume that $\mathbf{n} \in E_\alpha$. In this case $W^u_{\mathbf{n}} + W^s_{\mathbf{n}} = V_{\mathbf{c}}$ and there exists, for every $a \in \mathcal{K}$, a unique point $w_a \in V_{\mathbf{c}}$ with $W^u_{\mathbf{n}} \cap (W^s_{\mathbf{n}} + \iota_{\mathbf{c}}(a)) = \{w_a\}$

and $\pi(w_a) \in \Delta_{\alpha^{\mathbf{n}}}(X)$. Conversely, if $y \in \Delta_{\alpha^{\mathbf{n}}}(X)$ and $w \in \pi^{-1}(\{y\})$, then the continuity of $\bar{\beta}_{\mathbf{c}}^{\mathbf{n}}$ implies that there exist points $a, a' \in \mathcal{K}$ with

$$\lim_{k\to\infty} \bar{\beta}_{\mathbf{c}}^{k\mathbf{n}}(w - \iota_{\mathbf{c}}(a)) = \lim_{k\to-\infty} \bar{\beta}_{\mathbf{c}}^{k\mathbf{n}}(w - \iota_{\mathbf{c}}(a')) = 0,$$

and hence with $w \in (W_{\mathbf{n}}^u + \iota_{\mathbf{c}}(a)) \cap (W_{\mathbf{n}}^s + \iota_{\mathbf{c}}(a'))$ or, equivalently, $w = w_{a'-a}$. This shows that $\Delta_{\alpha^{\mathbf{n}}}(X) \cong \mathcal{K}$. The isomorphism $\Delta_{\alpha^{\mathbf{n}}}(X) \cong \mathcal{K}$ is obviously an R_d-module isomorphism, as claimed in (2).

The closure $Z = \overline{\Delta_{\alpha^{\mathbf{n}}}(X)}$ of the group of $\alpha^{\mathbf{n}}$-homoclinic points in X is a closed, infinite, α-invariant subgroup of X, and hence $Z = X$ by irreducibility. For the proof of (4) we observe that

$$X^* = \widehat{\mathcal{K}} \cong \widehat{\iota_{\mathbf{c}}(\mathcal{K})} = V_{\mathbf{c}}/\iota_{\mathbf{c}}(\mathcal{K})^{\perp},$$

and that α^* is algebraically conjugate to the quotient action on $V_{\mathbf{c}}/\iota_{\mathbf{c}}(\mathcal{K})^{\perp}$ induced by $\bar{\beta}$; here $\iota_{\mathbf{c}}(\mathcal{K})^{\perp}$ is the annihilator of $\iota_{\mathbf{c}}(\mathcal{K})$ in $\widehat{V_{\mathbf{c}}} \cong V_{\mathbf{c}}$. As above we see that $\Delta_{(\alpha^*)^{\mathbf{n}}} \cong \iota_{\mathbf{c}}(\mathcal{K})^{\perp}$ and hence that $\widehat{X} \cong \iota_{\mathbf{c}}(\mathcal{K})^{\perp} \subset V_{\mathbf{c}}$. This isomorphism carries the dual action $\hat{\alpha}$ of $\mathbb{Z}^d$ on $\widehat{X}$ to $\widehat{\alpha^{**}}$ on $\widehat{X^{**}} = \Delta_{(\alpha^*)^{\mathbf{n}}}$, as claimed in (4).

For (3) we use the same argument as in [11]. If $\mathbf{m}, \mathbf{n} \in E_\alpha$ satisfy that $\omega_v(\mathbf{m})\omega_v(\mathbf{n}) > 0$ for every $v \in S_{\mathbf{c}}$, then $W_{\mathbf{m}}^u = W_{\mathbf{n}}^u$, $W_{\mathbf{m}}^s = W_{\mathbf{n}}^s$, and the homoclinic points of $\alpha^{\mathbf{m}}$ and $\alpha^{\mathbf{n}}$ coincide.

If $\omega_v(\mathbf{m})\omega_v(\mathbf{n}) < 0$ for every $v \in S_{\mathbf{c}}$, then $W_{\mathbf{m}}^u = W_{\mathbf{n}}^s$ and $W_{\mathbf{m}}^s = W_{\mathbf{n}}^u$, and the homoclinic points of $\alpha^{\mathbf{m}}$ and $\alpha^{\mathbf{n}}$ again coincide.

If $\mathbf{m}$ and $\mathbf{n}$ satisfy neither of these conditions then we can interchange $\mathbf{m}$ and $\mathbf{n}$ and replace $\mathbf{n}$ by $-\mathbf{n}$, if necessary, and assume that $W_{\mathbf{m}}^u + W_{\mathbf{n}}^u \neq V_{\mathbf{c}}$, $W_{\mathbf{m}}^s + W_{\mathbf{n}}^s \neq V_{\mathbf{c}}$ and $W_{\mathbf{m}}^s + W_{\mathbf{n}}^u \neq V_{\mathbf{c}}$.

If a point $x \in X$ is homoclinic both for $\alpha^{\mathbf{m}}$ and $\alpha^{\mathbf{n}}$ then there exist elements $w \in V_{\mathbf{c}}$ and $a_1, a_2, a_3 \in \mathcal{K}$ with $\pi(w) = x$ and

$$\{w\} = W_{\mathbf{m}}^s \cap (W_{\mathbf{m}}^u - \iota_{\mathbf{c}}(a_1)) = (W_{\mathbf{n}}^s - \iota_{\mathbf{c}}(a_2)) \cap (W_{\mathbf{n}}^u - \iota_{\mathbf{c}}(a_3)).$$

Hence

$$\iota_{\mathbf{c}}(a_2) \in W_{\mathbf{m}}^s + W_{\mathbf{n}}^s = \bigoplus_{\{v \in S_{\mathbf{c}} : |\mathbf{c}^{\mathbf{m}}|_v < 1 \text{ and } |\mathbf{c}^{\mathbf{n}}|_v < 1\}} K_v \subsetneq \bigoplus_{v \in S_{\mathbf{c}}} K_v = V_{\mathbf{c}}$$

and from the definition of $\iota_{\mathbf{c}}(a_2)$ in (3.6) we conclude that $a_2 = 0$. Similarly we see that $a_1 = a_3 = 0$, since

$$\iota_{\mathbf{c}}(a_3) \in W_{\mathbf{m}}^s + W_{\mathbf{n}}^u \neq V_{\mathbf{c}}, \ \ \iota_{\mathbf{c}}(a_1) \in W_{\mathbf{m}}^u + W_{\mathbf{n}}^u \neq V_{\mathbf{c}}.$$

This shows that $w \in W_{\mathbf{m}}^s \cap W_{\mathbf{m}}^u = \{0\}$ and $\Delta_{\alpha^{\mathbf{m}}}(X) \cap \Delta_{\alpha^{\mathbf{n}}}(X) = \{0\}$.

Finally we note that $\Delta_{\alpha^{\mathbf{n}}}(X) = \{0\}$ if $\alpha^{\mathbf{n}}$ is nonexpansive. Indeed, if $\mathbf{n} \notin E_\alpha$, then $|\mathbf{c}^{\mathbf{n}}|_v = 1$ for some $v \in S_{\mathbf{c}}$. If $x \in \Delta_{\alpha^{\mathbf{n}}}(X)$ and $w \in \pi^{-1}(\{x\})$ then $w \in W_{\mathbf{n}}^s \cap (W_{\mathbf{n}}^u - \iota_{\mathbf{c}}(a))$ for some $a \in \mathcal{K}$ with $a \in W_{\mathbf{n}}^s \cap W_{\mathbf{n}}^u$. However, $W_{\mathbf{n}}^s + W_{\mathbf{n}}^u \neq V_{\mathbf{c}}$, and the same argument as in the last paragraph shows that $a = 0$, $w \in W_{\mathbf{n}}^u \cap W_{\mathbf{n}}^s = \{0\}$, and $x = \pi(w) = 0$. □

In Theorem 5.1 we have used implicitly the following terminology from [4].

Definition 5.2. Let α be an algebraic $\mathbb{Z}^d$-action on an infinite compact connected abelian group X of expansive rank one. The *adjoint action* of α is the algebraic $\mathbb{Z}^d$-action $\alpha^* = \alpha_{\Delta_{\alpha^{\mathbf{m}}}(X)}$ on the compact connected abelian group $X^* = \widehat{\Delta_{\alpha^{\mathbf{m}}}(X)}$ in (2.3) determined by the R_d-module $\Delta_{\alpha^{\mathbf{m}}}(X)$ for some (and hence, by Theorem 5.1 or Theorem 5.5 below, for any) expansive automorphism $\alpha^{\mathbf{m}}$, $\mathbf{m} \in \mathbb{Z}^d$.

According to Theorem 5.1 (4), the adjoint action α^* of an algebraic $\mathbb{Z}^d$-action α on an infinite compact connected abelian group X of expansive rank one again has expansive rank one and the compact group X^* is infinite and connected. We define the *second adjoint* action $\alpha^{**} = (\alpha^*)^*$ of α as before and obtain the following corollary.

Corollary 5.3. *Let α be an irreducible algebraic $\mathbb{Z}^d$-action on a compact connected abelian group X. Then the second adjoint α^{**} of α on $X^{**} = \widehat{\Delta_{\alpha^{\mathbf{n}}(X)}}$ is algebraically conjugate to α.*

Proof. Theorem 5.1 (4) shows that the dual modules $\widehat{X}$ and $\widehat{X^{**}} = \Delta_{(\alpha^*)^{\mathbf{n}}(X)}$ with $\mathbf{n} \in E_\alpha = E_{\alpha^*}$ are isomorphic, which proves that α and α^{**} are algebraically conjugate. □

In the setting of Theorem 3.8 the adjoint action α^* can be described explicitly. The proof is left to the reader as an exercise.

Corollary 5.4. *Suppose that K is an algebraic number field, $\mathbf{c} \in (K^\times)^d$ a vector of nonzero algebraic numbers with $K = \mathbb{Q}(\mathbf{c})$, and let $S_{\mathbf{c}} \subset P^{(K)}$ be the set of places defined by (3.5). For every nonzero ideal $\mathcal{I} \subset \mathcal{R}_{\mathbf{c}}$ the adjoint action $\alpha^*_{(\mathbf{c},\mathcal{I})}$ of the irreducible algebraic $\mathbb{Z}^d$-action $\alpha_{(\mathbf{c},\mathcal{I})}$ described in Theorem 3.8 is of the form $\alpha^*_{(\mathbf{c},\mathcal{I})} = \alpha_{(\mathbf{c}\mathcal{I}^{-1})}$, where $\mathcal{I}^{-1} = \{a \in K : a\mathcal{I} \subset \mathcal{R}_{\mathbf{c}}\}$.*

We end this section with a discussion of the module of homoclinic points for expansive elements in a general algebraic $\mathbb{Z}^d$-action of expansive rank one. The description of such actions in Theorem 4.5 allows to use almost exactly the same argument as in the irreducible case.

Theorem 5.5. *Let α be an algebraic action on an infinite compact connected abelian group X of expansive rank one. The homoclinic modules*

$$\Delta_{\alpha^{\mathbf{n}}} \cong \Delta_{\alpha^{\mathbf{m}}}$$

are isomorphic as R_d-modules for any two expansive automorphisms $\alpha^{\mathbf{m}}, \alpha^{\mathbf{n}}$, $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$.

Proof. By Theorem 4.5 we may assume that α is the $\mathbb{Z}^d$-action induced by $\bar{\beta}$ on $Y = \bar{M}/\mathcal{N}$ for some discrete, co-compact, $\bar{\beta}$-invariant subgroup $\mathcal{N} \subset \bar{M}$. Here $\bar{M}$ is a product of finite-dimensional vector spaces M_p over $\mathbb{Q}_p$ for $p \in S \cup \{\infty\}$, where S is a finite set of rational primes.

Fix $\mathbf{n} \in \mathbb{Z}^d$ such that $\alpha^{\mathbf{n}}$ is expansive. We claim that $\Delta_{\alpha^{\mathbf{n}}} \cong \mathcal{N}$.

By expansiveness, $\bar{M}$ is a sum of the stable and unstable subgroups

$$W^s = \{v \in \bar{M} : \bar{\beta}^{k\mathbf{n}} v \to 0 \text{ as } k \to \infty\},$$
$$W^u = \{v \in \bar{M} : \bar{\beta}^{-k\mathbf{n}} v \to 0 \text{ as } k \to \infty\},$$
$$\bar{M} = W^s \oplus W^u.$$

If $a \in \mathcal{N}$ we let $w_a \in \bar{M}$ be the element uniquely determined by $w_a \in W^s \cap (\{a\} + W^u)$. Then $x_a = w_a + \mathcal{N}$ is a homoclinic point for α. Conversely, if $x \in \Delta_{\alpha^{\mathbf{n}}}$ is homoclinic and $w \in M_{\mathbf{c}}$ is a point with $w + \mathcal{N} = x$, then there exists $a_1, a_2 \in \mathcal{N}$ such that $w - a_1 \in W^s$ and $w - a_2 \in W^u$. In other words, $x = x_{a_2 - a_1}$. It can easily be checked that the map $a \mapsto x_a$ is an R_d-module isomorphism. □

The proof of the following corollary is completely analogous to that of Theorem 5.1 (4) and Corollary 5.3.

Corollary 5.6. *Let α be an algebraic $\mathbb{Z}^d$-action on an infinite compact connected abelian group X of expansive rank one with adjoint action α^* on X^*. Then α^* has expansive rank one, and the adjoint action α^{**} of α^* is algebraically conjugate to α.*

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