LECTURES ON COCYCLES OF ERGODIC TRANSFORMATION GROUPS

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November, 1976

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Preface

These notes cover the material of a course jointly arranged by the Indian Statistical Institute and the Indian Institute of Technology, New Delhi, during the academic year 1975/76, and a series of seminars given at a number of Indian Universities and Research Institutes. My aim in these lectures was to present some recent developments in Ergodic Theory, namely the study of the first cohomology of ergodic transformation groups. Very regretfully I have restricted myself to the purely measure theoretic aspects of the theory without ever mentioning the closely related fascinating results achieved by A.Connes, W.Krieger and others in the theory of von Neumann algebras. But even within the relatively small scope of these notes there are many serious omissions, of which I am well aware and for which I apologize. To name just one example, I have left out a discussion of the criteria for hyperfiniteness, the problem of classification of both hyperfinite and nonhyperfinite actions, and in particular W.Krieger's complete description of weak equivalence classes of hyperfinite actions. I have also made no mention of the higher cohomology of actions.

Having indicated what is not in these notes, let me now turn to the contents of this volume.

PART I:

§ 1 contains some basic definitions and properties concerning actions of countable groups on (standard) measure spaces. In particular we introduce H.A.Dye's notions of full groups and of weak equivalence of group actions.

§ 2 gives an introductory discussion of the first cohomology of a countable group action and gives two equivalent definitions of the group of cocycles.

§ 3 is devoted to two cohomology invariants which will play a fundamental role throughout the following sections, namely the set of essential values (or the asymptotic range) and the notion of recurrence of a cocycle. In a series of lemmas we prove the basic properties of these invariants.

§ 4 shows that Radon Nikodym derivatives are always recurrent (Theorem 4.2) and derives some further properties of this important class of cocycles.

§ 5 introduces skew products, which are important both as a tool and as an application in the study of cocycles. There we investigate the connection between properties of the skew product arising from a cocycle and between the invariants defined in § 3.

§ 6 studies the ergodic decomposition of a countable group action. We give a purely measure theoretic proof of the existence of such a decomposition, using skew products. The section is completed by proving some well known results on the uniqueness and further properties of these decompositions.

§ 7 is one of the most crucial sections of these notes. Here we give a detailed analysis of cocycles whose set of essential values is equal to $\{0, \infty\}$. The structure of such cocycles is described in Theorem 7.22. Its proof depends on a close investigation of the ergodic decomposition for the skew product defined by such cocycles (Lemma 7.24). The somewhat technical analysis of this section derives its justification from two corollaries at the end of § 7. These corollaries (7.23 and 7.25) establish a connection between recurrent cocycles with essential values $\{0, \infty\}$ on one side and certain classes of nonatomic, σ-finite and ergodic measures for the underlying group action on the other side. This connection allows a systematic approach to the following problem: Given an ergodic action of a countable group G on a measure space $(X, \mathcal{S}, \mu)$, what other measures do there exist on $(X, \mathcal{S})$ which are also quasi-invariant and ergodic under G? We will investigate this question further in § 10. At this stage one should bear in mind that we have not yet proved the existence of recurrent cocycles with essential values $\{0, \infty\}$!

PART II:

§ 8 During the first seven sections we have always looked at actions of arbitrary countable groups. From now on we specialize our investigation to those actions whose orbits arise from a single automorphism of the measure space. This brings us closer to classical ergodic theory which is, after all, mainly interested in single ergodic transformations. In this section we show that hyperfiniteness can be characterized by the existence of transient real valued cocycles for the action (Theorem 8.7). The remaining part of § 8 describes a general model for hyperfinite actions. We introduce a standard Borel space $(\Omega, \mathcal{F})$ and a countable group Γ of Borel automorphisms of $(\Omega, \mathcal{F})$ such that every hyperfinite action is weakly equivalent to the action of Γ on $(\Omega, \mathcal{F}, \nu)$ for some suitably chosen probability measure ν (Theorem 8.15).

As a corollary one obtains the celebrated result of H.A.Dye which states that all finite measure preserving ergodic automorphisms are weakly equivalent. § 8 is completed by showing that every ergodic action of a countable group contains an ergodic automorphism in its full group (Theorem 8.22).

§ 9 describes the cohomology of a hyperfinite action. In particular we prove that 'most' cocycles of a hyperfinite action give rise to an ergodic skew product.

§ 10 In this section we study the following problem: Let $(X, \mathcal{S})$ be a standard Borel space, V a Borel automorphism of $(X, \mathcal{S})$, and $f: X \to \mathbb{R}$ a Borel map. When does there exist a nonatomic, σ-finite measure μ on $(X, \mathcal{S})$ which is quasi-invariant and ergodic under V and which satisfies $\log \frac{d\mu V}{d\mu}(x) = f(x)$ for μ-a.e. $x \in X$? In particular one can put $f = 0$ and ask for invariant measures. Using the tools developed in § 7 we give a partial solution to the general problem and prove among other things that every nonsingular ergodic automorphism of a nonatomic measure space has non-atomic, infinite, σ-finite, invariant ergodic measures. These results are based on two deep consequences of a certain recurrence property. The first one is that every sufficiently recurrent cocycle for a hyperfinite action is a coboundary for a large number of nonatomic and ergodic measures for the group action, and the second one is a characterization of Radon Nikodym derivatives purely in terms of recurrence - thus giving a partial converse to Theorem 4.2. For the precise statements of these results we refer to Theorem 10.5 and Corollary 10.6.

§ 11 Now we really turn to classical ergodic theory and look at cocycles for for finite measure preserving ergodic transformations. Here the characterization of recurrence takes a much simpler form (Corollaries 11.2 and 11.3), and becomes particularly nice for integrable cocycles (Theorem 11.4). Proposition 11.5 deals again with the ergodic decomposition of skew products and shows that (for example) a real valued cocycle is a coboundary if and only if its skew product decomposes into finite measure preserving transformations. As a consequence we get the probably simplest criterion for a cocycle to be a coboundary: A cocycle is a coboundary if and only if its sequence of distributions is uniformly tight (Theorem 11.8).

§ 12 deals with some examples. We look at certain cocycles arsising from random walks and from uniform distribution (modulo 1), and give an explicit computation of their essential values.

Many of the results in these notes have not been published elsewhere. After § 12 there is a short list of comments indicating the sources of the results presented here. At the end of most sections there are some exercises and an occasional problem. Some of these exercises are really quite trivial and are included just as illustrations of definitions and theorems preceding them.

I have tried to keep these notes fairly self contained, so that anybody with a good background in measure theory should be able to read them. I am, however, assuming familiarity of the reader with standard Borel spaces and with some results of Kuratowski on Borel maps. The first two chapters in [34] contain all the necessary material (and it is not necessary to understand all the proofs in Chapter I of [34] !).

Finally I would like to thank the Indian Statistical Institute and in particular its Director, Professor C.R.Rao, for inviting me to their New Delhi Campus and for their generous and warm assistance during my stay in Delhi, Calcutta, and during my travels in India. I am also grateful for the pleasant hospitality I encountered on visits to the Universities of Bombay, Chidambaram, Madurai, Mysore and Trivandrum, and to the research centres at Bangalore and TIFR, Bombay. These notes have greatly benefited from comments, suggestions and criticism made by a number of Indian mathematicians, and in particular by Professor K.R.Parthasarathy, Mr.Bhatia, Mr.Rana, and Mr.Subramaniam, all of IIT, New Delhi.

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PART I. THE FIRST COHOMOLOGY OF A COUNTABLE GROUP ACTION

§ 1 Introduction

In these notes the term <u>measure space</u> will always stand for a triple $(X, \mathcal{S}, \mu)$, where $(X, \mathcal{S})$ is a standard (and, in general, uncountable) Borel space and where μ is a σ-finite measure on $(X, \mathcal{S})$. A <u>probability space</u> is a measure space $(X, \mathcal{S}, \mu)$ with $\mu(X) = 1$. The measure space $(X, \mathcal{S}, \mu)$ is said to be <u>nonatomic</u> if μ is nonatomic, i.e. if $\mu(\{x\}) = 0$ for every $x \in X$. Let V be a Borel automorphism of $(X, \mathcal{S})$. We call V an <u>automorphism</u> of $(X, \mathcal{S}, \mu)$ if μ is quasi-invariant under V. $\mathcal{A}(X, \mathcal{S}, \mu)$ will denote the group of all automorphisms of $(X, \mathcal{S}, \mu)$. If $B_1, B_2 \in \mathcal{S}$ we say that $B_1 = B_2$ (mod 0) (or more precisely, (μ-mod 0)) if $\mu(B_1 \Delta B_2) = 0$. $B_1 = B_2$ indicates that the sets are actually equal. Similarly we shall assume relations between funtions to hold <u>everywhere</u> except when explicitly stated otherwise. If $B \in \mathcal{S}$ is a set of positive measure we write $\mathcal{S}_B = \{C \cap B : C \in \mathcal{S}\}$ and μ_B for the restriction of μ to $\mathcal{S}_B$. The same symbol μ_B will be used to denote the measure on $(X, \mathcal{S})$ given by $C \to \mu(C \cap B)$, $C \in \mathcal{S}$.

Consider now two measure spaces $(X_i, \mathcal{S}_i, \mu_i)$, $i = 1,2$. A map $\phi : X_1 \to X_2$ is called an isomorphism from $(X_1, \mathcal{S}_1, \mu_1)$ to $(X_2, \mathcal{S}_2, \mu_2)$ if

(a) ϕ is a Borel map, (1.1)

(b) there exist subsets $N_i \in \mathcal{S}_i$ with $\mu_i(N_i) = 0$ such that ϕ is a Borel isomorphism from $X_1 \setminus N_1$ onto $X_2 \setminus N_2$, (1.2)

(c) $\mu_1 \phi^{-1}$ is equivalent to μ_2 (in symbols: $\mu_1 \phi^{-1} \sim \mu_2$). (1.3)

If both X_1 and X_2 are uncountable, we can modify ϕ on a set $N \in \mathcal{S}_1$ of μ_1-measure zero to get the additional condition

(d) ϕ is a Borel isomorphism from X_1 onto X_2. (1.4)

Let now $(X, \mathcal{S}, \mu)$ be a fixed measure space and let G be a countable group with identity element e. An <u>action</u> T_G of G on $(X, \mathcal{S}, \mu)$ is a homomorphism $g \to T_g$, $g \in G$, from G into $\mathcal{A}(X, \mathcal{S}, \mu)$. If $x \in X$ and $B \in \mathcal{S}$, we put $T_G x = \{T_g x : g \in G\}$ and $T_G B = \bigcup_{g \in G} T_g B$. If the action of G on $(X, \mathcal{S}, \mu)$ is understood we shall write gx, Gx and GB instead of $T_g x$, $T_G x$ and $T_G B$.

Consider now two actions $T_G^{(i)}$ of a countable group G on measure spaces $(X_i, \mathcal{S}_i, \mu_i)$, i=1,2, respectively. We call these actions <u>conjugate</u> if there exists an isomorphism $\phi: (X_1, \mathcal{S}_1, \mu_1) \to (X_2, \mathcal{S}_2, \mu_2)$ (cf. (1.1) - (1.3)) with

$$\phi(T_g^{(1)} x) = T_g^{(2)} \phi(x) \tag{1.5}$$

for every $g \in G$ and for μ_1-a.e.x. Again, if X_1 and X_2 are uncountable sets, we may assume ϕ to satisfy (1.4).

We now return to a fixed action T_G of G on $(X, \mathcal{S}, \mu)$. A set $B \in \mathcal{S}$ is called T_G-invariant if $T_G B = B$. If $B \in \mathcal{S}$ has the property that $\mu(B \Delta T_G B) = 0$, then there exists a T_G-invariant set $B_0 \in \mathcal{S}$ with $\mu(B \Delta B_0) = 0$. T_G is said to be <u>ergodic</u> if every T_G-invariant set $B \in \mathcal{S}$ satisfies either $\mu(B) = 0$ or $\mu(\complement B) = 0$. Instead of saying that T_G is ergodic on $(X, \mathcal{S}, \mu)$ we shall often refer to μ as an ergodic measure for T_G (or T_G-ergodic measure) on $(X, \mathcal{S})$. T_G is called <u>conservative</u> if, for every $B \in \mathcal{S}$ with $\mu(B) > 0$,

$$\mu\left(\bigcup_{g \in G} (B \cap T_g^{-1} B \cap \{x : T_g x \neq x\})\right) > 0. \tag{1.6}$$

Using a theorem of Kuratowski on the existence of Borel cross-sections (cf. [34], § I.4) one can show that (1.6) is equivalent to the following condition: For every $B \in \mathcal{S}$ with $\mu(B) > 0$, and for μ-a.e. $x \in B$, the set

$$T_G x \cap B \tag{1.7}$$

is infinite. The opposite extreme of a conservative action is a <u>type I</u> action: T_G is called a type I action if there exists a set $D \in \mathcal{S}$ with

$$\bigcup_{g \in G} (D \cap T_g^{-1} D \cap \{x : T_g x \neq x\}) = \emptyset \tag{1.8}$$

and

$$\mu(X \setminus T_G D) = 0. \tag{1.9}$$

Note that every ergodic action on a nonatomic measure space must be conservative. In general, if T_G is an action of G on $(X, \mathcal{S}, \mu)$, we can find complementary T_G-invariant sets $B_1, B_2 \in \mathcal{S}$ with the following properties:

(a) There exists a Borel set $D \subset B_1$ which satisfies (1.8) and $T_G D = B_1$, (1.10)

(b) if $\mu(B_2) > 0$, then the (obvious) restriction of T_G to $(B_2, \mathcal{S}_{B_2}, \mu_{B_2})$ is conservative. (1.11)

Another important notion is that of a <u>free</u> action: T_G is said to be free if

$$\mu(\{x : T_g x = x\}) = 0 \tag{1.12}$$

for every $g \in G$, $g \neq e$.

For the following we shall consider a fixed action T_G of G on $(X, \mathcal{S}, \mu)$. Let

$$R(T_G) = \{(x, T_g x) : x \in X,\ g \in G\}. \tag{1.13}$$

$R(T_G)$ is a Borel subset of $X \times X$ and is called the <u>equivalence relation</u> of T_G. For $B \in \mathcal{S}$, put

$$R_B(T_G) = R(T_G) \cap B \times B. \tag{1.14}$$

We also define

$$R^{(2)}(T_G) = \{(x, T_g x, T_h x) : x \in X,\ g \in G,\ h \in G\}, \tag{1.15}$$

and

$$R_B^{(2)}(T_G) = R^{(2)}(T_G) \cap B \times B \times B. \tag{1.16}$$

One can define natural measures on $R(T_G)$ and $R^{(2)}(T_G)$, but we only need the following simpler concept: Let $B \in \mathcal{S}$ with $\mu(B) > 0$. A property is said to hold μ-a.e. on $R_B(T_G)$ $(R_B^{(2)}(T_G))$ if there exists $N \in \mathcal{S}$ with $\mu(N) = 0$ such that the property holds everywhere on $R_{B \setminus N}(T_G)$ $(R^{(2)}_{B \setminus N}(T_G))$. (1.17)

The full group $[T_G]$ of T_G is defined by

$$[T_G] = \{V \in \mathcal{A}(X, \mathcal{S}, \mu): (x, Vx) \in R(T_G) \text{ for every } x \in X\}. \tag{1.18}$$

If $B \in \mathcal{S}$ and $\mu(B) > 0$, the induced full group $[T_G]_B$ is given by

$$[T_G]_B = \{V \in \mathcal{A}(B, \mathcal{S}_B, \mu_B): (x, Vx) \in R_B(T_G) \text{ for every } x \in B\}. \tag{1.19}$$

We now come to one of our most important definitions. Let $T^{(i)}_{G_i}$ be an action of a countable group G_i on a measure space $(X_i, \mathcal{S}_i, \mu_i)$ for i=1,2. $T^{(1)}_{G_1}$ and $T^{(2)}_{G_2}$ are called weakly equivalent if there exists an isomorphism $\phi: (X_1, \mathcal{S}_1, \mu_1) \rightarrow (X_2, \mathcal{S}_2, \mu_2)$ such that

$$\phi(T^{(1)}_{G_1} x) = T^{(2)}_{G_2} \phi(x) \tag{1.20}$$

for μ_1-a.e. $x \in X_1$.

We conclude this section with a series of exercises which may illustrate the definitions.

Exercise 1.1. Let T_G be an action of G on $(X, \mathcal{S}, \mu)$. Show that the following conditions are equivalent.

(1) $V \in [T_G]$,

(2) there exists a Borel partition $\{B_g,\ g\in G\}$ of X such that $\{T_gB_g,\ g\in G\}$ is again a Borel partition of X, and $Vx = T_gx$ for every $x\in B_g$, $g\in G$.

<u>Exercise 1.2</u>. Suppose μ is invariant under T_G. Show that μ is invariant under every $V\in [T_G]$.

<u>Exercise 1.3</u>. Let T_G be an action of G on $(X,\mathcal{S},\mu)$, and let V be a Borel automorphism of $(X,\mathcal{S})$ such that

$$(Vx,x)\in R(T_G)$$

for every $x\in X$. Show that $V\in [T_G]$.

<u>Exercise 1.4</u>. Let T_G be an action of G on $(X,\mathcal{S},\mu)$, and let $Q\subset R(T_G)$ be a nonempty Borel set. Suppose Q is an equivalence relation, that is

(a) $(x,y)\in Q$ implies (y,x) and $(x,x)\in Q$,

(b) (x,y) and $(y,z)\in Q$ implies $(x,z)\in Q$.

Show that there exists a countable group $H(T_G,Q) \subset [T_G]_{\{x:(x,x)\in Q\}}$ with

$$\{(x,Vx):\ (x,x)\in Q,\ V\in H(T_G,Q)\} = Q. \tag{1.21}$$

In particular, let $B\in\mathcal{S}$ with $\mu(B)>0$, and put

$$Q_B = R_B(T_G). \tag{1.22}$$

Then there exists a countable group

$$T(G,B) = H(T_G, Q_B) \subset [T_G]_B \tag{1.23}$$

of automorphisms of the measure space $(B, \mathcal{S}_B, \mu_B)$ satisfying

$$[T(G,B)] = [T_G]_B. \tag{1.24}$$

Abusing notation we shall call $T(G,B)$ the action induced by T_G on B. Show that the following is true:

(c) $T(G,B)$ is ergodic on $(B, \mathcal{S}_B, \mu_B)$ whenever T_G is ergodic on $(X, \mathcal{S}, \mu)$.

(d) $T(G,B)$ is conservative whenever T_G is conservative.

(e) Suppose that T_G is ergodic. If μ_B is equivalent to a $T(G,B)$-invariant measure, show that μ is equivalent to a T_G-invariant measure.

(f) Suppose that $\mu(X \setminus T_G B) = 0$ and that $T(G,B)$ is ergodic. Show that T_G is ergodic on $(X, \mathcal{S}, \mu)$.

<u>Exercise 1.5</u>. Let $T_{G_i}^{(i)}$ be an action of G_i on $(X_i, \mathcal{S}_i, \mu_i)$, $i=1,2$. Show that the following conditions are equivalent:

(a) $T_{G_1}^{(1)}$ and $T_{G_2}^{(2)}$ are weakly equivalent,

(b) there exists an isomorphism $\phi : (X_1, \mathcal{S}_1, \mu_1) \to (X_2, \mathcal{S}_2, \mu_2)$ and sets $N_i \in \mathcal{S}_i$ with $\mu_i(N_i) = 0$ such that

i) ϕ is a Borel isomorphism from $X_1 \setminus N_1$ to $X_2 \setminus N_2$,

ii) $\phi\,[T^{(1)}_{G_1}]_{X_1 \setminus N_1}\,\phi^{-1} = [T^{(2)}_{G_2}]_{X_2 \setminus N_2}$, (1.25)

(c) there exists an isomorphism $\phi : (X_1, \mathcal{S}_1, \mu_1) \to (X_2, \mathcal{S}_2, \mu_2)$ and sets $N_i \in \mathcal{S}_i$ with $\mu_i(N_i) = 0$ such that

$$\{(\phi(x), \phi(y)) : (x,y) \in R(T^{(1)}_{G_1})_{X_1 \setminus N_1}\} = R(T^{(2)}_{G_2})_{X_2 \setminus N_2}. \qquad (1.26)$$

<u>Exercise 1.6</u>. Let T_G be an action of G on $(X, \mathcal{S}, \mu)$. Show that the following statements are invariants of weak equivalence.

(a) T_G is ergodic,

(b) T_G is conservative,

(c) T_G is type I,

(d) μ is equivalent to a T_G-invariant probability measure,

(e) μ is equivalent to an infinite, σ-finite, T_G-invariant measure.

<u>Exercise 1.7</u>. Let T_G be a conservative action of G on a measure space $(X, \mathcal{S}, \mu)$. Show that μ is nonatomic.

<u>Exercise 1.8</u>. Let T_G be an action of G on a measure space $(X, \mathcal{S}, \mu)$ and suppose that μ is purely atomic. Show that T_G is type I.

§ 2 The first cohomology of an action

Let T_G be an action of a countable group G on a measure space $(X, \mathcal{S}, \mu)$, and let A be a locally compact second countable abelian group with identity 0 and addition as composition.

Definition 2.1. A Borel map $a: G \times X \to A$ is called a cocycle for T_G if the following conditions are satisfied.

(1) For every $g_1, g_2 \in G$ and for every $x \in X$ we have

$$a(g_1, Tg_2 x) - a(g_1 g_2, x) + a(g_2, x) = 0. \tag{2.1}$$

(2) $$\mu\Big(\bigcup_{g \in G}(\{x : T_g x = x\} \cap \{x : a(g,x) \neq 0\})\Big) = 0. \tag{2.2}$$

A cocycle $a: G \times X \to A$ is called a coboundary if there exists a Borel map $c: X \to A$ with

$$a(g,x) = c(T_g x) - c(x) \tag{2.3}$$

for every $g \in G$ and for μ-a.e. $x \in X$. Two cocycles a_1 and a_2 for T_G are cohomologous if their difference is a coboundary, and equivalent if

$$\mu\Big(\bigcup_{g \in G}\{x : a_1(g,x) \neq a_2(g,x)\}\Big) = 0. \tag{2.4}$$

We shall write

$$a_1 \equiv a_2 \quad \text{if } a_1 \text{ and } a_2 \text{ are equivalent,} \tag{2.5}$$

and

$$a_1 \sim a_2 \quad \text{if } a_1 \text{ and } a_2 \text{ are cohomologous.} \tag{2.6}$$

To motivate (2.2) we first remark that it is trivially satisfied whenever T_G is a free action. The simplest and most important example of a cocycle for a general action gives another reason for imposing (2.2): There exists a cocycle $a_\mu : G \times X \to \mathbb{R}$ such that

$$a_\mu(g,x) = \log \frac{d\mu T_g}{d\mu}(x) \tag{2.7}$$

for μ-a.e. $x \in X$ and for every $g \in G$. If a_μ and a'_μ both satisfy (2.7) we must clearly have $a_\mu \equiv a'_\mu$. If we replace μ by an equivalent measure ν, then $a_\mu \sim a_\nu$. Moreover, a_μ is a coboundary if and only if μ is equivalent to a σ-finite invariant measure on $(X, \mathcal{S})$.

The real reason for (2.2) will be clear from the following discussion. Let $a: G \times X \to A$ be a cocycle for T_G. Then there exists a cocycle $a_0 : G \times X \to A$ with

$$a_0 \equiv a \tag{2.8}$$

and with

$$\{x: a_0(g,x) \neq 0\} \cap \{x: T_g x = x\} = \emptyset \tag{2.9}$$

for every $g \in G$. For every $(x,y) \in R(T_G)$, put

$$u_{a_0}(x,y) = a_0(g,y) \tag{2.10}$$

whenever $g \in G$ satisfies

$$x = T_g y. \tag{2.11}$$

(2.9) shows that u_{a_o} is defined unambiguously, and u_{a_o}: $R(T_G) \to A$ is easily seen to be a Borel map. For every $(x,y,z) \in R^{(2)}(T_G)$ we have

$$u_{a_o}(x,y) - u_{a_o}(x,z) + u_{a_o}(y,z) = 0 \qquad (2.12)$$

from (2.1).

<u>Definition 2.2.</u> A Borel map $u: R(T_G) \to A$ is called an <u>orbital cocycle</u> for T_G if

$$u(x,y) - u(x,z) + u(y,z) = 0 \qquad (2.13)$$

for every $(x,y,z) \in R^{(2)}(T_G)$. An orbital cocycle is said to be a <u>coboundary</u> if there exists a Borel map $c: X \to A$ such that

$$u(x,y) = c(x) - c(y) \qquad (2.14)$$

μ-a.e. on $R(T_G)$. Two orbital cocycles are <u>cohomologous</u> if their difference is a coboundary, and <u>equivalent</u> if they coincide μ-a.e. on $R(T_G)$. We write

$$u_1 \equiv u_2 \quad \text{if } u_1 \text{ and } u_2 \text{ are equivalent,} \qquad (2.15)$$

and

$$u_1 \sim u_2 \quad \text{if } u_1 \text{ and } u_2 \text{ are cohomologous.} \qquad (2.16)$$

<u>Proposition 2.3.</u> Let T_G be an action of G on $(X, \mathcal{S}, \mu)$ and let $a: G \times X \to A$ be a cocycle for T_G. Then there exists an orbital cocycle $u_a: R(T_G) \to A$ such that

$$u_a(T_g x, x) = a(g,x) \qquad (2.17)$$

for μ-a.e.$x\in$ and for every $g\in G$. u_a is determined up to equivalence. Furthermore, u_a is a coboundary if and only if a is a coboundary. Conversely, let $u:R(T_G)\to A$ be an orbital cocycle for T_G. Then there exists a cocycle $a_u:G\times X\to A$ such that

$$a_u(g,x) = u(T_g x,x) \tag{2.18}$$

for every $g\in G$ and for μ-a.e.$x\in X$. a_u is again determined up to equivalence, and it is a coboundary if and only if u is a coboundary.

Proof: The first statement follows from (2.8) - (2.12), and the rest is evident.

<u>Remark 2.4</u>. Proposition 2.3 establishes an isomorphism between equivalence classes of cocycles for T_G on one side and equivalence classes of orbital cocycles on the other side. This isomorphism carries coboundaries to coboundaries. We have thus two equivalent definitions of cocycles for T_G, one in terms of G itself, and one purely in terms of the orbits of T_G. While Definition 2.1 seems to be preferable in ergodic theory, where one is interested in specific actions of a particular group G, Definition 2.2 is more elegant, allows many proofs to be shortened, and is particularly useful for applications of cocycles in the theory of von Neumann algebras. In the following sections we shall freely change over from one notion to the other if such a change offers any advantage.

We conclude this section by showing that the first cohomology of a group action is an invariant of weak equivalence. Let $\mathcal{B}(X,\mu,A)$ stand for the set of all Borel maps from X to A where we identify maps which coincide μ-a.e. on X. $\mathcal{B}(X,\mu,A)$ is a group under pointwise addition of functions.

Lemma 2.5. Let μ_o be a probability measure on $(X,\mathcal{S})$ with $\mu_o \sim \mu$, and consider $\mathcal{B}(X,\mu_o,A) = \mathcal{B}(X,\mu,A)$ under the topology of convergence in μ_o-measure. Then $\mathcal{B}(X,\mu_o,A)$ is a complete separable metric topological group. Moreover the topology of $\mathcal{B}(X,\mu_o,A)$ is independent of the choice of μ_o.

Proof: Since A is a locally compact second countable abelian group, A is metrizable, and we can find an invariant metric δ on A which is bounded by 1 (invariant means that $\delta(\alpha,\beta) = (\alpha+\gamma,\beta+\gamma)$ for every $\alpha,\beta,\gamma \in A$). For $\phi_1, \phi_2 \in \mathcal{B}(X,\mu,A)$, put

$$d_{\mu_o}(\phi_1,\phi_2) = \int \delta(\phi_1(x), \phi_2(x))\, d\mu_o(x). \tag{2.19}$$

Clearly d_{μ_o} is a metric, and $\mathcal{B}(X,\mu,A)$ is a complete separable space under d_{μ_o}. Since the metric d_{μ_o} is invariant on the group $\mathcal{B}(X,\mu,A)$, $\mathcal{B}(X,\mu,A)$ becomes a topological group. If μ_1 is another probability measure on $(X,\mathcal{S})$ with $\mu_1 \sim \mu_o$ and if $(\phi_k, k \geq 0)$ is a sequence in $\mathcal{B}(X,\mu,A)$ with $d_{\mu_o}(\phi_k,\phi_o) \to 0$, it is easy to see that $d_{\mu_1}(\phi_k,\phi_o) \to 0$, so that the topology is indeed independent of the measure μ_o. The proof is complete.

We now fix a metric $d = d_{\mu_o}$ on $\mathcal{B}(X,\mu,A)$. $Z^1(T_G,\mathcal{B}(X,\mu,A))$ will stand for the group of all (equivalence classes of) cocycles $a:G\times X \to A$ for T_G under addition. We define a topology on $Z^1(T_G,\mathcal{B}(X,\mu,A))$ by introducing the following notion of convergence. If $(a_k,\ k \geq 0)$ is a sequence in $Z^1(T_G,\mathcal{B}(X,\mu,A))$, we say that

$$\lim_k a_k = a_o \tag{2.20}$$

if

$$\lim_k d(a_k(g,.),a_o(g,.)) = 0 \tag{2.21}$$

for every $g\in G$. Again $Z^1(T_G,\mathcal{B}(X,\mu,A))$ is a complete separable metric group in this topology. In future we shall always think of $Z^1(T_G,\mathcal{B}(X,\mu,A))$ as a topological group with the topology just defined. $B^1(T_G,\mathcal{B}(X,\mu,A))$ will denote the subgroup of coboundaries in $Z^1(T_G,\mathcal{B}(X,\mu,A))$. If no confusion is possible, we shall abbreviate $Z^1(T_G,\mathcal{B}(X,\mu,A))$ and $B^1(T_G,\mathcal{B}(X,\mu,A))$ by $Z^1(T_G,A)$ and $B^1(T_G,A)$, respectively. When the action of G is understood, we shall also write $Z^1(G,A)$ and $B^1(G,A)$.

Lemma 2.6. Let $a:G\times X\to A$ be a cocycle for T_G. Then there exists a map $\tilde{a}: [T_G]\times X\to A$ such that

(1) $a(V,.):X\to A$ is a Borel map for every $V\in [T_G]$,

(2) $a(T_g,x) = a(g,x)$ for every $g\in G$ and for μ-a.e.x, (2.22)

(3) $a(V_1,V_2x) - a(V_1V_2,x) + a(V_2,x) = 0$ (2.23)

for every $V_1,V_2 \in [T_G]$ and for every $x\in X$.

Proof: Choose an orbital cocycle u_a as in (2.17) and put

$$a(V,x) = u_a(Vx,x) \tag{2.24}$$

for every $V \in [T_G]$ and $x \in X$. a will then satisfy (1) - (3), by (2.13) and (2.17).

Let us again consider two actions $T^{(i)}_{G_i}$ of groups G_i on measure spaces $(X_i, \mathcal{S}_i, \mu_i)$ respectively. Suppose $T^{(1)}_{G_1}$ and $T^{(2)}_{G_2}$ are weakly equivalent, and choose an isomorphism $\phi:(X_1,\mathcal{S}_1,\mu_1) \to (X_2,\mathcal{S}_2,\mu_2)$ satisfying (1.20). If $u:R(T^{(1)}_{G_1}) \to A$ is an orbital cocycle, we can find an orbital cocycle $\tilde{\phi}(u):R(T^{(2)}_{G_2}) \to A$ satisfying

$$\tilde{\phi}(u)\ (\phi(x), \phi(y)) = u(x,y) \tag{2.25}$$

for μ_1-a.e. $(x,y) \in R(T^{(1)}_{G_1})$. $\tilde{\phi}(u)$ is determined up to equivalence by (2.25). Similarly, if $a:G_1 \times X_1 \to A$ is a cocycle for $T^{(1)}_{G_1}$, we can use (2.17) and (2.18) to obtain a cocycle $\tilde{\phi}(a):G_2 \times X_2 \to A$ for $T^{(2)}_{G_2}$, by setting

$$\tilde{\phi}(a) = a\,\tilde{\phi}(u_a) \tag{2.26}$$

Again we note that $\tilde{\phi}(a)$ is determined up to equivalence, and that

$$\tilde{\phi}(a_1) \equiv \tilde{\phi}(a_2) \quad \text{whenever} \quad a_1 \equiv a_2,$$

and

$$\tilde{\phi}(a_1) \sim \tilde{\phi}(a_2) \quad \text{whenever} \quad a_1 \sim a_2.$$

The following assertion is easily verified.

Proposition 2.7. $\tilde{\phi}: Z^1(T_{G_1}^{(1)}, A) \to Z^1(T_{G_2}^{(2)}, A)$ is a topological group isomorphism, which carries coboundaries to coboundaries. In particular, if T_G is an action of a countable group G on a measure space $(X, \mathcal{S}, \mu)$, the groups

$$Z^1(T_G, \mathcal{B}(X, \mu, A)) = Z^1(T_G, A)$$

$$B^1(T_G, \mathcal{B}(X, \mu, A)) = B^1(T_G, A)$$

and hence

$$H^1(T_G, \mathcal{B}(X, \mu, A)) = Z^1(T_G, A)/B^1(T_G, A) \tag{2.27}$$

are invariants of weak equivalence.

Remark 2.8. The quotient group $H^1(T_G, \mathcal{B}(X, \mu, A)) = H^1(T_G, A)$ in (2.27) is called the first cohomology group of the action T_G. Since $B^1(T_G, A)$ is in general not a closed subgroup of $Z^1(T_G, A)$, $H^1(T_G, A)$ is usually considered as a purely algebraic group without any topology or Borel structure.

Exercise 2.9. Let T_G be an ergodic action of a countable group G on $(X, \mathcal{S}, \mu)$. If μ has an atom, show that $Z^1(T_G, A) = B^1(T_G, A)$.

Exercise 2.10. Let T_G be an action of G on $(X, \mathcal{S}, \mu)$, and let $B \in \mathcal{S}$ with $\mu(B) > 0$. We write

$$u_B = u/R_B(T_G) \tag{2.28}$$

for the restriction of u to $R_B(T_G)$. If T(G,B) denotes the group defined in Exercise 1.4, show that u_B is an orbital cocycle for $T(G,B)$. Moreover, if we put

$$a_B(V,x) = u_B(Vx,x) \tag{2.29}$$

for every $x \in B$, $V \in T(G,B)$, then $a_B : T(G,B) \times B \to A$ is a cocycle for $T(G,B)$ on $(B, \mathcal{S}_B, \mu_B)$ (more precisely: a_B is a cocycle for the obvious action of $T(G,B)$ on $(B, \mathcal{S}_B, \mu_B)$). Conversely, suppose that $u_B : R_B(T_G) \to A$ is an orbital cocycle for $T(G,B)$ on $(B, \mathcal{S}_B, \mu_B)$. Show that there exists an orbital cocycle $u : R(T_G) \to A$ for T_G on $(X, \mathcal{S}, \mu)$ such that

$$u(x,y) = u_B(x,y)$$

for μ_B-a.e. $(x,y) \in R_B(T_G)$.

<u>Exercise 2.11.</u> Let T_G be an action of G on a measure space $(X, \mathcal{S}, \mu)$, and let a_μ be the cocycle (2.7). Denote by $u_\mu = u_{a_\mu}$ the orbital cocycle (2.17) and by $\tilde{a}_\mu$ the extension of a_μ to $[T_G]$ defined in Lemma 2.6. Show that, for every $W \in [T_G]$,

$$u_\mu(Wx,x) = \tilde{a}_\mu(W,x) = \log \frac{d\mu W}{d\mu}(x) \tag{2.30}$$

for μ-a.e. $x \in X$.

§ 3 Some cohomology invariants

Let A be a locally compact second countable group. If A is noncompact, we put $\bar{A} = A \cup \{\infty\}$, the one point compactification of A. For compact A let $\bar{A} = A$. We fix a countable group G and a nonatomic measure space $(X, \mathcal{S}, \mu)$.

Definition 3.1. Let T_G be an ergodic action of G on $(X, \mathcal{S}, \mu)$ and let $a: G \times X \to A$ be a cocycle for T_G. An element $\alpha \in \bar{A}$ is called an essential value of a if, for every neighbourhood $N(\alpha)$ of α in $\bar{A}$, and for every $B \in \mathcal{S}$ with $\mu(B) > 0$,

$$\mu\left(\bigcup_{g \in G} (B \cap T_g^{-1} B \cap \{x : a(g,x) \in N(\alpha)\} \right) > 0. \tag{3.1}$$

The set of all essential values of a will be denoted by $\bar{E}(a)$, and we put $E(a) = \bar{E}(a) \cap A$.
If $u: R(T_G) \to A$ is an orbital cocycle we put $\bar{E}(u) = \bar{E}(a_u)$ and $E(u) = E(a_u)$, where a_u is given by (2.18).

To determine the properties of $\bar{E}(a)$ we need a few lemmas. Until these assumptions are changed explicitly, T_G will denote an ergodic action of G on $(X, \mathcal{S}, \mu)$ and $a: G \times X \to A$ a fixed cocycle for T_G.

Lemma 3.2. Let $b: G \times X \to A$ be a coboundary. Then $\bar{E}(a+b) = \bar{E}(a)$.

Proof: Because of symmetry we only have to show that $\bar{E}(a) \subset \bar{E}(a+b)$.

We choose a Borel map $c:X \to A$ such that $b(g,x) = c(T_g x) - c(x)$ for every $g \in G$ and for μ-a.e. $x \in X$. Let $\alpha \in \bar{E}(a)$, $N(\alpha)$ a neighbourhood of α in $\bar{A}$, and let $B \in \mathcal{S}$ with $\mu(B) > 0$. We choose neighbourhoods $N_1(\alpha) \subset \bar{A}$ and $N_2(0) \subset A$ with $N_1(\alpha) + N_2(0) \subset N(\alpha)$ (we adopt the convention $\beta + \infty = \infty$ for every $\beta \in A$). Next we choose $C \subset B$ with $\mu(C) > 0$ such that $c(x) - c(y) \in N_2(0)$ for every $x,y \in C$. Let $g_0 \in G$ be given with $\mu(C \cap T_{g_0}^{-1} C \cap \{x: a(g_0,x) \in N_1(\alpha)\}) > 0$. We can find $D \in \mathcal{S}$ such that $\mu(D) > 0$, $D \cup T_{g_0} D \subset C$, and $a(g_0,x) \in N_1(\alpha)$ for all $x \in D$, and get

$$a(g_0,x)+b(g_0,x) = a(g_0,x)+c(T_{g_0}x)-c(x) \in N_1(\alpha)+N_2(0) \subset N(\alpha)$$

for μ-a.e. $x \in D$. This shows that $\alpha \in \bar{E}(a+b)$ and proves the lemma.

Lemma 3.3. $E(a)$ is a closed subgroup of A.

Proof: Clearly we have $0 \in E(a)$, so that $E(a) \neq \emptyset$. Since $E(a)$ is obviously closed, it will be enough to show that, for every $\alpha, \beta \in E(a)$, we also have $\alpha - \beta \in E(a)$. Indeed, let α, $\beta \in E(a)$ and let $B \in \mathcal{S}$ with $\mu(B) > 0$. Suppose $N(0)$ is a neighbourhood of 0 in A. We choose a symmetric neighbourhood $N_1(0)$ with $N_1(0)+N_1(0) \subset N(0)$ and an element $g_1 \in G$ for which $\mu(B \cap T_{g_1}^{-1} B \cap \{x: a(g_1,x) \in N_1(0)+\alpha\}) > 0$. If $C = B \cap T_{g_1}^{-1} B \cap \{x: a(g_1,x) \in N_1(0)+\alpha\}$, we have $\mu(C) > 0$, $C \cup T_{g_1} C \subset B$, and $a(g_1,x) \in N_1(0)+\alpha$ for $x \in C$. Applying the same argument to C and β, we choose an element $g_2 \in G$ and a Borel set D with $\mu(D) > 0$, $D \cup T_{g_2} D \subset C$, and $a(g_2,x) \in N_1(0)+\beta$ for every $x \in D$. Put $D' = T_{g_2}D$. Then $\mu(D') > 0$, $D' \cup T_{g_1 g_2^{-1}}D' \subset B$, and

$$a(g_1 g_2^{-1}, y) = a(g_1, T_{g_2}^{-1} y) + a(g_2^{-1}, y) \in N_1(0) + \alpha + (N_1(0) - \beta)$$
$$\subset N(0) + \alpha - \beta$$

for every $y \in D'$. Since B and $N(0)$ were arbitrary, we have proved that $\alpha - \beta \in E(a)$. The proof is complete.

We now modify our assumptions and assume for the following four lemmas that T_G is an arbitrary (not necessarily ergodic) action of G on $(X, \mathcal{S}, \mu)$, and that $u: R(T_G) \rightarrow A$ is an orbital cocycle.

<u>Lemma 3.4.</u> Let $\phi: X \rightarrow X$ be a Borel map with

$$\phi(x) \in T_G x \quad \text{for } \mu\text{-a.e. } x \in X. \tag{3.2}$$

Then there exists an orbital cocycle $u_\phi: R(T_G) \rightarrow A$ such that

$$u_\phi(x,y) = u(\phi(x), \phi(y)) \tag{3.3}$$

μ-a.e. on $R(T_G)$. u_ϕ is unique up to equivalence, and it is cohomologous to u.

Proof: Put $N_1 = \{x : \phi(x) \notin T_G x\}$ and $N = T_G N_1$. The map $c: X \rightarrow A$ with

$$c(x) = \begin{cases} u(x, \phi(x)) & \text{for } x \in X \setminus N, \\ 0 & \text{otherwise}, \end{cases} \tag{3.4}$$

is Borel. We define

$$u_\phi(x,y) = \begin{cases} u(\phi(x), \phi(y)) & \text{for } (x,y) \in R_{X \setminus N}(T_G), \\ 0 & \text{on } R_N(T_G). \end{cases} \quad (3.5)$$

$u_\phi : R(T_G) \to A$ is an orbital cocycle, and

$$u_\phi(x,y)+c(x)-c(y) = u(x, \phi(x))+u(\phi(x), \phi(y))+u(\phi(y),y)$$

$$= u(x,y) \quad (3.6)$$

for every $(x,y) \in R_{X \setminus N}(T_G)$. Since $\mu(N) = 0$, (3.6) implies that u_ϕ is cohomologous to u. (3.3) shows that u_ϕ is unique up to equivalence. The proof is complete.

<u>Lemma 3.5</u>. Let $B_1, B_2 \in \mathcal{S}$ with $\mu(B_1 \cap T_G B_2) > 0$, and let W be a set and N(0) a neighbourhood of 0 in A. Suppose that

$$u(x,y) \in W \quad (3.7)$$

for every $(x,y) \in R_{B_1}(T_G)$. Then there exists a set $C \subset B_2$ with $\mu(C) > 0$, $C \in \mathcal{S}$, and

$$u(x,y) \in W+N(0) \quad (3.8)$$

for every $(x,y) \in R_C(T_G)$.

Proof: By assumption we can find $D \in \mathcal{S}$ and $g_o \in G$ with $\mu(D) > 0$, $D \subset B_2$, and $T_{g_o} D \subset B_1$. Let $C \subset D$ be a Borel set of positive measure such that $u(x,T_{g_o}x)-u(y,T_{g_o}y) \in N(0)$ for all $x,y \in C$. We get

$$u(x,y) = u(T_{g_o}x, T_{g_o}y) + u(x, T_{g_o}x) - u(y, T_{g_o}y) \in W + N(0)$$

for every $(x,y) \in R_C(T_G)$. The proof is complete.

Lemma 3.6. Let $B_o \in \mathcal{S}$ with $\mu(X \setminus T_G B_o) = 0$. Then there exists a Borel map $\phi_{B_o}: X \to X$ such that

$$\phi_{B_o}(x) \in T_G x \cap B_o \tag{3.9}$$

for μ-a.e. $x \in X$.

Proof: Let $(g_o = e, g_1, g_2, \dots)$ be an enumeration of G. Put $B_o = B$, and

$$B_k = T_{g_k} B \setminus \bigcup_{i=0}^{k-1} T_{g_i} B$$

for $k \geq 1$. By assumption we have $\mu(X \setminus \bigcup_k B_k) = 0$, and it is clear that $B_i \cap B_j = \emptyset$ for $i \neq j$. Put

$$\phi_{B_o}(x) = \begin{cases} T_{g_k}^{-1} x & \text{for } x \in B_k,\ k = 0,1,\dots, \\ x_o & \text{for } x \in X \setminus \bigcup_k B_k, \end{cases} \tag{3.10}$$

where x_o is an arbitrary, but fixed, point in B_o. ϕ_{B_o} then satisfies (3.9).

Lemma 3.7. Let W_1, W_2 be subsets of A, and suppose the following is true:

(1) $u(x,y) \in W_1$ for μ-a.e. $(x,y) \in R(T_G)$, (3.11)

(2) for every $C \in \mathcal{S}$ with $\mu(C) > 0$ there exists a

$B \in \mathcal{S}$ with $\mu(B) > 0$, $B \subset C$, and with

$$u(x,y) \notin W_2 \text{ for every } (x,y) \in R_B(T_G). \tag{3.12}$$

Then there exists a Borel map $c: X \to A$ such that

(3) $$u'(x,y) = u(x,y) + c(x) - c(y) \notin W_2 \tag{3.13}$$

for μ-a.e. $(x,y) \in R(T_G)$, and

(4) $$c(x) \in W_1 \text{ for } \mu\text{-a.e. } x \in X. \tag{3.14}$$

Proof: We define a sequence $(B_k, k \geq 1) \subset \mathcal{S}$ by induction. Put $C = X$ and apply (2) to find $B_1 \in \mathcal{S}$ with $\mu(B_1) > 0$ and with $u(x,y) \notin W_2$ for all $(x,y) \in R_{B_1}(T_G)$. Suppose we have constructed $B_1, \ldots, B_k \in \mathcal{S}$ with $\mu(B_i) > 0$, $T_G B_i \cap T_G B_j = \emptyset$, and $u(x,y) \notin W_2$ for all $(x,y) \in R_{B_i}(T_G)$, $1 \leq i,j \leq k$, $i \neq j$. Put $C = X \setminus \bigcup_{i=1}^{k} T_G B_i$. If $\mu(C) > 0$ we apply (2) to find $B = B_{k+1}$ satisfying (3.12). This procedure either terminates, or we can find a sequence $(B_k, k \geq 1)$ with $\mu(B_k) > 0$, $T_G B_k \cap T_G B_l = \emptyset$ for $k \neq l$, and $u(x,y) \notin W_2$ for all $(x,y) \in R_{B_k}(T_G)$ and $k = 1,2,\ldots$ An exhaustion argument shows that we may assume $\mu(X \setminus \bigcup_k T_G B_k) = 0$. Put $B_o = \bigcup_{k=1}^{\infty} B_k$ (or, if the procedure terminates after n steps, $B_o = \bigcup_{k=1}^{n} B_k$). B_o satisfies the following conditions:

$$\mu(X \setminus T_G B_o) = 0 \tag{3.15}$$

and

$$u(x,y) \notin W_2 \text{ for every } (x,y) \in R_{B_o}(T_G). \tag{3.16}$$

By (3.15) and by Lemma 3.6 we can find a Borel map $\phi = \phi_{B_o}$: $X \to X$ satisfying (3.9). Applying now Lemma 3.4, we choose an orbital cocycle $u_\phi : R(T_G) \to A$ such that $u_\phi(x,y) = u(\phi(x),\phi(y))$ μ-a.e. on $R(T_G)$. From (3.6) we see that

$$u_\phi(x,y) = u(x,y)+c(x)-c(y)$$

with

$$c(x) \in W_1 \quad \text{for } \mu\text{-a.e.} x \in X.$$

The definition of u_ϕ together with (3.16) shows that

$$u_\phi(x,y) \notin W_2 \text{ for } \mu\text{-a.e.} (x,y) \in R(T_G).$$

We now put $u' = u_\phi$, and the lemma is proved.

<u>Proposition 3.8</u>. Let T_G be an ergodic action of G on $(X, \mathcal{S}, \mu)$ and let $a: G \times X \to A$ be a cocycle for T_G. Suppose $K \subset A$ is a compact set with $K \cap E(a) = \phi$. Then the following is true:

(1) For every $C \in \mathcal{S}$ with $\mu(C) > 0$ there exists a Borel set $B \subset C$ with $\mu(B) > 0$ and with

$$B \cap T_g^{-1} B \cap \{x : a(g,x) \in K\} = \emptyset \tag{3.17}$$

for every $g \in G$.

(2) There exists a coboundary $b: G \times X \to A$ for T_G such that

$$\mu\left(\bigcup_{g \in G} \{x : a(g,x)+b(g,x) \in K\}\right) = 0. \tag{3.18}$$

Moreover, if $W \subset A$ is a set with

$$\mu\left(\bigcup_{g \in G} \{x : a(g,x) \notin W\}\right) = 0 \tag{3.19}$$

we can choose b of the form

$$b(g,x) = c(T_g x) - c(x) \tag{3.20}$$

with

$$c(x) \in W \tag{3.21}$$

for every $x \in X$.

Proof: Let u_a be given by (2.17). According to the definition of $E(a)$ we can find, for every $\alpha \in K$, a neighbourhood $M_\alpha(0)$ in A and a Borel set B_α with $\mu(B_\alpha) > 0$ and with

$$\bigcup_{g \in G} (B_\alpha \cap T_g^{-1} B_\alpha \cap \{x : a(g,x) \in M_\alpha(0) + \alpha\}) = \phi.$$

Equivalently, we have

$$u_a(x,y) \notin M_\alpha(0) + \alpha$$

for every $(x,y) \in R_{B_\alpha}(T_G)$. Choose, for each $\alpha \in K$, a neighbourhood $N_\alpha(0)$ with $N_\alpha(0) + N_\alpha(0) \subset M_\alpha(0)$, and use the compactness of K to select finitely many $\alpha_1, \ldots, \alpha_m \in K$ with $\bigcup_{i=1}^{m} (N_{\alpha_i}(0) + \alpha_i) \supset K$. Let $C \in \mathcal{S}$ with $\mu(C) > 0$. By Lemma 3.5 we can find a Borel set $C_1 \subset C$ with $\mu(C_1) > 0$ such that

$$u_a(x,y) \notin N_{\alpha_1}(0) + \alpha_1$$

for every $(x,y) \in R_{C_1}(T_G)$. Repeating this procedure we construct a decreasing sequence $C_1 \supset C_2 \supset \ldots \supset C_m$ with $C_i \in \mathcal{S}$ and $\mu(C_m) > 0$ such that

$$u_a(x,y) \notin \bigcup_{k=1}^{l} N_{\alpha_k}(0) + \alpha_k$$

on $R_{C_l}(T_G)$, $l=1,\dots,m$. In particular we get

$$u_a(x,y) \notin K$$

for $(x,y) \in R_{C_m}(T_G)$. If we put $C_m=B$, we have proved (3.17) and hence (1). (2) follows now from Lemma 3.7, and (3.19) - (3.21) is clear from (3.11), (3.13) and (3.14). The proof is complete.

We can now state the first theorem in this section.

<u>Theorem 3.9</u>. Let T_G be an ergodic action of a countable group G on a nonatomic measure space $(X, \mathcal{S}, \mu)$, and let A be a locally compact second countable abelian group. For every cocycle $a: G \times X \to A$ for T_G we define $\bar{E}(a)$ according to Definition 2.1. Then the following conditions hold.

(1) $\bar{E}(a)$ is a closed nonempty subset of $\bar{A}$,

(2) $E(a) = \bar{E}(a) \cap A$ is a closed subgroup of A.

(3) If $a_1 \sim a_2$, we have $\bar{E}(a_1) = \bar{E}(a_2)$.

(4) a is a coboundary (i.e. $a \sim 0$) if and only if $\bar{E}(a) = \{0\}$.

Proof: (1) is obvious from the definition of $\bar{E}(a)$ and from the fact that $0 \in \bar{E}(a)$. (2) is Lemma 3.3, (3) is Lemma 3.2, and one half of (4) also follows from Lemma 3.2. To complete the proof we have to show that $a \sim 0$ whenever $\bar{E}(a) = \{0\}$.

If $\bar{E}(a) = \{0\}$, we can find a compact set $K \subset A$ and a $B \in \mathcal{S}$ with $\mu(B) > 0$ such that $u_a \in K$ μ-a.e. on $R_B(T_G)$, where u_a is given by (2.17). Let now δ be an invariant metric on A and let $N_k(0) = \{\alpha \in A: \delta(\alpha,0) < 2^{-k}\}$. By Lemma 3.4 and 3.6 there exists a Borel map $c_0 : X \to A$ such that

$$u_0(x,y) = u_a(x,y)+c_0(x)-c_0(y) \in K$$

for μ-a.e. $(x,y) \in R(T_G)$. Now put $K_1 = K \setminus N_1(0)$. Applying Proposition 2.8 we find a Borel map $c_1 : X \to A$ with $c_1(X) \subset K$ and with $u_1(x,y) = u_a(x,y)+c_0(x)+c_1(x)-c_0(y)-c_1(y) \in N_1(0)$ μ-a.e. on $R(T_G)$. Proceeding by induction, we define a sequence $(c_k : X \to A,\ k \geq 0)$ of Borel functions such that

$$c_k(X) \subset N_{k-1}(0) \quad \text{for } k=1,2,\ldots$$

and

$$u_k(x,y) = u_a(x,y)+\sum_{j=0}^{k} c_j(x)-\sum_{j=0}^{k} c_j(y) \in N_k(0) \tag{3.22}$$

for μ-a.e. $(x,y) \in R(T_G)$ and for every $k=1,2,\ldots$. Since $\delta(c_k(x),0) < 2^{-k+1}$ for every x, the series

$$c = \sum_{k=0}^{\infty} c_k$$

converges to a Borel function $c : X \to A$. Moreover we have from (3.22) that

$$u_a(x,y) = c(x)-c(y)$$

μ-a.e. on $R(T_G)$. Proposition 2.3 now implies that a is a coboundary. The proof is complete.

<u>Lemma 3.10.</u> Let T_G be ergodic and let $a: G \times X \to A$ be a cocycle for T_G. Consider the cocycle $a^*: G \times X \to A/E(a)$ given by

$$a^*(g,x) = a(g,x)+E(a). \qquad (3.23)$$

Then $E(a^*) = \{0\}$.

Proof: If $E(a) = A$, there is nothing to be proved. Assume therefore that $E(a) \neq A$ and that there exists an $\alpha_0 \in A$, $\alpha_0 \notin E(a)$ with $\alpha_0^* = \alpha_0+E(a) \in E(a^*)$. Let $B \in \mathcal{S}$ with $\mu(B) > 0$ and let $N(0)$ be a neighbourhood of 0 in A. We choose a symmetric neighbourhood $N_1(0)$ in A with $N_1(0)+N_1(0)+N_1(0) \subset N(0)$. By assumption, there exists a $g_0 \in G$ for which the set $C = B \cap T_{g_0}^{-1} B \cap \{x: a(g_0,x) \in N_1(0)+\alpha_0+E(a)\}$ has positive measure. There exists a Borel map $\eta: C \to E(a)$ such that, for every $x \in C$, $a(g_0,x) - \eta(x) \in N_1(0)+\alpha_0$. We choose a Borel set $D \subset C$ of positive measure such that $\eta(x) - \eta(y) \in N_1(0)$ for all $x,y \in D$. Let $x_0 \in D$ be fixed. Since $-\eta(x_0) \in E(a)$, we can find $D_1 \subset D$ with $\mu(D_1) > 0$, and an element $g_1 \in G$ such that $T_{g_1} D_1 \subset D$ and $a(g_1,x) \in N_1(0) - \eta(x_0)$ for all $x \in D_1$. We then have

$$a(g_0 g_1,x) = a(g_0,T_{g_1}x)+a(g_1,x) \subset N_1(0)+N_1(0)+N_1(0)+\alpha_0$$

$$\subset \alpha_0+N(0)$$

for every $x \in D_1$. Moreover, we see that $D_1 \cup T_{g_0 g_1} D_1 \subset B$. This shows that $\alpha_0 \in E(a)$, which is absurd. The contradiction shows that $E(a^*) = \{0\}$, and the proof is complete.

<u>Definition 3.11</u>. Suppose T_G is ergodic. Let $a:G\times X\to A$ be a cocycle for T_G and let $a^*:G\times X\to A/E(a)$ be given by (3.23). The cocycle a is called <u>regular</u> if $\bar{E}(a^*) = \{0\}$, and <u>nonregular</u> if $\bar{E}(a^*) = \{0,\infty\}$.

<u>Proposition 3.12</u>. Let T_G be ergodic and let $a:G\times X\to A$ be a cocycle for T_G. The following conditions are equivalent:

(1) a is regular,

(2) there exists a coboundary $b:G\times X\to A$ for T_G with

$$a(g,x)+b(g,x)\in E(a) \qquad (3.24)$$

for every $g\in G$, $x\in X$.

Proof: Let a^* be given by (3.23). If follows from Theorem 3.9 that a^* is a coboundary if and only if a is regular. If a^* is a coboundary, we choose a Borel map $c^*:X\to A/E(a)$ with $a^*(g,x) = c^*(T_gx)-c^*(x)$ for all $g\in G$ and for μ-a.e.$x\in X$. By using a result of Kuratowski (cf.[34], § I.6) we can find a Borel map $c:X\to A$ with $c^*(x) = c(x)+E(a)$ for every x. Define $a':G\times X\to A$ by

$$a'(g,x) = a(g,x)-c(T_gx)+c(x).$$

a' will then satisfy (3.24). The converse is trivial.

We now come to a second cohomology invariant which arises from the theory of random walks.

Definition 3.13. Let T_G be a conservative action of a countable group G on a nonatomic measure space $(X, \mathcal{S}, \mu)$. A cocycle $a: G \times X \to A$ for T_G is called recurrent if, for every $B \in \mathcal{S}$ with $\mu(B) > 0$, and for every neighbourhood N(0) in A,

$$\mu\Big(\bigcup_{g \in G} (B \cap T_g^{-1} B \cap \{x: a(g,x) \in N(0)\} \cap \{x: T_g x \neq x\})\Big) > 0. \qquad (3.25)$$

If a is not recurrent, we shall call it transient. An orbital cocycle $u: R(T_G) \to A$ will be called recurrent or transient according as a_u is recurrent or transient (cf. (2.18)).

Proposition 3.14. Let T_G be a conservative action of a countable group G on a nonatomic measure space $(X, \mathcal{S}, \mu)$, and let $a_i: G \times X \to A$, i=1,2, be cohomologous cocycles for T_G. Then a_1 and a_2 are either both recurrent or both transient.

Proof: We have to show the following: If a is a cocycle and b a coboundary for T_G, and if a is recurrent, then a+b is recurrent. Let $c: X \to A$ be a Borel map with $b(g,x) = c(T_g x) - c(x)$, and let $B \in \mathcal{S}$ be a set with positive measure. If N(0) is a neighbourhood of 0 in A, choose a symmetric neighbourhood $N_1(0)$ of 0 with $N_1(0) + N_1(0) \subset N(0)$. We now select a Borel set $B_1 \subset B$ with $\mu(B_1) > 0$ such that $c(x) - c(y) \in N_1(0)$ for all x,y $\in B_1$. Since a is recurrent, we can find a $g_0 \in G$ with

$$\mu(B_1 \cap T_{g_0}^{-1} B_1 \cap \{x: a(g_0,x) \in N_1(0)\} \cap \{x: T_{g_0} x \neq x\}) > 0.$$

Hence we get

$$\mu(B_1 \cap T_{g_0}^{-1} B_1 \cap \{x: a(g_0,x) + b(g_0,x) \in N(0)\} \cap \{x: T_{g_0} x \neq x\}) > 0.$$

Since $B_1 \subset B$ and since B and $N(0)$ were arbitrary, a+b is recurrent. The proof is complete.

Proposition 3.15. Let T_G be an ergodic action of a countable group G on a nonatomic measure space $(X, \mathcal{S}, \mu)$, and let $a: G \times X \to A$ be a transient cocycle for T_G. Then

$$\bar{E}(a) = \{0, \infty\} .$$

Proof: a cannot be a coboundary, since coboundaries are recurrent by Proposition 3.14. If $E(a)$ is not equal to $\{0\}$, choose $\alpha_0 \neq 0$ in $E(a)$, and let $B \in \mathcal{S}$ with $\mu(B) > 0$. We fix a neighbourhood $N(0)$ of 0 in A and a symmetric neighbourhood $N_1(0)$ with $N_1(0)+N_1(0) \subset N(0)$. Since $\alpha_0 \neq 0$ and $\alpha_0 \in E(a)$, we can find a $g_0 \in G$ for which $\mu(B \cap T_{g_0}^{-1} B \cap \{x: a(g_0,x) \in N_1(0)+\alpha_0\} \cap \{x: T_{g_0} x \neq x\}) > 0$ (cf. (2.2)). Hence there exists a Borel set $D \subset B$ with $\mu(D) > 0$ such that $D \cup T_{g_0} D \subset B$, $D \cap T_{g_0} D = \emptyset$, and $a(g_0,x) \in N_1(0)+\alpha_0$ for all $x \in D$. Since $E(a)$ is a group, $-\alpha_0 \in E(a)$, and we can apply the same argument to $T_{g_0} D$ and $-\alpha_0$ to find a Borel set D_1 of positive measure and a $g_1 \in G$ with $D_1 \cup T_{g_1} D_1 \subset T_{g_0} D$, $D_1 \cap T_{g_1} D_1 = \emptyset$, and with $a(g_1,y) \in N_1(0)-\alpha_0$ for all $y \in D_1$. Now put $C = T_{g_0}^{-1} D_1$. Then $\mu(C) > 0$, $C \cup T_{g_1 g_0} C \subset B$, $C \cap T_{g_1 g_0} C = \emptyset$, and $a(g_1 g_0,x) = a(g_1, T_{g_0} x)+a(g_0,x) \in N_1(0)+N_1(0) \subset N(0)$. a is thus recurrent, contrary to our assumption. The proof is complete, since $\bar{E}(a)$ must be equal to $\{0, \infty\}$.

The last statement shows that these invariants are also invariants of weak equivalence.

<u>Theorem 3.16</u>. Let $T_{G_i}^{(i)}$ be weakly equivalent actions of countable groups G_i on nonatomic measure spaces $(X_i, \mathcal{S}_i, \mu_i)$, $i=1,2$, and let A be a locally compact second countable abelian group. Let $\phi:(X_1, \mathcal{S}_1, \mu_1) \to (X_2, \mathcal{S}_2, \mu_2)$ be an isomorphism satisfying (1.20), and let $\tilde{\phi}: Z^1(T_{G_1}^{(1)}, \mathcal{B}(X_1, \mu_1, A)) \to Z^1(T_{G_2}^{(2)}, \mathcal{B}(X_2, \mu_2, A))$ be the isomorphism (2.26). Then the following is true:

(1) If $T_{G_1}^{(1)}$ (and hence $T_{G_2}^{(2)}$) is conservative, and if $a \in Z^1(T_{G_1}^{(1)}, \mathcal{B}(X_1, \mu_1, A))$ is recurrent, then $\tilde{\phi}(a)$ is recurrent.

(2) If $T_{G_1}^{(1)}$ (and hence $T_{G_2}^{(2)}$) is ergodic, and if $a \in Z^1(T_{G_1}^{(1)}, \mathcal{B}(X_1, \mu_1, A))$, then $\bar{E}(\tilde{\phi}(a)) = \bar{E}(a)$.

The proof is left as an exercise.

We conclude this section with some more exercises.

<u>Exercise 3.17</u>. Let T_G be a *type I* action of G on a measure space $(X, \mathcal{S}, \mu)$. Show that every cocycle for T_G is a coboundary.

<u>Exercise 3.18</u>. Let T_G be an action of G on a nonatomic measure space $(X, \mathcal{S}, \mu)$, and let $B \in \mathcal{S}$ with $\mu(B) > 0$. If $u: R(T_G) \to A$ is an orbital cocycle, define u_B to be the restriction of u to $R_B(T_G)$ as in (2.28) in Exercise 2.10.

(a) Suppose T_G is conservative and u is recurrent. Show that u_B is a recurrent orbital cocycle for $T(G,B)$.

(b) If T_G is ergodic, show that $\bar{E}(u) = \bar{E}(u_B)$. In particular, if u_1 and u_2 are cohomologous orbital cocycles for T_G, u_{1_B} and u_{2_B} will be cohomologous, and vice versa.

(c) If T_G is ergodic, and if u_B is recurrent, show that u is recurrent.

<u>Exercise 3.19</u>. Let T_G be an ergodic action of G on a nonatomic measure space $(X, \mathcal{S}, \mu)$. Let, for any cocycle $a: G \times X \to A$ for T_G,

$$R(a) = \{\alpha \in \bar{A}: \mu(\bigcup_{g \in G} \{x: a(g,x) \in N(\alpha)\}) > 0 \text{ for every neighbourhood } N(\alpha) \text{ of } \alpha \text{ in } \bar{A}\}. \tag{3.26}$$

Then R(a) is a closed subset of $\bar{A}$. Show that

$$\bar{E}(a) = \bigcap_{b \in B^1(T_G, \mathcal{B}(X, \mu, A))} R(a+b). \tag{3.27}$$

<u>Exercise 3.20</u>. Let T_G be an ergodic action of G on a nonatomic measure space $(X, \mathcal{S}, \mu)$, and let $a: G \times X \to A$ be a transient cocycle for T_G. Then there exists a set $B \in \mathcal{S}$ with $\mu(B) > 0$ and a neighbourhood N(0) of 0 in A with

$$\mu(\bigcup_{g \in G} B \cap T_g^{-1} B \cap \{x: a(g,x) \in N(0)\} \cap \{x: T_g x \neq x\}) = 0.$$

Let $N_1(0)$ and $N_2(0)$ be any neighbourhoods of 0 in A with $N_1(0)+N_2(0) \subset N(0)$. If $C \in \mathcal{S}$ with $\mu(C) > 0$, show that the following is true: There exists a Borel set $C_1 \subset C$ with $\mu(C_1) > 0$ and with

$$\bigcup_{g \in G} (C_1 \cap T_g^{-1} C_1 \cap \{x : a(g,x) \in N_1(0)\} \cap \{x : T_g x \neq x\}) = \emptyset. \qquad (3.28)$$

§ 4 Recurrence of Radon-Nikodym derivatives

Let $(X, \mathcal{S}, \mu)$ be a nonatomic measure space, G a countable group, and let T_G be a conservative action of G on $(X, \mathcal{S}, \mu)$.

Lemma 4.1. Let $u: R(T_G) \to \mathbb{R}$ be a real valued orbital cocycle. Suppose there exists an $\varepsilon > 0$ and a $B \in \mathcal{S}$ with $\mu(B) > 0$ such that

$$|u(x,y)| \geq \varepsilon \tag{4.1}$$

whenever

$$(x,y) \in R_B(T_G) \quad \text{and} \quad x \neq y. \tag{4.2}$$

Then we can find a $V \in [T_G]$ such that

(1) $VB = B$ and $Vx = x$ for every $x \in X \setminus B$, (4.3)

(2) $u(Vx,x) \geq \varepsilon$ for μ-a.e.$x \in B$, (4.4)

(3) $\{V^k x : k \in \mathbb{Z}\} = T_G x \cap B$ for μ-a.e.$x \in B$. (4.5)

Proof: (2.13) and (4.1) imply that

$$|u(x_1,y) - u(x_2,y)| \geq \varepsilon \tag{4.6}$$

for every $x_1, x_2 \in T_G y$ with $x_1 \neq x_2$ and with $x_1, x_2, y \in B$. The set

$$S_y = \{u(x,y) : x \in T_G y \cap B\} \tag{4.7}$$

is thus a disrete subset of $\mathbb{R}$ for every $y \in B$. Put

$$B_+ = \{ y \in B : S_y \cap (-\infty, 0) = \phi \} , \qquad (4.8)$$

and

$$B_- = \{ y \in B : S_y \cap (0, \infty) = \phi \} .$$

Both B_+ and B_- are Borel sets. Suppose that $\mu(B_+) > 0$. Since T_G is conservative, we can find $x_o \in B_+$ and $g_o \in G$ with $T_{g_o} x_o \neq x_o$ and $T_{g_o} x_o \in B_+$. (4.1) and (4.8) imply that $u(T_{g_o} x_o, x_o) \geq \varepsilon$. Put $y_o = T_{g_o} x_o$. Then $y_o \in B_+$, $T_{g_o}^{-1} y_o = x_o \in B_+$, and $u(T_{g_o}^{-1} y_o, y_o) = u(x_o, y_o) = -u(y_o, x_o) \leq -\varepsilon$, which violates (4.8). This contradiction shows that $\mu(B_+) = 0$, and similarly one proves that $\mu(B_-) = 0$. Put

$$B' = B \setminus T_G(B_+ \cup B_-). \qquad (4.9)$$

Then $\mu(B \setminus B') = 0$, and $S_y \cap (0, \infty)$ as well as $S_y \cap (-\infty, 0)$ is nonempty for every $y \in B'$. We define Borel maps $\psi_\pm : B' \to \mathbb{R}$ by

$$\psi_+(y) = \min \{ t \in S_y : t > 0 \} \qquad (4.10)$$

and

$$\psi_-(y) = \max \{ t \in S_y : t < 0 \} . \qquad (4.11)$$

By (4.6) and (4.7) there exist, for every $y \in B'$, unique elements $x_+(y)$ and $x_-(y)$ in $T_G y \cap B = T_G y \cap B'$ with

$$u(x_\pm(y), y) = \psi_\pm(y) \qquad (4.12)$$

Define Borel maps $V_\pm : B' \to B'$ by

$$V_\pm y = x_\pm(y) \qquad (4.13)$$

for every $y \in B'$. (2.13) and (4.10) - (4.11) show that

$$\psi_+(y) + \psi_-(V_+y) = \psi_-(y) + \psi_+(V_-y) = 0 \tag{4.14}$$

for every $y \in B'$. (4.14) and (4.6) now imply

$$V_+V_-y = V_-V_+y = y$$

for every $y \in B'$. Finally we put

$$Vx = \begin{cases} V_+x & \text{for } x \in B', \\ x & \text{otherwise.} \end{cases} \tag{4.15}$$

Exercise 1.3 shows that $V \in [T_G]$. Furthermore we have

$$u(Vx,x) \geq \varepsilon$$

for every $x \in B'$ and hence for μ-a.e. $x \in B$. We have proved (4.3) and (4.4). From (4.6), (4.7) and (4.10) it is also clear that

$$\{V_+^k x : k \in \mathbb{Z}\} = T_G y \cap B'$$

for every $y \in B'$, which in turn proves (4.5).

We can now state the main result of this section.

<u>Theorem 4.2</u>. Let T_G be a conservative action of a countable group G on a nonatomic measure space $(X, \mathcal{S}, \mu)$. Let $a_\mu : G \times X \rightarrow \mathbb{R}$ be a cocycle for T_G with

$$a_\mu(g,x) = \log \frac{d\mu T_g}{d\mu}(x) \tag{4.16}$$

for every $g \in G$ and for μ-a.e. $x \in X$. Then a_μ is recurrent.

Proof: Let $u = u_{a_\mu}$ be an orbital cocycle arising from (2.17). If a_μ is transient, there exists a set $B \in \mathcal{S}$ with $0 < \mu(B) < \infty$ and an $\varepsilon > 0$ such that

$$|u(x,y)| \geq \varepsilon \tag{4.17}$$

whenever $(x,y) \in R_B(T_G)$ and $x \neq y$. The conditions of Lemma 4.1 are satisfied, and we conclude the existence of a $V \in [T_G]$ satisfying (4.3) - (4.5). (4.4) and Exercise 2.11 together give

$$\log \frac{d\mu V}{d\mu}(x) \geq \varepsilon$$

for μ-a.e.$x \in B$. We get

$$\infty > \mu(B) = \mu(VB) = \int_B \frac{d\mu V}{d\mu}(x)\, d\mu(x) \geq$$

$$\int_B e^{\varepsilon}\, d\mu(x) = e^{\varepsilon} \cdot \mu(B) > \mu(B) > 0,$$

which is impossible. This contradiction shows that a_μ is recurrent. The proof is complete.

Corollary 4.3. Let $(X, \mathcal{S}, \mu)$ be a probability space and let $V \in \mathcal{A}(X, \mathcal{S}, \mu)$. Consider the $\mathbb{Z}$-action $n \to V^n$ on $(X, \mathcal{S}, \mu)$ and choose a cocycle $a_o : \mathbb{Z} \times X \to \mathbb{R}$ for this action with

$$a_o(n,x) = \log \frac{d\mu V^n}{d\mu}(x) \tag{4.18}$$

for every $n \in \mathbb{Z}$ and for μ-a.e.$x \in X$. Then there exist four disjoint V-invariant Borel sets X_o, X_1, X_2, N in X with the following properties.

(1) $X_0 \cup X_1 \cup X_2 \cup N = X,$ (4.19)

(2) $X_0 = \bigcup_{n \geq 1} \{x : V^n x = x\},$ (4.20)

(3) for every $x \in X_1$, we have

$$\liminf_{n \to +\infty} |a_0(n,x)| = \liminf_{n \to -\infty} |a_0(n,x)| = 0. \tag{4.21}$$

(4) For every $x \in X_2$, we get

$$\lim_{|n| \to \infty} a_0(n,x) = -\infty. \tag{4.22}$$

(5) $\mu(N) = 0.$ (4.23)

Proof: Define X_0 by (4.20) and put $X' = X \setminus X_0$. If $\mu(X') > 0$, and if V (or, more precisely, the $\mathbb{Z}$-action $n \to V^n$) is conservative on X', we can easily deduce from Theorem 4.2 that (4.21) holds everywhere on X' except for a Borel set $N' \subset X'$ of measure zero. Put $X_1 = X' \setminus \bigcup_{n \in \mathbb{Z}} V^n N'$ and $N = \bigcup_{n \in \mathbb{Z}} V^n N'$. (4.19) - (4.23) is then satisfied with $X_2 = \emptyset$. Suppose now that $\mu(X') > 0$ and that $n \to V^n$ is a type I action of $\mathbb{Z}$ on $(X', \mathcal{S}_{X'}, \mu_{X'})$. Choose a set $D \in \mathcal{S}_{X'}$ with $D \cap V^k D = \emptyset$ for $k \neq 0$ and with $\mu(X' \setminus \bigcup_{n \in \mathbb{Z}} V^n D) = 0$. For every integer $m \geq 0$, let

$$Y_m = \{x \in X' : \limsup_{|n| \to \infty} a_0(n,x) > -m\}. \tag{4.24}$$

If $\mu(Y_m) > 0$, we must have $\mu(Y_m \cap V^n D) > 0$ for some n, and hence

$$1 \geq \mu(X') \geq \mu\Big(\bigcup_{k \in \mathbb{Z}} V^k(Y_m \cap V^n D)\Big) =$$

$$= \sum_{k \in \mathbb{Z}} \mu(V^k(Y_m \cap V^n D)) = \infty,$$

by (4.24). This shows that $\mu(Y_m) = 0$ for every m. Put $X_1 = \phi$, $N = \bigcup_{n \in \mathbb{Z}} \bigcup_{m \geq 1} V^n Y_m$ and $X_2 = X' \setminus N$. Again (4.19) - (4.23) will be satisfied. Finally, if $n \to V^n$ is neither conservative nor type I on X' and $\mu(X') > 0$, one can find disjoint V-invariant Borel sets X_1' and X_2' in X' with $X_1' \cup X_2' = X'$, $\mu(X_1') \cdot \mu(X_2') \neq 0$, such that the restriction of the $\mathbb{Z}$-action $n \to V^n$ to $(X_1', \mathcal{S}_{X_1'}, \mu_{X_1'})$ is conservative and that to $(X_2', \mathcal{S}_{X_2'}, \mu_{X_2'})$ type I. By the preceding arguments we can choose disjoint Borel sets N_1 and X_1 in X_1', both of which are V-invariant, such that $\mu(N_1) = 0$, $X_1' = X_1 \cup N_1$, and (4.21) holds on X_1. Similarly we can decompose X_2' into disjoint V-invariant Borel sets N_2 and X_2 with $X_2' = N_2 \cup X_2$, $\mu(N_2) = 0$, and with (4.22) holding on X_2. We put $N = N_1 \cup N_2$ and have again proved (4.19) - (4.23). The only remaining case $\mu(X') = 0$ is trivial, so that the proof is complete.

§ 5 Skew products

Let $(X, \mathcal{S}, \mu)$ be a nonatomic measure space and let A be a locally compact second countable abelian group, with Haar measure λ and with Borel field $\mathcal{B}$. We form a new measure space $(\tilde{X}, \tilde{\mathcal{S}}, \tilde{\mu})$ with

$$\tilde{X} = X \times A, \tag{5.1}$$

$$\tilde{\mathcal{S}} = \mathcal{S} \times \mathcal{B} \quad , \tag{5.2}$$

$$\tilde{\mu} = \mu \times \lambda \quad . \tag{5.3}$$

For every $\alpha \in A$ we define an automorphism R_α of $(\tilde{X}, \tilde{\mathcal{S}}, \tilde{\mu})$ by

$$R_\alpha(x, \beta) = (x, \alpha + \beta) \tag{5.4}$$

for every $(x, \beta) \in \tilde{X}$. Clearly R_α preserves $\tilde{\mu}$, and

$$R_\alpha R_\beta = R_{\alpha+\beta} \tag{5.5}$$

for every $\alpha, \beta \in A$. Suppose now that T_G is an action of a countable group G on $(X, \mathcal{S}, \mu)$. Any cocycle $a: G \times X \rightarrow A$ for T_G gives a new action T_G^a of G on $(\tilde{X}, \tilde{\mathcal{S}}, \tilde{\mu})$ as follows: For every $g \in G$, $(x, \alpha) \in \tilde{X}$, put

$$T_g^a(x, \alpha) = (T_g x, \alpha + a(g,x)). \tag{5.6}$$

The action T_G^a of G on $(\tilde{X}, \tilde{\mathcal{S}}, \tilde{\mu})$ is called a <u>skew product</u> of T_G.

Lemma 5.1. Let $a_i : G \times X \to A$, i=1,2, be cohomologous cocycles for T_G. Then the actions $T_G^{a_i}$, i=1,2, are conjugate.

Proof: Let $c : X \to A$ be a Borel map with $a_1(g,x) - a_2(g,x) = c(T_g x) - c(x)$ for every $g \in G$ and for μ-a.e. $x \in X$. We define an automorphism ϕ_c of $(\tilde{X}, \tilde{S}, \tilde{\mu})$ by

$$\phi_c(x, \alpha) = (x, \alpha + c(x)). \tag{5.7}$$

ϕ_c preserves $\tilde{\mu}$, and we have, for every $g \in G$ and for $\tilde{\mu}$-a.e. $(x, \alpha) \in \tilde{X}$,

$$T_g^{a_1} \phi_c(x, \alpha) = \phi_c T_g^{a_2}(x, \alpha) = (T_g x, \alpha + c(x) + a_1(g,x)),$$

by a straightforward computation. (1.5) shows that the proof is complete.

Lemma 5.1 shows that T_G^a depends essentially only on the cohomology class of the cocycle a. Our first aim is to relate the cohomology invariants of § 3 to properties of T_G^a. We put $\mathcal{B}_0 = \{ B \in \mathcal{B} : \lambda(B) < \infty \}$ and identify elements of $\mathcal{B}_0$ which differ by a null set. For every $B_1, B_2 \in \mathcal{B}_0$, put

$$\eta(B_1, B_2) = \lambda(B_1 \triangle B_2). \tag{5.8}$$

$(\mathcal{B}_0, \eta)$ is then a complete separable metric space. The maps from $\mathcal{B}_0 \times \mathcal{B}_0$ to $\mathcal{B}_0$ given by

$$(B_1, B_2) \to B_1 \cup B_2 \tag{5.9}$$

and

$$(B_1, B_2) \to B_1 \cap B_2 \tag{5.10}$$

are both continuous, and so is the map from $A \times \mathcal{B}_0$ to $\mathcal{B}_0$ defined by

$$(\alpha, B) \to \alpha + B = \{\alpha + \beta : \beta \in B\} . \tag{5.11}$$

Assume now the action T_G to be ergodic. We fix a cocycle $a: G \times X \to A$ and define T_G^a by (5.6). Since R_α commutes with T_g^a for every $\alpha \in A$ and for every $g \in G$, $R_\alpha B$ will be a T_G^a-invariant set whenever $B \in \tilde{\mathcal{S}}$ is T_G^a-invariant. We write $\mathcal{Z}$ for the family of T_G^a-invariant elements in $\tilde{\mathcal{S}}$ and put

$$I(a) = \{\alpha \in A : R_\alpha B = B \pmod 0 \text{ for every } B \in \mathcal{Z}\} . \tag{5.12}$$

The following theorem gives a geometric interpretation of E(a).

<u>Theorem 5.2</u>. Let T_G be an ergodic action of a countable group G on a nonatomic measure space $(X, \mathcal{S}, \mu)$, and let $a: G \times X \to A$ be a cocycle for T_G. Then $I(a) = E(a)$, where E(a) is given by Definition 2.1.

Proof: We first show that $E(a) \supset I(a)$. If $\alpha_0 \notin E(a)$, there exists a set $C \subset X$ with $C \in \mathcal{S}$, $\mu(C) > 0$, and a neighbourhood N(0) of 0 in A such that $C \cap T_g^{-1} C \cap \{x : a(g,x) \in N(0) + \alpha_0\} = \emptyset$ for every $g \in G$. Choose a neighbourhood $N_1(0)$ in A with $N_1(0) + N_1(0) \subset N(0)$ and define

$$B_1 = T_G^a(C \times N_1(0))$$

and

$$B_2 = T_G^a(C \times (N_1(0) + \alpha_0)) = R_{\alpha_0} B_1 .$$

Then $B_1 \cap B_2 = \phi$, and $\tilde{\mu}(B_1) = \tilde{\mu}(B_2) > 0$. Hence $\alpha_0 \notin I(a)$, which proves our first assertion.
For the converse we assume that $\alpha_0 \notin I(a)$. In this case we can find disjoint sets $B_1, B_2 \in \mathcal{Z}$ with $\tilde{\mu}(B_1) = \tilde{\mu}(B_2) \neq 0$ and with $R_{\alpha_0} B_1 = B_2$. Put

$$D_x = \{\alpha \in A: (x, \alpha) \in B_1\}, \quad x \in X. \tag{5.13}$$

Then $\lambda(D_x) \neq 0$ for μ-a.e.$x \in X$ (in fact, the map $x \to \lambda(D_x)$ is constant a.e.). We choose a Borel map $\theta: X \to \mathcal{B}_0$ such that, for μ-a.e.$x \in X$,

$$\begin{aligned} &(1) \quad 0 < \lambda(\theta(x)) < \infty, \\ &(2) \quad \theta(x) \subset D_x. \end{aligned} \tag{5.14}$$

For the purpose of this proof we replace μ by an equivalent probability measure μ' and identify $(X, \mathcal{S})$ with the closed unit interval $[0,1]$ in $\mathbb{R}$. We can find a compact set $Y_1 \subset X$ with $\mu'(Y_1) > 0$ such that the map θ is continuous on Y_1 and that $\lambda(\theta(x)) \geq \varepsilon_1 > 0$ for all $x \in Y_1$, where $\varepsilon_1 > 0$ is chosen suitably. Let $\omega: Y_1 \times Y_1 \times A \to \mathbb{R}$ be the map

$$\omega(x, y, \alpha) = \lambda(\theta(x) \cap (\theta(y) + \alpha)).$$

(5.9) - (5.11) imply that ω is continuous, and we have $\omega(x,x,0) \geq \varepsilon_1$ for every $x \in Y_1$. Hence there exists a compact set $Y \subset Y_1$ and a neighbourhood $N_1(0)$ of 0 in A such that $\omega(x,y,\alpha) \geq \varepsilon_1/2$ for every $x, y \in Y$ and for every $\alpha \in N_1(0)$, and such that $\mu'(Y) > 0$. Since B_1 and B_2 are disjoint, (5.14) implies that

$$\alpha_0 + \Theta(T_g x) \cap \Theta(x) + a(g,x) = \phi \tag{5.15}$$

for every $g \in G$ and for μ'-a.e. $x \in Y$ with $T_g x \in Y$. (5.15) in turn shows that, for μ'-a.e. $x \in Y$, we have $a(g,x) - \alpha_0 \notin N_1(0)$ whenever $T_g x \in Y$. From this it is clear that $\alpha_0 \notin E(a)$ and hence that $I(a) \supset E(a)$. The proof is complete.

We can apply Theorem 5.2 to find a criterion for the ergodicity of T_G^a, but first we need a simple lemma.

<u>Lemma 5.3.</u> Let $D_i \in \mathcal{B}$, i=1,2, be sets of positive Haar measure. Then there exists an open set $\mathcal{O} \subset A$ with

$$\psi(\alpha) = \lambda(D_1 \cap (D_2 + \alpha)) > 0 \tag{5.16}$$

for every $\alpha \in \mathcal{O}$.

Proof: It will obviously be enough to prove the lemma under the additional assumption that $\lambda(D_i) < \infty$ for i=1,2. We write χ_{D_i} for the characteristic function of the set D_i and get

$$\int \lambda(D_1 \cap (D_2 + \alpha))\, d\lambda(\alpha) =$$

$$\iint \chi_{D_1}(\beta) \cdot \chi_{D_2}(\beta - \alpha)\, d\lambda(\alpha)\, d\lambda(\beta) =$$

$$\lambda(D_1) \cdot \lambda(D_2) > 0. \tag{5.17}$$

ψ is nonnegative and continuous by (5.10) and (5.11), and (5.17) shows that ψ does not vanish identically. This implies (5.16) and proves the lemma.

Corollary 5.4. T_G^a is ergodic if and only if $E(a) = A$.

Proof: If T_G^a is ergodic, we must have $I(a) = A$. Theorem 5.2 now shows that $E(a) = A$. Suppose now that $E(a) = A$. Then $I(a) = A$, by Theorem 5.2. We choose $B_1 \in \mathcal{Z}$ with $\tilde{\mu}(B_1) > 0$ and define D_x, $x \in X$, by (5.13). Again we note that the map $x \to \lambda(D_x)$ is T_G-invariant and hence constant a.e. on X. We can choose a dense sequence $(\alpha_1, \alpha_2, \ldots)$ in A and a set $N \in \mathcal{S}$ with $\mu(N) = 0$ such that

$$\lambda(D_x) > 0 \tag{5.18}$$

and

$$\lambda(D_x \triangle D_x + \alpha_i) = 0 \tag{5.19}$$

for every $x \in X \setminus N$ and for every $i = 1, 2, \ldots$. We fix $x \in X \setminus N$ for the moment. If $\lambda(A \setminus D_x) > 0$, we can apply Lemma 5.3 to find an open set $\mathcal{O}_x$ with $\lambda(D_x + \alpha \cap \complement D_x) > 0$ for every $\alpha \in \mathcal{O}_x$. Hence we can find a positive integer $i(x)$ with $\lambda(D_x + \alpha_{i(x)} \cap \complement D_x) > 0$. But this violates (5.18) and (5.19), so that $\lambda(A \setminus D_x) = 0$ for all $x \in X \setminus N$. Fubini's Theorem now implies that $\tilde{\mu}(\complement B_1) = 0$, and the proof is complete.

Theorem 5.5. Let T_G be a conservative action of a countable group G on a nonatomic measure space $(X, \mathcal{S}, \mu)$ and let $a: G \times X \to A$ be a cocycle for T_G. T_G^a is conservative if and only if a is recurrent.

Proof: If a is transient, there exists a Borel set $B \subset X$ with $\mu(B) > 0$ and a neighbourhood $N(0)$ of 0 in A such that

$B \cap T_g^{-1} B \cap \{x: a(g,x) \in N(0)\} \cap \{x: T_g x \neq x\} = \emptyset$ for every $g \in G$.
We choose a neighbourhood $N_1(0)$ in A with $N_1(0)+N_1(0) \subset N(0)$ and put $D_o = B \times N_1(0)$. Our assumptions imply that

$$\tilde{\mu}(D_o \cap T^a_{g^{-1}} D_o \cap \{(x, \alpha): T^a_g(x, \alpha) \neq (x, \alpha)\}) = 0 \qquad (5.20)$$

for every $g \in G$, so that T^a_G is not conservative.
Assume now that T^a_G is not conservative, and choose a set $B_1 \subset X$ with $\tilde{\mu}(\bigcup_{g \in G} B_1 \cap T^a_{g^{-1}} B_1 \cap \{(x, \alpha): T^a_g(x, \alpha) \neq (x, \alpha)\}) = 0$ and $\tilde{\mu}(B_1) > 0$. We define D_x, $x \in X$, as in (5.13). As in the proof of Theorem 5.2 we can find a set $Y \subset X$ with $\mu(Y) > 0$ and a neighbourhood $N(0)$ of 0 in A such that $\lambda(D_x \cap (D_y + \alpha)) > 0$ for every $x, y \in Y$ and for every $\alpha \in N(0)$. Our choice of B_1 shows that $a(g,x) \notin N(0)$ whenever $x \in Y$, $T_g x \in Y$, and $T_g x \neq x$. In other words, a is transient, and the theorem is proved.

<u>Corollary 5.6.</u> Let T_G be an ergodic action of G on $(X, \mathcal{S}, \mu)$, and let $a: G \times X \to A$ be a cocycle for T_G. Then T^a_G is a type I action if and only if a is transient.

Proof: From Theorem 5.5 it is clear that T^a_G can only be type I if a is transient. Conversely, if a is transient, we apply the first part of the proof of Theorem 5.5 to find a set $B \subset X$ with $\mu(B) > 0$ and a neighbourhood $N_1(0)$ of 0 in A such that $D_o = B \times N_1(0)$ satisfies (5.20). Let $(\alpha_1, \alpha_2, \dots)$ be a dense sequence in A and put

$$F_o = T^a_G D_o,$$

$$F_i = R_{\alpha_i} F_o, \quad i=1,2,\ldots,$$

and

$$C_i = \bigcup_{k=0}^{i-1} F_k, \quad i=1,2,\ldots$$

Finally we let

$$D_i = R_{\alpha_i} D_o \setminus C_i$$

for $i=1,2,\ldots$. Then $D_i \cap D_j = \emptyset$ for $i \neq j$, and

$$D' = T_G^a\left(\bigcup_{i=0}^{\infty} D_i\right) \supset B \times A.$$

Since $D' \in \mathcal{Z}$, and since T_G is ergodic, we must have $\tilde{\mu}(\tilde{X} \setminus D') = 0$. From the definition of the sequence $D_0, D_1, \ldots$ it is clear that

$$D = \bigcup_{i=0}^{\infty} D_i$$

satisfies (1.8) and (1.9) with T_G^a replacing T_G. Hence T_G^a is type I and the proof is complete.

§ 6 An application of skew products to ergodic decomposition

Let T_G be an action of a countable group G on a probability space $(X, \mathcal{S}, \mu)$. We put

$$\mathcal{S}^* = \{B \in \mathcal{S} : T_G B = B\} \tag{6.1}$$

and write $L^2(X, \mathcal{S}^*, \mu)$ for the space of (equivalence classes of) $\mathcal{S}^*$-measurable real valued μ-square integrable functions on X. Choose a sequence $(z_k, k=1,2,\dots)$ of real valued $\mathcal{S}^*$-measurable functions on $(X, \mathcal{S})$ which satisfy

(a) $0 \leq z_k(x) \leq 1$ for every $k=1,2,\dots,\ x \in X$,

(b) the linear span of $(z_k, k=1,2,\dots)$ is dense in $L^2(X, \mathcal{S}^*, \mu)$.

We denote by $I = [0,1]$ the closed unit interval in $\mathbb{R}$ and by $Z = \prod_{k=1}^{\infty} I_k$ the cartesian product with $I_k = I$ for all k. A typical element $z \in Z$ is of the form $z = (z_1, z_2, \dots)$ with $0 \leq z_k \leq 1$ for all k. Z is a compact metric space in the product topology, and we write $\mathcal{F}_1$ for the Borel field of Z. $\mathcal{F}_2$ will stand for the Borel field of I, Ω for the product space $Z \times I$, and $\mathcal{F}$ for the product Borel field $\mathcal{F}_1 \times \mathcal{F}_2$ on Ω. $\pi_1 : \Omega \to Z$ and $\pi_2 : \Omega \to I$ denote the two coordinate projections. We define a Borel map $\phi_1 : X \to Z$ by

$$\phi_1(x) = (z_1(x), z_2(x), \dots), \tag{6.2}$$

and choose and fix an injective Borel map $\phi_2: X \to I$ (This is obviously possible if X is countable. If X is uncountable, we may even assume ϕ_2 to be a Borel isomorphism.). If we now put

$$\phi(x) = (\phi_1(x), \phi_2(x)), \ x \in X, \tag{6.3}$$

we get an injective Borel map $\phi : X \to \Omega$. Note that

$$\Omega' = \phi(X) \in \mathcal{F} \tag{6.4}$$

(cf. Theorem 3.9 in [34]). For every $g \in G$, $\omega \in \Omega$, define

$$T_g^* \omega = \begin{cases} \phi(T_g x) & \text{whenever } \omega \in \Omega' \text{ and } \phi(x) = \omega, \\ \omega & \text{otherwise.} \end{cases} \tag{6.5}$$

We put

$$\nu = \mu \phi^{-1} \tag{6.6}$$

and see that T_G^* is an action of G on the probability space $(\Omega, \mathcal{F}, \nu)$, which is conjugate to T_G. Let

$$\mathcal{F}^* = \{B \in \mathcal{F} : T_G^* B = B\} \tag{6.7}$$

and

$$\tilde{\mathcal{F}} = \{\pi_1^{-1}(E) : E \in \mathcal{F}_1\}. \tag{6.8}$$

It is clear that $\tilde{\mathcal{F}} \subset \mathcal{F}^*$. On the other hand it is easy to see that for every $B \in \mathcal{F}^*$, there exists a $C \in \tilde{\mathcal{F}}$ with $\nu(B \Delta C) = 0$. We shall express this by saying that

$$\tilde{\mathcal{F}} = \mathcal{F}^* \pmod 0. \tag{6.9}$$

Similarly, if a function is $\mathcal{F}^*$-measurable, it will coincide ν-a.e. with an $\tilde{\mathcal{F}}$-measurable function.

Lemma 6.1. Let $\rho = \nu\pi_1^{-1}$. There exists a family $\{p_z : z \in Z\}$ of probability measures on $(\Omega, \mathcal{F})$ such that

(1) $z \to p_z(B)$ is an $\mathcal{F}_1$-measurable map on Z for every $B \in \mathcal{F}$, (6.10)

(2) $\nu(B \cap C) = \int_C p_z(B)\, d\rho(z)$ for every $B \in \mathcal{F}$, $C \in \mathcal{F}^*$, (6.11)

(3) $p_z(\pi_1^{-1}(\{z\})) = 1$ for every $z \in Z$. (6.12)

Moreover, (1) - (3) determines p_z uniquely for ρ-a.e. $z \in Z$.

Proof: This is an application of the well known theorem on regular conditional probability.

We now choose a cocycle $a_\nu : G \times \Omega \to \mathbb{R}$ for T_G^* with

$$a_\nu(g,x) = \log \frac{d\nu T_g^*}{d\nu}(\omega) \tag{6.13}$$

for every $g \in G$ and for ν-a.e. $\omega \in \Omega$. Fix $g \in G$ for the moment, put $V = T_g^*$, and write $a_0 : \mathbb{Z} \times \Omega \to \mathbb{R}$ for the function

$$a_0(n,\omega) = a_\nu(g^n, \omega).$$

Applying Corollary 4.3 to V and a_0, we obtain a decomposition $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2 \cup N$ of Ω into disjoint V-invariant Borel sets satisfying (4.19) - (4.23) with Ω and Ω_i replacing X and X_i. To emphasize the dependence of this decomposition on g, we put $\Omega_i = \Omega_i^{(g)}$ and $N = N^{(g)}$. Varying g now, we define

$$N_o = T_G^*\left(\bigcup_{g \in G} N^{(g)}\right). \tag{6.14}$$

By (4.23), $\nu(N_o) = 0$. Consider the cocycle $\tilde{a}: G \times \Omega \to \mathbb{R}$ for T_G^* given by

$$\tilde{a}(g,\omega) = \begin{cases} a_\nu(g,\omega) & \text{for every } g \in G \text{ and } \omega \in \Omega \setminus N_o, \\ 0 & \text{otherwise.} \end{cases} \tag{6.15}$$

$\tilde{a}$ has the following properties (cf. (4.19) - (4.23)).

(1) $\tilde{a} \equiv a_\nu$, (6.16)

(2) for every $g \in G$ and for every $\omega \in \Omega$ with

$$T^*_{g^n}\omega \neq \omega \quad \text{for } n \neq 0,$$

we have either

$$\liminf_{n \to \infty} |\tilde{a}(g^n,\omega)| = \liminf_{n \to -\infty} |\tilde{a}(g^n,\omega)| = 0, \tag{6.17}$$

or

$$\lim_{|n| \to \infty} \tilde{a}(g^n,\omega) = -\infty. \tag{6.18}$$

(3) If $g \in G$ and $\omega \in \Omega$ satisfies $T^*_{g^{n_o}}\omega = \omega$ for some $n_o > 0$, we have either

$$\tilde{a}(g^{k+n_o},\omega) = \tilde{a}(g^k,\omega) \text{ for every } k \in \mathbb{Z}, \tag{6.19}$$

or there exists a real number c with

$$\tilde{a}(g^k,\omega) = k.c \text{ for every } k\in\mathbb{Z}. \tag{6.20}$$

We denote by $\tilde{P}$ the set of all probability measures σ on $(\Omega,\mathcal{F})$ which are quasi-invariant under T_G^* and which satisfy

$$\log \frac{d\sigma T_g^*}{d\sigma}(\omega) = \tilde{a}(g,\omega) \tag{6.21}$$

for σ-a.e. $\omega\in\Omega$ and for every $g\in G$.

<u>Lemma 6.2.</u> $\nu\in\tilde{P}$. Furthermore there exists a set $N_1\in\mathcal{F}_1$ with $\rho(N_1) = 0$ and with $p_z\in\tilde{P}$ whenever $z\in Z\setminus N_1$.

Proof: Since $\tilde{a}\equiv a_\nu$, we have $\nu\in\tilde{P}$ (cf. (6.13), (6.16) and (6.21)). (6.11) shows that, for every nonnegative Borel function f on Ω and for every $C\in\tilde{\mathcal{F}}$,

$$\int_C f(\omega)\, d\nu(\omega) = \int_C f(\omega)\, dp_z(\omega)\, d\rho(z).$$

Furthermore,

$$\int_C f(T^*_{g^{-1}}\omega)\, d\nu(\omega) = \int_C f(\omega).e^{\tilde{a}(g,\omega)}\, d\nu(\omega),$$

since $\nu\in\tilde{P}$. We get

$$\int_C\int f(T^*_{g^{-1}}\omega)\, dp_z(\omega)\, d\rho(z) = \int_C\int f(\omega).e^{a(g,\omega)}\, d\nu(\omega). \tag{6.22}$$

Let $(\phi_k,\ k=1,2,\dots)$ be a sequence of nonnegative continuous functions on Ω whose linear span is dense in the space $C(\Omega)$ of all continuous real valued functions on Ω in the maximum norm. (6.22) shows that there exists a set $N_1\in\mathcal{F}_1$ with

$\rho(N_1) = 0$ and with

$$\int \phi_k(T^*_{g^{-1}}\omega)\, dp_z(\omega) = \int \phi_k(\omega).e^{\tilde{a}(g,\omega)}\, dp_z(\omega) \qquad (6.23)$$

for every k and for every $z \notin N_1$. It follows immediately that (6.23) holds for all continuous realvalued functions ϕ on Ω, and for every $g \in G$, $z \in Z \setminus N_1$. Hence $p_z \in \tilde{P}$ for $z \in Z \setminus N_1$, and the lemma is proved.

Remark 6.3. If ν is invariant under T^*_G, we can choose $\tilde{a} = 0$ and see from (6.23) that p_z is T^*_G-invariant for every $z \in Z \setminus N_1$. If ν is equivalent to a σ-finite T^*_G-invariant measure we can choose $\tilde{a}$ to be of the form $\tilde{a}(g,\omega) = c(T^*_g\omega) - c(\omega)$ for every $g \in G$ and every $\omega \in \Omega$, where $c: \Omega \to \mathbb{R}$ is a Borel map. (6.23) now implies that p_z is equivalent to a σ-finite T^*_G-invariant measure for every $z \in Z \setminus N_1$.

We now have to prove that p_z is ergodic under T^*_G for ρ-a.e. $z \in Z$, but we need some preparation for that. Put

$$\Delta = \Omega \times \mathbb{R}$$

and

$$\Delta_n = \Omega \times (-n,n) \quad \text{for } n \geq 1,$$

where $(-n,n) = \{t \in \mathbb{R} : -n < t < n\}$. $\mathcal{D}$ and $\mathcal{D}_n$ will denote the Borel fields of Δ and Δ_n, respectively. Every $\sigma \in \tilde{P}$ defines a measure $\tilde{\sigma}$ on $(\Delta, \mathcal{D})$ by

$$d\tilde{\sigma}(\omega,t) = d\sigma(\omega).e^{-t}dt. \qquad (6.24)$$

We denote by $\tilde{\sigma}_n$ the restriction of $\tilde{\sigma}$ to $(\Delta_n, \mathcal{D}_n)$, multiplied by the constant $(e^n - e^{-n})^{-1}$ to make $\tilde{\sigma}_n$ a probability measure. For every $(\omega,t) \in \Delta$, $g \in G$, put

$$V_g(\omega,t) = (T_g^* \omega, t+\tilde{a}(g,\omega)). \tag{6.25}$$

If σ is any measure in $\tilde{P}$, V_G is an action of G on $(\Delta, \mathcal{D}, \tilde{\sigma})$ which preserves $\tilde{\sigma}$. We now fix $g \in G$ and $n \geq 1$. For every $(\omega,t) \in \Delta_n$, put

$$m_+(\omega,t) = \begin{cases} \min \{m \geq 1: V_g^m(\omega,t) \in \Delta_n\} & \text{if } \{m \geq 1: V_g^m(\omega,t) \in \Delta_n\} \neq \emptyset, \\ \infty & \text{otherwise.} \end{cases} \tag{6.26}$$

and

$$m_-(\omega,t) = \begin{cases} \min \{m \leq 0: V_g^m(\omega,t) \in \Delta_n\} & \text{if } \{m \leq 0: V_g^m(\omega,t) \in \Delta_n\} \text{ is a finite set,} \\ -\infty & \text{otherwise.} \end{cases} \tag{6.27}$$

(6.17) - (6.20) imply that $m_-(\omega,t) > -\infty$ whenever $m_+(\omega,t) = \infty$. We can thus define a Borel automorphism $V(g,n)$ of $(\Delta_n, \mathcal{D}_n)$ by

$$V(g,n)(\omega,t) = \begin{cases} V_g^{m_+(\omega,t)}(\omega,t) & \text{if } m_+(\omega,t) < \infty, \\ V_g^{m_-(\omega,t)}(\omega,t) & \text{if } m_+(\omega,t) = \infty. \end{cases} \tag{6.28}$$

Since $V(g,n) \in [V_G]_{\Delta_n}$ by Exercise 1.3, $\tilde{\sigma}_n$ is invariant under $V(g,n)$ (Exercise 1.2). Moreover we have

$$\Delta_n \cap \{V_g^k(\omega,t): k \in \mathbb{Z}\} = \{V(g,n)^k(\omega,t): k \in \mathbb{Z}\} \tag{6.29}$$

for every $(\omega,t) \in \Delta_n$. We put

$$\mathcal{D}^*(g,n) = \{B \in \mathcal{D}_n : V(g,n)B=B\}. \tag{6.30}$$

Let $(g_1, g_2, \ldots)$ be a fixed enumeration of G, and let

$$\mathcal{D}^*(i,n) = \bigcap_{k=1}^{i} \mathcal{D}^*(g_k,n). \tag{6.31}$$

We now define

$$\mathcal{F}^*(i,n) = \{B \in \mathcal{F} : B \times (-n,n) \in \mathcal{D}^*(i,n)\}. \tag{6.32}$$

For fixed n, $\mathcal{F}^*(i,n)$ decreases as $i \to \infty$, and for fixed i, $\mathcal{F}^*(i,n)$ decreases as $n \to \infty$. In fact we easily see from (6.29) that

$$\bigcap_{n \geq 1} \bigcap_{i \geq 1} \mathcal{F}^*(i,n) = \mathcal{F}^*. \tag{6.33}$$

Let now $f: \Delta_n \to \mathbb{R}$ be a bounded Borel function, where $n \geq 1$ is again fixed. For every $k \geq 1$, put

$$\Theta_k(f)\ (\omega,t) = \lim_m \sup \frac{1}{m} \sum_{i=0}^{m-1} f(V(g_k,n)^i(\omega,t)). \tag{6.34}$$

$\Theta_k(f): \Delta_n \to \mathbb{R}$ is again a Borel map. For every $j \geq 1$, let

$$\Lambda_j(f) = \Theta_1(\Theta_2(\ldots \Theta_{j-1}(\Theta_j(\Theta_{j-1}(\ldots(\Theta_1(f))\ldots)))\ldots)). \tag{6.35}$$

Proceeding by induction, we put

$$\Lambda_j^{(1)}(f) = \Lambda_j(f)$$

and

$$\Lambda_j^{(l)}(f) = \Lambda_j(\Lambda_j^{(l-1)}(f))$$

for $l \geq 2$. Finally we set

$$\tilde{\Lambda}_j(f,n)\ (\omega,t) = \lim_m \sup \frac{1}{m} \sum_{l=1}^{m} \Lambda_j^{(l)}(f)\ (\omega,t). \tag{6.36}$$

$\tilde{\Lambda}_j(f,n): \Delta_n \to \mathbb{R}$ is again a bounded Borel map.

<u>Lemma 6.4</u>. For every bounded Borel map $f: \Delta_n \to \mathbb{R}$, every $j \geq 1$, and for every $\sigma \in \tilde{P}$, we have

$$\tilde{\Lambda}_j(f,n) = E_{\tilde{\sigma}_n}(f \,|\, \mathcal{D}^*(j,n)) \tag{6.37}$$

$\tilde{\sigma}_n$-a.e. on Δ_n, where $E_{\tilde{\sigma}_n}$ denotes the coditional expectation with respect to $\tilde{\sigma}_n$.

Proof: We use a minor modification of an elegant proof of the ergodic theorem in [3]. Throughout this proof we shall assume relations between functions and between sets to hold only modulo sets of $\tilde{\sigma}_n$-measure zero. The L^2-ergodic theorem implies that Λ_j defines a bounded, linear, self-adjoint operator on $L^2(\Delta_n, \mathcal{D}_n, \tilde{\sigma}_n)$. We apply the spectral theorem to this operator and see that a square integrable (and, in particular, a bounded) function f satisfies $\Lambda_j(f) = f$ if and only if f is $\mathcal{D}^*(j,n)$-measurable. Let now $f: \Delta_n \to \mathbb{R}$ be a bounded measurable function, and put

$$f' = f - E_{\tilde{\sigma}_n}(f \,|\, \mathcal{D}^*(j,n)). \tag{6.38}$$

If

$$\tilde{f} = \lim_m \sup \frac{1}{m} \sum_{l=1}^{m} \Lambda_j^{(l)}(f'),$$

we have to show that $\tilde{f} = 0$, and that will prove the lemma. Fix $\varepsilon > 0$, and put

$$B_\varepsilon = \{(\omega,t): \tilde{f}(\omega,t) > \varepsilon\} .$$

Since $\tilde{f}$ is invariant under Λ_j, we have $B_\varepsilon \in \mathcal{D}^*(j,n)$. We write χ_{B_ε} for the characteristic function of B_ε and put

$$f^* = (f' - \varepsilon) . \chi_{B_\varepsilon} .$$

For $m \geqslant 1$, let

$$S_m = f^* + \Lambda_j(f^*) + \ldots + \Lambda_j^{(m-1)}(f^*)$$

and

$$M_m = \max\,(0, S_1, \ldots, S_m).$$

We shall first prove that, for every $m \geqslant 1$,

$$\int_{C_m} f^* \, d\,\tilde{\sigma}_n \geqslant 0, \tag{6.39}$$

where

$$C_m = \{(\omega,t) : M_m(\omega,t) > 0\} .$$

Indeed, if $1 \leqslant l \leqslant m$, we have

$$M_m \geqslant S_l,$$

hence

$$\Lambda_j(M_m) \geqslant \Lambda_j(S_l)$$

and

$$f^* + \Lambda_j(M_m) \geqslant f^* + \Lambda_j(S_l) = S_{l+1}. \tag{6.40}$$

Furthermore we note that $\Lambda_j(M_m) \geqslant 0$, and hence

$$f^* \geqslant S_1 - \Lambda_j(M_m). \tag{6.41}$$

Combining (6.40) and (6.41) gives

$$f^* \geqslant \max\,(S_1, \ldots, S_m) - \Lambda_j(M_m).$$

On the set C_m we have

$$\max (S_1,\dots,S_m) = M_m,$$

and hence

$$\int_{C_m} f^* \, d\tilde{\sigma}_n \geq \int_{C_m} M_m - \Lambda_j(M_m) \, d\tilde{\sigma}_n \geq \int M_m - \Lambda_j(M_m) \, d\tilde{\sigma}_n = 0. \qquad (6.42)$$

The last equality in (6.42) holds because Λ_j is a composition of conditional expectations. Having proved (6.39), we put

$$C_o = \{(\omega,t): \sup_{k \geq 1} S_k(\omega,t) > 0\} = \bigcup_{k \geq 1} C_k.$$

Since

$$C_o = \{(\omega,t): \sup_{k\geq 1} \frac{1}{k} \cdot S_k(\omega,t) > 0\}$$
$$= \{(\omega,t): \sup_{k\geq 1} \frac{1}{k} \cdot \sum_{l=1}^{k} \Lambda_j^{(l)}(f') > \varepsilon\} \cap B_\varepsilon ,$$

and since

$$\sup_k \frac{1}{k} \cdot \sum_{l=1}^{k} \Lambda_j^{(l)}(f') \geq \tilde{f},$$

we get

$$C_o = B_\varepsilon .$$

The boundedness of f implies the boundedness of f' and f . We can therefore use the dominated convergence theorem to get

$$\lim_k \int_{C_k} f^* \, d\tilde{\sigma}_n = \int_B f^* \, d\tilde{\sigma}_n,$$

and (6.39) shows that

$$\int_{B_\varepsilon} f^* \, d\tilde{\sigma}_n \geq 0.$$

We get

$$0 \leq \int_{B_\varepsilon} f^* \, d\tilde{\sigma}_n = \int_{B_\varepsilon} f' \, d\tilde{\sigma}_n - \varepsilon.\, \tilde{\sigma}_n(B_\varepsilon)$$
$$\int_{B_\varepsilon} E_{\tilde{\sigma}_n}(f' | \mathcal{D}^*(j,n)) \, d\tilde{\sigma}_n - \varepsilon.\, \tilde{\sigma}_n(B_\varepsilon)$$
$$= -\varepsilon.\, \tilde{\sigma}_n(B_\varepsilon),$$

by (6.38). $\tilde{\sigma}_n(B_\varepsilon)$ must thus be zero for every $\varepsilon > 0$, which proves that $\tilde{f} \leq 0$. The same argument, applied to $-f'$, gives

$$0 \geq \lim_m \sup \frac{1}{m}.\sum_{l=1}^{m} \Lambda_j^{(l)}(-f') = -\lim_m \inf \frac{1}{m}.\sum_{l=1}^{m} \Lambda_j^{(l)}(f'),$$

so that

$$f = \lim_m \sup \frac{1}{m}.\sum_{l=1}^{m} \Lambda_j^{(l)}(f') = \lim_m \inf \frac{1}{m}.\sum_{l=1}^{m} \Lambda_j^{(l)}(f') = 0.$$

The lemma is proved.

We choose a sequence $(\phi_k,\ k=1,2,\ldots)$ of continuous real valued functions on Ω which is dense in $C(\Omega)$ in the maximum norm. For every $n \geq 1$ and $(\omega,t) \in \Delta_n$, we put

$$\phi_k^{(n)}(\omega,t) = \phi_k(\omega).$$

If $k,j,n \geq 1$, define

$$h(k,j,n)\,(\omega) = \frac{1}{e^n - e^{-n}} \int_{-n}^{n} \tilde{\Lambda}_j(\phi_k^{(n)},n)\,(\omega,t)\,dt$$

and

$$h_k(\omega) = \lim_n \sup \lim_j \sup h(k,j,n)\,(\omega).$$

$h_k : \Omega \to \mathbb{R}$ is a bounded Borel map for every k. From Lemma 6.4, (6.32), and from the dominated convergence theorem we conclude that, for every $\sigma \in \tilde{P}$, and for every $k \geq 1$ and every j,n,

$$h(k,j,n) = E_\sigma(\phi_k / \mathcal{F}^*(j,n)) \quad \sigma\text{-a.e.}$$

The martingale theorem and (6.33) now imply for every $k \geq 1$ and for every $\sigma \in \tilde{P}$

$$h_k = E_\sigma(\phi_k / \mathcal{F}^*) \quad \sigma\text{-a.e.} \tag{6.43}$$

Note that h_k is defined independently of any measure!

<u>Lemma 6.5</u>. There exists a set $N_2 \in \mathcal{F}_1$ with $\rho(N_2) = 0$ such that, for every $z \in Z \setminus N_2$,

(1) $p_z \in \tilde{P}$,

(2) p_z is ergodic under T_G^*.

Proof: Since $\nu \in \tilde{P}$ by Lemma 6.2, we can apply (6.43) with $\sigma = \nu$. From the relation (6.9) between $\mathcal{F}^*$ and $\tilde{\mathcal{F}}$ it is clear that there exists a sequence $(\tilde{h}_k,\ k=1,2,\ldots)$ of bounded $\tilde{\mathcal{F}}$-measurable functions on Ω and a set $\tilde{N} \in \mathcal{F}$ with $\nu(\tilde{N}) = 0$ and with

$$h_k(\omega) = \tilde{h}_k(\omega)$$

for every $k \geq 1$ and for every $\omega \in \Omega \setminus \tilde{N}$. Let $N_2' = \{z \in Z: p_z(\tilde{N}) \neq 0\}$. (6.10) implies that $N_2' \in \mathcal{F}_1$ and $\rho(N_2') = 0$. (6.11) now shows that, for every $z \in Z \setminus N_2'$, and for every $k=1,2,\ldots$, h_k is equal to a constant function p_z-a.e. Let $N_2 = N_1 \cup N_2'$, where N_1 is given in Lemma 6.2, and fix $z \in Z \setminus N_2$. Then $p_z \in \tilde{P}$, and, applying (6.43) to p_z, we see that $E_{p_z}(\phi_k / \mathcal{F}^*)$ is p_z-a.e. equal to a constant function for every $k=1,2,\ldots$. Since $(\phi_k, k=1,2,\ldots)$ is dense in $C(\Omega)$ we conclude that $L^2(\Omega, \mathcal{F}^*, p_z)$ is one-dimensional. But this is the same as saying that p_z is ergodic for T_G^*. The proof is complete.

We can now state the main result of this section.

Theorem 6.6 (Existence of ergodic decomposition). Let T_G be an action of a countable group G on a probability space $(X, \mathcal{S}, \mu)$. Then there exists a standard Borel space $(Y, \mathcal{T})$, a surjective Borel map $\psi : X \to Y$, and a family $\{q_y : y \in Y\}$ of probability measures on $(X, \mathcal{S})$ with the following properties:

(1) $y \to q_y(B)$ is a Borel map on Y for every $B \in \mathcal{S}$. (6.44)

(2) $\mu = \int q_y \, d\rho(y)$, where $\rho = \mu\psi^{-1}$. (6.45)

(3) Every q_y, $y \in Y$, is quasi-invariant and ergodic under T_G. (6.46)

(4) If μ is invariant under T_G, then so is every q_y. (6.47)

(5) If μ is equivalent to a σ-finite T_G-invariant measure, then so is every q_y. (6.48)

(6) $q_y(\psi^{-1}(\{y\})) = 1$ for every $y \in Y$. In particular, q_y and $q_{y'}$ are mutually singular whenever $y \neq y'$. (6.49)

(7) Let $\mathcal{S}^*$ denote the σ-algebra of T_G-invariant sets in $\mathcal{S}$, and let $\psi^{-1}(\mathcal{T}) = \{\psi^{-1}(C): C \in \mathcal{T}\}$. Then $\psi^{-1}(\mathcal{T})$ is a subalgebra of $\mathcal{S}^*$ which is equal to $\mathcal{S}^*$ modulo sets of μ-measure zero. (6.50)

Proof: We define $(Z, \mathcal{F}_1)$, $(\Omega, \mathcal{T})$, $\phi: X \to \Omega$, T_G^*, ν, $\mathcal{F}^*$ and $\tilde{\mathcal{T}}$ as in (6.2) - (6.8). By Lemma 6.1 there exists a family $\{p_z: z \in Z\}$ of probability measures on $(\Omega, \mathcal{T})$ satisfying (6.10) - (6.12). Lemma 6.5 implies the existence of a set $N_2 \in \mathcal{F}_1$ with $\rho(N_2) = 0$ (ρ is given in Lemma 6.1) such that p_z is quasi-invariant and ergodic under T_G^* for every $z \in Z \setminus N_2$. Remark 6.3 shows that we may in addition assume that the measures p_z, $z \in Z$, are invariant under T_G^* (equivalent to σ-finite T_G^*-invariant measures) whenever ν is T_G^*-invariant (equivalent to a σ-finite T_G^*-invariant measure). Let $N_3 = \{z \in Z: p_z(\Omega') \neq 1\}$, where $\Omega' = \phi(X)$. (6.10) shows that $N_3 \in \mathcal{F}_1$, and (6.11) implies $\rho(N_3) = 0$. We put $Y = Z \setminus (N_2 \cup N_3)$ and $\mathcal{T} = \mathcal{F}_{1_Y}$ (i.e. the restriction of $\mathcal{F}_1$ to Y). The restriction of ρ to $(Y, \mathcal{T})$ will again be denoted by ρ. We now put, for every $y \in Y$, $B \in \mathcal{S}$,

$$q_y(B) = p_y(\phi(B)). \tag{6.51}$$

$\{q_y: y \in Y\}$ is a family of probability measures on $(X, \mathcal{S})$ which satisfies (6.44) - (6.48). To define $\psi: X \to Y$, let y_o be an arbitrary, but fixed, point in Y, and let

$$\psi(x) = \begin{cases} \pi_1 \cdot \phi(x) & \text{whenever } x \in (\pi_1 \cdot \phi)^{-1}(Y), \\ y_o & \text{otherwise.} \end{cases} \tag{6.52}$$

ψ is clearly surjective and Borel. (6.49) is clear from (6.12), and (6.50) follows from (6.9). The proof is complete.

<u>Theorem 6.7</u> (Uniqueness of the ergodic decomposition). Let T_G be an action of a countable group G on a probability space $(X, \mathcal{S}, \mu)$ and let $\mathcal{S}^* = \{B \in \mathcal{S} : T_G B = B\}$. Let furthermore $(Y_i, \mathcal{T}_i, \rho_i)$, i=1,2, be two probability spaces and $\{q_y^{(i)} : y \in Y_i\}$, i=1,2, two families of probability measures on $(X, \mathcal{S})$ which satisfy the following conditions for i=1,2:

(1) For every $B \in \mathcal{S}$, the map $y \to q_y^{(i)}(B)$ is Borel on $(Y_i, \mathcal{T}_i)$. (6.53)

(2) $$\mu = \int_{Y_i} q_y^{(i)} \, d\rho_i(y). \tag{6.54}$$

(3) $q_y^{(i)}$ is quasi-invariant and ergodic under T_G for every $y \in Y_i$. (6.55)

(4) For every $C \in \mathcal{S}^*$, let

$$C_{Y_i} = \{y \in Y_i : q_y^{(i)}(C) = 1\}.$$

Then $\mathcal{S}^*_{Y_i} = \{C_{Y_i} : C \in \mathcal{S}^*\}$ is a sub-σ-algebra of $\mathcal{T}_i$, which is equal to $\mathcal{T}_i$ modulo sets of ρ_i-measure zero. (6.56)

Then there exists a measure preserving isomorphism

$$\Theta : (Y_1, \mathcal{T}_1, \rho_1) \to (Y_2, \mathcal{T}_2, \rho_2) \tag{6.57}$$

with

$$q^{(2)}_{\Theta(y)} = q_y^{(1)} \tag{6.58}$$

for ρ_1-a.e. $y \in Y_1$.

Proof: From the ergodicity of every $q_y^{(i)}$ it is clear that, for every $C \in \mathcal{S}^*$, i=1,2, $q_y^{(i)}(C)$ is either equal to zero or to one. We conclude that, for $C_1, C_2 \in \mathcal{S}^*$,

$$(C_1 \cup C_2)_{Y_i} = C_{1_{Y_i}} \cup C_{2_{Y_i}}$$

and

$$(C_1 \cap C_2)_{Y_i} = C_{1_{Y_i}} \cap C_{2_{Y_i}} .$$

Moreover we have, for every $C \in \mathcal{S}^*$, $B \in \mathcal{S}$, i=1,2,

$$\mu(B \cap C) = \int_{C_{Y_i}} q_y^{(i)}(B)\, d\rho_i(y). \tag{6.59}$$

A standard measure theoretic argument due to von Neumann [46] shows the existence of an isomorphism $\Theta: (Y_1, \mathcal{T}_1, \rho_1) \to (Y_2, \mathcal{T}_2, \rho_2)$ with

$$\rho_2(\Theta(C_{Y_1}) \Delta C_{Y_2}) = 0 \tag{6.60}$$

for every $C \in \mathcal{S}^*$ (cf. e.g. [10], p. 325). (6.59) and (6.60) now imply that $\rho_1(C_{Y_1}) = \rho_2(C_{Y_2}) = \rho_2(\Theta(C_{Y_1}))$ for every $C \in \mathcal{S}^*$, and (6.56) shows that Θ is measure preserving. Since $\mathcal{S}$ is countably generated, one can conclude (6.56) from (6.59). The proof is complete.

<u>Remark 6.8.</u> The space $(Y, \mathcal{T}, \rho)$ and the family $\{q_y : y \in Y\}$ arising in Theorem 6.6 satisfy (6.53) - (6.56).

<u>Corollary 6.9</u>. Let T_G be an action of a countable group G on a measure space $(X, \mathcal{S}, \mu)$. Then there exists a measure space $(Y, \mathcal{T}, \rho)$ and a family $\{q_y : y \in Y\}$ of σ-finite measures on $(X, \mathcal{S})$ such that

(1) $y \to q_y(B)$ is a Borel map from Y to $\overline{\mathbb{R}}$ for every $B \in \mathcal{S}$, (6.61)

(2) $\mu(B) = \int q_y(B)\, d\rho(y)$ for every $B \in \mathcal{S}$, (6.62)

(3) every q_y is quasi-invariant and ergodic under T_G, (6.63)

(4) if μ is invariant under T_G, then so is every q_y, (6.64)

(5) q_y and $q_{y'}$ are mutually singular whenever $y \neq y'$. (6.65)

(6) For every $C \in \mathcal{S}^* = \{B \in \mathcal{S} : T_G B = B\}$, let $C_Y = \{y \in Y : q_y(C) \neq 0\}$. Then $\mathcal{S}_Y^* = \{C_Y : C \in \mathcal{S}^*\}$ is a subalgebra of $\mathcal{T}$ which is equal to $\mathcal{T}$ modulo sets of ρ-measure zero. (6.66)

(7) If $(Y', \mathcal{T}', \rho')$ is another measure space and $\{q'_y : y \in Y'\}$ a family of σ-finite measures on $(X, \mathcal{S})$ satisfying (1) - (6), then there exists an isomorphism

$$\Theta : (Y, \mathcal{T}, \rho) \to (Y', \mathcal{T}', \rho') \tag{6.67}$$

with

$$q'_{\Theta(y)} \sim q_y \tag{6.68}$$

for ρ-a.e. $y \in Y$.

Proof: Left as an exercise.

We conclude this section with an analysis of the ergodic decomposition for conservative and for type I actions. For the following lemmas we assume X to be uncountable and identify it with the closed unit interval $I = [0,1]$ with the usual Borel structure.

Lemma 6.10. Under the assumptions and in the notation of Theorem 6.6, the set

$$D = \{ y \in Y : q_y \text{ is atomic} \} \tag{6.69}$$

lies in $\mathcal{T}$.

Proof: Put $B_{n,k} = [k.2^{-n},(k+1).2^{-n})$ for $n=1,2,\ldots$, $k=0,\ldots,2^n-2$, and $B_{n,2^n-1} = [(2^n-1).2^{-n},1]$. If ν is a nonatomic probability measure on $X = I$, we can find, for every $\varepsilon > 0$ and for every $x \in X$, a neighbourhood of the form $(x-2^{-n_0(x)},x+2^{-n_0(x)})$ with $\nu((x-2^{-n_0(x)},x+2^{-n_0(x)})) < \varepsilon/2$. The compactness of X shows that we can choose finitely many $x_1,\ldots,x_k$ with $X \subset \bigcup_{i=1}^{k}(x_i-2^{-n_0(x_i)},x_i+2^{-n_0(x_i)})$. Put $n(\varepsilon) = \max_{i=1,\ldots,k} n_o(x_i)+1$. Then

$$\max_{0 \leq k \leq 2^{n(\varepsilon)}-1} \nu(B_{n(\varepsilon),k}) < \varepsilon.$$

We have shown that

$$a(\nu) = \lim_n \max_{0 \leq k \leq 2^n-1} \nu(B_{n,k}) = 0 \tag{6.70}$$

for every nonatomic measure ν. If ν is atomic, we obviously have

$$a(\nu) = \lim_n \max_{0 \leq k \leq 2^n-1} \nu(B_{n,k}) > 0. \tag{6.71}$$

(6.71) shows that

$$D = \{y: \lim_n \max_{0 \leq k \leq 2^n-1} q_y(B_{n,k}) > 0\}.$$

Since $y \to q_y(B_{n,k})$ is Borel for every n,k, we have proved the lemma.

Lemma 6.11. There exists a Borel map $\sigma: D \to X$ such that

$$q_y(\{\sigma(y)\}) > 0 \text{ for every } y \in D. \tag{6.72}$$

Proof: Let $\bar{B}_{n,k}$ denote the closure of the interval $B_{n,k}$ in the proof of Lemma 6.10. For every $y \in D$, we define $a(q_y)$ by (6.70) or (6.71). Put, for $n \geq 1$, $y \in D$,

$$k_n(y) = \min\{k: q_y(\bar{B}_{n,k}) \geq a(q_y)\}$$

and

$$\sigma_n(y) = k_n(y).2^{-n}.$$

$\sigma_n: D \to X$ is a Borel map for every n, by (6.45). It is easy to see that $(\sigma_n(y), n=1,2,\ldots)$ is a Cauchy sequence for every $y \in D$, and that

$$\lim_n \sigma_n(y) = \min\{x \in X: p_y(\{x\}) = a(q_y)\}.$$

Hence $\sigma = \lim \sigma_n$ is a Borel map and satisfies (6.68).

Lemma 6.12. Suppose $\rho(D) > 0$, and put, for every $B \in \mathcal{S}$,

$$\mu^D(B) = \int_D q_y(B)\, d\rho(y).$$

Then T_G is a type I action on $(X, \mathcal{S}, \mu^D)$.

Proof: Put $E = \{\sigma(y) : y \in D\}$, where σ has been constructed in Lemma 6.11. From (6.49) we see that, for every $x \in E$, and for every $g \in G$ with $T_g x \in E$, we must have $T_g x = x$. On the other hand,

$$T_G E = \bigcup_{y \in D} T_G \sigma(y) = \bigcup_{y \in D} \{x \in X : q_y(\{x\}) > 0\},$$

which means that

$$\mu^D(X \setminus T_G E) = 0.$$

We have proved the lemma.

Lemma 6.13. Put $C = Y \setminus D$, and let $C_1 \subset C$ be a Borel set with $\rho(C_1) > 0$. Put, for every $B \in \mathcal{S}$,

$$\mu^{C_1}(B) = \int_{C_1} q_y(B)\, d\rho(y).$$

Then T_G is a conservative action on $(X, \mathcal{S}, \mu^{C_1})$.

Proof: Since q_y is nonatomic for every $y \in C$, and since T_G is ergodic on $(X, \mathcal{S}, q_y)$, T_G is conservative on $(X, \mathcal{S}, q_y)$. For every $B \in \mathcal{S}$ with $\mu^{C_1}(B) > 0$, and for every $g \in G$, put

$$F(g) = \{y \in C_1 : q_y(B \cap T_g^{-1} B \cap \{x : T_g x \neq x\}) > 0\}.$$

Since T_G is conservative on every $(X, \mathcal{S}, q_y)$, $y \in C$, we have

$$\bigcup_{g \in G} F(g) = \{y \in C_1 : q_y(B) > 0\}.$$

From (6.45) we get $\rho(\bigcup_{g \in G} F(g)) > 0$ and hence $\rho(F(g_0)) > 0$ for some $g_0 \in G$. Applying again (6.45), we get

$$\mu^{C_1}(B \cap T_{g_0}^{-1}B \cap \{x : T_{g_0}^{-1}x \neq x\}) =$$

$$\int_{F(g_0)} q_y(B \cap T_{g_0}^{-1}B \cap \{x : T_{g_0} x \neq x\})\, d\rho(y) > 0.$$

We have proved that T_G is conservative on $(X, \mathcal{S}, \mu^{C_1})$.

Combining Lemma 6.12 with Lemma 6.13 we get

<u>Theorem 6.14.</u> Let T_G be an action of a countable group G on a measure space $(X, \mathcal{S}, \mu)$, and let $(Y, \mathcal{T}, \rho)$ be a measure space and $\{q_y : y \in Y\}$ a family of σ-finite measures on $(X, \mathcal{S})$ satisfying (6.61) - (6.66). Then the following is true:

(1) T_G is type I if and only if q_y is purely atomic for ρ-a.e. $y \in Y$,

(2) T_G is conservative if and only if q_y is nonatomic for ρ-a.e. $y \in Y$.

Proof: The statement of the theorem will not be affected if we replace μ by an equivalent probability measure and if we assume $(Y, \mathcal{T}, \rho)$ and $\{q_y : y \in Y\}$ to satisfy (6.44) - (6.50). Lemma 6.12 shows that T_G is type I whenever ρ-a.e.

q_y is atomic, and Lemma 6.13 implies that T_G is conservative if ρ-a.e. q_y is nonatomic.

Suppose now that T_G is type I, but that $\rho\{y \in Y: q_y \text{ is nonatomic}\}$ is nonzero. Lemma 6.13 then shows that we can write μ as $\mu_1 + \mu_2$ with μ_1 and μ_2 mutually singular, nonzero, and quasi-invariant under T_G, and such that T_G is conservative on $(X, \mathcal{S}, \mu_1)$. But this implies that the restriction of the type I action T_G to some T_G-invariant set of positive measure is conservative, which is impossible. This contradiction proves (1) completely, and (2) is proved similarly.

<u>Exercise 6.15</u>. Let T_G be an action of a countable group G on a measure space $(X, \mathcal{S}, \mu)$, and let μ' be a measure on $(X, \mathcal{S})$ with $\mu' \sim \mu$. Suppose $(Y, \mathcal{T}, \rho)$ is a measure space and $\{q_y : y \in Y\}$ a family of σ-finite measures on $(X, \mathcal{S})$ which satisfy (6.61) - (6.66). Suppose furthermore that $(Y', \mathcal{T}', \rho')$ is a measure space and $\{q'_y : y \in Y'\}$ a family of σ-finite measures on $(X, \mathcal{S})$ which fulfil (6.61) - (6.66) with μ' replacing μ. Show that there exists an isomorphism $\theta : (Y, \mathcal{T}, \rho) \to (Y', \mathcal{T}', \rho')$ such that

$$q'_{\theta(y)} \sim q_y \tag{6.73}$$

for ρ-a.e. $y \in Y$.

§ 7 Skew products with nonregular cocycles

Let T_G be an ergodic action of a countable group G on a nonatomic measure space $(X, \mathcal{S}, \mu)$, and let A be a locally compact second countable abelian group with Borel field $\mathcal{B}$ and Haar measure λ. In this section we shall analyze skew products T_G^a, where $a:G \times X \to A$ is a cocycle with $\bar{E}(a) = \{0, \infty\}$.

Definition 7.1. A cocycle $a:G \times X \to A$ for T_G is called lacunary if there exists a neighbourhood N(0) of 0 in A with

$$\bigcup_{g \in G} \{x: a(g,x) \in N(0) \setminus \{0\}\} = \emptyset. \tag{7.1}$$

An orbital cocycle $u:R(T_G) \to A$ will be called lacunary if a_u is lacunary (cf. (2.18)).

Proposition 7.2. Let $a:G \times X \to A$ be a cocycle for T_G. Then a is cohomologous to a lacunary cocycle if and only if there exists a neighbourhood $N_1(0)$ of 0 in A with

$$N_1(0) \cap E(a) = \{0\}. \tag{7.2}$$

Proof: Without loss in generality we may assume that $N_1(0)$ has compact closure. Let δ be an invariant metric on A. For every $\varepsilon > 0$, we write $B(0, \varepsilon) = \{\alpha \in A: \delta(\alpha, 0) < \varepsilon\}$ and $\bar{B}(0, \varepsilon) = \{\alpha \in A: \delta(\alpha, 0) \leq \varepsilon\}$. We now fix $\varepsilon > 0$ with

$\bar{B}(0,5\varepsilon) \subset N_1(0)$. Put $K = \bar{B}(0,5\varepsilon) \setminus B(0,\varepsilon)$. K is compact, $K \cap E(a) = \emptyset$, and we can apply Proposition 3.8 to get a coboundary $b_0 : G \times X \to A$ for T_G with

$$\mu\left(\bigcup_{g \in G} \{x : a(g,x) + b_0(g,x) \in K\}\right) = 0.$$

By Proposition 2.3 we can find an orbital cocycle $u_0 : R(T_G) \to A$ with

$$u_0(T_g x, x) = a(g,x) + b_0(g,x) \tag{7.3}$$

for every $g \in G$ and for μ-a.e. $x \in X$, and such that

$$u_0(x,y) \notin K \tag{7.4}$$

for all $(x,y) \in R(T_G)$. Let

$$Q = \{(x,y) \in R(T_G) : u_0(x,y) \in B(0,\varepsilon)\}. \tag{7.5}$$

$Q \subset R(T_G)$ is a Borel set. (7.4) implies that Q is also an equivalence relation, so that we can apply Exercise 1.4 to show the existence of a countable group $H_0 = H(T_G, Q) \subset [T_G]$ with $R(H_0) = Q$ (cf. (1.21)). We restrict u_0 to Q and denote this restriction by $v_0 : Q \to A$. v_0 is an orbital cocycle for H_0, and we want to prove that v_0 is in fact a coboundary. One can proceed in two steps:

(a) Let $0 < \eta \leq \varepsilon$, and let $f : X \to A$ be a Borel map with $v_0(x,y) + f(x) - f(y) \in B(0,\eta)$ for μ-a.e. $(x,y) \in Q$. Then there exists a Borel map $c : X \to A$ with $c(x) \in B(0,\eta)$ for μ-a.e. $x \in X$, and with $v_0(x,y) + f(x) - f(y) + c(x) - c(y) \in B(0, \eta/2)$ μ-a.e. on Q.

Proof of (a): Put $u_1(x,y) = u_o(x,y)+f(x)-f(y)$ for every $(x,y) \in R(T_G)$ and $a_1(g,x) = u_1(T_g x,x)$ for $x \in X$, $g \in G$. a_1 is cohomologous to a, and the set $K_1 = \bar{B}(0,\eta) \setminus B(0,\eta/2)$ is compact and satisfies $K_1 \cap E(a_1) = \emptyset$. Proposition 3.8 shows that the following is true: For every $C \in \mathcal{S}$ with $\mu(C) > 0$ there exists a Borel set $B \subset C$ with $\mu(B) > 0$ and with $a_1(g,x) \notin K_1$ for all $g \in G$, $x \in B \cap T_g^{-1}B$. In particular we see that $u_1(x,y) \notin K_1$ for $(x,y) \in Q \cap B \times B = R_B(H_o)$. Applying now Lemma 3.7 with $W_1 = \bar{B}(0,\eta)$ and $W_2 = A \setminus B(0,\eta/2)$ to the orbital cocycle $v_1 : Q \rightarrow A$ for H_o given by $v_1(x,y) = u_1(x,y)$, $(x,y) \in Q$, we get a Borel map $c : X \rightarrow A$ satisfying the required conditions.

(b) v_o is a coboundary for H_o.

Proof of (b): We apply (a) to $\eta = \varepsilon$ and f=0 to find a Borel map $f_1 = c : X \rightarrow A$ with $c(x) \in B(0,\varepsilon)$ and $v_o(x,y)+c(x) -c(y) \in B(0,\varepsilon/2)$ μ-a.e. on Q. Proceeding by induction, we assume to have found Borel maps $f_1,\dots,f_n$ from X to A with $\delta(f_i(x),0) < \varepsilon . 2^{-i+1}$, $i=1,\dots,n$, and with $v_o(x,y) + \sum_{k=1}^{n} f_k(x) - \sum_{k=1}^{n} f_k(y) \in B(0,\varepsilon . 2^{-n})$ μ-a.e. on Q. Put $f = \sum_{k=1}^{n} f_k$, $\eta = \varepsilon . 2^{-n}$, and apply (a) to find $c : X \rightarrow A$

with $c(x) \in B(0,\varepsilon . 2^{-n})$ and $v_o(x,y)+f(x)-f(y)+c(x)-c(y) \in B(0,\varepsilon . 2^{-n-1})$ μ-a.e. on Q. This process gives a sequence $(f_k,\ k=1,2,\dots)$ of Borel maps from X to A. We put

$$f(x) = \begin{cases} \sum_{k=1}^{\infty} f_k(x) & \text{if the series converges} \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\sum_{k=1}^{\infty} f_k(x)$ converges for μ-a.e. $x \in X$, so that

$$v_o(x,y)+f(x)-f(y) = 0 \tag{7.6}$$

for μ-a.e. $(x,y) \in Q$. v_o is thus a coboundary for H_o.

We can now complete the proof of Proposition 7.2. Our definition of f implies that

$$f(x) \in B(0,2\varepsilon) \tag{7.7}$$

for μ-a.e. $x \in X$. We choose a T_G-invariant (and hence H_o-invariant) Borel set N of measure zero such that

$$v_o(x,y)+f(x)-f(y) = 0 \tag{7.8}$$

and

$$f(x) \in B(0,2\varepsilon) \tag{7.9}$$

for all (x,y) in $Q \setminus N \times N$, and put

$$u'(x,y) = \begin{cases} u_o(x,y)+f(x)-f(y) & \text{for } (x,y) \in R_{X \setminus N}(T_G), \\ 0 & \text{otherwise.} \end{cases}$$

If $u'(x,y) \in B(0,\varepsilon)$ for some $(x,y) \in R(T_G)$, we have either $(x,y) \in N \times N$, or we see from (7.9) that $u_o(x,y) \in B(0,5\varepsilon)$. (7.4) now implies $(x,y) \in Q$, and (7.8) gives $u'(x,y) = 0$.

Moreover, $u' \sim u_o$. Put $a_2(g,x) = u'(T_g x,x)$ for $x \in X$, $g \in G$, and apply Proposition 2.3 to show that $a_2 : G \times X \to A$ is a cocycle for T_G which is cohomologous to a. Since $u'(x,y) \notin B(0,\varepsilon) \setminus \{0\}$ for all $(x,y) \in R(T_G)$, we get

$$\bigcup_{g \in G} \{x : a_2(g,x) \in B(0,\varepsilon) \setminus \{0\}\} = \emptyset,$$

so that a_2 is lacunary. The proof is complete.

For the rest of this section we fix an orbital cocycle $u_o : R(T_G) \to A$ and a neighbourhood $N_o(0)$ in A such that

$$\bar{E}(u_o) = \{0, \infty\} \tag{7.10}$$

and

$$u_o(x,y) \notin N_o(0) \setminus \{0\} \quad \text{for all } (x,y) \in R(T_G). \tag{7.11}$$

Put

$$Q_o = \{(x,y) \in R(T_G) : u_o(x,y) = 0\}. \tag{7.12}$$

By Exercise 1.4 we can find a countable group

$$H_o = H(T_G, Q_o) \subset [T_G] \tag{7.13}$$

with

$$R(H_o) = Q_o. \tag{7.14}$$

We write

$$\mathcal{S}^* = \{B \in \mathcal{S} : H_o B = B\}. \tag{7.15}$$

Choose and fix a probability measure μ' on $(X, \mathcal{S})$ with

$$\mu' \sim \mu \tag{7.16}$$

and apply Theorem 6.6 to find a standard Borel space $(Y, \mathcal{T})$, a surjective Borel map $\psi : X \to Y$, and a family $\{q_y : y \in Y\}$ of probability measures on $(X, \mathcal{S})$ satisfying

(1) each q_y is quasi-invariant and ergodic under H_o, (7.17)

(2) for every $B \in \mathcal{S}$, the map $y \to q_y(B)$ is Borel on $(Y, \mathcal{T})$, and $\mu'(B) = \int q_y(B)\, d\rho(y)$, where $\rho = \mu' \psi^{-1}$, (7.18)

(3) $q_y(\psi^{-1}(\{y\})) = 1$ for every $y \in Y$, (7.19)

(4) $\psi^{-1}(\mathcal{T}) = \{\psi^{-1}(C) : C \in \mathcal{T}\}$ is contained in $\mathcal{S}^*$, and is equal to $\mathcal{S}^*$ modulo sets of μ'-measure zero, (7.20)

(5) if μ' is invariant under H_o (equivalent to a σ-finite H_o-invariant measure) then every q_y is invariant under H_o (equivalent to a σ-finite H_o-invariant measure). (7.21)

Finally we put, for every $g \in G$, $x \in X$,

$$a_o(g,x) = u_o(T_g x, x) \tag{7.22}$$

to obtain a lcunary cocycle $a_o : G \times X \to A$ for T_G with $\bar{E}(a_o) = \{0, \infty\}$.

The aim of the following analysis is a description of the skew product $T_G^{a_o}$. In particular we want to investigate the ergodic decomposition of $\tilde{\mu} = \mu \times \lambda$ with respect to $T_G^{a_o}$. Since this decomposition will turn out to be in terms of $(Y, \mathcal{T}, \rho)$, ψ, and $\{q_y : y \in Y\}$, we shall start by discussing the ergodic decomposition of μ' with respect to H_0 in some more detail.

<u>Lemma 7.3</u>. ρ is nonatomic.

Proof: Suppose there exists a $y_0 \in Y$ with $\rho(\{y_0\}) > 0$. By (7.20), $B_0 = \psi^{-1}(\{y_0\})$ is H_0-invariant, and every H_0-invariant Borel set $C \subset B_0$ satisfies $\mu'(B_0 \setminus C).\mu'(C) = 0$. Moreover, $\mu'(B_0) = \rho(\{y_0\}) \neq 0$, by (7.18). It follows that the restriction of H_0 to $(B_0, \mathcal{S}_{B_0}, \mu'_{B_0})$ is ergodic. Since a_0 is not a coboundary and hence not everywhere zero on B_0, we can find a set $C \in \mathcal{S}$ with $\mu'(C) > 0$ and an element $g_0 \in G$ such that $C \cup T_{g_0} C \subset B_0$ and $a_0(g_0, x) \neq 0$ for all $x \in C$. Hence there exists an $\alpha_0 \neq 0$ in A with $\mu'\{x \in C: a_0(g_0,x) \in M(\alpha_0)\} > 0$ for every neighbourhood $M(\alpha_0)$ of α_0 in A. We fix such a neighbourhood $M(\alpha_0)$ and a Borel set $D \subset B_0$ with $\mu'(D) > 0$. Put $C_0 = \{x \in C: a_0(g_0,x) \in M(\alpha_0)\}$. As the restriction of H_0 to B_0 is ergodic, we can find $V_1 \in H_0$ with $\mu'(V_1 D \cap C_0) > 0$. Since $V_1 \in [T_G]$, we can make use of Exercise 1.1 to select a Borel set $D_1 \subset D$ with $\mu'(D_1) > 0$ and a $g_1 \in G$ with

$$V_1 x = T_{g_1} x \quad \text{for every } x \in D_1$$

and

$$T_{g_1} D_1 \subset C_0.$$

Put $E = T_{g_0 g_1} D_1$ and apply the same argument to find $g_2 \in G$

and a Borel set $E_1 \subset E$ of positive measure such that

$$T_{g_2} x \in H_o x \quad \text{for every } x \in E_1,$$

and

$$T_{g_2} E_1 \subset D.$$

Finally, let

$$D_2 = T^{-1}_{g_o g_1} E_1.$$

Then $\mu'(D_2) > 0$, $D_2 \cup T_{g_2 g_o g_1} D_2 \subset D$, and

$$a_o(g_2 g_o g_1, x) = a_o(g_2, T_{g_o g_1} x) + a_o(g_o, T_{g_1} x) + a_o(g_1, x)$$
$$= a_o(g_o, T_{g_1} x) \in M(\alpha_o),$$

for every $x \in D_2$. We have proved that every subset $D \subset B_o$ of positive measure and every neighbourhood $M(\alpha_o)$ of α_o in A satisfy

$$\mu'\Big(\bigcup_{g \in G} (D \cap T_g^{-1} D \cap \{x : a_o(g,x) \in M(\alpha_o)\}\Big) > 0.$$

From Proposition 3.8 it is now clear that $\alpha_o \in E(a_o)$, contrary to our assumption $E(a_o) = \{0\}$. This contradiction proves that ρ is nonatomic.

Lemma 7.4. H_o is conservative whenever a_o is recurrent, and type I whenever a_o is transient.

Proof: Suppose a_o is recurrent. We take $B \in \mathcal{S}$ with $\mu'(B) > 0$ and choose a symmetric neighbourhood $N_1(0)$ in A with

$N_1(0)+N_1(0) \subset N_o(0)$, where $N_o(0)$ is given in (7.11). Put $\tilde{B} = B \times N_1(0)$. Since $T_G^{a_o}$ is conservative on $(\tilde{X}, \tilde{S}, \tilde{\mu}')$ by Theorem 5.5, we can find a $g_o \in G$ with

$$\tilde{\mu}'(\tilde{B} \cap T^{a_o}_{g_o^{-1}}\tilde{B} \cap \{(x,\alpha): T^{a_o}_{g_o}(x,\alpha) \neq (x,\alpha)\}) > 0.$$

But for every $(x,\alpha) \in \tilde{B} \cap T^{a_o}_{g_o^{-1}}\tilde{B}$, we must have

$$a_o(g_o,x) \in N_1(0)+N_1(0) \subset N_o(0),$$

so that

$$a_o(g_o,x) = u_o(T_{g_o}x,x) = 0.$$

If, in addition,

$$(x,\alpha) \neq T^{a_o}_{g_o}(x,\alpha) = (T_{g_o}x, \alpha+a_o(g_o,x)) = (T_{g_o}x, \alpha),$$

we get

$$T_{g_o}x \neq x.$$

Together this implies

$$\mu'(B \cap T^{-1}_{g_o}B \cap \{x: T_{g_o}x \neq x\} \cap \{x: a_o(g_o,x)=0\}) > 0.$$

This in turn shows that

$$\mu'(B \cap V^{-1}B \cap \{x: Vx \neq x\}) > 0$$

for some $V \in H_o$. Since B was arbitrary, H_o is conservative. To prove the converse, assume a_o to be transient. Combining (3.28) with (7.11), we see that the following is true: For every $C \in \mathcal{S}$ with $\mu'(C) > 0$ there exists a $C_1 \subset C$ with $\mu'(C_1) > 0$ and with

$$\bigcup_{g\in G}(C_1 \cap T_g^{-1}C_1 \cap \{x: a_o(g,x)=0\} \cap \{x: T_g x \neq x\}) = \emptyset . \qquad (7.23)$$

We first put C=X to find C_1 satisfying (7.23), and set $D_1=C_1$. Suppose we have constructed Borel sets $D_1,\ldots,D_n$ with $\mu'(D_i) > 0$ and $H_oD_i \cap H_oD_j = \emptyset$ for all $i\neq j$, such that

$$\bigcup_{g\in G}(D_i \cap T_g^{-1}D_i \cap \{x: a_o(g,x)=0\} \cap \{x: T_g x\neq x\}) = \emptyset \qquad (7.24)$$

for i=1,..,n. We put $C = X \setminus \bigcup_{i=1}^{n} H_oD_i$. If $\mu'(C)>0$, we can again find C_1 satisfying (7.23) with $\mu'(C_1)>0$, and we put $D_{n+1}=C_1$. In this fashion one constructs a sequence $(D_k,\ k=1,2,\ldots)$ of Borel sets of positive measure (which may terminate), each of which satisfies (7.24), such that $H_oD_i \cap H_oD_j = \emptyset$ for $i\neq j$. An exhaustion argument shows that we may assume $\mu'(X \setminus \bigcup_{i\geq 1} H_oD_i) = 0$. Put $D_o = \bigcup_{i\geq 1} D_i$. Then it is clear that

$$\bigcup_{V\in H_o}(D_o \cap V^{-1}D_o \cap \{x: Vx\neq x\}) = \emptyset$$

and

$$\mu'(X \setminus H_oD_o) = 0.$$

We have proved that H_o is type I.

<u>Corollary 7.5</u>. If a_o is recurrent, ρ-a.e.q_y is nonatomic. If a_o is transient, q_y is purely atomic for ρ-a.e.$y\in Y$.

Proof: Theorem 6.14 and Lemma 7.4.

We define a Borel map $i: X \to \tilde{X}$ $(=X\times A)$ by setting

$$i(x) = (x,0),\ x\in X, \qquad (7.25)$$

and put

$$\sigma_y(B) = q_y(i^{-1}(B)) \tag{7.26}$$

for every $y \in Y$ and $B \in \tilde{\mathcal{S}}$. Choose a probability measure λ' on $(A, \mathcal{B})$ with

$$\lambda' \sim \lambda \tag{7.27}$$

and write

$$\tilde{\tilde{\mu}} = \mu' \times \lambda'. \tag{7.28}$$

$\tilde{\tilde{\mu}}$ is a probability measure on $(\tilde{X}, \tilde{\mathcal{S}})$, and

$$\tilde{\tilde{\mu}} \sim \tilde{\mu} = \mu \times \lambda. \tag{7.29}$$

For every $V \in H_o$ we define an automorphism $\tilde{V}$ of $(\tilde{X}, \tilde{\mathcal{S}}, \tilde{\tilde{\mu}})$ by

$$\tilde{V}(x, \alpha) = (Vx, \alpha), \quad (x, \alpha) \in \tilde{X}. \tag{7.30}$$

Let

$$\tilde{H}_o = \{\tilde{V} : V \in H_o\} . \tag{7.31}$$

An explicit ergodic decomposition of $\tilde{\tilde{\mu}}$ with respect to H_o can be written down as follows: Put

$$\begin{aligned} \tilde{Y} &= Y \times A, \\ \tilde{\mathcal{T}} &= \mathcal{T} \times \mathcal{B}, \\ \tilde{\rho} &= \rho \times \lambda', \end{aligned} \tag{7.32}$$

and define $\tilde{\psi} : \tilde{X} \to \tilde{Y}$ by

$$\tilde{\psi}(x, \alpha) = (\psi(x), \alpha). \tag{7.33}$$

If

$$\tilde{\mathcal{S}}^* = \{B \in \tilde{\mathcal{S}} : \tilde{H}_o B = B\}, \tag{7.34}$$

and

$$\sigma_{(y,\alpha)} = \sigma_y R^{-1}, \ (y,\alpha) \in \tilde{Y}, \tag{7.35}$$

then the map $\tilde{\psi} : \tilde{X} \to \tilde{Y}$, the probability space $(\tilde{Y}, \tilde{\mathcal{T}}, \tilde{\rho})$, and the family $\{\sigma_{(y,\alpha)} : (y,\alpha) \in \tilde{Y}\}$ satisfy all the conditions (6.44) - (6.50) in Theorem 6.6. In particular, each $\sigma_{(y,\alpha)}$ is quasi-invariant and ergodic under H_o,

$$\tilde{\tilde{\mu}} = \int_{\tilde{Y}} \sigma_{(y,\alpha)} \, d\tilde{\rho}(y,\alpha), \tag{7.36}$$

and

$\sigma_{(y,\alpha)}$ is singular to $\sigma_{(y',\alpha')}$ whenever $(y,\alpha) \neq (y',\alpha')$. (7.37)

All this is an immediate consequence of (7.17) - (7.20) and (7.35). Turning now to the action $T_G^{a_o}$ on $(\tilde{X}, \tilde{\mathcal{S}}, \tilde{\tilde{\mu}})$, we choose and fix an enumeration

$$(g_1 = e, g_2, g_3, \ldots) \tag{7.38}$$

of G. For every $y \in Y$, $B \in \tilde{\mathcal{S}}$, let

$$\tau_y(B) = \sum_{k=1}^{\infty} 2^{-k} \sigma_y(T_{g_k}^{a_o} B) \tag{7.39}$$

and put, for $(y,\alpha) \in \tilde{Y}$,

$$\tau_{(y,\alpha)} = \tau_y R_\alpha^{-1}. \tag{7.40}$$

Each $\tau_{(y,\alpha)}$ is a probability measure on $(\tilde{X}, \tilde{\mathcal{S}})$.

Lemma 7.6. Let $N_1(0)$ be a fixed symmetric neighbourhood of 0 in A with

$$N_1(0)+N_1(0) \subset N_0(0) \tag{7.41}$$

(cf. (7.11)), and let $(y,\alpha)\in Y\times N_1(0)$. Then the measure

$$B \longrightarrow \tau_{(y,\alpha)}(B\cap(X\times N_1(0))),\ B\in\tilde{\mathcal{S}},$$

is equivalent to $\sigma_{(y,\alpha)}$.

Proof: This is obvious from (7.11) and from the quasi-invariance of $\sigma_{(y,\alpha)}$ under $\tilde{H}_0$.

Lemma 7.7. For every $(y,\alpha)\in\tilde{Y}$, $\tau_{(y,\alpha)}$ is quasi-invariant and ergodic under $T_G^{a_0}$. Moreover, $\tau_{(y,\alpha)}$ is quasi-invariant under $\tilde{H}_0$.

Proof: The quasi-invariance of $\tau_{(y,\alpha)}$ under $T_G^{a_0}$ is obvious from (7.39) and (7.40). We now fix $(\tilde{X},\tilde{\mathcal{S}},\tau_{(y,\alpha)})$ as our measure space. Since $(\tilde{V}(x,\alpha),(x,\alpha))\in R(T_G^{a_0})$ for every $\tilde{V}\in\tilde{H}_0$ and for every $(x,\alpha)\in\tilde{X}$, Exercise 1.3 shows that $\tilde{H}_0\subset[T_G^{a_0}]$ and in particular that $\tau_{(y,\alpha)}$ is quasi-invariant under $\tilde{H}_0$. Since R_α commutes with $T_g^{a_0}$ for every α, g, it will be enough to prove that $\tau_{(y,0)}=\tau_y$ is ergodic under $T_G^{a_0}$ for every $y\in Y$, and this will imply the ergodicity of every $\tau_{(y,\alpha)}$. On the measure space $(\tilde{X},\tilde{\mathcal{S}},\tau_y)$, consider the restriction $\tilde{H}_0^C$ of $\tilde{H}_0$ to the $\tilde{H}_0$-invariant set $C=X\times N_1(0)$. (7.11) shows that

$$[\tilde{H}_o^C] = [\tilde{H}_o]_C = [T_G^{a_o}]_C.$$

The restriction τ_{y_C} of τ_y to C is equivalent to $\sigma_{(y,0)} = \sigma_y$ by Lemma 7.6. Hence the action of $\tilde{H}_o^C$ is ergodic on $(C, \tilde{S}_C, \tau_{y_C})$. Since we also have

$$\tau_y(\tilde{X} \setminus T_G^{a_o} C) = 0$$

by (7.39), we can apply (f) in Exercise 1.4 to prove that τ_y is $T_G^{a_o}$-ergodic. The proof is complete.

<u>Corollary 7.8.</u> Let $N_1(0)$ be given by (7.41) and let (y, α), $(y', \alpha') \in \tilde{Y}$ with $(y, \alpha) \neq (y', \alpha')$ and with $\alpha - \alpha' \in N_1(0)$. Then $\tau_{(y,\alpha)}$ and $\tau_{(y',\alpha')}$ are mutually singular.

Proof: It will be enough to prove that $\tau_{(y,\alpha)}$ and $\tau_{y'}$ are mutually singular whenever $\alpha \in N_1(0)$ and $(y, \alpha) \neq (y', 0)$. Indeed, by Lemma 7.6, the restrictions of $\tau_{(y,\alpha)}$ and $\tau_{y'}$ to $X \times N_1(0)$ are equivalent to $\sigma_{(y,\alpha)}$ and to $\sigma_{y'}$, respectively. Since $(y, \alpha) \neq (y', 0)$, (7.37) shows that $\tau_{(y,\alpha)}$ and $\tau_{y'}$ are not equivalent. But both measures are ergodic by Lemma 7.7, and this implies that the measures are mutually singular. The proof is complete.

<u>Lemma 7.9.</u> Let $\pi_2 : \tilde{X} \to A$ be the second coordinate projection

$$\pi_2(x, \alpha) = \alpha, \quad (x, \alpha) \in \tilde{X},$$

and let, for every $y \in Y$,

$$m_y = \tau_y \pi_2^{-1}. \tag{7.42}$$

Then the following is true.

(1) For every $y \in Y$, the measure m_y on $(A, \mathcal{B})$ is purely atomic.

(2) For every $y \in Y$, put

$$A(y) = \{\alpha \in A : m_y(\{\alpha\}) > 0\}. \tag{7.43}$$

Then

$$\text{card}\,(A(y) \cap (N_1(0) + \alpha)) \leq 1 \tag{7.44}$$

for every $\alpha \in A$ and for every $y \in Y$.

Proof: Let m_y^c denote the continuous part of m_y and assume that $m_y^c \neq 0$ for some $y \in Y$. Choose $\alpha \in A$ with $m_y^c(\alpha + N_1(0)) > 0$. Since m_y^c is nonatomic, we can find disjoint Borel sets $B_1, B_2 \subset \alpha + N_1(0)$ with $m_y^c(B_1) \cdot m_y^c(B_2) \neq 0$. Hence $m_y(B_1) \cdot m_y(B_2) \neq 0$, and the ergodicity of τ_y (Lemma 7.7) implies that

$$\bigcup_{g \in G} \{x : a_0(g, x) \in B_1 - B_2\} \neq \emptyset.$$

It now suffices to observe that $B_1 - B_2 \subset N(0) \setminus \{0\}$, so that (7.11) is violated. This shows that m_y is purely atomic. The second assertion is proved similarly.

<u>Lemma 7.10</u>. $$\Omega = \{(y, \alpha) : \alpha \in A(y), y \in Y\} \tag{7.45}$$

is a Borel subset of $\tilde{Y}$.

Proof: From (7.18) and (7.26) we see that $\{y: \sigma_y(B) > 0\} \in \mathcal{T}$ for every $B \in \tilde{\mathcal{S}}$. Applying this to sets of the form $B = T_g^{a_o}(X \times C)$, $g \in G$, $C \in \mathcal{B}$, we get

$$\bigcup_{g \in G} \{y: \sigma_y(T_g^{a_o}(X \times C)) > 0\} = \{y \in Y: \tau_y(X \times C) > 0\} \in \mathcal{T}. \qquad (7.46)$$

Let $(\alpha_k, k=1,2,\dots)$ be a dense sequence in A, and put, for $i \geq 1$,

$$Y_i = \{y \in Y: \tau_y(X \times (\alpha_i + N_1(0))) > 0\}.$$

By (7.46), $Y_i \in \mathcal{T}$, and Lemma 7.9 implies the existence of a unique element $\eta_i(y) \in A(y) \cap (\alpha_i + N_1(0))$ for every $y \in Y_i$ and every i. Applying (7.46) once again, we get

$$\{y \in Y_i: \eta_i(y) \in C\} = \{y \in Y: \tau_y((\alpha_i + N_1(0)) \cap C) > 0\}$$

for every $C \in \mathcal{B}$. Hence $\eta_i: Y_i \to A$ is a Borel map for every i. We conclude the proof by noting that

$$\Omega = \{(y, \alpha): \alpha \in A(y), y \in Y\} = \bigcup_{i=1}^{\infty} \{(y, \eta_i(y)): y \in Y_i\} \in \tilde{\mathcal{T}}.$$

We recall the definition of $N_1(0)$ in (7.41) and choose a symmetric neighbourhood $N_3(0)$ of 0 in A with

$$N_3(0)+N_3(0)+N_3(0) \subset N_1(0). \qquad (7.47)$$

Put

$$N_3(0)+N_3(0) = N_2(0). \qquad (7.48)$$

For every $\beta \in A$, let $N_i(\beta) = N_i(0)+\beta$, i=1,2,3, and define

$$X(\beta) = \bigcup_{g\in G} \{x\in X: a_o(g,x)\in N_3(\beta)\}, \tag{7.49}$$

$$C(\beta) = X(\beta)\times N_3(0), \tag{7.50}$$

and

$$D(\beta) = T_G^{a_o} C(\beta)\cap (X\times N_2(\beta)). \tag{7.51}$$

Note that $X(\beta)\in \mathfrak{S}^*$ and $C(\beta), D(\beta)\in \tilde{\mathfrak{S}}^*$. Recalling (7.17) - (7.21), (7.26) and (7.35), we put

$$Y(\beta) = \{y\in Y: q_y(X(\beta)) = 1\}, \tag{7.52}$$

$$C^*(\beta) = Y(\beta)\times N_3(0) = \{(y,\alpha)\in \tilde{Y}: \sigma_{(y,\alpha)}(C(\beta))=1\}, \tag{7.53}$$

and

$$D^*(\beta) = \{(y,\alpha)\in \tilde{Y}: \sigma_{(y,\alpha)}(D(\beta))=1\}. \tag{7.54}$$

It is again clear that $Y(\beta)\in \mathcal{T}$, $C^*(\beta)$ and $D^*(\beta)\in \tilde{\mathcal{T}}$, and $D^*(\beta)\subset Y\times N_2(\beta)$.

Lemma 7.11. For every $\beta\in A$, we have the identity

$$Y(\beta) = \{y\in Y: A(y)\cap N_3(\beta)\neq \phi\}. \tag{7.55}$$

Proof: If $y\in Y(\beta)$, then $q_y(\bigcup_{g\in G}\{x: a_o(g,x)\in N_3(\beta)\}) = \sigma_y(\bigcup_{g\in G} T_{g^{-1}}^{a_o}(X\times N_3(\beta)))=1$. But this shows that $\tau_y(X\times N_3(\beta))>0$, and hence that $A(y)\cap N_3(\beta)\neq\phi$ (cf. Lemma 7.9). Conversely, if $A(y)\cap N_3(\beta)\neq\phi$ for some $y\in Y$, we get $\tau_y(X\times N_3(\beta))>0$ and hence $q_y(\bigcup_{g\in G}\{x\in X: a_o(g,x)\in N_3(\beta)\} = \sigma_y(T_G^{a_o}(X\times N_3(\beta))) = 1$, as in the first part of the proof.

Lemma 7.12. Let $\beta \in A$ be fixed and define $Y(\beta)$ by (7.52) or (7.55). For every $y \in Y(\beta)$, we denote by $\eta(\beta,y)$ the unique element in $A(y) \cap N_3(\beta)$ (cf. Lemma 7.9). Then there exists a Borel set $N_\beta \subset Y(\beta)$ with $\rho(N_\beta) = 0$ and an injective Borel map $\varphi'_\beta : Y(\beta) \setminus N_\beta \to Y$ with

$$\tau_{\varphi'_\beta(y)} \sim \tau_{(y,-\eta(\beta,y))} \tag{7.56}$$

for every $y \in Y'(\beta) = Y(\beta) \setminus N_\beta$. Moreover, if E is any Borel subset of $Y(\beta) \setminus N_\beta$, then $\varphi'_\beta(E)$ is a Borel set, and

$$\rho(E) > 0 \text{ if and only if } \rho(\varphi'_\beta(E)) > 0. \tag{7.57}$$

Proof: If $\mu'(X(\beta)) = \rho(Y(\beta)) = 0$, the statement is trivial. We shall therefore assume that $\rho(Y(\beta)) > 0$. (7.49) - (7.51) show that $\overset{a}{T}{}_G^{\,o} C(\beta) = \overset{a}{T}{}_G^{\,o} D(\beta)$. We put

$$\kappa_1 = c_1 \cdot \tilde{\tilde{\mu}}_{C(\beta)}, \tag{7.58}$$

and

$$\kappa_2 = c_2 \cdot \tilde{\tilde{\mu}}_{D(\beta)}, \tag{7.59}$$

where $c_1, c_2 > 0$ are chosen to make κ_1 and κ_2 probability measures. Let $(g_1, g_2, g_3, \ldots)$ be the enumeration (7.38) of G, and let

$$\tilde{\kappa}_i(B) = \sum_{k=1}^{\infty} 2^{-k} \kappa_i(\overset{a}{T}{}_{g_k}^{\,o} B), \; B \in \tilde{\mathcal{S}}, \; i=1,2. \tag{7.60}$$

$\tilde{\kappa}_i$, i=1,2, are probability measures on $(\tilde{X}, \tilde{\mathcal{S}})$, and it is easy to see that

$$\tilde{\varkappa}_1 \sim \tilde{\varkappa}_2 \sim \tilde{\tilde{\mu}}_{T_G^{a_0} C(\beta)}. \tag{7.61}$$

Since $\tilde{\varkappa}_1$ is quasi-invariant under $T_G^{a_0}$, we can find its ergodic decomposition. Put

$$\tilde{Y}_1 = C^*(\beta),$$
$$\tilde{\mathcal{F}}_1 = \tilde{\mathcal{F}}_{\tilde{Y}_1},$$
$$\tilde{\rho}_1 = c_1 \tilde{\rho}_{\tilde{Y}_1},$$

with c_1 as in (7.57), and

$$q^{(1)}_{(y,\alpha)} = \tau_{(y,\alpha)}, \quad (y,\alpha) \in \tilde{Y}_1. \tag{7.61.a}$$

Each $q^{(1)}_{(y,\alpha)}$ is quasi-invariant and ergodic under $T_G^{a_0}$ (Lemma 7.7), $q^{(1)}_{(y,\alpha)}$ and $q^{(1)}_{(y',\alpha')}$ are mutually singular whenever $(y,\alpha) \neq (y',\alpha')$ (Corollary 7.8), and

$$\tilde{\varkappa}_1(B) = \int_{\tilde{Y}_1} q^{(1)}_{(y,\alpha)}(B) \, d\, \tilde{\rho}_1(y,\alpha) \tag{7.61.b}$$

for every $B \in \tilde{\mathcal{S}}$, by (7.36), (7.58), (7.39) and (7.60). In fact, $(\tilde{Y}_1, \tilde{\mathcal{F}}_1, \tilde{\rho}_1)$ and $\{q^{(1)}_{(y,\alpha)} : (y,\alpha) \in \tilde{Y}_1\}$ satisfy all the conditions (6.53) - (6.56) of an ergodic decomposition of $\tilde{\varkappa}_1$ with respect to $T_G^{a_0}$. Let us now decompose $\tilde{\varkappa}_2$. We set

$$\tilde{Y}_2 = D^*(\beta),$$
$$\tilde{\mathcal{F}}_2 = \tilde{\mathcal{F}}_{\tilde{Y}_2},$$
$$\tilde{\rho}_2 = c_2 \cdot \tilde{\rho}_{\tilde{Y}_2},$$

and

$$q^{(2)}_{(y,\alpha)} = \tau_{(y,\alpha)}, \quad (y,\alpha) \in \tilde{Y}_2.$$

Again one verifies quite easily, that $(\tilde{Y}_2, \tilde{\mathcal{F}}_2, \tilde{\rho}_2)$ and $\{ q^{(2)}_{(y,\alpha)} : (y,\alpha) \in \tilde{Y}_2 \}$ satisfy all the conditions (6.53) - (6.56) for an ergodic decomposition of $\tilde{K}_2$ with respect to $T_G^{a_o}$. Applying Exercise 6.13 and (7.61) we conclude the existence of an isomorphism $\Theta : (\tilde{Y}_1, \tilde{\mathcal{F}}_1, \tilde{\rho}_1) \to (\tilde{Y}_2, \tilde{\mathcal{F}}_2, \tilde{\rho}_2)$ with

$$q^{(2)}_{(y,\alpha)} \sim q^{(1)}_{(y,\alpha)} \tag{7.62}$$

for $\tilde{\rho}_1$-a.e. $(y,\alpha) \in \tilde{Y}_1$. Lemma 7.3 implies that $\tilde{\rho}_1$ and $\tilde{\rho}_2$ are nonatomic and hence that $\tilde{Y}_1$ and $\tilde{Y}_2$ are uncountable. We may thus assume Θ to be a Borel isomorphism from $(\tilde{Y}_1, \tilde{\mathcal{F}}_1)$ to $(\tilde{Y}_2, \tilde{\mathcal{F}}_2)$. Fubini's theorem now implies the existence of an $\alpha_o \in N_3(0)$ with

$$q^{(2)}_{\Theta(y,\alpha_o)} \sim q^{(1)}_{(y,\alpha_o)}$$

for ρ-a.e. $y \in Y(\beta)$. We put

$$N_\beta = \{ y \in Y(\beta) : q^{(2)}_{\Theta(y,\alpha_o)} \not\sim q^{(1)}_{(y,\alpha_o)} \} .$$

It is not difficult to verify that N_β is a Borel set. We put

$$Y'(\beta) = Y(\beta) \setminus N_\beta$$

and write

$$\Theta(y,\alpha_o) = (\phi'(y), \psi(y))$$

for every $y \in Y'(\beta)$ with $\phi'(y) \in Y$ and $\psi(y) \in A$. If

$$\psi'(y) = \psi(y) - \alpha_0, \tag{7.63}$$

we get

$$\tau_{(\phi'(y),\, \psi'(y))} \sim \tau_{(y,0)}$$

for every $y \in Y'(\beta)$. It follows that

$$\tau_{\phi'(y)} = \tau_{(\phi'(y),0)} \sim \tau_{(y,-\psi'(y))} = \tau_y R_{\psi'(y)} \tag{6.64}$$

for every $y \in Y'(\beta)$. (7.64) implies in particular

$$\tau_{(y,-\psi'(y))}(X \times \{0\}) = \tau_y(X \times \{\psi'(y)\}) > 0$$

and hence

$$\psi'(y) \in A(y).$$

Since

$$\Theta(y, \alpha_0) = (\phi'(y), \psi(y)) \in D^*(\beta) \subset Y \times N_2(\beta)$$

for every $y \in Y'(\beta)$, (7.62) implies

$$\psi'(y) \in N_2(\beta) + N_3(\beta) \subset N_1(\beta).$$

Lemma 7.9 now shows that

$$\psi'(y) = \eta(\beta, y), \tag{7.65}$$

since both $\psi'(y)$ and $\eta(\beta, y)$ lie in $A(y) \cap N_1(\beta)$ for every $y \in Y'(\beta)$. We can now put $\phi' = \phi'_\beta$ and obtain (7.56) from (7.64) and (7.65). Corollary 7.8 implies immediately that ϕ'_β is injective.

To prove (7.57) we proceed as follows: For every (y,α) $\in Y'(\beta)\times N_3(0)$, put

$$\Theta'(y,\alpha) = (\phi'_\beta(y), \eta(\beta,y)+\alpha).$$

Then $\Theta'(y,\alpha)\in Y\times N_2(\beta)$, and $q^{(2)}_{\Theta'(y,\alpha)} \sim q^{(1)}_{(y,\alpha)}$ for every $(y,\alpha)\in Y'(\beta)\times N_3(0)$. Comparing this with (7.62) and applying once again Corollary 7.8, we get

$$\Theta'(y,\alpha) = \Theta(y,\alpha)$$

for $\tilde{\rho}_1$-a.e. $(y,\alpha)\in Y'(\beta)\times N_3(0)$. It follows immediately that for every Borel set $F\subset Y'(\beta)\times N_3(0)$, $\tilde{\rho}_1(F)>0$ if and only if $\tilde{\rho}_2(\Theta'(F))>0$. Note that $\Theta'(F)$ is a Borel set, since Θ' is injective. If we now choose F of the form $F = E\times N_3(0)$ with $E\in\mathcal{S}$ and $E\subset Y'(\beta)$, we get (7.57). The proof is complete.

Lemma 7.13. There exists a Borel set $N_0\subset Y$ with $\rho(N_0) = 0$, which satisfies the following conditions: Put $Y_0 = Y\setminus N_0$ and

$$\Omega_0 = \{(y,\alpha): y\in Y_0,\ \alpha\in A(y)\}. \tag{7.66}$$

Then there exists a Borel map $\phi:\Omega_0\to Y_0$ with

$$\tau_{\phi(y,\alpha)} \sim \tau_y R_\alpha \tag{7.67}$$

for every $(y,\alpha)\in\Omega_0$. ϕ is uniquely determined by (7.67).

Proof: Choose and fix a dense sequence $(\beta_1,\beta_2,\dots)$ in A. For every β_i we define $Y(\beta_i)$, N_{β_i}, $\eta(\beta_i,y)$ and φ'_{β_i}

by Lemma 7.12. Put $N' = \bigcup_{i \geq 1} N_{\beta_i}$ and $\Omega_i' = \{(y, \alpha):$ $y \in Y \setminus N', \alpha \in A(y) \cap N_3(\beta_i)\}$. Since $\Omega_i' = \{(y, \eta(\beta_i, y)):$ $y \in Y(\beta_i) \setminus N'\}$, we can define injective Borel maps $\phi_i' : \Omega_i' \to Y$ by

$$\phi_i'(y, \eta(\beta_i, y)) = \phi'_{\beta_i}(y), \quad (y, \eta(\beta_i, y)) \in \Omega_i'. \tag{7.68}$$

From (7.56) we get

$$\tau_{\phi_i'(y, \alpha)} \sim \tau_y R_\alpha \tag{7.69}$$

for every i=1,2,... and for every $(y, \alpha) \in \Omega_i'$. If $(y, \alpha) \in \Omega_i' \cap \Omega_j'$, $i \neq j$, we can once again apply Corollary 7.8 to see that

$$\phi_i'(y, \alpha) = \phi_j'(y, \alpha). \tag{7.70}$$

Let now $\Omega' = \bigcup_{i \geq 1} \Omega_i' = \{(y, \alpha) : y \in Y \setminus N', \alpha \in A(y)\}$, and define $\phi' : \Omega' \to Y$ by

$$\phi'(y, \alpha) = \phi_i'(y, \alpha) \text{ whenever } (y, \alpha) \in \Omega_i'. \tag{7.71}$$

(7.70) shows that ϕ' is defined unambiguously on Ω', and (7.69) and (7.71) together imply

$$\tau_{\phi'(y, \alpha)} \sim \tau_y R_\alpha \tag{7.72}$$

for every $(y, \alpha) \in \Omega'$. We now put

$$Y_o = \phi'(\Omega').$$

Each ϕ_i' is injective on Ω_i', so that $\phi_i'(\Omega_i')$ is a Borel set. Hence $Y_o = \bigcup_{i \geq 1} \phi_i^!(\Omega_i^!)$ is also a Borel set, and its complement $N_o = Y \setminus Y_o$ will have ρ-measure zero. We now want to extend the map ϕ' to $\Omega_o = \{(y, \alpha): y \in Y_o, \alpha \in A(y)\}$. If $(y, \alpha) \in \Omega'$, we define

$$\phi(y, \alpha) = \phi'(y, \alpha).$$

Suppose now that $y' \in Y_o \setminus Y'$ and $\alpha' \in A(y')$. Then there exists $(y, \alpha) \in \Omega'$ with $\phi'(y, \alpha) = y'$. (7.72) implies

$$\tau_{y'} \sim \tau_{y} R_{\alpha}$$

and hence

$$\tau_{y'} R_{\alpha'} \sim \tau_{y} R_{\alpha + \alpha'}.$$

Since $\alpha' \in A(y')$, we must have $\alpha + \alpha' \in A(y)$. Applying (7.72) once again, we get

$$\tau_{y'} R_{\alpha'} \sim \tau_{\phi'(y, \alpha + \alpha')}.$$

If we now set

$$\phi(y', \alpha') = \phi'(y, \alpha + \alpha'),$$

we have constructed a map $\phi: \Omega_o \to Y_o$ satsifying (7.67). Note that the definition of ϕ is unambiguous and uniquely determined by (7.67), again as a consequence of Corollary 7.8. We leave it to the reader to verify that ϕ is a Borel map. The proof is complete.

Lemma 7.11. Let $\beta \in A$, and let

$$Y_o(\beta) = \{y \in Y_o : A(y) \cap N_3(\beta) \neq \phi\}. \qquad (7.73)$$

For every $y \in Y_o(\beta)$ we define $\eta_o(y,\beta)$ to be the unique element in $A(y) \cap N_3(\beta)$. Consider the map $\phi_\beta : Y_o(\beta) \to Y_o$ given by

$$\phi_\beta(y) = \phi(y, \eta_o(y,\beta)), \; y \in Y_o(\beta). \qquad (7.74)$$

Then ϕ_β is an injective Borel map, and for every Borel subset $D \subset Y_o(\beta)$ we have

$$\rho(D) > 0 \text{ if and only if } \rho(\phi_\beta(D)) > 0. \qquad (7.75)$$

Proof: If $\pi_1 : \Omega_o \to Y_o$ denotes the projection onto the first coordinate, then π_1 is a Borel map and injective when restricted to $\Omega_o \cap Y_o \times N_3(\beta)$. Hence the restriction π_1^* of π_1 to $\Omega_o \cap Y_o \times N_3(\beta)$ is a Borel isomorphism between $\Omega_o \cap Y_o \times N_3(\beta)$ and $Y_o(\beta)$. Since ϕ is also injective on $\Omega_o \cap Y_o \times N_3(\beta)$ by Corollary 7.8, and since $\phi_\beta = \phi \cdot \pi_1^{-1}$, ϕ_β is indeed an injective Borel map on $Y_o(\beta)$. Let now D be a Borel subset of $Y_o(\beta)$, and let $Y'(\beta)$ be defined as in Lemma 7.12. Then $\rho(D) = \rho(D \cap Y'(\beta))$. The uniqueness of ϕ shows that $\phi_\beta(y) = \phi_\beta'(y)$ for every $y \in Y'(\beta)$ (cf.(7.56)), and (7.57) now implies that $\rho(D \cap Y'(\beta)) > 0$ if and only if $\rho(\phi_\beta(D \cap Y'(\beta))) = \rho(\phi_\beta'(D \cap Y'(\beta))) > 0$. We conclude that $\rho(\phi_\beta(D)) > 0$ whenever $\rho(D) > 0$. To prove the converse, assume that $\rho(\phi_\beta(D)) > 0$. From the definition of ϕ, ϕ_β and $Y_o(\beta)$ it is clear that

$$\phi_\beta(Y_o(\beta)) = Y_o(-\beta)$$

and

$$\phi_{-\beta}(\phi_\beta(y)) = y$$

for every $y \in Y_o(\beta)$. We apply the first part of the proof to $-\beta$ and to the Borel set $\phi_\beta(D) \subset Y_o(-\beta)$ to see that $\rho(D) > 0$ whenever $\rho(\phi_\beta(D)) > 0$. The proof is complete.

We now put

$$\mathcal{T}_o = \mathcal{T}_{Y_o} \tag{7.76}$$

and

$$\rho_o = \rho_{Y_o}. \tag{7.77}$$

<u>Proposition 7.15</u>. There exists a countable group Γ of automorphisms of the measure space $(Y_o, \mathcal{T}_o, \rho_o)$ with

$$\Gamma y = \{\phi(y, \alpha) : \alpha \in A(y)\} \tag{7.78}$$

for every $y \in Y_o$.

Proof: Choose and fix a dense sequence $(\beta_1, \beta_2, \ldots)$ in A. For every $i \geq 1$, put $E_o^{(i)} = \{y \in Y_o(\beta_i) : \phi_{\beta_i}(y) = y\}$, where $Y_o(\beta_i)$ and ϕ_{β_i} are defined as in Lemma 7.14. Next we choose a sequence $(E_k^{(i)}, k=1,2,\ldots)$ of Borel subsets of $Y_o(\beta_i)$ such that $\phi_{\beta_i}(E_k^{(i)}) \cap E_k^{(i)} = \emptyset$ for every $k \geq 1$, and

$$\bigcup_{k\geq 0} E_k^{(i)} = Y_o(\beta_i),\ i\geq 1.$$

For every i,k, let $V_{(i,k)}: Y_o \to Y_o$ be the Borel automorphism given by

$$V_{(i,k)}y = \begin{cases} y & \text{if } y \notin E_k^{(i)} \cup \phi_{\beta_i}(E_k^{(i)}), \\ \phi_{\beta_i}(y) & \text{for } y \in E_k^{(i)}, \\ \phi_{\beta_i}^{-1}(y) & \text{for } y \in \phi_{\beta_i}(E_k^{(i)}). \end{cases}$$

(7.75) shows that $V_{(i,k)}$ is an automorphism of the measure space $(Y_o, \mathcal{T}_o, \rho_o)$. Let Γ be the countable subgroup of $\mathcal{A}(Y_o, \mathcal{T}_o, \rho_o)$ generated by $\{V_{(i,k)}: i\geq 1, k\geq 1\}$. It is clear that Γ will satisfy (7.76). The proof is complete.

<u>Lemma 7.16.</u> Let $y \in Y_o$, let $(\alpha_1, \alpha_2, \ldots)$ be an enumeration of $A(y)$, and let $(g_1, g_2, \ldots)$ be the enumeration (7.38) of G. Then the following statements hold:

(1) τ_y is equivalent to the measure

$$B \to \sum_{k\geq 1} 2^{-k}\, \sigma_{\phi(y,\alpha_k)}(R_{\alpha_k}^{-1}B),\ B \in \tilde{\mathcal{S}}. \tag{7.79}$$

(2) The measures on $(X, \mathcal{S})$ given by

$$\Theta_y^{(1)}(B) = \sum_{k\geq 1} 2^{-k} q_{\phi(y,\alpha_k)}(B) \tag{7.80}$$

and

$$\Theta_y^{(2)}(B) = \sum_{k\geq 1} 2^{-k} q_y(T_{g_k}B), \tag{7.81}$$

$B \in \mathcal{S}$, are equivalent, and they are quasi-invariant and ergodic under T_G.

Proof: (1) is clear from Lemma 7.6 and Lemma 7.14. (2) follows by applying (1) to two sets of the form $B = B_1 \times A$ with $B_1 \in \mathcal{S}$.

Proposition 7.17. Let Γ be the group of automorphisms of $(Y_o, \mathcal{T}_o, \rho_o)$ constructed in Proposition 7.15. Then Γ is ergodic.

Proof: Let $C \subset Y_o$ be a Borel set with $\Gamma C = C$. If $\rho_o(C) \cdot \rho_o(\complement C) \neq 0$, we can define two measures on $(X, \mathcal{S})$ by

$$m_1 = \int_C q_y \, d\rho(y)$$

and

$$m_2 = \int_{\complement C} q_y \, d\rho(y).$$

m_1 and m_2 are mutually singular, and $m_1 + m_2 = \mu'$ (cf. (7.17) - (7.21)). (7.80), (7.81) and (7.78) together imply that $m_1 T_{g_1}^{-1}$ and $m_2 T_{g_2}^{-1}$ are mutually singular for every $g_1, g_2 \in G$. It follows that the measures

$$m_1^*(B) = \sum_{k \geq 1} 2^{-k} m_1(T_{g_k} B)$$

and

$$m_2^*(B) = \sum_{k \geq 1} 2^{-k} m_2(T_{g_k} B),$$

$B \in \mathcal{S}$, are mutually singular. On the other hand it is clear that

$$m_1^* + m_2^* = \sum_{k \geq 1} 2^{-k} \mu' T_{g_k} \sim \mu',$$

so that we are violating the ergodicity of μ'. This contradiction proves the proposition.

<u>Corollary 7.18.</u> A(y) is an infinite set for ρ_0-a.e. $y \in Y_0$.

Proof: If A(y) is finite for some $y \in Y$, Γy is a finite set. Since Γ is ergodic on a nonatomic measure space, ρ_0-a.e. $y \in Y_0$ must have an infinite orbit. The proof is complete.

<u>Lemma 7.19.</u> The set $F_0 = \{ y \in Y_0 : \phi(y, \alpha) = y$ for some $\alpha \neq 0$ in $A(y) \}$ is a Borel set, and $\rho_0(F_0) = 0$.

Proof: For every $\beta \in A$ we define $\phi_\beta : Y_0(\beta) \to Y_0$ as in Lemma 7.14 by (7.73) and (7.74). If $W \subset A$ is a Borel set, put $F(W, \beta) = \{ y \in Y_0(\beta) : \eta_0(y, \beta) \in W$ and $\phi_\beta(y) = y \}$. It is clear that $F(W, \beta)$ is a Borel set for every $W \in \mathcal{B}$ and for every $\beta \in A$. We choose a sequence $(\beta_1, \beta_2, \ldots)$ which is dense in A. The set

$$F(W) = \bigcup_{i \geq 1} F(W, \beta_i)$$

is again Borel, and we have

$$F(W) = \{ y \in Y_0 : \phi(y, \alpha) = y \text{ for some } \alpha \in A(y) \cap W \}.$$

Let now $y \in F(W)$ and $\alpha' \in A(y)$. From (7.67) we get

$$\tau_{\phi(y,\alpha')} \sim \tau_y R_{\alpha'}.$$

By assumption, there exists an $\alpha \in A(y) \cap W$ for which

$$\tau_y \sim \tau_y R_\alpha .$$

Consequently one has

$$\tau_{\phi(y,\alpha')} \sim \tau_y R_{\alpha'} \sim \tau_y R_{\alpha+\alpha'} \sim \tau_{\phi(y,\alpha')} R_\alpha ,$$

which shows that $\phi(y,\alpha') \in F(W)$. (7.78) now implies that $\Gamma F(W) = F(W)$ for every $W \in \mathcal{B}$, and Proposition 7.17 yields $\rho_0(F(W)).\rho_0(\complement F(W)) = 0$.

Turning now to the assertion of this lemma, we assume that $\rho_0(F_0) \neq 0$. Since $F_0 = F(A \setminus \{0\})$, we conclude from the first part of this proof that $\rho_0(F_0) = 1$. Let Λ be an invariant metric on A. We choose sequences $(\beta_k^{(1)}, k=1,2,\ldots)$ of points in A and $(c_k, k=1,2,\ldots)$ of positive real numbers such that the closed balls $\bar{B}(\beta_k^{(1)}, c_k) = \{\alpha \in A : \Lambda(\alpha, \beta_k^{(1)}) \leq c_k\}$ together cover $A \setminus \{0\}$ for every k, but that $0 \notin \bar{B}(\beta_k^{(1)}, c_k)$ for every $k \geq 1$. Since $F(A \setminus \{0\}) = \bigcup_{k \geq 1} F(\bar{B}(\beta_k^{(1)}, c_k))$, and since the measure of every $F(\bar{B}(\beta_k^{(1)}, c_k))$ is either zero or one, there exists a $k_1 \geq 1$ with $\rho_0(F(\bar{B}(\beta_{k_1}^{(1)}, c_{k_1})) = 1$. We can again find a sequence $(\beta_k^{(2)}, k=1,2,\ldots)$ in A such that $\bigcup_{k \geq 1} \bar{B}(\beta_k^{(2)}, c_{k_1}/2) \supset \bar{B}(\beta_{k_1}^{(1)}, c_{k_1})$, and conclude the existence of an integer $k_2 \geq 1$ with $\rho_0(F(\bigcap_{l=1}^{2} \bar{B}(\beta_{k_l}^{(l)}, c_{k_1}.2^{-l+1})))$ $=1$. In this fashion we construct a sequence

$$(\beta_{k_1}^{(1)}, \beta_{k_2}^{(2)}, \ldots) \subset A$$

for which

$$\rho_o(F(\bigcap_{l=1}^{m}\bar{B}(\beta_{k_l}^{(l)},c_{k_l}.2^{-l+1}))) = 1$$

for every $m\geq 1$. The completeness of A implies the existence of a point $\alpha_o \in \bigcap_{k=1}^{\infty}\bar{B}(\beta_{k_l}^{(l)},c_{k_l}.2^{-l+1})$. It follows immediately that

$$\rho_o(F(B(\alpha_o, \varepsilon))) = 1$$

for every $\varepsilon > 0$, where $B(\alpha_o, \varepsilon) = \{\alpha \in A: d(\alpha, \alpha_o) < \varepsilon\}$. Note that $\alpha_o \neq 0$ by construction. We now claim that $\alpha_o \in E(a)$. Indeed, if $\alpha_o \notin E(a)$, there exists a set $D \in \mathcal{S}$ with $\mu'(D) > 0$ and an $\varepsilon > 0$ for which

$$\bigcup_{g\in G} D \cap T_g^{-1}D \cap \{x: a_o(g,x) \in B(\alpha_o, \varepsilon)\} = \phi.$$

By (7.18) there exists a $y_1 \in F(B(\alpha_o, \varepsilon))$ with $q_{y_1}(D) > 0$. We choose an element $\alpha_1 \in B(\alpha_o, \varepsilon)$ for which $\tau_{y_1} R_{\alpha_1} \sim \tau_{y_1}$. If $\tilde{D} = D \times \{0\} \subset \tilde{X}$, we see that $\sigma_{y_1}(\tilde{D}) > 0$ and hence that $\tau_{y_1}(\tilde{D}) > 0$. It follows that $\tau_{y_1}(R_{\alpha_1}\tilde{D}) > 0$. The ergodicity of τ_{y_1} under $T_G^{a_o}$ implies the existence of a point $(x,0) \in \tilde{D}$ and of a $g_1 \in G$ for which $T_{g_1}^{a_o}(x,0) \in R_{\alpha_1}\tilde{D}$. In other words, we get $x \in D$, $T_{g_1}x \in D$, and $a_o(g,x) = \alpha_1 \in B(\alpha_o, \varepsilon)$. This violates our choice of D and shows that α_o is an essential value of a_o. We have proved that the assumption $\rho_o(F_o) > 0$ leads to a contradiction to condition (7.10) and (7.22). Hence $\rho_o(F_o) = 0$, and the lemma is proved.

Lemma 7.20. Let F_o be given by Lemma 7.19 and let $Y_1 = Y_o \setminus \Gamma F_o$. We put $\mathcal{T}_1 = \mathcal{T}_{Y_1}$, $\rho_1 = \rho_{Y_1}$. The restriction of Γ to $(Y_1, \mathcal{T}_1, \rho_1)$ will be denoted by Γ_1, and $R(\Gamma_1)$ will stand for the equivalence relation of Γ_1. Then the following is true:

(1) For every $(y_1,y_2) \in R(\Gamma_1)$ there exists a unique element $\alpha(y_1,y_2) \in A(y_2)$ with $y_1 = \phi(y_2, \alpha(y_1,y_2))$.

(2) The map $u^*: R(\Gamma_1) \to A$ given by

$$u^*(y_1,y_2) = \alpha(y_1,y_2), \quad (y_1,y_2) \in R(\Gamma_1), \tag{7.82}$$

is a transient orbital cocycle for Γ_1.

Proof: (1) follows from (7.78) and from the definition of Y_1. That u^* in (7.82) is Borel and an orbital cocycle is obvious. Lemma 7.9 shows that $u^*(y_1,y_2) \notin N_1(0)$ whenever $y_1 \neq y_2$, which implies that u^* is transient. The proof is complete.

Lemma 7.21. Let $\psi: X \to Y$ be given as in (7.17) - (7.21) and let Γ_1 and $(Y_1, \mathcal{T}_1, \rho_1)$ be defined by Lemma 7.20. we fix a point $y_1 \in Y_1$, and put

$$\psi_1(x) = \begin{cases} \psi(x) & \text{for } x \in \psi^{-1}(Y_1), \\ y_1 & \text{otherwise.} \end{cases} \tag{7.83}$$

Then $\psi_1: X \to Y_1$ is a Borel map, and $\rho_1 = \mu' \psi_1^{-1}$.

Furthermore there exists a Borel set $N^* \subset X$ with $\mu'(N^*) = 0$ such that

(1) $\psi_1(T_G x) = \Gamma_1 \psi_1(x)$ for every $x \in X \setminus N^*$, (7.84)

(2) $a_o(g,x) = u^*(\psi_1(T_g x), \psi_1(x))$ for every $x \in X \setminus N^*$ and every $g \in G$, where u^* comes from (7.82). (7.85)

Proof: This proof will consist of several parts. We note that the properties of ψ_1 are obvious. Our first aim is to show that

$$\psi_1(T_G x) \subset \Gamma_1 \psi_1(x) \text{ for } \mu'\text{-a.e.} x \in X. \quad (7.86)$$

For every $g \in G$, the set $B_g = \{x \in X : (\psi_1(T_g x), \psi_1(x)) \notin R(\Gamma_1)\}$ is Borel. Hence

$$N_1^* = \{x \in X : \psi_1(T_G x) \not\subset \Gamma_1 \psi_1(x)\} = \bigcup_{g \in G} B_g$$

is Borel. For every $y \in Y_1$, we have $q_y(\psi_1^{-1}(\{y\})) = 1$ from (7.19). Let $\Theta_y^{(1)}$ be the measure on $(X, \mathcal{S})$ given by (7.80). Since $\Theta_y^{(1)}$ is quasi-invariant under T_G by Lemma 7.16, we get

$$\Theta_y^{(1)}(\bigcup_{\alpha \in A(y)} \psi_1^{-1}(\{\phi(x,\alpha)\})) =$$

$$\Theta_y^{(1)}(\{x \in X : T_G x \subset \bigcup_{\alpha \in A(y)} \psi_1^{-1}(\{\phi(y,\alpha)\})\}) = 1,$$

and hence, by (7.78), $\Theta_y^{(1)}(N_1^*) = 0$ for every $y \in Y_1$. (7.80) implies $q_y(N_1^*) = 0$ for every $y \in Y_1$, and (7.18) gives $\mu'(N_1^*) = 0$. This proves (7.86). If we now put $N_2^* = T_G N_1^*$, we still have $\Theta_y^{(1)}(N_2^*) = 0$ for every $y \in Y_1$, and hence $q_y(N_2^*) = 0$ for every $y \in Y_1$.

We now turn to the proof of (7.85). For every $y \in Y_1$, put

$$E(y) = \bigcup_{\alpha \in A(y)} \{(x,\alpha): \psi(x) = \phi(y,\alpha)\}.$$

(7.79) gives $\tau_y(E(y)) = 1$ for every $y \in Y_1$, and the quasi-invariance of τ_y shows that $\tau_y(\{(x',\alpha') \in \tilde{X}: T_G^{a_o}(x',\alpha') \subset E(y)\} = 1$. (7.79) now implies

$$\sigma_y(\{(x',\alpha') \in \tilde{X}: T_G^{a_o}(x',\alpha') \subset E(y)\})$$

$$= q_y(\{x' \in X: T_G^{a_o}(x',0) \subset E(y)\}) = 1$$

Since $q_y(N_2^*) = 0$, we even get $q_y(\{x' \in X \setminus N_2: T_G^{a_o}(x',0) \subset E(y)\}) = 1$ for every $y \in Y_1$. Let now $y \in Y_1$ be fixed, and let $x' \in X \setminus N_2$ satisfy $T_G^{a_o}(x',0) \subset E(y)$, and hence $\psi(x') = y$. For every $g \in G$, there exists a unique $\alpha(g) \in A(y)$ with $\psi(T_g x') = \phi(y, \alpha(g))$ (cf. Lemma 7.20). Moreover, since $(T_g x', a_o(g,x')) \in E(y)$, we must have $a_o(g,x') = \alpha(g)$. A glance at Lemma 7.20 now shows that $a_o(g,x') = u^*(\psi(T_g x'), \psi(x'))$. We have thus proved that

$$F^* = \bigcap_{g \in G} \{x \in X \setminus N_2^*: a_o(g,x) = u^*(\psi(T_g x), \psi(x))\} \supset$$

$$\{x \in X \setminus N_2^*: T_G^{a_o}(x',0) \subset E(y)\}$$

for every $y \in Y_1$. It follows immediately that $q_y(F^*) = 1$ for every $y \in Y_1$. F^* is clearly a Borel subset of $X \setminus N_2$, so that $\mu'(F^*) = 1$ by (7.18).

The remaining statement to be proved is (7.84). We have already shown that (7.86) holds. Since

$$\{x \in F^*: \psi_1(T_G x) \neq \Gamma_1 \psi_1(x)\} \subset$$

$$\bigcup_{i \geq 1} (\{x \in F^*: A(\psi(x)) \cap N_3(\beta_i) \neq \phi\} \cap$$

$$\bigcap_{g \in G} \{x \in F^*: a_0(g,x) \notin N_3(\beta_i)\},$$

where $(\beta_1, \beta_2, \ldots)$ is a fixed dense sequence in A, we only have to show that each set

$$E_i = \{x \in F^*: A(\psi(x)) \cap N_3(\beta_i) \neq \phi\} \cap$$

$$\bigcap_{g \in G} \{x \in F^*: a_0(g,x) \in N_3(\beta_i)\}$$

$$= \{x \in F^*: A(\psi(x)) \cap N_3(\beta_i) \neq \phi\}$$

$$\bigcap_{g \in G} \{x \in F^*: u^*(\psi(T_g x), \psi(x)) \notin N_3(\beta_i)\}$$

has μ'-measure zero. But it is quite easy to show (the details are left to the reader) that $q_y(E_i) = 0$ for every $y \in Y_1$, and for every $i=1,2,\ldots$. (7.18) once again implies that $\psi_1(T_G x) = \Gamma_1 \psi_1(x)$ for μ'-a.e. $x \in F^*$ and hence for μ'-a.e. $x \in X$. The proof is complete.

We can now formulate the first main result of this section.

Theorem 7.22. Let $(X, \mathcal{S}, \mu)$ be a nonatomic probability space, T_G an ergodic action of a countable group G on $(X, \mathcal{S}, \mu)$, and let A be a locally compact second countable abelian group. Assume $a: G \times X \to A$ is a cocycle for T_G with $\bar{E}(a) = \{0, \infty\}$. Then the following is true.

(1) There exists a coboundary $b: G \times X \to A$ for T_G such that $a_o = a+b$ is lacunary (Definition 7.1).

(2) There exists a standard Borel space $(Y_1, \mathcal{T}_1)$, a surjective Borel map $\psi_1: X \to Y_1$ for which $\rho_1 = \mu \psi_1^{-1}$ is nonatomic, and a countable ergodic group Γ_1 of automorphisms of $(Y_1, \mathcal{T}_1, \rho_1)$ which satisfies

$$\psi_1(T_G x) = \Gamma_1 \psi_1(x) \text{ for } \mu\text{-a.e.} x \in X. \tag{7.87}$$

(3) There exists a transient orbital cocycle $u^*: R(\Gamma_1) \to A$ for Γ_1 such that

$$a_o(g,x) = u^*(\psi_1(T_g x), \psi_1(x)) \tag{7.88}$$

for every $g \in G$ and for μ-a.e. $x \in X$.

(4) Let H be any countable subgroup of T_G with

$$[H] = \{V \in [T_G]: \psi_1 \cdot V = \psi_1\} . \tag{7.89}$$

Then a_o (and hence a) is recurrent if and only if H is conservative, and transient if and only if H is type I.

(5) Conversely, let $(Y_1, \mathcal{T}_1)$, $\psi_1 : X \to Y_1$, $\rho_1 = \mu\psi_1^{-1}$ and Γ_1 satisfy all the conditions in (2), and let $u^* : R(\Gamma_1) \to A$ be a transient orbital cocycle for Γ_1. Then there exists a cocycle $a_1 : G \times X \to A$ for T_G with

$$a_1(g,x) = u^*(\psi_1(T_g x), \psi_1(x))$$

for every $g \in G$ and for μ-a.e. $x \in X$. a_1 is unique up to equivalence, and $\bar{E}(a_1) = \{0, \infty\}$. Moreover, if H is any countable subgroup of $[T_G]$ satisfying (7.89), then a_1 is recurrent if and only if H is conservative, and transient if and only if H is type I.

Proof: (1) follows from Proposition 7.2. We choose an orbital cocycle $u_o : R(T_G) \to A$ with $a_o = a + b \equiv a_{u_o}$ and put $\mu' = \mu$ in (7.16). Replacing b by an equivalent coboundary we may in fact assume that $a_o = a_{u_o}$. Let now $(Y_1, \mathcal{T}_1, \rho_1)$ and Γ_1 be given as in Lemma 7.20 and let $\psi_1 : X \to Y_1$ be the map (7.83) in Lemma 7.21. From (7.84) we get (7.87), and Proposition 7.17 proves the ergodicity of Γ_1. This completes the proof of (2). If $u^* : R(\Gamma_1) \to A$ is the transient orbital cocycle (7.82) for Γ_1, we have (7.88) from (7.85) in Lemma 7.21. Having proved (3), we define $H_o \subset [T_G]$ by (7.13) and (7.14). From the definition of $\psi_1 : X \to Y_1$ it is clear that $\psi_1 \cdot V = \psi_1$ for every $V \in [H_o]$.

Conversely, if $V \in [T_G]$ satisfies $\psi_1 \cdot V = \psi_1$, we see from (7.88) that $u_o(Vx,x)$ is μ-a.e.equal to $u^*(\psi_1(Vx), \psi_1(x)) = 0$. Hence V coincides μ-a.e. with some element of $[H_o]$. It now follows from (7.89) that H is conservative or type I according as H_o is conservative or type I. Lemma 7.4 completes the proof of (4). (5) is left to the reader as an exercise. The theorem is proved.

<u>Corollary 7.23</u>. Let $(X, \mathcal{S}, \mu)$ be a nonatomic probability space, T_G an ergodic action of a countable group G on $(X, \mathcal{S}, \mu)$, and let A be a locally compact second countable abelian group. Suppose $a: G \times X \to A$ is a recurrent cocycle with $\bar{E}(a) = \{0, \infty\}$. Then there exists an uncountable family of mutually inequivalent nonatomic probability measures $\{M_\xi : \xi \in \Xi\}$ (Ξ is some uncountable set) such that

(1) for every $\xi \in \Xi$, M_ξ is quasi-invariant and ergodic under T_G,

(2) if $B \in \mathcal{S}$ is a set with $M_\xi(B) = 0$ for all $\xi \in \Xi$, then $\mu(B) = 0$,

(3) for every $\xi \in \Xi$, a is a coboundary for the action T_G of G on $(X, \mathcal{S}, M_\xi)$.

Proof: As in the beginning of the proof of Theorem 7.22 we put $\mu' = \mu$ in (7.16) and choose a coboundary $b: G \times X \to A$

and an orbital cocycle $u_o: R(T_G) \to A$ with $a_o = a_{u_o} = a+b$ lacunary. Let $f: X \to A$ be a Borel function with $b(g,x) = f(T_g x) - f(x)$ for every $g \in G$ and for μ-a.e. $x \in X$, and let $N_1 = \bigcup_{g \in G} \{x \in X : b(g,x) \neq f(T_g x) - f(x)\}$. Put $N_1^* = T_G N_1$. Using the notation of Theorem 7.22 we put $N_2 = \{x \in X: \psi_1(T_G x) \neq \Gamma_1 \psi_1(x)\}$ and $N_3 = \bigcup_{g \in G} \{x \in X \setminus N_2 : a_o(g,x) \neq u^*(\psi_1(T_g x), \psi_1(x))\}$. Next we define $N^* = N_1^* \cup T_G(N_2 \cup N_3)$ and $M^* = \Gamma_1(\{y \in Y_1 : q_y(N^*) \neq 0\} \cup \{y \in Y_1 : q_y \text{ is atomic}\})$. It is easy to see from (7.18), Corollary 7.5, and from the quasi-invariance of ρ_1 under Γ_1, that $\mu(N^*) = \rho_1(M^*) = 0$. We denote by $\pi_1: \tilde{X} \to X$ the projection onto the first coordinate, and put $M_y = \tau_y \pi_1^{-1}$ for every $y \in Y_1 \setminus M^*$. From Lemma 7.7 we see that each M_y is a probability measure on $(X, \mathcal{S})$ which is quasi-invariant and ergodic for T_G. The choice of M^* together with (7.79) implies that $M_y(N^*) = 0$ for every $y \in Y_1 \setminus M^*$. In particular, we get, for every $y \in Y_1 \setminus M^*$,

$$a_o(g,x) = u^*(\psi_1(T_g x), \psi_1(x)) \tag{7.90}$$

for every $g \in G$ and for M_y-a.e. $x \in X$. We now fix $y \in Y_1 \setminus M^*$, and put

$$c_y(x) = \begin{cases} u^*(\psi_1(x), y) & \text{for } x \in \psi_1^{-1}(\Gamma_1 y) \setminus N_2, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that

$$u^{*}(\psi_1(T_g x), \psi_1(x)) = c_y(T_g x) - c_y(x) \tag{7.91}$$

for every $g \in G$ and for every $x \in \psi_1^{-1}(\Gamma_1 y) \setminus N_2$. Another look at (7.79) shows that $M_y(\psi_1^{-1}(\Gamma_1 y)) = 1$. (7.90) and (7.91) now imply that a_o is a coboundary for the action T_G on $(X, \mathcal{S}, M_y)$ with $y \in Y_1 \setminus M^{*}$. From the definition of $N_1 \subset N^{*}$ we see that $a: G \times X \rightarrow A$ is again a coboundary for T_G on $(X, \mathcal{S}, M_y)$ for $y \in Y_1 \setminus M^{*}$. We use Zorn's lemma to find a set $\Xi \subset Y_1 \setminus M$ which intersects each orbit of Γ_1 in exactly one point. Ξ will be uncountable, and (7.79) shows that M_y and $M_{y'}$ are mutually singular whenever $y \neq y'$ and $y, y' \in \Xi$. So far we have proved (1) and (3). To prove (2) we remark that (7.79) shows the following: If $B \in \mathcal{S}$ satisfies $M_y(B) = 0$ for every $y \in \Xi$, then we must also have $q_y(B) = 0$ for every $y \in Y_1 \setminus M^{*}$. From $\rho_1(M^{*}) = 0$ and from (7.18) we get $\mu(B) = 0$, which proves (2). The corollary is proved completely.

<u>Lemma 7.24</u>. Let T_G, $(X, \mathcal{S}, \mu)$ and A be given as in Theorem 7.23, and let $\mathcal{B}$ denote the Borel field and λ the Haar measure on A. Let furthermore $a_o: G \times X \rightarrow A$ be a lacunary cocycle for T_G with $\bar{E}(a_o) = \{0, \infty\}$, and let $T_G^{a_o}$ denote the skew product action of G on $(X, \mathcal{S}, \mu)$ as defined in (5.1) - (5.3) and (5.6). We define a probability space $(Y_1, \mathcal{T}_1, \rho_1)$, a Borel map $\psi_1: X \rightarrow Y_1$, and a countable group Γ_1 of automorphisms of $(Y_1, \mathcal{T}_1, \rho_1)$, and a transient orbital cocycle $u^{*}: R(\Gamma_1) \rightarrow A$ for Γ_1 as in Theorem 7.22 and satisfying (7.87) and (7.88). Put $\tilde{Y}_1 = Y_1 \times A$,

$\tilde{\mathcal{F}}_1 = \mathcal{F}_1 \times \mathcal{B}$. On the standard Borel space $(\tilde{Y}_1, \tilde{\mathcal{F}}_1)$ we define an equivalence relation $\sim$ by

$$(y, \alpha) \sim (y', \alpha') \tag{7.92}$$

if and only if there exists a $V \in \Gamma_1$ with

$$Vy = y'$$

and (7.93)

$$u^*(y',y) = \alpha - \alpha'.$$

For every $(y, \alpha) \in \tilde{Y}$, we write

$$[y, \alpha] = \{(Vy, u^*(Vy,y) + \alpha) : V \in \Gamma_1\}$$

for the equivalence class containing (y, α). Put

$$Y_1^* = \{[y, \alpha] : (y, \alpha) \in \tilde{Y}_1\}$$

and define

$$j : \tilde{Y}_1 \to Y_1^*$$

by

$$j(x, \alpha) = [x, \alpha].$$

We write $\mathcal{F}_1^*$ for the quotient Borel field on Y_1^* ($\mathcal{F}_1^*$ is the largest σ-algebra of subsets of Y_1^* which makes j measurable with respect to $\tilde{\mathcal{F}}_1$ and $\mathcal{F}_1^*$). Then $(Y_1^*, \mathcal{F}_1^*)$ is again a standard Borel space. We choose and fix a probability measure λ' on A with $\lambda' \sim \lambda$, and put $\tilde{\tilde{\rho}}_1 = \rho_1 \times \lambda'$.
If

$$\rho_1^* = \tilde{\tilde{\rho}}_1 \cdot j^{-1},$$

$(Y_1^*, \mathcal{T}_1^*, \rho_1^*)$ becomes a nonatomic probability space. Furthermore there exists a family $\{q_{y^*}^* : y^* \in Y_1^*\}$ of σ-finite measures on $(\tilde{X}, \tilde{\mathcal{S}})$ which satisfies the following conditions.

(1) For every $B \in \tilde{\mathcal{S}}$, the map $y \to q_{y^*}^*(B)$ is Borel from Y_1^* to $\overline{\mathbb{R}}$.

(2) For every $B \in \tilde{\mathcal{S}}$, we have

$$\tilde{\mu}(B) = \int q_{y^*}^*(B) \, d\rho_1^*(y^*).$$

(3) For every $(y, \alpha) \in \tilde{Y}_1$, we have

$$\overline{\tau}_{(y,\alpha)} \sim q^*_{j(y,\alpha)}.$$

In particular, every $q_{y^*}^*$, $y^* \in Y_1^*$, is quasi-invariant and ergodic under $T_G^{a_o}$, and $q_{y^*}^*$ and $q_{y'^*}^*$ are mutually singular whenever $y^* \neq y'^*$.

(4) Let $\mathcal{Z} = \{B \in \tilde{\mathcal{S}} : T_G^{a_o} B = B\}$. For every $B \in \mathcal{Z}$, put $B_{Y_1^*} = \{y^* \in Y_1^* : q_{y^*}^*(B) > 0\}$. Then $\mathcal{Z}_{Y_1^*} = \{B_{Y_1^*} : B \in \mathcal{Z}\}$ is equal to $\mathcal{T}_1^*$ modulo sets of ρ_1^*-measure zero.

Proof: We shall first prove the existence of a Borel set $D \subset \tilde{Y}_1$ which intersects each equivalence class $[y, \alpha]$ in exactly one point. For every $V \in \Gamma_1$, let V^{u^*} be the Borel automorphism of $(Y_1^*, \mathcal{T}_1^*)$ given by

$$V^{u^*}(y, \alpha) = (Vy, \alpha + u^*(Vy, y)),$$

and let $\Gamma_1^{u^*} = \{V^{u^*} : V \in \Gamma_1\}$. $\Gamma_1^{u^*}$ is simply the skew product of Γ_1 defined by the cocycle u^*. From (7.44) and from Lemma 7.20 we see that $u^*(Vy,y) \notin N_1(0)$ whenever $Vy \neq y$. We choose a dense sequence $(\beta_1, \beta_2, \ldots)$ in A, define $\mathcal{O}_i = Y_1 \times N_3(\beta_i)$ (cf. (7.47) - (7.48)), and get $V^{u^*}\mathcal{O}_i \cap \mathcal{O}_i \cap \{(y, \alpha) : V^{u^*}(y, \alpha) \neq (y, \alpha)\} = \emptyset$ for every $V^{u^*} \in \Gamma_1^{u^*}$ and $i \geq 1$. Put

$$D_1 = \mathcal{O}_1$$

and

$$D_i = \mathcal{O}_i \setminus \bigcup_{k=1}^{i-1} \Gamma_1^{u^*} \mathcal{O}_k.$$

The set $D = \bigcup_{i \geq 1} D_i$ will then intersect each equivalence class $[y, \alpha]$ in exactly one point. Hence the restriction j_D of j to D is a Borel isomorphism from D to $(Y_1^*, \mathcal{T}_1^*)$, so that Y_1^* is standard Borel. The Borel isomorphism j_D will carry the measure $\tilde{\rho}_{1_D}$ on $(D, \tilde{\mathcal{T}}_{1_D})$ to a measure equivalent to ρ_1^* on $(Y_1^*, \mathcal{T}_1^*)$. Consider now, for any $B \in \tilde{\mathcal{S}}$, the map $(y, \alpha) \longrightarrow \tau_{(y, \alpha)}(B)$, $(y, \alpha) \in D$. The properties of $\{\sigma_{(y, \alpha)} : (y, \alpha) \in \tilde{Y}_1\}$ (cf. (7.35) - (7.37)) together with the definition of $\tau_{(y, \alpha)}$ (cf. (7.38) - (7.40)) imply that this map is Borel. This immediately shows that the map $[y, \alpha] \longrightarrow \tau_{j_D^{-1}[y, \alpha]} = q^{**}_{[y, \alpha]}$ is Borel on $(Y_1^*, \mathcal{T}_1^*)$. Furthermore, each $q^{**}_{a[y, \alpha]}$, $[y, \alpha] \in Y_1^*$, is quasi-invariant and ergodic for T_{G^o} (Lemma 7.7). The equivalence relation (7.92) - (7.93)

satisfies $(y,\alpha) \sim (y',\alpha')$ if and only if $\tau_{(y,\alpha)} \sim \tau_{(y',\alpha')}$ (cf. Lemmas 7.13 and 7.20). Since D intersects each equivalence class in only one point, we see that $q^{**}_{[y,\alpha]} \not\sim q^{**}_{[y',\alpha']}$ whenever $[y,\alpha] \neq [y',\alpha']$.
Let now $B \in \mathcal{Z}$. (7.79) implies that $\{(y,\alpha) \in \tilde{Y}_1 : \sigma_{(y,\alpha)}(B)=1\}$ is $\Gamma_1^{u^*}$-invariant. Moreover, we have $\{(y,\alpha) \in D : \rho_{(y,\alpha)}(B)=1\} = \{(y,\alpha) \in D : \sigma_{(y,\alpha)}(B)=1\}$. In particular we see from (7.29) and (7.36) that $\tilde{\mu}(B)>0$ if and only if $\tilde{\rho}_{1_D}(\{(y,\alpha) \in D : \rho_{(y,\alpha)}(B)=1\})>0$ and hence if $\rho_1^*(\{[y,\alpha] \in Y_1^* : q^{**}_{[y,\alpha]}(B)=1\}) >0$. Let, for every $B \in \tilde{\mathcal{S}}$,

$$m^*(B) = \int q^{**}_{[y,\alpha]}(B)\, d\rho_1([y,\alpha]).$$

It is clear that m^* is equivalent to the measure

$$\int_D \tau_{(y,\alpha)}\, d\tilde{\rho}_{1_D}(y,\alpha)$$

and hence absolutely continuous with respect to $\tilde{\mu}$. If $m^* \not\sim \tilde{\mu}$, there exists a set $C \subset \tilde{X}$ with $\tilde{\mu}(C)>0$ and $m^*(C)=0$. Since both $\tilde{\mu}$ and m^* are quasi-invariant under $T_G^{a_0}$, we may assume that $C \in \mathcal{Z}$. But we have just proved that $C \in \mathcal{Z}$ and $\tilde{\mu}(C)>0$ implies $\rho_1^*(\{[y,\alpha] \in Y_1^* : q^{**}_{[y,\alpha]}(C)=1\}) = m^*(C)>0$ and the resulting contradiction implies $m^* \sim \tilde{\mu}$.
Finally, let $\tilde{\psi}_1 : \tilde{X} \to \tilde{Y}_1$ denote the map $\tilde{\psi}_1(x,\alpha) = (\psi_1(x),\alpha)$. If $C \in \mathcal{S}_1^*$, put $C_1 = j^{-1}(C)$ and $B = \psi_1^{-1}(\Gamma_1^{u^*} C_1)$. Applying once again (7.79), the ergodicity of the measures $\tau_{(y,\alpha)}$, and the fact that $\sigma_{(y,\alpha)}(\tilde{\psi}_1^{-1}(\{(y,\alpha)\}))=1$ for every $(y,\alpha) \in \tilde{Y}_1$, we see that, for every $(y,\alpha) \in \tilde{Y}_1$, $\tau_{(y,\alpha)}(T_G^{a_0}B)=1$

if and only if $\tau_{(y, \alpha)}(B) = 1$. This in turn implies

$$\{[y,\alpha] \in Y_1^*: q^{**}_{[y,\alpha]}(T_G^{a_o}B)=1\} =$$

$$= j(\{(y,\alpha) \in D: \tau_{(y,\alpha)}(B)=1\})$$

$$= j(D \cap \Gamma_1^{u^*} C_1) = j(C_1) = C.$$

We have thus proved the following:

(a) For every $B \in \widetilde{\mathcal{S}}$, the map $[y,\alpha] \to q^{**}_{[y,\alpha]}(B)$ is Borel on $(Y_1^*, \mathcal{T}_1^*)$.

(b) The measure $m^* = \int q^{**}_{[y,\alpha]} \, d\rho_1^*([y,\alpha])$ is equivalent to $\tilde{\mu}$.

(c) $q^{**}_{[y,\alpha]}$ is quasi-invariant and ergodic under $T_G^{a_o}$ for every $(y,\alpha) \in \widetilde{Y}_1$.

(d) $q^{**}_{[y,\alpha]}$ and $q^{**}_{[y',\alpha']}$ are mutually singular whenever $[y,\alpha] \neq [y',\alpha']$.

(e) For every $B \in \mathcal{Z}$, put $B_{Y_1^*} = \{[y,\alpha] \in Y_1^*: q^{**}_{[y,\alpha]}(B) = 1\}$. Then $\{B_{Y_1^*}: B \in \mathcal{Z}\}$ is equal to $\mathcal{T}_1^*$.

The assertion of the lemma now follows by choosing a Borel map $f: \widetilde{X} \to \mathbb{R}$ with $\frac{d\tilde{\mu}}{dm^*}(x) = e^{f(x)}$ for m^*-a.e. $x \in X$, and by setting

$$dq^*_{[y,\alpha]}(x) = e^{f(x)} . dq^{**}_{[y,\alpha]}(x)$$

for every $x \in X$ and for every $[y,\alpha] \in Y_1^*$. The proof is complete.

Corollary 7.25. Let $(X, \mathcal{S}, \mu)$ be a nonatomic measure space and let T_G be an ergodic measure preserving action of a countable group G on $(X, \mathcal{S}, \mu)$. Suppose there exists a locally compact second countable abelian group A and a recurrent cocycle $a: G \times X \to A$ for T_G with $\bar{E}(a) = \{0, \infty\}$. Then there exists an uncountable family $\{\bar{M}_\xi : \xi \in \Xi\}$ of mutually inequivalent nonatomic σ-finite measures on $(X, \mathcal{S})$ such that

(1) for every $\xi \in \Xi$, $\bar{M}_\xi$ is invariant and ergodic under T_G,

(2) if $B \in \mathcal{S}$ satisfies $\bar{M}_\xi(B) = 0$ for every $\xi \in \Xi$, then $\mu(B) = 0$,

(3) for every $\xi \in \Xi$, the cocycle a is a coboundary for the action T_G on $(X, \mathcal{S}, \bar{M}_\xi)$.

Proof: If there exists a cocycle $a: G \times X \to A$ with the properties stated, then there will also exist a lacunary orbital cocycle $u_0: R(T_G) \to A$ which is recurrent, and which satisfies $a_{u_0} \sim a$ and hence $\bar{E}(u_0) = \{0, \infty\}$. We put $a_0 = a_{u_0}$. Let us assume for the moment that μ is a probability measure, and put $\mu = \mu'$ in (7.16). Applying (7.17) - (7.21), we find a probability space $(Y, \mathcal{T}, \rho)$ and a family $\{q_y : y \in Y\}$ of H_0-invariant and H_0-ergodic measures on $(X, \mathcal{S})$. In this case we put $\bar{q}_y = q_y$ for every $y \in Y$. Turning now to the case where μ is infinite, we choose an equivalent probability measure μ' in (7.16) and apply again (7.17) - (7.21) to construct $(Y, \mathcal{T}, \rho)$ and $\{q_y : y \in Y\}$. Now each of the measures q_y will be equivalent to some σ-finite H_o-invariant measure $\bar{q}_y$ on $(X, \mathcal{S})$. Having defined $(Y, \mathcal{T}, \rho)$ and $\{\bar{q}_y : y \in Y\}$ both in the case where μ is totally finite and where it is infinite, we let μ again be an arbitrary σ-finite T_G-invariant and T_G-ergodic measure. Lemma 7.24, Corollary 6.9 and Exercise 6.13 together imply that for ρ_1-a.e.$(y, \alpha) \in \widetilde{Y}_1$, the measure $\tau_{(y,\alpha)}$ is equivalent to a σ-finite $\overset{a}{T}_G{}^{o}$-invariant measure on $(\widetilde{X}, \widetilde{\mathcal{S}})$. An application of Fubini's theorem theorem shows that in fact for ρ_1-a.e.$y \in Y_1$ τ_y is equivalent to a σ-finite $\overset{a}{T}_G{}^{o}$-invariant measure on $(X, \mathcal{S})$. Without loss in generality we assume that every τ_y, $y \in Y_1$, is equivalent to such a σ-finite $\overset{a}{T}_G{}^{o}$-invariant measure $\bar{\tau}_y$. Since $H_o \subset [\overset{a}{T}_G{}^{o}]$, each $\bar{\tau}_y$ will also be invariant under H_o (cf. Exercises 1.2 and 1.3). For every $(y, \alpha) \in \widetilde{Y}_1$, we define $\bar{\sigma}_{(y,\alpha)}$ as in (7.26) and (7.35) with $\bar{q}_y$ replacing q_y. From the invariance of $\bar{\tau}_y$ under H_o, and from (7.79) we conclude that, for every $y \in Y_1$, there exist positive real numbers $\{c_y(\alpha) : \alpha \in A(y)\}$, $y \in Y_1$, with

$$\bar{\tau}_y = \sum_{\alpha \in A(y)} c_y(\alpha) \cdot \bar{\sigma}(\phi(y, \alpha), \alpha). \tag{7.94}$$

Since $\phi(y, \alpha_1) \neq \phi(y, \alpha_2)$ whenever $\alpha_1, \alpha_2 \in A(y)$, $\alpha_1 \neq \alpha_2$, and $y \in Y_1$, the measures $\bar{q}_{\phi(y,\alpha_1)}$ and $\bar{q}_{\phi(y,\alpha_2)}$ will be mutually singular. If $\pi_1 : \tilde{X} \to X$ is the first coordinate projection, we put

$$\bar{M}_y = \bar{\tau}_y \pi_1^{-1} = \sum_{\alpha \in A(y)} c_y(\alpha) . \bar{q}_{\phi(y, \alpha)}, \tag{7.95}$$

and note that $\bar{M}_y$ is σ-finite for every $y \in Y_1$, since it is a sum of mutually singular σ-finite measures on $(X, \mathcal{S})$. Clearly each $\bar{M}_y$ is T_G-invariant and T_G-ergodic, and $\bar{M}_y$ and $\bar{M}_{y'}$ will be mutually singular whenever $\Gamma_1 y \neq \Gamma_1 y'$.
The proof is completed in the same way as in Corollary 7.23.

PART II. THE COHOMOLOGY OF A HYPERFINITE ACTION

§ 8 Hyperfinite actions

The main interest of ergodic theory has always been concentrated on the analysis of single automorphisms V of a measure space $(X, \mathcal{S}, \mu)$. Since every $V \in \mathcal{A}(X, \mathcal{S}, \mu)$ defines a $\mathbb{Z}$-action $n \to V^n$, $n \in \mathbb{Z}$, we are led to consider $\mathbb{Z}$-actions on $(X, \mathcal{S}, \mu)$. To simplify our notation, we denote by $[V]$ the full group of the action $n \to V^n$, $n \in \mathbb{Z}$, and by R(V) its equivalence relation. If T_G is an action of a countable group G on a measure space $(X', \mathcal{S}', \mu')$ which is weakly equivalent to $n \to V^n$, $n \in \mathbb{Z}$, we shall say that T_G is weakly equivalent to V. The statement that $V \in \mathcal{A}(X, \mathcal{S}, \mu)$ and $V' \in \mathcal{A}(X', \mathcal{S}', \mu')$ are weakly equivalent has to be read similarly. We call V ergodic, conservative, type I or aperiodic according as $n \to V^n$, $n \in \mathbb{Z}$, is ergodic, conservative, type I or free.

<u>Definition 8.1</u>. Let T_G be an action of a countable group G on a measure space $(X, \mathcal{S}, \mu)$. T_G is called <u>hyperfinite</u> if there exists a $V \in [T_G]$ with $\{V^k x : k \in \mathbb{Z}\} = T_G x$ for μ-a.e. $x \in X$. (8.1)

Obviously hyperfiniteness is an invariant of weak equivalence.

The aim of Part II is an analysis of hyperfinite actions and of their cohomology. We start with some measure theoretic lemmas which will often be used subsequently. Until we change our assumptions explicitly, T_G will denote an ergodic and not necessarily hyperfinite action of a countable group G on a nonatomic measure space $(X, \mathcal{S}, \mu)$. We fix an orbital cocycle $u_\mu : R(T_G) \to \mathbb{R}$ with

$$u_\mu (T_g x, x) = \log \frac{d\mu T_g}{d\mu}(x) \tag{8.2}$$

for μ-a.e. $x \in X$ and for every $g \in G$, and define the cocycle $a_\mu : G \times X \to \mathbb{R}$ in (2.7) by

$$a_\mu (g,x) = u_\mu (T_g x, x). \tag{8.3}$$

Lemma 8.2. Let $B, C \in \mathcal{S}$ with $\mu(B) > 0$, $\mu(C) > 0$. Suppose there exist Borel sets $B_0 \subset B$ and $C_0 \subset C$ with $\mu(B \setminus B_0) = \mu(C \setminus C_0) = 0$ and injective Borel maps

$$U : B_0 \to C,$$
$$V : C_0 \to B$$

such that

$$Ux \in T_G x \quad \text{for every } x \in B_0$$

and

$$Vx \in T_G x \quad \text{for every } x \in C_0.$$

Then there exists a Borel set $B_1 \subset B_0$ with $\mu(B \setminus B_1) = 0$

and an injective Borel map

$$W: B_1 \to C$$

with

$$Wx \in T_G x \quad \text{for every } x \in B_1$$

and

$$\mu(C \setminus WB_1) = 0.$$

Proof: Let $N_1 = \{x \in B_o : Ux \notin C_o\} \cup \{x \in C_o : Vx \notin B_o\}$, and let $N = T_G N_1$. We define $B_1 = B_o \setminus N$ and $B_2 = V(C_o \setminus N)$. Proceeding by induction, we set $B_{n+2} = VUB_n$ for every $n \geq 1$. The sequence $(B_k, k=1,2,\ldots)$ is nonincreasing, so that

$$B_1 = \bigcup_{n \geq 1} (B_n \setminus B_{n+1}) \cup \bigcap_{n \geq 1} B_n$$

and

$$B_2 = \bigcup_{n \geq 2} (B_n \setminus B_{n+1}) \cup \bigcap_{n \geq 1} B_n.$$

Since we also have $VU(B_n \setminus B_{n+1}) = B_{n+2} \setminus B_{n+3}$ for every n, we can define a Borel isomorphism $W_1 : B_1 \to B_2$ by

$$W_1 x = \begin{cases} VUx & \text{for } x \in \bigcup_{n \geq 1} (B_{2n-1} \setminus B_{2n}) \\ x & \text{for } x \in \bigcup_{n \geq 1} (B_{2n} \setminus B_{2n+1}) \cup \bigcap_{n \geq 1} B_n. \end{cases}$$

Clearly we have $W_1 x \in T_G x$ for every $x \in B_1$. If we now put $Wx = V^{-1} W_1 x$ for every $x \in B_1$, we get a Borel isomorphism $W: B_1 \to C_o \setminus N$ with $Wx \in T_G x$ for every $x \in B_1$. It is clear that W has all the required properties. The proof is complete.

Lemma 8.3. Suppose μ is not equivalent to any σ-finite T_G-invariant measure on $(X,\mathcal{S})$. Let $B,C\in\mathcal{S}$ with $\mu(B)>0$, $\mu(C)>0$. Then there exists a Borel set $B_o\subset B$ with $\mu(B\setminus B_o)=0$ and an injective Borel map $U:B_o\to C$ with $Ux\in T_G x$ for every $x\in B_o$.

Proof: The statement of the lemma will not be affected if we replace μ by an equivalent probability measure μ'. If $E\subset B$, $F\subset C$ are Borel sets with $\mu'(E).\mu'(F)\neq 0$, we can choose a Borel set $E_1\subset E$ with $\mu'(E_1)>0$ and a $g_1\in G$ such that $T_{g_1}E_1\subset F$. Decreasing E_1 if necessary, we may also assume that $a_{\mu'}(g_1,x)\leq m_o$ for some $m_o\in\mathbb{R}$ and for all $x\in E_1$. Our assumption on μ implies that a_μ and hence $a_{\mu'}$ is not a coboundary. Consequently one can choose a Borel set $E_2\subset E_1$ and a $g_2\in G$ such that $\mu'(E_2)>0$, $T_{g_2}E_2\subset E_1$, and $a_{\mu'}(g_2,x)\leq \log\mu'(C)-\log\mu'(B)-m_o$ for every $x\in E_2$. We now put $H=E_2$, $h=g_1g_2$, and get $\mu'(H)>0$, $H\subset E$, $T_hH\subset F$, and $a_{\mu'}(h,x)\leq\log\mu'(C)-\log\mu'(B)$ for every $x\in H$. The last condition implies

$$\mu'(T_hH)=\int_H\frac{d\mu'T_h}{d\mu'}\,d\mu'\leq\mu'(H).\,\mu'(C)/\mu'(B).$$

We have proved the following intermediate assertion:

(I_1) Let $E\subset B$ and $F\subset C$ be Borel sets of positive measure. Then there exists a Borel set $H\subset E$ and an element $h\in G$ such that $\mu'(H)>0$, $T_hH\subset F$, and $\mu'(T_hH)\leq\mu'(H).\mu'(C)/\mu'(B)$.

Using induction and an exhaustion argument, (I_1) allows us to construct a sequence of disjoint Borel sets $(H_i,\ i=1,2,\dots)$ and a sequence $(h_i,\ i=1,2,\dots)$ in G such that $\mu'(H_i)>0$, $T_{h_i}H_i \subset C$, $T_{h_i}H_i \cap T_{h_j}H_j = \emptyset$ for $i\neq j$, $\mu'(T_{h_i}H_i) \leq \mu'(H_i)\mu'(C)/\mu'(B)$, and $\mu'(B \setminus \bigcup H_i) = 0$. Put $B_o = \bigcup H_i$ and define an injective Borel map $U: B_o \to C$ by

$$Ux = T_{h_i}x \quad \text{whenever } x \in H_i,\ i \geq 1.$$

U will satisfy our conditions, and the proof is complete.

Lemma 8.4. Suppose μ is not equivalent to any σ-finite T_G-invariant measure on $(X, \mathcal{S})$. If $B, C \in \mathcal{S}$ are disjoint sets with $\mu(B)>0$ and $\mu(C)>0$, then there exists a $W \in [T_G]$ with

$$WB = C \pmod 0 \qquad (8.4)$$

and

$$W^2 = \text{identity}. \qquad (8.5)$$

Proof: We apply Lemma 8.3 to find a Borel set $B_o \subset B$ with $\mu(B \setminus B_o) = 0$ and an injective Borel map $U: B_o \to C$ with $Ux \in T_G x$ for every $x \in B_o$. If we reverse the roles of B and C, we get a Borel set $C_o \subset C$ and an injective Borel map $V: C_o \to B$ with $\mu(C \setminus C_o) = 0$ and with $Vx \in T_G x$ for every $x \in C_o$. Lemma 8.2 now implies the existence of a set $B_1 \subset B$ with $\mu(B \setminus B_1) = 0$ and of an injective Borel map $W_1: B_1 \to C$ such that $W_1 x \in T_G x$ for every $x \in B_1$, and $\mu(C \setminus W_1 B_1) = 0$. We conclude the proof by setting

$$Wx = \begin{cases} W_1 x & \text{for } x \in B_1, \\ W_1^{-1} x & \text{for } x \in W_1 B_1, \\ x & \text{otherwise.} \end{cases}$$

<u>Lemma 8.5</u>. Suppose μ is invariant under T_G and $B, C \in \mathcal{S}$ are disjoint sets with $0 < \mu(B) = \mu(C)$. Then there exists a $W \in [T_G]$ with $WB = C$ (mod 0) and $W^2 =$ identity.

Proof: We first assume that $0 < \mu(B) = \mu(C) < \infty$. One can then immediately see that the following is true:

(I_2) For every pair of Borel sets $E \subset B$, $F \subset C$ of positive measure there exists a Borel set $H \subset E$ and an element $h \in G$ such that $\mu(H) > 0$ and $T_h H \subset F$.

As in Lemma 8.3, we can use (I_2), induction, and an exhaustion argument to construct a sequence (H_i, $i=1,2,\ldots$) of disjoint Borel sets of positive measure, and a sequence (h_i, $i=1,2,\ldots$) in G such that $H_i \subset B$, $T_{h_i} H_i \subset C$, $T_{h_i} H_i \cap T_{h_j} H_j = \emptyset$ for $i \neq j$, and $\mu(B \setminus \bigcup H_i) = 0$. We put $B_0 = \bigcup H_i$ and set

$$W_1 x = T_{h_i} x \text{ whenever } x \in H_i,\ i \geq 1.$$

It is clear that $W_1 x \in T_G x$ for every $x \in B_0$, and that $\mu(C \setminus W_1 B_0) = 0$. If we now put

$$Wx = \begin{cases} W_1 x & \text{for } x \in B_0, \\ W_1^{-1} x & \text{for } x \in W_1 B_0, \\ x & \text{otherwise,} \end{cases}$$

we have proved the lemma in this special case.
Turning now to the situation where $\mu(B) = \mu(C) = \infty$, we choose sequences $(B^{(k)}, k=1,2,\dots)$ and $(C^{(k)}, k=1,2,\dots)$ in $\mathcal{S}$ such that $B^{(k)} \subset B$, $C^{(k)} \subset C$, $B^{(k)} \cap B^{(l)} = C^{(k)} \cap C^{(l)} = \emptyset$ for every $k \neq l$, $\mu(B^{(k)}) = \mu(C^{(k)}) = 1$, $B = \bigcup_k B^{(k)}$, $C = \bigcup_k C^{(k)}$,
Using the first part of this proof, we construct Borel sets $B_o^{(k)} \subset B^{(k)}$, $k=1,2,\dots$ with $\mu(B^{(k)} \setminus B_o^{(k)}) = 0$ for every $k \geq 1$, and injective Borel maps $W_k : B_o^{(k)} \to C^{(k)}$ satisfying $W^{(k)}x \in T_G x$ for every $x \in B_o^{(k)}$ and every $k \geq 1$.
If we now define

$$Wx = \begin{cases} W_k x & \text{for } x \in B_o^{(k)},\ k \geq 1, \\ W_k^{-1} x & \text{for } x \in W_k B_o^{(k)},\ k \geq 1, \\ x & \text{for } x \notin \bigcup_{k \geq 1} (B_o^{(k)} \cup W_k B_o^{(k)}), \end{cases}$$

we have proved the lemma.

We can now prove an important property of hyperfiniteness.

<u>Proposition 8.6.</u> Let T_G be an ergodic action of a countable group G on a nonatomic measure space $(X, \mathcal{S}, \mu)$. Suppose there exists a set $B \in \mathcal{S}$ with $\mu(B) > 0$ such that the action $T(G,B)$ (defined in Exercise 1.4) is hyperfinite. Then T_G is hyperfinite. Conversely, if T_G is hyperfinite, then $T(G,B)$ is hyperfinite for every Borel set B in X of positive measure.

Proof: We first assume that T_G is hyperfinite and show that $T(G,B)$ is hyperfinite for every $B \in \mathcal{S}$ with $\mu(B) > 0$. Indeed, if T_G is hyperfinite, we can choose $V \in [T_G]$ with $\{V^k x : k \in \mathbb{Z}\} = T_G x$ for μ-a.e. $x \in X$. V will be ergodic by Exercise 1.6, and hence conservative. Let B be a Borel set of positive measure, and let $N_+ = \{x \in B : V^n x \notin B$ for every $n \geq 1\}$, $N_- = \{x \in B : V^n x \notin B$ for every $n \leq -1\}$, and $N = T_G(N_+ \cup N_-)$. For every $x \in B$, put

$$n(x) = \begin{cases} \min \{n \geq 1 : V^n x \in B\} & \text{if } x \in B \setminus N, \\ 0 & \text{otherwise,} \end{cases} \tag{8.6}$$

and

$$V_B x = V^{n(x)} x. \tag{8.7}$$

V_B is the transformation induced by V on B. It is easy to see that $V_B \in [T(G,B)] = [T_G]_B$, and that $\{V_B^k x : k \in \mathbb{Z}\} = T(G,B)x$ for μ_B-a.e. $x \in B$. Hence $T(G,B)$ is hyperfinite. To prove the converse, assume that $T(G,B)$ is hyperfinite for some Borel set B of positive measure. The first part of the proof implies that $T(G,B_1)$ is hyperfinite for every Borel set $B_1 \subset B$ with $\mu(B_1) > 0$. We now have to distinguish between three cases.

(a) Assume that μ is equivalent to an infinite T_G-invariant measure μ'. We choose a Borel set $B_1 \subset B$ with $0 < \mu'(B_1) < \infty$. Scaling μ' if necessary, we assume that $\mu'(B_1) = 1$. Let $f : B_1 \to \mathbb{Z}$ be a Borel map with $f(x) \geq 1$

for all $x \in B_1$ and with $\int f \, d\mu'_{B_1} = \infty$. We can find Borel sets $\{B_{k,l}: k=1,2,\ldots, l=1,2,\ldots,k\}$ in X such that

$$B_{k,l} \cap B_{k',l'} = \emptyset \text{ whenever } (k,l) \neq (k',l'),$$

$$B_{k,l} \cap B_1 = \emptyset \text{ for all } k,l,$$

$$\mu'(B_{k,l}) = \mu'_{B_1}(f^{-1}(\{k\})) \text{ for all } k,l,$$

and

$$X = B_1 \cup \bigcup_{k,l} B_{k,l}.$$

For every $k \geq 1$, we put $B_{k,0} = f^{-1}(\{k\}) \subset B_1$. Applying Lemma 8.5 we find maps $W_{k,l} \in [T_G]$, $k=1,2,\ldots$, $l=1,\ldots,k$, with

$$W_{k,l}B_{k,0} = B_{k,l} \pmod 0,$$

and

$$W_{k,l}^2 = \text{identity}.$$

We now choose a $V \in [T(G,B_1)]$ with $\{V^k x: k \in \mathbb{Z}\} = T(G,B_1)x$ for μ_{B_1}-a.e.$x \in B_1$, and put $N_{k,l} = B_{k,0} \,\Delta\, W_{k,l}B_{k,l}$ for every k,l. Let $N = T_G(\bigcup_{k,l} N_{k,l})$, and define

$$Wx = \begin{cases} W_{k,1}x & \text{if } x \in B_{k,0} \setminus N,\ k \geq 1, \\ W_{k,l+1}W_{k,l}x & \text{if } x \in B_{k,l} \setminus N,\ k \geq 1,\ 1 \leq l \leq k-1, \\ VW_{k,k}x & \text{if } x \in B_{k,k} \setminus N,\ k \geq 1, \\ x & \text{if } x \in N. \end{cases}$$

Then $W \in [T_G]$ and $\{W^k x: k \in \mathbb{Z}\} = T_G x$ for μ'-a.e.$x \in X$. The proof is complete under assumption (a).

(b) Suppose μ is equivalent to a T_G-invariant probability measure μ'. We choose a subset $B_1 \subset B$ with $\mu'(B_1) = 1/n$ for some positive integer n. There exist disjoint Borel sets $B_2,\dots,B_n$ in X with $\mu'(B_i) = 1/n$ for $i=2,\dots,n$, and with $X = \bigcup_{i=1}^{n} B_i$. Applying Lemma 8.5 we choose maps $W_i \in [T_G]$, $i=2,\dots,n$, with

$$W_i B_1 = B_i \pmod 0$$

and

$$W_i^2 = \text{identity}$$

for $i=2,\dots,n$. If $V \in [T(G,B_1)]$ satisfies $\{V^k x: k \in \mathbb{Z}\} = T(G,B_1)x$ for μ'_{B_1}-a.e.x, and if $N_i = B_1 \Delta W_i B_i$, $i=2,\dots,n$, and $N = T_G(\bigcup_{i=2}^{n} N_i)$, we define

$$Wx = \begin{cases} W_2 x & \text{for } x \in B_1 \setminus N, \\ W_{i+1} W_i x & \text{for } x \in B_i \setminus N,\ i=2,\dots,n-1, \\ VW_n x & \text{for } x \in B_n \setminus N, \\ x & \text{for } x \in N. \end{cases}$$

Again we have $W \in [T_G]$ and $\{W^k x: k \in \mathbb{Z}\} = T_G x$ for μ'-a.e. $x \in X$, so that T_G is hyperfinite.

(c) Suppose finally that μ is not equivalent to any σ-finite T_G-invariant measure on $(X, \mathcal{S})$. To avoid trivialities we assume that $\mu(X \setminus B) > 0$. By Lemma 8.4 there exists a $W_1 \in [T_G]$ with $W_1 B = \complement B \pmod 0$ and with $W_1^2 = \text{identity}$. Choose $V \in [T(G,B)]$ with $\{V^k x: k \in \mathbb{Z}\} =$

$T(G,B)x$ for μ_B-a.e. $x \in B$, and put $N = T_G(B \triangle W_1 B)$. Again we define

$$Wx = \begin{cases} W_1 x & \text{if } x \in B \setminus N, \\ VW_1 x & \text{if } x \in \complement B \setminus N, \\ x & \text{if } x \in N. \end{cases}$$

Then $W \in [T_G]$, and $\{W^k x : k \in \mathbb{Z}\} = T_G x$ for μ-a.e. $x \in X$. Hence T_G is hyperfinite, and the proposition is proved completely.

The following result gives an interesting charcterization of hyperfinite actions in terms of their cohomology.

<u>Theorem 8.7</u>. Let T_G be an ergodic action of a countable group G on a nonatomic measure space $(X, \mathcal{S}, \mu)$. Then T_G is hyperfinite if and only if there exists a transient cocycle $a: G \times X \longrightarrow \mathbb{R}$ for T_G.

Proof: Assume that T_G is hyperfinite, and choose a $V \in [T_G]$ with $\{V^k x : k \in \mathbb{Z}\} = T_G x$ for μ-a.e. $x \in X$. V will be ergodic. Hence the set $N_1 = \{x \in X : V^k x = x \text{ for some } k \neq 0\}$ has measure zero. Put $N_2 = \{x \in X : \{V^k x : k \in \mathbb{Z}\} \neq T_G x\}$ and $N = T_G(N_1 \cup N_2)$. We define an orbital cocycle $u_1 : R(V)$ by

$$u_1(x,y) = \begin{cases} n & \text{if } y \in X \setminus N \text{ and } x = V^n y, \\ 0 & \text{if } (x,y) \in R_N(V). \end{cases}$$

Clearly u_1 is a transient cocycle for V. Hence

$$a(g,x) = \begin{cases} u_1(T_g x, x) & \text{if } x \in X \setminus N,\ g \in G, \\ 0 & \text{if } x \in N, \end{cases}$$

defines a transient cocycle $a: G \times X \to \mathbb{R}$ for T_G. To prove the converse, assume that T_G has a transient real valued cocycle a. By Exercise 3.20, there exists a Borel set B with $\mu(B) > 0$ and an $\varepsilon > 0$ such that

$$\bigcup_{g \in G} (B \cap T_g^{-1} B \cap \{x : |a(g,x)| < \varepsilon\} \cap \{x : T_g x \neq x\}) = \emptyset.$$

Lemma 4.1 now implies the existence of a $V \in [T_G]$ with $VB = B$ and with $\{V^k x : k \in \mathbb{Z}\} = T_G x \cap B = T(G,B)x$ for μ_B-a.e. $x \in B$. Hence $T(G,B)$ is hyperfinite, and Proposition 8.6 completes the proof of this theorem.

For the rest of this section we shall be concerned with some structure theorems for hyperfinite actions.

<u>Definition 8.8.</u> Let $(X, \mathcal{S}, \mu)$ be a nonatomic measure space and let $V \in \mathcal{A}(X, \mathcal{S}, \mu)$. V is called <u>set periodic</u> with period k if there exists a Borel partition $\mathcal{D}(V) = \{D_1, \dots, D_k\}$ of X with $D_i = V^{i-1} D_1$ (mod 0) for all $i=1,\dots,k$. V is called <u>weak von Neumann</u> (w.v.N), if there exists a sequence $\mathcal{D}_n(V) = \{D_1^{(n)}, \dots, D_{2^n}^{(n)}\}$ of Borel partitions of X with

$$(1) \quad D_i^{(n)} = D_i^{(n+1)} \cup D_{i+2^n}^{(n+1)} \quad \text{for all } i,n, \qquad (8.8)$$

(2) $D_i^{(n)} = V^{i-1} D_1^{(n)}$ (mod 0) for all i,n. (8.9)

If there exists a sequence $(\mathcal{D}_n(V),\ n=1,2,\ldots)$ of Borel partitions satisfying (8.8), (8.9), and

(3) the σ-algebra generated by $\{\mathcal{D}_n(V),\ n \geq 1\}$ is equal to $\mathcal{S}$ (mod 0) (8.10)

we shall call V a <u>von Neumann</u> (v.N.) transformation.

<u>Lemma 8.9</u>. Let $(X, \mathcal{S}, \mu)$ be a nonatomic measure space and let $V \in \mathcal{A}(X, \mathcal{S}, \mu)$ be ergodic. For every $0 < \varepsilon < \mu(X)$ there exists a $B \in \mathcal{S}$ with $B \cap VB = \emptyset$ and with $\mu(X \setminus (B \cup VB)) = \varepsilon$.

Proof: We choose a set $E \in \mathcal{S}$ with $\mu(E) \leq \varepsilon/2$ and with $\log \frac{d\mu V}{d\mu}(x) \leq 0$ for μ-a.e. $x \in E$. That this is possible follows from Theorem 4.2. For every $i \geq 1$, let $E_i = \{x \in E : V^k x \notin E$ for $1 \leq k < i$, but $V^i x \in E\}$. Since V is ergodic and hence conservative, we have $\mu(E \setminus \bigcup_{i=1}^{\infty} E_i) = \mu(X \setminus \bigcup_{i=1}^{\infty} \bigcup_{k=0}^{i-1} V^k E_i) = 0$. Put

$$B_1 = \bigcup_{i=2}^{\infty} \bigcup_{k=0}^{[\frac{i}{2}]-1} V^{2k} E_i,$$

where $[\frac{i}{2}]$ denotes the integral part of $\frac{i}{2}$. Then $B_1 \cap VB_1 = \emptyset$, and

$$\mu(X \setminus (B_1 \cup VB_1)) \leq \mu(V^{-1}E) = \int_E \frac{d\mu V}{d\mu}\, d\mu \leq \mu(E) \leq \varepsilon/2.$$

If we now choose a Borel set $B \subset B_1$ with $\mu(X \setminus (B \cup VB)) = \varepsilon$, we have proved the lemma.

For the following lemmas we assume $(X, \mathcal{S}, \mu)$ to be a non-atomic probability space until explicitly stated otherwise.

Lemma 8.10. Let $V \in \mathcal{A}(X, \mathcal{S}, \mu)$ be ergodic and assume that either

(1) μ is invariant under V,

or

(2) μ is not equivalent to any σ-finite V-invariant measure on $(X, \mathcal{S})$.

Then there exists a w.v.N. transformation $W \in [V]$ with $[W] = [V]$.

Proof: By Lemma 8.9 we choose a set $B_1 \in \mathcal{S}$ with $B_1 \cap VB_1 = \emptyset$ and $\mu(C_1) = 3/4$, where $C_1 = B_1 \cup VB_1$. Let $\tilde{V}_1$ and V_1 denote the transformations induced by V on B_1 and on C_1, respectively, i.e. $\tilde{V}_1 = V_{B_1}$ and $V_1 = V_{C_1}$. Then V_1 is set periodic with period 2. We continue by induction. Let $V_0 = V$ and suppose that, for each i=1,2,...,n, we have chosen sets B_i, C_i together with the corresponding induced transformations $\tilde{V}_i = V_{B_i}$, $V_i = V_{C_i}$ such that

(a) $C_i = \bigcup_{k=0}^{2^i-1} V_{i-1}^k B_i$,

(b) $V_{i-1}^k B_i \cap V_{i-1}^l B_i = \emptyset$ for $0 \leq k < l \leq 2^i-1$,

(c) $\mu(C_i) = 2^{-i-1} + 2^{-1}$

(d) $B_{i-1} \supset B_i \cup \tilde{V}_{i-1} B_i$, for every $i=1,\ldots,n$. In particular we note that V_i is set periodic with period 2^i for every $i=1,\ldots,n$. To choose B_{n+1} and C_{n+1} we proceed as follows. First we note that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\mu\left(\bigcup_{k=0}^{2^{n+1}-1} V_n^k D\right) > 2^{-1}+2^{-n-1}-\varepsilon$$

whenever

$$D \cap \tilde{V}_n D = \emptyset\ ,\ D \cup \tilde{V}_n D \subset B_n,$$

and

$$\mu(B_n \setminus (D \cup \tilde{V}_n D)) < \delta.$$

In particular we can choose $\varepsilon = 2^{-n-2}$ and apply Lemma 8.9 to the measure space $(B_n, \mathcal{S}_{B_n}, \mu_{B_n})$, the ergodic transformation $V_n = V_{B_n}$, and the number δ corresponding to $\varepsilon = 2^{-n-2}$. We obtain a set $B'_{n+1} \subset B_n$ with $B'_{n+1} \cap V_n B'_{n+1} = \emptyset$, and $\mu(B_n \setminus (B'_{n+1} \cup V_n B'_{n+1})) < \delta$. Next we choose a Borel set $B_{n+1} \subset B'_{n+1}$ for which

$$C_n = \bigcup_{k=0}^{2^{k+1}-1} V_n^k B_{n+1}$$

has measure $2^{-1}+2^{-k-2}$. Put $\tilde{V}_{n+1} = V_{B_{n+1}}$, $V_{n+1} = V_{C_{n+1}}$. Then V_{n+1} is set periodic with period 2^{n+1} and satisfies (a) - (d) with $i = n+1$. Let now $C = \bigcap C_n$. Then $\mu(C) = 1/2$ and the induced transformation V_C is set periodic with period 2^n for every n. If $\tilde{B}_n = B_n \cap C$ and $\mathcal{D}_n(V_C) = \{\tilde{B}_n,\ldots,V_C^{2^n-1}\tilde{B}_n\}$, $n=1,2,\ldots$, we get a sequence of partitions satsifying (8.8) and (8.9). V_C is thus w.v.N. on $(C, \mathcal{S}_C, \mu_C)$. By Lemma 8.4 or 8.5 we can construct a map $J \in [V]$ with

$JC = \complement C \pmod 0$ and J^2 = identity. Let $N = \bigcup_{k \in \mathbb{Z}} V^k(C \triangle J \complement C)$ and

$$Wx = \begin{cases} Jx & \text{for } x \in C \setminus N, \\ V^C Jx & \text{for } x \in \complement C \setminus N, \\ x & \text{for } x \in N. \end{cases}$$

W is again w.v.N., and $\{W^k x : k \in \mathbb{Z}\} = \{V^k x : k \in \mathbb{Z}\}$ for μ-a.e. $x \in X$. Modifying W on a set of measure zero we achieve $[W] = [V]$, and the proof is complete.

<u>Lemma 8.11</u>. Let $V \in \mathcal{A}(X, \mathcal{S}, \mu)$ be ergodic and assume that μ is either invariant under V or inequivalent to every σ-finite V-invariant measure on $(X, \mathcal{S})$. Suppose $V_1 \in [V]$ is set periodic with period 2^K and $\mathcal{D}(V_1) = (D_1, \ldots, D_{2^K})$ is a Borel partition of X such that $V_1^{l-1} D_1 = D_l \pmod 0$ for every $l = 1, \ldots, 2^K$. Then for every $\varepsilon > 0$, every integer $Q > 0$, and every set $E \in \mathcal{S}$ with $\mu(E) > 0$ there exists a w.v.N. transformation $V_2 \in [V]$ and an integer $L > 0$ such that

(1) $[V_2] = [V]$.

(2) There exists a sequence of partitions $(\mathcal{D}_n(V_2), n = 1, 2, \ldots)$ of X, satisfying (8.8) - (8.9) (with V_2 replacing V) such that $\mathcal{D}_K(V_2) = \mathcal{D}(V_1) \pmod 0$.

(3) $\{x : V_2 x \neq V_1 x\} \subset D_{2^K} \in \mathcal{D}(V_1) \pmod 0$.

(4) For every $n \geq 1$, let

$$E_n' = \bigcup_{D \in \mathcal{D}_n'} D \quad \text{with } \mathcal{D}_n' = \{D \in \mathcal{D}_n(V_2) : D \subset E \pmod 0\}$$

and

$$E_n'' = \bigcup_{D \in \mathcal{D}_n''} D \quad \text{with } \mathcal{D}_n'' = \{D \in \mathcal{D}_n(V_2) : \mu(D \cap E) > 0\}.$$

Then $\mu(E_n'' \setminus E_n') < \varepsilon$ for every $n \geq L$.

(5) $\mu(\bigcup_{k=0}^{Q} D_k^{(n)} \cup \bigcup_{k=2^n-Q+1}^{2^n} D_k^{(n)}) < \varepsilon$ for every $n \geq L$,

where $\mathcal{D}_n(V_2) = \{D_1^{(n)}, \ldots, D_{2^n}^{(n)}\}$ for every $n \geq 1$.

Proof: Let $\mathcal{E}$ denote the partition $\{E, \complement E\}$ of X, and let $\mathcal{D}^* = \mathcal{D}(V_1) \vee \mathcal{E} \vee V_1^{-1}\mathcal{E} \vee \ldots \vee V_1^{-2^K+1}\mathcal{E}$ be the partition generated by $\mathcal{D}(V_1)$ and $V_1^{-k}\mathcal{E}$, $k=0,\ldots,2^K-1$. $\mathcal{D}^*$ induces a partition $\mathcal{D}^{**} = \{D \in \mathcal{D}^* : D \subset D_1\}$ of $D_1 \in \mathcal{D}(V_1)$. It is clear that $\mathcal{D}^{**}$ has at most 2^{2^K} elements, and we can choose an enumeration $\{B_1, \ldots, B_p\}$ of $\mathcal{D}^{**}$ with $p \leq 2^{2^K}$. We now have to distinguish between two cases.

(a) Suppose μ is invariant under V. We choose an integer $L > 0$ such that $2.Q.2^K.2^{2^K}.2^{-L} < \varepsilon$. For each $i=1,\ldots,p$ there exists an integer $q_i \geq 0$ and disjoint sets B_{ij}, $j=0,\ldots,q_i$ in $\mathcal{S}$ with $B_i = \bigcup_{j=0}^{q_i} B_{ij}$, $\mu(B_{ij}) = 2^{-L}$ for $j=1,\ldots,q_i$, and $\mu(B_{i0}) < 2^{-L}$. Let $B_0 = \bigcup_{i=0}^{p} B_{i0}$. Then $\mu(B_0) < p.2^{-L} < \varepsilon.2^{-K}$. Since $\mu(D_1) = 2^{-K}$, there exists an integer $q_0 \geq 0$ and disjoint Borel sets B_{0k}, $k=0,\ldots,q_0$, such that $B_o = \bigcup_{j=0}^{q_o} B_{oj}$ with $\mu(B_{oj}) = 2^{-L}$ for $j=1,\dots,q_o$, and $\mu(B_{oo}) = 0$. Hence

$$D_1 = B_{oo} \cup \bigcup_{i=0}^{p} \bigcup_{j=1}^{q_i} B_{ij},$$

where

$$\mu(B_{ij}) = 2^{-L} \quad \text{for } j=1,\dots,q_i \text{ and } i=1,\dots,p,$$

and

$$\mu(B_{oo}) = 0.$$

Let $\{C_1',\dots,C'_{2^{L-K}}\}$ be an enumeration of $\{B_{ij}: i=0,\dots,p, j=1,\dots,q_i\}$. By Lemma 8.5 we can choose maps $J_i \in [V]$ such that $J_i^2 = $ identity and $J_i V_1^{2^K-1} C_i' = C_{i+1}'$ (mod 0) for every $i=1,\dots,2^{L-K}-1$. Put

$$N_1 = \bigcup_{i=1}^{2^{L-K}-1} (J_i V_1^{2^K-1} C_i' \,\Delta\, C_{i+1}')$$

and

$$N = \bigcup_{k \in \mathbb{Z}} V^k(N_1 \cup B_{oo} \cup \bigcup_{i=1}^{2^K-1} V_1^i D_1 \,\Delta\, D_{i+1}).$$

For every $i=1,\dots,2^{L-K}$, let $C_i = C_i' \setminus N$. Applying Lemma 8.10 we find a w.v.N. transformation W on $(C_1, \mathcal{S}_{C_1}, \mu_{C_1})$ with $[W] = [V_{C_1}]$, where V_{C_1} is the transformation induced by V on C_1. We choose a sequence $(\tilde{\mathcal{D}}_n(W),\ n=1,2,\dots)$ of Borel partitions of C_1 satisfying (8.8) - (8.9) for W on $(C_1, \mathcal{S}_{C_1}, \mu_{C_1})$, where $\tilde{\mathcal{D}}_n(W) = \{\tilde{D}_1^{(n)},\dots,\tilde{D}_{2^n}^{(n)}\}$ for every $n \geq 1$. Define

$$V_2x = \begin{cases} V_1x & \text{if } x \in V_1^j C_i \text{ with } 0 \leq j \leq 2^K-2 \\ & \quad\text{and } 1 \leq i \leq 2^{L-K}, \\ J_i x & \text{if } x \in V_1^{2^K-1} C_i \text{ with } 1 \leq i < 2^{L-K}, \\ WV_2^{-2^L+1}x & \text{if } x \in V_1^{2^K-1} C_{2^{L-K}}, \\ x & \text{if } x \in N, \end{cases}$$

and set

$$\tilde{\tilde{D}}^{(n+L)}_{(i-1).2^L+l+1} = V_2^l \tilde{D}_i^{(n)}$$

for $n=1,2,\ldots$, $i=1,\ldots,2^n$, $l=0,1,\ldots,2^L-1$. For every $k \geq 1$, $i=1,\ldots,2^{n+L}$, we now put

$$D_i^{(n+L)} = \begin{cases} \tilde{\tilde{D}}_1^{(n+L)} \cup N & \text{if } i=1, \\ \tilde{\tilde{D}}_i^{(n+L)} & \text{otherwise.} \end{cases}$$

Finally, if $1 \leq m \leq L$ and $1 \leq i \leq 2^m$, let

$$D_i^{(m)} = \bigcup_{l=1}^{2^{L+1-m}} D^{(L+1)}_{i+(l-1).2^m}.$$

Then $(\mathcal{D}_n(V_2) = \{D_1^{(n)},\ldots,D_{2^n}^{(n)}\}$, $n \geq 1)$, is a sequence of partitions of X satisfying (8.8) - (8.9) for V_2 on $(X, \mathcal{S}, \mu)$, so that V_2 is w.v.N. . Morover, $(\mathcal{D}_n(V_2)$, $n=1,2,\ldots)$ will satisfy (2) in the statement of this lemma. Since our construction of V_2 implies that $\{V_2^k x : k \in \mathbb{Z}\} = \{V^k x : k \in \mathbb{Z}\}$ for μ-a.e. $x \in X$, we may modify V_2 on a set of measure zero to get $[V_2] = [V]$. Clearly, V_2 will still be w.v.N. after this modification. (3) is obvious from the construction of V_2. Let now E'_n and E''_n be given as in (4), and let $D \in \mathcal{D}_L(V_2)$. If we have $\mu(D \cap E \setminus E'_n) > 0$ or $\mu(D \cap E''_n \setminus E) > 0$ for some $n \geq L$, then

$$D \subset \bigcup_{i=0}^{2^K-1} V_1^i B_o \pmod 0,$$

and (4) follows from $\mu(B_o) < \varepsilon . 2^{-K-1}$ and from the invariance of μ under V_2. (5) follows from

$$\mu\Big(\bigcup_{k=1}^{Q} D_k^{(n)} \cup \bigcup_{k=2^n-Q+1}^{2^n} D_k^{(n)}\Big) = 2Q.2^{-L} < \varepsilon,$$

and the proof is complete in the measure preserving case.

(b) Suppose μ is not equivalent to any σ-finite V-invariant measure. Then there exists a $\delta > 0$ such that

$$\mu\Big(\bigcup_{i=0}^{2^K-1} V_1^i C\Big) < \varepsilon$$

whenever $C \subset D_1$ is a Borel set with $\mu(C) < \delta$. Let $Q_1 = [Q.2^{-K}]+1$, where [ ] again denotes the integral part, and let $L > 0$ be an integer satisfying $2.Q.2^K.2^{2^K}.2^{-L} < \varepsilon.\delta$. We choose a Borel partition $\{C'_1,\dots,C'_{2^{L-K}}\}$ of D_1 into sets of positive measure such that

(i) $\mu\Big(\bigcup_{i=1}^{Q_1} C_i \cup \bigcup_{i=2^{L-K}-Q_1+1}^{2^{L-K}} C_i\Big) < \delta$

and

(ii) $\{C'_1,\dots,C'_{2^{L-K}}\}$ refines $\mathcal{D}^{**}$ (mod 0) - i.e. for every $D \in \mathcal{D}^{**}$ and for every $j=1,\dots,2^{L-K}$ we have $\mu(D \cap C'_j).\mu(D \cap \complement C'_j) = 0$.

Applying Lemma 8.4 we find transformations $J_i \in [V]$ with J_i^2 = identity and such that

$$J_i V_1^{2^K-1} C_i' = C_{i+1}' \pmod 0 \text{ for every } i=1,2,\dots,2^{L-K}-1.$$

Again we put

$$N = \bigcup_{k \in \mathbb{Z}} \Big(\bigcup_{i=1}^{2^{L-K}-1} (J_i V_1^{2^K-1} C_i' \,\Delta\, C_{i+1}') \Big),$$

and the rest of the proof goes exactly as in the measure preserving case (a). The only difference is that (4) can now be replaced by the stronger assertion $\mu(E_n'' \setminus E_n') = 0$ for $n \geqslant L$, and (5) is proved as follows: Let $n \geqslant L$. Then

$$\bigcup_{k=0}^{Q} D_k^{(n)} \cup \bigcup_{k=2^n-Q+1}^{2^n} D_k^{(n)} \subset \bigcup_{k=0}^{Q} D_k^{(L)} \cup \bigcup_{k=2^L-Q+1}^{2^L} D_k^{(L)}$$

$$\subset \bigcup_{k=0}^{2^K-1} V_1^k \Big(\bigcup_{i=1}^{Q_1} C_i \cup \bigcup_{i=2^{L-K}-Q_1+1}^{2^{L-K}} C_i \Big) \pmod 0.$$

(i) together with the choice of δ in the beginning of part (b) now implies (5), and the proof is complete.

<u>Proposition 8.12</u>. Let $V \in \mathcal{A}(X, \mathcal{S}, \mu)$ be ergodic and assume that μ is a probability measure on $(X, \mathcal{S})$ which is either invariant under V or otherwise inequivalent to every σ-finite V-invariant measure on $(X, \mathcal{S})$. Then there exists a v.N. transformation $W \in [V]$ with $[W] = [V]$.

Proof: Let η be the pseudometric on $\mathcal{S}$ given by $\eta(B,C) =$

$\mu(B \Delta C)$, $B, C \in \mathcal{S}$. Modulo sets of measure zero, $(\mathcal{S}, \eta)$ is a complete separable metric space, and we can choose a dense sequence $(B_i,\ i=1,2,\ldots)$ in $\mathcal{S}$ with $\mu(B_i) > 0$ for every i. We now take a sequence $(C_i,\ i=1,2,\ldots)$ in $\mathcal{S}$ with $\mu(C_i)\ \ 0$ for every i, in which every B_i occurs infinitely often. By induction we define sequences $(U_i,\ i=1,2,\ldots)$ of w.v.N. transformations on $(X, \mathcal{S}, \mu)$ and $(P_i,\ i=1,2,\ldots)$ of integers as follows. Let $P_1 = 1$ and apply Lemma 8.10 to construct a w.v.N. transformation U_1 with $[U_1] = [V]$. Suppose we have chosen $U_1, \ldots, U_n$ and $P_1, \ldots, P_n$. We regard U_n as set periodic with period 2^{P_n} and apply Lemma 8.11 to $V_1 = U_n$, $K = P_n$, $Q = 2^n$, $\varepsilon = 2^{-n}$, and $E = C_n$. This gives a w.v.N. transformation $U_{n+1} = V_2$ and an integer $P_{n+1} = L$ satisfying (1) - (5) in Lemma 8.11. This induction process gives a sequence $(U_i,\ i=1,2,\ldots)$ of w.v.N. transformations on $(X, \mathcal{S}, \mu)$ with $[U_i] = [V]$ for every i. (3) and (5) in Lemma 8.11 show that, for μ-a.e. $x \in X$, there exists a smallest integer $q(x)$ with $U_n x = U_{q(x)} x$ for every $n \geq q(x)$. If no such integer exists, we put $q(x) = \infty$. The map $q: X \to \mathbb{Z}$ is Borel, and we can define a $W \in \mathcal{A}(X, \mathcal{S}, \mu)$ by putting

$$N = \bigcup_{k \in \mathbb{Z}} V^k \{x : q(x) = \infty\}$$

and

$$Wx = \begin{cases} U_{q(x)} x & \text{for } x \in X \setminus N, \\ x & \text{for } x \ \ N. \end{cases}$$

It is easy to see that W is w.v.N., that $W \in [V]$, and that W is in fact a v.N.transformation. We shall now prove that $\{W^k x : k \in \mathbb{Z}\} = \{V^k x : k \in \mathbb{Z}\}$ for μ-a.e. $x \in X$. Since $[U_n] = [V]$ for every n, we can find a sequence of Borel maps m_n: $X \to \mathbb{Z}$ such that $U_n^{m_n(x)} x = Vx$ for every $n \geq 1$ and every $x \in X$. Note that we may assume $m_n(x) = m_{q(x)}(x)$ whenever $n \geq q(x)$. We put

$$C(n,x) = \begin{cases} \{x, U_n x, \ldots, U_n^{m_n(x)} x\} & \text{if } m_n(x) \geq 0, \\ \{x, U_n^{-1} x, \ldots, U_n^{m_n(x)} x\} & \text{if } m_n(x) < 0. \end{cases}$$

Let now $\varepsilon > 0$ be fixed, and choose an integer $N > 0$ with

$$\mu\{x : q(x) > N\} < \varepsilon/3,$$

$$\mu\{x : m_{q(x)}(x) > 2^N\} < \varepsilon/3,$$

and

$$2^{-N} < \varepsilon/3.$$

Since

$$\{x : Vx \notin \{W^k x : k \in \mathbb{Z}\}\} \subset \{x : C(N,x) \cap D_{2^{P_N}}^{(P_N)} \neq \emptyset\}$$

$$\subset \bigcup_{k=1}^{2^N} D_k^{(P_N)} \cup \bigcup_{k=2^{P_N}-2^N+1}^{2^{P_N}} D_k^{(P_N)} \cup \{x : |m_N(x)| > 2^N\}$$

and

$$\{x : |m_N(x)| > 2^N\} \subset \{x : q(x) > N\} \cup \{x : |m_{q(x)}(x)| > 2^N\}$$

we have

$$\mu\{x : Vx \in \{W^k x : k \in \mathbb{Z}\}\} \leq 2^{-N} + \varepsilon/3 < \varepsilon,$$

by (3) and (5) in Lemma 8.11. Hence $Vx \in \{W^k x : k \in \mathbb{Z}\}$ for μ-a.e. $x \in X$, which shows that $\{V^k x : k \in \mathbb{Z}\} = \{W^k x : k \in \mathbb{Z}\}$ for μ-a.e. $x \in X$. Changing W on a set of measure zero, we get $[W] = [V]$, and the proof is complete.

Lemma 8.13. Let $(X, \mathcal{S}, \mu)$ be a nonatomic probability space and let $V \in \mathcal{A}(X, \mathcal{S}, \mu)$ be an ergodic v.N. transformation which does not preserve any σ-finite measure equivalent to μ. Then there exists a v.N. transformation V' on $(X, \mathcal{S}, \mu)$ with $[V'] = [V]$, a sequence $(\mathcal{D}_n(V') = \{D_1'^{(n)}, \ldots, D_{2^n}'^{(n)}\}$, $n=1,2,\ldots)$ of Borel partitions satisfying (8.8) - (8.10) for V', and a probability measure $\mu' \sim \mu$ such that $\log \frac{d\mu' V'}{d\mu'}$ is essentially bounded on $\bigcup_{i=1}^{2^n-1} D_i'^{(n)}$ for every $n \geq 1$.

Proof: Let $(\mathcal{D}_n(V) = \{D_1^{(n)}, \ldots, D_{2^n}^{(n)}\}$, $n=1,2,\ldots)$ be a sequence of Borel partitions of X satisfying (8.8) - (8.10) for V. We choose a Borel set $\tilde{D}_1^{(1)} \subset D_1^{(1)}$ such that $V\tilde{D}_1^{(1)} \cap \tilde{D}_1^{(1)} = \phi$, $\log \frac{d\mu V}{d\mu}$ is essentially bounded on $\tilde{D}_1^{(1)}$, and

$$\mu(\tilde{D}_1^{(1)} \cup V\tilde{D}_1^{(1)}) = 3/4.$$

Proceeding by induction, we define a sequence of Borel sets $\tilde{D}_1^{(k)} \subset D_1^{(k)}$ with

$$\tilde{D}_1^{(k)} \cup V^{2^{k-1}} \tilde{D}_1^{(k)} \subset \tilde{D}_1^{(k-1)},$$

$$\tilde{D}_1^{(k)} \cap V^{2^{k-1}} D_1^{(k)} = \phi,$$

and

$$\mu \Big(\bigcup_{i=0}^{2^k-1} V^i \tilde{D}_1^{(k)} \Big) = 2^{-1} + 2^{-k-1},$$

such that

$$\log \frac{d\mu V^{2^k}}{d\mu} \text{ is essentially bounded on } D_1^{(k)},$$

for every $k \geq 1$. Let

$$C = \bigcap_{k=1}^{\infty} \bigcup_{i=0}^{2^k-1} V^i D_1^{(k)},$$

and let V_C be the transformation induced by V on C. Since $\mu(C) = 1/2$, we can apply Lemma 8.4 to find a $J \in [V]$ with J^2 = identity and with $JC = \complement C \pmod 0$. Put $N = \bigcup_{k \in \mathbb{Z}} V^k (JC \,\Delta\, \complement C)$ and

$$V'x = \begin{cases} Jx & \text{for } x \in C \setminus N, \\ VJx & \text{for } x \in \complement C \setminus N, \\ x & \text{for } x \in N. \end{cases}$$

One sees immediately that V' is a v.N.transformation, and that we can modify V' on a set of measure zero to achieve $[V'] = [V]$. To describe the sequence $(\mathcal{D}_n(V'),\ n=1,2,\ldots)$, we put

$$D_1'^{(1)} = C \cup N, \quad D_2'^{(1)} = \complement C \setminus N.$$

If $n > 1$, let

$$D_1'^{(n)} = (D_1'^{(n-1)} \cap C) \cup N$$

and

$$D_{k+1}'^{(n)} = V'^k D_1'^{(n)} \setminus N \quad \text{for } k=1,\ldots,2^n-1.$$

Finally we define a probability measure μ' on $(X,\mathcal{S})$ by

$$\mu'(B) = \mu(B \cap C) + \mu(JB \cap C),\ B \in \mathcal{S}.$$

The conditions of the lemma are all satisfied, and the proof is complete.

Let now $\mathbb{Z}_2$ be the group $\{1,-1\}$ under multiplication and let

$$\Omega = \mathbb{Z}_2^{\mathbb{N}_+}$$

be the cartesian product of countably many copies of $\mathbb{Z}_2$, where $\mathbb{N}_+ = \{1,2,\ldots\}$. Ω is a compact abelian group in the product topology, and we write $\mathcal{F}$ for the Borel field of Ω. $\pi_i: \Omega \to \mathbb{Z}_2$ will denote the i-th coordiante projection for $i=1,2,\ldots$, and $\omega \in \Omega$ will also be written as $\omega = (\omega_1, \omega_2, \ldots)$ with $\omega_i = \pi_i(\omega)$. The dual group $\Gamma = \hat{\Omega}$ of Ω is the countable discrete group consisting of all $\gamma = (\gamma_1, \gamma_2, \ldots)$ in Ω with $\gamma_i = -1$ for only finitely many $i \geq 1$. Γ acts on Ω by multiplication, since it is identified with a countable dense subgroup of Ω. This action will be denoted by $(\gamma, \omega) \to \gamma\omega$, $\gamma \in \Gamma$, $\omega \in \Omega$. 1 will stand for the identity element both in Γ and in Ω.

The following result connects actions of Γ on Ω with v.N. transformations.

Proposition 8.14. Let $(X, \mathcal{S}, \mu)$ be a nonatomic probability space and let $V \in \mathcal{A}(X, \mathcal{S}, \mu)$ be an ergodic v.N. transformation. Then there exists a probability measure ν on $(\Omega, \mathcal{F})$ which is nonatomic, quasi-invariant and ergodic under Γ, and such that the action of Γ on $(\Omega, \mathcal{F}, \nu)$ is weakly equivalent to V. If μ is invariant under V, ν can be chosen as the Haar measure of Ω. If μ is not equivalent to a V-invariant probability measure, ν can be chosen such that $\log \frac{d\nu\gamma}{d\nu}$ is essentially bounded for every $\gamma \in \Gamma$.

Proof: Let $(\mathcal{D}_n(V) = \{D_1^{(n)}, \ldots, D_{2^n}^{(n)}\}, n \geq 1)$ be a sequence of partitions satisfying (8.8) - (8.10). If μ is inequivalent to every σ-finite V-invariant measure we may assume that $\log \dfrac{d\mu V}{d\mu}$ is essentially bounded on $\int D_{2^n}^{(n)}$ for every $n \geq 1$, by Lemma 8.13. If μ is equivalent to a σ-finite invariant measure μ', we shall assume $\mu = \mu'$ for the sake of simplicity. (8.10) implies that, for μ-a.e.x X, there exists a sequence $i(n,x)$ of positive integers with $i(n,x) \leq 2^n$ and with

$$\bigcap_n D_{i(n,x)}^{(n)} = \{x\} .$$

For every such $x \in X$ we define an element $\phi(x) \in \Omega$ by

$$\phi(x) = (\omega_1(x), \omega_2(x), \ldots) \tag{8.11.a}$$

with

$$\omega_k(x) = \begin{cases} +1 & \text{if } i(k,x) \leq 2^{n-1}, \\ -1 & \text{if } i(k,x) > 2^{n-1}, \end{cases} \tag{8.11.b}$$

for every $k \geq 1$. The μ-a.e. defined map $\phi: X \to \Omega$ is Borel and a.e. injective. We put $\nu = \mu\phi^{-1}$ and extend ϕ to an isomorphism from $(X, \mathcal{S}, \mu)$ to $(\Omega, \mathcal{F}, \nu)$. ν will be nonatomic. Furthermore we see that $\phi(Vx) \neq \phi(x)$ for μ-a.e. $x \in X$. On the other hand, $\phi(x)$ and $\phi(Vx)$ will differ in at most n coordinates for μ-a.e. $x \in \complement D^{(n)}_{2^n}$. Assumption (8.10) implies that $\lim_n \mu(D^{(n)}_{2^n}) = 0$, so that $\phi(Vx) \in \Gamma\phi(x)$ for μ-a.e. $x \in X$.

To prove the converse relation, we choose a V-invariant set $N \in \mathcal{S}$ of measure zero such that $\phi(x)$ is given by (8.11) for every $x \in X \setminus N$, and

(a) ϕ is injective on $X' = X \setminus N$,

(b) $\bigcap_n D^{(n)}_{2^n} \subset N$,

and

(c) if $D^{(n)}_i = D^{(n)}_i \cap X'$, we have $V^{i-1}D^{(n)}_1 = D^{(n)}_i$ for every $n \geq 1$ and $i = 1, \dots, 2^n$.

Let $x^{(i)} \in X'$, $i=1,2$, and let $\omega^{(i)} = \phi(x^{(i)}) = (\omega^{(i)}_1, \omega^{(i)}_2, \dots)$. We assume that $\omega^{(2)} \in \Gamma\omega^{(1)}$. Then $\omega^{(1)}$ and $\omega^{(2)}$ will differ in only finitely many coordinates. (c) and the invariance of N under V imply that there exists an $n_0 \geq 1$

with $\omega_{n_o}^{(1)} = \omega_{n_o}^{(2)} = 1$ and with $\omega_n^{(1)} = \omega_n^{(2)}$ for $n \geq n_o$.
Put

$$r_k^{(i)} = \begin{cases} 0 & \text{according as } \omega_k^{(i)} = +1 \\ 1 & \phantom{\text{according as }} \omega_k^{(i)} = -1 \end{cases} \quad \text{for } k=1,2,\ldots,n_o-1,$$

and

$$l^{(i)} = 1 + \sum_{k=1}^{n_o-1} r_k^{(i)}.2^k.$$

By construction, we have $x^{(i)} \in D_{l^{(i)}}^{(n_o-1)}$, $i=1,2$. We assume $l^{(1)} \leq l^{(2)}$ and get

$$V^{l^{(2)}-l^{(1)}} x^{(1)} = x^{(2)}.$$

Hence $\Gamma \phi(x) \subset \phi\{V^k x : k \in \mathbb{Z}\}$ for μ-a.e.x X. This proves the weak equivalence of Γ and V, and the properties of ν are clear from its definition.

We can now state the second main result of this section.

Theorem 8.15. Let V be an ergodic automorphism of a non-atomic measure space $(X, \mathcal{S}, \mu)$. Then there exists a probability measure ν on $(\Omega, \mathcal{F})$ which is nonatomic, quasi-invariant and ergodic under Γ, and such that

(1) V is weakly equivalent to the action of Γ on $(\Omega, \mathcal{F}, \nu)$,

(2) $\log \frac{d\nu\gamma}{d\nu}$ is essentially bounded for every $\gamma \in \Gamma$.

Furthermore, if V preserves a probability measure $\mu' \sim \mu$, we may choose ν to be the Haar measure of Ω.

Proof: If μ is equivalent to a V-invariant probability measure or if μ is inequivalent to every σ-finite V-invariant measure on $(X, \mathcal{S})$, the result follows from Proposition 8.12, and Proposition 8.14. We are thus left with the case where μ is equivalent to an infinite V-invariant measure μ'. Let $\tilde{\Omega} = \{\omega \in \Omega : \pi_{2i}(\omega) = 1 \text{ for all } i\}$, and let $\tilde{\Gamma} = \Gamma \cap \tilde{\Omega}$. $\tilde{\Gamma}$ is a subgroup of Γ which is isomorphic to Γ, and $\tilde{\Omega}$ is a closed subgroup of Ω isomorphic to Ω. Let $\tilde{\lambda}$ denote the (normalized) Haar measure of $\tilde{\Omega}$, and let $\tilde{\mathcal{F}}$ denote the Borel field of $\tilde{\Omega}$. Furthermore we define $\Gamma^* = \{\gamma \in \Gamma : \gamma_{2i+1} = +1 \text{ for } i=0,1,2,\dots\}$. Again Γ^* is a subgroup of Γ. Let $\{1 = \gamma^{(1)}, \gamma^{(2)}, \dots\}$ be an enumeration of Γ^*. We choose a set $B \in \mathcal{S}$ with $\mu'(B) = 1$ and consider the induced tranformation V_B on $(B, \mathcal{S}_B, \mu'_B)$. By the first part of the proof there exists an isomorphism $\phi_1 : (B, \mathcal{S}_B, \mu'_B) \to (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\lambda})$ with $\tilde{\lambda} = \mu'_B \phi_1^{-1}$ and with $\phi_1(\{V_B^k x : k \in \mathbb{Z}\}) = \tilde{\Gamma}\phi_1(x)$ for μ'_B-a.e. $x \in B$. We choose a sequence $(B_1, B_2, \dots)$ of disjoint Borel sets in X with $B_1 = B$, $\mu'(B_i) = 1$ for all i, and with $X = \bigcup_i B_i$. By Lemma 8.5 we can find maps $J_i \in [V]$ with $J_i^2 = \text{identity}$ and with $J_i B_1 = B_i \pmod 0$ for all $i \geq 2$. Ignoring a set of measure zero, we put

$$\phi(x) = \gamma^{(i)} \phi_1(J_i x)$$

whenever $x \in B_i$. After modifying ϕ suitably on a set of measure zero, ϕ will be a Borel isomorphism from $(X, \mathcal{S})$ to $(\Omega, \mathcal{F})$. Let λ^* be the infinite measure $\lambda^* = \sum_{i=1}^{\infty} \tilde{\lambda}\gamma^{(i)}$ on Ω. Then ϕ is an isomorphism from $(X, \mathcal{S}, \mu')$ to $(\Omega, \mathcal{F}, \lambda^*)$ with $\mu'\phi^{-1} = \lambda^*$, and with $\phi(\{V^k x : k \in \mathbb{Z}\}) = \Gamma\phi(x)$ for μ'-a.e. $x \in X$. Finally we put $\nu = \sum_{i=1}^{\infty} 2^{-1}\tilde{\lambda}\gamma^{(i)}$. Then ν is a probability measure on $(\Omega, \mathcal{S})$ which satisfies condition (2) in the statement of this theorem. The proof is complete.

<u>Corollary 8.16</u>. Let $(X_i, \mathcal{S}_i, \mu_i)$ be nonatomic probability spaces, and let $V_i \in \mathcal{A}(X_i, \mathcal{S}_i, \mu_i)$ be ergodic measure preserving transformations for i=1,2. Then V_1 and V_2 are weakly equivalent.

Proof: Each V_i is weakly equivalent to the action of Γ on $(\Omega, \mathcal{F}, \nu_0)$, where ν_0 is the Haar measure on Ω.

<u>Corollary 8.17</u>. Let $(X_i, \mathcal{S}_i, \mu_i)$ be nonatomic infinite measure space and let $V_i \in \mathcal{A}(X_i, \mathcal{S}_i, \mu_i)$ be ergodic measure preserving transformations for i=1,2. Then V_1 and V_2 are weakly equivalent.

Proof: Each V_i is weakly equivalent to the action of Γ on $(\Omega, \mathcal{F}, \lambda^*)$ with λ^* given in the proof of Theorem 8.15.

The next result is a simple converse of Theorem 8.15.

<u>Theorem 8.18.</u> Let ν be a probability measure on $(\Omega, \mathcal{F})$ which is nonatomic, quasi-invariant and ergodic under Γ. Then the action of Γ on $(\Omega, \mathcal{F}, \nu)$ is hyperfinite.

Proof: Let $\beta_i : \Omega \to \mathbb{R}$, i=0,1,2,... be the sequence of Borel maps given by

$$\beta_i(\omega) = \begin{cases} 2^i & \text{if } \pi_1(\omega) = +1, \\ 0 & \text{otherwise.} \end{cases} \tag{8.12}$$

For every $\gamma \in \Gamma$, $\omega \in \Omega$, we put

$$a(\gamma, \omega) = \sum_{k=0}^{\infty} \beta_k(T^k \gamma\omega) - \beta_k(T^k \omega), \tag{8.13}$$

where $T: \Omega \to \Omega$ is the shift

$$T(\omega_1, \omega_2, \ldots) = (\omega_2, \omega_3, \ldots) \tag{8.14}$$

for every $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$. $a: \Gamma \times \Omega \to \mathbb{R}$ is a cocycle for Γ which is easily seen to be transient. The proof is completed by applying Theorem 8.7.

We conclude this section by proving that every ergodic action of a countable group on a nonatomic measure space contains an ergodic transformation in its full group.
For the following lemmas we fix a nonatomic probability space $(X, \mathcal{S}, \mu)$ and an ergodic action T_G of a countable group G on $(X, \mathcal{S}, \mu)$ such that μ is either invariant or

inequivalent to every σ-finite T_G-invariant measure.

Lemma 8.19. There exists a w.v.N.transformation $W \in [T_G]$.

Proof: We choose a sequence $(B_k, k=1,2,\dots) \in \mathcal{S}$ such that $B_1 \supset B_2 \supset B_3 \supset \dots$ and $\mu(B_k) = 2^{-k}$ for all k. Setting $B_0=X$, we choose a sequence $(W_k, k=1,2,\dots)$ in $[T_G]$ such that W_k^2 = identity, $W_k x=x$ for every $x \notin B_{k-1}$, and $W_k B_k = B_{k-1} \setminus B_k$ (mod 0) for every $k \geq 1$. Let $N = T_G(\bigcap_k B_k \cup \bigcup_k (W_k B_k \Delta (B_{k-1} \setminus B_k)))$, and let $B_k' = B_k \setminus N$ for every $k \geq 0$. For every $l \geq 1$, let

$$V_l x = \begin{cases} W_1 W_2 \dots W_l x & \text{for } x \in X \setminus N, \\ x & \text{otherwise.} \end{cases}$$

It is clear that V_l is set periodic with period 2^l, since $B_l', V_l B_l', \dots, V_l^{2^l-1} B_l'$ are all disjoint, and $\mu(X \setminus \bigcup_{k=0}^{2^l-1} V_l^k B_l') = 0$. Moreover, V_l and V_{l+1} will coincide on $\complement B_l'$. Since $\bigcap_k B_k' = \phi$, we can define a Borel map $u: X \to \mathbb{Z}$ with

$$V_l x = V_{u(x)} x$$

for every $l \geq u(x)$. If we now put

$$Wx = V_{u(x)} x, \; x \in X,$$

W will be a w.v.N. transformation, and the lemma is proved.

Lemma 8.20. Suppose $W \in [T_G]$ is a w.v.N. transformation, $(\mathcal{D}_n(W) = \{D_1^{(n)}, \dots, D_{2^n}^{(n)}\}$, $n=1,2,\dots)$ a sequence of

Borel partitions of X satisfying (8.8) and (8.9) for W, ε a positive real number, K a positive integer, and E a Borel set in X of positive measure. Then there exists a w.v.N. transformation $W' \in [T_G]$ and an integer $L > 0$ such that the following is true.

(1) There exists a sequence $(\mathcal{D}_n(W'), n=1,2,\ldots)$ of Borel partitions of X satisfying (8.8) and (8.9) for W' with $\mathcal{D}_n(W') = \mathcal{D}_n(W)$ (mod 0) for $1 \leq n \leq K$.

(2) $\{x : Wx \neq W'x\} \subset D^{(K)}_{2^K}$.

(3) For every $n \geq 1$, let

$$E'_n = \bigcup_{D \in \mathcal{D}'_n} D \quad \text{with } \mathcal{D}'_n = \{D \in \mathcal{D}_n(W') : D \subset E \text{ (mod 0)}\}$$

and

$$E''_n = \bigcup_{D \in \mathcal{D}''_n} D \quad \text{with } \mathcal{D}''_n = \{D \in \mathcal{D}_n(W') : \mu(D \cap E) > 0\}.$$

Then $\mu(E''_n \setminus E'_n) < \varepsilon$ for every $n \geq L$.

(4) $\mu(D^{(L)}_{2^L}) = 2^{-L}$.

Proof: This proof is very similar to the proof of Lemma 8.11. Let $\mathcal{E}$ denote the partition $\{E, \complement E\}$ of X, and let $\mathcal{D}^* = \mathcal{D}_K(W) \vee \mathcal{E} \vee \ldots \vee W^{-2^K+1}\mathcal{E}$ be the partition generated by $\mathcal{D}_K(W)$ and by $W^{-l}\mathcal{E}$, $l=0,\ldots,2^K-1$. $\mathcal{D}^*$ induces a partition $\mathcal{D}^{**}$ of $D^{(K)}_1$, which has at most 2^{2^K} elements, and we choose an enumeration $\{B_1,\ldots,B_p\}$ of $\mathcal{D}^{**}$ with $p \leq 2^{2^K}$. Again we have to distinguish between two cases.

(a) Suppose μ is invariant under T_G. We choose an integer $L > 0$ such that $2.2^K.2^{2^K}.2^{-L} < \varepsilon$. For each $i=1,\ldots,p$ there exists an integer $q_i \geq 0$ and disjoint Borel sets B_{ij}, $j=0,\ldots,q_i$, with $B_i = \bigcup_j B_{ij}$, $\mu(B_{ij}) = 2^{-L}$ for $j=1,\ldots,q_i$, and $\mu(B_{i0}) = 0$. Let

$$B_o = \bigcup_{i=0}^{p} B_{i0}.$$

Then $\mu(B_o) < p.2^{-L} < \varepsilon .2^{-K}$. Since $\mu(D_1) = 2^{-K}$, there exists an integer $q_o \geq 0$ and disjoint Borel sets B_{0k}, $k=0,\ldots,q_o$, such that $B_o = \bigcup_k B_{0k}$, $\mu(B_{0k}) = 2^{-L}$ for $k=1,\ldots,q_o$, and $\mu(B_{00}) = 0$. We get

$$D_1^{(K)} = B_{00} \cup \bigcup_{i=0}^{p} \bigcup_{j=1}^{q_i} B_{ij},$$

where

$$\mu(B_{ij}) = 2^{-L} \text{ for all } j=1,\ldots,q_i \text{ and } i=0,\ldots,p,$$

and

$$\mu(B_{00}) = 0.$$

Let $\{C'_1,\ldots,C'_{2^{L-K}}\}$ be an enumeration of $\{B_{ij}: i=0,\ldots,p, j=1,\ldots,q_i\}$. By Lemma 8.5 we choose maps $J_i \in [T_G]$ such that $J_i^2 = \text{identity}$ and $J_i W^{2^K-1} C'_i = C'_{i+1} \pmod 0$ for every $i=1,\ldots,2^{L-K}-1$. Applying Lemma 8.19 we find a w.v.N. transformation V on $(C'_1, \mathcal{S}_{C'_1}, \mu_{C'_1})$ with $V \in [T_G]_{C'_1}$. The remainder of the proof is completely analogous to the end of part (a) of the proof of Lemma 8.11, and we leave it to the reader.

(b) Suppose μ is inequivalent to every σ-finite T_G-invariant measure. The argument in this case is again the same as in part (b) of the proof of Lemma 8.11 with the minor modifications sketched in part (a) of this proof.

Lemma 8.21. There exists an ergodic automorphism $V \in [T_G]$.

Proof: Let η be the pseudometric on $\mathcal{S}$ given by $\eta(B,C) = \mu(B \Delta C)$, $B, C \in \mathcal{S}$, and let $(B_i, i=1,2,\dots)$ be a dense sequence in $(\mathcal{S}, \eta)$ with $\mu(B_i) > 0$ for every i. Let $(C_i, i=1,2,\dots)$ be a sequence in $\mathcal{S}$ in which every B_i occurs infinitely often, and such that $\mu(C_i) > 0$ for every i. By induction we define sequences of w.v.N. transformations $(W_i, i=1,2,\dots)$ and integers $(L_i, i=1,2,\dots)$ as follows: Put $L_1=1$ and define a w.v.N. transformation $W = W_1$ on $(X, \mathcal{S}, \mu)$ by Lemma 8.19 with $W_1 \in [T_G]$. Suppose we have chosen $W_1, \dots, W_n$ and $L_1, \dots, L_n$. We apply Lemma 8.20 to $W = W_n$, $K = L_n$, $\varepsilon = 2^{-n}$, and $E = C_n$. This gives a w.v.N. transformation $W' = W_{n+1}$ and an integer $L = L_{n+1}$, satisfying (1) - (4) in Lemma 8.20. This procedure gives a sequence $(W_i, i=1,2,\dots)$ of w.v.N. transformations in $[T_G]$. (2) and (4) in Lemma 8.20 imply that, for μ-a.e. $x \in X$, there exists an integer q(x) with $W_n x = W_{q(x)} x$ for every n q(x). If no such integer exists, we put $q(x) = \infty$. Furthermore we can assume that $q: X \to \bar{\mathbb{Z}}$ is a Borel map. Let $N = T_G \{x : q(x) = \infty\}$ and

$$Vx = \begin{cases} W_{q(x)}x & \text{for } x \in X \setminus N, \\ x & \text{for } x \in N. \end{cases}$$

Then V is a v.N. transformation, and hence ergodic (Exercise 8.24). The proof is complete.

Theorem 8.22. Let T_G be an ergodic action of a countable group G on a nonatomic measure space $(X, \mathcal{S}, \mu)$. Then there exists an ergodic automorphism $V \in [T_G]$.

Proof: We have already proved the result whenever μ is equivalent to a T_G-invariant probability measure or when μ is inequivalent to every σ-finite T_G-invariant measure (Lemma 8.21). We leave it to the reader to extend the proof to the case where μ is equivalent to an infinite T_G-invariant measure. (One can use the method described in part (a) of the proof of Proposition 8.6).

Exercise 8.23. Let T_G be an action of a countable group G on a nonatomic measure space $(X, \mathcal{S}, \mu)$, and let $T'_{G'}$ be another action of a countable group G' on $(X, \mathcal{S}, \mu)$ with $R(T'_{G'}) \subset R(T_G)$. Show that $T'_{G'}$ is hyperfinite whenever T_G is hyperfinite.

Exercise 8.24. Show that every v.N. transformation on a nonatomic measure space is ergodic.

§ 9 The cohomology of a hyperfinite action

Let $(X, \mathcal{S}, \mu)$ be a nonatomic measure space and let T_G be a hyperfinite action of a countable group G on $(X, \mathcal{S}, \mu)$. If A is a locally compact second countable abelian group, we are interested in $Z^1(T_G, \mathcal{B}(X, \mu, A))$. Proposition 2.7, Theorem 3.16 and Theorem 8.15 together imply that there exists a probability measure ν on $(\Omega, \mathcal{F})$ which is quasi-invariant and ergodic under Γ such that $Z^1(T_G, \mathcal{B}(X, \mu, A))$ and $Z^1(\Gamma, \mathcal{B}(\Omega, \nu, A))$ are topologically isomorphic, and the isomorphism will preserve all the invariants defined in § 3. In this section we study $Z^1(\Gamma, \mathcal{B}(\Omega, \nu, A))$, where A is a fixed locally compact second countable abelian group, and where ν is a probability measure on $(\Omega, \mathcal{F})$ which is nonatomic and quasi-invariant and ergodic under Γ. We define maps $S, T: \Omega \to \Omega$ by

$$S(\omega_1, \omega_2, \dots) = (1, \omega_1, \omega_2, \dots) \tag{9.1}$$

and

$$T(\omega_1, \omega_2, \dots) = (\omega_2, \omega_3, \dots) \tag{9.2}$$

for every $\omega = (\omega_1, \omega_2, \dots) \in \Omega$. $\pi_i: \Omega \to \mathbb{Z}_2$ will again stand for the i-th coordinate projection $\pi_i(\omega) = \omega_i$ on Ω. Since Γ is embedded as a dense subgroup of Ω, we can consider the restrictions of S,T and π_i to Γ. These will be denoted by $\hat{S}, \hat{T}$ and $\hat{\pi}_i$, respectively.

Theorem 9.1. Let $a: \Gamma \times \Omega \to A$ be a cocycle for Γ on $(\Omega, \mathcal{F}, \nu)$. Then there exists a sequence of Borel maps $(\beta_n, n=0,1,2,\dots)$ from Ω to A such that

$$a(\gamma, \omega) = \sum_{k=0}^{\infty} (\beta_k(T^k \gamma \omega) - \beta_k(T^k \omega)) \tag{9.3}$$

for every $\gamma \in \Gamma$, $\omega \in \Omega$. Conversely, if $(\beta_n, n=0,1,\dots)$ is any sequence of Borel maps from Ω to A, and if $a: \Gamma \times \Omega \to A$ is given by (9.3), then a is a cocycle for Γ on $(\Omega, \mathcal{F}, \nu)$.

Proof: For every $n \geq 1$, consider the subgroup $\Gamma_n = \{\gamma \in \Gamma : \hat{\pi}_k(\gamma) = 1 \text{ for } k \geq n+1\}$ of Γ. If $\omega \in \Omega$ and $n \geq 1$, we put

$$\omega^{(n)} = (\omega_1, \omega_2, \dots, \omega_n, 1, 1, \dots) \in \Gamma_n$$

and

$$\omega_{(n)} = (1, 1, \dots, 1, \omega_{n+1}, \omega_{n+2}, \dots) \in \Omega.$$

For every $n \geq 1$, we define a map $\alpha_n : \Omega \to A$ by

$$\alpha_n(\omega) = a(\omega^{(n)}, \omega_{(n)}).$$

The cocycle identity implies

$$a(\gamma, \omega) = \alpha_n(\gamma\omega) - \alpha_n(\omega)$$

for every $\gamma \in \Gamma_n$ and $\omega \in \Omega$. If we fix n for the moment, we get

$$a(\gamma, \omega) = \alpha_n(\gamma\omega) - \alpha_n(\omega) = \alpha_{n+1}(\gamma\omega) - \alpha_{n+1}(\omega)$$

for every $\gamma \in \Gamma_n$. Hence there exists a Borel map β_n: $\Omega \to A$ with

$$\beta_n(T^n\omega) = \alpha_{n+1}(\omega) - \alpha_n(\omega).$$

If we now put $\beta_0 = \alpha_1$, we get, for every $\gamma \in \Gamma$, $\omega \in \Omega$,

$$a(\gamma,\omega) = \sum_{k=0}^{\infty}(\beta_k(T^k\gamma\omega) - \beta_k(T^k\omega)),$$

which proves (9.3). The converse is obvious.

Corollary 9.2. Let A_0 be a dense subgroup of A, and let $F(\Omega, A_0)$ denote the set of all Borel maps f: $\Omega \to A_0$ which depend only on finitely many coordinates. Let furthermore $Z^*(A_0)$ stand for the set of all cocycles a: $\Gamma \times \Omega \to A$ of the form $a(\gamma,\omega) = \sum_{k=0}^{\infty}(\beta_k(T^k\gamma\omega) - \beta_k(T^k\omega))$ with $\beta_k \in Z^*(A_0)$ for every $k \geq 0$. Then Z (A_0) is dense in $Z^1(\Gamma, A) = Z^1(\Gamma, \mathcal{B}(\Omega,\nu,A))$. Moreover, if a: $\Gamma \times \Omega \to A$ is any cocycle for Γ, then there exists an $a' \in Z^*(A_0)$ with $a' \sim a$.

Proof: Let a: $\Gamma \times \Omega \to A$ be any cocycle for Γ, and let $(\beta_k$, k=0,1,...) be a sequence of Borel functions satisfying (9.3). We recall the definition of the metric d_ν on $\mathcal{B}(\Omega,\nu,A)$ from (2.19). Since $F(\Omega, A_0)$ is dense in $\mathcal{B}(\Omega,\nu,A)$, we can find, for every $\varepsilon > 0$, and for every finite set $\{\gamma^{(1)},\dots,\gamma^{(1)}\} \subset \Gamma$, a sequence $(\beta'_k, k=0,1,\dots)$ in $F(\Omega, A_0)$ such that $d_\nu(\beta_k T^k, \beta'_k T^k) < \varepsilon . 2^{-k-2}$ and $d_\nu(\beta_k T^k \gamma^{(i)}, \beta'_k T^k \gamma^{(i)}) < \varepsilon . 2^{-k-2}$ for every $k \geq 0$, i=1,...,l.

We put $c = \sum_{k=0}^{\infty} \beta_k T^k - \beta'_k T^k$. c is a well defined element of $\mathcal{B}(\Omega, \nu, A)$. If we now put $a'(\gamma, \omega) = \sum_{k=0} (\beta'_k(T^k \gamma\omega) - \beta'_k(T^k\omega))$, we get $a(\gamma, \omega) = a'(\gamma, \omega) + c(\gamma\omega) - c(\omega)$ for every $\gamma \in \Gamma$ and for ν-a.e. $\omega \in \Omega$. Hence a and a' are cohomologous. Moreover we have, for every $i=1,\dots,l$, $d_\nu(a(\gamma^{(i)},.), a'(\gamma^{(i)},.)) = d_\nu(c(\gamma^{(i)}.), c(.)) < \varepsilon$. Hence $Z^*(A_0)$ is dense, and the proof is complete.

Theorem 9.1 allows explicit examples of cocycles with specified properties.

Example 9.3. Let ν_0 be the Haar measure on Ω and consider the action of Γ on $(\Omega, \mathcal{F}, \nu_0)$. If A is any locally compact second countable abelian group, let $(\alpha_i, i=0,1,2,\dots)$ be a dense sequence in A and let $(\tilde{\alpha}_i, i=0,1,2,\dots)$ be a sequence in A in which every α_i occurs infinitely often. Put, for every $k \geq 0$,

$$\beta_k(\omega) = \begin{cases} \tilde{\alpha}_k & \text{if } \pi_1(\omega) = \omega_1 = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$a_1(\gamma, \omega) = \sum_{k=0}^{\infty} (\beta_k(T^k \gamma\omega) - \beta_k(T^k\omega)). \tag{9.4}$$

$a_1 : \Gamma \times \Omega \to A$ is a cocycle with $E(a) = A$.

Suppose now that A_1 is a noncompact, compactly generated, locally compact second countable abelian group. Then A_1 will contain a closed subgroup $\{n\alpha_0 : n \in \mathbb{Z}\}$ which is isomorphic to $\mathbb{Z}$. For every $k \geq 0$, we define

$$\beta'_k(\omega) = \begin{cases} 2^k \cdot \alpha_0 & \text{if } \pi_1(\omega) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Put

$$a_2(\gamma, \omega) = \sum_{k=0}^{\infty} (\beta'_k(T^k \gamma \omega) - \beta'_k(T^k \omega)) \tag{9.5}$$

and

$$a_3(\gamma, \omega) = \sum_{k=0}^{\infty} (\beta'_k(T^{2k} \gamma \omega) - \beta'_k(T^{2k} \omega)). \tag{9.6}$$

Then a_2 is transient, a_3 is recurrent, and $\bar{E}(a_2) = \bar{E}(a_3) = \{0, \infty\}$. This example also implies that every ergodic measure preserving automorphism V of a nonatomic probability space $(X, \mathcal{S}, \mu)$ has a recurrent cocycle $a: \mathbb{Z} \times X \to A_1$ for the $\mathbb{Z}$-action $n \to V^n$ with $\bar{E}(a) = \{0, \infty\}$. The relevance of such cocycles has been discussed in § 7.

We now want to show that there always exist cocycles $a: \Gamma \times \Omega \to A$ with $E(a) = A$. We start with a few lemmas on cocycles for general ergodic group actions. Let T_G be an ergodic action of a countable group G on a nonatomic probability space $(X, \mathcal{S}, \mu)$, and let $a: G \times X \to A$ be a cocycle for T_G. We write $\tilde{a}: [T_G] \times X \to A$ for the map defined in Lemma 2.6.

Lemma 9.4. Suppose that $\alpha_0 \in E(a)$. If $B \in \mathcal{S}$ is a set of positive measure, and if $N(0)$ is a symmetric neighbourhood of 0 in A, there exists a $V \in [T_G]$ with

$$VB = B \pmod 0 \tag{9.7}$$

and

$$\tilde{a}(V,x) \in N(0)+\alpha_0 \cup N(0)-\alpha_0 \tag{9.8}$$

for μ-a.e. $x \in B$.

Proof: The assertion is obvious if $\alpha_0 = 0$. Let us therefore assume that $\alpha_0 \neq 0$. We apply Definition 3.1 to find a Borel set $B_1 \subset B$ and a $g_1 \in G$ with $\mu(B_1) > 0$, $T_{g_1}B_1 \cap B_1 = \emptyset$, $T_{g_1}B_1 \subset B$, and with $a(g_1,x) \in N(0)+\alpha_0$ for every $x \in B_1$. Replacing now B by $B \setminus (B_1 \cup T_{g_1}B_1)$, we choose $B_2 \in \mathcal{S}$ and $g_2 \in G$ with $\mu(B_2) > 0$, $B_2 \cup T_{g_2}B_2 \subset B \setminus (B_1 \cup T_{g_1}B_1)$, $B_2 \cap T_{g_2}B_2 = \emptyset$, and $a(g_2,x) \in N(0)+\alpha_0$ on B_2. Using induction and an exhaustion argument we construct (finite or infinite) sequences $(B_i,\ i=1,2,\ldots)$ of Borel sets of positive measure, $(g_i,\ i=1,2,\ldots) \subset G$, such that $B_i \cup T_{g_i}B_i \subset B$, $B_i \cap B_j = B_i \cap T_{g_i}B_i = B_i \cap T_{g_j}B_j = T_{g_i}B_i \cap T_{g_j}B_j = \emptyset$ for all i,j with $i \neq j$, $\mu(B \setminus \bigcup_i (B_i \cup T_{g_i}B_i)) = 0$, and $a(g_i,x) \subset N(0)+\alpha_0$ for every $i \geq 1$ and every $x \in B_i$. Finally we put

$$Vx = \begin{cases} T_{g_i}x & \text{for } x \in B_i,\ i \geq 1, \\ T_{g_i}^{-1}x & \text{for } x \in T_{g_i}B_i,\ i \geq 1, \\ x & \text{for } x \in X \setminus \bigcup(B_i \cup T_{g_i}B_i), \end{cases}$$

and the lemma is proved.

Lemma 9.5. Let $\alpha_0 \in E(a)$. Then there exists a set $B \in \mathcal{S}$ with $\mu(B) > 0$ and a symmetric neighbourhood $N(0)$ of 0 in A such that

$$\mu(B \cap V^{-1}B \cap \{x : a(V,x) \in N(0) + \alpha_0 \cup N(0) - \alpha_0\}) = 0 \qquad (9.9)$$

for every $V \in [T_G]$.

Proof: Trivial.

Lemma 9.6. Let $B \in \mathcal{S}$ with $\mu(B) > 0$, $\alpha_0 \in A$, and $\delta > 0$. For every syymetric neighbourhood $N(0)$ of 0 in A, the set

$$U(\alpha_0, N(0), B, \delta) = \{a \in Z^1(T_G, A) : \qquad (9.10)$$

$$\sup_{V \in [T_G]} \mu(B \cap V^{-1}B \cap \{x : a(V,x) \in N(0) + \alpha_0 \cup N(0) - \alpha_0\}) > \delta\}$$

is open in $Z^1(T_G, A) = Z^1(T_G, \mathcal{B}(X, \mu, A))$.

Proof: Left as an exercise.

Lemma 9.7. Let $(B_k,\ k=1,2,\ldots)$ be a sequence of Borel sets in X which is dense in $\mathcal{S}$ in the pseudometric $\eta(B,C) = \mu(B \Delta C)$, $B, C \in \mathcal{S}$, and let $\{N_k(0) : k \in \mathbb{N}\}$ be a basis of symmetric neighbourhoods of 0 in A. For every $\alpha_0 \in A$, let

$$U(\alpha_0) = \{a \in Z^1(T_G, A) : \alpha_0 \in E(a)\}. \qquad (9.11)$$

Then

$$U(\alpha_0) = \bigcap_{k,l\geq 1} U(\alpha_0, N_k(0), B_l, \frac{\mu(B_l)}{2}). \tag{9.12}$$

In particular, $U(\alpha_0)$ is a G_δ in $Z^1(T_G, A)$.

Proof: This follows from the Lemmas 9.4 - 9.6.

Lemma 9.8. Let

$$U(A) = \{a \in Z^1(T_G, A) : E(a) = A\}. \tag{9.13}$$

Then

$$U(A) = \bigcap_{k=1} U(\alpha_k), \tag{9.14}$$

where $(\alpha_k, k=1,2,\ldots)$ is a dense sequence in A and $U(\alpha_i)$ is given by (9.11). In particular, $U(A)$ is a G_δ in $Z^1(T_G, A)$.

The proof is again left as an exercise.

We now return to the action of Γ on $(\Omega, \mathcal{F}, \nu)$, where ν is a fixed nonatomic probability measure which is quasi-invariant and ergodic for Γ.

Lemma 9.9. Let $B \in \mathcal{F}$ be a set of positive measure. For every $k \geq 0$, there exists a $\gamma \in \hat{S}^k\Gamma$ such that $\gamma \neq 1$, but

$$\nu(B \cap \gamma B) > 0. \tag{9.15}$$

Proof: Left as an exercise.

Lemma 9.10. For every $B \in \mathcal{F}$ with $\nu(B) > 0$, every $\alpha_0 \in A$, and for every symmetric neighbourhood $N(0)$ of 0 in A, the set $U(\alpha_0, N(0), B, \nu(B)/2)$ is dense in $Z^1(\Gamma, A)$ (cf. (9.10)).

Proof: If $U(\alpha_0, N(0), B, \nu(B)/2)$ is not dense, there exists a nonempty open set $\mathcal{O} \subset Z^1(\Gamma, A)$ which does not intersect $U(\alpha_0, N(0), B, \nu(B)/2)$. Corollary 9.2 implies in particular that $Z^*(A)$ is dense in $Z^1(\Gamma, A)$. Hence there exists a cocycle $a_0 \in Z^*(A) \cap \mathcal{O}$. We choose a sequence of Borel maps $(\beta_k, k=0,1,\ldots)$ in $F(\Omega, A)$ such that $a_0(\gamma, \omega) = \sum_{k=0}^{\infty} (\beta_k(T^k \gamma\omega) - \beta_k(T^k \omega))$ for every γ, ω. Since $\mathcal{O}$ is open, there exists an $\varepsilon > 0$ and elements $\gamma^{(1)}, \ldots, \gamma^{(1)} \in \Gamma$ such that

$$\{a \in Z^1(\Gamma, A) : d_\nu(a(\gamma^{(i)}, \cdot), a_0(\gamma^{(i)}, \cdot)) < \varepsilon \text{ for } i=1,\ldots,l\} \subset \mathcal{O},$$

where d_ν is given by (2.19). We may choose positive integers $M_1 < M_2$ such that

$$a_0(\gamma^{(i)}, \omega) = \sum_{k=0}^{N_1} (\beta_k(T^k \gamma\omega) - \beta_k(T^k \omega))$$

for every $i=1,\ldots,l$ and for every $\omega \in \Omega$, and that

$$a_0(\gamma^{(i)}, \cdot) \text{ is invariant under } \hat{S}^{M_2}\Gamma \text{ for } i=1,\ldots,l.$$

By Lemma 9.9, there exists a $\tilde{\gamma}^{(1)} \in \hat{S}^{M_2}\Gamma$ and a set $B_1 \in \mathcal{F}$ with $\nu(B_1) > 0$, $B_1 \cap \tilde{\gamma}^{(1)} B_1 = \emptyset$, and $B_1 \cup \tilde{\gamma}^{(1)} B_1 \subset B$.

Since $\tilde{\gamma}^{(1)} \neq 1$, we can choose a $k_1 > M_2$ with $\hat{\pi}_{k_2}(\tilde{\gamma}^{(1)}) = -1$. Moreover there will exist an integer m_1 for which $\hat{\pi}_n(\tilde{\gamma}^{(1)}) = 1$ whenever $n \geq m_1$. We put $C_1 = B \setminus (B_1 \cup \tilde{\gamma}^{(1)} B_1)$. If $\nu(C_1) > 0$, we can again apply Lemma 9.9 to find $\gamma^{(2)} \in \hat{S}^{m_1}\Gamma$ and $B_2 \subset C_1$ with $\nu(B_2) > 0$, $B_2 \cup \tilde{\gamma}^{(2)} B_2 \subset C_1$, and $B_2 \cap \tilde{\gamma}^{(2)} B_2 = \emptyset$. Again we choose integers $m_2 > k_2 > m_1$ with $\pi_{k_2}(\tilde{\gamma}^{(2)}) = -1$ and with $\hat{\pi}_n(\tilde{\gamma}^{(2)}) = 1$ for $n \geq m_2$. In this fashion we construct sequences $(B_i,\ i=1,2,\ldots) \subset \mathcal{F}$, $(\tilde{\gamma}^{(i)},\ i=1,2,\ldots) \subset \Gamma$, $(k_i,\ i=1,2,\ldots) \in \mathbb{N}$, and $(m_i,\ i=1,2,\ldots) \subset \mathbb{N}$ such that $B_i \cup \tilde{\gamma}^{(i)} B_i \subset B$, $B_i \cap B_j = B_i \cap \tilde{\gamma}^{(i)} B_i = B_i \cap \tilde{\gamma}^{(j)} B_j = \tilde{\gamma}^{(i)} B_i \cap \tilde{\gamma}^{(j)} B_j = \emptyset$ for every i,j with $i \neq j$, $\hat{\pi}_{k_i}(\tilde{\gamma}^{(i)}) = -1$, $\hat{\pi}_n(\tilde{\gamma}^{(i)}) = 1$ for all $n \geq m_i$, and $k_i < m_i < k_{i+1} < m_{i+1}$ for all $i \geq 1$. An exhaustion argument shows that we may assume $\nu(B \setminus (\bigcup_{i \geq 1} (B_i \cup \tilde{\gamma}^{(i)} B_i))) = 0$. We now put

$$Vx = \begin{cases} x & \text{if } x \notin \bigcup_{i \geq 1} (B_i \cup \tilde{\gamma}^{(i)} B_i), \\ \tilde{\gamma}^{(i)} x & \text{if } x \in B_i \cup \tilde{\gamma}^{(i)} B_i,\ i \geq 1. \end{cases}$$

Then $V \in [\Gamma]$. We define a Borel map $\beta : \Omega \to A$ by

$$\beta(\omega) = \begin{cases} \alpha_0 \\ 0 \end{cases} \quad \text{according as} \quad \begin{matrix} \pi_1(\omega) = 1, \\ \pi_1(\omega) = -1, \end{matrix}$$

and put

$$a'(\gamma,\omega) = \sum_{k=0}^{N_1} (\beta_k(T^k \gamma\omega) - \beta_k(T^k \omega)) + \sum_{i \geq 1} (\beta(T^{k_i} \gamma\omega) - \beta(T^{k_i}\omega)).$$

It is clear that

$$a'(\gamma^{(i)},.) = a_0(\gamma^{(i)},.) \qquad (9.16)$$

for i=1,...,l. If $\tilde{a}'$ denotes the map from $[\Gamma]\times\Omega$ to A arising from a' through Lemma 2.6, we get

$$\tilde{a}'(V,\omega) = \sum_{k=0}^{N_1}(\beta_k(T^kV\omega)-\beta_k(T^k\omega)) + \sum_{i\geq 1}(\beta(T^{k_i}V\omega)-\beta(T^{k_i}\omega)) \qquad (9.17)$$

$$\in \{\alpha_0,-\alpha_0\}$$

for ν-a.e. $\omega\in$ B. (9.17) shows that $a'\in U(\alpha_0,N(0),B,\nu(B)/2)$, and (9.16) implies $a'\in\mathcal{O}$, which is absurd. This contradiction shows that $U(\alpha_0,N(0),B,\nu(B)/2)$ is dense, and the lemma is proved.

<u>Theorem 9.11.</u> Let T_G be an ergodic hyperfinite action of a countable group G on a nonatomic measure space $(X,\mathcal{S},\mu)$, and let A be a locally compact second countable abelian group. Let furthermore $U(A) = \{a\in Z^1(T_G,A):E(a) = A\}$. Then U(A) is a dense G_δ in $Z^1(T_G,A)$. Furthermore, $B^1(T_G,A)$ is dense in $Z^1(T_G,A)$.

Proof: From the discussion at the beginning of this section it is clear that one only has to prove the corresponding statements for every action of Γ on $(\Omega,\mathcal{F},\nu)$, where ν is a nonatomic probability measure which is quasi-invariant and ergodic under Γ. But there the first assertion follows from the Lemmas 9.7, 9.8 and 9.10, and from the fact that $Z^1(\Gamma,A)$ is a complete metric space and hence of second category. That $B^1(\Gamma,A)$ is dense in $Z^1(\Gamma,A)$ is an immediate consequence of Corollary 9.2. The proof is complete.

<u>Problem 9.11.</u> Is the statement of Theorem 9.10 true for every ergodic action of a countable group on a nonatomic measure space?

§ 10 On Radon Nikodym derivatives of quasi-invariant measures

In this section we apply the results of § 7 and § 9 to the following problem. Let $(X, \mathcal{S})$ be a standard Borel space, V a Borel automorphism of $(X, \mathcal{S})$, and $f: X \to \mathbb{R}$ a Borel map. When does there exist a nonatomic σ-finite measure μ on $(X, \mathcal{S})$ which is quasi-invariant and ergodic under V and which satisfies

$$\log \frac{d\mu V}{d\mu}(x) = f(x) \tag{10.1}$$

for μ-a.e. $x \in X$? If such a measure exists we shall say that (10.1) has a solution. We give two examples where (10.1) fails to have a solution.

(1) Let $X = \mathbb{R}$ and let $Vx = x+1$. Every σ-finite measure on X which is quasi-invariant and ergodic under V is atomic. (10.1) can thus never have a solution.

(2) Let V be a Borel automorphism of a standard Borel space $(X, \mathcal{S})$ and let $f: X \to \mathbb{R}$ be given by $f(x) = 1$ for every x. If there exists a solution μ for (10.1), then $a_\mu(n,x) = \log \frac{d\mu V^n}{d\mu}(x) = n$ for every n. This implies that a_μ is transient, contrary to Theorem 4.2. Again we conclude that (10.1) cannot have a solution for f=1.

In this section we shall not give a complete solution of the problem, but we shall derive some sufficient conditions for (10.1) to have a solution. We start with some notation.
Let $(\alpha_0, \alpha_1, \dots)$ be a sequence of rationally independent irrationals in $\mathbb{R}$ and let, for every $i \geq 0, n \geq 1$, $A_i^{(n)}$ be the countable dense subgroup of $\mathbb{R}^n$ given by

$$A_i^{(n)} = \{(k_1\alpha_{2i}+l_1\alpha_{2i+1}, \dots, k_n\alpha_{2i}+l_n\alpha_{2i+1}): (k_1,\dots,k_n),(l_1,\dots,l_n) \in \mathbb{Z}^n\}.$$

For every nonempty subset $E \subset \mathbb{N}$ we write $A^{(n)}(E)$ for the group generated by $\{A_i^{(n)} : i \in E\}$. Each $A^{(n)}(E)$ is a countable dense subgroup of $\mathbb{R}^n$, and $A^{(n)}(E_1) \cap A^{(n)}(E_2) = \{0\}$ whenever $E_1 \cap E_2 = \emptyset$. We also put $A^{(n)} = A^{(n)}(\mathbb{N})$.
Let now Γ and $(\Omega, \mathcal{F})$ be defined as in § 8, and let ν be a nonatomic probability measure on $(\Omega, \mathcal{F})$ such that

(a) ν is quasi-invariant and ergodic under Γ,

(b) $\log \dfrac{d\nu\gamma}{d\nu}$ is ν-essentially bounded for every $\gamma \in \Gamma$.

If $\nu' \sim \nu$ is any probability measure on $(\Omega, \mathcal{F})$, we define as before

$$a_{\nu'}(\gamma, \omega) = \log \frac{d\nu'\gamma}{d\nu'}(\omega)$$

for every $\gamma \in \Gamma$ and ν-a.e. $\omega \in \Omega$.

<u>Lemma 10.1.</u> There exists a probability measure $\nu' \sim \nu$ on $(\Omega, \mathcal{F})$ and a sequence of Borel maps $(\beta'_k,\ k=0,1,2,\dots)$ such that

(1) $\beta'_k \in F(\Omega, A_k^{(1)})$ for every $k \geq 0$, where $F(\Omega, A_k^{(1)})$ is defined as in Corollary 9.2, (10.2)

(2) $a_{\nu'}(\gamma, \omega) = \sum_{k=0}^{\infty} (\beta'_k(T^k \gamma\omega) - \beta'_k(T^k\omega))$ for every $\gamma \in \Gamma$ and for ν-a.e. $\omega \in \Omega$. (10.3)

Proof: By Theorem 9.1 there exists a sequence $(\beta_k,\ k=0,1,\dots)$ of real valued Borel maps on Ω such that

$$a_{\nu}(\gamma, \omega) = \sum_{k=0}^{\infty} (\beta_k(T^k\gamma\omega) - \beta_k(T^k\omega))$$

for every γ, ω. From the proof of Theorem 9.1 it is also clear that each β_k may be assumed to be bounded. The Stone-Weierstrass theorem implies that $F(\Omega, A_k^{(1)})$ is uniformly dense in the set of continuous real valued functions on Ω, for every $k \geq 0$. Since the contiuous functions lie dense in $\mathcal{B}(\Omega, \nu, \mathbb{R})$ in the topology of convergence in measure, we can choose a sequence $(\beta'_k,\ k=0,1,\dots)$ such that

(a) $\beta'_k \in F(\Omega, A_k^{(1)})$ for every k,

(b) $\sup_{\omega\in\Omega} |\beta'_k(\omega) - \beta_k(\omega)| \leq 2.\sup_{\omega\in\Omega} |\beta_k(\omega)| = 2m_k$,

and

(c) $\nu\{\omega : |\beta'_k(T^k\omega) - \beta_k(T^k\omega)| \geq 2^{-k-1}\} < e^{-2(\sum_{i=0}^{k+1} m_i + k)}$

for every $k \geq 0$. Since

$$\int e^{\sum_{k=0}^{\infty}(\beta'_k(T^k\omega)-\beta_k(T^k\omega))}\,d\nu(\omega) \leq$$

$$\int e^{\sum_{k=0}^{\infty}|\beta'_k(T^k\omega)-\beta_k(T^k\omega)|}\,d\nu(\omega) \leq$$

$$e^{2m_o} + \sum_{l=0}^{\infty} \nu\{\omega : e^{\sum_{k=0}^{\infty}|\beta'_k(T^k\omega)-\beta_k(T^k\omega)|} \geq e^{2.\sum_{i=0}^{l} m_i + l}\} . e^{2.\sum_{i=0}^{l+1} m_i + l + 1} =$$

$$e^{2m_o} + \sum_{l=0}^{\infty} \nu\{\omega : \sum_{k=0}^{\infty}|\beta'_k(T^k\omega)-\beta_k(T^k\omega)| \geq 2.\sum_{i=0}^{l} m_i + l\} . e^{2.\sum_{i=0}^{l+1} m_i + l + 1} \leq$$

$$e^{2m_o} + \sum_{l=0}^{\infty}\Big(\sum_{k=l+1}^{\infty} \nu\{\omega : |\beta'_k(T^k\omega)-\beta_k(T^k\omega)| \geq 2^{-k-1}\}\Big) . e^{2.\sum_{i=0}^{l+1} m_i + l + 1} \leq$$

$$e^{2m_o} + \sum_{l=0}^{\infty}\Big(\sum_{k=l+1}^{\infty} e^{-2(\sum_{i=0}^{k+1} m_i + k) + 2.\sum_{i=0}^{l+1} m_i + l + 1}\Big)$$

$$e^{2m_o} + \sum_{l=0}^{\infty} \sum_{k=l+1}^{\infty} e^{-k} < \infty ,$$

the function

$$e^{\sum_{k=0}^{\infty}(\beta'_k(T^k\omega)-\beta_k(T^k\omega))}$$

is integrable. Hence we can define a probability measure ν' on $(\Omega, \mathcal{F})$ by

$$d\nu'(\omega) = \text{const.}\, e^{\sum_{k=0}^{\infty}(\beta'_k(T^k\omega)-\beta_k(T^k\omega))}\, d\nu(\omega). \qquad (10.4)$$

It is clear that

$$a_{\nu'}(\gamma,\omega) = \sum_{k=0}^{\infty}(\beta'_k(T^k\gamma\omega) - \beta'_k(T^k\omega))$$

for every $\gamma \in \Gamma$ and for ν'-a.e. $\omega \in \Omega$, and the lemma is proved.

<u>Lemma 10.2.</u> Let $(\beta_k,\ k=0,1,\ldots)$ be a sequence in $F(\Omega, \mathbb{R}^n)$ and let $d\colon \Gamma \times \Omega \to \mathbb{R}^n$ be the cocycle

$$d(\gamma,\omega) = \sum_{k=0}^{\infty}(\beta_k(T^k\gamma\omega) - \beta_k(T^k\omega))$$

for Γ on $(\Omega, \mathcal{F}, \nu)$. If d is recurrent, there exists a cocycle $d'\colon \Gamma \times \Omega \to \mathbb{R}^n$ with the following properties:

(1) $d'(\gamma,\omega) \in A^{(n)}$ for every $\gamma \in \Gamma$, $\omega \in \Omega$, (10.5)

(2) for every $\alpha \in A^{(n)}$, $\alpha \neq 0$, and for every $\omega \in \Omega$, there exists an integer $k \geq 0$ with $d'(\gamma,\omega) \neq \alpha$ for every $\gamma \in \hat{S}^k\Gamma$, (10.6)

(3) for ν-a.e. $\omega \in \Omega$, the set $\{\gamma : d'(\gamma,\omega) = 0\}$ is infinite, (10.7)

(4) $\sup_{\gamma \in \Gamma} \sup_{\omega \in \Omega} \| d(\gamma,\omega) - d'(\gamma,\omega) \| \leq 1/2$, where $\|\cdot\|$ denotes the usual norm on $\mathbb{R}^n$, (10.8)

(5) d' is cohomologous to d. (10.9)

Proof: Let $(\tilde{\beta}_k : k=0,1,\ldots)$ be a sequence of Borel maps from Ω to $A^{(n)}$ such that

$$\tilde{\beta}_k \in F(\Omega, A_k^{(n)}) \tag{10.10}$$

and

$$\sup_{\omega\in\Omega} \|\tilde{\beta}_k(\omega)-\beta_k(\omega)\| < 2^{-k-4} \tag{10.11}$$

for every $k \geq 0$. We put

$$m(k) = \min \{ m \geq 0 : \tilde{\beta}_k \text{ is invariant under } \hat{S}^m \Gamma \} . \tag{10.12}$$

For every pair of integers $0 < M \leq N$, let

$$\Gamma(M,N) = \{ \gamma \in \Gamma : \hat{\pi}_k(\gamma) = 1 \text{ for } k < M \text{ and } k > N \} .$$

The cocycle

$$\tilde{d}(\gamma, \omega) = \sum_{k=0}^{\infty} (\tilde{\beta}_k(T^k \gamma\omega) - \tilde{\beta}_k(T^k\omega))$$

satisfies

$$\sup_{\gamma\in\Gamma} \sup_{\omega\in\Omega} \|\tilde{d}(\gamma,\omega)-d(\gamma,\omega)\| < 1/4, \tag{10.13}$$

by (10.11). Theorem 3.9 implies immediately that $\tilde{d} \sim d$.

We now assume d to be recurrent. $\tilde{d}$ will then also be recuurent.

Hence we have, for every $M \geq 1$, and for every $\delta > 0$,

$$\nu\Big(\bigcup_{\gamma \in \hat{S}^{M-1}\Gamma \setminus \{1\}} \{ \omega : \|\tilde{d}(\gamma,\omega)\| < \delta \} \Big) = 1.$$

This implies the following: For every $\delta, \varepsilon > 0$ and for every $M > 0$ there exists an integer $N(M, \delta, \varepsilon) \geq M$ with

$$\nu\Big(\bigcup_{\gamma \in \Gamma(M,N(M,\delta,\varepsilon)) \setminus \{1\}} \{ \omega : \|\tilde{d}(\gamma,\omega)\| < \delta \} \Big) > 1 - \varepsilon.$$

We now define a sequence (N_k, $k=0,1,\ldots$) by induction. Put $N_o = 0$, $N_1 = 1$ and $N_2 = N(1,2^{-4},2^{-2})+1$. If $N_o,\ldots,N_{2l}$ have already been chosen, put

$$N_{2l+1} = \max \{m(k)+k+1 : 0 \leq k \leq N_{2l}\}$$

and

$$N_{2l+2} = N(N_{2l+1}, 2^{-l-4}, 2^{-l}) + 1.$$

This determines an infinite sequence (N_k, $k=0,1,2,\ldots$). For every $k=0,1,2,\ldots$, we let

$$B_k = \bigcup_{\gamma \in \Gamma(N_{2k+1},N_{2k+2}) \setminus \{1\}} \{\omega : \|\tilde{d}(\gamma,\omega)\| < 2^{-k-4}\}$$

and

$$b_k(\omega) = \sum_{i=N_{2k}+1}^{N_{2k+2}} \tilde{\beta}_i(T^i\omega).$$

It is clear that

$$\nu(B_k) > 1-2^{-k}$$

and

$$\tilde{d}(\gamma,\omega) = b_k(\gamma\omega)-b_k(\omega)$$

for every $k \geq 0$ and for every $\gamma \in \Gamma(N_{2k+1},N_{2k+2})$, $\omega \in \Omega$. We now fix k, put

$$\Omega^{(k)} = \{\omega : \pi_i(\omega)=1 \text{ for } N_{2k+1} \leq i \leq N_{2k+2}\}$$

and choose an enumeration

$$1 = \gamma_1^{(k)}, \gamma_2^{(k)}, \ldots, \gamma_{L_k}^{(k)}$$

of $\Gamma(N_{2k+1},N_{2k+2})$, where $L_k = 2^{N_{2k+2}-N_{2k+1}+1}$.

For l,m with $1 \leq l,m \leq L_k$, let

$$B_{l,m}^{(k)} = \{ \omega \in \Omega^{(k)} : \| b_k(\gamma_l^{(k)}\omega) - b_k(\gamma_m^{(k)}\omega) \| < 2^{-k-4} \}.$$

Using these sets we define a Borel map $\tilde{b}_k : \Omega \to \mathbb{R}^n$ by an induction process. Let us first define b_k on $\Omega^{(k)}$: For every $\omega \in \Omega^{(k)}$, we set

$$\tilde{b}_k(\omega) = \begin{cases} b_k(\gamma_r^{(k)}\omega) & \text{whenever } \omega \in \bigcup_{m>1} B_{1,m}^{(k)}, \text{ where} \\ & r = \min\{m : 1 < m \leq L_k \text{ and } \omega \in B_{1,m}^{(k)}\}, \\ b_k(\omega) & \text{otherwise.} \end{cases} \qquad (10.16)$$

Suppose we have defined b_k on $\bigcup_{i=1}^{s-1} \gamma_i^{(k)}\Omega^{(k)}$, where $s > 1$. Let

$$b_k(\quad_s^{(k)}\quad) = \begin{cases} b_k(\gamma_s^{(k)}\omega) & \text{whenever } \omega \notin \bigcup_{m \neq s} B_{m,s}^{(k)}, \text{ or when-} \\ & \text{ever } b_k(\gamma_s^{(k)}\omega) = \tilde{b}_k(\gamma_i^{(k)}\omega) \\ & \text{for some } i < s. \\ b_k(\gamma_r^{(k)}\omega) & \text{whenever } b_k(\gamma_s^{(k)}\omega) \neq \tilde{b}_k(\gamma_i^{(k)}\omega) \\ & \text{for all } i < s \text{ and } \omega \in \bigcup_{m>s} B_{m,s}^{(k)}. \\ & \text{Here } r = \min\{m : s < m \leq L_k \text{ and } \omega \in B_{s,m}^{(k)}\}. \\ b_k(\gamma_t^{(k)}\omega) & \text{whenever } b_k(\gamma_s^{(k)}\omega) \neq \tilde{b}_k(\gamma_i^{(k)}\omega) \\ & \text{for all } i < s \text{ and } \omega \notin \bigcup_{m>s} B_{m,s}^{(k)}, \\ & \text{but } \omega \in \bigcup_{m<s} B_{m,s}^{(k)}. \text{ Here } t = \min\{m: \\ & 1 \leq m < s,\ \omega \in B_{m,s}^{(k)}, \text{ and } b_k(\gamma_m^{(k)}\omega) = \\ & \tilde{b}_k(\gamma_m^{(k)}\omega)\}. \end{cases} \qquad (10.17)$$

We claim that (10.17) defines $\tilde{b}_k(\gamma_s^{(k)}\cdot)$ everywhere on $\Omega^{(k)}$. To verify this we have to show that for every $\omega \in \bigcup_{m \neq s} B_{m,s}^{(k)}$ with $b_k(\gamma_s^{(k)}\omega) \neq \tilde{b}_k(\gamma_i^{(k)}\omega)$ for $1 \leq i < s$ and with $\omega \notin \bigcup_{m > s} B_{m,s}^{(k)}$, there exists an $m_o < s$ for which $\tilde{b}_k(\gamma_{m_o}^{(k)}\omega) = b_k(\gamma_{m_o}^{(k)}\omega)$. Let $\{m_1, \dots, m_p\}$ be the set of all $m < s$ for which $\omega \in B_{m,s}^{(k)}$, and assume that $1 \leq m_1 < m_2 < \dots < m_p < s$. If $\tilde{b}_k(\gamma_{m_p}^{(k)}\omega) \neq b_k(\gamma_{m_p}^{(k)}\omega)$ this means that $b_k(\gamma_{m_p}^{(k)}\omega) \neq \tilde{b}_k(\gamma_i^{(k)}\omega)$ for all $i < m_p$. On the other hand we have $s = \min\{m : m_p < m \leq L_k$, and $\omega \in B_{m_p,m}^{(k)}\}$. By our induction hypothesis we see that $\tilde{b}_k(\gamma_{m_p}^{(k)}\omega) = b_k(\gamma_s^{(k)}\omega)$, which is absurd. This contradiction implies that $\tilde{b}_k(\gamma_{m_p}^{(k)}\omega) = b_k(\gamma_{m_p}^{(k)}\omega)$. We conclude that $\tilde{b}_k(\gamma_s^{(k)}\cdot)$ is indeed defined everwhere on $\Omega^{(k)}$, for every $s=1,\dots,L_k$. We have thus constructed a Borel map $\tilde{b}_k : \Omega \to \mathbb{R}^n$. Obviously we have

$$\sup_{\omega \in \Omega} \| \tilde{b}_k(\omega) - b_k(\omega) \| < 2^{-k-4} \tag{10.18}$$

and

$$\tilde{b}_k(\omega) \quad A^{(n)}(\{N_{2k}, \dots, N_{2k+2}\}). \tag{10.19}$$

We now claim that

$$B_k \subset \bigcup_{\gamma \in \Gamma(N_{2k+1}, N_{2k+2}) \setminus \{1\}} \{\omega : \tilde{b}_k(\gamma\omega) = \tilde{b}_k(\omega)\}. \tag{10.20}$$

To prove (10.20), take $\omega \in B_k$. Then there exists an s with $1 \leq s \leq L_k$ such that $\omega \in \gamma_s^{(k)} \Omega^{(k)}$. Hence $\gamma_s^{(k)^{-1}} \omega = \omega_o \in \bigcup_{m \neq s} B_{m,s}^{(k)}$. A glance at (10.16) and (10.17) shows that in this case $\tilde{b}_k(\gamma_s^{(k)} \omega_o) = \tilde{b}_k(\gamma_{s'}^{(k)} \omega_o)$ for some $s' \neq s$, so that ω is contained in the right hand side of (10.20).

Varying k now, we obtain a sequence $(\tilde{b}_k,\ k=0,1,2,\ldots)$ of Borel maps satisfying (10.18) - (10.20) for every $k \geq 0$.

Let

$$d'(\gamma, \omega) = \tilde{d}(\gamma, \omega) - \sum_{k=0}^{\infty} (b_k(\gamma \omega) - \tilde{b}_k(\gamma \omega)) + \sum_{k=0}^{\infty} (b_k(\omega) - \tilde{b}_k(\omega)). \tag{10.21}$$

(10.18) implies that

$$\sup_{\gamma \in \Gamma} \sup_{\omega \in \Omega} \| d'(\gamma, \omega) - \tilde{d}(\gamma, \omega) \| < 2. \sum_{k=0}^{\infty} 2^{-k-4} = 1/4,$$

and (10.13) gives

$$\sup_{\gamma \in \Gamma} \sup_{\omega \in \Omega} \| d'(\gamma, \omega) - d(\gamma, \omega) \| \leq 1/2.$$

Once again Theorem 3.9 implies $d \sim d'$, and we have proved (10.8) and (10.9). It is clear that d' satisfies (10.5) and (10.6). To prove (10.7), consider the sets

$$C_k = \bigcup_{\gamma \in \Gamma(N_{2k+1}, N_{2k+2}) \setminus \{1\}} \{\omega : d'(\gamma, \omega) = 0\}.$$

(10.20) shows that $C_k \supset B_k$ for every k, and (10.14) gives

$$\nu(\lim_k \sup C_k) \geq \nu(\lim_k \sup B_k) = 1.$$

Since $\Gamma(N_{2k+1}, N_{2k+2}) \cap \Gamma(N_{2k'+1}, N_{2k'+2}) = \{1\}$ whenever $k \neq k'$, we have proved (10.7). The lemma is proved completely.

<u>Lemma 10.3.</u> Let a: $\Gamma \times \Omega \to \mathbb{R}^n$ be a cocycle for Γ on $(\Omega, \mathcal{F}, \nu)$, and let c: $\Gamma \times \Omega \to \mathbb{R}^{n+1}$ be the cocycle

$$c(\gamma, \omega) = (a_\nu(\gamma, \omega), a(\gamma, \omega)). \tag{10.22}$$

Suppose that c is recurrent. Then there exists a probability measure $\nu_1 \sim \nu$ on $(\Omega, \mathcal{F})$ and a cocycle $a_1 \sim a$ such that the following is true for a suitable choice of a_{ν_1}:

(1) $$c_1(\gamma, \omega) = (a_{\nu_1}(\gamma, \omega), a_1(\gamma, \omega)) \in A^{(n+1)} \tag{10.23}$$
for every $\gamma \in \Gamma$ and for every $\omega \in \Omega$.

(2) Let $A_d^{(n+1)}$ denote the group $A^{(n+1)}$ in the discrete topology. c_1 can then be considered as a cocycle taking values in $A_d^{(n+1)}$. To emphasize this we define $\hat{c}_1: \Gamma \times \Omega \to A_d^{(n+1)}$ to be the cocycle
$$\hat{c}_1(\gamma, \omega) = c_1(\gamma, \omega) \in A_d^{(n+1)}. \tag{10.24}$$
Then $\hat{c}_1$ is recurrent, and $\bar{E}(\hat{c}_1) = \{0, \infty\}$.

Proof: We choose a probability measure $\nu' \sim \nu$ on $(\Omega, \mathcal{F})$ which satisfies (10.2) and (10.3) in Lemma 10.1. Applying Corollary 9.2 to a, we find a sequence $(\beta_k, k=0,1,\dots)$ in $F(\Omega, \mathbb{R}^n)$ such that the cocycle

$$b(\gamma, \omega) = \sum_{k=0}^{\infty} (\beta_k(T^k \gamma\omega) - \beta_k(T^k \omega))$$

is cohomologous to a. Define d: $\Gamma \times \Omega \to \mathbb{R}^{n+1}$ by

$$d(\gamma,\omega) = (a_{\gamma'}(\gamma,\omega), b(\gamma,\omega)).$$

d will satisfy the conditions of Lemma 10.2. Hence there exists a cocycle d': $\Gamma \times \Omega \to \mathbb{R}^{n+1}$ fulfilling (10.5) - (10.9). We write

$$d(\gamma,\omega) = (d^{(1)}(\gamma,\omega),\ldots,d^{(n+1)}(\gamma,\omega))$$

and

$$d'(\gamma,\omega) = (d'^{(1)}(\gamma,\omega),\ldots,d'^{(n+1)}(\gamma,\omega))$$

for the n+1 coordinates of d and of d' respectively. Since d and d' are cohomologous, and from (10.11), (10.13) and (10.21) we see that there exists a Borel map f: $\Omega \to \mathbb{R}^{n+1}$ such that

$$d'(\gamma,\omega) = d(\gamma,\omega) + f(\gamma\omega) - f(\omega)$$

and

$$\| f(\omega) \| < 1/4 \tag{10.25}$$

for every $\gamma \in \Gamma$ and for every $\omega \in \Omega$. Again we write

$$f(\omega) = (f^{(1)}(\omega),\ldots,f^{(n+1)}(\omega))$$

for the coordinates of f. By (10.25), we can define a probability measure ν_1 on $(\Omega,\mathcal{F})$ with

$$d\nu_1(\omega) = \text{const.}\, e^{f_1(\omega)} \cdot d\nu'(\omega). \tag{10.26}$$

Furthermore we put

$$a'(\gamma,\omega) = (d'^{(2)}(\gamma,\omega),\ldots,d'^{(n+1)}(\gamma,\omega)). \tag{10.27}$$

Since $d'(\gamma,\omega) \in A^{(n+1)}$ for every γ,ω, we can define a cocycle $\hat{d}: \Gamma \times \Omega \to A_d^{(n+1)}$ with $\hat{d}(\gamma,\omega) = d'(\gamma,\omega)$ for every γ,ω. We choose an orbital cocycle $u: R(\Gamma) \to A_d^{(n+1)}$ for Γ on $(\Omega, \mathcal{F}, \nu_1)$ with $a_u \equiv \hat{d}$ (cf. (2.18)) and apply Theorem 8.18 to find a $V \in [\Gamma]$ with $\{V^k\omega : k \in \mathbb{Z}\} = \Gamma\omega$ for ν_1-a.e. $\omega \in \Omega$. Let

$$q(\omega) = \begin{cases} \min\{k>0 : u(V^k\omega,\omega)=0\} & \text{if } \{k>0 : u(V^k\omega,\omega)=0\} \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

and put

$$N = \Gamma(\{\omega : q(\omega)=0\} \cup \bigcup_{\gamma\in\Gamma} \{\omega : \hat{d}(\gamma,\omega) \neq u(\gamma\omega,\omega)\}).$$

(10.7) and a standard argument imply that $\nu_1(N) = 0$. We now define

$$W\omega = \begin{cases} V^{q(\omega)}\omega & \text{for } \omega \in \Omega \setminus N, \\ \omega & \text{otherwise.} \end{cases} \tag{10.28}$$

It is clear that $\{W^k\omega : k \in \mathbb{Z}\} = \{\gamma\omega : \gamma \in \Gamma$ and $u(\gamma\omega,\omega)=0\}$ for ν_1-a.e. $\omega \in \Omega$. In particular we see that W preserves the probability measure ν_1 and that ν_1-a.e. orbit of W is infinite. This forces W to be conservative, and we conclude that $\hat{d}$ is recurrent. From (10.6) we get $E(d) = \{0\}$, so that $\hat{d}$ is either a coboundary or a recurrent cocycle with $\bar{E}(\hat{d}) = \{0, \infty\}$.

Let us first assume that $\hat{d}$ is a coboundary. This implies in particular that $d': \Gamma \times \Omega \to \mathbb{R}^{n+1}$ is a coboundary and hence that $d'^{(1)}(\gamma, \omega) = \log \frac{d\nu_1\gamma}{d\nu_1}(\omega)$ is a coboundary. Hence ν_1 is equivalent to a σ-finite Γ-invariant measure on $(\Omega, \mathcal{F})$. In the following Lemma 10.4 we shall prove that there exists a cocycle $a^*: \Gamma \times \Omega \to A_d^{(n)}$ with the following properties:

(i) a^* is recurrent, and $\bar{E}(a^*) = \{0, \infty\}$,

(ii) a^* is a coboundary when considered as a cocycle taking values in $\mathbb{R}^n$ - i.e. there exists a Borel map $\phi: \Omega \to \mathbb{R}^n$ with $\phi(\gamma\omega) - \phi(\omega) = a^*(\gamma, \omega)$ for every $\gamma \in \Gamma$ and for ν_1-a.e. $\omega \in \Omega$.

We put $a_1(\gamma, \omega) = a'(\gamma, \omega) + a^*(\gamma, \omega)$, where a' is given by (10.27), and the measure ν_1 and the cocycle $a_1: \Gamma \times \Omega \to \mathbb{R}^n$ together will satisfy (1) and (2) in the statement of this lemma, so that the proof is complete in this case.
If d is not a coboundary, we simply set $a_1 = a'$, where a' is given by (10.21). Again ν_1 and a_1 will have the required properties, and the proof is complete.

<u>Lemma 10.4</u>. Let ν_0 be a nonatomic σ-finite measure on $(\Omega, \mathcal{F})$ which is invariant and ergodic under Γ. Then there exists a cocycle $a^*: \Gamma \times \Omega \to A_d^{(n)}$ for Γ on $(\Omega, \mathcal{F}, \nu_0)$ with the following properties.

(1) a^* is recurrent, and $\bar{E}(a^*) = \{0, \infty\}$,

(2) there exists a Borel map $\phi : \Omega \to R^n$ such that $\phi(\gamma\omega) - \phi(\omega) = a^*(\gamma, \omega)$ for every $\gamma \in \Gamma$, and for every $\omega \in \Omega$.

Proof: Theorem 8.18 shows that Γ is hyperfinite on $(\Omega, \mathcal{F}, \nu_0)$. Applying Theorem 8.15 we see that the action of Γ on $(\Omega, \mathcal{F}, \nu_0)$ is weakly equivalent to the action of Γ on $(\Omega, \mathcal{F}, \nu^*)$, where ν^* is either the Haar measure of Ω or equal to the infinite measure λ^* defined in the proof of Theorem 8.15. It will thus be enough to prove the result for Γ acting on $(\Omega, \mathcal{F}, \nu^*)$ with ν^* as above. We choose a sequence $(\alpha_0^*, \alpha_1^*, \ldots)$ of rationally independent irrationals in $A^{(1)}$ such that

$$\alpha_i^* \in A_i^{(1)}$$

and (10.29)

$$|\alpha_i^*| < 2^{-i}$$

for every $i = 0, 1, 2, \ldots$. Let $\beta_i^* : \Omega \to A_i^{(n)}$, $i = 0, 1, \ldots$ be given by

$$\beta_i^*(\omega) = \begin{cases} (\alpha_i^*, \ldots, \alpha_i^*) \in A_i^{(n)} & \text{if } \pi_1(\omega) = 1, \\ (0, \ldots, 0) \in A_i^{(n)} & \text{otherwise,} \end{cases} \tag{10.30}$$

and put

$$a^*(\gamma, \omega) = \sum_{k=0}^{\infty} (\beta_k^*(T^{4k}\gamma\omega) - \beta_k^*(T^{4k}\omega)). \tag{10.31}$$

As in Example 9.3 one can check that $a^*: \Gamma \times \Omega \to A_d^{(n)}$ is a recurrent cocycle with $\bar{E}(a^*) = \{0, \infty\}$. We now consider a^* as a cocycle taking values in $\mathbb{R}^n$. Explicitly we define $a^{**}: \Gamma \times \Omega \to \mathbb{R}^n$ by $a^{**}(\gamma, \omega) = a^*(\gamma, \omega)$ for every γ, ω. From (10.29)-(10.30) we see that

$$\sup_{\gamma \in \Gamma} \sup_{\omega \in \Omega} \| a^{**}(\gamma, \omega) \| \leq \sum_{k=0}^{\infty} n^{\frac{1}{n}}.2^{-k} = 2.n^{\frac{1}{n}},$$

so that a^{**} must be a coboundary by Theorem 3.9. The lemma is proved.

We can now state the first main result of this section.

<u>Theorem 10.5.</u> Let T_G be an ergodic hyperfinite action of a countable group G on a nonatomic measure space $(X, \mathcal{S}, \mu)$, and let $a: G \times X \to \mathbb{R}^n$, $n \geq 1$ and fixed, be a cocycle for T_G. We choose a cocycle $a_\mu: G \times X \to \mathbb{R}$ with

$$a_\mu(g,x) = \log \frac{d\mu T_g}{d\mu}(x) \tag{10.32}$$

for every $g \in G$ and for μ-a.e. $x \in X$, and define $c: G \times X \to \mathbb{R}^{n+1}$ by

$$c(g,x) = (a_\mu(g,x), a(g,x)). \tag{10.33}$$

If c is recurrent, there exists an uncountable family $\{\bar{M}_\xi : \xi \in \Xi\}$ of mutually inequivalent nonatomic σ-finite measures on $(X, \mathcal{S})$ such that

(1) for every $\xi \in \Xi$, $\bar{M}_\xi$ is invariant and ergodic for T_G,

(2) if $B \in \mathcal{S}$ satisfies $\bar{M}_{\xi}(B) = 0$ for every $\xi \in \Xi$, then $\mu(B) = 0$,

(3) for every $\xi \in \Xi$, a is a coboundary for T_G on $(X, \mathcal{S}, \bar{M}_{\xi})$.

Proof: Using Theorem 8.15, we find a probability measure ν on $(\Omega, \mathcal{F})$ such that T_G is weakly equivalent to Γ on $(\Omega, \mathcal{F}, \nu)$, and that $\log \frac{d\nu\gamma}{d\nu}$ is essentially bounded for every $\gamma \in \Gamma$. Applying now Lemma 10.3 and Theorem 3.16 we construct a probability measure $\mu_1 \sim \mu$ on $(X, \mathcal{S})$ and a cocycle $a_1 \sim a$ such that $c_1(g,x) = (a_{\mu_1}(g,x), a_1(g,x)) \in A^{(n+1)}$ for every $g \in G$, $x \in X$, where a_{μ_1} satisfies (10.32) with μ_1 replacing μ, and that $\hat{c}_1 : G \times X \to A_d^{(n+1)}$ is recuurent with $\bar{E}(\hat{c}_1) = \{0, \infty\}$. Here $\hat{c}_1$ is given by $\hat{c}_1(g,x) = c_1(g,x)$ for every g,x. We now have to consider the ergodic decomposition of the skew product $T_G^{\hat{c}_1}$ and appeal to § 7 for many of the notions introduced there. Put $\tilde{X} = X \times A_d^{(n+1)}$, write $\tilde{\mathcal{S}}$ for the product Borel field, and let λ be the Haar (i.e. the counting -) measure on $A_d^{(n+1)}$. We set $\tilde{\mu}_1 = \mu_1 \times \lambda$ and apply Lemma 7.24 to obtain the ergodic decomposition of $\tilde{\mu}_1$ with respect to $T_G^{\hat{c}_1}$. Let $(Y_1^*, \mathcal{T}_1^*, \rho_1^*)$ be the nonatomic probability space and $\{q_{y^*}: y^* \in Y_1^*\}$ the family of σ-finite measures on $(\tilde{X}, \tilde{\mathcal{S}})$ which satisfy (1) - (4) in Lemma 7.24. Consider now the measure on $(\tilde{X}, \tilde{\mathcal{S}})$ given by

$$d\tilde{\mu}_2(x,\alpha) = e^{-\alpha^{(1)}} d\mu_1(x)\, d\lambda(\alpha)$$

for every $x \in X$ and $\alpha = (\alpha^{(1)}, \dots, \alpha^{(n+1)}) \in A_d^{(n+1)}$. Since $\tilde{\mu}_2 \sim \tilde{\mu}_1$, and since $\tilde{\mu}_2$ is invariant under the skew product $T_G^{\hat{c}_1^2}$, we can apply Corollary 6.9 and Exercise 6.13 to show that ρ_1^*-a.e. $q_y^* \varkappa$ is equivalent to a σ-finite $T_G^{\hat{c}_1}$-invariant measure. Condition (3) in Lemma 7.24 and Fubini's theorem now imply that, for ρ_1-a.e. $y \in Y_1$, τ_y is equivalent to a σ-finite $T_G^{\hat{c}_1}$-invariant measure $\bar{\tau}_y$ on $(\tilde{X}, \tilde{\mathcal{S}})$ (for the definitions of these terms we refer to (7.25),(7.26), and (7.38) - (7.40)). Without loss in generality we assume that every τ_y, $y \in Y_1$, is equivalent to such an invariant measure $\bar{\tau}_y$. Exactly as in the proof of Corollary 7.25 one can now show that $\bar{M}_y = \bar{\tau}_y \pi_1^{-1}$ is a σ-finite measure on $(X, \mathcal{S})$ for every $y \in Y_1$, where $\pi_1 : \tilde{X} \to X$ is the first coordinate projection. It is clear that each $\bar{M}_y$ is invariant and ergodic under T_G.

We now have to digress for a moment. Since the cocycles c and c_1 are cohomologous, there exists a Borel map $f : X \to \mathbb{R}^{n+1}$ with $c_1(g,x) = c(g,x) + f(T_g x) - f(x)$ for every $g \in G$ and for μ_1-a.e. $x \in X$. We choose a T_G-invariant Borel set N of μ_1-measure zero such that $c_1(g,x) = c(g,x) + f(T_g x) - f(x)$ for every $g \in G$, $x \in X \setminus N$. (7.95) and (7.18) (with $\mu' = \mu_1$) imply that $\bar{M}_y(N) = 0$ for ρ_1-a.e. $y \in Y_1$, and once again we assume it to be true for every $y \in Y_1$. From the proof of Corollary 7.23 we see that $\hat{c}_1$ is a coboundary for every action T_G on $(X, \mathcal{S}, \bar{M}_y)$, $y \in Y_1$. Since we have for every $y \in Y_1$, every $g \in G$, and for $\bar{M}_y$-a.e. $x \in X$, $c_1(g,x) = c(g,x) + f(T_g x) - f(x)$, we see that c is also a coboundary for T_G on $(X, \mathcal{S}, \bar{M}_y)$. As in the proof of Corollary 7.23 we note that each $\bar{M}_y$ is equivalent to at most countably many other $\bar{M}_{y'}$, $y' \in Y_1$. Hence we can choose a maximal family $\{\bar{M}_\xi : \xi \in \Xi\}$ of mutually singular measures among the $\{M_y : y \in Y_1\}$, and Ξ will be uncountable. The proof is complete.

Corollary 10.6. Let T_G be an ergodic hyperfinite action of a countable group G on a nonatomic measure space $(X, \mathcal{S}, \mu)$ and let $a: G \times X \to \mathbb{R}$ be a cocycle for T_G. We define $a_\mu: G \times X \to \mathbb{R}$ and $c: G \times X \to R^2$ as in (10.32) and (10.33). If c is recurrent, there exists an uncountable family $\{M_\xi : \xi \in \Xi\}$ of mutually inequivalent nonatomic σ-finite measures on $(X, \mathcal{S})$ such that

(1) for every $\xi \in \Xi$, M_ξ is equivalent to a T_G-invariant and T_G-ergodic measure,

(2) if $B \in \mathcal{S}$ satisfies $M_\xi(B) = 0$ for every $\xi \in \Xi$, then $\mu(B) = 0$,

(3) for every $\xi \in \Xi$, and for M_ξ-a.e. $x \in X$, we have

$$\log \frac{dM_\xi T_g}{dM_\xi}(x) = a(g,x)$$

for every $g \in G$.

Proof: Let $\{\bar{M}_\xi : \xi \in \Xi\}$ be the family of measures arising in Theorem 10.5. Since c is a coboundary for T_G on every $(X, \mathcal{S}, \bar{M}_\xi)$, $\xi \in \Xi$, we can find Borel maps $f_\xi : X \to A_d^{(2)}$ with $c(g,x) = f_\xi(T_g x) - f_\xi(x)$ for every $g \in G$ and for $\bar{M}_\xi$ - a.e. $x \in X$. We write

$$f_\xi(x) = (f_\xi^{(1)}(x), f_\xi^{(2)}(x)) \in \mathbb{R}^2$$

for the two coordinates of f_ξ and put

$$dM_\xi(x) = e^{f_\xi^{(2)}(x)} d\bar{M}(x).$$

Clearly we have, for every $\xi \in \Xi$, every $g \in G$, and for M_ξ - a.e. $x \in X$,

$$\log \frac{dM_\xi T_g}{dM_\xi}(x) = a(g,x).$$

The proof is complete.

<u>Corollary 10.7</u>. Let T_G be an ergodic hyperfinite action of a countable group G on a nonatomic measure space $(X, \mathcal{S}, \mu)$ and let $a_\mu : G \times X \to \mathbb{R}$ be given by (10.32). Then there exists an uncountable family $\{M_\xi : \xi \in \Xi\}$ of mutually inequivalent nonatomic σ-finite measures on $(X, \mathcal{S})$ such that

(1) M_ξ is equivalent to a σ-finite T_G-invariant and T_G-ergodic measure on $(X, \mathcal{S})$ for every $\xi \in \Xi$,

(2) $\log \dfrac{dM_\xi T_g}{dM_\xi}(x) = a_\mu(g,x)$ for every $\xi \in \Xi$, $g \in G$, and for M_ξ -a.e. $x \in X$.

Proof: Theorem 4.2 shows that a_μ is a recurrent cocycle. Hence $(a_\mu, a_\mu): G \times X \to \mathbb{R}^2$ is recurrent, and we can apply Corollary 10.6 with $a = a_\mu$.

<u>Proposition 10.8.</u> Let $(X, \mathcal{S}, \mu)$ be a nonatomic probability space and let $V \in \mathcal{A}(X, \mathcal{S}, \mu)$ be ergodic and measure preserving. Then there exists a V-invariant Borel set $N \subset X$ such that

(1) $\mu(N) = 0$,

(2) if $\mu_1 \neq \mu$ is any probability measure on $(X, \mathcal{S})$ which is invariant and ergodic for V, then $\mu_1(N) = 1$.

Proof: Let $(X', \mathcal{S}', \mu')$ denote the unit interval $X' = [0,1)$ with its usual Borel field $\mathcal{S}'$ and with Lebesgue measure μ'. We define an ergodic measure preserving automorphism W of $(X', \mathcal{S}', \mu')$ by $Wx = x + \alpha \pmod 1$, where α is a fixed irrational. It is well known that μ' is the only W-invariant probability measure on $(X', \mathcal{S}')$. By Corollary 8.16 V and W are weakly equivalent. Hence there exists a Borel isomorphism $\phi: X \to X'$ with $\mu' = \mu\phi^{-1}$ and with $\phi\{V^k x : k \in \mathbb{Z}\} = \{W^k\phi(x) : k \in \mathbb{Z}\}$ for μ-a.e. $x \in X$. We choose a V-invariant Borel set $N \subset X$ such that $\mu(N) = 0$ and $\phi\{V^k x : k \in \mathbb{Z}\} = \{W^k \phi(x) : k \in \mathbb{Z}\}$ for every $x \in X \setminus N$. Since ϕ is an isomorphism, $N' = \phi(N) \in \mathcal{S}'$ in W-invariant in X', and $\mu'(N') = 0$. The restriction of W to $(X' \setminus N', \mathcal{S}_{X' \setminus N'})$

has $\mu'_{X'\setminus N'}$ as its only invariant probability measure. From Exercise 1.2 it follows that the restriction of V to $(X\setminus N, \mathcal{S}_{X\setminus N})$ has a unique invariant probability measure, namely $\mu_{X\setminus N}$. But this is equivalent to the statement we wanted to prove.

<u>Corollary 10.9</u>. Let $(X,\mathcal{S},\mu)$ be a nonatomic probability space and let T_G be an ergodic measure preserving action of a countable group G on $(X,\mathcal{S},\mu)$. Then there exists a T_G-invariant Borel set $N'\subset X$ such that

(1) $\mu(N') = 0$,

(2) if $\mu_1\neq\mu$ is any probability measure on $(X,\mathcal{S})$ which is invariant and ergodic under T_G, then $\mu_1(N') = 1$.

Proof: By Theorem 8.22 there exists an ergodic automorphism $V\in[T_G]$. We apply Proposition 10.8 and choose a V-invariant Borel set $N\subset X$ which satisfies (1) and (2) in Proposition 10.8. Put $N'=T_GN$. It it easy to see that N' will satisfy (1) and (2) in the statement of this corollary.

<u>Theorem 10.10</u>. Let V be a Borel automorphism of a standard Borel space $(X,\mathcal{S})$ and suppose that there exists a nonatomic σ-finite measure on $(X,\mathcal{S})$ which is quasi-invariant and ergodic under V. Then there exists an uncountable family of mutually inequivalent nonatomic, σ-finite, infinite

measures on $(X, \mathcal{S})$ each of which is invariant and ergodic under V.

Proof: Let μ be a nonatomic σ-finite measure on $(X, \mathcal{S})$ which is quasi-invariant and ergodic under V. We define the cocycle $a_\mu : \mathbb{Z} \times X \to \mathbb{R}$ for V (i.e. for the $\mathbb{Z}$-action $n \to V^n$) by (10.32) and choose a second coycle $a: \mathbb{Z} \times X \to \mathbb{R}$ for V by setting $a(n,x)=0$ for every n,x. Theorem 4.2 shows that $(a_\mu, a): \mathbb{Z} \times X \to \mathbb{R}^2$ is recurrent. Hence we can apply Corollary 10.7 to find an uncountable family $\{M_\xi : \xi \in \Xi\}$ of mutually inequivalent nonatomic σ-finite V-invariant and V-ergodic measures on $(X, \mathcal{S})$. If uncountably many of the measures $\{M_\xi : \xi \in \Xi\}$ are infinite, the theorem is proved. Otherwise, let M_{ξ_0}, $\xi_0 \in \Xi$, be a totally finite measure. We assume $M_0 = M_{\xi_0}$ to be a probability measure and apply Proposition 10.8 to find a **Borel** set $N_0 \subset X$ with $VN_0 = N_0$, and such that $\mu_1(N_0) = 1$ for every V-invariant and V-ergodic probability measure $\mu_1 \neq M_0$. Put $X' = X \setminus N_0$, $\mathcal{S}' = \mathcal{S}_{X'}$, $M' = M_{0_{X'}}$, and write V' for the restriction of V to X'. We apply Corollary 10.7 to find an uncountable number of mutually inequivalent nonatomic σ-finite V'-invariant and V'-ergodic measures on $(X', \mathcal{S}')$. The choice of X' implies that at most one of these measures can be equivalent to a probability measure, so that uncountably many among them must be infinite. We put these measures back onto $(X, \mathcal{S})$ in the obvious way, and the proof is complete.

We can now present a sufficient condition for equation (10.1) to have a solution.

<u>Theorem 10.11.</u> Let V be a Borel automorphism of a standard Borel space $(X, \mathcal{S})$ and let $f: X \to \mathbb{R}$ be a Borel map. Put

$$B_f = \{x \in X : \liminf_n \left| \sum_{k=0}^{n} f(V^k x) \right| = 0\}.$$

If there exists a nonatomic V-invariant and V-ergodic probability measure μ on $(X, \mathcal{S})$ with $\mu(B_f)=1$, then there also exists a nonatomic σ-finite measure μ' on $(X, \mathcal{S})$ which is quasi-invariant and ergodic for V, and which satisfies

$$\log \frac{d\mu' V}{d\mu'}(x) = f(x)$$

for μ'-a.e. $x \in X$.

Proof: Since μ is nonatomic and ergodic, we have $\mu\{x : V^n x = x\} = 0$ for every $n \neq 0$. Hence there exists a V-invariant Borel set $N \subset X$ with $VN=N$, $\mu(N)=0$, and such that $V^n x \neq x$ for every $x \in X \setminus N$ and every $n \neq 0$. Let $X' = X \setminus N$, $\mathcal{S}' = \mathcal{S}_{X'}$, $\mu' = \mu_{X'}$, V' the restriction of V to X', and f' the restriction of f to X'. Put

$$a'(n,x) = \begin{cases} \sum_{k=0}^{n-1} f'(V'^k x) & \text{for } n > 0,\ x \in X', \\ 0 & \text{for } n = 0,\ x \in X', \\ -a'(-n, V^n x) & \text{for } n < 0,\ x \in X'. \end{cases}$$

Then $a' : \mathbb{Z} \times X' \to \mathbb{R}$ is a cocycle for V' (i.e. for $n \to V'^n$) on $(X', \mathcal{S}', \mu')$, and we shall prove in Corollary 11.2

that a' is recurrent. Applying Corollary 10.7 to $a_{\mu'} = 0$ and a', we get an uncountable family of measures $\{M_\xi : \xi \in \Xi\}$, each of which satisfies

$$\log \frac{dM_\xi V'}{dM_\xi}(x) = f'(x)$$

for M_ξ-a.e. $x \in X'$. If we put any of these measures back onto $(X, \mathcal{S})$, we have proved the theorem.

§ 11 Cocycles for single measure preserving transformations

Let $(X,\mathcal{S},\mu)$ be a measure space and let A be a locally compact second countable abelian group. If $V\in\mathcal{A}(X,\mathcal{S},\mu)$ is ergodic, we can study cocycles for the $\mathbb{Z}$-action $n\to V^n$, $n\in\mathbb{Z}$, on $(X,\mathcal{S},\mu)$. Such cocycles are simply called <u>cocycles for V</u>. A cocycle $a:\mathbb{Z}\times X\to A$ for V is thus a Borel map satisfying

$$a(n,V^m x)-a(n+m,x)+a(m,x) = 0$$

for every $x\in X$, $n,m\in\mathbb{Z}$. Any such cocycle is uniquely determined by the function $f = a(1,.)$. Indeed we have, for every $x\in X$,

$$a(n,x) = \begin{cases} \sum_{k=0}^{n-1} f(V^k x) & \text{for } n\geq 1, & (11.1)\\ 0 & \text{for } n=0, & (11.2)\\ -a(-n,V^n x) & \text{for } n\leq -1. & (11.3)\end{cases}$$

Conversely, if $f:X\to A$ is any Borel map, formulae (11.1) - (11.3) define a cocycle a for V. a will be a coboundary if and only if

$$f(x) = c(Vx)-c(x) \qquad (11.4)$$

for μ-a.e.$x\in X$, where $f:X\to A$ is a Borel map.

Throughout this section we shall assume that μ is a non-atomic probability measure which is invariant and ergodic for V.

Theorem 11.1. Let V be an ergodic measure preserving automorphism of a nonatomic probability space $(X, \mathcal{S}, \mu)$ and let $a: \mathbb{Z} \times X \to A$ be a cocycle for V. Then the following conditions are equivalent:

(1) a is recurrent,

(2) for every neighbourhood N(0) of 0 in A,

$$\mu\left(\bigcup_{n \geq 1} \{x : a(n,x) \in N(0)\}\right) = 1. \qquad (11.5)$$

Proof: Let $\tilde{X} = X \times A$, put $\tilde{\mathcal{S}}$ equal to the product Borel field, and set $\tilde{\mu} = \mu \times \lambda$, where λ is the Haar measure on A. As in § 5 we define a skew product transformation V_a on $(\tilde{X}, \tilde{\mathcal{S}}, \tilde{\mu})$ by

$$V_a(x, \alpha) = (Vx, \alpha + a(1,x)) \qquad (11.6)$$

for every $(x, \alpha) \in \tilde{X}$. V_a preserves $\tilde{\mu}$ and is conservative if and only if a is recurrent (Theorem 5.5). Assume now a to be recurrent. We fix a neighbourhood N(0) of 0 in A and choose a neighbourhood $N_1(0)$ with $N_1(0) + N_1(0) \subset N(0)$. If $\mathcal{O} = X \times N_1(0)$, the recurrence of a implies that, for μ-a.e. $(x, \alpha) \in \mathcal{O}$, we have $V_a^k(x, \alpha) \in \mathcal{O}$ for infinitely many $k \in \mathbb{Z}$ (cf. (1.7)). Let $D = \{(x, \alpha) \in \mathcal{O} : V_a^k(x, \alpha) \notin \mathcal{O}$ for every $k > 0\}$. It is easy to see that $V_a^k D \cap V_a^l D = \emptyset$ for $k \neq l$. Since V_a is conservative, we have $\mu(D) = 0$. Fubinis theorem now implies that for μ-a.e. $x \in X$ there exists a $k > 0$ (depending on x) such that $a(k,x) \in N(0)$,

so that we have proved that (1) implies (2).
To prove the converse, assume that (2) is satisfied. We choose an invariant metric δ on A. If a is transient, there exists a $B \in \mathcal{S}$ with $\mu(B) > 0$ and an $\varepsilon > 0$ such that

$$B \cap V^{-n}B \cap \{x: \delta(a(n,x),0) < \varepsilon\} = \emptyset \tag{11.7}$$

for every $n \neq 0$. Decreasing ε if necessary, we may also assume that

$$B(0, \varepsilon/4) = \{\alpha \in A: \delta(\alpha,0) < \varepsilon/4\}$$

has compact closure. Put $\mathcal{O} = X \times B(0, \varepsilon/4)$. Then $0 < \tilde{\mu}(\mathcal{O}) < \infty$. We shall first prove that, for $\tilde{\mu}$-a.e.$(x, \alpha) \in \mathcal{O}$, $V_a^k(x, \alpha) \in \mathcal{O}$ for infinitely many $k \in \mathbb{Z}$. Indeed, let

$$F_{k,l} = \bigcup_{m \in \mathbb{Z}} \bigcap_{n \geq 1} \{x: \delta(a(n,V^m x),0) \geq 1/l.2^k\}$$

for every $k,l \geq 1$. (11.5) implies that $\mu(F_{k,l}) = 0$ for every k,l. Put $F_l = \bigcup_{k,l} F_{k,l}$. For every $x \in X \setminus F_l$, there exist

$$n_1 \geq 1 \quad \text{with} \quad \delta(a(n_1,x),0) < 1/l.2,$$

$$\vdots$$

$$n_k \geq 1 \quad \text{with} \quad \delta(a(n_k,V^{n_1+\dots+n_{k-1}}x),0) < 1/l.2^k,$$

$$\vdots$$

etc.

Hence we have, for every $x \in X \setminus F_l$ and for every k=1,2,...,

$$\delta(a(n_1+\dots+n_k,x),0) \leq \sum_{l=1}^{k} \delta(a(n_l, V^{n_1+\dots+n_{l-1}}x),0) < 1/l,$$

which in turn implies that

$$\mu\Big(\bigcap_{k\geq 1}\bigcup_{n\geq k}\{x: \delta(a(n,x),0) < 1/l\}\Big) = 1 \qquad (11.8)$$

for every $l \geq 1$. From (11.8) it is clear that, for $\tilde{\mu}$-a.e. $(x,\alpha) \in \mathcal{O}$, $V_a^k(x,\alpha) \in \mathcal{O}$ for infinitely many $k > 0$. Having established this, consider the set

$$D = B \times B(0, \varepsilon/4),$$

with B given by (11.7). It is clear that $V_a^k D \cap V_a^{k'} D = \emptyset$ whenever $k \neq k'$. On the other hand we have $V_a^k(x,\alpha) \in \mathcal{O}$ for infinitely many $k > 0$, for μ-a.e. $(x,\alpha) \in D$. Combining these two statements, we get $\tilde{\mu}(\mathcal{O}) = \infty$, which is impossible. Hence a must be recurrent, and the theorem is proved.

<u>Corollary 11.2.</u> Let V be an ergodic measure preserving automorphism of a nonatomic probability space $(X, \mathcal{S}, \mu)$, and let $a: \mathbb{Z} \times X \to A$ be a cocycle for V. Then the following conditions are equivalent:

(1) a is recurrent,

(2) for every neighbourhood N(0) of 0 in A, we have

$$\mu\Big(\bigcap_{k=1}^{\infty}\bigcup_{n\geq k}\{x: a(n,x) \in N(0)\}\Big) = 1, \qquad (11.9)$$

(3) $\liminf_n \delta(a(n,x),0) = 0$ for μ-a.e. $x \in X$. (11.10)

Proof: If a is recurrent, a will satisfy (11.5). Hence a will satisfy (11.8), which in turn implies (11.9). (11.10) is an immediate consequence of (11.9). (11.10) implies (11.5), and hence, by Theorem 11.1, the recurrence of a. The proof is complete.

Corollary 11.3. Let V be an ergodic measure preserving automorphism of a nonatomic probability space $(X, \mathcal{S}, \mu)$ and let $a: \mathbb{Z} \times X \to A$ be a cocycle for V. Then the following conditions are equivalent:

(1) a is transient,

(2) for every compact set $K \subset A$,

$$\mu\left(\bigcap_{k=1}^{\infty} \bigcup_{|n| \geq k} \{x : a(n,x) \in K\}\right) = 0, \tag{11.11}$$

(3) $\lim_n a(n,x) = \infty$ for μ-a.e. $x \in X$. (11.12)

Proof: (2) and (3) are equivalent and imply (1) by Corollary 11.2. Assume now that a is transient. Let $K \subset A$ be a compact set and let δ be an invariant metric on A. We choose $\varepsilon > 0$ such that $\lambda\{\alpha \in A : \delta(\alpha, K) < \varepsilon\} < \infty$. Put

$$\sigma = X \times \{\alpha : \delta(\alpha, K) < \varepsilon\},$$

$$B = \bigcap_{k \geq 1} \bigcup_{|n| \geq k} \{x : a(n,x) \in K\},$$

and

$$C = B \times \{\alpha \in A : \delta(\alpha, 0) < \varepsilon\}.$$

Our assumptions imply that $0 < \tilde{\mu}(\sigma) < \infty$. If $\mu(B) > 0$, we can find a subset $D \subset B$ with $\mu(D) > 0$ and a $\xi > 0$ such that

$$D \cap V^{-n} D \cap \{x: \delta(a(n,x),0) < \xi\} = \phi$$

whenever $n \neq 0$ (cf. Exercise 3.20). We may assume that $\xi < \varepsilon$ and put

$$E = D \times \{\alpha \in A: \delta(\alpha,0) < \xi/4\}.$$

Then $V_a^k E \cap V_a^{k'} E = \phi$ whenever $k \neq k'$, $\tilde{\mu}(E) > 0$, and $V_a^n(x,\alpha)$ will lie in $\mathcal{O}$ for infinitely many n, for $\tilde{\mu}$-a.e. $(x,\alpha) \in E$. Again this is impossible, since it would force $\mathcal{O}$ to have infinite measure. So B must be a null set, and the corollary is proved.

For real valued cocycles one can prove a much more useful result:

Theorem 11.4. Let V be an ergodic measure preserving automorphism of a nonatomic probability space $(X, \mathcal{S}, \mu)$ and let $a: \mathbb{Z} \times X \to \mathbb{R}$ be a cocycle for V. Suppose that

$$\int |a(1,x)| \, d\mu(x) < \infty. \tag{11.13}$$

Then the following conditions are equivalent.

(1) a is recurrent, (11.14)

(2) $\int a(1,x) \, d\mu(x) = 0$.

Proof: Suppose $a(1,.)$ is integrable and $\int a(1,x) \, d\mu(x) = c_0 \neq 0$.

From the individual ergodic theorem we get

$$\lim_n \frac{a(n,x)}{n} = c_0$$

and hence

$$\lim_n |a(n,x)| = \infty$$

for μ-a.e. $x \in X$. Corollary 11.3 shows that a is transient. To prove the converse, assume that $\int a(1,x)\, d\mu(x) = 0$, but that a is transient. We choose $\varepsilon > 0$ and $B \in \mathcal{S}$ with $\mu(B) > 0$ and with

$$B \cap V^{-n}B \cap \{x : |a(n,x)| < \varepsilon\} = \emptyset \qquad (11.16)$$

for $n \neq 0$. As in (8.6) we set, for every $x \in B$,

$$n(x) = \begin{cases} \min \{n \geq 1 : V^n x \in B\} & \text{if } \{n \geq 1 : V^n x \in B\} \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

and put

$$N_1 = \bigcup_{k \in \mathbb{Z}} V^k \{x : n(x) = 0\},$$

$$N_2 = \bigcup_{k \neq 0} \{x : V^k x = x\},$$

and

$$C = B \setminus N_1 \cup N_2.$$

For every $x \in C$ we define the induced transformation V_C by

$$V_C x = V^{n(x)} x.$$

For every $x \in C$ and for every $k \geq 1$, there exists a unique integer $n(k,x)$ with

$$V^{n(k,x)}x = V_C^k x.$$

In fact, $n(k,x)$ is given recursively by

$$n(1,x) = n(x)$$

and

$$n(k,x) = n(V_C^{k-1}x)+n(k-1,x).$$

The ergodic theorem implies that for μ_C-a.e. $x \in C$,

$$\lim_k \frac{k}{n(k,x)} = \mu(C) \quad (= \mu(B)).$$

We can thus find a set $C_1 \subset C$ with $\mu(C_1)>0$ and an integer $N>0$ with

$$\frac{k}{n(k,x)} \geq \frac{\mu(C)}{2} \tag{11.17}$$

whenever $x \in C_1$ and $k \geq N$. (11.16) implies that

$$|a(n(k,x),x)| \geq \varepsilon$$

for every $x \in C$ and $k \geq 1$. It follows that for every $x \in C$, and every $k_1 \neq k_2$,

$$|a(n(k_1,x),x)-a(n(k_2,x),x)| \geq \varepsilon \ .$$

We conclude that, for every $x \in C$ and for every $k \geq 1$, there exists an integer $k_0(x)$ with $0 < k_0(x) \leq k$ and

$$|a(n(k_o(x),x),x)| \geq \frac{\varepsilon . k}{2}.$$

In particular we get

$$\lim_k \sup \frac{|a(n(k,x),x)|}{k} \geq \varepsilon/2 \qquad (11.18)$$

for every $x \in C$. Combining (11.17) and (11.18) we get, for every $x \in C_1$,

$$\lim_n \sup \frac{|a(n,x)|}{n} \geq \lim_k \sup \frac{|a(n(k,x),x)|}{n(k,x)} \geq \frac{\varepsilon . \mu(C)}{4}.$$

But the individual ergodic theorem implies

$$\lim_n \frac{a(n,x)}{n} = \int a(1,x)\, d\mu(x) = 0$$

for μ-a.e. $x \in X$, and this contradiction shows that a is recurrent. The proof is complete.

We now turn to the problem of determining whether a given cocycle for an ergodic measure preserving automorphism of a nonatomic probability space is a coboundary. We start with a general result about measure preserving group actions.

Proposition 11.5. Let $(X, \mathcal{S}, \mu)$ be a nonatomic probability space, let T_G be an ergodic measure preserving action of a countable group G on $(X, \mathcal{S}, \mu)$, and let A be a locally compact second countable abelian group whose only compact subgroup is $\{0\}$. Let $a: G \times X \to A$ be a cocycle for T_G, and form the skew product T_G^a on $(\tilde{X}, \tilde{\mathcal{S}}, \tilde{\mu})$ as defined in

(5.1) - (5.3) and (5.6). We consider the ergodic decomposition of $\tilde{\mu}$ with respect to T_G^a. Let $(Y, \mathcal{T}, \rho)$ be a measure space and let $\{q_y : y \in Y\}$ be a family of σ-finite measures on $(\tilde{X}, \tilde{\mathcal{S}})$ which satisfy the following conditions.

(1) For every $B \in \tilde{\mathcal{S}}$, the map $y \to q_y(B)$ is Borel from Y to $\overline{\mathbb{R}}$.

(2) For every $B \in \tilde{\mathcal{S}}$, $\tilde{\mu}(B) = \int q_y(B)\,d\rho(y)$.

(3) Every q_y, $y \in Y$, is invariant and ergodic under T_G^a.

(4) For every $y \neq y'$, q_y and $q_{y'}$ are mutually singular.

(5) Let $\tilde{\mathcal{S}}^* = \{B \in \tilde{\mathcal{S}} : T_G^a B = B\}$. For every $B \in \tilde{\mathcal{S}}^*$, we put $B_Y = \{y \in Y : q_y(B) > 0\}$. Then $\tilde{\mathcal{S}}_Y^* = \{B_Y : B \in \tilde{\mathcal{S}}^*\}$ is equal to $\mathcal{T}$ modulo sets of ρ-measure zero.

Then the following is true:

(6) $\bar{E}(a) = \{0\}$ (i.e. a is a coboundary) if and only if q_y is totally finite for ρ-a.e. $y \in Y$.

(7) $\bar{E}(a) \neq \{0\}$ if and only if q_y is infinite for ρ-a.e. $y \in Y$.

Proof: Suppose that $\bar{E}(a) = \{0\}$. Then there exists a Borel map $b: X \to A$ with $a(g,x) = b(T_g x) - b(x)$ for every $g \in G$ and for μ-a.e. $x \in X$. Let $Y = A$ and choose $\mathcal{T}$ to be the Borel field and $\rho = \lambda$ the Haar measure of A. For every $\alpha \in A$

we define a map $\phi_\alpha : X \to \tilde{X}$ by setting $\phi_\alpha(x) = (x, \alpha + b(x))$ for every $x \in X$. If we now put, for every $\alpha \in A$,

$$q_\alpha = \mu \phi_\alpha^{-1},$$

we obtain a family $\{q_\alpha : \alpha \in Y\}$ of probability measures on $(\tilde{X}, \tilde{\mathcal{S}})$. It is easy to see that $(Y, \mathcal{T}, \rho)$ and $\{q_\alpha : \alpha \in Y\}$ satisfy (1) - (5). The uniqueness of the ergodic decomposition now implies that we have proved the following: If a is a coboundary, and if $(Y, \mathcal{T}, \rho)$ and $\{q_y : y \in Y\}$ satisfy (1) - (5), the q_y is totally finite for ρ-a.e. $y \in Y$. Next we assume that $\bar{E}(a) = \{0, \infty\}$. It is clear that the statement of Proposition 11.5 is not affected if we remove a T_G-invariant set of measure zero from X. Using Corollary 10.9 we may thus assume that μ is the only probability measure on $(X, \mathcal{S})$ which is invariant and ergodic under T_G. The ergodic decomposition of $\tilde{\mu}$ with respect to T_G^a is described in Lemma 7.24, and as there we denote it by $(Y_1^*, \mathcal{T}_1^*, \rho_1^*)$ and $\{q_{y^*}^* : y^* \in Y_1^*\}$, where we assume $q_{y^*}^*$ to be invariant under T_G^a for every $y^* \in Y_1^*$ (cf. Corollary 6.9). We write $\pi_1 : \tilde{X} \to X$ for the projection onto the first coordinate. As in the proof of Corollary we show that $q_{y^*}^* \pi_1^{-1}$ is a σ-finite T_G-invariant and T_G-ergodic measure on $(X, \mathcal{S})$ for every $y^* \in Y_1^*$. The assumption that μ is the only T_G-invariant probability measure on $(X, \mathcal{S})$ implies that the measures $q_{y^*}^* \pi_1^{-1}$ are infinite for ρ_1^*-a.e. $y^* \in Y_1^*$. We have thus proved – using the uniqueness of the ergodic decomposition – that whenever $\bar{E}(a) = \{0, \infty\}$ and whenever $(Y, \mathcal{T}, \rho)$ and $\{q_y : y \in Y\}$ satisfy (1) – (5), then q_y is infinite for ρ-a.e. $y \in Y$.

Finally we have to deal with the case where $E(a) \neq \{0\}$. Since A does not have any nontrivial compact subgroups, the Haar measure λ_0 of E(a) will be infinite. We put $A^* = A/E(a)$ and write λ^* for the Haar measure of A^*. There exists a Borel map $\psi : A^* \to A$ with $\psi(\alpha^*) + E(a) = \alpha^*$ for every $\alpha^* \in A^*$ (cf. Lemma I.5.1 in [34]), and we may assume the measures λ_0 and λ^* to satisfy

$$\lambda(B) = \int_{A^*} \int_{E(a)} \chi_B(\psi(\alpha^*) + \beta)\, d\lambda_0(\beta)\, d\lambda^*(\alpha^*)$$

for every Borel set $B \subset A$, where χ_B is the characteristic function of B. As in Lemma 3.10 we write $a^* : G \times X \to A^*$ for the cocycle $a^*(g,x) = a(g,x) + E(a)$. Suppose we have found a measure space $(Y, \mathcal{T}, \rho)$ and a family of σ-finite measures $\{q_y^* : y \in Y\}$ on $X \times A^*$ satisfying (1) – (5) for the cocycle a^*. For every $y \in Y$ we define a measure q_y on $(\tilde{X}, \tilde{\mathcal{S}})$ by

$$q_y(B) = \int_{X \times A^*} \int_{E(a)} \chi_B(x, \psi(\alpha^*) + \beta)\, d\lambda_0(\beta)\, dq_y^*(x, \alpha^*)$$

for every $B \in \tilde{\mathcal{S}}$. Every q_y will be infinite and σ-finite, and one can easily check that $(Y, \mathcal{T}, \rho)$ and $\{q_y : y \in Y\}$ satisfies (1) – (5) for the cocycle a. Together with the uniqueness of the ergodic decomposition this completes the proof of the proposition.

<u>Lemma 11.6.</u> Let $(X, \mathcal{S}, \mu)$ be a nonatomic infinite measure space and let V be a measure preserving automorphism of $(X, \mathcal{S}, \mu)$. Suppose $(Y, \mathcal{T}, \rho)$ is a measure space and $\{q_y : y \in Y\}$ a family of σ-finite measures on $(X, \mathcal{S})$ which form an ergodic decomposition of μ with respect to V:

(1) For every $B \in \mathcal{S}$, the map $y \to q_y(B)$ is Borel from Y to $\overline{\mathbb{R}}$,

(2) For every $B \in \mathcal{S}$, $\mu(B) = \int q_y(B)\, d\rho(y)$.

(3) Every q_y, $y \in Y$, is invariant and ergodic under V.

(4) For every $y' \neq y$, $q_{y'}$ and q_y are mutually singular.

(5) Let $\mathcal{S}^* = \{B \in \mathcal{S} : VB = B\}$. For every $B \in \mathcal{S}^*$, put $B_Y = \{y \in Y : q_y(B) \neq 0\}$. Then $\mathcal{S}^*_Y = \{B_Y : B \in \mathcal{S}^*\}$ is equal to $\mathcal{T}$ modulo sets of ρ-measure zero.

Suppose furthermore that q_y is infinite for ρ-a.e. $y \in Y$. Then we have, for every ρ-integrable real valued Borel map f on X,

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} f(V^k x) = 0 \text{ for } \mu\text{-a.e. } x \in X.$$

Proof: The set $C = \{x \in X : \lim_n \sup \left| \frac{1}{n} \sum_{k=0}^{n-1} f(V^k x) \right| \neq 0\}$ is Borel, and the individual ergodic theorem implies that $q_y(C) = 0$ for ρ-a.e. $y \in Y$. Hence $\mu(C) = 0$, by (2), and the proof is complete.

<u>Lemma 11.7.</u> Let $(X, \mathcal{S}, \mu)$ be a nonatomic probability space, V an ergodic measure preserving automorphism of $(X, \mathcal{S}, \mu)$, and $a: \mathbb{Z} \times X \to \mathbb{R}$ a cocycle for V. For every $n \in \mathbb{Z}$, we define a probability measure τ_n^a on $\mathbb{R}$ by

$$\tau_n^a(B) = \mu\{x \in X : a(n,x) \in B\} = \mu a(n,.)^{-1}(B) \qquad (11.19)$$

for every $B \in \mathcal{S}$. Then the following conditions are equivalent.

(1) The family $\{\tau_n^a : n \in \mathbb{Z}\}$ is uniformly tight,

(2) the family $\{\tau_n^a : n \geq 1\}$ is uniformly tight,

(3) a is a coboundary.

Proof: The equivalence of (1) and (2) is obvious. Assume now that a is a coboundary, and choose a Borel map $b: X \to \mathbb{R}$ with $a(n,x) = b(V^n x) - b(x)$ for every n and for μ-a.e. $x \in X$. Let $\nu = \mu b^{-1} = \mu(b.V^n)^{-1}$. For every fixed $\varepsilon > 0$, we choose a $c(\varepsilon) > 0$ such that $\nu\{t : |t| > c(\varepsilon)\} < \varepsilon/2$. For every $n \in \mathbb{Z}$ we get

$$\mu\{x : |a(n,x)| > 2c(\varepsilon)\} \leq \mu\{x : |b(x)| > c(\varepsilon)\} + \mu\{x : |b(V^n x)| > c(\varepsilon)\} < \varepsilon .$$

Hence $\tau_n^a\{t : |t| > 2c(\varepsilon)\} < \varepsilon$ for every n, which shows that $\{\tau_n^a : n \in \mathbb{Z}\}$ is uniformly tight.

To prove the converse, assume that $\{\tau_n^a : n \geq 1\}$ is uniformly tight, and choose an $\varepsilon > 0$. Then there exists a $c(\varepsilon) > 0$

with $\tau_n^a\{t:|t|>c(\varepsilon)\}<\varepsilon$ for every $n\geq 1$. We consider the skew product V_a on $(\tilde{X},\tilde{\mathcal{S}},\tilde{\mu})$ defined as in (10.6). Let

$$E = X\times[-\tfrac{1}{2},\tfrac{1}{2}]\subset\tilde{X},$$

where $[-\frac{1}{2},\frac{1}{2}] = \{t\in\mathbb{R}:-1/2\leq t\leq 1/2\}$, and let

$$F = X\times[-\tfrac{1}{2}-c(\varepsilon),\tfrac{1}{2}+c(\varepsilon)].$$

Then $\tilde{\mu}(E) = 1$ and $\tilde{\mu}(F) = 2c(\varepsilon)+1$. Since $\tau_n^a\{t:|t|>c(\varepsilon)\}<\varepsilon$, we get

$$\mu(V_a^{-n}E\cap F) > 1-\varepsilon$$

for every $n\in\mathbb{Z}$. Hence

$$\int(\frac{1}{n}\sum_{k=0}^{n-1}\chi_E(V_a^k(x,\alpha)).\chi_F(x,\alpha))\,d\tilde{\mu}(x,\alpha) > 1-\varepsilon$$

for every $n\geq 1$. Since χ_F is integrable, this shows that

$$\tilde{\mu}\{(x,\alpha)\in\tilde{X}:\lim_n\sup\frac{1}{n}|\sum_{k=0}^{n-1}\chi_E(V_a^k(x,\alpha))|\neq 0\}>0.$$

Lemma 11.6 and Proposition 11.5 now imply that a is a coboundary. The proof is complete.

Theorem 11.8. Let $(X,\mathcal{S},\mu)$ be a nonatomic probability space, V an ergodic measure preserving automorphism, and let A be a locally compact second countable abelian group of the form $A = \mathbb{Z}^k\times\mathbb{R}^l$, where $k,l\geq 0$. If $a:\mathbb{Z}\times X\to A$ is a cocycle for V, we define probability measures τ_n^a, $n\in\mathbb{Z}$, on A by

$$\tau_n^a(B) = \mu\{x \in X: a(n,x) \in B\} \tag{11.20}$$

for every Borel set $B \subset A$. Then the following conditions are equivalent.

(1) The family $\{\tau_n^a : n \in \mathbb{Z}\}$ is uniformly tight,

(2) the family $\{\tau_n^a : n \geq 1\}$ is uniformly tight,

(3) a is a coboundary.

Proof: Let $A = \mathbb{Z}^k \times \mathbb{R}^l$. We write every point $\alpha \in A$ as

$$\alpha = (\alpha_1, \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_{k+l})$$

where $\alpha_i \in \mathbb{Z}$ for $i=1,\ldots,k$ and $\alpha_i \in \mathbb{R}$ for $i=k+1,\ldots,k+l$. Furthermore we define maps $\phi_i : A \to \mathbb{R}$ by

$$\phi_i(\alpha) = \alpha_i$$

for every $\alpha \in A$ and for every $i=1,\ldots,k+l$. If $a: \mathbb{Z} \times X \to A$ is a cocycle for V, let $a_i : \mathbb{Z} \times X \to \mathbb{R}$ be given by

$$a_i(n,x) = \phi_i(a(n,x))$$

for $i=1,\ldots,k+l$, and put

$$\tau_n^{a_i} = \tau_n^a \phi_i^{-1}$$

for every $n \in \mathbb{Z}$, and for $i=1,\ldots,k+l$. If the family $\{\tau_n^a : n \in \mathbb{Z}\}$ is uniformly tight, each family $\{\tau_n^{a_i} : n \in \mathbb{Z}\}$, $i=1,\ldots,k+l$, will again be uniformly tight, and we can apply

Lemma 11.7 to show that each a_i, $i=1,\ldots,k+1$, is a coboundary. Hence a is a coboundary. The converse is left to the reader.

Problem 11.9. Does Theorem 11.8 hold for every ergodic measure preserving action of a countable group on a non-atomic probability space?

Exercise 11.10. Generalize Theorem 11.8 to the case of an ergodic measure preserving action of a countable amenable group G on a nonatomic probability space $(X,\mathcal{S},\mu)$.

§ 12 Some examples

If a is a cocycle for an ergodic automorphism of a measure space $(X, \mathcal{S}, \mu)$, the computation of $\bar{E}(a)$ is usually a problem of considerable difficulty. In this section we shall compute $\bar{E}(a)$ in some special cases.

The first cocycle will be a random walk on the integers. Let ν be a probability measure on $\mathbb{Z}$ with

$$\int k \, d\,\nu(k) = 0. \tag{12.1}$$

We put

$$X = \mathbb{Z}^{\mathbb{Z}} \tag{12.2}$$

for the cartesian product of countably many copies of $\mathbb{Z}$, denote by $\mathcal{S}$ the product Borel field, and write $\mu = \prod_{k \in \mathbb{Z}} \nu_k$ with $\nu_k = \nu$ for all $k \in \mathbb{Z}$. A point $x \in X$ will be written as

$$x = (\ldots, x_{-1}, x_0, x_1, \ldots)$$

with $x_i \in \mathbb{Z}$ for all i. We introduce the shift V on X by setting

$$V(\ldots, x_{-1}, x_0, x_1, \ldots) = (\ldots, x_0, x_1, x_2, \ldots) \tag{12.3}$$

for every $x = (\ldots, x_{-1}, x_0, x_1, \ldots)$ in X. It is well known (and easy to prove) that V is an ergodic measure preserving automorphism of $(X, \mathcal{S}, \mu)$. Let $f: X \to \mathbb{Z}$ be given by

$$f(x) = x_0 \tag{12.4}$$

for every $x = (\dots, x_{-1}, x_0, x_1, \dots) \in X$, and define a cocycle $a: \mathbb{Z} \times X \to \mathbb{Z}$ for V by (11.1) - (11.3). For every $n \geq 1, x \in X$, we have

$$a(n,x) = \sum_{k=0}^{n-1} x_k. \tag{12.5}$$

If we consider the sequence $(a(n,.),\ n=1,2,\dots)$ as a sequence of integer valued random variables on the probability space $(X, \mathcal{S}, \mu)$, we obtain the random walk on $\mathbb{Z}$ given by ν. (12.1) and Theorem 11.4 imply that a is recurrent. If we define, for every $x \in X$,

$$p'(x) = \begin{cases} \min \{n \geq 1 : a(n,x)=0\} & \text{if } \{n \geq 1 : a(n,x)=0\} \neq \emptyset, \\ 0 \text{ otherwise,} & \end{cases}$$

we see that $\mu\{x : p'(x)=0\} = 0$. Put $N = \bigcup_{k \in \mathbb{Z}} V^n \{x : p'(x)=0\}$ and let

$$p(x) = \begin{cases} 1 & \text{if } x \in N, \\ p'(x) & \text{if } x \in X \setminus N. \end{cases}$$

Consider the automorphism

$$Wx = V^{p(x)}x, \quad x \in X.$$

Clearly, $W \in [V]$. Using (11.1) - (11.3) we define a cocycle

$p:\mathbb{Z}\times X \to \mathbb{Z}$ for W with $p(1,x) = p(x)$ for every $x\in X$. Let

$$\Theta = \bigcup_{n\geq 1} \mathbb{Z}^n.$$

For every $x\in X$, $k\in\mathbb{Z}$, we define

$$Z_k(x) = (x_{p(k-1,x)}, x_{p(k-1,x)+1}, \dots, x_{p(k,x)-1}) \in \Theta$$

Regarding Θ as a discrete space, we note that $Z_k : X \to \Theta$ is a Borel map for every k, and that

$$Z_{k+1}(x) = Z_k(Wx) \tag{12.6}$$

for every k,x. We consider $\{Z_k : k\in\mathbb{Z}\}$ as a two sided stationary stochastic process on the probability space $(X, \mathcal{S}, \mu)$ - the stationarity being obvious from (12.6).

Lemma 12.1. The random variables $\{Z_k : k\in\mathbb{Z}\}$ are independent.

Proof: This is a consequence of the strong Markov property. We include a proof for the sake of completeness. Let $k_1 < k_2 < \dots < k_m \in \mathbb{Z}$ with $m\geq 2$. Since $(Z_{k_1},\dots,Z_{k_m})$ are independent if and only if $(Z_{k_1}\cdot W^{-k_1+1},\dots,Z_{k_m}\cdot W^{-k_m+1})$ are independent, we may assume that $k_1=1$. For every $n\geq 1$ we put $B_n = \{x: p(1,x)=n\}$. Then B_n is independent of the coordinates $n, n+1, \dots$, and the same is true for the set $B_n \cap \{x: Z_1(x)\in F_1\}$, where F_1 is any subset of Θ. Let now $F_1,\dots,F_m \subset \Theta$. Then we have

$$\mu\left(\bigcap_{i=1}^{m}\{x:Z_{k_i}(x)\in F_i\}\right)=\sum_{n\geq 1}\mu\left(\bigcap_{i=1}^{m}\{x:Z_{k_i}(x)\in F_i\}\cap B_n\right)$$

$$=\sum_{n\geq 1}\mu\left(\bigcap_{i=2}^{m}\{x:Z_{k_i}(x)\in F_i\}\cap B_n\cap\{x:Z_1(x)\in F_1\}\right).$$

For i=2,...,m we have

$$\{x:Z_{k_i}(x)\in F_i\}\cap B_n=\{x:(x_{p(k_i-2,V^nx)+n},\ldots,$$

$$x_{p(k_i-1,V^nx)+n})\in F_i\}\cap B_n. \qquad (12.7)$$

The first term on the right hand side of (12.7) depneds only on the coordinates n,n+1,..., and is thus independent of B_n as well as of $B_n\cap\{x:Z_1(x)\in F_1\}$. Hence

$$\bigcap_{i=2}^{m}\{x:(x_{p(k_i-2,V^nx)+n},\ldots,x_{p(k_i-1,V^nx)+n})\in F_i\}$$

is independent of $B_n\cap\{x:Z_1(x)\in F_1\}$. We get

$$\mu\left(\bigcap_{i=1}^{m}\{x:Z_{k_i}(x)\in F_i\}\right)=\sum_{n\geq 1}\mu\left(\bigcap_{i=2}^{m}\{x:Z_{k_i}(x)\in F_i\}\cap\right.$$

$$\left.\{x:Z_1(x)\in F_1\}\cap B_n\right)$$

$$=\sum_{k\geq 1}\mu\left(\bigcap_{i=2}^{m}\{x:(x_{p(k_i-2,V^nx)+n},\ldots,x_{p(k_i-1,V^nx)+n})\in F_i\}\right.$$

$$\left.\cap\{x:Z_1(x)\in F_1\}\cap B_n\right)$$

$$= \sum_{n \geq 1} \mu\left(\bigcap_{i=2}^{m} \{x : (x_{p(k_i-2, V^n x)+n}, \ldots, x_{p(k_i-1, V^n x)+n}) \in F_i\}\right)$$

$$\cdot \mu(\{x : Z_1(x) \in F_1\} \cap B_n)$$

$$= \mu\left(\bigcap_{i=2}^{m} \{x : Z_{k_i-1}(x) \in F_i\}\right) \cdot \mu(\{x : Z_1(x) \in F_1\})$$

$$= \mu\left(\bigcap_{i=2}^{m} \{x : Z_{k_i}(x) \in F_i\}\right) \cdot \mu(\{x : Z_1(x) \in F_1\}).$$

Induction now proves the independence of $(Z_{k_1}, \ldots, Z_{k_m})$. The proof is complete.

Lemma 12.2. Let $\mathcal{S}_1$ be the smallest σ-algebra of subsets of X with respect to which the random variables $\{Z_k : k \in \mathbb{Z}\}$ are measurable. Then $\mathcal{S}_1 = \mathcal{S}$.

Proof: This is obvious.

Lemma 12.3. W is ergodic on $(X, \mathcal{S}, \mu)$.

Proof: W is the shift of a stationary sequence of independent random variables which generate the σ-algebra $\mathcal{S}$. It is well known that this implies the ergodicity of W.

We can now determine $\bar{E}(a)$.

Theorem 12.4. Let $(X, \mathcal{S}, \mu)$ and V be given by (12.1) - (12.3), and let $a : \mathbb{Z} \times X \to \mathbb{Z}$ be the cocycle (12.5). Then $\bar{E}(a)$ is equal to the smallest subgroup of $\mathbb{Z}$ containing $\{k: \nu(k) > 0\}$. Moreover $a \cong 0$ whenever $E(a) = \{0\}$.

Proof: Let $u: R(V) \to \mathbb{Z}$ be an orbital cocycle with $a \cong a_u$, and let $B \subset X$ be a Borel set with $\mu(B) > 0$. If $\nu(\{k\}) > 0$ there exists a Borel set C with $\mu(C) > 0$ such that $a(1,x) = u(Vx,x) = k$ for every $x \in C$. Since W is ergodic we can choose a Borel set $D \subset B$ of positive measure and integers n_1, n_2 such that $W^{n_1}D \subset C$ and $W^{n_2}VW^{n_1}D \subset B$. For every $x \in D$, we have $u(W^{n_2}VW^{n_1}x,x) = u(W^{n_2}VW^{n_1}n, VW^{n_1}x) + u(VW^{n_1}x, W^{n_1}x) + u(W^{n_1}x,x)$. The first and last terms are equal to zero for μ-a.e. $x \in X$, and the middle term is equal to k. It follows that, for every Borel set B with $\mu(B) > 0$, there exists a $W_1 \in [V]$ with $\mu(B \cap W_1^{-1}B \cap \{x: u(W_1x,x) = k\}) > 0$. Lemma 9.5 shows that $k \in E(a)$. Hence $E(a)$ contains the smallest subgroup of $\mathbb{Z}$ containing $\{k: \nu(k) > 0\}$. It is also easy to see that no other integer can lie in $E(a)$, and the first assertion is proved. The second assertion is trivial.

Theorem 12.4 can be generalized quite easily, but we shall instead turn to an example arising in the theory of uniform distribution (mod 1). Let $X = \mathbb{R}/\mathbb{Z} = [0,1)$ denote the additive group of real numbers (mod 1), and let $\mathcal{S}$ be the Borel field and μ the Lebesgue measure on X. For every $\alpha_1, \alpha_2 \in X$, we write $\alpha_1 + \alpha_2$ for $\alpha_1 + \alpha_2$ (mod 1). let now $\alpha \in X$ be irrational, and put

$$V_\alpha x = x+\alpha \tag{12.8}$$

for every $x \in X$. For every irrational $\alpha \in X$, V_α is ergodic and measure preserving on $(X, \mathcal{S}, \mu)$. We fix α for the moment. For every real number β with $0 \leq \beta \leq 1$, let

$$f_\beta(x) = \begin{cases} 1-\beta & \text{if } 0 \leq x < \beta, \\ -\beta & \text{if } \beta \leq x < 1. \end{cases} \tag{12.9}$$

We write

$$a_{(\alpha,\beta)} : \mathbb{Z} \times X \to \mathbb{R} \tag{12.10}$$

for the cocycle for V_α arising from f_β through (11.1) - (11.3). Theorem 11.4 shows that $a_{(\alpha,\beta)}$ is recurrent for every (α,β) with α irrational and with $0 \leq \beta \leq 1$.

<u>Proposition 12.5</u>. $a_{(\alpha,\beta)}$ is a coboundary for V_α if and only if

$$\alpha = n\alpha \pmod 1 \tag{12.11}$$

for some $n \in \mathbb{Z}$.

Proof: Suppose that $a_{(\alpha,\beta)}$ is a coboundary. By (11.4) there exists a Borel map $c: X \to \mathbb{R}$ such that

$$f_\beta(x) = c(x+\alpha) - c(x)$$

for μ-a.e.x. Hence

$$e^{2\pi i f_\beta(x)} = e^{-2\pi i\beta} = e^{2\pi i(c(x+\alpha)-c(x))}$$

for μ-a.e.$x \in X$. In other words, $e^{2\pi i c(\cdot)}$ is an eigenfunction for V_α with eiegenvalue $e^{2\pi i\beta}$. Since every eigenvalue of V_α is of the form $e^{2\pi i n\alpha}$ for some $n \in \mathbb{Z}$, we have shown that β satisfies (12.11).

To prove the converse, assume that β satisfies (12.11). Expanding f_β in a Fourier series, we get

$$f_\beta(x) = \sum_{k\neq 0} (2\pi ik)^{-1}.(e^{2\pi ikn\alpha}-1).e^{2\pi ikx}.$$

For every $k\neq 0$ we put

$$c_k = (2\pi ik.(e^{2\pi ik\alpha}-1))^{-1}(e^{2\pi ikn\alpha}-1).$$

Then $\sum_{k\neq 0} |c_k|^2 < \infty$, so that

$$c(x) = \sum_{k\neq 0} c_k.e^{2\pi ikx}$$

lies in $L^2(X,\mathcal{S},\mu)$. Moreover we have

$$c(x+\alpha) - c(x) = \sum_{k\neq 0} (2\pi ik.(e^{2\pi ik\alpha}-1))^{-1}.$$

$$.(e^{2\pi ikn\alpha}-1).(e^{2\pi ik\alpha}-1).e^{2\pi ikx}$$

$$= f_\beta(x) \qquad (12.12)$$

for μ-a.e.$x \in X$. (12.12) shows that $a_{(\alpha,\beta)}$ is a coboundary, and the proof is complete.

We now turn to the special case $\beta = 1/2$. We fix the irrational number $\alpha \in X$ and put, for every $n \in \mathbb{Z}$, $x \in X$,

$$a_o(n,x) = 2.a_{(\alpha,1/2)}(n,x). \tag{12.13}$$

Since $a_o(n,x) \in \mathbb{Z}$ for every n,x, we shall consider

$$a_o: \mathbb{Z} \times X \to \mathbb{Z} \tag{12.14}$$

as an integral valued cocycle for V. We denote by $[a_o;a_1,a_2,\ldots]$ the continued fraction expansion of α and write p_k/q_k, $k \geq 0$ for the k-th convergent of α. Recall the well known formulae

$$p_k/q_k = [a_o;a_1,\ldots,a_k], \tag{12.15}$$

$$|\alpha - p_k/q_k| < (q_k q_{k+1})^{-1} < q_k^{-2}, \tag{12.16}$$

$$q_k p_{k-1} - p_k q_{k-1} = (-1)^k, \tag{12.17}$$

$$p_k = a_k p_{k-1} + p_{k-2}, \tag{12.18}$$

and

$$q_k = a_k q_{k-1} + q_{k-2} \tag{12.19}$$

for every $k \geq 0$, where $p_{-1}=q_{-2}=1$ and $p_{-2}=q_{-1}=0$. For details we refer to [4, 21].

Lemma 12.6. $E(a_o) \cap \{1,3\} \neq \emptyset$. (12.20)

Proof: The inequality of Denjoy and Koksma (cf. [8, 9]) implies that, for every $k \geq 0$,

$$\sup_{x \in X} |a_o(q_k,x)| = \sup_{x \in X} \left| \sum_{l=0}^{q_k-1} 2.f_{1/2}(x+l\alpha) \right|$$

$$2.\mathrm{Var}\ f_{1/2} = 4,$$

where Var stands for variation. (12.17) implies that q_{k+1} is odd whenever q_k is even. In particular we conclude the existence of an infinite sequence k_1 k_2 ... such that q_{k_i} is odd for every i. It is easy to see that $a_o(q_{k_i},x)$ is odd for every i and for every $x \in X$. Hence $|a_o(q_{k_i},x)|$ is either equal to 1 or equal to 3. Since we also have

$$\lim_i |q_{k_i}\alpha - p_{k_i}| = 0$$

from (12.16), we get

$$\lim_i \mu(B \cap (B - q_{k_i}.\alpha)) = \mu(B)$$

for every Borel set $B \subset X$. It follows that

$$\lim_i \mu(B \cap V^{-q_{k_i}}B \cap \{x: a_o(q_{k_i},x) \in \{1,3\}\}) = \mu(B)$$

for every $B \in \mathcal{S}$. Proposition 3.8 now shows that $E(a) \cap \{1,3\} \neq \emptyset$, and the proof is complete.

<u>Corollary 12.7</u>. The cocycle a_o is regular, and E(a) is either equal to $\mathbb{Z}$ or to $3.\mathbb{Z}$.

Proof: Trivial.

Theorem 12.8. Let $X = \mathbb{R}/\mathbb{Z}$ denote the additive group of real numbers (mod 1), and let $\mathcal{S}$ and μ stand for the Borel field and the Lebesgue measure on X, respectively. We fix an irrational number α and denote by V_α the automorphism of $(X, \mathcal{S}, \mu)$ given by

$$V_\alpha x = x + \alpha \pmod 1.$$

Let furthermore $a_o: \mathbb{Z} \times X \to \mathbb{Z}$ denote the cocycle for V_α given by

$$a_o(1,x) = \begin{cases} +1 & \text{for } 0 \leq x < 1/2, \\ -1 & \text{for } 1/2 \leq x < 1. \end{cases}$$

Then $E(a_o) = \mathbb{Z}$.

Proof: If $E(a_o) \neq \mathbb{Z}$, Corollary 12.7 shows that $E(a_o) = 3\mathbb{Z}$. We put $\mathbb{Z}/3\mathbb{Z} = \mathbb{Z}_3 = \{0,1,2\}$, and consider the cocycle $a_o^*: \mathbb{Z} \times X \to \mathbb{Z}_3$ given by

$$a_o^*(n,x) = a_o(n,x) + E(a_o). \tag{12.21}$$

Corollary 12.7 implies that a_o is a coboundary. This in turn implies that for every sequence $(t_k,\ k=1,2,\ldots)$ of integers with

$$\lim_k \langle\langle t_k \alpha \rangle\rangle = 0 \tag{12.22}$$

we have

$$\lim_k \mu\{x: a_0(t_k,x) \neq 0\} = 0. \tag{12.23}$$

Here $\langle\langle x \rangle\rangle$ denotes the distance of a real number x from the closest integer. Let now

$$\alpha' = 2\alpha \pmod 1,$$

and let $[a_0';a_1',a_2',\ldots]$ and $(p_k'/q_k',\ k=0,1,2,\ldots)$ stand for the continued fraction expansion and the convergents of α', respectively. Since at least every other p_k' is odd (cf. (12.17)), we can find an infinite sequence $k_1 < k_2 < k_3\ \ldots$ of nonnegative integers such that p_{k_i}' is odd for every i. Now we put, for every i,

$$l_i = k_{2i}$$

and

$$t_i = q_{l_i}' - q_{k_i}'.$$

It is easy to conclude from (12.19) that

$$\lim_i t_i/q_{l_i}' = 1. \tag{12.24}$$

(12.16) shows that

$$\lim_i \langle\langle q_{k_i}'\alpha - 1/2 \rangle\rangle = 0$$

and

$$\lim_i \langle\langle q_{l_i}'\alpha - 1/2 \rangle\rangle = 0.$$

Hence the sequence $(t_i,\ i=1,2,\ldots)$ satisfies (12.22), and consequently (12.23). We now look at the functions $a(t_i,.)$, $i \geq 1$. Each $a(t_i,.)$ is discontinuous exactly at the elements of the set

$$S_i = \{-n\alpha + m/2 \pmod 1 : 0 \leq n < t_i,\ m=1,2\}.$$

We arrange the elements of S_i in increasing order,

$$0 = \beta_0^{(i)} < \beta_1^{(i)} < \dots < \beta_{2t_i-1}^{(i)} < 1,$$

say, and put

$$\beta_{2t_i}^{(i)} = 1.$$

Keeping i fixed, we note that $a(t_i,.)$ is constant on each interval $\beta_m^{(i)} \leq x < \beta_{m+1}^{(i)}$, $m=0,\dots,2t_i-1$, and that

$$\lim_{h \to 0} \left(a_o(t_i,\beta_m^{(i)}+h) - a_o(t_i,\beta_m^{(i)}-h)\right) \equiv \pm 1 \pmod 3.$$

Hence $a_o(t_i,.)$ is zero in at most t_i of the intervals $\beta_m^{(i)} \leq x < \beta_{m+1}^{(i)}$, $m=0,\dots,2t_i-1$. We now estimate the minimal length of these intervals. An easy argument shows that

$$h_i = \min_{0 \leq m < 2t_i} (\beta_{m+1}^{(i)} - \beta_m^{(i)}) =$$

$$\frac{1}{2} \cdot \min_{0 < l < t_i} \langle\langle 2l\alpha \rangle\rangle .$$

Since $q'_{l_i-1} < t_i < q'_{l_i}$, we can apply Theorem 17 in [21] to show that

$$h_i = \frac{1}{2} \min_{0<l<t_i} \langle\langle l\alpha' \rangle\rangle = \frac{1}{2} |q'_{l_{i}-1}\alpha' - p'_{l_{i}-1}| >$$

$$\frac{1}{2}\cdot(q'_{l_{i}-1}+q'_{l_i})^{-1} > \frac{1}{4q'_{l_i}} .$$

Hence

$$\mu\{x:a_o(t_i,x)\neq 0\} \geq \frac{t_i}{4q'_{l_i}} . \qquad (12.25)$$

(11.24) and (11.25) together imply

$$\lim_i \mu\{x:a_o(t_i,x)\neq 0\} \geq 1/4,$$

which contradicts (12.23). This shows that $E(a_o) = \mathbb{Z}$, and the theorem is proved.

Exercise 12.10. Let V_α and $(X, \mathcal{S}, \mu)$ be given as in Theorem 12.8, and let β be a rational number. Show that $E(a_{(\alpha,\beta)})$ is equal to the smallest subgroup of $\mathbb{R}$ containing 1 and β .

Problem 12.11. Compute $\bar{E}(a_{(\alpha,\beta)})$ for every pair (α,β), where α is irrational and $0 \leq \beta \leq 1$.

Comments

§ 1: For the background on standard Borel spaces we refer to [29, 34]. The 'equivalence relation' of a group action was introduced and discussed in [14]. For the notions 'full group' and 'weak equivalence' we refer to [11, 12 and 22].

§ 2: Definitions 2.1 and 2.2 are taken from [39] and [14], respectively. Proposition 2.7 already occurs in [45].

§ 3: All results in this section are taken from [39] with some minor modifications. Definition 3.1 also occurs in [14, 25], and we refer to these papers for a deeper discussion of the origins of this definition. A much more general exposition of the underlying ideas can be found in [30]. The set $\bar{E}(a)$ is called 'asymptotic range' in [14]. Theorem 3.9 is also proved in [14, 26]. A cohomology invariant which is closely related to $\bar{E}(a)$ can be found in [16,17], and - in slight disguise - in [32]. Definition 3.13 is due to [39].

§ 4: Theorem 4.2 was first proved in [26] for the hyperfinite case. Corollary 4.3 is a simple consequence of Theorem 4.2, but I have not seen it in the literature.

§ 5: This section is again taken from [39] with some modifications, since the proofs there dealt with the hyperfinite case only. Theorem 5.2 appears in a special case in [4] and in full generality in [14]. There is a great deal of literature on skew products defined by cocycles taking values in a compact abelian group. See [32,33,18] for further references.

§ 6: The proof of Theorem 6.6 is a generalization of the corresponding proof in [44] for single automorphisms. There exist many proofs of Theorem 6.6 using operator algebras, and we mention [38] as just one example. The results 6.7 - 6.15 are all well known, but many of them are rarely proved explicitly.

§ 7: This section is a slightly more general and considerably expanded version of § 5 in [39], where the same results are proved for hyperfinite actions. Many of the methods described here are closely related to parts of the analysis in [26]. In particular, Definition 7.1 and Proposition 7.2 occur there in a special case. On closer look the reader will also find [26] to contain special cases of Proposition 7.15 and Theorem 7.22 in disguise, but the proofs are somewhat different.

§ 8: All the ideas in this section are contained in [11], even though they were proved there in the finite measure preserving case. Theorem 8.7, however, does not seem to appear in the literature. Theorem 8.15 (1) is due to

[24], and (2) has been remarked in [5]. The proof of Theorem 8.15 used here is a modification of a proof in [15] for the measure preserving case. Corollary 8.16 appears in [11], and Corollary 8.17 in [22]. Theorem 8.18 is a very special case of a result in [22]. Theorem 8.22 is again due to [11] in the finite measure preserving case.

§ 9: All the results in this section are taken from [35]. A related result for cocycles taking values in a compact group appears in [32].

§ 10: All the results come from [43]. The problem of finding further ergodic measures for nonsingular automorphisms of a given measure space has received attention in many papers (cf. [20 , 27 , 28 , 31 , 41 , 42]). Among these [28] is of particular interest, since it contains the following result: Let $(X, \mathcal{S}, \mu)$ be a nonatomic measure space and let V be a nonsingular ergodic automorphism of $(X, \mathcal{S}, \mu)$. Suppose that $(X', \mathcal{S}', \mu')$ is a further nonatomic measure space and V' a nonsingular ergodic automorphism of $(X', \mathcal{S}', \mu')$ such that μ' is not equivalent to *any* finite V'-invariant measure. Then there exists a nonatomic probability measure ν on the standard Borel space $(X, \mathcal{S})$ which is quasi-invariant and ergodic under V, and such that V on $(X, \mathcal{S}, \nu)$ is weakly equivalent to V'.

§ 11: All the results and proofs are from [39] with the exception of Theorem 11.4, which is due to [1], and Proposition 11.5 and Theorem 11.8, both of which are new as far as I know.

§ 12: Theorem 12.4 is well known and holdsin greater generality than indicated here. The proofs of the Lemmas 12.1 - 12.3 are taken from [3]. Proposition 12.5 appears in [36]. A special case of Theorem 12.8 was proved in [41], and [7] contains a more general result. Partial progress in solving Problem 12.11 was made in [2].

The above acknowledgements and references are only intended as a guideline, and I apologize for any omissions and errors. The following list of references is again not intended to be complete, but it should enable the reader to pursue further any interest or problem raised by these notes.

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