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Measurable Rigidity of Algebraic $\mathbb{Z}^d$-Actions

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1. Introduction

Let X be a compact abelian group with identity element 0_X, Borel field $\mathcal{B}_X$ and normalized Haar measure λ_X. An *algebraic $\mathbb{Z}^d$-action* on X is a group homomorphism $\alpha\colon \mathbf{n} \mapsto \alpha^{\mathbf{n}}$ from $\mathbb{Z}^d$, $d \geq 1$, into the group $\mathrm{Aut}(X)$ of continuous automorphisms of X. An algebraic $\mathbb{Z}^d$-action α on X is *expansive* if there exists an open set $\mathcal{O} \subset X$ with $\bigcap_{\mathbf{n}\in\mathbb{Z}^d} \alpha^{-\mathbf{n}}(\mathcal{O}) = \{0_X\}$, where 0_X is the identity element of X, and *irreducible* if every closed, α-invariant subgroup $Y \subsetneq X$ is finite. The action α is *ergodic* or *mixing* if the Haar measure λ_X of X is ergodic or mixing under α.

If α is an algebraic $\mathbb{Z}^d$-action on X and $Y \subset X$ a closed, α-invariant subgroup, then we denote by α_Y and $\alpha_{X/Y}$ the $\mathbb{Z}^d$-action induced by α on Y and X/Y, respectively.

Algebraic $\mathbb{Z}^d$-actions can be classified algebraically, topologically or measurably. In order to introduce the appropriate notions of conjugacy we assume that α_i is an algebraic $\mathbb{Z}^d$-action on a compact abelian group X_i with Haar measure λ_{X_i}, where $i = 1, 2$. A surjective Borel map $\phi\colon X_1 \longrightarrow X_2$ is a *measurable factor map* of α_1 and α_2 if

$$\lambda_{X_1}\phi^{-1} = \lambda_{X_2}, \tag{1.1}$$

and if

$$\phi \circ \alpha_1^{\mathbf{n}}(x) = \alpha_2^{\mathbf{n}} \circ \phi(x) \tag{1.2}$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and λ_{X_1}-a.e. $x \in X_1$. A bijective measurable factor map $\phi\colon X_1 \longrightarrow X_2$ is a *measurable conjugacy* of α_1 and α_2.

A continuous surjective group homomorphism $\phi\colon X_1 \longrightarrow X_2$ is an *algebraic factor map* of α_1 and α_2 if it satisfies (1.2) for every $\mathbf{n} \in \mathbb{Z}^d$ and $x \in X_1$. A bijective algebraic factor map $\phi\colon X_1 \longrightarrow X_2$ is an *algebraic conjugacy* of α_1 and α_2. Topological factor maps and topological conjugacy are defined similarly.

The action α_2 is a *measurable*, *algebraic* or *topological factor* of α_1 if there exists a measurable, algebraic or topological factor map $\phi\colon X_1 \longrightarrow X_2$ of α_1 and α_2. The actions α_1, α_2 are *measurably*, *algebraically* or *topologically conjugate* if there exists a measurable, algebraic or topological conjugacy $\phi\colon X_1 \longrightarrow X_2$ of α_1 and α_2, and they are *weakly measurably* (resp. *weakly algebraically*) *conjugate* if each of them is a measurable (resp. algebraic) factor of the other.

Finally we call a map $\phi\colon X_1 \longrightarrow X_2$ *affine* if it is of the form

$$\phi(x) = \psi(x) + x' \tag{1.3}$$

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for every $x \in X_1$, where $\psi\colon X_1 \longrightarrow X_2$ is a continuous surjective group homomorphism and $x' \in X_2$.

For expansive algebraic $\mathbb{Z}^d$-actions on compact connected abelian groups the topological and algebraic classifications coincide (i.e. topological conjugacy implies algebraic conjugacy), by [**20**, (4.10) and Theorem 5.9]. For algebraic $\mathbb{Z}^d$-actions on disconnected groups this is no longer true: let A be a finite abelian group, and let $\Omega = A^{\mathbb{Z}^d}$ be the compact abelian group consisting of all maps $\omega\colon \mathbb{Z}^d \longrightarrow A$, furnished with the product topology and coordinate-wise addition. We write every $\omega \in \Omega$ as $\omega = (\omega_{\mathbf{n}})$ with $\omega_{\mathbf{n}} \in A$ for every $\mathbf{n} \in \mathbb{Z}^d$ and define the *shift-action* σ of $\mathbb{Z}^d$ on Ω by

$$(\sigma_{\mathbf{m}}\omega)_{\mathbf{n}} = \omega_{\mathbf{m}+\mathbf{n}} \tag{1.4}$$

for every $\omega \in \Omega$ and $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$. Clearly, σ is an expansive algebraic $\mathbb{Z}^d$-action on Ω. If A' is a second abelian group with the same cardinality as A, then the resulting shift-actions σ and σ' of $\mathbb{Z}^d$ on $\Omega = A^{\mathbb{Z}^d}$ and $\Omega' = A'^{\mathbb{Z}^d}$ are obviously topologically, but not necessarily algebraically, conjugate.

For $d = 1$, any algebraic $\mathbb{Z}$-action is determined by the powers of a single group automorphism α. If α is ergodic, then it is Bernoulli (cf. e.g. [**1**], [**2**], [**7**], [**12**], [**14**]), which implies that two such actions with equal entropy are measurably conjugate even if they are algebraically non-conjugate. If $d > 1$ and α_1, α_2 are algebraic $\mathbb{Z}^d$-actions on compact abelian groups X_1, X_2 with completely positive entropy (with respect to Haar measure), then they are Bernoulli by [**17**], and can thus again be measurably conjugate without being algebraically conjugate. However, if these actions have zero entropy, measurable conjugacy may have much stronger implications.

Theorem 1.1 ([**6**], [**11**]). *Let $d > 1$, and let α_1 and α_2 be mixing and expansive algebraic $\mathbb{Z}^d$-actions on compact abelian groups X_1 and X_2, respectively. If α_1 is irreducible, and if $\phi\colon X_1 \longrightarrow X_2$ is a measurable conjugacy of α_1 and α_2, then α_2 is irreducible and ϕ is λ_{X_1}-a.e. equal to an affine map.*

Corollary 1.2. *Let $d > 1$, and let α_1 and α_2 be measurably conjugate irreducible, mixing and expansive algebraic $\mathbb{Z}^d$-actions on compact abelian groups X_1 and X_2, respectively. Then α_1 and α_2 are algebraically conjugate.*

Actions with completely positive entropy and irreducible actions lie at opposite ends of the spectrum of ergodic algebraic $\mathbb{Z}^d$-actions. As mentioned above, the kind of isomorphism rigidity described in Theorem 1.1 is impossible for actions with completely positive entropy. However, it could conceivably hold for all mixing algebraic $\mathbb{Z}^d$-actions with zero entropy (i.e. without Bernoulli factors — cf. [**13**]). In this context it is worth pointing out the similarity between Theorem 1.1 and M. Ratner's rigidity theorem for horocycle flows (or their time-1-maps) in [**15**]: in both situations any measurable conjugacy carries with it the ambient algebraic structure. However, the underlying dynamical properties of the systems differ considerably: horocycle flows have zero entropy and are uniquely ergodic, whereas the algebraic $\mathbb{Z}^d$-actions under consideration here have a dense set of periodic orbits and their individual automorphisms are Bernoulli.

Theorem 1.1 implies that every Haar-measure-preserving invertible transformation T on a compact abelian group X which commutes with an irreducible, expansive and mixing $\mathbb{Z}^d$-action α on X is affine. This makes it possible — at least in principle — to determine the measurable centralizer of α. In order to provide the necessary notation we assume that α is an algebraic $\mathbb{Z}^d$-action on a compact abelian group X and write $\mathrm{Aff}(X)$ for the group of all continuous surjective affine bijections

$\psi\colon X \longrightarrow X$. Put

$$\begin{aligned}\bar{Z}(\alpha) &= \{\psi \in \mathrm{Aff}(X) : \phi \circ \alpha^{\mathbf{n}} = \alpha^{\mathbf{n}} \circ \psi \text{ for every } \mathbf{n} \in \mathbb{Z}^d\},\\ Z(\alpha) &= \{\phi \in \mathrm{Aut}(X) : \phi \circ \alpha^{\mathbf{n}} = \alpha^{\mathbf{n}} \circ \phi \text{ for every } \mathbf{n} \in \mathbb{Z}^d\}.\end{aligned} \tag{1.5}$$

Corollary 1.3. *Let α be an irreducible, expansive and mixing algebraic $\mathbb{Z}^d$-action on a compact abelian group X, and let $Z_{\lambda_X}(\alpha)$ be the* measurable centralizer *of α, i.e. the group of all λ_X-preserving invertible transformations $T\colon X \longrightarrow X$ with $T \circ \alpha^{\mathbf{n}} = \alpha^{\mathbf{n}} \circ T$ λ_X-a.e., for every $\mathbf{n} \in \mathbb{Z}^d$. Then the following properties hold.*

(1) *Every $T \in Z_{\lambda_X}(\alpha)$ coincides λ_X-a.e. with an element $\psi \in \bar{Z}(\alpha)$;*
(2) *$\bar{Z}(\alpha)$ is countable and has an abelian subgroup of finite index.*

Proof. The first assertion is clear from Theorem 1.1. For the second assertion we have to appeal to terminology and results from later sections. Proposition 2.2 shows that α is weakly algebraically equivalent to a $\mathbb{Z}^d$-action of the form $\alpha_{\mathfrak{R}_d/\mathfrak{p}}$ for some prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$. Since α is irreducible, the algebraic factor maps $\chi\colon X \longrightarrow X_{\mathfrak{R}_d/\mathfrak{p}}$ and $\chi'\colon X_{\mathfrak{R}_d/\mathfrak{p}} \longrightarrow X$ between α and $\alpha_{\mathfrak{R}_d/\mathfrak{p}}$ are finite-to-one. Hence there exists a subgroup of finite index $Z' \subset \bar{Z}(\alpha)$ such that every $\phi \in Z'$ acts trivially on $\ker(\chi)$ and thus induces an element $\phi' \in \bar{Z}(\alpha_{\mathfrak{R}_d/\mathfrak{p}})$. As $\bar{Z}(\alpha_{\mathfrak{R}_d/\mathfrak{p}})$ is abelian this implies the second assertion of this corollary (cf. [**6**] and [**11**]). Note that the abelian subgroup of finite index $A \subset \bar{Z}(\alpha)$ obtained in this manner is isomorphic to a subgroup of finite index in $\bar{Z}(\alpha_{\mathfrak{R}_d/\mathfrak{p}})$. □

This brief survey is organized as follows. In Section 2 we give a description of irreducible algebraic $\mathbb{Z}^d$-actions. Section 3 deduces Theorem 1.1 from a result about invariant measures of algebraic $\mathbb{Z}^d$-actions (Theorem 3.3). The Sections 4 and 5 contain brief sketches of the proofs of Theorem 3.3 in the two relevant cases of connected and zero-dimensional groups, respectively. The last section illustrates Theorem 1.1 with examples.

2. Irreducible $\mathbb{Z}^d$-actions

Following [**8**], [**18**] and [**20**] we denote by $\mathfrak{R}_d = \mathbb{Z}[u_1^{\pm1},\dots,u_d^{\pm1}]$ the ring of Laurent polynomials with integral coefficients in the commuting variables $u_1,\dots,u_d$, and write every $f \in \mathfrak{R}_d$ as $f = \sum_{\mathbf{m}\in\mathbb{Z}^d} c_f(\mathbf{m})u^{\mathbf{m}}$ with $u^{\mathbf{m}} = u_1^{m_1}\cdots u_d^{m_d}$ and $c_f(\mathbf{m}) \in \mathbb{Z}$ for every $\mathbf{m} = (m_1,\dots,m_d) \in \mathbb{Z}^d$.

Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X. We denote by $\widehat{X}$ the additively written dual group of X, write $\langle a,x\rangle$ the value of a character $a \in \widehat{X}$ at a point $x \in X$, and define the *dual action* $\hat{\alpha}\colon \mathbf{n} \mapsto \hat{\alpha}^{\mathbf{n}}$ of $\mathbb{Z}^d$ on $\widehat{X}$ by

$$\langle \hat{\alpha}^{\mathbf{n}} a, x\rangle = \langle a, \alpha^{\mathbf{n}} x\rangle$$

for every $\mathbf{n} \in \mathbb{Z}^d$, $x \in X$ and $a \in \widehat{X}$. For $f \in \mathfrak{R}_d$, $x \in X$ and $a \in \widehat{X}$, we set

$$f(\alpha)(x) = \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n})\alpha^{\mathbf{n}}x, \qquad f(\hat{\alpha})(a) = \sum_{\mathbf{n}\in\mathbb{Z}^d} c_f(\mathbf{n})\hat{\alpha}^{\mathbf{n}}a, \tag{2.1}$$

and note that $f(\alpha)\colon X \longrightarrow X$ is a group homomorphism with dual homomorphism $\widehat{f(\alpha)} = f(\hat{\alpha})\colon \widehat{X} \longrightarrow \widehat{X}$,

The group $\widehat{X}$ is a module over the ring $\mathfrak{R}_d$ with operation

$$f \cdot a = f(\hat{\alpha})(a) \tag{2.2}$$

for $f \in \mathfrak{R}_d$ and $a \in \widehat{X}$. In particular,

$$u^{\mathbf{m}} \cdot a = \hat{\alpha}^{\mathbf{m}} a \tag{2.3}$$

for $\mathbf{m} \in \mathbb{Z}^d$ and $a \in \mathfrak{M}$. This *dual module* $\mathfrak{M} = \widehat{X}$ of α is Noetherian (and hence countable) whenever α is expansive (cf. [**18**, Proposition 5.4]).

Conversely, if $\mathfrak{M}$ is an $\mathfrak{R}_d$-module, we define an algebraic $\mathbb{Z}^d$-action $\alpha_{\mathfrak{M}}$ on the compact abelian group

$$X_{\mathfrak{M}} = \widehat{\mathfrak{M}} \tag{2.4}$$

by setting

$$\hat{\alpha}_{\mathfrak{M}}^{\mathbf{m}} a = u^{\mathbf{m}} \cdot a \tag{2.5}$$

for every $\mathbf{m} \in \mathbb{Z}^d$ and $a \in \mathfrak{M}$.

A prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ is *associated* with an $\mathfrak{R}_d$-module $\mathfrak{M}$ if there exists an $a \in \mathfrak{M}$ with $\mathfrak{p} = \operatorname{ann}(a) = \{f \in \mathfrak{R}_d : f \cdot a = 0\}$. The set $\operatorname{asc}(\mathfrak{M})$ of all prime ideals associated with $\mathfrak{M}$ has the property that

$$\bigcup_{\mathfrak{p} \in \operatorname{asc}(\mathfrak{M})} \mathfrak{p} = \{f \in \mathfrak{R}_d : \text{multiplication by } f \text{ on } \mathfrak{M} \text{ is not injective}\}. \tag{2.6}$$

According to [**18**], [**13**] and [**20**], many properties of an algebraic $\mathbb{Z}^d$-action α can be expressed in terms of the prime ideals associated with the dual module $\mathfrak{M} = \widehat{X}$ of α.

PROPOSITION 2.1. *Let $\mathfrak{M}$ a countable $\mathfrak{R}_d$-module.*

(1) *For any $\mathbf{n} \in \mathbb{Z}^d$, the following conditions are equivalent.*
   (a) *$\alpha_{\mathfrak{M}}^{\mathbf{n}}$ is ergodic;*
   (b) *$\alpha_{\mathfrak{R}_d/\mathfrak{p}}^{\mathbf{n}}$ is ergodic for every prime ideal $\mathfrak{p}$ associated with $\mathfrak{M}$;*
   (c) *No prime ideal $\mathfrak{p}$ associated with $\mathfrak{M}$ contains a polynomial of the form $u^{l\mathbf{n}} - 1$ with $l \geq 1$.*
(2) *The following conditions are equivalent.*
   (a) *$\alpha_{\mathfrak{M}}$ is ergodic;*
   (b) *$\alpha_{\mathfrak{R}_d/\mathfrak{p}}$ is ergodic for every prime ideal $\mathfrak{p}$ associated with $\mathfrak{M}$;*
   (c) *No prime ideal $\mathfrak{p}$ associated with $\mathfrak{M}$ contains a set of the form $\{u^{l\mathbf{n}} - 1 : \mathbf{n} \in \mathbb{Z}^d\}$ with $l \geq 1$.*
(3) *The following conditions are equivalent.*
   (a) *$\alpha_{\mathfrak{M}}$ is mixing;*
   (b) *$\alpha_{\mathfrak{M}}^{\mathbf{n}}$ is ergodic for every non-zero $\mathbf{n} \in \mathbb{Z}^d$;*
   (c) *$\alpha_{\mathfrak{M}}^{\mathbf{n}}$ is mixing for every non-zero $\mathbf{n} \in \mathbb{Z}^d$;*
   (d) *$\alpha_{\mathfrak{R}_d/\mathfrak{p}}$ is mixing for every prime ideal $\mathfrak{p}$ associated with $\mathfrak{M}$;*
   (e) *None of the prime ideals associated with $\mathfrak{M}$ contains a polynomial of the form $u^{\mathbf{n}} - 1$ with $\mathbf{0} \neq \mathbf{n} \in \mathfrak{R}_d$.*
(4) *The following conditions are equivalent.*
   (a) *α is expansive;*
   (b) *$\mathfrak{M}$ is Noetherian and $\alpha_{\mathfrak{R}_d/\mathfrak{p}}$ is expansive for every prime ideal $\mathfrak{p}$ associated with $\mathfrak{M}$;*
   (c) *For every prime ideal $\mathfrak{p}$ associated with $\mathfrak{M}$, $V_{\mathbb{C}}(\mathfrak{p}) \cap \mathbb{S}^d = \varnothing$, where*
   $$V_{\mathbb{C}}(\mathfrak{p}) = \{c \in (\mathbb{C} \smallsetminus \{0\})^d : f(c) = 0 \text{ for every } f \in \mathfrak{p}\}.$$
(5) *The following conditions are equivalent.*
   (a) *$\alpha_{\mathfrak{M}}$ is has positive entropy;*
   (b) *$\alpha_{\mathfrak{R}_d/\mathfrak{p}}$ has positive entropy for some prime ideal $\mathfrak{p}$ associated with $\mathfrak{M}$;*
   (c) *At least one prime ideal $\mathfrak{p}$ associated with $\mathfrak{M}$ has the property that $\mathfrak{p}$ is principal and $\alpha_{\mathfrak{R}_d/\mathfrak{p}}$ is mixing.*

For a proof we refer to [**20**, Propositions 6.6 and 19.4, and Theorem 6.5]. The next result is contained in [**20**, Proposition 6.6 and Theorem 29.2], or in [**6**] and [**11**].

PROPOSITION 2.2. *Let α be an irreducible and ergodic algebraic $\mathbb{Z}^d$-action on a compact zero-dimensional abelian group X. Then there exists a unique prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ with the following properties.*

(1) *$\alpha_{\mathfrak{R}_d/\mathfrak{p}}$ is ergodic;*
(2) *For every ideal $I \supsetneq \mathfrak{p}$ in $\mathfrak{R}_d$, $\mathfrak{R}_d/I$ is finite;*
(3) *α is weakly algebraically conjugate to $\alpha_{\mathfrak{R}_d/\mathfrak{p}}$;*
(4) *The group X is zero-dimensional if and only if $\mathfrak{p}$ contains a rational prime $p > 1$, and connected otherwise.*

Conversely, if $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal satisfying the conditions (1)–(2) *above, then the $\mathbb{Z}^d$-action $\alpha = \alpha_{\mathfrak{R}_d/\mathfrak{p}}$ on $X_{\mathfrak{R}_d/\mathfrak{p}}$ is irreducible and ergodic.*

PROPOSITION 2.3. *Let α be an irreducible and ergodic algebraic $\mathbb{Z}^d$-action on a compact abelian group X. Then $h(\alpha^{\mathbf{n}}) < \infty$ for every $\mathbf{n} \in \mathbb{Z}^d$, where $h(\cdot)$ denotes topological entropy.*

Conversely, if α is an expansive, ergodic (*and mixing*) *algebraic $\mathbb{Z}^d$-action on a compact abelian group X with $h(\alpha^{\mathbf{n}}) < \infty$ for every $\mathbf{n} \in \mathbb{Z}^d$, and if $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal associated with the module $\mathfrak{M} = \widehat{X}$, then the $\mathbb{Z}^d$-action $\alpha_{\mathfrak{R}_d/\mathfrak{p}}$ on $X_{\mathfrak{R}_d/\mathfrak{p}}$ is irreducible, ergodic* (*and mixing*).

The proof of Proposition 2.3 is similar to that of Corollary 2.3 in [**11**].

REMARKS 2.4. (1) Let α be an irreducible and ergodic algebraic $\mathbb{Z}^d$-action on a compact abelian group X, and let $\mathfrak{p} \subset \mathfrak{R}_d$ be the prime ideal satisfying the conditions (1)–(3) in Proposition 2.2. Proposition 2.1 shows that α is mixing if and only if $\alpha^{\mathbf{m}} \neq \mathrm{id}_X$ whenever $\mathbf{0} \neq \mathbf{m} \in \mathbb{Z}^d$, where id_X is the identity automorphism of X.

(2) If $\mathfrak{q} \subset \mathfrak{R}_d$ is a prime ideal satisfying the conditions (1)–(2) in Proposition 2.2, and if $\alpha = \alpha_{\mathfrak{R}_d/\mathfrak{q}}$, then the the prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ satisfying the conditions (1)–(3) in Proposition 2.2 is equal to $\mathfrak{q}$.

Motivated by Proposition 2.2 we take a closer look at $\mathbb{Z}^d$-actions of the form $\alpha_{\mathfrak{R}_d/\mathfrak{p}}$, where $\mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and let $\Omega = \mathbb{T}^{\mathbb{Z}^d}$, furnished with the product topology and component-wise addition. We write every $\omega \in \Omega$ as $\omega = (\omega_{\mathbf{n}})$ with $\omega_{\mathbf{n}} \in \mathbb{T}$ for every $\mathbf{n} \in \mathbb{Z}^d$ and define the shift-action σ of $\mathbb{Z}^d$ on Ω by (1.4). The pairing

$$\langle h, \omega \rangle = e^{2\pi i \sum_{\mathbf{n} \in \mathbb{Z}^d} c_h(\mathbf{n}) \omega_{\mathbf{n}}}, \quad h \in \mathfrak{R}_d, \omega \in \Omega, \tag{2.7}$$

identifies the additive group $\mathfrak{R}_d$ with the dual group $\widehat{\Omega}$ of Ω. With this identification the automorphism $\hat{\sigma}^{\mathbf{n}}$ of $\mathfrak{R}_d$ dual to the shift $\sigma^{\mathbf{n}}$ on Ω consists of multiplication by $u^{\mathbf{n}}$. If $I \subset \mathfrak{R}_d$ is an ideal, then

$$I^{\perp} = \widehat{\mathfrak{R}_d/I} = X_{\mathfrak{R}_d/I} = \{\omega \in \Omega : \langle h, \omega \rangle = 1 \text{ for every } h \in I\} \tag{2.8}$$

is a closed, shift-invariant subgroup of Ω, and $\alpha_{\mathfrak{R}_d/I}$ is the restriction of σ to $X_{\mathfrak{R}_d/I}$.

Now consider the special case where $I = \mathfrak{p} \subset \mathfrak{R}_d$ is a prime ideal containing a rational prime $p > 1$. For any $k \geq 1$ we denote by F_{p^k} the field with p^k elements and write

$$\mathfrak{R}_d^{(p^k)} = F_{p^k}[u_1^{\pm 1}, \dots, u_d^{\pm 1}] \tag{2.9}$$

for the ring of Laurent polynomials in the variables $u_1, \dots, u_d$ with coefficients in F_{p^k}. Again we write every $h \in \mathfrak{R}_d^{(p^k)}$ as $h = \sum_{\mathbf{n} \in \mathbb{Z}^d} c_h(\mathbf{n}) u^{\mathbf{n}}$ with $c_h(\mathbf{m}) \in F_{p^k}$ for every $\mathbf{m} \in \mathbb{Z}^d$.

We define a ring homomorphism $f \mapsto f_{/p}$ from $\mathfrak{R}_d$ to $\mathfrak{R}_d^{(p)}$ by reducing each coefficient of f modulo p. If

$$\overline{\mathfrak{p}} = \{f_{/p} : f \in \mathfrak{p}\}, \tag{2.10}$$

then $\overline{\mathfrak{p}} \subset \mathfrak{R}_d^{(p)}$ is again a prime ideal, and the map $f \mapsto f_{/p}$ induces an $\mathfrak{R}_d$-module isomorphism

$$\mathfrak{R}_p/\mathfrak{p} \cong \mathfrak{R}_d^{(p)}/\overline{\mathfrak{p}}. \tag{2.11}$$

We set $\Omega_p = F_p^{\mathbb{Z}^d}$, write every $\omega \in \Omega_p$ as $\omega = (\omega_{\mathbf{n}})$ with $\omega_{\mathbf{n}} \in F_p$ for every $\mathbf{n} \in \mathbb{Z}^d$, and use the pairing

$$\langle h, \omega \rangle = e^{2\pi i \left(\sum_{\mathbf{n} \in \mathbb{Z}^d} c_h(\mathbf{n}) \omega_{\mathbf{n}}\right)/p}, \quad h \in \mathfrak{R}_d^{(p)}, \omega \in \Omega_p,$$

to identify $\mathfrak{R}_d^{(p)}$ with the dual group $\widehat{\Omega}_p$ of Ω_p. Then

$$\widehat{\mathfrak{R}_d/\mathfrak{p}} = \widehat{\mathfrak{R}_d^{(p)}/\overline{\mathfrak{p}}} = X_{\mathfrak{R}_d^{(p)}/\overline{\mathfrak{p}}} = \{\omega \in \Omega_p : \langle h, \omega \rangle = 1 \text{ for every } h \in \mathfrak{p}\}, \tag{2.12}$$

and $\alpha_{\mathfrak{R}_d^{(p)}/\overline{\mathfrak{p}}}$ is the restriction to $X_{\mathfrak{R}_d^{(p)}/\overline{\mathfrak{p}}}$ of the shift-action σ of $\mathbb{Z}^d$ on Ω_p.

3. Mixing sets and invariant measures

Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X, and let μ be an α-invariant probability measure on the Borel field $\mathcal{B}_X$ of X. Following [**9**]–[**10**] we call a nonempty finite set $S \subset \mathbb{Z}^d$ *μ-mixing* under α if

$$\lim_{k \to \infty} \mu\biggl(\bigcap_{\mathbf{n} \in S} \alpha^{-k\mathbf{n}}(B_{\mathbf{n}})\biggr) = \prod_{\mathbf{n} \in S} \mu(B_{\mathbf{n}})$$

for every collection $(B_{\mathbf{n}}, \mathbf{n} \in S)$ in $\mathcal{B}_X$, and *μ-nonmixing* otherwise. A nonempty finite set $S \subset \mathbb{Z}^d$ is *minimal μ-nonmixing* if it is μ-nonmixing, but every nonempty subset $S' \subsetneq S$ is μ-mixing. A λ_X-(non-)mixing set is called a (non-)mixing set of α.

Measurably conjugate algebraic $\mathbb{Z}^d$-actions obviously have the same nonmixing sets. Furthermore, the nonmixing sets of an algebraic $\mathbb{Z}^d$-actions on a compact abelian group X are determined by the prime ideals $\mathfrak{p} \subset \mathfrak{R}_d$ associated with the dual module $\mathfrak{M} = \widehat{X}$ of α.

Proposition 3.1 ([**20**], [**10**]). *Let α be an algebraic $\mathbb{Z}^d$-actions on a compact abelian group X with dual module $\mathfrak{M} = \widehat{X}$. For every nonempty finite set $S \subset \mathbb{Z}^d$ the following conditions are equivalent.*

(1) *S is a mixing set of α;*
(2) *S is mixing for every $\alpha_{\mathfrak{R}/\mathfrak{p}}$ with $\mathfrak{p} \in \operatorname{asc}(\mathfrak{M})$.*

If the group X is connected and λ_X is mixing under α, then α is mixing of every order, and hence every nonempty finite set $S \subset \mathbb{Z}^d$ is mixing under α, by [**21**]. If X is disconnected, then a mixing algebraic $\mathbb{Z}^d$-action α on X has nonmixing sets if and only if λ_X does not have completely positive entropy (cf. [**11**], [**20**, Section 27] and Proposition 3.2 below). In particular, if α is an irreducible and mixing algebraic $\mathbb{Z}^d$-action on a compact zero-dimensional abelian group X, then α has nonmixing sets.

Proposition 3.2 ([**11**]). *Let $d > 1$, α an irreducible and ergodic algebraic $\mathbb{Z}^d$-action on a compact zero-dimensional abelian group X, and let $\mathfrak{p} \subset \mathfrak{R}_d$ be the prime ideal and $1 < p \in \mathfrak{p}$ the rational prime described in Proposition 2.2. We define the prime ideal $\overline{\mathfrak{p}} \subset \mathfrak{R}_d^{(p)}$ by (2.10).*

(1) *The $\mathbb{Z}^d$-actions α and $\alpha_{\mathfrak{R}_d/\mathfrak{p}} = \alpha_{\mathfrak{R}_d^{(p)}/\overline{\mathfrak{p}}}$ have the same nonmixing sets;*
(2) *For every nonzero element $h \in \overline{\mathfrak{p}} \subset \mathfrak{R}_d^{(p)}$, the* support

$$\mathcal{S}(h) = \{\mathbf{n} \in \mathbb{Z}^d : c_h(\mathbf{n}) \neq 0\}$$

of h is a nonmixing set of $\alpha_{\mathfrak{R}_d/\mathfrak{p}}$ and hence of α.

In this terminology we can express Theorem 1.1 as a consequence of the following result.

THEOREM 3.3. *Let $d > 1$, α_1 and α_2 irreducible, mixing and expansive algebraic $\mathbb{Z}^d$-actions on compact abelian groups X_1 and X_2, respectively, and let $\phi\colon X_1 \longrightarrow X_2$ be a measurable factor map of α_1 and α_2. We set $X = X_1 \times X_2$, denote by $\alpha = \alpha_1 \times \alpha_2$ the product-action of $\mathbb{Z}^d$ on X, and assume that one of the following two conditions is satisfied.*

(1) *α_1 is mixing of every order;*
(2) *α_1 and α_2 share a minimal nonmixing set $S \subset \mathbb{Z}^d$.*

Then every joining ν of λ_{X_1} and λ_{X_2} such that the $\mathbb{Z}^d$-action α on (X, ν) is measurably conjugate to α_1 on (X_1, λ_{X_1}), is a translate of the Haar measure of a closed α-invariant subgroup $Y \subset X$.

PROOF OF THEOREM 1.1, USING THEOREM 3.3. Let $\phi\colon X_1 \longrightarrow X_2$ be a measurable conjugacy of α_1 and α_2. We choose a prime ideal $\mathfrak{q} \subset \mathfrak{R}_d$ which is associated with $\mathfrak{M}_2 = \widehat{X}_2$ and an element $a \in \mathfrak{M}_2$ with $\mathfrak{q} = \operatorname{ann}(a)$. The inclusion $\mathfrak{N} = \mathfrak{R}_d \cdot a \subset \mathfrak{M}_2$ determines a dual factor map $\psi\colon X_2 \longrightarrow X_2' = \widehat{\mathfrak{N}}$ of α_2 and $\alpha_2' = \alpha_{\mathfrak{N}}$. We set $\phi' = \psi \circ \phi\colon X_1 \longrightarrow X_2'$, $X' = X_1 \times X_2'$, $\alpha' = \alpha_1 \times \alpha_2'$, and denote by ν' the α'-invariant probability measure on $\Gamma(\phi') = \{(x, \phi'(x)) : x \in X_1\} \subset X'$ which satisfies that $\nu'\pi_1^{-1} = \lambda_{X_1}$ and $\nu'\pi_2'^{-1} = \lambda_{X_2'}$, where $\pi_1\colon X \longrightarrow X_1$ and $\pi_2'\colon X \longrightarrow X_2'$ are the coordinate projections.

Since α_1 is irreducible, $h(\alpha_2^{\mathbf{n}}) = h(\alpha_1^{\mathbf{n}}) < \infty$ for every $\mathbf{n} \in \mathbb{Z}^d$ by Corollary 1.2. Hence α_2' is irreducible, mixing and expansive by the Propositions 2.1 and 2.2.

If X_1 is connected, then α_1 is mixing of every order by [**21**], and Theorem 3.3 implies that ν' is a translate of the Haar measure of a closed, α_2'-invariant subgroup of X'. It follows that ϕ' coincides λ_{X_1}-*a.e.* with an affine map.

We denote by $\psi'\colon X_1 \longrightarrow X_2'$ the homomorphism part of ϕ' (cf. (1.3)). As α_1 is irreducible, ψ' is finite-to-one. Hence $\phi' = \psi \circ \phi$ and ψ are both finite-to-one, and α_2 is irreducible by Proposition 2.2. We repeat the first part of the proof with α_2' and X_2' replaced by α_2 and X_2 and obtain that ϕ coincides λ_{X_1}-*a.e.* with an affine map.

If X_1 is not connected, it is zero-dimensional by irreducibility, and α_1 has nonmixing sets by Proposition 3.2. We fix a minimal nonmixing set $S \subset \mathbb{Z}^d$ for α_1 and α_2. Proposition 3.1 allows us to choose the associated prime ideal $\mathfrak{q} \subset \mathfrak{R}_d$ considered above such that S is minimal nonmixing for $\alpha_{\mathfrak{R}_d/\mathfrak{q}}$. Theorem 3.3 implies that ν' is a translate of the Haar measure of a closed, α_2'-invariant subgroup of X'. It follows that ϕ' coincides λ_{X_1}-*a.e.* with an affine map, and the proof is completed as in the connected case. □

4. The proof of Theorem 3.3 in the absence of nonmixing sets

If α_1 has no nonmixing sets, then it is mixing of every order, and the same is true for α_2, and X_1 and X_2 are connected (cf. the Propositions 2.1–2.3 and 3.1–3.2). In this case Theorem 3.3 is in essence a consequence of a result by Katok and Spatzier ([**5**]) on invariant measures for commuting toral automorphisms.

DEFINITION 4.1. An algebraic $\mathbb{Z}^d$-action α on a compact abelian group X is *semi-irreducible* if there exist an integer $n \geq 1$, irreducible algebraic $\mathbb{Z}^d$-actions α_i on compact abelian groups X_i, $i = 1, \dots, n$, and a continuous, surjective, finite-to-one group homomorphism $\phi\colon X \longmapsto \bar{X} = X_1 \times \cdots \times X_n$ such that $\phi \cdot \alpha^{\mathbf{n}} = \bar{\alpha}^{\mathbf{n}} \cdot \phi$ for every $\mathbf{n} \in \mathbb{Z}^d$, where $\bar{\alpha} = \alpha_1 \times \cdots \times \alpha_n$ is the product-action of $\mathbb{Z}^d$ on $\bar{X}$.

The class of semi-irreducible actions is easily seen to be closed under taking direct products and algebraic factors. The following version of the result by Katok and Spatzier is suited for proving Theorem 3.3.

Theorem 4.2. *Let $d > 1$, and let α be an semi-irreducible expansive and mixing algebraic $\mathbb{Z}^d$-action on a compact connected abelian group X and μ an α-invariant and mixing probability measure on X. Then there exists a closed α-invariant subgroup $Y \subset X$ with the following properties:*

(1) *μ is invariant under translation by Y,*
(2) *The projection of μ onto X/Y has zero entropy under $\alpha_{X/Y}^{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{Z}^d$, where $\alpha_{X/Y}$ is the action induced by α on X/Y.*

Theorem 4.2 clearly implies Theorem 3.3 for connected groups, since μ has completely positive entropy under each $\alpha^{\mathbf{n}}$. By using an induction argument we may reduce the proof of Theorem 4.2 to the case where μ is not invariant under translation by any infinite subgroup of X.

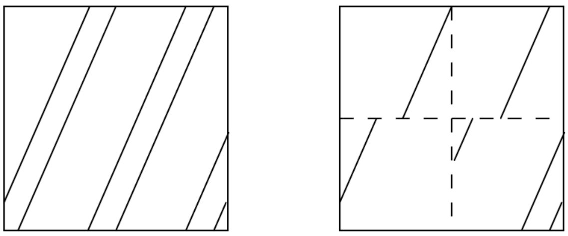
In order to explain the proof of Theorem 4.2 in this special case we assume initially that $X = \mathbb{T}^n$ for some $n \geq 3$ (no smaller n is possible, since $d > 1$ and α is mixing), and that the matrices $\alpha^{\mathbf{m}} \in \mathrm{GL}(n, \mathbb{Z})$, $\mathbf{m} \in \mathbb{Z}^d$, can be diagonalized simultaneously over $\mathbb{C}$. Then there exists a decomposition of $\mathbb{R}^n$ into eigenspaces $V_1, \dots, V_k$ of the linear action $\bar{\alpha}$ of α on $\mathbb{R}^n$ with eigenvalues

$$\eta_i \colon \mathbb{Z}^d \longrightarrow \mathbb{R}^\times \text{ if } V_i \cong \mathbb{R}, \;\; \eta_i \colon \mathbb{Z}^d \longrightarrow \mathbb{C}^\times \text{ if } V_i \cong \mathbb{C},$$

where $\bar{\alpha}^{\mathbf{n}} v = \eta_i(\mathbf{n}) v$ for every $i = 1, \dots, k$, $v \in V_i$ and $\mathbf{n} \in \mathbb{Z}^d$.

Fix $\mathbf{m} \in \mathbb{Z}^d$ such that $\alpha^{\mathbf{m}}$ is hyperbolic, set $P^-(\mathbf{n}) = \{i : |\eta_i(\mathbf{n})| < 1\}$, and assume without loss in generality that $P^-(\mathbf{n}) = \{1, \dots, l\}$, $l < k$.

We intersect the leaves of the foliation $\mathcal{F}$ of $X = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ by the cosets of $V_1 \subset X$ with a finite partition $\mathcal{P}$ of X into 'nice' convex sets: the resulting measurable partition generates a countably generated Borel sigma-algebra $\mathcal{A} \subset \mathcal{B}_X$. The following illustration shows part of a typical leaf L of $\mathcal{F}$ (on the left), and some of the atoms of $\mathcal{A}$ into which L is decomposed by $\mathcal{P}$ (on the right).



Let $\{\mu_x^{\mathcal{A}} : x \in X\}$ be the decomposition of μ with respect to $\mathcal{A}$, and let $J_\mu^{\mathcal{A}} : \mathbb{Z}^d \times X \longrightarrow \mathbb{R}$ be the *information cocycle* of $\mathcal{A}$, given by

$$J_\mu^{\mathcal{A}}(\mathbf{n}, x) = \log \frac{d\mu_x^{\mathcal{A}}}{d\mu_x^{\alpha^{-\mathbf{n}}(\mathcal{A})}}(x) = \log \frac{\mu_x^{\alpha^{-\mathbf{n}}(\mathcal{A})}([x]_{\mathcal{A} \vee \alpha^{-\mathbf{n}}(\mathcal{A})})}{\mu_x^{\mathcal{A}}([x]_{\mathcal{A} \vee \alpha^{-\mathbf{n}}(\mathcal{A})})}$$

for every $\mathbf{n} \in \mathbb{Z}^d$ and $x \in X$. The cocycle equation

$$J_\mu^{\mathcal{A}}(\mathbf{n}, \alpha^{\mathbf{n}'} x) + J_\mu^{\mathcal{A}}(\mathbf{n}', x) = J_\mu^{\mathcal{A}}(\mathbf{n} + \mathbf{n}', x)$$

is just the chain-rule of Radon-Nikodym derivatives. If

$$\mathbf{n} \in S = \{\mathbf{k} \in \mathbb{Z}^d : |\eta_1(\mathbf{k}) - 1| < 1/2\},$$

then each atom of $\mathcal{A}$ intersects only a bounded number of atoms of $\alpha^{-\mathbf{n}}(\mathcal{A})$ and vice versa. This implies that $\int |J_\mu^{\mathcal{A}}(\mathbf{n}, \cdot)| \, d\mu$ is bounded on the infinite set S. As μ is mixing under α this can be used to show that there exists an $\mathcal{A}$-measurable Borel map $b \colon X \longrightarrow \mathbb{R}$ with

$$b(\alpha^{\mathbf{n}} x) + J_\mu^{\mathcal{A}}(\mathbf{n}, x) - b(x) = \theta(\mathbf{n})$$

μ-a.e. for every $\mathbf{n} \in \mathbb{Z}^d$, where $\theta\colon \mathbb{Z}^d \longrightarrow \mathbb{R}$ is a group homomorphism.

We 'correct' the measures $\mu_x^{\mathcal{A}}$ by setting

$$\nu_x = e^{-b(x)} \mu_x^{\mathcal{A}}$$

for every $x \in X$. Then

$$\log \frac{d\nu_x}{d\nu_{\alpha^{\mathbf{n}} x} \alpha^{-\mathbf{n}}}(x) = \theta(\mathbf{n})$$

for every $\mathbf{n}, x$.

Fix $x \in X$ for the moment and combine the measures ν_y, $y \in x + V_1 \subset X$, to a sigma-finite measure $\bar{\rho}_x$ on $x + V_1$. The bijection $x + v \mapsto v$ from $x + V_1$ to V_1 sends $\bar{\rho}_x$ to a sigma-finite measure ρ_x on V_1. This map $x \mapsto \rho_x$ has the following properties:

(1) $\rho_x = e^{\theta(\mathbf{n})} \rho_{\alpha^{\mathbf{n}} x} M_{\eta_1(\mathbf{n})}$
(2) $\rho_{x+v} = \rho_v T_{-v}$,

where M_t is multiplication by t and T_v translation by v on V_1.

Ergodicity implies that there are only two possibilities: either each ρ_x is concentrated in a single atom, or the map $x \mapsto \rho_x$ is constant and each ρ_x is translation-invariant (some caution is needed in this part of the argument).

The latter possibility is excluded, since it would imply the invariance of μ under V_1 and hence under the closure of V_1 in X. In the former case the measure μ is concentrated on a set which intersects each leaf of $\mathcal{F}$ in at most one point.

We repeat the argument with V_2 replacing V_1. After l steps we obtain that μ is supported on a set which intersects each coset of $V_1 + \cdots + V_l$ in at most one point, which implies that $\alpha^{\mathbf{n}}$ has zero entropy.

In order to deal with the general case, where X is not necessarily a finite-dimensional torus, we have to give an alternative description of irreducible $\mathbb{Z}^d$-actions on compact connected abelian groups, taken from [**18**]–[**20**]; the algebraic background can be found in [**3**] and [**22**].

Let $d \geq 1$, $c = (c_1, \dots, c_d) \in (\overline{\mathbb{Q}}^{\times})^d$, and let $K = \mathbb{Q}(c)$ be the algebraic number field generated by $\{c_1, \dots, c_d\}$. We write P^K, P_f^K, and P_∞^K, for the sets of places (= equivalence classes of valuations), finite places and infinite places of K. For every $v \in P^K$ we write K_v for the completion of K at the place v, and choose a Haar measure λ_v on the locally compact additive group K_v. Since K_v is a topological ring, we can define a distinguished valuation $|\cdot|_v\colon K_v \longmapsto \mathbb{R}$ in v by setting

$$\lambda_v(aB) = |a|_v \lambda_v(B)$$

for every $a \in K_v$ and every compact set $B \subset K_v$. The set

$$R_v = \{r \in K_v : |r|_v \leq 1\}$$

is a compact subset of K_v. If $v \in P_f^K$, then R_v is, in addition, open, and is the unique maximal compact subring of K_v. Furthermore,

$$\mathfrak{o}_K = \bigcap_{v \in P_f^K} R_v \tag{4.1}$$

is the ring of integral elements in K. Let

$$F = P_\infty^K \cup \{v \in P_f^K : |c_i|_v \neq 1 \text{ for some } i \in \{1, \dots, d\}\}, \tag{4.2}$$

Then F is finite by Theorem III.3 in [**22**], and we denote by

$$\iota\colon K \longmapsto V = \prod_{v \in F} K_v \tag{4.3}$$

the diagonal embedding $r \mapsto (r, \dots, r)$, $r \in K$, and put

$$R = \{a \in K : |a|_v \leq 1 \text{ for every } v \in P_f^K \smallsetminus F\} \supset \mathfrak{o}_K. \tag{4.4}$$

Then $\iota(R)$ is a discrete, co-compact, additive subgroup of V, and we consider the compact quotient group

$$Y_c = \widehat{R} = V \,/\, \iota(R) \cong \Bigl(\prod_{v \in P_\infty^K} K_v \times \prod_{v \in F \cap P_f^K} R_v \Bigr) \Big/ \iota(\mathfrak{o}_K), \tag{4.5}$$

where $\widehat{R}$ is the Pontryagin dual of R, and write

$$\pi \colon V \longmapsto Y_c \tag{4.6}$$

for the quotient map. In the special case where $F = P_\infty^K$ (i.e. where each c_i, $i = 1, \dots, d$, is a unit in $\mathfrak{o}_K$),

$$R = \mathfrak{o}_K, \qquad V \cong \mathbb{R}^{r(K)}, \qquad Y_c = \widehat{\mathfrak{o}_K} \cong \mathbb{T}^{r(K)}, \tag{4.7}$$

with

$$r(K) = |\{v \in P_\infty^K : K_v = \mathbb{R}\}| + 2|\{v \in P_\infty^K : K_v = \mathbb{C}\}|. \tag{4.8}$$

For every $a \in R$ and $b = (b_v,\ v \in F) \in V$ we set $a \cdot b = (ab_v,\ v \in F)$. Since $a \cdot \iota(R) \subset \iota(R)$, multiplication by a defines a surjective group homomorphism $\theta_a \colon Y_c \longmapsto Y_c$ by

$$\theta_a(b + \iota(R)) = a \cdot b + \iota(R), \tag{4.9}$$

for every $b \in V_F$. Then $\theta_{aa'} = \theta_a \cdot \theta_{a'}$ for all $a, a' \in R_F$. In particular, if $R_F^\times \subset R_F$ is the group of units (i.e. invertible elements) in R_F, then θ_a is a continuous group automorphism of Y_F for every $a \in R_F^\times$. This allows us to define a $\mathbb{Z}^d$-action β_c by automorphisms of Y_c by setting

$$\beta_c^{\mathbf{n}} y = \theta_{c^{\mathbf{n}}}(y) \tag{4.10}$$

for every $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d$, where

$$c^{\mathbf{n}} = c_1^{n_1} \cdots c_d^{n_d}.$$

Proposition 4.3 (Proposition 7.2 in [**20**]). *Suppose that $d \geq 1$, and that α is a mixing and expansive algebraic $\mathbb{Z}^d$-action on a compact connected abelian group X. Then α is irreducible if and only if there exist a point $c = (c_1, \dots, c_d) \in (\overline{\mathbb{Q}}^\times)^d$ with the following properties.*

(i) *$|c^{\mathbf{m}}| \neq 1$ for every nonzero $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$,*
(ii) *α is weakly algebraically conjugate to the $\mathbb{Z}^d$-action β_c on Y_c.*

Furthermore,

$$h(\alpha^{\mathbf{n}}) = h(\beta_c^{\mathbf{n}}) = \sum_{v \in F} \log \max \{1, |c^{\mathbf{n}}|_v\} \tag{4.11}$$

for every $\mathbf{n} \in \mathbb{Z}^d$, where F appears in (4.2).

We return to the proof of Theorem 4.2 in the general case. Equation (4.5) allows us to assume the following (perhaps after modifying α and X by a finite-to-one algebraic factor map):

(1) $X = \prod_{i=1}^n K_i \,/\, D$, where each K_i is a locally compact abelian field with characteristic zero and $D \subset \prod_{i=1}^n K_i$ is a discrete co-compact subgroup;
(2) There exist homomorphisms $\gamma_i \colon \mathbb{Z}^d \longrightarrow K_i^\times$, $i = 1, \dots, d$, where $K_i^\times$ is the multiplicative group $K_i \smallsetminus \{0\}$, with the following property: if

$$\theta^{\mathbf{m}}(a_, \dots, a_n) = (\gamma_1(\mathbf{m})a_1, \dots, \gamma_n(\mathbf{m})a_n)$$

for every $\mathbf{m} \in \mathbb{Z}^d$ and $(a_1, \dots, a_n) \in \prod_{i=1}^n K_i$, then $\theta^{\mathbf{m}}(D) = D$ and $\alpha^{\mathbf{m}}$ is the automorphism of X induced by $\theta^{\mathbf{m}}$.

This description of α and X is sufficiently similar to the toral case described earlier to allow essentially the same proof as before.

5. The proof of Theorem 3.3 in the presence of nonmixing sets

If α_1 and α_2 share a nonmixing set $S \subset \mathbb{Z}^d$ then the following lemma allows us to synchronize the 'nonmixing times' of α_1 and α_2.

LEMMA 5.1 ([**11**]). *Let α_1 and α_2 be algebraic $\mathbb{Z}^d$-actions on compact abelian groups X_1 and X_2, respectively. If α_2 is a measurable factor of α_1, and if $S \subset \mathbb{Z}^d$ is a nonempty finite set which is minimal nonmixing for both α_1 and α_2, then there exist nonzero elements $a_i(\mathbf{n}) \in \widehat{X}_i$, $\mathbf{n} \in S$, $i = 1, 2$, such that*

$$\sum_{\mathbf{n}\in S} \hat{\alpha}_1^{k\mathbf{n}}(a_1(\mathbf{n})) = \sum_{\mathbf{n}\in S} \hat{\alpha}_2^{k\mathbf{n}}(a_2(\mathbf{n})) = 0 \tag{5.1}$$

for every k in an infinite subset $K \subset \mathbb{N}$.

For the proof of Theorem 3.3 in the zero-dimensional case (i.e. in the presence of nonmixing sets) we use Lemma 5.1 to find nonzero elements $a_i(\mathbf{n}) \in \widehat{X}_i$, $\mathbf{n} \in S$, $i = 1, 2$, such that (5.1) holds for every k in an infinite subset $K \subset \mathbb{N}$. It follows that

$$\sum_{\mathbf{n}\in S} \hat{\alpha}_1^{k\mathbf{n}}\bigl(f_1(\hat{\alpha}_1)(a_1(\mathbf{n}))\bigr) = \sum_{\mathbf{n}\in S} \hat{\alpha}_2^{k\mathbf{n}}\bigl(f_2(\hat{\alpha}_2)(a_2(\mathbf{n}))\bigr) = 0 \tag{5.2}$$

for every $k \in K$ and $f_1, f_2 \in \mathfrak{R}_d$.

Put $X = X_1 \times X_2$ and denote by $\alpha = \alpha_1 \times \alpha_2$ the product-$\mathbb{Z}^d$-action on X. We fix a nonempty finite set S which is minimal nonmixing simultaneously for λ_{X_1}, λ_{X_2} and ν. As every proper subset of S is ν-mixing, the Fourier transform $\hat{\nu}\colon \widehat{X} = \widehat{X}_1 \times \widehat{X}_2 \longrightarrow \mathbb{C}$ satisfies that

$$\begin{aligned}
&\overline{\hat{\nu}\bigl(f_1(\hat{\alpha}_1)(a_1(\mathbf{m})), f_2(\hat{\alpha}_2)(a_2(\mathbf{m}))\bigr)} \\
&\quad = \hat{\nu}\bigl(-\hat{\alpha}_1^{k\mathbf{m}}(f_1(\hat{\alpha}_1)(a_1(\mathbf{m}))), -\hat{\alpha}_2^{k\mathbf{m}}(f_2(\hat{\alpha}_2)(a_2(\mathbf{m})))\bigr) \\
&\quad = \lim_{\substack{k\to\infty \\ k\in K}} \hat{\nu}\Biggl(\sum_{\mathbf{n}\in S\smallsetminus\{\mathbf{m}\}} \bigl(\hat{\alpha}_1^{k\mathbf{n}}(f_1(\hat{\alpha}_1)(a_1(\mathbf{n}))), \hat{\alpha}_2^{k\mathbf{n}}(f_2(\hat{\alpha}_2)(a_2(\mathbf{n})))\bigr)\Biggr) \\
&\quad = \prod_{\mathbf{n}\in S\smallsetminus\{\mathbf{m}\}} \hat{\nu}\bigl(f_1(\hat{\alpha}_1)(a_1(\mathbf{n})), f_2(\hat{\alpha}_2)(a_2(\mathbf{n}))\bigr)
\end{aligned}$$

for every $\mathbf{m} \in S$. By varying $\mathbf{m} \in S$ we see that

$$\bigl|\hat{\nu}\bigl(f_1(\hat{\alpha}_1)(a_1(\mathbf{m})), f_2(\hat{\alpha}_2)(a_2(\mathbf{m}))\bigr)\bigr| \in \{0, 1\}$$

for every $\mathbf{m} \in S$ and $f_1, f_2 \in \mathfrak{R}_d$.

We fix an element $\mathbf{n} \in S$, consider the $\hat{\alpha}$-invariant subgroup

$$\mathfrak{N} = \bigl\{\bigl(f_1(\hat{\alpha}_1)(a_1(\mathbf{n})), f_2(\hat{\alpha}_2)(a_2(\mathbf{n}))\bigr) : f_1, f_2 \in \mathfrak{R}_d\bigr\} \subset \widehat{X}.$$

Since $|\hat{\nu}(a)| \in \{0, 1\}$ for every $a \in \mathfrak{N}$, the projection of ν onto the quotient group $Z = X/\mathfrak{N}^\perp = \widehat{\mathfrak{N}}$ is a translate of the Haar measure of a closed, α_Z-invariant subgroup, where α_Z is the $\mathbb{Z}^d$-action induced by α on Z. Since the map $\pi\colon X \longrightarrow Z$ is finite-to-one by irreducibility and ν is ergodic, an elementary skew-product argument shows that ν is also a translate of the Haar measure of a closed, α-invariant subgroup $Y \subset X$. This proves Theorem 3.3 in the zero-dimensional case.

REMARK 5.2. There is a significant difference between the connected and zero-dimensional cases in Theorem 3.3. Assume for simplicity that $d > 1$, and that α is an irreducible, expansive and mixing algebraic $\mathbb{Z}^d$-action on a compact abelian group X. If X is connected, then it is not known whether there exists a nonatomic probability measure $\mu \neq \lambda_X$ on X which is invariant and ergodic under α; in some special cases it has even been conjectured that no such measures exist (cf. e.g. [**4**], [**16**]). Even the existence of nonatomic mixing measures $\mu \neq \lambda_X$ has not been ruled out. However, if μ is an α-invariant mixing measure with $h_\mu(\alpha^{\mathbf{n}}) > 0$ for some expansive $\alpha^{\mathbf{n}}$, then $\mu = \lambda_X$ by [**5**].

If X is zero-dimensional, then there always exist nonatomic α-invariant and ergodic probability measures $\mu \neq \lambda_X$ with $h_\mu(\alpha^{\mathbf{n}}) > 0$ for some nonzero $\mathbf{n} \in \mathbb{Z}^d$. One can also construct nonatomic α-invariant and ergodic probability measures μ with $h_\mu(\alpha^{\mathbf{n}}) = 0$ for every $\mathbf{n} \in \mathbb{Z}^d$. We refer to [**9**] and [**19**] for examples.

6. Examples

EXAMPLE 6.1. The matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & 0 \end{pmatrix} \in \mathrm{GL}(3, \mathbb{Z})$$

has the irreducible characteristic polynomial

$$f = x^3 - 3x - 1$$

with roots

$$a_1 = -1.532\ldots,\ a_2 = -0.3473\ldots,\ a_3 = 1.879\ldots$$

Hence A has a two-dimensional expanding subspace, but the expanding subspace of A^{-1} has dimension 1. It follows that A and A^{-1} are not algebraically conjugate, although they are measurably conjugate. Since the characteristic polynomial of A is irreducible, A does not have an invariant subtorus and the $\mathbb{Z}$-action $n \mapsto A^n$ is irreducible.

EXAMPLE 6.2. Let

$$A = \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix},\ B = \begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix},\ C = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}.$$

If

$$M = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix},\ M' = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix},$$

then $AM = MB$ and $M \in \mathrm{SL}(2, \mathbb{Z})$, i.e. A and B are algebraically conjugate.

Also, $CM' = M'B$, but $M' \notin \mathrm{GL}(2, \mathbb{Z})$. Since there is no $M'' \in \mathrm{GL}(2, \mathbb{Z})$ with $CM'' = M''B$, B and C are not algebraically conjugate, but they are measurably conjugate: they are Bernoulli with equal entropy. The $\mathbb{Z}$-actions $n \mapsto A^n$, $n \mapsto B^n$, $n \mapsto C^n$ are all irreducible.

Theorem 1.1 shows that for irreducible, expansive and mixing algebraic $\mathbb{Z}^d$-actions with $d > 1$ topological and algebraic conjugacy are equivalent. The following elementary observations, taken from [**6**], are sometimes useful for checking that two given $\mathbb{Z}^d$-actions are algebraically nonconjugate.

Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X. We say that $\hat{\alpha}$ is *cyclic* if the dual module $\mathfrak{M} = \widehat{X}$ is singly generated, i.e. if $\mathfrak{M} = \mathfrak{R}_d \cdot a$ for some $a \in \mathfrak{M}$. In this case $\mathfrak{M} \cong \mathfrak{R}_d/I$ for some ideal $I \subset \mathfrak{R}_d$.

PROPOSITION 6.3. *Let α be an irreducible and mixing algebraic $\mathbb{Z}^d$-action on a compact connected abelian group X. Then $\hat{\alpha}$ is cyclic if and only if α is algebraically conjugate to $\alpha_{\mathfrak{R}_d/\mathfrak{p}}$ for some prime ideal $\mathfrak{p} \subset \mathfrak{R}_d$ (cf. Proposition 2.2).*

If $\hat{\alpha}$ is cyclic, and if β is an algebraic $\mathbb{Z}^d$-action on a compact abelian group Y which is weakly algebraically conjugate to α, then β is algebraically conjugate to α if and only if $\hat{\beta}$ is cyclic.

COROLLARY 6.4. *Let α be an irreducible algebraic $\mathbb{Z}^d$-action on a compact abelian group X with the property that $\hat{\alpha}^{\mathbf{n}}$ is cyclic for some $\mathbf{n} \in \mathbb{Z}^d$. If β is an algebraic $\mathbb{Z}^d$-action on a compact abelian group Y which is weakly algebraically conjugate to α, but for which $\hat{\beta}^{\mathbf{n}}$ is not cyclic, then $\hat{\beta}$ is not cyclic.*

COROLLARY 6.5. *Suppose that α and β are algebraic $\mathbb{Z}^d$-actions on compact abelian groups such that $\hat{\alpha}$ is cyclic, but not $\hat{\beta}$. Then α is not algebraically conjugate to any re-parametrization of β (i.e. to any $\mathbb{Z}^d$-action of the form $\mathbf{n} \mapsto \beta^{A\mathbf{n}}$ with $A \in GL(d,\mathbb{Z})$).*

We illustrate Proposition 6.3 and its corollaries with some examples.

EXAMPLE 6.6. Let α be the $\mathbb{Z}^2$-action on $\mathbb{T}^3$ given by $\alpha^{\mathbf{n}} = A^{n_1}B^{n_2}$ for every $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$, where

$$A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -4 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 & -1 \\ -1 & -2 & 2 \\ 2 & 5 & -2 \end{pmatrix}.$$

The matrices A and B have the same irreducible characteristic polynomial

$$g = x^3 + 3x^2 - 6x + 1,$$

A is irreducible and hyperbolic, and α is irreducible, expansive and mixing.

Let

$$V = \begin{pmatrix} 2 & -2 & -1 \\ 0 & -3 & 0 \\ 1 & -4 & -2 \end{pmatrix},$$

$$A' = V^{-1}AV = \begin{pmatrix} 2 & -4 & -1 \\ 1 & -4 & -1 \\ 1 & -5 & -1 \end{pmatrix},$$

$$B' = V^{-1}BV = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 6 & -3 \end{pmatrix},$$

and denote by α' the irreducible $\mathbb{Z}^2$-action $\mathbf{n} \mapsto A'^{n_1}B'^{n_2}$. It is clear that $\alpha^{\mathbf{n}}$ and $\alpha'^{\mathbf{n}}$ are measurably conjugate for every $\mathbf{n} \in \mathbb{Z}^d$, but α and α' are not algebraically conjugate, since $\hat{B}'$ is cyclic, but $\hat{B}$ is not. Hence they are measurably non-conjugate by Theorem 1.1.

EXAMPLE 6.7. Denote by σ the shift-action (1.4) of $\mathbb{Z}^d$ on $\bar{X} = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^d}$. For every nonempty finite set $E \subset \mathbb{Z}^d$ we denote by $X_E \subset \bar{X}$ the closed shift-invariant subgroup consisting of all $x \in \bar{X}$ whose coordinates sum to 0 in every translate of E in $\mathbb{Z}^d$. If E has at least two points then X_E is uncountable and the restriction σ_E of σ to X_E is an expansive algebraic $\mathbb{Z}^d$-action.

For $d = 2$ and the subset

$$E = \{(0,0), (1,0), (0,1)\} \subset \mathbb{Z}^2,$$

the $\mathbb{Z}^2$-action σ_E on X_E is called *Ledrappier's example*: σ_E is mixing and expansive, but not mixing of order 3 (for every $n \geq 0$, $x_{(0,0)} + x_{(2^n,0)} + x_{(0,2^n)} = 0$).

We also consider the subsets

$$\begin{aligned} E_1 =& \{(0,0), (1,0), (2,0), (1,1), (0,2)\}, \\ E_2 =& \{(0,0), (2,0), (0,1), (1,1), (0,2)\}, \\ E_3 =& \{(0,0), (1,0), (2,0), (0,1), (1,1), (0,2)\}. \end{aligned}$$

of $\mathbb{Z}^2$. The shift-actions $\sigma_i = \sigma_{E_i}$ of $\mathbb{Z}^2$ on $X_i = X_{E_i}$ are mixing, irreducible and expansive.

For every $\mathbf{n} \in \mathbb{Z}^2$, the automorphisms $\sigma_i^{\mathbf{n}}$ are measurably conjugate, but for $i \neq j$ the $\mathbb{Z}^2$-actions σ_i and σ_j are measurably nonconjugate by Theorem 1.1.

The shift-action σ_4 of $\mathbb{Z}^2$ on X_{E_4} with

$$E_4 = \{(0,0), (2,0), (0,2)\}$$

is also mixing and expansive, but reducible: the map $\phi\colon X_{E_4} \to X_E$, given by

$$\phi(x)_{(m,n)} = x_{m,n} + x_{(m+1,n)} + x_{(m,n+1)},$$

is a shift-commuting surjective group homomorphism with kernel X_E.

Example 6.8 (Conjugacy of $\mathbb{Z}^2$-actions with positive entropy). We define the sets $E_1, E_2, E_3 \subset \mathbb{Z}^2$ as in Example 6.7, set

$$Y_i = \Bigl\{ x = (x_{\mathbf{n}}) \in (\mathbb{Z}/4\mathbb{Z})^2 : \sum_{\mathbf{m} \in E_i} x_{\mathbf{m}+\mathbf{n}} = 0 \pmod 2 \text{ for every } \mathbf{n} \in \mathbb{Z}^2 \Bigr\},$$

and denote by τ_i the restriction to Y_i of the shift-action (1.4) of $\mathbb{Z}^2$ on $\Omega = (\mathbb{Z}/4\mathbb{Z})^{\mathbb{Z}^2}$. The entropy formula in [**13**, Theorem 4.2 and Lemma 4.2] (or a fairly straightforward direct calculation) shows that $h(\tau_i) = \log 2$ for $i = 1, 2, 3$. Theorem 6.5 in [**13**] implies that the Pinsker algebra $\pi(\tau_i)$ of τ_i is the sigma-algebra $\mathcal{B}_{Y_i/Z_i}$ of Z_i-invariant Borel sets in Y_i, where

$$Z_i = \{x = (x_{\mathbf{n}}) \in Y_i : x_{\mathbf{n}} = 0 \,(\mathrm{mod}\, 2) \text{ for every } \mathbf{n} \in \mathbb{Z}^2\}.$$

Then the $\mathbb{Z}^2$-action τ_i' induced by τ_i on Y_i/Z_i is algebraically conjugate to the shift-action σ_i on the group $X_i = X_{E_i}$ in Example 6.7.

Since any measurable conjugacy of τ_i and τ_j would map $\pi(\tau_i)$ to $\pi(\tau_j)$ and induce a conjugacy of τ_i' and τ_j' and hence of σ_i and σ_j, Example 6.7 implies that τ_i and τ_j are measurably nonconjugate for $1 \le i < j \le 3$.

Example 6.9. Let $f = 1 + \omega u_1 + \omega^2 u_2 \in \mathfrak{R}_2^{(4)}$, where $\omega \in F_4$ and $1 + \omega + \omega^2 = 0$ (cf. (2.9)). The inclusion $\mathfrak{R}_2^{(2)} \subset \mathfrak{R}_2^{(4)}$ induces an injective $\mathfrak{R}_2$-module homomorphism $\jmath\colon \mathfrak{M} = \mathfrak{R}_2^{(2)}/\overline{\mathfrak{p}} \longrightarrow \mathfrak{M}' = \mathfrak{R}_2^{(4)}/\overline{\mathfrak{q}}$, where

$$\overline{\mathfrak{p}} = (1 + u_1 + u_2 + u_1^2 + u_1u_2 + u_2^2)\mathfrak{R}_2^{(2)} \subset \mathfrak{R}_2^{(2)}$$

and

$$\overline{\mathfrak{q}} = f\mathfrak{R}_2^{(4)}.$$

Since $\jmath(\mathfrak{M})$ has index 2 in $\mathfrak{M}'$, there exists a two-to-one surjective dual homomorphism $\psi\colon X_{\mathfrak{M}'} = \widehat{\mathfrak{M}} \longrightarrow X' = \widehat{\mathfrak{M}'}$ with $\alpha_{\mathfrak{M}}^{\mathbf{n}} \cdot \psi = \psi \cdot \alpha_{\mathfrak{M}'}^{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{Z}^2$.

Since $\hat{\alpha}_{\mathfrak{M}}$ is cyclic, but $\hat{\alpha}_{\mathfrak{M}'}$ is not, Proposition 6.3 implies that the irreducible and mixing $\mathbb{Z}^2$-actions $\alpha_{\mathfrak{M}}$ and $\alpha_{\mathfrak{M}'}$ are not measurably conjugate, although they are weakly algebraically conjugate.

Further examples can be found in [**6**] and [**11**], where we also compute explicitly the measurable centralizers of such actions.

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