TAIL-FIELDS OF PRODUCTS OF RANDOM VARIABLES AND ERGODIC EQUIVALENCE RELATIONS

KLAUS SCHMIDT

ABSTRACT. We prove the following result: Let G be a countable discrete group with finite conjugacy classes, and let $(X_n,\, n \in \mathbb{Z})$ be a two-sided, strictly stationary sequence of G-valued random variables. Then $\mathcal{T}_\infty = \mathcal{T}_\infty^*$, where $\mathcal{T}_\infty$ is the two-sided tail-sigma-field $\bigcap_{M \ge 1} \sigma(X_m : |m| \ge M)$ of (X_n) and T_∞^* the tail-sigma-field $\bigcap_{M \ge 0} \sigma(Y_{m,n} : m,n \ge M)$ of the random variables $(Y_{m,n},\, m,n \ge 0)$ defined as the products $Y_{m,n} = X_n \cdots X_{-m}$. This statement generalises a number of results in the literature concerning tail triviality of two-sided random walks on certain discrete groups.

1. INTRODUCTION

Let G be a countable discrete group, $(X_n,\, n \in \mathbb{Z})$ a strictly stationary two-sided sequence of G-valued random variables, and let

$$\mathcal{T}_\infty = \bigcap_{M \ge 0} \sigma(X_m : |m| \ge M) \tag{1.1}$$

be the two-sided tail-sigma-field of (X_n). For every $m, n \in \mathbb{N}$ we consider the product

$$Y_{m,n} = X_n \cdots X_0 \cdots X_{-m}.$$

The tail-sigma-field

$$\mathcal{T}_\infty^* = \bigcap_{M \ge 0} \sigma(Y_{m,n} : m,n \ge M) \supset \mathcal{T}_\infty \tag{1.2}$$

of the G-valued random variables $(Y_{m,n},\, m,n \in \mathbb{N})$ has received some attention in the literature. For example, if $G = \mathbb{Z}$ and the process (X_n) is independent, then $(Y_{m,n},\, m,n \in \mathbb{Z})$ is essentially the two-sided random walk on G associated with (X_n), and the Hewitt-Savage zero-one law states that $\mathcal{T}_\infty = \mathcal{T}_\infty^*$ is trivial ([7]). The analogous result for finite state Markov chains was proved by Blackwell and Freedman ([3]) for Markov measures, and by Georgii for Gibbs states ([5], [6]). Berbee and den Hollander showed that, for an arbitrary integer valued strictly stationary process (X_n), the triviality of $\mathcal{T}_\infty$ implies that of $\mathcal{T}_\infty^*$ whenever the random variable X_0 has finite entropy (cf. [1]; their proof actually shows that $\mathcal{T}_\infty = \mathcal{T}_\infty^*$ even if $\mathcal{T}_\infty$ is nontrivial). In [11] the results by Blackwell, Freedman and Georgii were extended to processes taking values in arbitrary discrete groups G with finite conjugacy classes: *Let X be a topologically mixing two-sided shift of finite type, μ the shift-invariant Gibbs measure arising from a function $\phi\colon X \longmapsto \mathbb{R}$ with summable variation, T the shift on X, and $f\colon X \longmapsto G$ a continuous map with*

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values in a discrete group G with finite conjugacy classes. If $X_n = f \cdot T^n$ for every $n \in \mathbb{Z}$, then the stationary process (X_n) satisfies that $\mathcal{T}_\infty = \mathcal{T}^*_\infty$ is trivial.

The main result of this paper (Theorem 3.1) has the following probabilistic formulation.

Theorem 1.1. *Let G be a countable discrete group with finite conjugacy classes, and let $(X_n,\, n \in \mathbb{Z})$ be a two-sided, strictly stationary sequence of G-valued random variables. Then $\mathcal{T}_\infty = \mathcal{T}^*_\infty$.*

If $G = \mathbb{Z}$ and X_0 has finite entropy, Theorem 1.1 essentially reduces to Theorem 1.3 in [1]. If G is an arbitrary discrete group with finite conjugacy classes, X a topologically mixing two-sided shift of finite type, T the shift transformation on X, μ the shift-invariant Gibbs state arising from a function $\phi\colon X \longmapsto \mathbb{R}$ with summable variation, $f\colon X \longmapsto G$ a continuous map, and $X_n = f \cdot T^n$ for every $n \in \mathbb{Z}$, then Theorem 1.1 becomes the statement that $\mathcal{T}_\infty = \mathcal{T}^*_\infty$ is trivial (Theorem 3.3 in [11]). If we assume in addition that $G = \mathbb{Z}$ we are in the setting of [6], and in the special case where X is a full shift and μ a Bernoulli product measure we are back to [7]. For abelian groups G, Theorem 1.1 is contained in a much more general result (Theorem 2.3 in [13]) which does not, however, appear to carry over to the nonabelian case.

Throughout this paper we consider only two-sided processes. For one-sided processes $(X_n,\, n \geq 1)$ we denote by

$$\begin{aligned} \mathcal{T}_{+\infty} &= \bigcap_{M \geq 1} \sigma(X_m : m \geq M), \\ \mathcal{T}^*_{+\infty} &= \bigcap_{M \geq 1} \sigma(X_m \cdots X_1 : m \geq M), \end{aligned} \tag{1.3}$$

the one-sided versions of the tail-sigma-fields (1.1) and (1.2). If the process (X_n) is not independent, the triviality of $\mathcal{T}^*_{+\infty}$ cannot be expected to imply that of $\mathcal{T}_{+\infty}$ without additional assumptions: in [3], [5]–[6] and [11] there are combinatorial obstructions to the triviality of $\mathcal{T}^*_{+\infty}$ even when $\mathcal{T}_{+\infty}$ is trivial, and in [1] a zero-two-law governs the triviality of $\mathcal{T}^*_{+\infty}$ in the case where $G = \mathbb{Z}$, X_0 has finite entropy, and $\mathcal{T}_{+\infty}$ is trivial.

If one drops the assumption that the group G in which the two-sided process (X_n) takes its values has finite conjugacy classes, then the coincidence of $\mathcal{T}_\infty$ and $\mathcal{T}^*_\infty$ is again no longer automatic, and depends on certain recurrence properties of the random variables $X_{n-1} \cdots X_0 g X_0^{-1} \cdots X_{n-1}^{-1}$, $g \in G$, $n \geq 0$, which are not easy to check.

The techniques used for proving Theorem 1.1 come from ergodic theory and are closely connected with those employed in [11] and [13]. The paper is organised as follows. In Section 2 we discuss briefly a few classical facts about ergodic equivalence relations. In Section 3 we state Theorem 1.1 in the language of ergodic equivalence relations (Theorem 3.1), prove it in that setting, and finally deduce Theorem 1.1 from Theorem 3.1. Section 4 contains three applications of the equivalent Theorems 1.1 and 3.1. The first of these (Example 4.1) shows that every countable generator of a measure preserving, ergodic automorphism of a probability space with trivial two-sided tail-sigma-field is *super-K* in the sense of [11] (Example 4.1). For an exact endomorphism one-sided generators are no longer automatically *super-K*, but every measure-preserving, exact endomorphism on a probability space with a finite generator also has a finite *super-K*-generator (cf. [12]). In Example 4.2 we consider what amounts to a finite collection $(X_n^{(i)})$, $i = 1, \dots, k$, of jointly stationary stochastic processes with values in countable discrete groups $G^{(i)}$ with finite conjugacy classes, set $Y_{m,n}^{(i)} = X_n^{(i)} \cdots X_{-m}^{(i)}$ for each $m, n \geq 0$ and $i = 1, \dots, k$, and observe that the sigma-algebras

$$\overline{\mathcal{T}}_\infty = \bigcap_{M \geq 0} \sigma(X_m^{(i)} : |m| \geq M,\ i = 1, \dots, k),$$

$$\overline{\mathcal{T}}^*_\infty = \bigcap_{M \geq 0} \sigma(Y_{m,n}^{(i)} : m, n \geq M,\ i = 1, \dots, k),$$

satisfy that

$$\overline{\mathcal{T}}_\infty = \overline{\mathcal{T}}^*_\infty. \tag{1.4}$$

In Example 4.3 we apply Theorem 3.1 to random deformations of certain infinite polygonal chains.

I am grateful to Frank den Hollander for bringing to my attention the paper [1], and to Sylvia Richardson for pointing out to me a curious application of Theorem 3.1 to molecules like DNA (cf. Example 4.3).

2. Equivalence relations defined by a function

Let $(X, \mathcal{S})$ be a standard Borel space. A subset $\mathbf{R} \subset X \times X$ is a *Borel equivalence relation* on X if $\mathbf{R}$ is a Borel set and an equivalence relation. If $\mathbf{R}$ is a Borel equivalence relation on X we write

$$\mathbf{R}(x) = \{y \in X : (x, y) \in \mathbf{R}\} \tag{2.1}$$

for the equivalence class of a point $x \in X$ and denote by

$$\mathbf{R}(B) = \bigcup_{x \in B} \mathbf{R}(x) \tag{2.2}$$

the *saturation* of a set $B \subset X$. For every $C \in \mathcal{S}$ we denote by

$$\mathbf{R}^{(C)} = \mathbf{R} \cap (C \times C) \tag{2.3}$$

the *restriction* of $\mathbf{R}$ to C. Following [4] we say that a Borel equivalence relation $\mathbf{R}$ on X is *discrete* if $\mathbf{R}(x)$ is countable for every $x \in X$. The following lemma is taken from [4].

Lemma 2.1. *Let $\mathbf{R}$ be a Borel equivalence relation on X. Then the saturation $\mathbf{R}(B)$ of every $B \in \mathcal{S}$ is an analytic set, and hence μ-measurable for every probability measure μ on $\mathcal{S}$. If $\mathbf{R}$ is discrete then $\mathbf{R}(B) \in \mathcal{S}$ for every $B \in \mathcal{S}$.*

Proof. We assume without loss of generality that X is a Polish space and $\mathcal{S} = \mathcal{B}_X$ is the Borel field of X. Denote by $\pi_i \colon X \times X \longmapsto X$, $i = 1, 2$, the two coordinate projections and observe that

$$\mathbf{R}(B) = \pi_2((B \times X) \cap \mathbf{R})$$

is the image of a Borel set in $X \times X$ under the continuous map $\pi_2 \colon X \times X \longmapsto X$, and hence an analytic subset of X. In particular, $\mathbf{R}(B)$ is measurable with respect to every probability measure on X (cf. [8]). If $\mathbf{R}$ is discrete then $\mathbf{R}(B) \in \mathcal{S}$ by Kunugui's theorem (cf. [4], [9]–[10]). $\square$

Lemma 2.1 justifies the following definitions.

Definition 2.2. Let $(X, \mathcal{S})$ be a standard Borel space and μ a probability measure on $\mathcal{S}$ (the resulting triple $(X, \mathcal{S}, \mu)$ is called a *standard probability space*). Suppose that $\mathbf{R}$ is a Borel equivalence relation on X.

(1) The equivalence relation $\mathbf{R}$ is *ergodic* (with respect to μ) if $\mu(\mathbf{R}(B)) \in \{0, 1\}$ for every $B \in \mathcal{S}$;

(2) The measure μ is *quasi-invariant* under $\mathbf{R}$ if $\mu(\mathbf{R}(B)) = 0$ for every $B \in \mathcal{S}$ with $\mu(B) = 0$.

Lemma 2.3. *Suppose that* $\mathbf{R}$ *is a discrete Borel equivalence relation on a standard probability space* $(X, \mathcal{S}, \mu)$. *Then there exists a countable group* $\Gamma = \Gamma_{\mathbf{R}}$ *of nonsingular Borel automorphisms of* $(X, \mathcal{S}, \mu)$ *with the following properties.*

(1) *There exists a set* $B \in \mathcal{S}$ *with* $\mu(B) = 1$ *such that*

$$\mathbf{R}_\Gamma = \{(x, \gamma x) : x \in X,\ \gamma \in \Gamma\} = \mathbf{R}^{(B)} \cup \{(x, x) : x \in X\} \subset \mathbf{R};$$

(2) *If*

$$\mathcal{S}^{\mathbf{R}} = \{\mathbf{R}(B) : B \in \mathcal{S}\} \subset \mathcal{S},$$

$$\mathcal{S}^{\mathbf{R}_\Gamma} = \{\mathbf{R}_\Gamma(B) : B \in \mathcal{S}\} = \{B \in \mathcal{S} : \gamma B = B \text{ for every } \gamma \in \Gamma\},$$

then

$$\mathcal{S}^{\mathbf{R}} = \mathcal{S}^{\mathbf{R}_\Gamma} \pmod{\mu}.$$

If T *is a measure preserving Borel automorphism of* $(X, \mathcal{S}, \mu)$ *which* preserves $\mathbf{R}$ *in the sense that*

$$(T \times T)(\mathbf{R}) = \mathbf{R}, \tag{2.4}$$

then the set $B \in \mathcal{S}$ *in* (1) *may be chosen to be* T*-invariant, and the group* Γ *to satisfy that* $T\Gamma T^{-1} = \Gamma$.

Proof. Since $\mathbf{R}$ is discrete we can apply Theorem 1 in [4] to find a countable group Δ of Borel automorphisms of X with

$$\mathbf{R} = \mathbf{R}_\Delta = \{(x, Sx) : x \in X,\ S \in \Delta\}.$$

Choose an enumeration $(S_1, S_2, \dots)$ of Δ and define a finite measure ν on $\mathcal{S}$ by

$$\nu = \sum_{n \geq 1} 2^{-n} \mu S_n.$$

Then ν is quasi-invariant under Δ (or, equivalently, under $\mathbf{R}$) and μ is absolutely continuous with respect to ν. We write ν as a sum of two finite measures $\nu = \nu_1 + \nu_2$ with $\nu_1 \sim \mu$ and $\nu_2 \perp \mu$, choose a set $B \in \mathcal{S}$ with $\nu_2(B) = \nu_1(X \smallsetminus B) = 0$, and set

$$\mathbf{R}' = \mathbf{R}^{(B)} \cup \{(x, x) : x \in X\}.$$

The quasi-invariance of ν under $\mathbf{R}$ implies that the restriction ν_1 of ν to B is quasi-invariant under $\mathbf{R}^{(B)}$. As ν_1 is equivalent to μ, μ is quasi-invariant under $\mathbf{R}^{(B)}$ and hence under $\mathbf{R}'$, and $\mathcal{S}^{\mathbf{R}'} = \mathcal{S}^{\mathbf{R}} \pmod{\mu}$. The proof is completed by applying Theorem 1 in [4] once again to choose a countable group Γ of nonsingular Borel automorphisms of $(X, \mathcal{S}, \mu)$ with $\mathbf{R}' = \mathbf{R}_\Gamma$ (cf. (1)).

If $\mathbf{R}$ is T-invariant in the sense of (2.4) we replace B by

$$B' = \bigcap_{n\in\mathbb{Z}} T^{-n}B \in \mathcal{S}^T = \{C \in \mathcal{S} : T^{-1}C = C\},$$

define $\mathbf{R}'$ as above with B' replacing B, and observe that $\mathbf{R}'$ is T-invariant. Finally we define Γ as above, denote by Γ' the group generated by $\{T^n\gamma T^{-n} : \gamma \in \Gamma,\, n \in \mathbb{Z}\}$, and obtain that $T\Gamma'T^{-1} = \Gamma'$ and $\mathbf{R}_{\Gamma'} = \mathbf{R}'$. □

Let T be a measure preserving, ergodic automorphism of our standard probability space $(X, \mathcal{S}, \mu)$, G a countable discrete group with identity element 1_G, and $f\colon X \longmapsto G$ a Borel map. The map f determines two equivalence relations $\mathbf{R}_f^* \subset \mathbf{R}_f$ on X. The first of these is given by

$$\begin{aligned}\mathbf{R}_f = \{(x,y) \in X \times X : f(T^n x) \neq f(T^n y) \\ \text{for only finitely many } n \in \mathbb{Z}\}.\end{aligned} \tag{2.5}$$

For the second relation we set, for every $(x,y) \in \mathbf{R}_f$ and $L \geq 1$,

$$\begin{gathered} a_f^+(x,y)^{(L)} = f(x)^{-1}\cdots f(T^{L-1}x)^{-1}\cdot f(T^{L-1}y)\cdots f(y), \\ a_f^-(x,y)^{(L)} = f(T^{-1}x)\cdots f(T^{-L}x)\cdot f(T^{-L}y)^{-1}\cdots f(T^{-1}y)^{-1}, \\ a_f^+(x,y) = \lim_{L\to\infty} a_f^+(x,y)^{(L)}, \\ a_f^-(x,y) = \lim_{L\to\infty} a_f^-(x,y)^{(L)}, \end{gathered} \tag{2.6}$$

and observe that $a_f^\pm\colon \mathbf{R}_f \longmapsto G$ are well defined Borel maps, and that

$$a_f^+(x,y)a_f^+(y,z) = a_f^+(x,z), \qquad a_f^-(x,y)a_f^-(y,z) = a_f^-(x,z) \tag{2.7}$$

for all $(x,y), (x,z) \in \mathbf{R}_f$. In particular,

$$\mathbf{R}_f^* = \{(x,y) \in \mathbf{R}_f \colon a_f^+(x,y) = a_f^-(x,y)\} \tag{2.8}$$

is an equivalence relation on X which is contained in (and hence a subrelation of) $\mathbf{R}_f$. We note in passing that the equation (2.7) is usually expressed by saying that a_f^+ and a_f^- are (1-)*cocycles* on the equivalence relation $\mathbf{R}_f$ with values in G (cf. [4]).

For the following elementary observations we fix a Borel map $f\colon X \longmapsto G$ and say that f *separates the points of* X (or *generates* $\mathcal{S}$) *under* T if, for every pair of points $x, y \in X$, $x = y$ if and only if $f(T^n x) = f(T^n y)$ for every $n \in \mathbb{Z}$.

Proposition 2.4. *The equivalence relations $\mathbf{R}_f$ and $\mathbf{R}_f^*$ are Borel and T-invariant in the sense of* (2.4). *If f generates $\mathcal{S}$ under T the relations $\mathbf{R}_f$ and $\mathbf{R}_f^*$ are discrete.*

Proof. For every $N \geq 0$ the set

$$\mathbf{R}_f^{(N)} = \{(x,y) \in X \times X : f(T^k x) = f(T^k y) \text{ whenever } k \in \mathbb{Z},\, |k| \geq N\}$$

is a Borel equivalence relation, and $\mathbf{R}_f = \bigcup_{N\ge 0} \mathbf{R}_f^{(N)}$ is Borel. Since $a_f^{\pm}\colon \mathbf{R}_f \longmapsto G$ are Borel maps on $\mathbf{R}_f$, (2.8) shows that $\mathbf{R}_f^* \subset X \times X$ is also a Borel set.

The T-invariance of $\mathbf{R}_f$ is obvious. In order to prove the T-invariance of $\mathbf{R}_f^*$ we note that, for every $(x,y) \in \mathbf{R}_f$,

$$\begin{aligned} a_f^+(Tx,Ty) &= f(Tx)^{-1}\cdots f(T^{m-1}x)^{-1}\cdot f(T^{m-1}y)\cdots f(Ty) \\ &= f(x)a_f^+(x,y)f(y)^{-1}, \\ a_f^-(Tx,Ty) &= f(x)\cdots f(T^{-m+1}x))\cdot f(T^{-m+1}y)^{-1}\cdots f(y)^{-1} \\ &= f(x)a_f^-(x,y)f(y)^{-1}, \end{aligned} \tag{2.9}$$

for every $(x,y) \in \mathbf{R}_f$ and every sufficiently large $m \ge 0$. Hence $(x,y) \in \mathbf{R}_f^*$ if and only if $(Tx,Ty) \in \mathbf{R}_f^*$, which proves the T-invariance of $\mathbf{R}_f^*$.

If f generates $\mathcal{S}$ under T, the equivalence class

$$\mathbf{R}_f^{(N)}(x) = \{y \in X : (x,y) \in \mathbf{R}_f^{(N)}\}$$

is countable for every $x \in X$ and $N \ge 0$, so that $\mathbf{R}_f(x) = \bigcup_{N\ge 0} \mathbf{R}_f^{(N)}(x)$ is countable for every $x \in X$. □

3. The main theorem and its proof

A countable group G has *finite conjugacy classes* if the conjugacy class $[g] = \{hgh^{-1} : h \in G\}$ of every $g \in G$ is finite.

Theorem 3.1. *Let T be a measure preserving and ergodic automorphism of a standard probability space $(X, \mathcal{S}, \mu)$, G a countable group with finite conjugacy classes, and $f\colon X \longmapsto G$ a Borel map. Then the equivalence relations $\mathbf{R}_f$ and $\mathbf{R}_f^*$ in* (2.5) *and* (2.8) *satisfy that $\mathcal{S}^{\mathbf{R}_f} = \mathcal{S}^{\mathbf{R}_f^*} \pmod{\mu}$.*

The remainder of this section is devoted to the proof of Theorem 3.1. Until further notice we assume that T is an ergodic automorphism of a standard probability space $(X, \mathcal{S}, \mu)$, G a countable discrete group and $f\colon X \longmapsto G$ a Borel map. Put, for every $n \in \mathbb{Z}$,

$$f(n,x) = \begin{cases} f(T^{n-1}x)\cdots f(x) & \text{if } n \ge 1, \\ 1_G & \text{if } n = 0, \\ f(T^{-n}x)^{-1}\cdots f(T^{-1}x)^{-1} & \text{if } n < 0. \end{cases} \tag{3.1}$$

The resulting map $f\colon \mathbb{Z} \times X \longmapsto G$ satisfies that

$$f(m, T^n x)\cdot f(n,x) = f(m+n,x) \tag{3.2}$$

for every $n \in \mathbb{Z}$ and $x \in X$, and is called a (1-)*cocycle* of T with values in G.

Lemma 3.2. *Suppose that G is finite with cardinality $|G|$. If $\mathbf{R}_T^*$ is the nonsingular Borel equivalence relation on $(X, \mathcal{S}, \mu)$ defined by*

$$\mathbf{R}_T^* = \{(x, T^n x) : x \in X \text{ and } f(n,x) = 1_G\}$$

then there exists a partition $\mathcal{Q} \subset \mathcal{S}$ into at most $|G|$ sets with $\mathcal{A}(\mathcal{Q}) = \mathcal{S}^{\mathbf{R}_T^*}$ (mod μ)*, where $\mathcal{A}(\mathcal{Q})$ is the algebra generated by $\mathcal{Q}$.*

Proof. Put $Y = X \times G$, write $\mathcal{T}$ for the product Borel field of Y, and set $\nu = \mu \times \lambda$, where λ is the normalised Haar (= counting) measure on G. We denote by $T_f \colon X \times G \longmapsto X \times G$ the skew-product transformation

$$T_f(x,g) = (Tx, f(x)g),$$

observe that T_f preserves ν, and set

$$\mathcal{T}^{T_f} = \{C \in \mathcal{T} : T_f C = C\}.$$

For every $B \in \mathcal{S}^{\mathbf{R}^*_T}$, $C \in \mathcal{T}^{T_f}$ and $h \in G$ we put

$$\begin{aligned} \bar{B}(h) &= \bigcup_{n\in\mathbb{Z}} T_f^n(B \times \{h\}) \in \mathcal{T}^{T_f} \\ C_h &= \{x \in X : (x,h) \in C\} \in \mathcal{S}^{\mathbf{R}^*_T}, \end{aligned} \tag{3.3}$$

and note that

$$\mathcal{S}^{\mathbf{R}^*_T} = \{C_h : C \in \mathcal{T}^{T_f}\} \tag{3.4}$$

for every $h \in G$: indeed, if $C \in \mathcal{T}^{T_f}$ and $h \in G$, then $C_h \in \mathcal{S}^{\mathbf{R}^*_T}$; conversely, if $B \in \mathcal{S}^{\mathbf{R}^*_T}$ and $h \in G$, then

$$B = \bar{B}(h)_h. \tag{3.5}$$

As T is ergodic and T_f commutes with the measure preserving action $R \colon h \mapsto R_h$ of G on Y defined by

$$R_h(x,g) = (x, gh)$$

for $x \in X$ and $g, h \in G$, the joint action of T_f and R is ergodic (i.e. $\nu(B) \in \{0,1\}$ for every $B \in \mathcal{T}$ which is invariant both under T_f and R). Hence

$$\mu\Bigl(\bigcup_{g\in G} \bar{B}(h)_g\Bigr) = \mu\Bigl(\bigcup_{n\in\mathbb{Z}} T^n B\Bigr) = 1 \tag{3.6}$$

for every $h \in G$ and $B \in \mathcal{S}$ with $\mu(B) > 0$.

Suppose that there exists a partition $\mathcal{Q} = \{Q(1), \dots, Q(|G|+1)\} \subset \mathcal{S}^{\mathbf{R}^*_T}$ of X into sets of positive μ-measure. We set $g_1 = 1_G$ and note that

$$\overline{Q(1)}(g_1)_{g_1} = Q(1),$$

by (3.5). According to (3.6) there exists an element $g_2 \in G \smallsetminus \{g_1\}$ with $\mu(\overline{Q(1)}(g_1)_{g_2} \cap Q(2)) > 0$. We replace $Q(2)$ by the possibly smaller set

$$Q'(2) = \overline{Q(1)}(g_1)_{g_2} \cap Q(2) \in \mathcal{S}^{\mathbf{R}^*_T},$$

choose $g_3 \in G \smallsetminus \{g_1, g_2\}$ with $\mu(\overline{Q'(2)}(g_2)_{g_3} \cap Q(3)) > 0$, and set

$$Q'(3) = \overline{Q(2)}(g_2)_{g_3} \cap Q(3) \in \mathcal{S}^{\mathbf{R}^*_T}.$$

Proceeding by induction, we obtain sets $Q'(1) = Q(1), Q'(2) \subset Q(2), \dots, Q'(|G|) \subset Q(|G|))$ of positive μ-measure in $\mathcal{S}^{\mathbf{R}^*_T}$ and an enumeration $g_1 = 1_G, g_2, \dots, g_{|G|}$ of G with the following properties:

(i) $\overline{Q'(j+1)}(g_{j+1}) \subset \overline{Q'(j)}(g_j)$ for $j = 1, \dots, |G|-1$,
(ii) $\overline{Q'(j)}(g_j)_{g_j} = Q'(j) \subset Q(j)$ for $j = 1, \dots, |G|$.

By setting $j = |G|$ we obtain that $\mu(Q'(|G|)) > 0$ and $\overline{Q'(|G|)}(g_{|G|})_{g_j} \subset Q(j)$ for $j = 1, \dots, |G|$. As $Q(j) \cap Q(|G|+1) = \varnothing$ for $j = 1, \dots, |G|$ this contradicts (3.6).

This shows that $\mathcal{S}^{\mathbf{R}_T^*}$ is purely atomic with at most $|G|$ atoms and proves the lemma. □

Lemma 3.3. *Suppose that G is finite and that $\mathcal{Q} \subset \mathcal{S}$ is the finite partition described in Lemma* 3.2. *For every $A \in \mathcal{Q}$ and $B, C \in \mathcal{S}$ with $B \cup C \subset A$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(B \cap T^{-n}C \cap \{x \in X : f(n,x) = 1_G\}) = \frac{\mu(B)\mu(C)}{|G|}.$$

Proof. We use the notation of the proof of Lemma 3.2 and assume that $\mu(A) > 0$. The set $\bar{A}(1_G)$ is T_f-invariant, $\nu(\bar{A}(1_G)) > 0$, and hence

$$\nu(\bigcup_{h\in G} R_h \bar{A}(1_G)) = 1$$

and

$$\nu(\bar{A}(1_G)) \geq 1/|G|.$$

Furthermore, since A is an atom of $\mathcal{S}^{\mathbf{R}_T^*}$, the restriction of ν to $\bar{A}(1_G)$ is ergodic under T_f, and the ergodic theorem guarantees that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \nu(B' \cap T_f^{-n} C') = \nu(B')\nu(C'),$$

where $B' = B \times \{1_G\}$ and $C' = C \times \{1_G\}$. This is precisely our assertion. □

For the definition of the equivalence relations $\mathbf{R}_f^* \subset \mathbf{R}_f \subset X \times X$ and the maps $a_f^{\pm} \colon \mathbf{R}_f \longmapsto G$, $f \colon \mathbb{Z} \times X \longmapsto G$ we refer to (2.5), (2.8), (2.6) and (3.1). The following Definition 3.4 (3) is taken from [13].

Definition 3.4. Let $\mathbf{R}$ be a discrete nonsingular Borel equivalence relation on a standard probability space $(X, \mathcal{S}, \mu)$.

(1) The *full group* $[\mathbf{R}]$ of $\mathbf{R}$ is the set of all nonsingular Borel automorphisms V of $(X, \mathcal{S}, \mu)$ with $(x, Vx) \in \mathbf{R}$ for every $x \in X$.

(2) A nonsingular automorphism V of $(X, \mathcal{S}, \mu)$ is an *automorphism* of $\mathbf{R}$ if there exists a μ-null set $N \in \mathcal{S}$ with $\mathbf{R}(N) = N$ and $(Vx, Vy) \in \mathbf{R}$ for every $(x, y) \in \mathbf{R}^{(X\smallsetminus N)}$.

(3) A measure preserving automorphism V of $\mathbf{R}$ is *asymptotically central* if

$$\lim_{n\to\infty} \mu(V^{-n}B \triangle WV^{-n}B) = 0 \tag{3.7}$$

for every $W \in [\mathbf{R}]$ and $B \in \mathcal{S}$.

Remarks 3.5. (1) A nonsingular automorphism V of $(X, \mathcal{S}, \mu)$ is an automorphism of $\mathbf{R}$ if and only if there exists, for every $W \in [\mathbf{R}]$, an element $W' \in [\mathbf{R}]$ with $V^{-1}WV = W' \pmod{\mu}$. The last condition is usually expressed by saying that V *normalises* $[\mathbf{R}]$ (modulo null sets).

(2) If V is an asymptotically central automorphism of a discrete, nonsingular equivalence relation $\mathbf{R}$ on $(X, \mathcal{S}, \mu)$ and $\mathbf{R}' \subset \mathbf{R}$ a V-invariant subrelation, then V is an asymptotically central automorphism of $\mathbf{R}'$.

(3) If V is an asymptotically central automorphism of a discrete, nonsingular, ergodic equivalence relation $\mathbf{R}$ on $(X, \mathcal{S}, \mu)$ then V is mixing (cf. [13]).

Lemma 3.6. *Suppose that f generates $\mathcal{S}$ under T and that $\mathbf{R} = \mathbf{R}_f$ is the equivalence relation* (2.5). *We choose a countable group $\Gamma = T\Gamma T^{-1}$ according to Lemma* 2.3 *and set $\mathbf{R}_\Gamma = \{(x, \gamma x) : x \in X, \gamma \in \Gamma\} \subset \mathbf{R}$. Then T and T^{-1} are asymptotically central automorphisms of $\mathbf{R}_\Gamma$.*

Proof. By assumption, the partition $\mathcal{P} = \{f^{-1}(\{g\}) : g \in G\}$ is a generator for T, i.e. $\mathcal{S} = \bigvee_{n\in\mathbb{Z}} T^{-n}(\mathcal{P})$.

Fix $W \in [\mathbf{R}_\Gamma]$ and set, for every $n \geq 1$,

$$C_n = \{x \in X : f(T^k Wx) = f(T^k x) \text{ for every } k \in \mathbb{Z}, |k| \geq n\}.$$

Since the set

$$\{n \in \mathbb{Z} : f(T^n Wx) \neq f(T^n x)\}$$

is finite for μ-a.e. $x \in X$, $\lim_{n\to\infty} \mu(C_n) = 1$. Furthermore, if

$$D \in \bigvee_{|k|\leq m} T^{-k}(\mathcal{P})$$

for some $m \geq 0$, then

$$\begin{aligned} &\mu(T^{-m-n}D \triangle WT^{-m-n}D) \\ &\qquad \leq \mu(T^{-m-n}(C_n \cap D) \triangle WT^{-m-n}(C_n \cap D)) \\ &\qquad\qquad + \mu(X \smallsetminus C_n) + \mu(X \smallsetminus WC_n) \\ &\qquad \leq 2\mu(X \smallsetminus C_n) + \mu(X \smallsetminus WC_n) + \mu(X \smallsetminus W^{-1}C_n) \to 0 \end{aligned} \tag{3.8}$$

as $n \to \infty$. In the second inequality in (3.8) we are using the fact that any point $x \in C_n \cap T^{-m-n}(C_n \cap D)$ satisfies that $Wx \in D$.

Similarly we obtain that $\mu(T^{m+n}D \triangle WT^{m+n}D) \to 0$ as $n \to \infty$.

For every $B \in \mathcal{S}$ with $\mu(B) > 0$ we can find a sequence $(B_m, m \geq 1)$ in $\mathcal{S}$ with $\lim_{m\to\infty} \mu(B \triangle B_m) = 0$ and $B_m \in \bigvee_{|k|\leq m} T^k(\mathcal{P})$ for every $m \geq 1$. Then

$$\begin{aligned} &\limsup_{|n|\to\infty} \big(\mu(T^n B \triangle WT^n B) \\ &\qquad \leq \limsup_{|n|\to\infty} \mu(T^n B_m \triangle WT^n B_m) + \mu(B \triangle B_m) + \mu(WT^n(B \triangle B_m))\big) \\ &\qquad = \mu(B \triangle B_m) + \limsup_{|n|\to\infty} \mu(WT^n(B \triangle B_m)), \end{aligned}$$

and by letting $m \to \infty$ we obtain that $\lim_{|n|\to\infty} \mu(T^n B \triangle WT^n B) = 0$. As B and W were arbitrary this proves that T and T^{-1} are asymptotically central. □

Since $\mathbf{R}_\Gamma \subset \mathbf{R}_f$ we can restrict the cocycles $a_f^\pm : \mathbf{R}_f \longmapsto G$ in (2.6) to $\mathbf{R}_\Gamma$, and we set

$$c_f(x, y) = a_f^-(x, y) a_f^+(y, x) \tag{3.9}$$

for every $(x, y) \in \mathbf{R}_\Gamma$. According to (2.9),

$$c_f(T^n x, T^n y) = f(n, x) c_f(x, y) f(n, x)^{-1} \tag{3.10}$$

for every $(x, y) \in \mathbf{R}_\Gamma$ and $\mathbf{n} \in \mathbb{Z}$.

Lemma 3.7. *Suppose that f generates $\mathcal{S}$ under T, that $\mathbf{R} = \mathbf{R}_f$ is the equivalence relation (2.5), and that $\Gamma = T\Gamma T^{-1}$ is the group of nonsingular automorphisms of $(X, \mathcal{S}, \mu)$ constructed in Lemma 2.3. Assume furthermore that $g \in G$ is an element with finite conjugacy class $[g] = \{hgh^{-1} : h \in G\}$, and that*

$$\mu\biggl(\bigcup_{\gamma\in\Gamma}\{x \in X : c_f(x,\gamma x) \in [g]\}\biggr) > 0, \tag{3.11}$$

where $c_f \colon \mathbf{R}_\Gamma \longmapsto G$ is defined in (3.9). We write $C(g) = \{h \in G : hgh^{-1} = g\}$ for the centraliser of g and put

$$H = \bigcap_{h\in G} hC(g)h^{-1}. \tag{3.12}$$

Then $H \subset G$ is a normal subgroup of finite index, and we set

$$f'(x) = f(x)H \in G' = G/H$$

for every $x \in X$. The map $f' \colon X \longmapsto G'$ is Borel and has values in the finite group G', and we denote by $\mathcal{Q} \subset \mathcal{S}$ the finite partition arising from f' in Lemma 3.2. Then the following conditions are satisfied for every $A \in \mathcal{Q}$ with $\mu(A) > 0$.

(1) *There exists an element $g' \in [g]$ with*

$$\mu\biggl(\bigcup_{\gamma\in\Gamma} B \cap \gamma^{-1}B \cap \{x \in X : c_f(x,\gamma x) = a_f^+(\gamma x, x) = g'\}\biggr) > 0 \tag{3.13}$$

for every $B \in \mathcal{S}$ with $B \subset A$ and $\mu(B) > 0$;

(2) *The set*

$$G(A) = \{g' \in G : g' \text{ satisfies (3.13) for every measurable set } B \subset A \text{ with } \mu(B) > 0\}$$

is a subgroup of G;

(3) *If*

$$\mu\biggl(\bigcup_{\gamma\in\Gamma}\{x \in A \cap \gamma^{-1}A : c_f(x,\gamma x) = g\}\biggr) > 0, \tag{3.14}$$

then $g \in G(A)$.

Proof. For the proof of (2) we note that, if $g', g'' \in G(A)$, then we can find, for every $B \subset A$ with positive measure, elements $\gamma', \gamma'' \in \Gamma$ and Borel sets B', B'' with $B' \cup \gamma' B' \subset B$, $B'' \cup \gamma'' B'' \subset \gamma' B'$, $\mu(B'') > 0$, $c_f(x, \gamma' x) = a_f^+(\gamma' x, x) = g'$ and $c_f(y, \gamma'' y) = a_f^+(\gamma'' y, y) = g''$ for every $x \in B'$ and $y \in B''$. Then

$$\begin{aligned} c_f(x, \gamma''\gamma' x) &= a_f^-(x, \gamma' x) c_f(\gamma' x, \gamma''\gamma' x) a_f^+(\gamma' x, x) \\ &= a_f^+(\gamma''\gamma' x, \gamma' x) a_f^+(\gamma' x, x) = g'' g' \end{aligned}$$

for every $x \in \gamma'^{-1} B'' \subset B'$. Since $G(A)$ is obviously closed under taking inverses this shows that $G(A)$ is a group.

In order to prove (1) we suppose that $g \in G$ satisfies (3.11), and that $A \in \mathcal{Q}$ has positive measure. Choose a set $C \in \mathcal{S}$ and a $\gamma \in \Gamma$ with $\mu(C) > 0$ and $c_f(x, \gamma x) = g' \in [g]$ for every $x \in C$. By decreasing C, if necessary, we may also assume that there exists an integer $L \geq 0$ with $f(T^l\gamma x) = f(T^l x)$ for every $x \in C$ and $|l| \geq L$.

Since T is ergodic and an asymptotically central automorphism of $\mathbf{R}_\Gamma$ by Lemma 3.6,

$$\lim_{|n|\to\infty} \mu(A \triangle T^n\gamma^{-1}T^{-n}A) = 0, \tag{3.15}$$

and hence

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \mu(A \cap T^n\gamma^{-1}T^{-n}A \cap T^{-n}C) &= \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \mu(A \cap T^{-n}C) \\ &= \mu(A)\mu(C). \end{aligned}$$

From (3.10) we also know that

$$c_f(x, T^{-n}\gamma T^n x) = f(n,x)^{-1} c_f(T^n x, \gamma T^n x) f(n,x) \in [g]$$

for every $x \in A \cap T^n\gamma^{-1}T^{-n}A \cap T^{-n}C$.

Since $|G/C(g)| = |[g]| < \infty$ and every subgroup of finite index contains a normal subgroup of finite index, the normal subgroup H in (3.12) has finite index in G.

Suppose that $B \in \mathcal{S}$, $B \subset A$ and $\mu(B) > 0$. According to Lemma 3.3,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \mu(B \cap T^{-n}C \cap \{x \in X : f(n,x) \in H\}) = \frac{\mu(B)\mu(C)}{|G/H|},$$

and by combining this with (3.15) (with A replaced by B) we see that

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \mu(B \cap T^{-n}\gamma^{-1}T^n B \cap T^{-n}C \cap \{x \in X : f(n,x) \in H\}) \\ = \frac{\mu(B)\mu(C)}{|G/H|}. \end{aligned}$$

It follows that

$$\mu(B \cap T^{-n}\gamma^{-1}T^n B \cap T^{-n}C \cap \{x \in X : f(n,x) \in H\}) > \frac{\mu(B)\mu(C)}{2|G/H|}$$

for infinitely many $n \geq N$, and (3.10) yields that

$$\begin{aligned} g' = c_f(T^n x, \gamma T^n x) &= f(n,x) c_f(x, T^{-n}\gamma T^n x) f(n,x)^{-1} \\ &= c_f(x, T^{-n}\gamma T^n x) = a_f^+(T^{-n}\gamma T^n x, x) \end{aligned}$$

for every $n > N$ and

$$x \in B \cap T^{-n}\gamma^{-1}T^n B \cap T^{-n}C \cap \{x \in X : f(n,x) \in H\},$$

which proves (1). If $g = g'$ satisfies (3.14) then this proof also shows that $g \in G(A)$, which proves (3). □

Proof of Theorem 3.1. Suppose that G has finite conjugacy classes, that $f\colon X \longmapsto G$ is a Borel map, and that $\mathbf{R}_f^* \subset \mathbf{R}_f$ are the equivalence relations defined in (2.5). We use Lemma 2.3 with $\mathbf{R} = \mathbf{R}_f$ to find a countable group $\Gamma = T\Gamma T^{-1}$ of nonsingular Borel automorphisms of $(X, \mathcal{S}, \mu)$ with $\mathcal{S}^{\mathbf{R}_f} = \mathcal{S}^{\mathbf{R}_\Gamma} \pmod{\mu}$.

Let $g \in G$ satisfy (3.11), and let $C(g)$ be the centraliser of g. As in Lemma 3.7 we consider the normal subgroup $H \subset G$ in (3.12), set $G' = G/H$ and

$f' = fH\colon X \longmapsto G'$, and denote by $\mathcal{Q} \subset \mathcal{S}$ the finite partition arising from f' in Lemma 3.2. If $A \in \mathcal{Q}$ has positive measure, then Lemma 3.7 implies that the set

$$G(A) = \{g' \in G : g' \text{ satisfies (3.13) for every measurable set } B \subset A \text{ with } \mu(B) > 0\}$$

is a subgroup of G. We put

$$\mathbf{R}_\Gamma^* = \{(x, \gamma x) : x \in X,\, \gamma \in \Gamma,\, c_f(x, \gamma x) = 1_G\} = \mathbf{R}_f^* \cap \mathbf{R}_\Gamma$$

and denote by $\mathbf{S}^* = (\mathbf{R}_\Gamma^*)^{(A)}$ and $\mathbf{S} = \mathbf{R}_\Gamma^{(A)}$ the restrictions of $\mathbf{R}_\Gamma^*$ and $\mathbf{R}_\Gamma$ to A. If $\mathcal{S}_A = \{B \cap A : B \in \mathcal{S}\}$ and $\mu_A(B) = \mu(B)/\mu(A)$ for every $B \in \mathcal{S}_A$ then we claim that

$$\mathcal{S}_A^{\mathbf{S}^*} = \mathcal{S}_A^{\mathbf{S}} \pmod{\mu_A}. \tag{3.16}$$

Indeed, if $B, C \in \mathcal{S}_A$ have positive measure, and if there exists a $\gamma \in \Gamma$ with $\mu(\gamma B \cap C) > 0$, then we can find a $g \in G$ and a subset $B' \subset B$ with $\mu(B') > 0$, $\gamma B' \subset C$ and $c_f(x, \gamma x) = g$ for every $x \in B'$. According to Lemma 3.7 (3) there exists a subset $B'' \subset B'$ and an element $\gamma' \in \Gamma$ with $\gamma' B'' \subset B'$ and $c_f(x, \gamma' x) = a_f^+(\gamma x, x) = g^{-1}$, and we conclude that

$$\begin{aligned} c_f(x, \gamma\gamma' x) &= a_f^-(x, \gamma' x) c_f(\gamma' x, \gamma' \gamma x) a_f^+(\gamma' x, x) \\ &= c_f(\gamma' x, \gamma\gamma' x) a_f^+(\gamma' x, x) = 1_G. \end{aligned}$$

Since $B'' \subset B$, $\gamma\gamma' B'' \subset C$ and $c_f(x, \gamma\gamma' x) = 1_G$ it follows that $\mathbf{S}^*(B) = \mathbf{S}(B) \pmod{\mu}$ for every $B \in \mathcal{S}$ with $B \subset A$, which implies (3.16). By varying $A \in \mathcal{Q}$ in (3.16) we see that

$$\mathcal{S}^{\mathbf{R}_\Gamma^*} \subset \mathcal{S}^{\mathbf{R}_\Gamma} \vee \mathcal{Q} \pmod{\mu}. \tag{3.17}$$

Standard decomposition theory allows us to find a $\mathcal{S}^{\mathbf{R}_\Gamma}$-measurable map $x \mapsto \nu_x$ from X into the set of probability measures on $\mathcal{S}$ with

$$\mu(B \cap C) = \int_C \nu_x(B)\, d\mu(x)$$

for every $B \in \mathcal{S}$ and $C \in \mathcal{S}^{\mathbf{R}_\Gamma}$, and the T-invariance of $\mathcal{S}^{\mathbf{R}_\Gamma}$ and μ enables us to assume in addition that

$$\nu_{Tx} = \nu_x T^{-1} \tag{3.18}$$

for every $x \in X$.

For every finite partition $\mathcal{R} \subset \mathcal{S}^{\mathbf{R}_\Gamma^*}$ denote by $\mathcal{A}(\mathcal{R})$ the algebra of sets generated by $\mathcal{R}$.

Let $\mathcal{D}(\mathcal{Q})$ be the collection of all partitions $\mathcal{P} \subset \mathcal{A}(\mathcal{Q})$. According to (3.17) we can find, for μ-a.e. $x \in X$, a partition $\mathcal{P} \in \mathcal{D}(\mathcal{Q})$ with

$$\mathcal{A}(\mathcal{P}) = \mathcal{S}^{\mathbf{R}_\Gamma^*} \pmod{\nu_x}. \tag{3.19}$$

For every fixed $\mathcal{P} \in \mathcal{D}(\mathcal{Q})$, the set $E(\mathcal{P}) = \{x \in X : x \text{ satisfies (3.19)}\}$ lies in $\mathcal{S}^{\mathbf{R}_\Gamma}$. We select a subset $\mathcal{D}' \subset \mathcal{D}(\mathcal{Q})$ with $\mu\big(\bigcup_{\mathcal{P} \in \mathcal{D}'} E(\mathcal{P})\big) = 1$ and $E(\mathcal{P}) \cap E(\mathcal{P}') = \varnothing$ whenever $\mathcal{P}, \mathcal{P}' \in \mathcal{D}'$ and $\mathcal{P} \neq \mathcal{P}'$, put

$$\mathcal{Q}' = \{E(\mathcal{P}) \cap B : B \in \mathcal{P} \in \mathcal{D}'\} \subset \mathcal{S}^{\mathbf{R}_\Gamma^*},$$

and observe that $\mu(\bigcup_{A\in\mathcal{Q}'} A) = 1$. After modifying the elements of $\mathcal{Q}'$ by null sets, if necessary, we may assume that $\mathcal{Q}' \subset \mathcal{S}^{\mathbf{R}_\Gamma^*}$ is a partition of X, that $\mu(A) > 0$ for every $A \in \mathcal{Q}'$, and that

$$\begin{aligned} \mathcal{S}^{\mathbf{R}_\Gamma} \vee \mathcal{Q}' &= \mathcal{S}^{\mathbf{R}_\Gamma^*} \pmod{\mu}, \\ \mathcal{A}(\mathcal{Q}') &= \mathcal{S}^{\mathbf{R}_\Gamma^*} \pmod{\nu_x} \text{ for } \mu\text{-a.e. } x \in X. \end{aligned} \tag{3.20}$$

The T-invariance of $\mathcal{S}^{\mathbf{R}_\Gamma^*}$, $\mathcal{S}^{\mathbf{R}_\Gamma}$ and (3.18) together imply that $T^{-n}(\mathcal{Q}')$ also satisfies (3.20) for every $n \in \mathbb{Z}$; in particular there exists, for every $n \in \mathbb{Z}$ and μ-a.e. $x \in X$, a bijection $\xi_{n,x} \colon \mathcal{Q}' \longmapsto \mathcal{Q}'$ with

$$\nu_x(A \triangle T^{-n}\xi_{n,x}(A)) = 0$$

for every $A \in \mathcal{Q}'$ (we are not assuming the maps $x \mapsto \xi_{n,x}$ to be measurable, although this can be achieved as well).

Since T is an asymptotically central automorphism of $\mathbf{R}_\Gamma$,

$$\begin{aligned} 0 = \lim_{n\to\infty} \sum_{A\in\mathcal{Q}'} \mu(T^{-n}A \triangle \gamma T^{-n}A) &= \lim_{n\to\infty} \sum_{A\in\mathcal{Q}'} \int \nu_x(T^{-n}A \triangle \gamma T^{-n}A)\, d\mu(x) \\ &= \lim_{n\to\infty} \int \sum_{A\in\mathcal{Q}'} \nu_x(T^{-n}A \triangle \gamma T^{-n}A)\, d\mu(x) \\ &= \lim_{n\to\infty} \int \sum_{A\in\mathcal{Q}'} \nu_x(T^{-n}\xi_{n,x}(A) \triangle \gamma T^{-n}\xi_{n,x}(A))\, d\mu(x) \\ &= \int \sum_{A\in\mathcal{Q}'} \nu_x(A \triangle \gamma A)\, d\mu(x) = \sum_{A\in\mathcal{Q}'} \int \nu_x(A \triangle \gamma A)\, d\mu(x) = \sum_{A\in\mathcal{Q}'} \mu(A \triangle \gamma A) \end{aligned}$$

for every $\gamma \in \Gamma$, and (3.20) implies that $\mathcal{S}^{\mathbf{R}_\Gamma^*} \subset \mathcal{S}^{\mathbf{R}_\Gamma} \pmod{\mu}$. It follows that $S^{\mathbf{R}_f} = \mathcal{S}^{\mathbf{R}_f^*} \pmod{\mu}$, which completes the proof of the theorem under the hypothesis that f generates $\mathcal{S}$ under T.

If f does not generate $\mathcal{S}$ under T we define a map $\eta \colon X \longmapsto G^{\mathbb{Z}} = Y$ by setting $\eta(x)_n = f(T^n x)$ for every $x \in X$ and $n \in \mathbb{Z}$, where a typical point in $Y = G^{\mathbb{Z}}$ is of the form $y = (y_n)$ with $y_n \in G$ for every $n \in \mathbb{Z}$. If T' is the shift

$$(T'y)_n = y_{n+1} \tag{3.21}$$

for every $y = (y_n) \in Y$ and $n \in \mathbb{Z}$, then $\eta \cdot T = T' \cdot \eta$ and $f = \pi_0 \cdot \eta$, where $\pi_0(y) = y_0$ for every $y = (y)_n \in Y$. We set $\nu = \mu\eta^{-1}$ and obtain a measure preserving, ergodic automorphism T' of the standard probability space $(Y, \mathcal{T}, \nu)$ and a Borel map $\pi_0 \colon Y \longmapsto G$ for which $(\eta \times \eta)^{-1}(\mathbf{R}_{\pi_0}) = \mathbf{R}_f$ and $(\eta \times \eta)^{-1}(\mathbf{R}_{\pi_0}^*) = \mathbf{R}_f^*$. The first part of this proof guarantees that $\mathcal{T}^{\mathbf{R}_{\pi_0}^*} = \mathcal{T}^{\mathbf{R}_{\pi_0}}$, which implies that $\mathcal{S}^{\mathbf{R}_f^*} = \mathcal{S}^{\mathbf{R}_f}$. □

Proof of Theorem 1.1. In order to realise the process (X_n) we set $X = G^{\mathbb{Z}}$, write $\mathcal{S} = \mathcal{B}_X$ for the product Borel field and T for the shift (3.21) on X, choose an appropriate shift-invariant probability measure μ on X, and put $X_n = f \cdot T^n$ for every $n \in \mathbb{Z}$, where $f = \pi_0 \colon X \longmapsto G$ is the zero coordinate projection. The resulting probability space $(X, \mathcal{S}, \mu)$ is standard and T is a measure preserving Borel automorphism of $(X, \mathcal{S}, \mu)$. If T is nonergodic we choose an ergodic decomposition $\mu = \int_X \mu'_x \, d\mu(x)$ of μ, where each μ'_x is a T-invariant and ergodic probability measure on $\mathcal{S}$, and where $x \to \mu'_x$

is a T-invariant Borel map from X into the space of T-invariant probability measures on $\mathcal{S}$ with their usual Borel structure.

For every $x \in X$, Theorem 3.1 implies that $\mathcal{S}^{\mathbf{R}_f} = \mathcal{S}^{\mathbf{R}_f^*} \pmod{\mu'_x}$, and by re-integrating we see that $\mathcal{T}_\infty = \mathcal{S}^{\mathbf{R}_f} = \mathcal{S}^{\mathbf{R}_f^*} = \mathcal{T}_\infty^* \pmod{\mu}$, as claimed. □

4. Examples

Example 4.1. (Super-K generators.) Let T be a measure-preserving, ergodic automorphism of a standard probability space $(X, \mathcal{S}, \mu)$, and let $\mathcal{P} \subset \mathcal{S}$ be a countable partition which generates $\mathcal{S}$ under T, and for which the two-sided tail

$$\mathcal{T}_\infty(\mathcal{P}) = \bigcap_{M \to \infty} \bigvee_{|m| \geq M} T^{-m}(\mathcal{P}) \tag{4.1}$$

is trivial. We write $G_\mathcal{P}$ for the free abelian group generated by $\mathcal{P}$ and define a Borel map $f_\mathcal{P} \colon X \longmapsto G_\mathcal{P}$ by

$$f_\mathcal{P}(x) = P(x) \in \mathcal{P} \subset G_\mathcal{P} \tag{4.2}$$

for every $x \in X$, where $P(x) \in \mathcal{P}$ is the unique element containing $x \in X$. Then the equivalence relation $\mathbf{R}^*_{f_\mathcal{P}}$ in (2.8), which is ergodic by Theorem 3.1, consists precisely of those elements $(x, y) \in X \times X$ for which $P(T^n x) = P(T^n y)$ for all but finitely many times $n \in \mathbb{Z}$, and for which the sequences $(P(T^n x),\ n \in \mathbb{Z})$ and $(P(T^n y),\ n \in \mathbb{Z})$, are finite permutations of each other. In the terminology of [11] this means that a countable generator $\mathcal{P} \subset \mathcal{S}$ is *super-K* whenever $\mathcal{T}_\infty(\mathcal{P})$ is trivial.

In the one-sided case things are different. If T is a measure-preserving, ergodic endomorphism of $(X, \mathcal{S}, \mu)$ and $\mathcal{P} \subset \mathcal{S}$ a countable partition we define $f_\mathcal{P} \colon X \longmapsto G_\mathcal{P}$ by (4.2) and put, for every $x \in X$,

$$f_\mathcal{P}(n, x) = \begin{cases} f_\mathcal{P}(T^{n-1}x) \cdots f_\mathcal{P}(x) & \text{if } n \geq 1, \\ 0_G \ \text{(the identity element of } G) & \text{if } n = 0. \end{cases}$$

Denote by

$$\begin{aligned} \mathcal{T}_{+\infty}(\mathcal{P}) &= \bigcap_{M \geq 0} \bigvee_{m \geq M} T^{-m}(\mathcal{P}) = \bigcap_{M \geq 0} \sigma(f_\mathcal{P} \cdot T^m : m \geq M), \\ \mathcal{T}^*_{+\infty}(\mathcal{P}) &= \bigcap_{M \geq 0} \sigma(f_\mathcal{P}(m, \cdot) : m \geq M) \end{aligned} \tag{4.3}$$

the one-sided analogues (1.3) of the tail sigma-fields (1.1)–(1.2). As mentioned in the introduction, the sigma-algebras $\mathcal{T}_{+\infty}(\mathcal{P})$ and $\mathcal{T}^*_{+\infty}(\mathcal{P})$ need not coincide even when $\mathcal{T}_{+\infty}(\mathcal{P})$ is trivial. However, in [12] it is shown that the existence of a finite partition $\mathcal{P} \subset \mathcal{S}$ with $\mathcal{T}_{+\infty}(\mathcal{P}) = \{\varnothing, X\} \pmod{\mu}$ implies the existence of some other finite partition $\mathcal{P}' \subset \mathcal{S}$ with

$$\mathcal{T}_{+\infty}(\mathcal{P}') = \mathcal{T}^*_{+\infty}(\mathcal{P}') = \{\varnothing, X\} \pmod{\mu}.$$

In the terminology of [12] this means that any measure preserving endomorphism T of $(X, \mathcal{S}, \mu)$ which is exact and has a finite (one-sided) generator also has has a *super-K*-generator, but that generators of T are not automatically *super-K* (*exact* means that $\bigcap_{n \geq 0} T^{-n}(\mathcal{S}) = \{\varnothing, X\} \pmod{\mu}$).

Example 4.2. (Equivalence relations generated by several functions.) Let T be a measure-preserving, ergodic automorphism of the probability space $(X, \mathcal{S}, \mu)$, and let $\mathcal{P} \subset \mathcal{S}$ be a countable partition for which $\mathcal{T}_\infty(\mathcal{P})$ is trivial. If $n \geq 1$, G a discrete group with finite conjugacy classes, $\tau\colon \mathcal{P}^n \longmapsto G$ an arbitrary map, and if $f\colon X \longmapsto G$ is defined by $f(x) = \tau(P(x), \dots, P(T^{n-1}x))$ for every $x \in X$, then Theorem 3.1 guarantees that $\mathbf{R}_f^* = \mathbf{R}_f$ is trivial. One class of examples of such maps is obtained by setting, for every $n \geq 1$, $G_{\mathcal{P}}^{(n)}$ equal to the free abelian group generated by $\mathcal{P}^n$ and $\tau(P_0, \dots, P_{n-1}) = (P_0, \dots, P_{n-1}) \in G_{\mathcal{P}}^{(n)}$. The ergodicity of $\mathbf{R}_f^*$ corresponds to the by now unsurprising statement that the equivalence relation on X obtained by finite permutations of n-names with respect to $\mathcal{P}$ is ergodic.

If $\mathcal{P}$ is finite we may put $n = 2$, write $G = S(\mathcal{P})$ for the symmetric (i.e. permutation) group of the alphabet $\mathcal{P}$, set $\tau(P_0, P_1)$ equal to the transposition $(P_0, P_1) \in S(\mathcal{P})$, and put $f(x) = \tau(P(x), P(Tx))$. According to Theorem 3.1, the resulting relation $\mathbf{R}_f^* \subset \mathbf{R}_f$ is again ergodic, and $\mathcal{S}^{\mathbf{R}_f^*}$ is therefore trivial.

As the class of discrete groups with finite conjugacy classes is closed under finite direct sums we obtain that, for any finite collection $f_i\colon X \longmapsto G^{(i)}$, $i = 1, \dots, k$, of Borel maps from X into groups $G^{(i)}$ with finite conjugacy classes,

$$\mathcal{S}^{\bigcap_{i=1}^k \mathbf{R}_{f_i}} = \mathcal{S}^{\bigcap_{i=1}^k \mathbf{R}_{f_i}^*} \pmod{\mu}.$$

By translating this into the probabilistic setting of Theorem 1.1 we obtain (1.4). In particular, if the maps f_i are of the form described at the beginning of this example, then

$$\mathcal{S}^{\bigcap_{i=1}^n \mathbf{R}_{f_i}^*} = \{\varnothing, X\} \pmod{\mu}.$$

Example 4.3. (Local variations in long molecules.) Let F be a finite set, $X \subset F^{\mathbb{Z}}$ a closed, shift-invariant subset, T the shift (3.21) on X, and put, for every $i \in F$, $[i]_0 = \{x = (x_n) \in X : x_0 = i\}$. We denote by $\mathcal{P}_0 = \{[i]_0 : i \in F\}$ the *state partition* of X and assume that μ is a shift-invariant probability measure on X for which the two-sided tail-sigma-field $\mathcal{T}_\infty(\mathcal{P})$ in (4.1) is trivial (if X is a shift of finite type, then every Markov measure and, more generally, every Gibbs measure on X arising from a function $\phi\colon X \longmapsto \mathbb{R}$ with summable variation has this property—cf. [2] or [11]).

As in Example 4.1 we fix $n \geq 1$, denote by $G^{(n)}$ the free abelian group generated by F^n, and define a continuous map $f\colon X \longmapsto G$ by $f(x) = (x_0, \dots, x_{n-1}) \in F^n \subset G^{(n)}$ for every $x \in X$. Then the equivalence relation $\mathbf{R}_f^*$ in (2.8) consists of all pairs $(x, y) \in X \times X$ which differ in only finitely many coordinates, and for which the n-blocks $((x_k, \dots x_{k+n-1}), k \in \mathbb{Z})$ and $((y_k, \dots y_{k+n-1}), k \in \mathbb{Z})$ occurring in x and y differ only by a finite permutation, and is ergodic by Example 4.1. This fact can be expressed by saying that, for a typical point $x \in X$, a $\mathbf{R}_f^*$-equivalent point y could lie anywhere in the space X.

If one were to interpret F as a finite set of molecules and X as a collection of two-sided infinite concatenations of these molecules, then the ergodicity of $\mathbf{R}_f^*$ would imply the unreliability of any chemical analysis of the structure of such a concatenation based on an investigation of substrings of a given length.

We remain with this example a little longer. Let $d \geq 2$, and consider the d-dimensional Euclidean group $E(d) = \mathbb{R}^d \times SO(d)$, furnished with the group operation

$$(v, A) \cdot (v', A') = (vA' + v', AA')$$

for all $v, v' \in \mathbb{R}^d$ and $A, A' \in SO(d)$ (the elements of $\mathbb{R}^d$ are written as row vectors). Fix a discrete subgroup $G \subset E(d)$ (or, more generally, a countable subgroup $G \subset E(d)$ with finite conjugacy classes, regarded as a discrete group) and a continuous map $f\colon X \longmapsto G$. Then the compactness of X guarantees that f takes only finitely many values on disjoint closed and open subsets of X, so that $\mathbf{R}_f$ is still ergodic. The ergodicity of the relation $\mathbf{R}_f^*$, which is a consequence of Theorem 3.1, has the following geometrical interpretation.

For every $x \in X$ and $n \in \mathbb{Z}$ we define $f(n, x) \in G$ by (3.1) and write $f(n, x)$ as $f(n, x) = (v_n(x), A_n(x))$ with $v_n(x) \in \mathbb{R}^d$ and $A_n(x) \in SO(d)$. If $v(x) = v_1(x)$ and $A(x) = A_1(x)$ then

$$v_n(x) = \begin{cases} v(x) + v(Tx)A(x) + \dots \\ \qquad\qquad + v(T^{n-1}x)A(T^{n-1}x) \cdots A(x) & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -v(T^{-n}x)A(T^{-n}x)^{-1} \cdots A(x)^{-1} - \dots \\ \qquad\qquad - v(T^{-1}x)A(x)^{-1} & \text{if } n < 0. \end{cases}$$

By connecting successive points in the sequence $(v_n(x),\, n \in \mathbb{Z})$ by straight line segments we obtain an infinite polygonal curve in $\mathbb{R}^d$ which may, of course, have self-intersections. If we call two such polygonal curves associated with $x, y \in X$ *equivalent* if they differ in only finitely many segments (i.e. if $v_n(x) \neq v_n(y)$ for only finitely many $n \in \mathbb{Z}$) then Theorem 3.1 implies that equivalent curves may be unrecognisably different, even if we insist in addition that the points x and y they arise from differ only by a permutation of finitely many coordinates. Note that a finite change, or even a finite permutation, of the coordinates of a point $x \in X$ will generally lead to a point $y \in X$ whose polygonal curve is inequivalent to that of x.

If we were to continue with our interpretation of points $x \in X$ as (highly idealised) chains of molecules then the map f and the resulting sequence of coordinates $(v_n(x),\, n \in \mathbb{Z})$ would correspond to a spatial arrangement of the chain x determined by its molecular structure, and Theorem 3.1 to a statement about quite dissimilar chains having spatial arrangements with only local differences.

Other examples can be found in [11] and [13].

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Mathematics Institute, University of Vienna, Strudlhofgasse 4, A-1090 Vienna, Austria,

and

Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, A-1090 Vienna, Austria.

E-mail address: `klaus.schmidt@univie.ac.at`