QUOTIENTS OF $\ell^\infty(\mathbb{Z},\mathbb{Z})$ AND SYMBOLIC COVERS OF TORAL AUTOMORPHISMS

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Dedicated to Anatole M. Vershik on the occasion of his 70th birthday

Abstract. This note gives an account of the algebraic construction of symbolic covers and representations of ergodic automorphisms of compact abelian groups. For expansive toral automorphisms this subject was initiated by A.M. Vershik.

1. Introduction

In [17], A.M. Vershik showed that the two-sided golden mean shift is a symbolic representation of the hyperbolic automorphism $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ of the two-torus $\mathbb{T}^2$. The construction underlying this result was subsequently extended to arbitrary hyperbolic automorphisms of $\mathbb{T}^2$ in [16], and the paper [4] describes an algebraic construction of finite-to-one sofic covers of arbitrary hyperbolic toral automorphisms by using an alphabet consisting of a suitable finite set of integers in an algebraic number field associated with the automorphism.

A closely related algebraic construction of symbolic covers of expansive group automorphisms (and, more generally, of expansive $\mathbb{Z}^d$-actions by automorphisms of compact abelian groups) in [2], based on the analysis of the 'homoclinic group' of such automorphisms, was developed further in [14], where it was shown that the two-sided beta-shift of any Pisot number β provides a finite-to-one symbolic cover of the toral automorphism defined by the companion matrix of the minimal polynomial of β (cf. also [15]).

The analogous problem of finding a connection between the two-sided beta-shift of a Salem number β and the corresponding toral automorphism was one of the principal motivations for the paper [9], which investigated to what extent irreducible nonhyperbolic toral automorphisms can have symbolic representations. Since such automorphisms have no nontrivial homoclinic points, any straightforward translation of the ideas in [17], [4] or [14]

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is doomed to failure. However, as was shown in [9], there exists a much weaker form of symbolic representation of such automorphisms. The key to understanding these symbolic 'representations' in the nonexpansive case lies in the study of certain quotients of the space $\ell^\infty(\mathbb{Z},\mathbb{Z})$ of all bounded two-sided integer sequences which are determined by an irreducible polynomial h with integer coefficients.

In order to explain this in a special case we assume that the polynomial h is of the form $h(u) = u^d + h_{d-1}u^{d-1} + \dots h_1 u \pm 1$ and write α_h for the automorphism of the d-dimensional torus $\mathbb{T}^d$ defined by the companion matrix of h. Let σ be the shift on $\ell^\infty(\mathbb{Z},\mathbb{Z})$ and consider the homomorphism $h(\sigma) = \sigma^d + h_{d-1}\sigma^{d-1} + \dots + h_1\sigma \pm \mathrm{Id}\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longrightarrow \ell^\infty(\mathbb{Z},\mathbb{Z})$. If the polynomial h is hyperbolic (i.e. has no roots of absolute value 1), then the quotient group $Q^{(h)} = \ell^\infty(\mathbb{Z},\mathbb{Z})/h(\sigma)(\ell^\infty(\mathbb{Z},\mathbb{Z}))$ is naturally isomorphic to $\mathbb{T}^d$, and this isomorphism carries the automorphism of $Q^{(h)}$ induced by the shift σ to α_h (cf. [14] and Theorem 3.1). If h is nonhyperbolic, but also noncyclotomic, then the appropriate quotient space turns out to be $Q^{(h)} = \ell^\infty(\mathbb{Z},\mathbb{Z})/(\ell^\infty(\mathbb{Z},\mathbb{Z}) \cap h(\sigma)(\ell^*(\mathbb{Z},\mathbb{Z}))$, where $\ell^*(\mathbb{Z},\mathbb{Z})$ is the space of two-sided integer sequences with at most linear growth; here $Q^{(h)}$ can be identified naturally with the quotient of $\mathbb{T}^d$ by the dense 'central' subgroup of $\mathbb{T}^d$ on which α_h acts isometrically as a rotation (cf. Theorem 4.1).

In this language the search for symbolic models of α_h translates into the search for appropriate 'symbolic covers' of the space $Q^{(h)}$ by closed, bounded, shift-invariant subsets of $\ell^\infty(\mathbb{Z},\mathbb{Z})$ (cf. Definition 2.2).

The existence and, if they exist, the combinatorial structure of such covers is an open problem (Problem 6.1). However, the more modest task of finding good 'partial' symbolic covers of $Q^{(h)}$, i.e. closed, bounded, shift-invariant subsets of $\ell^\infty(\mathbb{Z},\mathbb{Z})$ which have small nonempty intersections with 'most' classes of $\Delta^{(h)}$, seems more manageable. In Theorem 6.4 we construct symbolic partial covers $V_L^{(h)} \subset \ell^\infty(\mathbb{Z},\mathbb{Z})$ of $Q^{(h)}$ with the following properties: the topological entropy $h(\sigma_{V_L^{(h)}})$ of the restriction of σ to $V_L^{(h)}$ is equal to $h(\alpha_h)$, and for every *weakly* d*-bounded* shift-invariant probability measure ν on $V_L^{(h)}$ (Definition 6.2) there exists a ν-*a.e.* countable-to-one equivariant map $\psi\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longrightarrow \mathbb{T}^d$ which sends ν to an α_h-invariant probability measure μ of equal entropy on $\mathbb{T}^d$ (the construction of α_h-invariant probability measures on $\mathbb{T}^d$ other than Lebesgue measure is of interest due to their somewhat exotic properties — cf. [9]). The existence of weakly d-bounded shift-invariant probability measures on $V_L^{(h)}$ with entropies arbitrarily close to $h(\alpha_h)$ has so far been verified only in some special cases in [9] (cf. Problem 6.8).

Most of the material in this note is based on [14] and [9], with an extension of some of the results in [9] about companion matrices of Salem numbers to arbitrary irreducible, nonhyperbolic and ergodic automorphisms of compact connected abelian groups.

2. Equivalence relations on $\ell^\infty(\mathbb{Z},\mathbb{Z})$ defined by polynomials

Let $R=\mathbb{Z}(u^{\pm1})$ be the ring of Laurent polynomials with integer coefficients in the variable u. We write every $f\in R$ as

$$f=\sum_{n\in\mathbb{Z}} f_n u^n \tag{2.1}$$

with $f_n\in\mathbb{Z}$ for every $n\in\mathbb{Z}$, set

$$\|f\|_1=\sum_{n\in\mathbb{Z}}|f_n|<\infty$$

and write

$$\mathsf{S}(f)=\{n\in\mathbb{Z}: f_n\neq 0\} \tag{2.2}$$

for the *support* of f. An element $f\in R$ is *irreducible* if it cannot be written as a product $f=f_1f_2$ with $\|f_i\|_1>1$ for $i=1,2$, and f is *hyperbolic* if it has no roots of absolute value 1.

Let

$$\ell^*(\mathbb{Z},\mathbb{Z})=\left\{w=(w_n)\in\mathbb{Z}^{\mathbb{Z}}:\sup_{n\in\mathbb{Z}}\frac{|w_n|}{|n|+1}<\infty\right\}$$
$$\supset \ell^\infty(\mathbb{Z},\mathbb{Z})=\left\{w=(w_n)\in\mathbb{Z}^{\mathbb{Z}}:\|w\|_\infty=\sup_{n\in\mathbb{Z}}|w_n|<\infty\right\}.$$

Both $\ell^*(\mathbb{Z},\mathbb{Z})$ and $\ell^\infty(\mathbb{Z},\mathbb{Z})$ will be furnished with the topology of coordinate-wise convergence. We write $\sigma\colon \ell^*(\mathbb{Z},\mathbb{Z})\longrightarrow\ell^*(\mathbb{Z},\mathbb{Z})$ for the *shift*, defined by

$$(\sigma w)_n=w_{n+1} \tag{2.3}$$

for every $w=(w_n)\in\ell^*(\mathbb{Z},\mathbb{Z})$.

For the remainder of this discussion we consider an irreducible polynomial

$$h=\sum_{n=0}^{d}h_nu^n\in R \ \text{ with } \ h_0>0,\ \ d>0 \ \text{ and } \ h_d\neq 0, \tag{2.4}$$

and define a group homomorphism $h(\sigma)\colon\ell^*(\mathbb{Z},\mathbb{Z})\longrightarrow\ell^*(\mathbb{Z},\mathbb{Z})$ by

$$h(\sigma)=\sum_{n\in\mathbb{Z}}h_n\sigma^n. \tag{2.5}$$

The *Mahler measure* of h is given by

$$\mathsf{M}(h)=\exp\left(\int_0^1\log|h(e^{2\pi it})|\,dt\right)=|h_d|\cdot\prod_{\{\gamma\in\mathbb{C}\,:\,h(\gamma)=0\}}\max\{|\gamma|,1\} \tag{2.6}$$

(cf. [6, (3-2)]). According to Kronecker's theorem [5], $\mathsf{M}(h)=1$ if and only if h is cyclotomic (i.e. if and only if h divides the polynomial u^n-1 for some $n\geq 1$).

Consider the shift-invariant subgroup

$$\ell_h^\infty(\mathbb{Z},\mathbb{Z})=\ell^\infty(\mathbb{Z},\mathbb{Z})\cap h(\sigma)(\ell^*(\mathbb{Z},\mathbb{Z})) \tag{2.7}$$

of $\ell^\infty(\mathbb{Z},\mathbb{Z})$, where $h(\sigma)$ is defined in (2.5). We write

$$Q^{(h)}=\ell^\infty(\mathbb{Z},\mathbb{Z})/\ell_h^\infty(\mathbb{Z},\mathbb{Z}) \tag{2.8}$$

for the corresponding quotient group and

$$q^{(h)}\colon\ell^\infty(\mathbb{Z},\mathbb{Z})\longrightarrow Q^{(h)} \tag{2.9}$$

for the quotient map. It will be convenient to write

$$\Delta^{(h)} = \{(w, w') \in \ell^\infty(\mathbb{Z}, \mathbb{Z}) \times \ell^\infty(\mathbb{Z}, \mathbb{Z}) : w - w' \in \ell_h^\infty(\mathbb{Z}, \mathbb{Z}))\} \tag{2.10}$$

for the equivalence relation arising from the partition of $\ell^\infty(\mathbb{Z}, \mathbb{Z})$ into cosets of $\ell_h^\infty(\mathbb{Z}, \mathbb{Z})$, and to denote by

$$\Delta^{(h)}(w) = w + \ell_h^\infty(\mathbb{Z}, \mathbb{Z}) \tag{2.11}$$

the $\Delta^{(h)}$-equivalence class of $w \in \ell^\infty(\mathbb{Z}, \mathbb{Z})$. In this notation $Q^{(h)}$ is simply the space of equivalence classes of $\Delta^{(h)}$.

Remarks 2.1. (1) It will be clear from (4.3)–(4.4) that, if w is a two-sided integer sequence with polynomial growth such that $h(\sigma)w$ is bounded, then w has at most linear growth and therefore lies in $\ell^*(\mathbb{Z}, \mathbb{Z})$. We could thus have defined $\Delta^{(h)}$ equivalently as the set of all pairs $(w, w') \in \ell^\infty(\mathbb{Z}, \mathbb{Z}) \times \ell^\infty(\mathbb{Z}, \mathbb{Z})$ whose difference is of the form $h(\sigma)w$ for some integer sequence w of polynomial (or, indeed, sub-exponential) growth.

(2) If the polynomial h is hyperbolic, Theorem 3.1 will show that $\ell_h^\infty(\mathbb{Z}, \mathbb{Z}) = h(\sigma)(\ell^\infty(\mathbb{Z}, \mathbb{Z}))$, so that

$$Q^{(h)} = \ell^\infty(\mathbb{Z}, \mathbb{Z})/h(\sigma)(\ell^\infty(\mathbb{Z}, \mathbb{Z})). \tag{2.12}$$

Definition 2.2. A closed, bounded, shift-invariant set $V \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ is a *symbolic partial cover* of $Q^{(h)}$. A symbolic partial cover $V \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ is a *symbolic cover* of $Q^{(h)}$ if

$$q^{(h)}(V) = Q^{(h)}. \tag{2.13}$$

V is a (partial) *finite-to-one* (or *countable-to-one*) symbolic cover of $Q^{(h)}$ if $V \cap \Delta^{(h)}(w)$ is finite (or countable) for every $w \in \ell^\infty(\mathbb{Z}, \mathbb{Z})$,

The main reason for trying to understand the quotient group $Q^{(h)}$ and to find covers of it is that these objects are intimately connected with symbolic covers — in the sense of [4], [9], [14] and [17] — of the irreducible ergodic automorphism α_h of the compact connected abelian group X_h in (3.2)–(3.3) determined by the polynomial h, and that any symbolic cover of $Q^{(h)}$ defines a kind of symbolic cover of α_h (cf. the Theorems 3.1 and 4.1). Even symbolic partial covers can be useful for constructing invariant probability measures of, for example, nonhyperbolic ergodic toral automorphisms (cf. [9]).

3. Expansive group automorphisms and quotients of ℓ^∞

Let $h \in R$ be an irreducible, noncyclotomic, and not necessarily hyperbolic, polynomial of the form (2.4). The following discussion describes the connection between the quotient space $Q^{(h)}$ in (2.8) and a certain irreducible automorphism α_h of a compact connected abelian group X_h (where *irreducible* means that every closed α_h-invariant subgroup $Y \subsetneq X_h$ is finite). Background and details of can be found in [2], [9], [12], [13], and [14].

We write $\tau\colon \mathbb{T}^\mathbb{Z} \longrightarrow \mathbb{T}^\mathbb{Z}$ for the shift

$$(\tau x)_n = x_{n+1}, \;\; x = (x_n) \in \mathbb{T}^\mathbb{Z}, \tag{3.1}$$

and define a closed, shift-invariant subgroup $X_h \subset \mathbb{T}^{\mathbb{Z}}$ by

$$X_h = \Bigl\{ x = (x_n) \in \mathbb{T}^{\mathbb{Z}} : \sum_{i=0}^{d} h_i x_{n+i} = 0 \pmod 1 \text{ for every } n \in \mathbb{Z} \Bigr\}. \tag{3.2}$$

The restriction

$$\alpha_h = \tau_{X_h} \tag{3.3}$$

of τ to X_h is a continuous, irreducible and ergodic automorphism of the compact abelian group X_h with entropy $\log \mathsf{M}(h)$. Furthermore, α_h is expansive if and only if h is hyperbolic (cf. [13, Theorem 7.1 and Propositions 7.2–7.3]).

In the case where $h_0 = |h_d| = 1$ in (2.4) and X_h is therefore isomorphic to $\mathbb{T}^d$, the automorphism α_h in (3.3) is algebraically conjugate to the companion matrix

$$M_h = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -h_0 & -h_1 & -h_2 & \dots & -h_{d-2} & -h_{d-1} \end{bmatrix}, \tag{3.4}$$

of h, acting on $\mathbb{T}^d$ from the left, where the isomorphism between X_h and $\mathbb{T}^d$ is the coordinate projection

$$x \mapsto \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{d-1} \end{bmatrix}.$$

More generally, every irreducible ergodic automorphism α of a compact connected abelian group X is finitely equivalent to such an automorphism α_h for an appropriate polynomial $h \in R$ (cf. [12], [13] or [8]–[9]).

We return to our polynomial h in (2.4), extend the shift σ on $\ell^*(\mathbb{Z},\mathbb{Z}) \subset \mathbb{Z}^{\mathbb{Z}}$ linearly to an automorphism $\bar{\sigma}$ of

$$\ell^*(\mathbb{Z},\mathbb{R}) = \Bigl\{ w = (w_n) \in \mathbb{R}^{\mathbb{Z}} : \sup_{n\in\mathbb{Z}} \frac{|w_n|}{|n|+1} < \infty \Bigr\}$$

and define

$$h(\bar{\sigma}) = \sum_{n=0}^{d} h_n \bar{\sigma}^n \colon \ell^*(\mathbb{Z},\mathbb{R}) \longrightarrow \ell^*(\mathbb{Z},\mathbb{R}) \tag{3.5}$$

as in (2.5). Denote by Ω_h the set of roots of h and set

$$\Omega_h^- = \{\omega \in \Omega_h : |\omega| < 1\}, \ \Omega_h^{(0)} = \{\omega \in \Omega_h : |\omega| = 1\}, \\ \Omega_h^+ = \{\omega \in \Omega_h : |\omega| > 1\}. \tag{3.6}$$

According to [9, (2.15)], the kernel

$$W_h^{(0)} = \ker h(\bar{\sigma}) \tag{3.7}$$

is the linear span of the vectors $\{w^{(1)}(\omega), w^{(2)}(\omega) : \omega \in \Omega_h^{(0)}\}$ with

$$w^{(1)}(\omega)_n = \operatorname{Re}(\omega^n), \qquad w^{(2)}(\omega)_n = \operatorname{Im}(\omega^n) \tag{3.8}$$

for every $n \in \mathbb{Z}$ and $\omega \in \Omega_h^{(0)}$, where Re and Im denote the real and imaginary parts.

Let $\rho\colon \ell^*(\mathbb{Z},\mathbb{R}) \longrightarrow \mathbb{T}^{\mathbb{Z}}$ be the map

$$\rho(w)_n = w_n \pmod 1, \quad w = (w_n) \in \ell^*(\mathbb{Z},\mathbb{R}). \tag{3.9}$$

Then

$$\rho \circ \bar{\sigma} = \tau \circ \rho,$$

and the set

$$W_h = h(\bar{\sigma})^{-1}(\ell^\infty(\mathbb{Z},\mathbb{Z})) \subset \rho^{-1}(X_h) \subset \ell^*(\mathbb{Z},\mathbb{R}) \tag{3.10}$$

is a closed and shift-invariant subgroup which contains both $\ell^\infty(\mathbb{Z},\mathbb{Z})$ and $W_h^{(0)}$ (cf. (3.16) and (4.5)).

Following [9] we write

$$\frac{1}{h(u)} = \frac{1}{h_d}\sum_{\omega\in\Omega_h}\frac{b_\omega}{u-\omega}$$

for the partial fraction decomposition of $1/h$ with $b_\omega \in \mathbb{C}$ for every $\omega \in \Omega_h$ and define elements $w^{\Delta_\pm}$ and w^{Δ_0} in $\ell^\infty(\mathbb{Z},\mathbb{R})$ by

$$\begin{aligned} w_n^{\Delta_+} &= \begin{cases} \frac{1}{h_d}\cdot\sum_{\omega\in\Omega_h^-} b_\omega\omega^{n-1} & \text{if } n\ge 1,\\ \frac{1}{h_d}\cdot\sum_{\omega\in\Omega_h^{(0)}\cup\Omega_h^+} -b_\omega\omega^{n-1} & \text{if } n\le 0,\end{cases}\\ w_n^{\Delta_-} &= \begin{cases} \frac{1}{h_d}\cdot\sum_{\omega\in\Omega_h^-\cup\Omega_h^{(0)}} b_\omega\omega_i^{n-1} & \text{if } n\ge 1,\\ \frac{1}{h_d}\cdot\sum_{\omega\in\Omega_h^+} -b_\omega\omega^{n-1} & \text{if } n\le 0,\end{cases}\\ w_n^{\Delta_0} &= \frac{1}{h_d}\cdot\sum_{\omega\in\Omega_h^{(0)}} b_\omega\omega^{n-1} \quad \text{for every } n\in\mathbb{Z}. \end{aligned} \tag{3.11}$$

Then

$$\begin{gathered} w^{\Delta_0} \in W_h^{(0)}, \qquad w^{\Delta_+} + w^{\Delta_0} = w^{\Delta_-} \in W_h,\\ h(\bar{\sigma})(w^{\Delta_+})_n = h(\bar{\sigma})(w^{\Delta_-})_n = v_n^{\Delta} =: \begin{cases} 1 & \text{if } n=0,\\ 0 & \text{if } 0\ne n\in\mathbb{Z},\end{cases} \end{gathered} \tag{3.12}$$

and

$$\begin{gathered} x^{\Delta_\pm} = \rho(w^{\Delta_\pm}) \in X_h, \;\; x^{\Delta_0} = \rho(w^{\Delta_0}) \in X_h^{(0)}, \;\; x^{\Delta_+} + x^{\Delta_0} = x^{\Delta_-},\\ \lim_{n\to\infty} w_n^{\Delta_+} = \lim_{n\to\infty} w_{-n}^{\Delta_-} = 0 \quad \text{exponentially fast.} \end{gathered} \tag{3.13}$$

Now suppose that h is hyperbolic. Then $w^{\Delta_0} = 0$ and we set

$$w^\Delta = w^{\Delta_+} = w^{\Delta_-}, \;\; x^\Delta = \rho(w^\Delta), \tag{3.14}$$

and define group homomorphisms

$$\bar{\xi}\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longrightarrow W_h, \qquad \xi = \rho\circ\bar{\xi}\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longrightarrow X_h \tag{3.15}$$

by

$$\bar{\xi}(v) = \sum_{n\in\mathbb{Z}} v_n\bar{\sigma}^{-n}w^\Delta, \qquad \xi(v) = \sum_{n\in\mathbb{Z}} v_n\alpha_h^{-n}x^\Delta$$

for every $v\in\ell^\infty(\mathbb{Z},\mathbb{Z})$. Then

$$h(\bar{\sigma})\circ\bar{\xi}(w) = w$$

for every $w \in \ell^\infty(\mathbb{Z},\mathbb{Z})$, and since $\ker h(\bar\sigma) = \{0\}$ we conclude that

$$\begin{gathered} W_h = h(\bar\sigma)^{-1}(\ell^\infty(\mathbb{Z},\mathbb{Z})) = \bar\xi(\ell^\infty(\mathbb{Z},\mathbb{Z})) \subset \ell^\infty(\mathbb{Z},\mathbb{R}), \\ h(\bar\sigma)(W_h) = \ell^\infty(\mathbb{Z},\mathbb{Z}), \\ \xi(\ell^\infty(\mathbb{Z},\mathbb{Z})) = X_h, \\ \ker\xi = \{w \in \ell^\infty(\mathbb{Z},\mathbb{Z}) : \xi(w) = 0\} = h(\sigma)(\ell^\infty(\mathbb{Z},\mathbb{Z})). \end{gathered} \tag{3.16}$$

Furthermore, ξ is equivariant in the sense that

$$\xi \circ \sigma = \alpha_h \circ \xi, \tag{3.17}$$

and induces an isomorphism

$$\xi'\colon \ell^\infty(\mathbb{Z},\mathbb{Z})/h(\sigma)(\ell^\infty(\mathbb{Z},\mathbb{Z})) \longrightarrow X_h, \tag{3.18}$$

which is again equivariant in the obvious sense. We obtain the following result.

Theorem 3.1 ([14]). *Let $h \in R$ be a nonconstant irreducible hyperbolic polynomial, and let α_h be the irreducible expansive automorphism of the compact abelian group X_h in* (3.2)–(3.3)*. If $h(\sigma)\colon \ell^*(\mathbb{Z},\mathbb{Z}) \longrightarrow \ell^*(\mathbb{Z},\mathbb{Z})$ is the homomorphism* (2.5) *and $Q^{(h)}$ is the quotient group* (2.8)*, then*

$$Q^{(h)} = \ell^\infty(\mathbb{Z},\mathbb{Z})/h(\sigma)(\ell^\infty(\mathbb{Z},\mathbb{Z})),$$

and the equivariant map $\xi\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longrightarrow X_h$ in (3.15) *has kernel*

$$\ker\xi = h(\sigma)(\ell^\infty(\mathbb{Z},\mathbb{Z}))$$

and thus induces an equivariant bijection

$$\xi'\colon Q^{(h)} \longrightarrow X_h.$$

Finally, if $V \subset \ell^\infty(\mathbb{Z},\mathbb{Z})$ is a symbolic cover of $Q^{(h)}$ in the sense of Definition 2.2*, then $\xi(V) = X_h$.*

Proof. The first inclusion in (3.16) shows that every $v \in \ell^\infty(\mathbb{Z},\mathbb{Z})$ with $h(\sigma)v \in \ell^\infty(\mathbb{Z},\mathbb{Z})$ must itself lie in $\ell^\infty(\mathbb{Z},\mathbb{Z})$, and all other statements follow from (3.16)–(3.18). □

4. Nonexpansive group automorphisms and quotients of ℓ^∞

In this section we assume that h is nonhyperbolic (but still noncyclotomic), and that $\Omega_h^{(0)}$ is therefore nonempty and $W_h^{(0)} = \ker h(\bar\sigma) \neq \{0\}$ (cf. (3.6)–(3.7)). Galois theory implies that the restriction of ρ to $W_h^{(0)}$ is injective, and the *central subgroup*

$$X_h^{(0)} = \rho(W_h^{(0)}), \tag{4.1}$$

on which α_h acts isometrically, is dense in X_h by irreducibility.

We define group homomorphisms $\bar\xi^*\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longrightarrow W_h$ and $\xi^*\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longrightarrow X_h$ by setting

$$\begin{gathered} \bar\xi^*(v) = \sum_{n\ge0} v_n\bar\sigma^{-n}(w^{\Delta_-}) + \sum_{n<0} v_n\bar\sigma^{-n}(w^{\Delta_+}), \\ \xi^*(v) = \rho\circ\bar\xi^*(v), \end{gathered} \tag{4.2}$$

for every $v = (v_n) \in \ell^\infty(\mathbb{Z},\mathbb{Z})$. Since the coordinates $w_n^{\Delta_+}$ and $w_{-n}^{\Delta_-}$ decay exponentially as $n \to \infty$ and $W_h \subset \ell^*(\mathbb{Z},\mathbb{R})$ is closed, $\bar{\xi}^*$ is well defined by (3.12). The second equation in (3.12) shows that

$$h(\bar{\sigma}) \circ \bar{\xi}^*(v) = v \tag{4.3}$$

for every $v \in \ell^\infty(\mathbb{Z},\mathbb{Z})$, and hence that

$$\bar{\xi}^* \circ h(\bar{\sigma})w - w \in W_h^{(0)} = \ker h(\bar{\sigma}) \tag{4.4}$$

for every $w \in W_h$. From (3.10) and (4.4) we see that

$$W_h = h(\bar{\sigma})^{-1}(\ell^\infty(\mathbb{Z},\mathbb{Z})) = \bar{\xi}^* \circ h(\bar{\sigma})(W_h) + W_h^{(0)} \tag{4.5}$$

(cf. (3.10)), and that

$$\bar{\xi}^* \circ h(\bar{\sigma})(W_h \cap \ell^\infty(\mathbb{Z},\mathbb{R})) \subset W_h \cap \ell^\infty(\mathbb{Z},\mathbb{R}). \tag{4.6}$$

The map $\bar{\xi}^* \colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longrightarrow W_h$ is obviously not shift-equivariant. Indeed,

$$\begin{aligned} \mathsf{d}(n,v) &= \bar{\sigma}^n \circ \bar{\xi}^*(v) - \bar{\xi}^* \circ \sigma^n(v) \\ &= \begin{cases} \sum_{j=0}^{n-1} v_j \bar{\sigma}^{n-j} w^{\Delta_0} & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\sum_{j=1}^{n} v_{-j} \bar{\sigma}^{j-n} w^{\Delta_0} & \text{if } n < 0. \end{cases} \end{aligned} \tag{4.7}$$

for every $n \in \mathbb{Z}$ and $v \in \ell^\infty(\mathbb{Z},\mathbb{Z})$, and the resulting map

$$\mathsf{d} \colon \mathbb{Z} \times \ell^\infty(\mathbb{Z},\mathbb{Z}) \longrightarrow W_h^{(0)} \tag{4.8}$$

satisfies the cocycle equation

$$\mathsf{d}(m,\sigma^n v) + \bar{\sigma}^m \mathsf{d}(n,v) = \mathsf{d}(m+n,v) \tag{4.9}$$

for every $m,n \in \mathbb{Z}$ and $v \in \ell^\infty(\mathbb{Z},\mathbb{Z})$. We put

$$\tilde{Y} = \ell^\infty(\mathbb{Z},\mathbb{Z}) \times W_h^{(0)} \tag{4.10}$$

and consider the continuous surjective maps $\tilde{\sigma} \colon \tilde{Y} \longrightarrow \tilde{Y}$, $\bar{\vartheta} \colon \tilde{Y} \longrightarrow W_h$ and $\vartheta \colon \tilde{Y} \longrightarrow X_h$ defined by

$$\begin{aligned} \tilde{\sigma}(v,w) &= (\sigma v, \bar{\sigma} w + \mathsf{d}(1,v)), \\ \bar{\vartheta}(v,w) &= \bar{\xi}^*(v) + w, \\ \vartheta(v,w) &= \rho \circ \bar{\vartheta}(v,w) \end{aligned} \tag{4.11}$$

for every $(v,w) \in \tilde{Y} = \ell^\infty(\mathbb{Z},\mathbb{Z}) \times W_h^{(0)}$. The map $\tilde{\sigma}$ is obviously a homeomorphism, and

$$\bar{\vartheta} \circ \tilde{\sigma} = \bar{\sigma} \circ \bar{\vartheta}, \quad \vartheta \circ \tilde{\sigma} = \alpha_h \circ \vartheta. \tag{4.12}$$

Since the restriction of ρ to $W_h^{(0)}$ is injective and $W_h^{(0)} \cap \ell^*(\mathbb{Z},\mathbb{Z}) = \{0\}$,

$$\vartheta(v,w) + X_h^{(0)} = \vartheta(v',w') + X_h^{(0)} \text{ if and only if } v - v' \in \ell_h^\infty(\mathbb{Z},\mathbb{Z}) \tag{4.13}$$

for all $(v,w),(v',w') \in \tilde{Y}$. We obtain the following result.

Theorem 4.1 ([9]). *Let $h \in R$ be an irreducible nonhyperbolic polynomial which is not cyclotomic, and let α_h be the irreducible, ergodic and nonexpansive automorphism of the compact abelian group X_h in* (3.2)–(3.3). *Let $h(\sigma)\colon \ell^*(\mathbb{Z},\mathbb{Z}) \longrightarrow \ell^*(\mathbb{Z},\mathbb{Z})$ be the homomorphism* (2.5), *and let $\ell_h^\infty(\mathbb{Z},\mathbb{Z}) \subset \ell^\infty(\mathbb{Z},\mathbb{Z})$ be the subgroup defined in* (2.7).

(1) *If $\xi^*\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longrightarrow X_h$ is the nonequivariant group homomorphism* (4.2), *then*

$$\xi^{*-1}(X_h^{(0)}) = \ell_h^\infty(\mathbb{Z},\mathbb{Z}),$$

and ξ^* induces an (*equivariant*) *bijection*

$$\xi^{*\prime}\colon Q^{(h)} \longrightarrow X_h/X_h^{(0)}.$$

(2) *If $V \subset \ell^\infty(\mathbb{Z},\mathbb{Z})$ is a symbolic cover of $Q^{(h)}$ in the sense of Definition* 2.2, *then $\xi^*(V) + X_h^{(0)} = X_h$.*

(3) *If $\vartheta\colon \tilde{Y} = \ell^\infty(\mathbb{Z},\mathbb{Z}) \times W_h^{(0)} \longrightarrow X_h$ is the equivariant map defined in* (4.11)–(4.12), *then two points $(v,w),(v',w') \in \tilde{Y}$ are mapped by ϑ to the same coset of $X_h^{(0)} \subset X_h$ if and only if $v - v' \in \ell_h^\infty(\mathbb{Z},\mathbb{Z})$.*

Remark 4.2. It is easy to find symbolic covers of $Q^{(h)}$ (irrespective of whether h is hyperbolic or not): if $Y_h = W_h \cap [0,1)^{\mathbb{Z}}$, then $\rho(Y_h) = X_h$, and $Z_h = \overline{h(\bar{\sigma})(Y_h)} \subset \ell^\infty(\mathbb{Z},\mathbb{Z})$ is a closed, bounded, shift-invariant subset with $\bar{\xi}^*(Z_h) + W_h^{(0)} \supset Y_h$ by (4.4). Hence $\xi^*(Z_h) + X_h^{(0)} = X_h$, and Theorem 4.1 shows that Z_h must intersect every coset of $\ell_h^\infty(\mathbb{Z},\mathbb{Z})$ in $\ell^\infty(\mathbb{Z},\mathbb{Z})$.

The following corollary of Theorem 4.1 suggests that the search for 'small' symbolic covers of $Q^{(h)}$ for nonhyperbolic h may be considerably more difficult than in the hyperbolic case (cf. Corollary 5.4 and Theorem 5.5).

Corollary 4.3. *Let $h \in R$ be an irreducible nonhyperbolic polynomial which is not cyclotomic. Then $Q^{(h)}$ has no finite-to-one symbolic cover* (*cf. Definition* 2.2).

Proof. We are claiming that there is no closed, bounded, shift-invariant set $V \subset \ell^\infty(\mathbb{Z},\mathbb{Z})$ which intersects every coset of $\ell_h^\infty(\mathbb{Z},\mathbb{Z})$ in a nonempty finite set.

Suppose that such a set V exists. Since $\ell^*(\mathbb{Z},\mathbb{Z})$ is sigma-compact in the topology of pointwise convergence, $h(\sigma)(\ell^*(\mathbb{Z},\mathbb{Z})) \subset \ell^*(\mathbb{Z},\mathbb{Z})$ is again sigma-compact. It follows that the equivalence relation

$$\Delta_V^{(h)} = \Delta^{(h)} \cap (V \times V)$$

is a Borel subset of $V \times V$ such that $\pi_1^{-1}(w) \cap \Delta_V^{(h)}$ is finite and nonempty for every $w \in V$ (here $\pi_1\colon V \times V \longrightarrow V$ is the first coordinate projection). In particular, $\pi_1(B) \subset V$ is Borel for every Borel set $B \subset \Delta_V^{(h)}$.

Since V must be uncountable and any two uncountable Borel sets are Borel isomorphic, there exists a Borel isomorphism $\phi\colon V \longrightarrow [0,1]$, and we use ϕ to pull the order on $[0,1]$ back to an order $\prec$ on V.

Let $E \subset \Delta_V^{(h)}$ be the uniquely determined Borel set with the following properties:

(i) The restriction of π_1 to E is a bijection of E and V,

(ii) for every $w \in V$, the unique element $(w, w') \in E$ satisfies that $w' \preceq w''$ for every $w'' \in \Delta^{(h)}(w) \cap V$.

The set E is the graph of a Borel map $\psi\colon V \longrightarrow V$ with the property that $\psi^{-1}(\{w\})$ is finite for every $w \in V$. Hence $B = \psi(E)$ is a Borel set, and our construction guarantees that $|B \cap \Delta^{(h)}(w)| = 1$ for every $w \in \ell^\infty(\mathbb{Z}, \mathbb{Z})$.

The continuous map $\xi^*\colon \ell^\infty(\mathbb{Z}, \mathbb{Z}) \longrightarrow X_h$ in (4.2) is injective on B, and hence $\xi^*(B) \subset X_h$ is a Borel set which intersects each coset of the dense subgroup $X_h^{(0)}$ in a single point (cf. Theorem 4.1). Since this is impossible we have proved the corollary by contradiction. □

5. Construction of symbolic partial covers of $Q^{(h)}$

We write $\ell^1(\mathbb{Z}, \mathbb{Z}) \subset \ell^\infty(\mathbb{Z}, \mathbb{Z}) \subset \ell^*(\mathbb{Z}, \mathbb{Z})$ for the set of all sequences with only finitely many nonzero terms. By viewing every $f = \sum_{n\in\mathbb{Z}} f_n u^n \in R$ as the element $(f_n) \in \ell^1(\mathbb{Z}, \mathbb{Z})$ we identify R with $\ell^1(\mathbb{Z}, \mathbb{Z})$.

Let $h \in R$ be an irreducible, nonconstant and noncyclotomic polynomial of the form (2.4). We define an equivalence relation $\Delta_1^{(h)}$ on $\ell^\infty(\mathbb{Z}, \mathbb{Z})$ by

$$\Delta_1^{(h)} = \bigl\{(w, w') \in \ell^\infty(\mathbb{Z}, \mathbb{Z}) \times \ell^\infty(\mathbb{Z}, \mathbb{Z}) : w - w' \in h(\sigma)(\ell^1(\mathbb{Z}, \mathbb{Z}))\bigr\} \tag{5.1}$$

(cf. (2.5)), and write

$$\Delta_1^{(h)}(w) = \{w' \in \ell^\infty(\mathbb{Z}, \mathbb{Z}) : (w, w') \in \Delta_1^{(h)}\} \tag{5.2}$$

for the equivalence class of $w \in \ell^\infty(\mathbb{Z}, \mathbb{Z})$.

We introduce a lexicographic order $\prec$ on the ring R by setting $0 \prec f$ if and only if $f_m > 0$ for the *smallest* $m \in \mathbb{Z}$ with $f_m \neq 0$ (cf. (2.1)), and by saying that $f \prec f'$ whenever $0 \prec f' - f$. The order $\prec$ on R induces a lexicographic order (again denoted by $\prec$) on each equivalence class of $\Delta_1^{(h)}$: if $(v, v') \in \Delta_1^{(h)}$ then $v - v' \in h(\sigma)(\ell^1(\mathbb{Z}, \mathbb{Z}))$, and $v \prec v'$ if and only if $v - v' = h(\sigma)f$ for some $f \prec 0$.

Let $V \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ be a closed, bounded, shift-invariant subset. We put $P = \{f \in R : 0 \prec f\}$ and set

$$\begin{aligned} V^{(h)} &= \bigcap_{f\in P} (V \smallsetminus (V - h(\sigma)f)) = V \smallsetminus \bigcup_{f\in P} (V - h(\sigma)f) \\ &= \{w \in V : w' \preceq w \text{ for every } w' \in V \cap \Delta^{(h)}(w)\}. \end{aligned} \tag{5.3}$$

Theorem 5.1. *Let $h \in R$ be an irreducible, nonconstant and noncyclotomic polynomial of the form* (2.4)*, L a positive integer,*

$$V_L = \{0, \dots, L-1\}^{\mathbb{Z}}, \tag{5.4}$$

and let $V_L^{(h)} \subset V_L$ be the subset defined by (5.3)*. Then $V_L^{(h)}$ is closed, shift-invariant and has the following properties.*

(1) $\bigl|V_L^{(h)} \cap \Delta_1^{(h)}(w)\bigr| \leq 1$ *for every $w \in \ell^\infty(\mathbb{Z}, \mathbb{Z})$;*

(2) *If h is hyperbolic then $V_L^{(h)}$ is a partial finite-to-one symbolic cover of $Q^{(h)}$.*

(3) *If $\sigma_{V_L^{(h)}}$ is the restriction to $V_L^{(h)}$ of the shift σ on $\ell^\infty(\mathbb{Z},\mathbb{Z})$ in* (2.3)*, then its topological entropy satisfies that $h(\sigma_{V_L^{(h)}}) \le \log \mathsf{M}(h)$ (cf.* (2.6)*).*

We start the proof of Theorem 5.1 with a lemma.

Lemma 5.2. *The polynomial h in* (2.4) *is hyperbolic if and only if there exists a constant $b > 0$ with*

$$\|h(\sigma)w\|_\infty \ge b \cdot \|w\|_\infty \text{ for every } w \in \ell^\infty(\mathbb{Z},\mathbb{Z}), \tag{5.5}$$

where $h(\sigma)$ is defined in (2.5)*.*

Proof. Since h is noncyclotomic, Galois theory implies that $\ker h(\sigma) = \{0\}$ (cf. [9, (2.15)]).

Suppose that h is hyperbolic. We define w^Δ by (3.14) and conclude from (3.13) there exist constants $\gamma \in (0,1)$ and $C > 0$ such that $|w_n^\Delta| \le C \cdot \gamma^{|n|}$ for every $n \in \mathbb{Z}$.

The shift-equivariant map $\bar{\xi}\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longrightarrow \ell^\infty(\mathbb{Z},\mathbb{R})$ in (3.15) satisfies that $\|\bar{\xi}(v)\|_\infty \le 2C \cdot \|v\|_\infty \cdot \sum_{n\ge 0}\gamma^n$ and $h(\sigma)\circ\bar{\xi}(w) = \bar{\xi}\circ h(\sigma)(w) = w$ for every $w \in \ell^\infty(\mathbb{Z},\mathbb{Z})$. Hence

$$\|w\|_\infty = \|\bar{\xi}(h(\sigma)w)\|_\infty \le 2C \cdot \|h(\sigma)w\|_\infty \cdot \sum_{n\ge 0}\gamma^n$$

for every $w \in \ell^\infty(\mathbb{Z},\mathbb{Z})$, which proves the existence of a lower bound $b > 0$ in (5.5).

If h is noncyclotomic and nonhyperbolic we choose $\theta \in \Omega_h^{(0)}$ (cf. (3.6)) and define, for every integer $j \ge 0$, a point $\omega^{(j)} = (\omega_n^{(j)}) \in \ell^\infty(\mathbb{Z},\mathbb{R})$ by setting

$$\omega_n^{(j)} = \begin{cases} \theta^n + \theta^{-n} & \text{if } n \ge j, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$(h(\bar{\sigma})\omega^{(j)})_n = 0$$

for $n < j - d$ and $n \ge j$, and $\|h(\bar{\sigma})\omega^{(j)}\|_\infty \le 2 \cdot \|h\|_1$.

For every $t \in \mathbb{R}$ we denote by $\lceil t \rceil$ the smallest integer $\ge t$, and we set, for every $M \ge 1$ and $n \in \mathbb{Z}$,

$$\tilde{\omega}^{(M)} = \sum_{j=0}^{M} \omega^{(3jd)}, \qquad w_n^{(M)} = \lceil \tilde{\omega}_n^{(M)} \rceil.$$

The resulting sequence $(w^{(M)},\ M \ge 1)$ in $\ell^\infty(\mathbb{Z},\mathbb{Z})$ satisfies that $\|w^{(M)}\|_\infty = M \cdot \|w^{(1)}\|_\infty$ and $\|h(\sigma)w^{(M)}\|_\infty \le 3 \cdot \sum_{i=0}^{d} |h_i|$ for every $M \ge 1$. This proves that there is no $b > 0$ satisfying (5.5). □

For the proof of Theorem 5.1 as well as that of Theorem 5.3 below we recall some facts from [12] (cf. also [1] or [18]). We fix $\gamma \in \Omega_h$ (cf. (3.6)), denote by $K = \mathbb{Q}(\gamma)$ the algebraic number field generated by γ, and write $P^{(K)}$, $P_f^{(K)}$

and $P_\infty^{(K)}$ for the sets of places (or equivalence classes of valuations), finite places and infinite places of K. For every place v of K we denote by K_v the *v-adic completion* of K and consider the valuation $|\cdot|_v \in v$ defined by

$$|a|_v \cdot \lambda_v(C) = \lambda_v(a \cdot C) \tag{5.6}$$

for every $a \in K$, where λ_v is a Haar measure on the additive group of the locally compact, metrizable field K_v, and where $C \subset K_v$ is a compact neighbourhood of 0. We write

$$\iota_v \colon K \longrightarrow K_v$$

for the embedding of K in K_v and set

$$R_v = \{a \in K : |a|_v \leq 1\}, \;\; \bar{R}_v = \{a \in K_v : |a|_v \leq 1\}. \tag{5.7}$$

Let

$$S = P_\infty^{(K)} \cup \{v \in P_f^{(K)} : |\gamma|_v \neq 1\}. \tag{5.8}$$

The set

$$K_S = \prod_{v \in S} K_v \tag{5.9}$$

is a locally compact abelian group with respect to coordinate-wise addition, and we write

$$\iota \colon K \longrightarrow K_S \tag{5.10}$$

for the diagonal embedding $a \mapsto \iota(a) = (\iota_v(a),\, v \in S)$, $a \in K$. If

$$R_S = \bigcap_{v \in P^{(K)} \smallsetminus S} R_v \tag{5.11}$$

is the ring of S-integers in K, then $\iota(R_S)$ is a discrete co-compact subgroup of K_S.

For every subset $F \subset \mathbb{Z}$ we write

$$\pi_F \colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longrightarrow \mathbb{Z}^F \tag{5.12}$$

for the projection onto the coordinates in F.

Proof of Theorem 5.1. For every $f \in P$ we set

$$V(f) = V_L \smallsetminus (V_L - h(\sigma)f). \tag{5.13}$$

Since $h(\sigma)f$ has only finitely many nonzero coordinates, $V(f) \subset V_L$ is closed and open, and hence $V_L^{(h)}$ is a closed — and obviously shift-invariant — subset of V_L.

If $V_L^{(h)}$ meets some equivalence class of $\Delta^{(h)}$ in more than one point, then there exist $w \in V_L^{(h)}$ and $f \in P$ such that $w' = w + h(\sigma)f \in V_L^{(h)}$. Hence $w' \in V_L \cap (V_L - h(\sigma)f)$, which contradicts the definition of $V_L^{(h)}$.

In order to prove (2) we set, for $N_1 < N_2$

$$\begin{aligned} B(N_1,N_2,K) = \{w \in \ell^*(\mathbb{Z},\mathbb{Z}) : |w|_n \leq K \\ \text{for } N_1 \leq n \leq N_1 + d \text{ and } N_2 \leq n \leq N_2 + d\}, \end{aligned} \tag{5.14}$$

where d appears in (2.4), and claim that

$$\big|\pi_{\{N_1,\dots,N_2\}}\big(\big(w+h(\sigma)(B(N_1,N_2,K))\big)\cap V_L^{(h)}\big)\big| \le (2K+1)^{2d+2} \tag{5.15}$$

for every $w\in V_L^{(h)}$, $K\ge 0$ and $N_1<N_2$ (cf. (5.12)).

Indeed, if (5.15) does not hold for some $w\in V_L^{(h)}$, $K\ge 0$ and $N_1<N_2$, then we can find elements $y,y'\in B(N_1,N_2,K)$ with the following properties:

(i) $y_r=y'_r$ for $r=N_1,\dots,N_1+d$ and $r=N_2,\dots,N_2+d$,

(ii) $(y_{N_1+d+1},\dots,y_{N_2-1})\ne(y'_{N_1+d+1},\dots,y'_{N_2-1})$,

(iii) $v=w+h(\sigma)y\in V_L^{(h)}$, $v'=w+h(\sigma)y'\in V_L^{(h)}$.

We define a Laurent polynomial $f\in R$ by

$$f_r=\begin{cases} y_r-y'_r & \text{for } r=N_1+d+1,\dots,N_2-1,\\ 0 & \text{otherwise}\end{cases}$$

(cf. (2.1)), and observe that the points $v-h(\sigma)f$ and $v'+h(\sigma)f$ both lie in $V_L^{(h)}$. By comparing this with (5.3) we obtain a contradiction in one of the two cases. This proves (5.15).

If h is hyperbolic then Lemma 5.2 implies that there exists a $K\ge 0$ such that

$$\{v\in\ell^\infty(\mathbb{Z},\mathbb{Z}): h(\sigma)v\in(V_L^{(h)}-V_L^{(h)})\}\subset\{-K,\dots,K\}^{\mathbb{Z}}.$$

Hence

$$V_L^{(h)}\cap\big(w+h(\sigma)(B(N_1,N_2,K))\big)=V_L^{(h)}\cap\Delta^{(h)}(w)$$

for every $w\in V_L^{(h)}$ and $N_1<N_2$, and (3) follows from (5.15).

For every $w\in\ell^\infty(\mathbb{Z},\mathbb{Z})$ and $r,s\in\mathbb{Z}$ with $r\le s$ we set

$$w_r^s(\gamma)=\sum_{i=r}^{s} w_i\gamma^i. \tag{5.16}$$

Let

$$\tilde h=u^d\cdot\sum_{n\in\mathbb{Z}}h_nu^{-n}\in R \tag{5.17}$$

be the *reversal* of h, and let $N\ge 1$. If $w,w'\in V_L^{(h)}$ and $w_0^{N-1}(\gamma)=w'^{N-1}_0(\gamma)$, then the Laurent polynomials $f(u)=\sum_{k=0}^{N-1}w_ku^k$ and $f'(u)=\sum_{k=0}^{N-1}w'_ku^k$ differ by a multiple $h(\sigma)f$ of $\tilde h$ which we may assume to lie in $P\tilde h$ (by interchanging w and w', if necessary). This implies that the point $w'+h(\sigma)f$ lies in V_L, and hence that $w'\in V_L\cap(V_L-h(\sigma)f)$. As in the preceding paragraph we obtain a contradiction to our hypothesis that $w'\in V_L^{(h)}$. This shows that $w_0^{N-1}(\gamma)\ne w'^{N-1}_0(\gamma)$ whenever $w,w'\in V_L^{(h)}$ and $\pi_{\{0,\dots,N-1\}}(w)\ne\pi_{\{0,\dots,N-1\}}(w')$ (cf. (5.16)).

For every $N\ge 1$, the set $\iota(\{w_0^{N-1}(\gamma):w\in V_L^{(h)}\})\subset K_S$ is contained in the set

$$F(N)=\prod_{v\in P_\infty^{(K)}}\big\{a\in K_v:|a|_v\le LN\cdot\max(1,|\gamma|_v^N)\big\}\cdot\prod_{v\in S\smallsetminus P_\infty^{(K)}}(\bar R_v\cup\gamma^N\bar R_v).$$

We fix a Haar measure λ on K_S with $\lambda\big(\prod_{v\in S}\bar{R}_v\big) = 1$ (cf. (5.7)). As N varies, the same calculation as in [7] shows that

$$\begin{aligned}\lambda(F(N)) &\le \prod_{\{v\in P_\infty^{(K)}:K_v=\mathbb{R}\}} LN\cdot\max\big(1,|\gamma|_v^N\big)\\ &\qquad\cdot\prod_{\{v\in P_\infty^{(K)}:K_v=\mathbb{C}\}} L^2N^2\cdot\max\big(1,|\gamma|_v^N\big)\cdot\prod_{v\in S\smallsetminus P_\infty^{(K)}}\max\big(1,|\gamma|_v^N\big)\\ &\le \mathsf{M}(\tilde{h})^N\cdot(LN)^{\mathsf{r}+2\mathsf{s}} = \mathsf{M}(h)^N\cdot(LN)^{\mathsf{r}+2\mathsf{s}},\end{aligned}$$

where r and s denote the numbers of real and complex places of K (i.e. the number of $v\in P_\infty^{(K)}$ with $K_v=\mathbb{R}$ and $K_v=\mathbb{C}$, respectively). As $\iota(R_S)$ is discrete and co-compact in K_S, there exists a constant $C>0$ such that, for every $N\ge 1$,

$$\big|\{w_0^{N-1}(\gamma): w\in V_L^{(h)}\}\big| = \big|\pi_{\{0,\dots,N-1\}}(V_L^{(h)})\big| \le C\cdot\mathsf{M}(h)^N\cdot(LN)^{\mathsf{r}+2\mathsf{s}},$$

where $|\cdot|$ denotes cardinality. This implies that

$$h(\sigma_{V_L^{(h)}}) = \lim_{N\to\infty}\frac{1}{N}\log\big|\pi_{\{0,\dots,N-1\}}(V_L^{(h)})\big| \le \log\mathsf{M}(h). \qquad \square$$

Theorem 5.3. *Let $h\in R$ be an irreducible, nonconstant and noncyclotomic polynomial of the form* (2.4)*, $L\ge 1$, and let $V_L^{(h)}\subset V_L$ be the closed, shift-invariant subset defined in* (5.3)*. If L is sufficiently large, then $h(\sigma_{V_L^{(h)}}) = \log\mathsf{M}(h)$.*

Corollary 5.4. *Suppose that the polynomial h is hyperbolic. If L is sufficiently large, then $V_L^{(h)}$ is a finite-to-one symbolic cover of $Q^{(h)}$.*

Proof of Corollary 5.4*, given Theorem* 5.3*.* If L is sufficiently large, then $h(\sigma_{V_L^{(h)}}) = \log\mathsf{M}(h)$ by Theorem 5.3. By Theorem 5.1 (2), $V_L^{(h)}$ is a partial finite-to-one symbolic cover of $Q^{(h)}$, which implies that the restriction to $V_L^{(h)}$ of the equivariant group homomorphism $\xi\colon \ell^\infty(\mathbb{Z},\mathbb{Z})\longrightarrow X_h$ in (3.15) is finite-to-one (cf. Theorem 3.1). In particular, $Y=\xi(V_L^{(h)})$ is a closed, α_h-invariant subset of X_h such that the restriction of α_h to Y has topological entropy $\log\mathsf{M}(h)=h(\alpha_h)$, and the uniqueness of the measure of maximal entropy of α_h implies that $\lambda_{X_h}(Y)=1$. Hence $Y=X_h$ and $V_L^{(h)}$ is a symbolic cover of $Q^{(h)}$. $\square$

Under the hypotheses of Corollary 5.4 we can even find almost one-to-one symbolic covers of $Q^{(h)}$:

Theorem 5.5. *Under the hypotheses of Theorem* 3.1 *there exists a closed, bounded, shift-invariant subset $V^*\subset\ell^\infty(\mathbb{Z},\mathbb{Z})$ with the following properties.*

(1) *V^* is a sofic shift;*
(2) *V^* is a finite-to-one symbolic cover of $Q^{(h)}$ (cf. Definition* 2.2*);*
(3) *$\lambda_{X_h}\big(\{x\in X_h : |\xi^{-1}(\{x\})\cap V^*|=1\}\big)=1$, where λ_{X_h} is the normalized Haar measure of X_h.*

Proof. This is [14, Theorem 5.1]. $\square$

For the proof of Theorem 5.3 we put, in the notation of (5.8),

$$S^- = \{v \in S : |\gamma|_v < 1\}, \ \ K_{S^-} = \prod_{v \in S^-} K_v, \tag{5.18}$$

and denote by $\iota^-\colon K \longrightarrow K_{S^-}$ the diagonal embedding of K in K_{S^-} (cf. (5.10)).

We set $\mathbb{N} = \{0, 1, 2, \dots\}$ and write $\ell^\infty(\mathbb{N}, \mathbb{Z})$ for the space of one-sided bounded integer sequences, furnished with the topology of coordinate-wise convergence, the maximum norm $\|\cdot\|_\infty$ and the one-sided shift σ_+ defined as in (2.3).

Lemma 5.6. *Let $\phi\colon \ell^\infty(\mathbb{N}, \mathbb{Z}) \longrightarrow K_{S^-}$ be defined by*

$$\phi(w) = \sum_{k \geq 0} w_k \cdot \iota^-(\gamma^k)$$

for every $w = (w_k) \in \ell^\infty(\mathbb{N}, \mathbb{Z})$. Then there exists an integer $L \geq 1$ such that $\phi(B_L^+)$ has nonempty interior in K_{S^-}, where

$$B_L^+ = \{-L, \dots, L\}^{\mathbb{N}}.$$

Proof. The basic idea for the proof of this lemma is due to Boris Solomyak (personal communication). We set $R(\gamma) = \mathbb{Z}[\gamma^{\pm 1}] = \{f(\gamma) : f \in R\} \subset R_S$, where $\gamma \in \Omega_h$ was chosen for the proof of Theorem 5.1. As discussed in [3] or [12], $R_S/R(\gamma)$ is finite, and $R(\gamma)^- = \iota^-(R(\gamma))$ is therefore dense in K_{S^-}.

We denote by $M_\gamma\colon K_{S^-} \longrightarrow K_{S^-}$ diagonal multiplication by γ on K_{S^-}. The set $Q = \prod_{v \in S^-} \bar{R}_v$ is a compact neighbourhood of $0 \in K_{S^-}$. As $R(\gamma)^-$ is dense in K_{S^-} and $M_\gamma Q \subset Q$ has nonempty interior, $\bigcup_{c \in R(\gamma)^-}(c + M_\gamma Q) \supset Q$, and the compactness of Q implies that there exists a finite set $F = \{f^{(1)}, \dots, f^{(l)}\} \subset R$ with

$$Q \subset \bigcup_{i=1}^{l} \bigl(\iota^-(f^{(i)}(\gamma)) + M_\gamma Q\bigr). \tag{5.19}$$

We fix $a \in Q$. By (5.19) there exists a $t_0(a) \in \{1, \dots, l\}$ with $a \in \iota^-(f^{(t_0(a))}(\gamma)) + M_\gamma Q$, and by repeating this argument we find a sequence $(t_0(a), t_1(a), t_2(a), \dots) \in \{1, \dots, l\}^{\mathbb{N}}$ with

$$a \in \sum_{i=0}^{s} M_\gamma^i[\iota^-(f^{(t_i(a))}(\gamma))] + M_\gamma^{s+1} Q \tag{5.20}$$

for every $s \geq 0$. As M_γ is a strict contraction on K_{S^-}, this implies that

$$a = \sum_{i=0}^{\infty} M_\gamma^i[\iota^-(f^{(t_i(a))}(\gamma))]. \tag{5.21}$$

There exist integers $J, J' \geq 0$ with $\mathsf{S}(f^{(i)}) \subset \{-J, \dots, J\}$ and $\|f^{(i)}\|_\infty \leq J'$ for every $i = 1, \dots, l$ (cf. (2.2)). We set $L = (2J+1)J'$ and obtain from (5.21) that there exists a sequence $w = (w_k) \in B_L^+$ with $a = \sum_{i \geq 0} w_k \cdot \iota^-(\gamma^{i-J})$. As $a \in Q$ was arbitrary this implies that

$$\phi(B_L^+) \supset M_\gamma^J Q,$$

which proves our claim. □

Lemma 5.7. *Let $L \geq 1$ and B_L^+ be chosen as in Lemma 5.6. Then there exists a $c > 0$ such that*

$$\left|\left\{\sum_{i=0}^{s-1} w_i \cdot \iota^-(\gamma^i) : w = (w_i) \in B_L^+\right\}\right| \geq c \cdot \mathsf{M}(h)^s \tag{5.22}$$

for every $s \geq 1$, where $|\cdot|$ denotes cardinality.

Proof. According to Lemma 5.6, $\phi(B_L^+) \subset K_{S^-}$ has nonempty interior and hence positive Haar measure $\lambda_{K_{S^-}}(\phi(B_L^+))$. The inclusion (5.20) implies that

$$\begin{aligned} Q &\subset \bigcup_{(j_0,\dots,j_{s-1})\in\{1,\dots,l\}^s} \sum_{i=0}^{s-1} M_\gamma^i(\iota^-(f^{(j_i)}(\gamma))) + M_\gamma^s Q \\ &\subset \bigcup_{w\in B_L^+} \sum_{i=0}^{s-1+2J} w_i \cdot \iota^-(\gamma^{i-J}) + M_\gamma^s Q \end{aligned}$$

for every $s \geq 1$. Since we need at least $\lambda_{K_{S^-}}(Q)/\lambda_{K_{S^-}}(M_\gamma^s Q) = \mathsf{M}(h)^s$ distinct translates of $M_\gamma^s Q$ to cover Q this implies that

$$\left|\left\{\sum_{i=0}^{s-1+2J} w_i \cdot \iota^-(\gamma^{i-J}) : w \in B_L^+\right\}\right| \geq \mathsf{M}(h)^s$$

for every $s \geq 1$, which proves (5.22). □

Lemma 5.8. *There exist an integer $L \geq 1$ and a $c > 0$ such that*

$$\left|\left\{\sum_{i=0}^{s-1} w_i \cdot \iota^-(\gamma^i) : w = (w_i) \in V_L^+\right\}\right| \geq c \cdot \mathsf{M}(h)^s \tag{5.23}$$

for every $s \geq 1$, where

$$V_L^+ = \{0, \dots, L-1\}^{\mathbb{N}}.$$

Proof. Let $L \geq 1$ be the integer appearing in Lemma 5.7. We set $L' = 2L+1$, $\bar{w} = (L, L, L, \dots)$ and $\bar{a} = \phi(\bar{w})$. Then $V_{L'}^+ = B_L^+ + \bar{w}$ and $\phi(V_{L'}^+) = \phi(B_L^+) + \bar{a}$. Equation (5.23) follows from (5.22) with L' replacing L. □

Proof of Theorem 5.3. For every quadruple of integers $s_1 \leq r_1 < r_2 \leq s_2$ we set

$$V^{(s_1,s_2)} = \bigcap_{\{f\in P:\, \mathsf{S}(h(\sigma)f)\subset\{s_1,\dots,s_2\}\}} (V_L \smallsetminus (V_L + h(\sigma)f)), \tag{5.24}$$

$$V_{(r_1,r_2)}^{(s_1,s_2)} = \pi_{\{r_1,\dots,r_2\}}(V^{(s_1,s_2)}) \subset \{0,\dots,L-1\}^{\{r_1,\dots,r_2\}},$$

$$N_{(r_1,r_2)}^{(s_1,s_2)} = \left|V_{(r_1,r_2)}^{(s_1,s_2)}\right|,$$

where $\mathsf{S}(h(\sigma)f)$ is the support of $h(\sigma)f$ (cf. (5.3)). Clearly,

$$\{w_{s_1}^{s_2}(\gamma) : w \in V^{(s_1,s_2)}\} = \{w_{s_1}^{s_2}(\gamma) : w \in V_L\},$$

and

$$
\begin{aligned}
N_{(s_1,s_2)}^{(s_1,s_2)} &= |\{w_{s_1}^{s_2}(\gamma) : w \in V^{(s_1,s_2)}\}| = |\{w_{s_1}^{s_2}(\gamma) : w \in V_L\}| \\
&= \left| \left\{ \sum_{i=s_1}^{s_2} w_i \cdot \iota^-(\gamma^{i-s_1})) : w \in V \right\} \right| \geq c \cdot \mathsf{M}(h)^{s_2-s_1+1}
\end{aligned}
$$

by (5.23). For $s_1' \leq s_1 \leq r_1 < r_2 \leq s_2 \leq s_2'$,

$$
V_{(r_1,r_2)}^{(s_1,s_2)} \supset V_{(r_1,r_2)}^{(s_1',s_2')} \text{ and hence } N_{(r_1,r_2)}^{(s_1,s_2)} \geq N_{(r_1,r_2)}^{(s_1',s_2')}. \tag{5.25}
$$

We fix $M \geq 1$. From (5.25) it is clear that

$$
N_{(kM,(k+1)M-1)}^{(-lM,(l+1)M-1)} \geq N_{((k+i)M,(k+i+1)M-1)}^{(-(l+j)M,(l+j+1)M-1)}
$$

for every $j \geq 1$, $i = -j, \dots, j$, $l \geq 1$ and $k = -l, \dots, l$. For fixed $l \geq 1$,

$$
\begin{aligned}
&c \cdot \mathsf{M}(h)^{(2l+2j+1)M} = c \cdot \mathsf{M}(\tilde{h})^{(2l+2j+1)M} \leq N_{(-(l+j)M,(l+j+1)M-1)}^{(-(l+j)M,(l+j+1)M-1)} \\
&\leq \prod_{k=-l-j}^{l+j} N_{(kM,(k+1)M-1)}^{(-(l+j)M,(l+j+1)M-1)} \\
&\leq \prod_{k=-l-j}^{-j-1} N_{(kM,(k+1)M-1)}^{(-(l+j)M,(l+j+1)M-1)} \cdot \prod_{k=-j}^{j} N_{(kM,(k+1)M-1)}^{(-(l+j)M,(l+j+1)M-1)} \\
&\qquad \cdot \prod_{k=j+1}^{l+j} N_{(kM,(k+1)M-1)}^{(-(l+j)M,(l+j+1)M-1)} \\
&\leq \left(\prod_{k=-l}^{-1} N_{(kM,(k+1)M-1)}^{(-lM,(l+1)M-1)} \right) \cdot \left[N_{(0,M-1)}^{(-lM,(l+1)M-1)} \right]^{2j+1} \\
&\qquad \cdot \left(\prod_{k=1}^{l} N_{(kM,(k+1)M-1)}^{(-lM,(l+1)M-1)} \right) \\
&= \left(\prod_{k=-l}^{l} N_{(kM,(k+1)M-1)}^{(-lM,(l+1)M-1)} \right) \cdot \left[N_{(0,M-1)}^{(-lM,(l+1)M-1)} \right]^{2j}
\end{aligned}
$$

for every $j \geq 1$, and by letting $j \to \infty$ we conclude that

$$
N_{(0,M-1)}^{(-lM,(l+1)M-1)} \geq \mathsf{M}(h)^M.
$$

As $l \to \infty$, $V_{(0,M-1)}^{(-lM,(l+1)M-1)}$ decreases to $\pi_{\{0,\dots,M-1\}}(V_L^{(h)})$, and hence

$$
\lim_{l \to \infty} N_{(0,M-1)}^{(-lM,(l+1)M-1)} = \left| \pi_{\{0,\dots,M-1\}}(V_L^{(h)}) \right| \geq \mathsf{M}(h)^M
$$

for every $M \geq 1$. By varying M we see that $h(\sigma_{V_L^{(h)}}) \geq \log \mathsf{M}(h)$, and the reverse inequality follows from Theorem 5.1. □

Remark 5.9. If the polynomial h is hyperbolic, the shift space $V_L^{(h)}$ in (5.3) is sofic for every $L \geq 1$ (cf. [2] and [14]). If h is irreducible and nonexpansive, the combinatorial structure of $V_L^{(h)}$ is not well understood.

6. Invariant measures on $X_h/X_h^{(0)}$

Remark 4.2 shows that we can always find symbolic covers of $Q^{(h)}$, and for hyperbolic h these covers can even be chosen to be finite-to-one. However, if h is nonhyperbolic, Corollary 4.3 raises a number of questions.

Problems 6.1. (1) Let $h \in R$ be irreducible and noncyclotomic. Is

$$\inf_V h(\sigma_V) = h(\alpha_h) = \log \mathsf{M}(h),$$

where the infimum is taken over all symbolic covers V of $Q^{(h)}$, and where $h(\sigma_V)$ is the topological entropy of the restriction of σ to V? For hyperbolic h the answer to this question is 'yes' (cf. Theorem 5.5).

(2) Does there always exist a symbolic cover V of $Q^{(h)}$ with $h(\sigma_V) = h(\alpha_h) = \log \mathsf{M}(h)$?

(3) Does there always exist a countable-to-one symbolic cover V of $Q^{(h)}$ (i.e. for which $\Delta^{(h)}(w) \cap V$ is countable for every $w \in V$)? A positive answer to this question would also solve (2).

(4) If there exist countable-to-one (or other nice) symbolic covers of $Q^{(h)}$, can one also find such covers with a relatively simple combinatorial structure? In the light of Section 7 the best one could probably hope for is covers which are factors of countable-state shifts of finite type with well-behaved factor maps (such as, for example, beta-shifts).

In the possible absence of good symbolic covers one can try to construct partial covers V of $Q^{(h)}$ which are 'large' in the sense that $h(\sigma_V) = h(\alpha_h) = \log \mathsf{M}(h)$, and which are 'small' in the sense that, for certain natural measures on V, the factor map $q^{(h)} \colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longrightarrow Q^{(h)}$ is countable-to-one *a.e.* As was shown in [9], these ideas can be used to construct invariant probability measures for irreducible nonexpansive group automorphisms.

Let $\mathsf{d} \colon \mathbb{Z} \times \ell^\infty(\mathbb{Z},\mathbb{Z}) \longrightarrow W_h^{(0)}$ be the cocycle describing the nonequivariance of $\bar{\xi}^*$ in (4.7)–(4.9).

Definition 6.2. A shift-invariant probability measure ν on $\ell^\infty(\mathbb{Z},\mathbb{Z})$ is *weakly* d*-bounded* if there exists, for every $\varepsilon > 0$, a compact subset $C_\varepsilon \subset W_h^{(0)}$ such that

$$\nu(\{v \in \ell^\infty(\mathbb{Z},\mathbb{Z}) : \mathsf{d}(k,v) \in C_\varepsilon\}) > 1 - \varepsilon \ \text{ for every } \ k \in \mathbb{Z}. \tag{6.1}$$

Theorem 6.3. *Let $h \in R_1$ be an irreducible nonhyperbolic polynomial which is not cyclotomic, α_h the ergodic and nonexpansive automorphism of the compact connected abelian group X_h defined in* (3.2)–(3.3)*, and let $\tilde{\sigma} \colon \tilde{Y} \longrightarrow \tilde{Y}$ be defined by* (4.10)–(4.11)*. For every σ-invariant probability measure ν on $\ell^\infty(\mathbb{Z},\mathbb{Z})$ the following conditions are equivalent.*

1. *ν is weakly d-bounded;*
2. *There exists a $\tilde{\sigma}$-invariant probability measure $\tilde{\nu}$ on $\tilde{Y}$ with $\tilde{\pi}_*\tilde{\nu} = \nu$, where $\tilde{\pi} \colon \tilde{Y} \longrightarrow \ell^\infty(\mathbb{Z},\mathbb{Z})$ is the first coordinate projection;*

(3) *There exists a Borel map* $\mathsf{b}\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longrightarrow W_h^{(0)}$ *with*

$$\mathsf{d}(1,v) = \mathsf{b}(\sigma v) - \bar{\sigma}\mathsf{b}(v) \text{ for } \nu\text{-a.e. } v \in \ell^\infty(\mathbb{Z},\mathbb{Z}). \tag{6.2}$$

If ν *satisfies these equivalent conditions, then the Borel maps* $\bar{\xi}^*_{\mathsf{b}}\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longrightarrow \ell^*(\mathbb{Z},\mathbb{R})$ *and* $\xi^*_{\mathsf{b}}\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longrightarrow X_h$*, defined by*

$$\bar{\xi}^*_{\mathsf{b}}(v) = \bar{\xi}^*(v) + \mathsf{b}(v) \text{ and } \xi^*_{\mathsf{b}}(v) = \xi^*(v) + \rho\circ\mathsf{b}(v) \tag{6.3}$$

for every $v \in \ell^\infty(\mathbb{Z},\mathbb{Z})$*, have the property that*

$$\begin{gathered} \xi^*_{\mathsf{b}}(v) - \xi^*(v) \in X_h^{(0)} \text{ for every } v \in \ell^\infty(\mathbb{Z},\mathbb{Z}), \\ \bar{\xi}^*_{\mathsf{b}}\circ\sigma = \bar{\sigma}\circ\bar{\xi}^*_{\mathsf{b}} \text{ and } \xi^*_{\mathsf{b}}\circ\sigma = \alpha_h\circ\xi^*_{\mathsf{b}} \ \nu\text{-a.e.,} \end{gathered} \tag{6.4}$$

and the probability measure

$$\mu = (\xi^*_{\mathsf{b}})_*\nu \tag{6.5}$$

on X_h *is* α_h*-invariant.*

Proof. This is [9, Theorem 4.13]. □

Theorem 6.4. *Let* $L \ge 1$*, and let* $V_L^{(h)} \subset V_L = \{0,\dots,L-1\}^{\mathbb{Z}} \subset \ell^\infty(\mathbb{Z},\mathbb{Z})$ *be defined by* (5.3)*. If* ν *a weakly* d*-bounded shift-invariant probability measure on* $V_L^{(h)}$*, and if* $\xi^*_{\mathsf{b}}\colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longrightarrow X_h$ *is the* ν*-a.e. equivariant map* (6.3)*, then the* α_h*-invariant probability measure* $\mu = (\xi^*_{\mathsf{b}})_*\nu$ *on* X_h *is singular with respect to Haar measure and satisfies that* $h_\nu(\sigma) = h_\mu(\alpha_h)$*.*

For the proof of Theorem 6.4 we need several lemmas. Let $\mathbf{R} = \Delta^{(h)} \cap (V_L^{(h)} \times V_L^{(h)})$ be the equivalence relation induced by $\Delta^{(h)}$ on $V_L^{(h)}$. Exactly as in the proof of Corollary 4.3 we see that $\mathbf{R}$ is a $\bar{\sigma}\times\bar{\sigma}$-invariant Borel set in $V_L^{(h)} \times V_L^{(h)}$.

Lemma 6.5. *Let* $Y \subset V_L^{(h)}$ *be a shift-invariant Borel set with* $\nu(Y) = 1$ *such that* (6.2) *holds for every* $v \in Y$*, and let*

$$\begin{gathered} Y(M) = \{y \in Y : \|\mathsf{b}(y)\|_\infty \le M\}, \\ L(M) = \{y \in \ell^*(\mathbb{Z},\mathbb{Z}) : \|y - \bar{\xi}^*\circ h(\sigma)(y)\|_\infty \le M\} \\ \mathbf{R}(M,w) = \bigl(w + h(\sigma)(L(M))\bigr) \cap V_L^{(h)} \subset \mathbf{R}(w) \end{gathered}$$

for every $M \ge 1$ *and* $w \in V_L^{(h)}$ *(cf.* (4.3)*). Then there exists a constant* $M_1 > 0$ *such that*

$$\begin{aligned} &\bigl|\pi_{\{0,\dots,n\}}\bigl(\mathbf{R}(K,w) \cap Y(M) \cap \sigma^{-n}(Y(M))\bigr)\bigr| \\ &\qquad\qquad \le (2M_1\beta + 4M + 2K + 1)^{2d+2}, \\ &\bigl|\pi_{\{-n,\dots,0\}}\bigl(\mathbf{R}(K,w) \cap Y(M) \cap \sigma^{n}(Y(M))\bigr)\bigr| \\ &\qquad\qquad \le (2M_1\beta + 4M + 2K + 1)^{2d+2}, \end{aligned} \tag{6.6}$$

for every $K, M \ge 1$*,* $w \in V_L^{(h)}$ *and* $n \ge 1$*.*

Proof. By (4.2) there exists a constant $M_1 > 0$ such that

$$\max_{j=0,\dots,d} |\bar{\xi}^*(w)_j| \le M_1 \cdot \|w\|_\infty$$

for every $w \in \ell^\infty(\mathbb{Z}, \mathbb{Z})$. As $(\bar{\sigma}^*)^n \circ \bar{\xi}^*(w) = \bar{\xi}^* \circ \sigma^n(w) + \mathsf{b}(\sigma^n w) - \bar{\sigma}^n \mathsf{b}(w)$ for every $n \in \mathbb{Z}$,

$$\max_{j=0,\dots,d} |\bar{\xi}^*(w)_{n+j}| \le M_1\beta + 2M$$

for every $M \ge 1$, $n \in \mathbb{Z}$ and $w \in Y(M) \cap \sigma^{-n}(Y(M))$. We fix $w \in V_L^{(h)}$ and obtain that, for every $v \in \mathbf{R}(K, w) \cap Y(M) \cap \sigma^{-n}(Y(M))$

$$\max_{j=0,\dots,d} |\bar{\xi}^*(v)_j| \le M_1\beta, \qquad \max_{j=0,\dots,d} |\bar{\xi}^*(v)_{n+j}| \le M_1\beta + 2M,$$

and that there exists a unique $y \in \ell^*(\mathbb{Z}, \mathbb{Z})$ with $v = w + h(\sigma)(y)$ and $\|y - \bar{\xi}^* \circ h(\sigma)(y)\|_\infty \le K$.

If v' is a second element in $\mathbf{R}(K, w) \cap Y(M) \cap \sigma^{-n}(Y(M))$ with $v' = w + h(\sigma)(y')$ for some $y' \in \ell^*(\mathbb{Z}, \mathbb{Z})$, then $\|y' - \bar{\xi}^* \circ h(\sigma)(y')\|_\infty \le K$, and hence

$$\max_{j=0,\dots,d} |y_j - y'_j| \le 2M_1\beta + 2K \quad\text{and}\quad \max_{j=0,\dots,d} |y_{n+j} - y'_{n+j}| \le 2M_1\beta + 4M + 2K.$$

The first inequality in (6.6) now follows from (5.15), and the proof of the second one is analogous. □

Lemma 6.6. *For ν-a.e. $w \in Y$, $\Delta^{(h)}(w) \cap Y$ is countable, and the map $\xi^*_{\mathsf{b}} \colon V_L^{(h)} \longrightarrow X_h$ is countable-to-one ν-a.e.*

Proof. In the notation of Lemma 6.5 we set $Y'(1) = Y(1)$ and $Y'(M) = Y(M) \smallsetminus Y(M-1)$ for every $M \ge 2$, and we define a map $q \colon Y \longrightarrow \mathbb{R}$ by setting $q(w) = 2^{-M}$ if $w \in Y'(M)$, $M \ge 1$. We fix an everywhere positive, Borel measurable and σ-invariant version $p = E_\nu(q|\mathcal{S}_Y^\sigma)$ of the conditional expectation of q, given the sigma-algebra $\mathcal{S}_Y^\sigma$ of σ-invariant Borel subsets of Y. After decreasing Y by a σ-invariant ν-null set, if necessary, we also assume that

$$\lim_{n\to\infty} \frac{1}{n} \cdot \sum_{j=1}^n q(\sigma^n w) = \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{j=1}^n q(\sigma^{-n} w) = p(w)$$

for every $w \in Y$.

We claim that

$$\sum_{v \in \mathbf{R}(K,w)\cap Y} q(v)p(v)^2 = \sup_{\substack{F \subset \mathbf{R}(K,w)\cap Y \\ F \text{ is finite}}} \sum_{v\in F} q(v)p(v)^2 < \infty \tag{6.7}$$

for every $w \in Y$ and $K \ge 1$.

Indeed, if $F \subset \mathbf{R}(K, w) \cap Y$ is finite, then

$$\sum_{v\in F} q(v)p(v)^2 = \lim_{n\to\infty} \sum_{v\in F} q(v) \cdot \frac{1}{n^2} \cdot \Biggl(\sum_{j=1}^n q(\sigma^{-j} v)\Biggr) \cdot \Biggl(\sum_{j'=1}^n q(\sigma^{j'} v)\Biggr)$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \cdot \sum_{j=1}^n \sum_{j'=1}^n \sum_{M\ge1} \sum_{M'\ge1} \sum_{M''\ge1} 2^{-M-M'-M''}$$

$$
\begin{aligned}
&\cdot \bigl|\pi_{\{-j,\dots,j'\}}\bigl(F\cap\sigma^{j}(Y'(M))\cap Y'(M'')\cap\sigma^{-j'}(Y'(M'))\bigr)\bigr|\\
&\le \lim_{n\to\infty}\frac{1}{n^2}\cdot\sum_{j=1}^{n}\sum_{j'=1}^{n}\sum_{M\ge 1}\sum_{M'\ge 1}\sum_{M''\ge 1}2^{-M-M'-M''}\\
&\qquad\cdot \bigl|\pi_{\{-j,\dots,0\}}\bigl(\mathbf{R}(K,w)\cap\sigma^{j}(Y'(M))\cap Y'(M'')\bigr)\bigr|\\
&\qquad\cdot \bigl|\pi_{\{0,\dots,j'\}}\bigl(\mathbf{R}(K,w)\cap\sigma^{-j'}(Y'(M'))\cap Y'(M'')\bigr)\bigr|\\
&\le \sum_{M\ge 1}\sum_{M'\ge 1}\sum_{M''\ge 1}2^{-M-M'-M''}\\
&\qquad\cdot \bigl[(2M_1\beta+2K+4\max\{M,M''\}+1)^{2d+2}\bigr]\\
&\qquad\cdot \bigl[(2M_1\beta+2K+4\max\{M',M''\}+1)^{2d+2}\bigr]<\infty
\end{aligned}
$$

by (5.15), which proves (6.7).

Since the maps $p,q\colon Y\longrightarrow\mathbb{R}$ are everywhere positive, (6.7) implies that the sets $\mathbf{R}(K,w)\cap Y$ and $\mathbf{R}(w)\cap Y=\bigcup_{K\ge 1}\mathbf{R}(K,w)\cap Y$ are countable for every $w\in Y$, and that the equivariant Borel map $\xi^*_{\mathsf{b}}\colon Y\longrightarrow X$ in (6.3) is therefore countable-to-one. □

Proof of Theorem 6.4. By Lemma 6.6 there exists a shift-invariant Borel set $Y\subset V_L^{(h)}$ with $\nu(Y)=1$ such that the Borel map $\xi^*_{\mathsf{b}}\colon Y\longrightarrow X_h$ in (6.3) is countable-to-one. Since countable-to-one factor maps do not decrease entropy, $h_\nu(\sigma)=h_\mu(\alpha_h)$. Furthermore, the Borel set $Z=\xi^*_{\mathsf{b}}(Y)\subset X_h$ is α_h-invariant with $\mu(Z)=1$ and intersects each coset of $X_h^{(0)}$ in a countable set. Hence $\lambda_{X_h}(Z)=0$, which proves that λ_{X_h} and μ are mutually singular. □

Corollary 6.7. *Let $L\ge 1$, and let $V_L^{(h)}\subset V_L=\{0,\dots,L-1\}^{\mathbb{Z}}\subset\ell^\infty(\mathbb{Z},\mathbb{Z})$ be defined by* (5.3). *If ν is a weakly* d*-bounded shift-invariant probability measure on $V_L^{(h)}$, then the map $\xi^{\#}\colon\ell^\infty(\mathbb{Z},\mathbb{Z})\longrightarrow X_h/X_h^{(0)}$ induced by the group homomorphism $\xi^*\colon\ell^\infty(\mathbb{Z},\mathbb{Z})\longrightarrow X_h$ has the following properties.*

(1) *The probability space $(X_h/X_h^{(0)},\mathcal{B}_{X_h/X_h^{(0)}},\xi^{\#}_*\nu)$ is standard, where $\mathcal{B}_{X_h/X_h^{(0)}}$ is the Borel field of $X_h/X_h^{(0)}$;*

(2) *If $\alpha_h^{\#}$ is the group automorphism of $X_h/X_h^{(0)}$ induced by α_h, then $\xi^{\#}_*\nu$ is $\alpha^{\#}$-invariant and $h_{\xi^{\#}_*\nu}(\alpha^{\#})=h_\nu(\sigma)$.*

Proof. By [8, Proposition 4.17] there exists a solution b' of (6.2) and an α_h-invariant Borel set $Z\subset X_h$ which intersects each coset of $X_h^{(0)}$ in at most one point, such that $(\xi^*_{\mathsf{b}'})_*\nu(Z)=1$. This implies all our assertions. □

Problem 6.8. The Theorems 5.3 and 6.4 raise the following question: is

$$
\sup_{\nu} h_\nu(V_L^{(h)})=h(\sigma_{V_L^{(h)}}),\tag{6.8}
$$

where the supremum in (6.8) is taken over all weakly bounded shift-invariant probability measures on $V_L^{(h)}$?

A shift-invariant probability measure ν on $V_L^{(h)}$ is *bounded* if there exists a compact subset $C \subset W_h^{(0)}$ such that $\mathsf{d}(n,w) \in C$ for every $n \in \mathbb{Z}$ and ν-a.e. $w \in V_L^{(h)}$. The following conditions can be shown to be equivalent.

(1) $\sup_{\nu \text{ bounded}} h_\nu(V_L^{(h)}) = h(\sigma_{V_L^{(h)}})$;
(2) $\sup_{N\ge 1} h(\sigma_{V_L^{(h)}\cap h(\sigma)(B_N)}) = h(\sigma_{V_L^{(h)}})$, where $B_N = \{-N,\dots,N\}^{\mathbb{Z}}$ for every $N \ge 0$.

If β is a Salem number with minimal polynomial $h \in R$, then $h = \tilde{h}$ (cf. (5.17)), and the positive answer to these equivalent questions follows from Proposition 7.2 below and [9, Theorem 7.1]. In the general case this question is still open.

7. Some examples

Example 7.1 (Beta-shifts)**.** Suppose that h has a single root $\gamma < 1$, that all other roots of h have absolute values ≥ 1, and that $h_0 = 1$ (i.e. that the inverse $\beta = \gamma^{-1}$ of γ is either an integer, a Pisot number or a Salem number). The S^- in (5.18) consists of a single real place.

Following [11] we consider the map

$$T_\beta x = \beta x \pmod 1 \tag{7.1}$$

from the unit interval $I = [0,1]$ to itself and define, for every $x \in I$, the *beta-expansion* $\omega_\beta(x) = (\omega_\beta(x)_n)$ of x by setting

$$\omega_\beta(x)_n = \beta T_\beta^n x - T_\beta^{n+1} x \tag{7.2}$$

for every $n \ge 0$. Note that $\omega_\beta(x)_n \in \{0,\dots,\lceil \beta - 1\rceil\}$ for every $n \ge 1$, where $\lceil \beta - 1\rceil$ is the smallest integer $\ge \beta - 1$, and that

$$x = \sum_{n\ge 0} \omega_\beta(x)_n \beta^{-n-1} \tag{7.3}$$

for every $x \in I$. We set

$$\omega_\beta^*(1) = \sup_{x\in[0,1)} \omega_\beta(x), \tag{7.4}$$

where the supremum ist taken with respect to the lexicographic order $\prec$ on $\ell^\infty(\mathbb{N},\mathbb{Z})$, and observe that

$$1 = \sum_{n\ge 0} \omega_\beta^*(1)_n \beta^{-n-1}. \tag{7.5}$$

Recall that $\sigma_+^k \omega_\beta^*(1) \ne (0,0,0,\dots)$ and

$$\sigma_+^k \omega_\beta^*(1) \preceq \omega_\beta^*(1) \tag{7.6}$$

for every $k \ge 1$ (cf. [11]). The restriction of σ_+ to the closed, shift-invariant set

$$V_\beta^+ = \{v \in \ell^\infty(\mathbb{N},\mathbb{N}) : \sigma_+^n v \preceq \omega_\beta^*(1) \text{ for every } n \ge 0\} \subset \{0,\dots,\lceil \beta - 1\rceil\}^{\mathbb{N}}$$

is called the *β-shift*. If we set

$$\eta_\beta(v) = \sum_{n\ge 0} v_n \beta^{-n-1} \tag{7.7}$$

for every $v \in V_\beta^+$, then $\eta_\beta\colon V_\beta^+ \longrightarrow [0,1]$ is surjective and at most two-to-one. Furthermore, if $v, v' \in V_\beta^+$ satisfy that $\eta_\beta(v) = \eta_\beta(v')$, and if $v \prec v'$, then there exists an integer $k \ge 0$ such that $v_n = v'_n$ for $n < k$, $v'_k = v_k + 1$, and $v'_n = 0$, $v_n = \omega^*_\beta(1)_{n-k-1}$ for $n > k$. Finally, if $v, v' \in V_\beta^+$, then $v \preceq v'$ if and only if $\eta_\beta(v) \le \eta_\beta(v')$ (cf. [11]).

In order to define the two-sided beta-shift space $V_\beta \subset \{0, \dots, \lceil \beta - 1\rceil\}^{\mathbb{Z}}$ we set $v^+ = (v_0, v_1, v_2, \dots) \in \ell^\infty(\mathbb{N}, \mathbb{Z})$ for every $v = (v_n) \in \ell^\infty(\mathbb{Z}, \mathbb{Z})$ and put

$$V_\beta = \{v \in \ell^\infty(\mathbb{Z}, \mathbb{Z}) : (\sigma^n v)^+ \in V_\beta^+ \text{ for every } n \in \mathbb{Z}\}. \tag{7.8}$$

From the description of the potential non-injectiveness of the map η_β in (7.7) it is clear that V_β intersects every equivalence class of $\Delta_1^{(h)}$ in at most one point (cf. (5.1)).

Proposition 7.2. *Suppose that the polynomial h in* (2.4) *satisfies that $h_0 = 1$, and that h has a root $\gamma < 1$ and all other roots of h have absolute values ≥ 1. Put $\beta = \gamma^{-1}$ and denote by $V_\beta \subset \{0, \dots, \lceil \beta - 1\rceil\}^{\mathbb{Z}}$ the two-sided beta-shift* (7.8)*. If $L > \beta$ then $V_L^{(h)} \supset V_\beta$ and $h(\sigma_{V_L^{(h)}}) = h(\sigma_{V_\beta}) = \log \beta$ (cf.* (5.3)*).*

Proof. In order to verify that $V_L^{(h)} \supset V_\beta$ we argue by contradiction and assume that there exists a $v \in V_\beta \smallsetminus V_L^{(h)}$. Then (5.3) and (5.13) show that there exists an $f \in P$ with $v \in V_L \cap (V_L - h(\sigma) f)$, i.e. that $v' = v + h(\sigma) f \in V_L$ for some $f \in P$. We set $n = \min \mathsf{S}(f)$ (cf. (2.2)) and assume without loss in generality that $n = 0$ (by shifting v and f, if necessary). According to (7.7)–(7.8), $\eta_\beta(v^+) = \eta_\beta(v'^+)$. As $v'_0 \ge v_0 + 1$, we conclude that $1 \le v'_0 - v_0 + \sum_{n\ge 0} v'_n \beta^{-n} = \sum_{n \ge 1} v_n \beta^{-n}$. Since $v \in V_\beta$ it follows that $v'_0 = v_0 + 1$, and that $v_n = \omega^*_\beta(1)_{n-1}$ and $v'_n = 0$ for every $n \ge 1$. This is clearly impossible, since v and v' differ in only finitely many coordinates.

The last identity follows from Theorem 5.3, since $h(\sigma_{V_\beta}) = \log \beta$. □

Take, for example, the polynomial $h(u) = 1 - u - u^2$ with roots $\gamma = \frac{2}{1+\sqrt{5}} < 1$ and $\gamma' = -1/\gamma$. If $V_2 = \{0,1\}^{\mathbb{Z}}$ (cf. (5.4)), then Proposition 7.2 shows that $V_2^{(h)} \supset V_\beta$, where $\beta = 1/\gamma = \frac{1+\sqrt{5}}{2}$, and $h(\sigma_{V_2^{(h)}}) = h(\sigma_{V_\beta}) = \log \beta$. One can check that every point $w \in V_2^{(h)} \smallsetminus V_\beta$ is either of the form

$$w_k = 1 \text{ for every } k \in \mathbb{Z},$$

or that there exists an $l \in \mathbb{Z}$ with

$$w_k = 1 \text{ for every } k < l,\ w_l = 0 \text{ and } (w_{l+1}, w_{l+2}, w_{l+3}, \dots) \in V_\beta^+.$$

We also have that $V_3^{(h)} \supset V_\beta$ and $h(\sigma_{V_3^{(h)}}) = \log \beta$, but neither of the spaces $V_2^{(h)}$ and $V_3^{(h)}$ contains the other: $(\dots,1,1,1,\dots) \in V_2^{(h)} \smallsetminus V_3^{(h)}$, whereas $(\dots,2,2,2,\dots) \in V_3^{(h)} \smallsetminus V_2^{(h)}$.

In this example, V_β is a shift of finite type and the spaces $V_L^{(h)}$, $L \geq 2$, are sofic by Remark 5.9.

Example 7.3 (The polynomial $h(u) = 5 - 6u + 5u^2$)**.** The roots of h are of the form $\gamma = \frac{3}{5} + i \cdot \frac{4}{5}$, $\bar{\gamma} = \frac{3}{5} - i \cdot \frac{4}{5}$ with absolute values equal to 1, and the set S in (5.8) consists of a single infinite complex place v_∞ (corresponding to γ and $\bar{\gamma}$) and two copies of the finite place 5 with $K_5 = \mathbb{Q}_5$, the 5-adic rationals (note that h has two roots $\gamma_1, \gamma_2 \in \mathbb{Q}_5$ with $|\gamma_1|_5 = 1/5$ and $|\gamma_2|_5 = 5$, where $|\cdot|_5$ is the 5-adic valuation). We write v_5 for the place of $\mathbb{Q}(\gamma)$ corresponding to γ_1 and obtain that $S^- = \{v_5\}$ and $K_{S^-} = \mathbb{Q}_5$ (cf. (5.18)).

Since every $t \in \mathbb{Z}_5 = \bar{R}_{v_5}$ can be expressed uniquely as

$$t = \sum_{n \geq 0} a_n \gamma_1^n$$

with $a_n \in \{0,1,2,3,4\}$ for every $n \geq 0$, we have that $V_5^{(h)} = V_5$, and the proofs of Lemma 5.7 and Theorem 5.3 show that $h(\sigma_{V_5^{(h)}}) = \log \mathsf{M}(h) = \log 5$ (cf. (2.6)). More generally, if $L \geq 5$, and if $V_L = \{0,\dots,L-1\}^{\mathbb{Z}}$, then the same argument as in Proposition 7.2 shows that $V_L^{(h)} \supset V_5^{(h)}$, and Theorem 5.1 guarantees that $h(\sigma_{V_L^{(h)}}) = h(\sigma_{V_5^{(h)}}) = \log 5$.

Example 7.4 (Reversing polynomials)**.** Let $h \in R$ be of the form (2.4), and let $g = \operatorname{sgn}(h_d)\tilde{h}$, where sgn stands for sign (cf. (5.17)). Then $\mathsf{M}(h) = \mathsf{M}(g)$ by (2.6), but the spaces $V_L^{(h)}$ and $V_L^{(g)}$ may not be reversals of each other (due to the possible sign-change involved in the definition of g).

For example, if $h(u) = 1 - u - u^2$ is the polynomial appearing at the end of Example 7.1, then $g(u) = 1 + u - u^2$, and $V_2^{(g)}$ is the set of all sequences in $V_2^{(h)}$, reversed and with zeros and ones interchanged.

Similarly, if $h(u) = 1 - u^2 - u^3$, then h has a single small root $\gamma = 0.75487\cdots < 1$ and two complex conjugate roots with absolute values > 1. If $\beta = \gamma^{-1} = 1.32472\cdots$, then $V_\beta \subset V_2$

The Examples 7.1 and 7.3–7.4 had the property that $|S^-| = 1$. Here is an example with $|S^-| \geq 2$ and $|S \smallsetminus S^-| \geq 2$.

Example 7.5 (The polynomial $h(u) = 1 - u^2 - u^4$)**.** The two roots of h of absolute value < 1 are given by $\gamma = \pm\sqrt{\frac{2}{1+\sqrt{5}}}$, and S^- consists of the two places corresponding to these roots.

Let $\beta = \frac{1+\sqrt{5}}{2}$, and let V_β be the corresponding two-sided beta-shift space consisting of all sequences $(v_n) \in \{0,1\}^{\mathbb{Z}}$ with $v_n v_{n+1} = 0$ for every $n \in \mathbb{Z}$. One can check as in Example 7.1 that $Y \subset V_2^{(h)}$, where Y is the shift of finite type determined by the condition that $y_n y_{n+2} = 0$ for every $n \in \mathbb{Z}$.

Note that Y consists of two interspersed copies of V_β, and that $h(\sigma_Y) = h(\sigma_{V_\beta}) = \log \beta = \mathsf{M}(h) = h(\sigma_{V_2^{(h)}})$.

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