

Problem Set 4  
Due Friday, October 26.

*Algebra*

Math 110A, Fall Quarter 2012

1. Do problems 3.2.2, 3.2.8, 3.2.12, 3.2.23, 3.2.38 in the textbook.
2. Do problems 3.3.9, 3.3.10, 3.3.14, 3.3.19, 3.3.33, 3.3.39 in the textbook.
3. Prove that the Binomial Theorem holds in any commutative ring  $R$  with identity: if  $n \geq 1$  and  $a, b \in R$ , then

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k.$$

Here we set  $a^0 := 1$  for any  $a \in R$ , and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = n(n-1)\cdots(n-k+1) \quad \text{for } 0 \leq k \leq n.$$

Also, for a positive integer  $n$  and  $a \in R$ ,  $na$  denotes the element  $a + a + \cdots + a$  ( $n$  many  $a$ 's) of  $R$ .

Hint: you may use that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \quad \text{for all } 1 \leq k \leq n.$$

4. Let  $R$  be a commutative ring with identity. An element  $a$  of  $R$  is called **nilpotent** if  $a^n = 0$  for some  $n \geq 1$ .
  - (a) Determine the nilpotent elements of  $\mathbb{Z}$ .
  - (b) Determine the nilpotent elements of  $\mathbb{Z}_{12}$ .
  - (c) Let  $a, b, c \in R$  where  $a$  and  $b$  are nilpotent. Show that then  $a + b$  and  $ac$  are nilpotent. (Hint: you may use Problem 3.)