Problem Set 5
Due Friday, November 3.

Algebra

Math 110A, Fall Quarter 2017

1. Do problems 3.3.9, 3.3.11, 3.3.14, 3.3.19, 3.3.33, 3.3.38 in the textbook.
2. Prove that the Binomial Theorem holds in any commutative ring R with identity: if $n \geq 1$ and $a, b \in R$, then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Here we set $a^0 := 1$ for any $a \in R$, and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = n(n-1)\cdots(n-k+1) \quad \text{for } 0 \leq k \leq n.$$

Also, recall that for a natural number n and $a \in R$, na denotes the element $a + a + \cdots + a$ (n many a's) of R.

Hint: you may use that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \qquad \text{for all } 1 \leq k \leq n.$$

3. Let R be a commutative ring with identity. An element a of R is called **nilpotent** if $a^n = 0$ for some $n \geq 1$.
   (a) Determine the nilpotent elements of $\mathbb{Z}$.
   (b) Determine the nilpotent elements of $\mathbb{Z}_{12}$.
   (c) Let $a, b, c \in R$ where a and b are nilpotent. Show that then $a + b$ and ac are nilpotent. (Hint: you may use Problem 2.)