Problem Set 4
Solutions

Mathematical Logic

Math 114L, Spring Quarter 2008

1. The countries are $C_1, C_2, \ldots$. We can use $\mathbf{A}_1$ to say that C_1 is red, $\mathbf{A}_2$ to say that C_1 is green, $\mathbf{A}_3$ to say that C_1 is blue, and $\mathbf{A}_4$ to say that C_1 is yellow. And then we can use $\mathbf{A}_5$–$\mathbf{A}_8$ to describe similarly the color of C_2, and so forth.

   Let's change the notation to something easier to read. Write $\mathbf{A}_{4i-3}$ as $\mathbf{R}_i$; use it to say C_i is red. Write $\mathbf{A}_{4i-2}$ as $\mathbf{G}_i$; use it to say C_i is green. Write $\mathbf{A}_{4i-1}$ as $\mathbf{B}_i$; use it to say C_i is blue. Write $\mathbf{A}_{4i}$ as $\mathbf{Y}_i$; use it to say C_i is yellow. For example, to say that country C_7 is red, we use $\mathbf{R}_7$, which is the same as $\mathbf{A}_{25}$. And the formula

   $$(\mathbf{R}_7 \wedge \neg\, \mathbf{G}_7 \wedge \neg\, \mathbf{B}_7 \wedge \neg\, \mathbf{Y}_7)$$

   says that C_7 is red and nothing but red.

   Then let Σ_1 consist of the following sentences, for $i = 1, 2, \ldots$:

   $$(\mathbf{R}_i \wedge \neg\, \mathbf{G}_i \wedge \neg\, \mathbf{B}_i \wedge \neg\, \mathbf{Y}_i) \vee (\neg\, \mathbf{R}_i \wedge \mathbf{G}_i \wedge \neg\, \mathbf{B}_i \wedge \neg\, \mathbf{Y}_i) \vee$$
   $$(\neg\, \mathbf{R}_i \wedge \neg\, \mathbf{G}_i \wedge \mathbf{B}_i \wedge \neg\, \mathbf{Y}_i) \vee (\neg\, \mathbf{R}_i \wedge \neg\, \mathbf{G}_i \wedge \neg\, \mathbf{B}_i \wedge \mathbf{Y}_i)$$

   These sentences say that each country C_i has exactly one color.

   Next, for a pair C_i, C_j of adjacent countries, we use the formula

   $$\neg(\mathbf{R}_i \wedge \mathbf{R}_j) \wedge \neg(\mathbf{B}_i \wedge \mathbf{B}_j) \wedge \neg(\mathbf{G}_i \wedge \mathbf{G}_j) \wedge \neg(\mathbf{Y}_i \wedge \mathbf{Y}_j)$$

   to say that C_i and C_j are not the same color. Let Σ_2 be the set of all such sentences, for each pair C_i, C_j of adjacent countries.

   Together, the formulas in the union $\Sigma_1 \cup \Sigma_2$ say that each country has exactly one color, and adjacent countries have different colors. Any truth assignment v that satisfies $\Sigma_1 \cup \Sigma_2$ gives us a proper coloring of the infinite map; we just do what v tells us: if $v(\mathbf{G}_7) = T$, then we color C_7 green. But is there any such v? That is, is the set $\Sigma_1 \cup \Sigma_2$ satisfiable?

   To show this fact, we use compactness and the four-color theorem. By compactness, it suffices to show that every finite subset of $\Sigma_1 \cup \Sigma_2$ is satisfiable. So consider an arbitrary finite subset. The sentence symbols in that subset refer to only finitely many countries; say C_M is mentioned somewhere in the subset but not C_i for any $i > M$.

   By the four-color theorem, there is a proper coloring of the finite map consisting of countries $C_1, C_2, \ldots, C_M$. From that coloring, make a truth assignment u. For example, if C_7 is blue, then $u(\mathbf{B}_7) = T$ and $u(\mathbf{R}_7) = u(\mathbf{G}_7) = u(\mathbf{Y}_7) = F$. The truth assignment u satisfies the finite subset.

2. (Thanks to all the people who alerted me to the typo in the statement of the problem!) We compute

$$\text{free}\left((= v_1 v_2 \wedge \forall v_1 (P v_1 v_2 \to P v_2 v_3))\right)$$

by following the recursive definition of free$(\cdots)$. But first we convert the given expression into a "legal" wff α by unwinding the abbreviations:

$$\alpha = (\neg(= v_1 v_2 \to (\neg \forall v_1 (P v_1 v_2 \to P v_2 v_3)))).$$

Now

$$\begin{aligned}
\text{free}(\alpha) &= \text{free}\left((= v_1 v_2 \to (\neg \forall v_1 (P v_1 v_2 \to P v_2 v_3)))\right) \\
&= \text{free}(= v_1 v_2) \cup \text{free}\left((\neg \forall v_1 (P v_1 v_2 \to P v_2 v_3))\right) \\
&= \{v_1, v_2\} \cup \text{free}\left(\forall v_1 (P v_1 v_2 \to P v_2 v_3)\right) \\
&= \{v_1, v_2\} \cup \text{free}\left((P v_1 v_2 \to P v_2 v_3)\right) \setminus \{v_1\} \\
&= \{v_1, v_2\} \cup \left(\text{free}(P v_1 v_2) \cup \text{free}(P v_2 v_3)\right) \setminus \{v_1\} \\
&= \{v_1, v_2\} \cup \left(\{v_1, v_2\} \cup \{v_2, v_3\}\right) \setminus \{v_1\} = \{v_1, v_2, v_3\}.
\end{aligned}$$

3. (a) "Zero is less than any number." $\forall x (Nx \to <\mathbf{0}x)$ But the translation of "Zero is less than any *other* number" is different.

   (b) "If any number is interesting, then zero is interesting." Probably this means "If *every* number is interesting, then zero is interesting." But conceivably it is like "If there's any man alive who can get the message through to Garcia, this man can do it." In the former case, we get $(\forall x (Nx \to Ix) \to I\mathbf{0})$. In the latter case, we get as a first approximation, $\exists x (Nx \wedge Ix) \to I\mathbf{0}$. Cleaned up, this becomes $((\neg \forall x (Nx \to (\neg Ix))) \to I\mathbf{0})$. This can be rewritten in a variety of ways. But the two cases are *not* equivalent.

   (c) "No number is less than zero." As a first approximation we obtain $\neg \exists x (Nx \wedge x < \mathbf{0})$. A legal version of this is $\forall x (Nx \to (\neg <x\mathbf{0}))$.

   (d) "Any uninteresting number with the property that all smaller numbers are interesting certainly is interesting."

   $$\forall x (Nx \to ((\neg Ix) \to (\forall y (Ny \to (<yx \to Iy)) \to Ix)))$$

   (e) "There is no number such all numbers are less than it." (As in (a), the speaker seems to have forgotten the word 'other.') A first approximation: $\neg \exists x (Nx \wedge \forall y (Ny \to y < x))$. A legal version:

   $$\forall x (Nx \to (\neg \forall y (Ny \to <yx)))$$

   (f) "There is no number such that no number is less than it." A first approximation is the sentence: $\neg \exists x (Nx \wedge \neg \exists y (Ny \wedge y < x))$. A legal version of this: $\forall x (Nx \to (\neg \forall y (Ny \to (\neg <yx))))$

4. "It is not the case that a is a member of every set, and it is also not the case that b is a member of every set:" $\neg(\forall x(a \in x) \vee \forall x(b \in x))$

5. (a) "You can fool some of the people all of the time." I take this to mean that there is at least one person who is so gullible that he or she can always be fooled (but other readings might be possible):

$$\exists x(Px \wedge \forall y(Ty \rightarrow Fxy))$$

   (b) "You can fool all of the people some of the time." Two inequivalent translations are

$$\exists y(Ty \wedge \forall x(Px \rightarrow Fxy)) \qquad \text{and} \qquad \forall x(Px \rightarrow \exists y(Ty \wedge Fxy)).$$

   One of the advantages of a precise formal language is that it clarifies the different possible readings of an English sentence.

   (c) "You can't fool all the people all of the time."

$$\neg \forall x \forall y(Px \wedge Ty \rightarrow Fxy)$$

6. Slightly rewritten, the sentences are these:

   (a) $\forall x \forall y \forall z(Pxy \wedge Pyz \rightarrow Pxz)$ (transitivity)
   (b) $\forall x \forall y(Pxy \wedge Pyx \rightarrow x = y)$ (antisymmetry)
   (c) $\forall x \exists y Pxy \rightarrow \exists y \forall x Pxy$

   I will use *finite* structures; in this particular case it is possible to do so.

   (a) Let $\mathfrak{A}$ be the structure $(\{0,1,2\}; \{\langle 0,1\rangle, \langle 1,2\rangle\})$. That is, $|\mathfrak{A}|$ is $\{0,1,2\}$ and to P we assign the binary relation $P^{\mathfrak{A}} = \{\langle 0,1\rangle, \langle 1,2\rangle\}$. (In the notation I used on the blackboard in class, we would denote $\mathfrak{A}$ by $\underline{A}$, and $|\mathfrak{A}|$ simply by A.) This structure can be pictured as the directed graph:

$$0 \rightarrow 1 \rightarrow 2$$

   Then (a) is false in $\mathfrak{A}$, (b) is vacuously true in $\mathfrak{A}$, and (c) is true (because $\forall x \exists y Pxy$ is false) in $\mathfrak{A}$. Hence $\{(\text{b}), (\text{c})\} \not\models (\text{a})$.

   (b) Take $\mathfrak{B}$ to be the two-element structure with $|\mathfrak{B}| = \{0,1\}$ and for the binary relation $P^{\mathfrak{B}}$ take the entire Cartesian product $\{0,1\} \times \{0,1\} = \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}$. Then (a) is true in $\mathfrak{B}$ and (c) is true in $\mathfrak{B}$ (because $\exists y \forall x Pxy$ is true in $\mathfrak{B}$). But (b) is false in $\mathfrak{B}$.

   (c) Take $\mathfrak{C}$ to be the two-element structure $(\{0,1\}; =)$. That is, $|\mathfrak{C}| = \{0,1\}$ and the binary relation is $\{\langle 0,0\rangle, \langle 1,1\rangle\}$. Then (a) and (b) are true in $\mathfrak{C}$ but (c) is false in $\mathfrak{C}$. (Everything is equal to something, but there is nothing that equals everything.)

7. One possibility for a first-order language appropriate for talking about vector spaces over the rational numbers has a constant symbol 0 (for the zero vector), a 2-place function symbol $+$ (for vector addition), and for each $q \in \mathbb{Q}$ a 1-place function symbol μ_q (for scalar multiplication by q).