Problem Set 6
Solutions

Mathematical Logic

Math 114L, Spring Quarter 2008

1. We will prove the stronger fact that if the language is finite (i.e., has finitely many parameters), then for a finite structure $\mathfrak{A}$, we can give a single sentence α that characterizes $\mathfrak{A}$ up to isomorphism. Assume that A has n elements, and let $A = \{a_1, \ldots, a_n\}$. Then α is $\exists\, \boldsymbol{v}_1 \cdots \exists\, \boldsymbol{v}_n \theta$, where θ is the conjunction of the following formulas:

   (1) $\boldsymbol{v}_i \neq \boldsymbol{v}_j$ for each $i < j$

   (2) $\forall\, \boldsymbol{v}_{n+1} \bigvee_{i \leq n} \boldsymbol{v}_{n+1} = \boldsymbol{v}_i$

   (3) $P\boldsymbol{v}_{i_1} \cdots \boldsymbol{v}_{i_k}$ for each k-place predicate parameter P and each k-tuple $\langle a_{i_1}, \ldots, a_{i_k} \rangle$ in $P^{\mathfrak{A}}$

   (4) $\neg\, P\boldsymbol{v}_{i_1} \cdots \boldsymbol{v}_{i_k}$ for each k-place predicate parameter P and each k-tuple $\langle a_{i_1}, \ldots, a_{i_k} \rangle$ *not* in $P^{\mathfrak{A}}$

   (5) $F\boldsymbol{v}_{i_1} \cdots \boldsymbol{v}_{i_k} = \boldsymbol{v}_j$ for each k-place function symbol F and each value $F^{\mathfrak{A}}(a_{i_1}, \ldots, a_{i_k}) = a_j$ of $F^{\mathfrak{A}}$.

   Clearly, α is true in $\mathfrak{A}$, and hence in any isomorph of $\mathfrak{A}$. Conversely, suppose $\mathfrak{B}$ is any model of α. Then for some $b_1, \ldots b_n$ in B we have $\mathfrak{B} \models \theta[\![b_1, \ldots, b_n]\!]$. Then the map $a_i \mapsto b_i$ is a one-to-one (by (1)) map of A onto (by (2)) B that preserves the relations (by (3) and (4)) and functions (by (5)). So $\mathfrak{A} \cong \mathfrak{B}$. For example,

   $$(a_i, a_j) \in P^{\mathfrak{A}} \;\Rightarrow\; P\boldsymbol{v}_i\boldsymbol{v}_j \text{ is in } \theta \;\Rightarrow\; \mathfrak{B} \models P\boldsymbol{v}_i\boldsymbol{v}_j[\![\vec{b}]\!] \;\Rightarrow\; (b_i, b_j) \in P^{\mathfrak{B}}$$

   and

   $$(a_i, a_j) \notin P^{\mathfrak{A}} \;\Rightarrow\; \neg\, P\boldsymbol{v}_i\boldsymbol{v}_j \text{ is in } \theta \;\Rightarrow\; \mathfrak{B} \not\models P\boldsymbol{v}_i\boldsymbol{v}_j[\![\vec{b}]\!] \;\Rightarrow\; (b_i, b_j) \notin P^{\mathfrak{B}}.$$

2. Here are the axioms for vector spaces over $\mathbb{Q}$, formulated as sentences (written slightly informally):

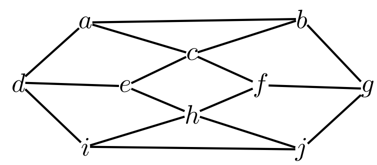
   $$\forall x \forall y \forall z((x + y) + z = x + (y + z))$$
   $$\forall x \forall y(x + y = y + x)$$
   $$\forall x(x + 0 = x)$$
   $$\forall x \exists y(x + y = 0)$$
   $$\forall x(\mu_r(\mu_s x) = \mu_{r \cdot s} x) \qquad \text{for each } r, s \in \mathbb{Q}$$
   $$\forall x(\mu_1 x = x)$$
   $$\forall x \forall y(\mu_r(x + y) = \mu_r x + \mu_r y) \qquad \text{for each } r \in \mathbb{Q}$$
   $$\forall x(\mu_{r+s} x = \mu_r x + \mu_s x) \qquad \text{for each } r, s \in \mathbb{Q}.$$

(Note that this is an *infinite* list of axioms. Also, there is nothing special about $\mathbb{Q}$: we could have used any coefficient field K as well and instead specified the axioms for vector spaces over K in a similar way.)

3. Here is a formula which expresses that R is the graph of a 1-place function:

$$\forall x \forall y_1 \forall y_2 (Rxy_1 \wedge Rxy_2 \rightarrow y_1 = y_2)$$

4. (a) We label the vertices of the given graph as follows:



This graph construed as a first-order structure is $\mathcal{G} = (G, R^{\mathcal{G}})$ where

$$G = \{a, b, c, d, e, f, g, h, i, j\}$$

and

$$\begin{aligned} R^{\mathcal{G}} = \big\{&(a,b),(b,a),(b,g),(g,b),(g,j),(j,g),\\ &(i,j),(j,i),(d,i),(i,d),(d,a),(a,d),\\ &(a,c),(c,a),(b,c),(c,b),(d,e),(e,d),\\ &(f,g),(g,f),(j,h),(h,j),(i,h),(h,i),\\ &(e,h),(h,e),(e,c),(c,e),(c,f),(f,c),(f,h),(h,f)\big\}. \end{aligned}$$

(b) Take an assignment s in $\mathcal{G}$ where $s(x) = c$, $s(y_1) = a$, $s(y_2) = b$, $s(y_3) = e$, $s(y_4) = f$. Then $\mathcal{G} \not\models \varphi[s]$.

(c) For (i) we can use the sentence

$$\varphi_1 = \forall x \exists y \exists z (Rxy \wedge Rxz \wedge \neg y = z).$$

We have $\mathfrak{A} \models \varphi_1$, but $\mathfrak{B} \models \neg\varphi_1$, so $\mathfrak{A}$ and $\mathfrak{B}$ cannot be isomorphic. For (ii) we can use the sentence

$$\varphi_2 = \exists x_1 \exists x_2 \exists x_3 \exists x_4 \left(\bigwedge_{1 \leq i < j \leq 4} \neg x_i = x_j \right)$$

to distinguish $\mathfrak{A}$ and $\mathfrak{B}$. Finally, for (iii) the sentence

$$\varphi_3 = \forall v \exists x \exists y \exists z \,(Rvx \wedge Rxy \wedge Ryz \wedge \neg v = z)$$

can be used.

(d) The two graphs are not isomorphic: In the first graph, there are exactly 2 vertices which have 4 edges adjacent to it, whereas in the second graph, there are 6 such vertices. The property of a graph having exactly 2 vertices with 4 adjacent edges can be expressed using a first-order sentence; hence its truth value must be preserved under isomorphism.

5. Assume for a contradiction that $\mathfrak{A} \equiv \mathfrak{B}$ but $\mathfrak{A} \not\cong \mathfrak{B}$. Hence no bijection $A \to B$ is an isomorphism $\mathfrak{A} \cong \mathfrak{B}$. By the definition of isomorphism, the failure of a bijection $\pi\colon A \to B$ being an isomorphism is witnessed by finitely many of the relation symbols, function symbols, and constant symbols in our language: π is *not* an isomorphism if and only if there exists a relation symbol R in our language and $a_1, \ldots, a_n \in A$ with $R^{\mathfrak{A}}(a_1, \ldots, a_n)$, but not $R^{\mathfrak{B}}(\pi(a_1), \ldots, \pi(a_n))$, or if there exists a function symbol f and $a_1, \ldots, a_n \in A$ with $\pi(f^{\mathfrak{A}}(a_1, \ldots, a_n)) \neq f^{\mathfrak{B}}(\pi(a_1), \ldots, \pi(a_n))$, or a constant symbol c with $\pi(c^{\mathfrak{A}}) \neq c^{\mathfrak{B}}$. Hence for any bijection $\pi\colon A \to B$ there exists a finite subset S of our parameters such that π is not an isomorphism of the structures $\mathfrak{A}|S$ and $\mathfrak{B}|S$. Here $\mathfrak{A}|S$ denotes the reduction of $\mathfrak{A}$ to S, i.e., the structure in the language with parameter set S which has the same universe A as $\mathfrak{A}$ and the same interpretations of the symbols in S as $\mathfrak{A}$. Note that this implies that for any finite subset S' of our parameters which contains S, π will also not be an isomorphism of the S'-structures $\mathfrak{A}|S'$ and $\mathfrak{B}|S'$. Since there are only finitely many bijections $A \to B$ (since A and B are finite sets), this means that there is some finite subset S' of S such that $\mathfrak{A}|S'$ and $\mathfrak{B}|S'$ are *not* isomorphic. But since $\mathfrak{A} \equiv \mathfrak{B}$, we clearly also have $\mathfrak{A}|S' \equiv \mathfrak{B}|S'$. So by the case of a finite language (Problem 1), we get $\mathfrak{A}|S' \cong \mathfrak{B}|S'$, a contradiction.