1. Calculate the limits, showing your work. (Do not use L'Hospital's Rule.)

[10 pts] (a) $\lim_{x\to 0} \frac{1}{2x} - \frac{1}{x(x+2)}$

[10 pts] (b) $\lim_{t\to 0} \frac{\cos t - 1}{\sin t}$

(a) $\lim_{x\to 0} \frac{1}{2x} - \frac{1}{x(x+2)} = \lim_{x\to 0} \frac{x(x+2) - 2x}{2x^2(x+2)}$

$= \lim_{x\to 0} \frac{\cancel{x^2}}{2\cancel{x^2}(x+2)} = \frac{1}{4}$

(b) $\lim_{t\to 0} \frac{\cos t - 1}{\sin t} \frac{\frac{1}{t}}{\frac{1}{t}} = \lim_{t\to 0} \frac{\frac{\cos t - 1}{t}}{\frac{\sin t}{t}}$

$= \frac{\lim_{t\to 0} \frac{\cos t - 1}{t}}{\lim_{t\to 0} \frac{\sin t}{t}} = \frac{0}{1} = 0$

2. Find $f'(2)$ for

$$f(x) = \int_1^{1/x} \sqrt{t^3+1}\,dt.$$

Let $A(x) = \int_1^x \sqrt{t^3+1}\,dt$ then $A'(x) = \sqrt{x^3+1}$

$f(x) = A(1/x)$ so

$f'(x) = A'(1/x)(-x^{-2}) = \sqrt{(1/x)^3+1}\,(-1/x^2)$

$f'(2) = \sqrt{(1/2)^3+1}\,(-1/4)$

3. Calculate dy/dx.

[10 pts] (a) $y = \sec(\sqrt{x^3 + 1})$

[10 pts] (b) $y^2 + \dfrac{x}{y} = 1$

(a) $\dfrac{dy}{dx} = \sec(\sqrt{x^3+1})\tan(\sqrt{x^3+1})\left(\tfrac{1}{2}(x^3+1)^{-\frac{1}{2}}\right)(3x^2)$

(b) $y^2 + xy^{-1} = 1$

$$2y\frac{dy}{dx} + y^{-1} + x(-y^{-2})\frac{dy}{dx} = 0$$

$$(2y - xy^{-2})\frac{dy}{dx} = -y^{-1}$$

$$\frac{dy}{dx} = -\frac{y^{-1}}{2y - xy^{-2}}$$

4. Let $f(x) = |x^2 - 1|$.

[pts] (a) Calculate

$$\int_0^3 f(x)\, dx.$$

[pts] (b) Find the (absolute) maximum and minimum of $f(x)$ on the interval $[-1/2, 2]$, showing your work.

$$f(x) = \begin{cases} 1 - x^2 & -1 \le x \le 1 \\ x^2 - 1 & x \le -1,\ x \ge 1 \end{cases}$$

(a) $\int_0^1 1 - x^2\, dx + \int_1^3 x^2 - 1\, dx = x - \frac{1}{3}x^3 \Big|_0^1 + \frac{1}{3}x^3 - x \Big|_1^3$

$= (1 - \frac{1}{3}) + ((9 - 3) - (\frac{1}{3} - 1))$

(b) $\frac{d}{dx}(1 - x^2) = -2x$, $\frac{d}{dx}(x^2 - 1) = 2x$ so no derivative at $x = 1$, $f'(0) = 0$ so $0, 1$ are critical points

$f(-\frac{1}{2}) = 3/4$, $f(0) = 1$, $f(1) = 0$, $f(2) = 3$
min max

5. Use the **definition** of the derivative to show that if

$$f(x) = \sqrt{2x+3}$$

then $f'(3) = 1/3$.

$$f'(3) = \lim_{x \to 3} \frac{\sqrt{2x+3} - 3}{x-3} \left(\frac{\sqrt{2x+3} + 3}{\sqrt{2x+3} + 3} \right)$$

$$= \lim_{x \to 3} \frac{(2x+3) - 9}{(x-3)(\sqrt{2x+3} + 3)}$$

$$= \lim_{x \to 3} \frac{2\cancel{(x-3)}}{\cancel{(x-3)}(\sqrt{2x+3} + 3)} = \frac{2}{3+3} = \frac{1}{3}$$

6. Evaluate the integrals.

[10 pts] (a) $\int \csc^3 x \cot x \, dx$

[10 pts] (b) $\int_1^2 x^3\sqrt{x^2-1}\, dx$

(a) $u = \csc x \quad du = -\csc x \cot x \, dx$

$\csc x \cot x \, dx = -du$

$$\int \csc^2 x \csc x \cot x \, dx = \int u^2(-du) = -\frac{1}{3}u^3 + C$$

$$= -\frac{1}{3}\csc^3 x + C$$

(b) $u = x^2 - 1 \quad x^2 = u+1 \quad du = 2x\,dx \quad x\,dx = \frac{1}{2}du$

$u(1) = 0 \quad u(2) = 3$

$$\int_1^2 x^3(x^2-1)^{1/2}dx = \int_1^2 x^2(x^2-1)^{1/2}x\,dx$$

$$= \int_0^3 (u+1)u^{1/2}\frac{1}{2}du = \frac{1}{2}\int_0^3 u^{3/2} + u^{1/2}\,du$$

$$= \frac{1}{2}\left(\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right)\Big|_0^3$$

$$= \frac{1}{5}(3)^{5/2} + \frac{1}{3}(3)^{3/2}$$

7. Find the intervals on which the function

$$f(x) = \frac{1}{\sqrt{2 + x - x^2}}$$

is increasing and decreasing, showing your work.

Domain: $2 + x - x^2 = (2-x)(1+x) > 0$

$2 - x > 0$, $1 + x > 0$ $\quad -1 < x < 2$

$f(x) = (2 + x - x^2)^{-1/2}$

$f'(x) = -\frac{1}{2}(2 + x - x^2)^{-3/2}(1 - 2x)$

$= \dfrac{2x-1}{2(2+x-x^2)^{3/2}}$

$2x - 1 > 0$ if $x > 1/2$, $2x - 1 < 0$ if $x < 1/2$

so decreasing on $(-1, 1/2)$

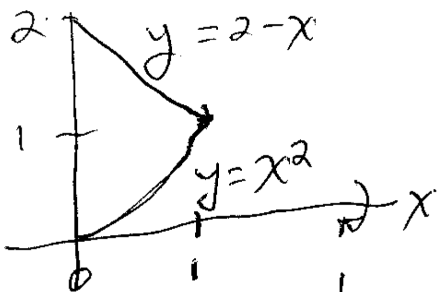
Increasing on $(1/2, 2)$

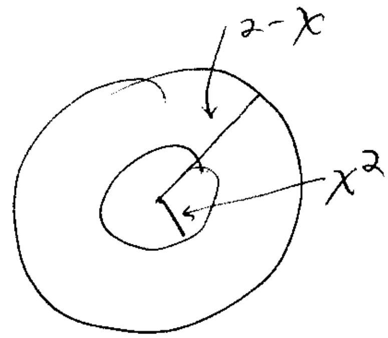
8. Set up, but do not evaluate, definite integrals for the volume of the solid obtained by rotating the portion of the first quadrant bounded by $y = 2 - x, y = x^2$ and the y-axis in the manner indicated.

pts] (a) Rotate about the x-axis.

pts] (b) Rotate about the line $x = 3$.

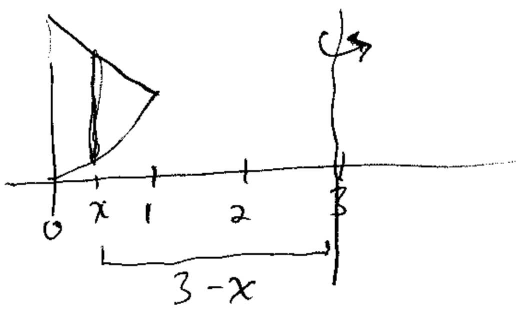
(a)





$$\pi \int_0^1 (2-x)^2 - (x^2)^2 \, dx$$

(b)



$$2\pi \int_0^1 (3-x)((2-x) - x^2) \, dx$$

pts] 9. (a) Prove that the equation

$$x^{5/3} + 2x = \cos x$$

has **at least** one solution.

pts] (b) Prove that the equation

$$x^{5/3} + 2x = \cos x$$

as **at most** one solution.

Let $f(x) = x^{5/3} + 2x - \cos x$, continuous

(a) $f(0) = -1 < 0$, $f(\pi/2) = (\pi/2)^{5/3} + \pi > 0$

so $f(x) = 0$ by IVT.

(b) If $f(a) = f(b) = 0$ then $f'(c) = 0$ some c in (a, b) by Rolle's Theorem, but

$$f'(x) = \frac{5}{3} x^{2/3} + 2 + \sin x \geq 0 + 2 + (-1) = 1 > 0$$

[5 pts] 10. (a) Find the equation of the tangent line to the curve $y = (x+1)^{-2}$ at $x = a$ for $a > 0$.

[5 pts] (b) Find the value of $a > 0$ such that the area of the triangle formed in the first quadrant by the x-axis, the y-axis and the tangent line to $y = (x+1)^{-2}$ at $x = a$ is a maximum. (You do not have to prove that it is the maximum.)

(a) $\frac{dy}{dx} = -2(x+1)^{-3}$ so $\frac{y-(a+1)^{-2}}{x-a} = -2(a+1)^{-3}$

$$y = -2(a+1)^{-3}(x-a) + (a+1)^{-2}$$

(b) $(y=0)$ $\quad 2(a+1)^{-3}(x-a) = (a+1)^{-2}$

$2(x-a) = a+1 \quad x = \frac{a+1}{2} + a = \frac{3a+1}{2}$

$(x=0)$ $\quad y = -2(a+1)^{-3}(-a) + (a+1)^{-2}$

$= \frac{2a+(a+1)}{(a+1)^3} = \frac{3a+1}{(a+1)^3}$

area $= A(a) = \frac{1}{2}\left(\frac{3a+1}{2}\right)\left(\frac{3a+1}{(a+1)^3}\right) = \frac{(3a+1)^2}{4(a+1)^3}$

$$A'(a) = \frac{2(3a+1)(3)4(a+1)^3 - (3a+1)^2(4)(3(a+1)^2)}{16(a+1)^6}$$

$$24(3a+1)(a+1)^3 = 12(3a+1)^2(a+1)^2$$

$2(a+1) = 3a+1 \quad 2a+2 = 3a+1 \quad a = 1$

1. Find dy/dx.

$$(a) \qquad y = x \tan y$$

$$(b) \qquad y = \int_{\cos x}^{1} \sqrt{1-t^2}\, dt$$

(a) $\dfrac{dy}{dx} = \tan y + x \sec^2 y \dfrac{dy}{dx}$

$(1 - x \sec^2 y)\dfrac{dy}{dx} = \tan y$

$\dfrac{dy}{dx} = \dfrac{\tan y}{1 - x \sec^2 y}$

(b) $y = -\int_1^{\cos x} \sqrt{1-t^2}\, dt = -g(\cos x)$

where $g(x) = \int_1^x \sqrt{1-t^2}\, dt$

$\dfrac{dy}{dx} = -g'(\cos x)(-\sin x) = \sqrt{1-\cos^2 x}\,(\sin x)$

2. Given that

$$f''(x) = \frac{1 + 3x^4}{x^3}$$

$f(1) = 1$ and $f'(1) = 3$, find $f(x)$.

$$f''(x) = x^{-3} + 3x$$

$$f'(x) = -\frac{1}{2}x^{-2} + \frac{3}{2}x^2 + C$$

$$3 = f'(1) = -\frac{1}{2} + \frac{3}{2} + C \qquad C = 2$$

$$f'(x) = -\frac{1}{2}x^{-2} + \frac{3}{2}x^2 + 2$$

$$f(x) = \frac{1}{2}x^{-1} + \frac{1}{2}x^3 + 2x + D$$

$$1 = f(1) = \frac{1}{2} + \frac{1}{2} + 2 + D \qquad D = -2$$

$$f(x) = \frac{1}{2}x^{-1} + \frac{1}{2}x^3 + 2x - 2$$

3. Find the absolute maximum and absolute minimum of the function

$$f(x) = x\sqrt{1-4x}$$

on the interval $[-1/4, 1/4]$.

$$f(x) = x(1-4x)^{1/2}$$

$$f'(x) = (1-4x)^{1/2} + x\left(\tfrac{1}{2}\right)(1-4x)^{-1/2}(-4)$$

$$= (1-4x)^{1/2} - \frac{2x}{(1-4x)^{1/2}}$$

$$= \frac{1-4x-2x}{(1-4x)^{1/2}} = \frac{1-6x}{(1-4x)^{1/2}}$$

$$= 0 \text{ if } x = 1/6$$

$f(-1/4) = -\frac{1}{4}\sqrt{2}$ absolute minimum

$f(1/6) = \frac{1}{6}\sqrt{\frac{1}{3}}$ absolute maximum

$f(1/4) = 0$

4. Evaluate the definite integrals.

$$(a) \qquad \int_0^4 |\sqrt{x} - 1|\, dx$$

$$(b) \qquad \int_0^1 x\sqrt{3 - x^2}\, dx$$

(a) $$|\sqrt{x} - 1| = \begin{cases} x^{1/2} - 1 & x^{1/2} \geq 1 \\ 1 - x^{1/2} & 0 \leq x^{1/2} < 1 \end{cases}$$

$$\int_0^1 1 - x^{1/2}\, dx + \int_1^4 x^{1/2} - 1\, dx$$

$$= x - \frac{2}{3}x^{3/2}\Big|_0^1 + \frac{2}{3}x^{3/2} - x\Big|_1^4$$

$$= \frac{1}{3} + \left(\frac{2}{3}(4)^{3/2} - 4\right) - \left(\frac{2}{3} - 1\right) \quad (= 2)$$

(b) $u = 3 - x^2 \quad du = -2x\,dx \quad x\,dx = -\frac{1}{2}\,du$

$u(0) = 3 \quad u(1) = 2$

$$\left(-\frac{1}{2}\right)\int_3^2 u^{1/2}\,du = \frac{1}{2}\int_2^3 u^{1/2}\,du$$

$$= \frac{1}{2}\left(\frac{2}{3}\right)u^{3/2}\Big|_2^3 = \frac{1}{3}\left(3^{3/2} - 2^{3/2}\right)$$

5. Calculate the limits, using only techniques from Math 31A.

$$(a) \qquad \lim_{x\to 0} \frac{\cos x - 1}{\sin x}$$

$$(b) \qquad \lim_{x\to -\infty} \frac{\sqrt{9x^6 - x}}{x^3+1}$$

(a) $$\lim_{x\to 0} \frac{\cos x - 1}{\sin x} = \lim_{x\to 0} \frac{\frac{\cos x - 1}{x}}{\frac{\sin x}{x}}$$

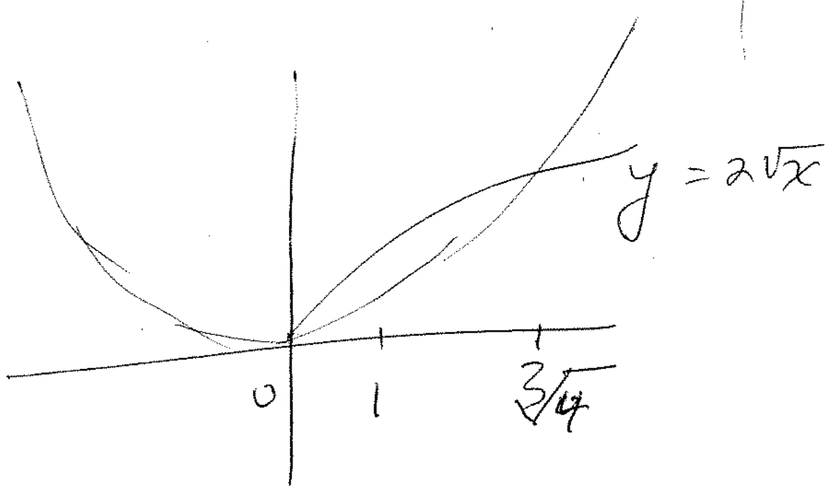
$$= \frac{\lim_{x\to 0} \frac{\cos x - 1}{x}}{\lim_{x\to 0} \frac{\sin x}{x}} = \frac{0}{1} = 0$$

(b) $$\lim_{x\to -\infty} \frac{\sqrt{9x^6 - x}}{x^3+1} = \lim_{x\to \infty} \frac{\sqrt{9(-x)^6 - (-x)}}{(-x)^3+1}$$

$$= \lim_{x\to \infty} \frac{\sqrt{9x^6+x}}{-x^3+1} = \lim_{x\to\infty} \frac{\sqrt{9 + 1/x^5}}{-1 + 1/x^3}$$

$$= \frac{\sqrt{9}}{-1} = -3$$

6. Calculate the area of the region of the plane bounded by the curves $y = x^2$ and $y = 2\sqrt{x}$.

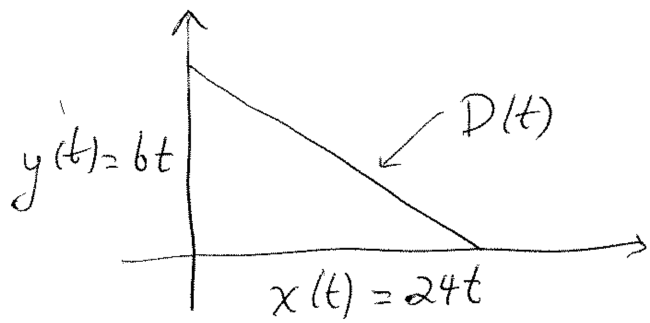


$x^2 = 2\sqrt{x}$ $\quad x^4 = 4x$ $\quad x(x^3 - 4) = 0$

$$\int_0^{\sqrt[3]{4}} 2\sqrt{x} - x^2 \, dx = 2\left(\frac{2}{3}\right)x^{3/2} - \frac{1}{3}x^3 \Big|_0^{4^{1/3}}$$

$$= \frac{4}{3}\left(4^{1/3}\right)^{3/2} - \frac{1}{3}\left(4^{1/3}\right)^3 = \frac{8}{3} - \frac{4}{3} = \frac{4}{3}$$

7. A blimp (motor-powered inflated airship) flying parallel to the ground at a speed of 24 mph releases a weather balloon that rises vertically at a rate of 6 mph. How fast is the distance between the blimp and the weather balloon increasing 5 minutes after the release?



$5 \text{ min} = \frac{1}{12} \text{ hr.}$

$$D\left(\tfrac{1}{12}\right) = \sqrt{\left(24\left(\tfrac{1}{12}\right)\right)^2 + \left(6\left(\tfrac{1}{12}\right)\right)^2} = \sqrt{4 + \tfrac{1}{4}} = \frac{\sqrt{17}}{2}$$

$$D(t)^2 = x(t)^2 + y(t)^2$$

$$\cancel{2}D(t)\frac{dD}{dt} = \cancel{2}x(t)\frac{dx}{dt} + \cancel{2}y(t)\frac{dy}{dt}$$

$$\frac{\sqrt{17}}{2}\frac{dD\left(\frac{1}{12}\right)}{dt} = (2)(24) + \left(\tfrac{1}{2}\right)(6) = 51$$

$$\frac{dD}{dt}\left(\tfrac{1}{12}\right) = \frac{102}{\sqrt{17}}$$

8. A function $f(x)$ is defined by

$$f(x) = \int_0^x \frac{1}{1+3t^2+2t^3}\,dt$$

Find the intervals on which $f(x)$ is concave upward and the intervals on which it is concave downward.

$$f'(x) = (1+3x^2+2x^3)^{-1}$$

$$f''(x) = -(1+3x^2+2x^3)^{-2}(6x+6x^2)$$

$$= \frac{-6x(1+x)}{(1+3x^2+2x^3)^2}$$

$$= 0 \text{ if } x = 0 \text{ or } -1$$

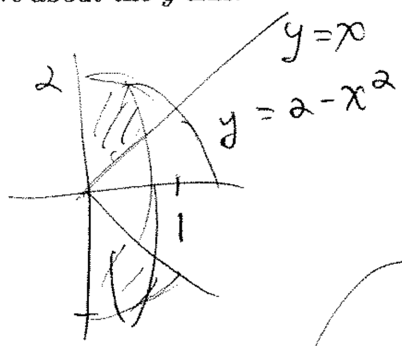
Interval	x	$1+x$	$f''(x)$	Concave
$(-\infty, -1)$	neg	neg	neg	downward
$(-1, 0)$	neg	pos	pos	upward
$(0, \infty)$	pos	pos	neg	downward

9. Set up, but do not evaluate, a definite integral for the volume of each of the solids obtained by rotating the portion of the first quadrant of the plane bounded by the curves $y = 2 - x^2, y = x$ and the y-axis in the following matter.

(a) Revolve about the x-axis.

(b) Revolve about the y-axis.

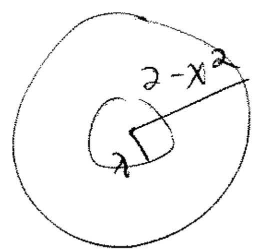
(a)



$$x = 2 - x^2$$

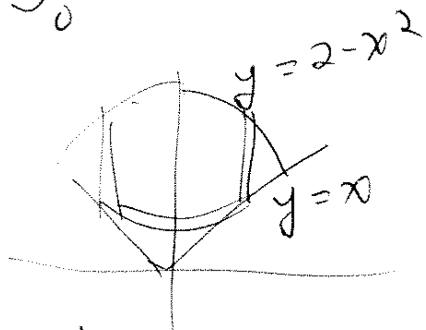
$$x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0$$



$$\int_0^1 \pi(2-x^2)^2 - \pi x^2 \, dx$$

(b)



$$\int_0^1 2\pi x \left((2-x^2) - x \right) dx$$

10. A function $f(x)$ has the properties $f(0) = 5$ and $f'(x) \leq -3$ for all x in $[0, 2]$. Prove that there is at least one solution to the equation $f(x) = 0$. Hint: What can you say about $f(2)$?

By the Mean Value Theorem

$$f(2) - f(0) = f'(c)(2-0)$$

$$f(2) - 5 = f'(c)(2) \leq -6$$

$$f(2) \leq -1 < 0$$

So $f(x) = 0$ has a solution by the Intermediate Value Theorem.

1. Find the critical numbers and the intervals of increase and decrease of the function

$$f(x) = \frac{x^2}{x+8}.$$

$$f'(x) = \frac{2x(x+8) - x^2}{(x+8)^2} = \frac{x^2+16x}{(x+8)^2} = \frac{x(x+16)}{(x+8)^2}$$

critical numbers: $x = 0, -16, -8$

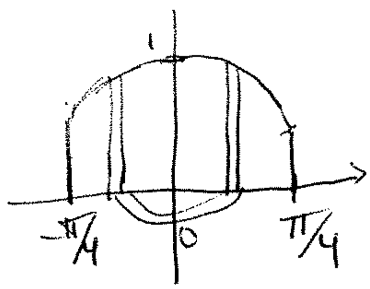
	x	$x+16$	$f'(x)$	
$(-\infty, -16)$	$-$	$-$	$+$	increasing
$(-16, -8)$	$-$	$+$	$-$	decreasing
$(-8, 0)$	$-$	$+$	$-$	decreasing
$(0, \infty)$	$+$	$+$	$+$	increasing

2. Let A be the region of the first quadrant bounded by the curves $y = \cos x$ and $x = \pi/4$. Write the volumes of the following solids as definite integrals but do not evaluate the integrals.

(10 points) (a) The solid formed by rotating A about the line $x = 0$.

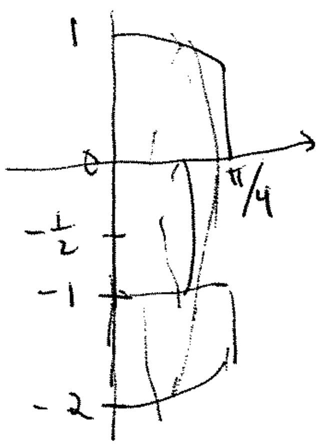
(10 points) (b) The solid formed by rotating A about the line $y = -1/2$.

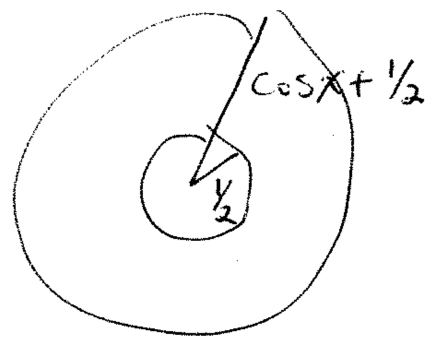
(a)



$$\int_0^{\pi/4} 2\pi x \cos x \, dx$$

(b)





$$\int_0^{\pi/4} \pi\left(\left(\cos x + \tfrac{1}{2}\right)^2 - \pi\left(\tfrac{1}{2}\right)^2\right) dx$$

3. Differentiate

(10 points) (a) $f(x) = \tan(\frac{1}{2}\sin(x^3 + 2x))$

(10 points) (b) $f(x) = \int_x^{x^3} \frac{t^{1/3}}{t^{2/3}+1}\,dt$

(a) $f'(x) = \sec^2(\frac{1}{2}\sin(x^3+2x))\frac{d}{dx}(\frac{1}{2}\sin(x^3+2x))$

$= \sec^2(\frac{1}{2}\sin(x^3+2x))\,\frac{1}{2}\cos(x^3+2x)(3x^2+2)$

(b) $f(x) = \int_x^0 \frac{t^{1/3}}{t^{2/3}+1}\,dt + \int_0^{x^3} \frac{t^{1/3}}{t^{2/3}+1}\,dt$

$= -\int_0^x \frac{t^{1/3}}{t^{2/3}+1}\,dt + \int_0^{x^3} \frac{t^{1/3}}{t^{2/3}+1}\,dt$

$f'(x) = -\frac{x^{1/3}}{x^{2/3}+1} + \frac{(x^3)^{1/3}}{(x^3)^{2/3}+1}(3x^2)$

(5 points) 4. (a) In the space just below, write in mathematical symbols the meaning of "$f(x)$ is *continuous* at $x = a$".

$$\lim_{x \to a} f(x) = f(a)$$

(15 points) (b) The meaning of "$f(x)$ is *differentiable* at $x = a$" is that the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. Since the limit of the denominator is zero and the limit of the quotient exists, the limit of the numerator must also be zero. Complete the proof that differentiability implies continuity by using properties of limits to prove that if

$$\lim_{x \to a} (f(x) - f(a)) = 0$$

then $f(x)$ is continuous at $x = a$.

$$0 = \lim_{x \to a} (f(x) - f(a))$$

$$= \lim_{x \to a} f(x) - \lim_{x \to a} f(a)$$

$$= \lim_{x \to a} f(x) - f(a)$$

$$\lim_{x \to a} f(x) = f(a)$$

5. Find the average value of the function

$$f(x) = x\sqrt{2x-3}$$

on the interval $[2, 6]$.

$$\text{Average Value} = \frac{1}{6-2}\int_2^6 x\sqrt{2x-3}\,dx$$

$u = 2x - 3 \qquad du = 2\,dx \qquad u(2) = 1$

$x = \frac{1}{2}(u+3) \qquad dx = \frac{1}{2}\,du \qquad u(6) = 9$

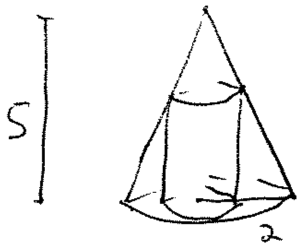
$$= \frac{1}{4}\int_1^9 \frac{1}{2}(u+3)\,u^{1/2}\,\frac{1}{2}\,du$$

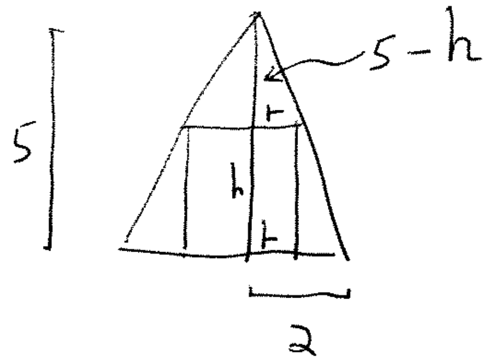
$$= \frac{1}{16}\int_1^9 u^{3/2} + 3u^{1/2}\,du$$

$$= \frac{1}{16}\left[\frac{2}{5}u^{5/2} + 2u^{3/2}\right]_1^9$$

$$= \frac{1}{16}\left[\left(\frac{2}{5}(9)^{5/2} + 2(9)^{3/2}\right) - \left(\frac{2}{5} + 2\right)\right]$$

6. Find the radius and height of the right circular cylinder of largest volume that can be inscribed in a cone whose height is 5 m and base radius is 2 m.





$$\frac{5-h}{r} = \frac{5}{2} \qquad 5-h = \frac{5}{2}r \qquad h = 5 - \frac{5}{2}r$$

$$V = \pi r^2 h = \pi r^2\left(5 - \frac{5}{2}r\right) = 5\pi r^2 - \frac{5}{2}\pi r^3$$

$$V'(r) = 10\pi r - \frac{15}{2}\pi r^2 = 0$$

$$r\left(10\pi - \frac{15}{2}\pi r\right) = 0 \qquad 10\pi - \frac{15}{2}\pi r = 0$$

$$\frac{15}{2}r = 10 \qquad r = \frac{20}{15} = \frac{4}{3}$$

$$h = 5 - \frac{5}{2}\left(\frac{4}{3}\right) = 5 - \frac{10}{3} = \frac{5}{3}$$

7. Show that the following inequalities are true.

(15 points) (a) $x\sqrt{x-x^2} \leq \frac{3\sqrt{3}}{16}$

(Hint: What is the maximum of that function?)

(5 points) (b) $\int_1^2 \sqrt{1+x^4}\,dx \geq \frac{7}{3}$

(a) $f(x) = x(x-x^2)^{1/2}$

$f'(x) = (x-x^2)^{1/2} + x\frac{1}{2}(x-x^2)^{-1/2}(1-2x)$

$= (x-x^2)^{1/2} + \dfrac{x(1-2x)}{2(x-x^2)^{1/2}}$

$= \dfrac{2(x-x^2)+x-2x^2}{2(x-x^2)^{1/2}} = \dfrac{3x-4x^2}{2(x-x^2)^{1/2}} = 0$

$3-4x=0 \quad x = 3/4$

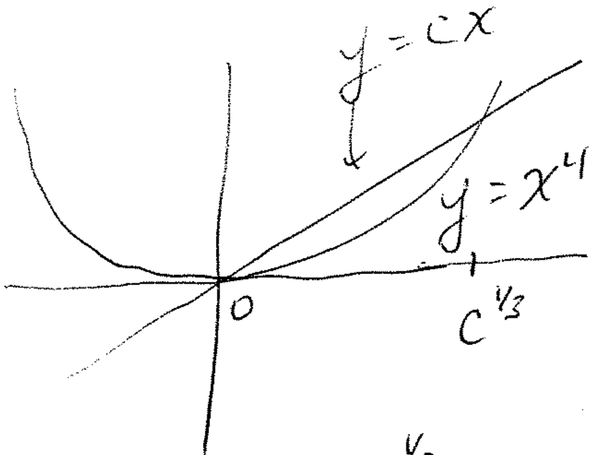
$f(x) \leq f(\frac{3}{4}) = (\frac{3}{4})\sqrt{\frac{3}{4}-\frac{9}{16}} = \frac{3}{4}\sqrt{\frac{3}{16}} = \frac{3\sqrt{3}}{16}$

(b) $\int_1^2 \sqrt{1+x^4}\,dx \geq \int_1^2 \sqrt{x^4}\,dx$

$= \int_1^2 x^2\,dx = \frac{1}{3}x^3\Big|_1^2 = \frac{7}{3}$

8

8. There is a number $c > 0$ such that the area between the curve $y = x^4$ and the line $y = cx$ is $96/10$. Find the value of c.



$$x^4 = cx$$
$$x(x^3 - c) = 0$$
$$x^3 = c \quad x = c^{1/3}$$

$$\frac{96}{10} = \int_0^{c^{1/3}} cx - x^4 \, dx = \frac{c}{2}x^2 - \frac{1}{5}x^5 \Big|_0^{c^{1/3}}$$

$$= \frac{c}{2}\left(c^{2/3}\right) - \frac{1}{5}c^{5/3} = c^{5/3}\left(\frac{1}{2} - \frac{1}{5}\right)$$

$$= c^{5/3}\left(\frac{3}{10}\right) \qquad c^{5/3} = \left(\frac{10}{3}\right)\frac{96}{10} = 32$$

$$c = (32)^{3/5} \left(= \left(2^5\right)^{3/5} = 8\right)$$

9. The equations below define curves y. For each equation, calculate the value of the second derivative y'' when $x = 0$.

(7 points) (a) $$y = \int_{-1}^{x} \frac{3t}{t^3+1}\,dt$$

(13 points) (b) $$\sin y = x^2$$

(a) $y' = \dfrac{3x}{x^3+1}$ $\quad y'' = \dfrac{3(x^3+1) - 3x(3x^2)}{(x^3+1)^2}$

if $x = 0$ then $y'' = 3$

(b) $\cos y \dfrac{dy}{dx} = 2x$

$\dfrac{dy}{dx} = \dfrac{2x}{\cos y}$ $\quad \dfrac{d^2y}{dx^2} = \dfrac{2(\cos y) - 2x(-\sin y)\frac{dy}{dx}}{(\cos y)^2}$

$= \dfrac{2(\cos y)}{(\cos y)^2} = \dfrac{2}{\cos y}$ when $x = 0$

(Note $\sin y = (0)^2$ so $\cos y = \pm 1$)

10. Suppose $f(x)$ is continuous on $[1,4]$ with $f(1) = -5$ and also $f'(x) \geq 2$ for all x in $(1,4)$.

(7 points) (a) Prove that there is at most one x in $[1,4]$ such that $f(x) = 0$.

(13 points) (b) Prove that there does exist x in $[1,4]$ such that $f(x) = 0$. (Hint: What can you conclude about $f(4) - f(1)$?)

(a) If $f(a) = f(b) = 0$ for some $a < b$ in $[1,4]$ then, by Rolle's Theorem, $f'(c) = 0$ for some c in (a,b), but $f'(x) \geq 2$.

(b) By the Mean Value Theorem

$$f(4) - f(1) = f'(c)(4-1)$$

for some c in $(1,4)$. Since $f(1) = -5$ and $f'(c) \geq 2$ then

$$f(4) - (-5) = f'(c)(3) \geq 6 \text{ so}$$

$f(4) \geq 1 > 0$ while $f(1) = -5 < 0$ so $f(x) = 0$ for some x in $(1,4)$ by the Intermediate Value Theorem.

1. Let

$$f(x) = \frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{3}{2}x^2 + 1.$$

(a) Find the critical numbers of f.

(b) Find the intervals of increase and decrease of f.

(a) $f'(x) = x^3 - 2x^2 - 3x = 0$

$x(x^2 - 2x - 3) = 0$

$x(x-3)(x+1) = 0$

$x = 0, 3, -1$

(b)

Interval	x	$x-3$	$x+1$	$f'(x)$	
$(-\infty, -1)$	$-$	$-$	$-$	$-$	dec.
$(-1, 0)$	$-$	$-$	$+$	$+$	increas.
$(0, 3)$	$+$	$-$	$+$	$-$	dec.
$(3, \infty)$	$+$	$+$	$+$	$+$	increas.

2. Calculate the limit, if it exists.

$$\lim_{x \to 0} \frac{\tan 4x}{x}.$$

$$\lim_{x \to 0} \frac{\tan 4x}{x} = \lim_{x \to 0} \frac{1}{x} \frac{\sin 4x}{\cos 4x}$$

$$= \lim_{x \to 0} \frac{4}{\cos 4x} \frac{\sin 4x}{4x}$$

$$= \lim_{x \to 0} \frac{4}{\cos 4x} \lim_{4x \to 0} \frac{\sin 4x}{4x} = \left(\frac{4}{1}\right)(1) = 4$$

3. Calculate

(a) the slope of the tangent to the curve

$$x \sin y = x + 1$$

at $(-1, 0)$.

(b) the derivative of

$$f(x) = \int_{-1}^{x^2} \sqrt{1 + t^4}\, dt$$

at $x = 2$.

(a) $\sin y + x \cos y \dfrac{dy}{dx} = 1$

$\sin 0 + (-1) \cos 0 \dfrac{dy}{dx} = 1$

$-\dfrac{dy}{dx} = 1 \qquad \dfrac{dy}{dx} = -1$

(b) $g(x) = \int_{-1}^{x} \sqrt{1+t^4}\,dt \qquad g'(x) = \sqrt{1+x^4}$

$f(x) = g(x^2) \qquad f'(x) = g'(x^2)(2x)$

$= \sqrt{1+(x^2)^4}\,(2x)$

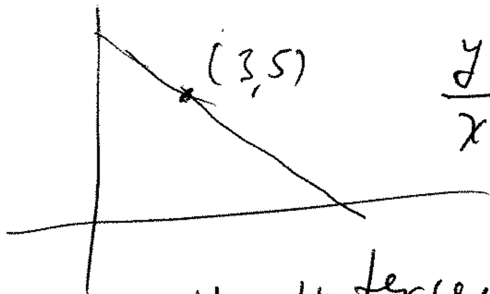
$f'(2) = \sqrt{1+(4)^4}\,(4)$

4. A line of negative slope through $(3,5)$ bounds a triangular region of the first quadrant of the plane.

(a) Write the area of the triangle as a function of the slope of the line.

(b) Find the slope that makes the area of the triangular region as small as possible.

(a)



$$\frac{y-5}{x-3} = m$$

y-intercept: $y-5=-3m \quad y=-3m+5$

x-intercept: $m(x-3)=-5 \quad x=-\frac{5}{m}+3$

$$A(m)=\frac{1}{2}\left(-\frac{5}{m}+3\right)(-3m+5)$$

$$=\frac{1}{2}\left(15-25m^{-1}-9m+15\right)$$

$$=15-\frac{25}{2}m^{-1}-\frac{9}{2}m$$

(b) $A'(m)=\frac{25}{2}m^{-2}-\frac{9}{2}=0$

$\frac{25}{2}m^{-2}=\frac{9}{2}, \quad m^{-2}=\frac{9}{25} \quad m^2=\frac{25}{9}$

$m=-\frac{5}{3}$ (negative!)

5. Evaluate

$$(a) \qquad \int_0^{\pi} \sqrt{x} + \sin x \, dx$$

$$(b) \qquad \int_{-2}^{0} |x+1| \, dx$$

(a) $\displaystyle\int_0^{\pi} x^{1/2} + \sin x \, dx = \frac{2}{3} x^{3/2} - \cos x \Big|_0^{\pi}$

$= \left(\frac{2}{3}\pi^{3/2} - \cos \pi\right) - (0 - \cos 0)$

(b) $\displaystyle\int_{-2}^{0} |x+1| \, dx = \int_{-2}^{-1} -(x+1) dx + \int_{-1}^{0} x+1 \, dx$

$= -\frac{1}{2}x^2 - x \Big|_{-2}^{-1} + \frac{1}{2}x^2 + x \Big|_{-1}^{0}$

$= \left(\left(-\frac{1}{2}(-1)^2 - (-1)\right) - \left(-\frac{1}{2}(-2)^2 - (-2)\right)\right)$

$\quad - \left(0 - \left(\frac{1}{2}(-1)^2 + (-1)\right)\right)$

$= \left(-\frac{1}{2} + 1\right) - \left(-\frac{1}{2}(4) + 2\right) - \left(\frac{1}{2}(1) - 1\right) = 1$

6. Find the value of x such that the tangent line to the curve $y = x^3 + 1$ at x passes through the point $(0, -1)$.

$y' = 3x^2$ $\qquad \dfrac{y-(-1)}{x-0} = 3x^2$

$y + 1 = 3x^3$

$(x^3+1)+1 = 3x^3 \qquad 2x^3 = 2$

$x^3 = 1 \qquad x = 1$

7. Evaluate the indefinite integrals

$$(a) \qquad \int \frac{x}{(x^2+1)^2}\,dx$$

$$(b) \qquad \int \sec^3 x \tan x\,dx$$

(a) $\int (x^2+1)^{-2}\, x\, dx$

$u = x^2+1 \quad du = 2x\,dx \quad x\,dx = \frac{1}{2}\,du$

$= \int u^{-2}\, \frac{1}{2}\, du = \frac{1}{2}(-u^{-1}) + C$

$= -\frac{1}{2}(x^2+1)^{-1} + C$

(b) $\int \sec^2 x \sec x \tan x\, dx$

$u = \sec x \quad du = \sec x \tan x\, dx$

$= \int u^2\, du = \frac{1}{3}u^3 + C = \frac{1}{3}\sec^3 x + C$

8. Prove that if $f(x)$ is continuous on $[a,b]$ and $f(a) < f(b)$, then there exists c in $[a,b]$ such that

$$f(c) = \frac{f(a)+f(b)}{2}.$$

$$f(a) < \frac{f(a)+f(b)}{2} < f(b) \quad \text{because}$$

$$f(a) = \frac{f(a)+f(a)}{2} < \frac{f(a)+f(b)}{2}$$

$$< \frac{f(b)+f(b)}{2} = f(b)$$

so c exists by the Intermediate Value Theorem

9. Find the value of a for which

$$\lim_{x \to 2} \frac{x^2 - 3x + a}{x - 2}$$

exists and evaluate the limit.

$$\begin{array}{r|l} & \quad x - 1 \\ \hline x - 2 & x^2 - 3x + a \\ & \underline{x^2 - 2x} \\ & \quad -x + a \\ & \quad \underline{-x + 2} \\ & \qquad a - 2 \end{array}$$

$a - 2 = 0$ if $a = 2$ then

$$\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x-1)}{\cancel{x-2}}$$

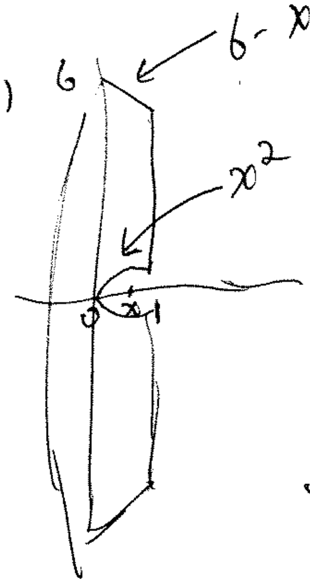
$$= \lim_{x \to 2} x - 1 = 1$$

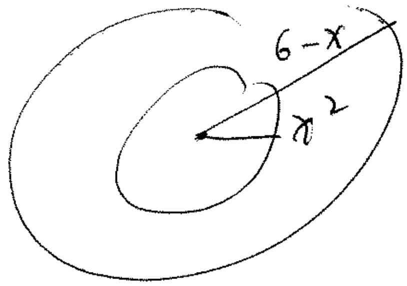
10. Let A be the region in the first quadrant bounded by the curves $x = 1, y = 6 - x$ and $y = x^2$. Write the volumes of the following solids as definite integrals but do NOT evaluate the integrals.

(a) The solid formed by rotating A about the line $y = 0$.

(b) The solid formed by rotating A about the line $x = 1$.

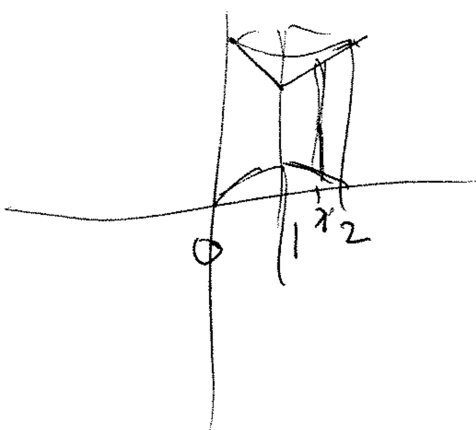
(a)





$$\int_0^1 \pi(6-x)^2 - \pi(x^2)^2 \, dx$$

(b)



$$\int_1^2 2\pi(x-1)((6-x)-x^2)\,dx$$