HOUR EXAM I

Math 180, Fall 2003

Calculus I

September 26, 2003

ANSWERS

Problem 1. Sketch the graph of a continuous function $y = f(x)$ with the following properties:

- $f(-3) = -1$, $f(0) = 2$, $f(3) = 0$;
- $f'(x)$ exists for all x, and $f'(x) > 0$ for $x < 0$, $f'(0) = 0$, $f'(x) < 0$ for $0 < x < 3$, $f'(x) > 0$ for $x > 3$;
- $\lim_{x\to\infty} f(x) = \infty$;
- $\lim_{x\to-\infty} f(x) = -\infty$.

Answer:
Here is an example of such a graph:

Problem 2. Consider the function $y = f(x)$ defined below. (Here k is a constant.)

$$f(x) = \begin{cases} \frac{x-2}{x+2} & \text{if } x < 2, \\ (x+k)^2 & \text{if } x \geq 2. \end{cases}$$

1. What is the domain of f?
2. Compute
$$\lim_{x \to 2^-} \frac{x-2}{x+2} \quad \text{and} \quad \lim_{x \to 2^+} (x+k)^2$$
3. Is there a number k such that the function $y = f(x)$ is continuous at $x = 2$? If yes, find such a k!
4. Is there a number k such that the function $y = f(x)$ is continuous for *all* values of x? If yes, find such a k!

Answer:

1. The domain of f consists of all real numbers except -2.
2. We have
$$\lim_{x \to 2^-} \frac{x-2}{x+2} = \frac{\lim_{x \to 2^-} x-2}{\lim_{x \to 2^-} x+2} = \frac{0}{4} = 0$$
and
$$\lim_{x \to 2^+} (x+k)^2 = (2+k)^2.$$
3. In order for $y = f(x)$ to be continuous at $x = 2$ we need
$$\lim_{x \to 2^-} \frac{x-2}{x+2} = \lim_{x \to 2^+} (x+k)^2,$$
that is, $(2+k)^2 = 0$. This is the case exactly for $k = -2$.
4. No, there is no such value for k, since the function $y = \frac{x-2}{x+2}$ has a vertical asymptote $x = -2$. Hence $y = f(x)$ has a vertical asymptote $x = -2$, regardless of k.

Problem 3. I leave a piece of cake sitting on my kitchen counter. Four of the ants living in my house find it, and each tells two of its friends about it. Each one of them in turn tells two of their friends about it, and so on. Let's assume that it takes each ant an hour to meet its two friends and exchange the news. After approximately how many hours do 2 million ants know about my cake?

Answer:
If $A(t)$ denotes the number of ants which know about the cake after t hours, then $A(t) = 4 \cdot 3^t$ for all t. We want to know the time t for which $A(t) = 2 \cdot 10^6$, that is, $4 \cdot 3^t = 2 \cdot 10^6$. We solve this equation by first dividing by 4 and then taking ln on both sides:

$$\ln(3^t) = \ln\left(\frac{1}{2} \cdot 10^6\right).$$

Using the rules for computing with ln this gives

$$t \ln 3 = \ln\left(\frac{1}{2} \cdot 10^6\right)$$

and hence $t = \ln\left(\frac{1}{2} \cdot 10^6\right) / \ln 3 \approx 11.944$, that is, roughly 12 hours.

Problem 4. Suppose $C(T)$ is the cost (in dollars per day) of heating my house when the outside temperature is T degrees Fahrenheit.

1. We have $C'(20) = -0.1$. What does this mean in practical terms?

2. Suppose in addition that $C(20) = 3$. Estimate $C(20.5)$!

Answer:

1. $C'(20) = -0.1$ means that if the current outside temperature is 20° F, then an increase in temperature by 1° F will reduce my heating costs by roughly 10 cents per day.

2. We have

$$\begin{aligned} -0.1 = C'(20) = \lim_{h\to 0} \frac{C(20+h) - C(20)}{h} &\approx \frac{C(20+0.5) - C(20)}{0.5} \\ &= \frac{C(20.5) - 3}{0.5}, \end{aligned}$$

so $C(20.5) = 3 + 0.5 \cdot (-0.1) = 2.95$.

Problem 5. Using the definition of the derivative as the limit of the difference quotient to find $\frac{dy}{dx}$ for the function

$$y = \frac{-3}{2-x}.$$

Determine whether there is a line with slope 1 which is tangent to the graph of this function.

Answer:
Write $y = f(x) = \frac{-3}{2-x}$. Then we have

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{-3}{2-(x+h)} - \frac{-3}{2-x}}{h}.$$

We simplify the fraction:

$$\frac{\frac{-3}{2-(x+h)} - \frac{-3}{2-x}}{h} = \frac{\frac{-3(2-x)-(-3)(2-x-h)}{(2-x-h)(2-x)}}{h} = \frac{-3}{(2-x-h)(2-x)}$$

Hence

$$f'(x) = \lim_{h\to 0} \frac{\frac{-3}{2-(x+h)} - \frac{-3}{2-x}}{h} = \lim_{h\to 0} \frac{-3}{(2-x-h)(2-x)} = \frac{-3}{(2-x)^2}.$$

There is no line with slope 1 which is tangent to the graph of $y = f(x)$, since the equation $f'(x) = 1$ has no solution: there is no x such that $(2-x)^2 = -1/3$, because $(2-x)^2 \geq 0$ for every x.