Problem Set 3
Due Monday, Oct. 11.

Formal Logic

Math 430, Fall 2004

1. Let $S = \{0, +\}$ where 0 is a constant symbol and + is a 2-place function symbol. Show that the set Φ consisting of the S-sentences

$$\forall x \forall y \forall z \big((x + y) + z = x + (y + z)\big),$$
$$\forall x (x + 0 = x \wedge 0 + x = x),$$
$$\forall x \forall y \big(x + y = 0 \rightarrow (x = 0 \wedge y = 0)\big)$$

is satisfiable.

2. Let S be a symbol set. A set Φ of S-sentences is called **independent** if there is no $\varphi \in \Phi$ such that $\Phi \setminus \{\varphi\} \models \varphi$. Suppose now that $S = \{E\}$ where E is a 2-place relation symbol. Show that the set of axioms for equivalence relations

$$\big\{\forall x Exx,\ \forall x \forall y (Exy \leftrightarrow Eyx),\ \forall x \forall y \forall z (Exy \wedge Eyz \rightarrow Exz)\big\}$$

is independent.

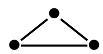
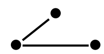
3. Let S be a symbol set. An S-formula which does not contain $\neg$, $\rightarrow$, $\leftrightarrow$ is called **positive.**
   (a) Give an inductive definition of the set $\mathcal{P}_S$ of positive S-formulas (similar to the definition of the set $\mathcal{F}_S$ of all S-formulas).
   (b) Show that every positive S-formula is satisfiable. (One may use an S-structure whose universe consists of a single element.)

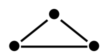
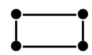
4. Let S be a symbol set and let $\mathcal{M}$ be an S-structure. Show:
   (a) If α and β are automorphisms of $\mathcal{M}$, then so is $\alpha \circ \beta$.
   (b) If α is an automorphism of $\mathcal{M}$, then so is α^{-1}.

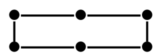
   (For those of you who know about groups: this yields that the set of all automorphisms of $\mathcal{M}$ forms a group, with $\circ$ as group operation, called the **automorphism group** of $\mathcal{M}$.)

5. Let $S = \{<\}$ where $<$ is a 2-place relation symbol. Without proof: what are all automorphisms of the S-structure $\mathcal{Z} = (\mathbb{Z}, <^{\mathcal{Z}})$, where $<$ is interpreted as the usual ordering on $\mathbb{Z}$?

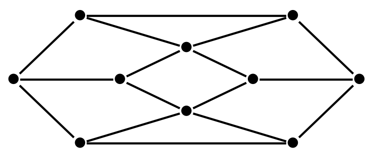
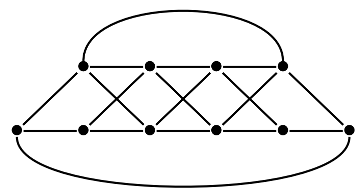
6. Recall the symbol set $S_{\text{graph}} = \{R\}$ appropriate for graphs introduced in Problem Set 2.
   (a) Show that the following graphs, construed as S_{graph}-structures, are not isomorphic (using 4.1.9 in the lecture notes):

i. $\mathcal{A} =$  and $\mathcal{B} =$ 

ii. $\mathcal{A} =$  and $\mathcal{B} =$ 

iii. $\mathcal{A} =$  and $\mathcal{B} =$ 

(b) Are the following graphs isomorphic?

7. (Extra credit.) A set A of natural numbers is called a **spectrum** if there is a symbol set S and an S-sentence φ such that

$$A = \big\{ n \in \mathbb{N} : \text{there is an } S\text{-structure } \mathcal{M} \text{ with } \mathcal{M} \models \varphi \\ \text{whose universe } M \text{ contains exactly } n \text{ elements} \big\}.$$

Show:

(a) Every finite subset of $\mathbb{N}^{>0} = \{1, 2, 3, \dots\}$ is a spectrum.

(b) For every $m \geq 1$, the set of positive integers which are divisible by m is a spectrum.

Which subsets of $\mathbb{N}^{>0}$ are spectra? This problem was asked by Heinrich Scholz in 1952, and it is still unsolved. For example, as far as I know, it is unknown whether the complement $\mathbb{N}^{>0} \setminus A$ of a spectrum A is also a spectrum.