Problem Set 5
Due Monday, Nov. 29.

Formal Logic

Math 430, Fall 2004

1. Which of the following rules are sound (proof or counterexample):

   (a) $$\frac{\begin{array}{ll}\Gamma\ \varphi_1 & \psi_1 \\ \Gamma\ \varphi_2 & \psi_2\end{array}}{\Gamma\ (\varphi_1 \vee \varphi_2) \quad (\psi_1 \vee \psi_2)}$$

   (b) $$\frac{\Gamma \quad \varphi\frac{y}{x}}{\Gamma \quad \forall x \varphi} \qquad \text{if } y \notin \text{fr}\,(\Gamma\ \forall x \varphi)$$

   (c) $$\frac{\Gamma \quad \varphi}{\Gamma \quad \forall x \varphi}$$

   (d) $$\frac{\Gamma \quad \exists x \varphi}{\Gamma \quad \forall x \varphi}$$

2. Let $S = \{f\}$ where f is a 2-place function symbol. Compute

$$\left((\exists v_2 f v_2 v_2 = v_0)\, \frac{v_2}{v_0} \right) \frac{f v_0 v_1}{v_2}.$$

3. Let S be a symbol set, let x be a variable, and let t be an S-term. Show that for every S-formula φ:

   (a) $\varphi\frac{x}{x} = \varphi$;

   (b) if $x \notin \text{fr}\,(\varphi)$, then $\varphi\frac{t}{x} = \varphi$.

   (Proceed by induction on the construction of φ.)

4. Using the rules of the sequent calculus, show that the following two rules are derivable. Here as usual "$\varphi \rightarrow \psi$" is shorthand for "$(\neg\varphi \vee \psi)$".

$$\frac{\Gamma\ \varphi \quad \psi}{\Gamma \quad \varphi \rightarrow \psi} \qquad \frac{\Gamma \quad \varphi \rightarrow \psi}{\Gamma\ \varphi \quad \psi}$$

(Please turn.)

5. Let $S = \{R\}$ where R is a 2-place relation symbol. Proof or counterexample:

   (a) $\forall y \exists x Rxy \vdash \exists x \forall y Rxy$

   (b) $\forall y \exists x Rxy \vdash \neg \exists x \forall y Rxy$.

6. (Extra credit.) The rules of the sequent calculus depend on the choice of a symbol set S. Suppose that R is a 1-place relation symbol not in S and let $\Gamma\ \varphi$ be a sequent over S (i.e., not involving the symbol R). Show: if $\Gamma\ \varphi$ is derivable in the sequent calculus for $S \cup \{R\}$, then $\Gamma\ \varphi$ is also derivable in the sequent calculus for S. (Hint: replace every occurrence of R in a derivation of $\Gamma\ \varphi$ inductively by the equation $x = x$.)