Course Proposal

Model Theory of the Real Exponential Function

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An **exponential set** in $\mathbb{R}^n$ is a set of the form

$$\{x \in \mathbb{R}^n : P(x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}) = 0\},$$

where P is a polynomial with real coefficients in $2n$ indeterminates, and a **subexponential set** in $\mathbb{R}^n$ is the image of an exponential set in $\mathbb{R}^{n+k}$ (for some k) under the projection $\mathbb{R}^{n+k} \to \mathbb{R}^n$ onto the first n coordinates. By a result of Khovanskii, every exponential set (and hence every subexponential set) has only finitely many connected components. There exists a subexponential set (in $\mathbb{R}^3$) which is not exponential, in fact, not even a Boolean combination thereof. However, in 1991 Wilkie proved (giving an answer to a question of Tarski): the complement of a subexponential set in $\mathbb{R}^n$ is subexponential, for any n. Hence every Boolean combination of a subexponential set is subexponential, or in model-theoretic terms: the theory of the field of real numbers equipped with the exponential function $x \mapsto e^x \colon \mathbb{R} \to \mathbb{R}$ is model-complete.

Since Wilkie's breakthrough, our understanding of the "logical" properties of the real exponential function has increased dramatically. The purpose of this course is to give an introduction to some of these results, assuming only basic knowledge of first-order model theory. Additional topics to be covered (time permitting) include: basic facts about o-minimal structures; their valuation theoretic properties, in particular Miller's dichotomy theorem; logarithmic-exponential series; decidability results.