Problem Set 1

Solutions

Model Theory

Math 506, Spring 2004.

1. Let $\mathcal{Q} = (\mathbb{Q}, +, \cdot)$ be the field of rational numbers considered as an $\mathcal{L}$-structure in the language $\mathcal{L} = \{+, \cdot\}$. Show that the set $\mathbb{Q}^{\geqslant 0}$ of non-negative rationals is $\emptyset$-definable in $\mathcal{Q}$. Hint: you may use the fact that every natural number can be written as the sum of four squares of natural numbers. (Lagrange's Theorem.) Show that $\mathbb{Q}^{\geqslant 0}$ is not $\emptyset$-definable if we drop the symbol $\cdot$ from $\mathcal{L}$.

   Solution. We claim that the formula $\varphi(x) = \exists a \exists b \exists c \exists d\, x = a^2 + b^2 + c^2 + d^2$ does the job, that is, given $r \in \mathbb{Q}$ we have

   $$\mathcal{Q} \vDash \varphi(r) \qquad \Leftrightarrow \qquad r \geqslant 0$$

   Here the direction $\Rightarrow$ is trivial. For the converse, suppose that $r \in \mathbb{Q}$ is non-negative, say $r = p/q$ with $p, q \in \mathbb{Z}$, $q \neq 0$. Then $p\,q$ is a non-negative integer, hence may be written as the sum of four squares of natural numbers: $p\,q = a^2 + b^2 + c^2 + d^2$ for some $a, b, c, d \in \mathbb{N}$. Hence

   $$r = \frac{p}{q} = \frac{p\,q}{q^2} = \left(\frac{a}{q}\right)^2 + \left(\frac{b}{q}\right)^2 + \left(\frac{c}{q}\right)^2 + \left(\frac{d}{q}\right)^2$$

   showing that $\mathcal{Q} \vDash \varphi(r)$. For the second part of the problem, it suffices to show that there exists an automorphism σ of $(\mathbb{Q}, +)$ such that $\sigma(\mathbb{Q}^{\geqslant 0}) \neq \mathbb{Q}^{\geqslant 0}$. Any map $\sigma\colon \mathbb{Q} \to \mathbb{Q}$ given by $\sigma(x) = \lambda\, x$ with $\lambda \in \mathbb{Q}$, $\lambda < 0$ works.

2. Write down an axiom system for the class of all groups:

   a) in the language $\mathcal{L}_1 = \{1, \cdot, ^{-1}\}$ by universal sentences;

   b) in the language $\mathcal{L}_2 = \{1, \cdot\}$ by $\forall\exists$-sentences;

   c) in the language $\mathcal{L}_3 = \{\cdot\}$ by $\exists\forall\exists$-sentences.

   Solution.

   a) $\forall x \forall y \forall z (x \cdot (y \cdot z) = (x \cdot y) \cdot z), \forall x (x \cdot 1 = 1 \cdot x = x), \forall x (x \cdot x^{-1} = x^{-1} \cdot x = 1)$

   b) $\forall x \forall y \forall z (x \cdot (y \cdot z) = (x \cdot y) \cdot z), \forall x (x \cdot 1 = 1 \cdot x = x), \forall x \exists y (x \cdot y = y \cdot x = 1)$

   c) $\forall x \forall y \forall z (x \cdot (y \cdot z) = (x \cdot y) \cdot z), \exists e \forall z \forall x \exists y (z \cdot e = e \cdot z = z \wedge x \cdot y = y \cdot x = e)$

3. Let $\mathcal{L} = \{+, -, 1\}$ where $+$ is a binary function symbol, $-$ is a unary function symbol, and 1 is a constant symbol. We construe $\mathbb{R}$ as a structure $\mathcal{R} = (\mathbb{R}, +, -, 1)$ in the natural way. Given an algebraic description of the class $\mathcal{F}$ of all functions $f\colon \mathbb{R}^n \to \mathbb{R}$ which are interpretations of $\mathcal{L}$-terms, i.e., for which there exists an $\mathcal{L}$-term $t(x_1, ..., x_n)$ such that $f(a_1, ..., a_n) = t^{\mathcal{R}}(a_1, ..., a_n)$ for all $a_1, ..., a_n \in \mathbb{R}$.

   Solution. We claim that $\mathcal{F}$ is the class of all functions $f\colon \mathbb{R}^n \to \mathbb{R}$ of the form

   $$f(x_1, ..., x_n) = a_0 + a_1\, x_1 + \cdots + a_n\, x_n$$

with $a_0, ..., a_n \in \mathbb{Z}$. To see this, note first that every such affine function is in $\mathcal{F}$: if $a_0 \neq 0$, then it is the interpretation of the term (in infix notation)

$$+(t_0, +(\cdot(t_1, x_{i_1}), +(\cdot(t_2, x_{i_2}), ...)))$$

where $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ are exactly the indices i with $a_i \neq 0$ and $t_i = +(1, +(1, ...))$ (a_i many times) if $a_i > 0$, $t_i = -(+(1, +(1, ...)))$ ($|a_i|$ many times) if $a_i < 0$. (Similarly if $a_0 \neq 0$.) As for the converse, we show by induction on terms that the interpretation of every term t has the desired form. This is clear if $t = x$ is a variable or $t = 1$. If s and t are terms and

$$s^{\mathcal{R}}(x_1, ..., x_n) = a_0 + a_1 x_1 + \cdots + a_n x_n$$

for some $a_0, ..., a_n \in \mathbb{Z}$,

$$t^{\mathcal{R}}(x_1, ..., x_n) = b_0 + b_1 x_1 + \cdots + b_n x_n$$

for some $b_0, ..., b_n \in \mathbb{Z}$, then

$$(+(s, t))^{\mathcal{R}}(x_1, ..., x_n) = (a_0 + b_0) + (a_1 + b_1) x_1 + \cdots + (a_n + b_n) x_n$$

is again of this form. Similarly for the term $-t$.

4. Let $\mathcal{L}$ be a language an $\mathcal{M}$ be an $\mathcal{L}$-structure. We say that $f: M^m \to M^n$ is ***A*-definable** (for some $A \subseteq M$) if the graph $\Gamma(f) := \{(a, f(a)): a \in M^m\}$ of f is A-definable in $\mathcal{M}$ (as a subset of M^{m+n}).

   a) Show that if $f: M^m \to M^n$ and $g: M^n \to M^l$ are A-definable, then so is $g \circ f$.

   b) Suppose that $f: M^m \to M^n$ is A-definable. Show that the image $f(M^m)$ of f is A-definable.

   Solution.

   a) Let $\varphi(x, y, u)$ and $\psi(y, z, v)$ with $x = (x_1, ..., x_m)$, $y = (y_1, ..., y_n)$, $z = (z_1, ..., z_l)$, $u = (u_1, ..., u_p)$, $v = (v_1, ..., v_q)$ be $\mathcal{L}$-formulas and $a \in A^p$, $a' \in A^q$ such that

   $$\begin{aligned}\Gamma(f) &= \{(b, c) \in M^m \times M^n: \mathcal{M} \vDash \varphi(b, c, a)\},\\ \Gamma(g) &= \{(c, d) \in M^n \times M^l: \mathcal{M} \vDash \psi(c, d, a')\}.\end{aligned}$$

   Then $\Gamma(g \circ f) = \{(b, d) \in M^m \times M^l: \mathcal{M} \vDash \theta(b, d, a, a')\}$ where

   $$\theta(x, z, u, v) := \exists y_1 \cdots \exists y_n (\varphi \wedge \psi)$$

   b) Let $\varphi(x, y, u)$ and $a \in A^p$ be as in (a). Then $f(M^m)$ is defined by $\gamma(y, a)$ where

   $$\gamma(y, u) := \exists x_1 \cdots \exists x_m (\varphi).$$

5. Let $\mathcal{L} = \{R\}$ where R is a binary relation symbol. A **graph** is an $\mathcal{L}$-structure $\mathcal{G} = (G, R^{\mathcal{G}})$ with the property that $(g, h) \in R^{\mathcal{G}} \Longleftrightarrow (h, g) \in R^{\mathcal{G}}$ and $(g, g) \notin R^{\mathcal{G}}$ for all $g, h \in G$. We say that **$g \in G$ and $h \in G$ are in the same connected component of $\mathcal{G}$** if there exist $g_0, ..., g_n \in G$ (for some $n \geqslant 0$) with $R^{\mathcal{G}}(g_i, g_{i+1})$ for all $i = 0, ..., n-1$ and $g_0 = g$, $g_n = h$. We say that $\mathcal{G}$ is **connected** if all g, $h \in G$ are in the same connected component of $\mathcal{G}$.

   a) Show that the class of all connected graphs is not elementary.

b) Deduce that there is no $\mathcal{L}$-formula $\varphi(x,y)$ with the property that for all graphs $\mathcal{G}$ and $g,h\in G$: $\mathcal{G}\vDash\varphi(g,h)\Longleftrightarrow g$ and h are in the same connected component of $\mathcal{G}$.

Solution. It is clearly enough to show (a): if there was a formula $\varphi(x,\,y)$ with the purported property in (b), then the class of connected graphs would be axiomatized by the conjunction of the sentence $\forall x\forall y\varphi(x,y)$, $\forall x\forall y(R(x,y)\leftrightarrow R(y,x))$ and $\forall x(\neg R(x,x))$.

As for (a), suppose for a contradiction that the class $\mathcal{C}$ of all connected graphs is elementary. For every $m\geqslant 0$, there do certainly exist connected graphs in which some elements can only be connected by a path of length $\geqslant m$: for example $\mathcal{G}_m=(G_m,R^{\mathcal{G}_m})$ with $G_m=\mathbb{Z}/2m\mathbb{Z}$ and $R^{\mathcal{G}_m}$ defined by: $(g,h)\in R^{\mathcal{G}m}\iff h=g+1$ or $g=h-1$. Then every path $g_0,\,\ldots.,\,g_n$ from $g_0=0$ to $g_n=m$ in $\mathcal{G}_m$ has length $n\geqslant m$. Let now $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$, and let $\mathcal{G}=(G,R^{\mathcal{G}})$ be the ultraproduct of the family $\{\mathcal{G}_m\}_{m\in\mathbb{N}}$ with respect to $\mathcal{U}$. By Łos' Theorem, $\mathcal{G}$ is a graph. Now consider the $\mathcal{L}$-formula $\varphi_m(x,y)$, for $m\in\mathbb{N}$, given by:

$$\varphi_m(x,y):=\forall x_0\cdots\forall x_m\left(x_0\neq x\vee x_m\neq y\vee\bigvee_{i=0}^{m-1}\neg R(x_i,x_{i+1})\right)$$

Let $g=(0)_{m\in\mathbb{N}}$ and $h=(m)_{m\in\mathbb{N}}$ (elements of $\prod_m G_m$). Then $\|\varphi_m(g,h)\|\supseteq\{m,m+1,\ldots\}$, hence $\mathcal{G}\vDash\varphi_m(g/\mathcal{U},\,h/\mathcal{U})$. Since this holds for every $m\in\mathbb{N}$, $g/\mathcal{U}$ and $h/\mathcal{U}$ are not in the same connected component of $\mathcal{G}$. Hence $\mathcal{G}$ is not connected, contradicting the fact that elementary classes are closed under ultraproducts. This contradiction shows that $\mathcal{C}$ is not elementary.

6. Let I be a non-empty set.

a) Show that a filter $\mathcal{U}$ on I is a principal ultrafilter if and only if $\mathcal{U}=\langle b\rangle$ for some $b\in I$.

b) Suppose that I is infinite, and let $\mathcal{F}$ be the Fréchet filter of cofinite subsets of I. Show that $\mathcal{F}$ is not principal and not an ultrafilter.

c) Show that an ultrafilter $\mathcal{U}$ on an infinite set I is non-principal if and only if $\mathcal{U}\supseteq\mathcal{F}$.

Solution.

a) Suppose first that $\mathcal{U}=\langle b\rangle$ for some $b\in I$. Then $\mathcal{U}$ is principal. If A is a subset of I with $A\notin\langle b\rangle$, that is, if $b\notin A$, then $b\in I\setminus A$ and hence $I\setminus A\in\langle b\rangle=\mathcal{U}$. So for every subset A of I, either $A\in\mathcal{U}$ or $I\setminus A\in\mathcal{U}$, that is, $\mathcal{U}$ is an ultrafilter. Conversely, suppose that $\mathcal{U}$ is a principal ultrafilter on I. Say $\mathcal{U}=\langle B\rangle$ for some $B\subseteq I$. Then $B\neq\emptyset$, and $\mathcal{U}\subseteq\langle b\rangle$ for every $b\in B$. Since $\mathcal{U}$ is maximal, $\mathcal{U}=\langle b\rangle$ for every $b\in B$.

b) If $\mathcal{F}=\langle B\rangle$ for some $B\subseteq I$ then $I\setminus B$ is finite; hence if we pick $b\in B$ arbitrary and set $B':=B\setminus\{b\}$, then $I\setminus B'=(I\setminus B)\cup\{b\}$ is also finite. But $B'\not\supseteq B$, a contradiction. This shows that $\mathcal{F}$ is not principal. To see that $\mathcal{F}$ is not an ultrafilter let A be any infinite subset of I whose complement $I\setminus A$ is also infinite; then neither A nor $I\setminus A$ belong to $\mathcal{F}$.

c) Suppose first that $\mathcal{U}$ is a principal ultrafilter on an infinite set I. Then $\mathcal{U}=\langle b\rangle$ for some $b\in I$, by (a). There certainly are cofinite subsets of I which do not contain b, for example $I\setminus\{b\}$. Hence $\mathcal{U}\not\supseteq\mathcal{F}$. Now suppose that $\mathcal{U}$ is an ultrafilter on an infinite set I, but $B\notin\mathcal{U}$ for some cofinite subset B of I. Then $I\setminus B\in\mathcal{U}$, since $\mathcal{U}$ is an ultrafilter. Now $I\setminus B$ is finite, say $I\setminus B=\{a_1,\ldots,a_n\}$ for pairwise distinct $a_1,\ldots,a_n\in I$. In order to show that $\mathcal{U}$ is principal, it is enough to see that $\mathcal{U}\subseteq\langle a_i\rangle$ for some i. Now if we had $\mathcal{U}\not\subseteq\langle a_i\rangle$ for all i, then for each i there exists $A_i\in\mathcal{U}$ with $a_i\notin A_i$. Hence $\emptyset=(I\setminus B)\cap A_1\cap\cdots\cap A_n\in\mathcal{U}$, a contradiction.

7. Let $\mathcal{L}$ be a language and $\{\mathcal{A}_i\}_{i\in I}$ be a family of $\mathcal{L}$-structures, $I\neq\emptyset$. Let $\mathcal{U}$ be a principal ultrafilter on I. Show that there exists $j\in I$ such that $\mathcal{A}_j\cong\prod_{i\in I}\mathcal{A}_i/\mathcal{U}$.

Solution. By 6.(a) there exists $j\in I$ such that $\langle j\rangle=\mathcal{U}$. For every $i\in I\setminus\{j\}$ pick an arbitrary element a_i of A_i, and consider the map h: $A_j\to\prod_{i\in I}A_i/\mathcal{U}$ given $h(x):=a/\mathcal{U}$ where a is given by $a(i)=a_i$ for $i\neq j$ and $a(j)=x$. We claim the h is an isomorphism $\mathcal{A}_j\to\prod_{i\in I}\mathcal{A}_i/\mathcal{U}$. Injectivity: if $h(x)=h(y)$ then the set of indices i with $(h(x))(i)=(h(y))(i)$ is an element of $\mathcal{U}$ and hence contains j; therefore $x=y$. Surjectivity: if $y/\mathcal{U}$ is an arbitrary element of $\prod_{i\in I}A_i/\mathcal{U}$, then let $a\in\prod_{i\in I}A_i$ be defined by $a(i)=a_i$ for $i\neq j$ and $a(j)=y(j)$. Then $h(x)=a/\mathcal{U}$ for $x=y(j)$, and $a(j)=y(j)$, so the set of indices $i\in I$ with $a(i)=y(i)$ is an element of $\mathcal{U}$; hence $y/\mathcal{U}=h(x)$. In a similar way one checks that h is an embedding of $\mathcal{L}$-structures.

8. Let $\{K_i\}_{i\in I}$ be a family of fields, considered as $\mathcal{L}$-structures in the language $\mathcal{L}=\{0,1,+,-,\cdot\}$, where $I\neq\emptyset$. Put $R:=\prod_{i\in I}K_i$, and for an ultrafilter $\mathcal{U}$ on I consider

$$\mathfrak{M}_{\mathcal{U}}:=\{r\in R:\|r=0\|\in\mathcal{U}\}.$$

By Łos' Theorem, the ultraproduct $\prod_{i\in I}K_i/\mathcal{U}$ is a field. Show the following:

a) $R/\mathfrak{M}_{\mathcal{U}}\cong\prod_{i\in I}K_i/\mathcal{U}$. (Hence $\mathfrak{M}_{\mathcal{U}}$ is a maximal ideal of R.)

b) For every maximal ideal $\mathfrak{m}$ of R there exists an ultrafilter $\mathcal{U}$ on I with $\mathfrak{m}=\mathfrak{M}_{\mathcal{U}}$.

Solution.

a) We have a natural surjective map $R\to\prod_{i\in I}K_i/\mathcal{U}$ given by $r=(r(i))\mapsto r/\mathcal{U}$, which we denote by h. In class we have already verified that h is a ring homomorphism; hence it is enough to show that $\ker h=\mathfrak{M}_{\mathcal{U}}$. We have

$$\begin{aligned} r=(r(i))\in\ker h &\iff r/\mathcal{U}=0\\ &\iff \{i\in I: r(i)=0\}\in\mathcal{U}\\ &\iff \|r=0\|\in\mathcal{U}\\ &\iff r\in\mathfrak{M}_{\mathcal{U}}. \end{aligned}$$

b) Let $\mathfrak{m}$ be a maximal ideal of R. Define $\mathcal{U}$ to be the collection of all subsets A of I such that $A=\|r=0\|$ for some $r\in\mathfrak{m}$. We claim that $\mathcal{U}$ is an ultrafilter on I. To see this, we first define, for every $r\in R$, an element $r'\in R$ by $r'(i)=0$ if $r(i)=0$ and $r'(i)=1$ otherwise, and an element $r''\in R$ by $r''(i)=0$ if $r(i)=0$ and $r''(i)=1/r(i)$ otherwise. Then $r=r\cdot r'$ and $r'=r\cdot r''$. It follows that $r\in\mathfrak{m}\iff r'\in\mathfrak{m}$. Note that $\emptyset\notin\mathcal{U}$, since $\|r=0\|=\emptyset$ with $r\in\mathfrak{m}$ implies $1=r'\in\mathfrak{m}$, which is impossible. So $\mathcal{U}$ satisfies axiom (F1) for filters. If $r,s\in\mathfrak{m}$ then $\|r'+s'-r's'=0\|=\|r=0\|\cap\|s=0\|$ (as one easily checks) hence $\mathcal{U}$ is closed under finite intersection (i.e., satisfies (F2)). Now let $A\subseteq I$ with $A\notin\mathcal{U}$. Define $r\in R$ by $r(i)=1$ if $i\in A$ and $R(i)=0$ otherwise. Then $\|r=0\|=A$, hence $r\notin\mathfrak{m}$, and $r\cdot(1-r)=0$; therefore $1-r\in\mathfrak{m}$, and $\|1-r=0\|=I\setminus A$; hence $I\setminus A\in\mathcal{U}$. This also shows that $\mathcal{U}$ is closed under supersets (and hence an ultrafilter on $\mathcal{U}$): if $A\supseteq B$ for some $B\in\mathcal{U}$ and $A\notin\mathcal{U}$, then $I\setminus A\in\mathcal{U}$ and hence $\emptyset=(I\setminus A)\cap B\in\mathcal{U}$, a contradiction to what we have already shown. It is clear that $\mathfrak{m}\subseteq\mathfrak{M}_{\mathcal{U}}$, by definition of $\mathcal{U}$, and hence $\mathfrak{m}=\mathfrak{M}_{\mathcal{U}}$, since $\mathfrak{m}$ is a maximal ideal.