Problem Set 1

Due February 6

Model Theory

Math 506, Spring 2004.

Do 7 of the following 8 problems!

1. Let $\mathcal{Q} = (\mathbb{Q}, +, \cdot)$ be the field of rational numbers considered as an $\mathcal{L}$-structure in the language $\mathcal{L} = \{+, \cdot\}$. Show that the set $\mathbb{Q}^{\geqslant 0}$ of non-negative rationals is $\emptyset$-definable in $\mathcal{Q}$. Hint: you may use the fact that every natural number can be written as the sum of four squares of natural numbers. (Lagrange's Theorem.) Show that $\mathbb{Q}^{\geqslant 0}$ is not $\emptyset$-definable if we drop the symbol $\cdot$ from $\mathcal{L}$.

2. Write down an axiom system for the class of all groups:

   a) in the language $\mathcal{L}_1 = \{1, \cdot, ^{-1}\}$ by universal sentences;

   b) in the language $\mathcal{L}_2 = \{1, \cdot\}$ by $\forall\exists$-sentences;

   c) in the language $\mathcal{L}_3 = \{\cdot\}$ by $\exists\forall\exists$-sentences.

3. Let $\mathcal{L} = \{+, -, 1\}$ where $+$ is a binary function symbol, $-$ is a unary function symbol, and 1 is a constant symbol. We construe $\mathbb{R}$ as a structure $\mathcal{R} = (\mathbb{R}, +, -, 1)$ in the natural way. Give an algebraic description of the class $\mathcal{F}$ of all functions $f: \mathbb{R}^n \to \mathbb{R}$ which are interpretations of $\mathcal{L}$-terms, i.e., for which there exists an $\mathcal{L}$-term $t(x_1, \ldots, x_n)$ such that $f(a_1, \ldots, a_n) = t^{\mathcal{R}}(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_n \in \mathbb{R}$.

4. Let $\mathcal{L}$ be a language and let $\mathcal{M}$ be an $\mathcal{L}$-structure. We say that $f: M^m \to M^n$ is ***A*-definable** (for some $A \subseteq M$) if the graph $\Gamma(f) := \{(a, f(a)): a \in M^m\}$ of f is A-definable in $\mathcal{M}$ (as a subset of M^{m+n}).

   a) Show that if $f: M^m \to M^n$ and $g: M^n \to M^l$ are A-definable, then so is $g \circ f$.

   b) Suppose that $f: M^m \to M^n$ is A-definable. Show that the image $f(M^m)$ of f is A-definable.

5. Let $\mathcal{L} = \{R\}$ where R is a binary relation symbol. A **graph** is an $\mathcal{L}$-structure $\mathcal{G} = (G, R^{\mathcal{G}})$ with the property that $(g, h) \in R^{\mathcal{G}} \Longleftrightarrow (h, g) \in R^{\mathcal{G}}$ and $(g, g) \notin R^{\mathcal{G}}$ for all $g, h \in G$. We say that $\boldsymbol{g \in G}$ **and** $\boldsymbol{h \in G}$ **are in the same connected component of** $\boldsymbol{\mathcal{G}}$ if there exist $g_0, \ldots, g_n \in G$ (for some $n \geqslant 0$) with $R^{\mathcal{G}}(g_i, g_{i+1})$ for all $i = 0, \ldots, n-1$ and $g_0 = g$, $g_n = h$. We say that $\mathcal{G}$ is **connected** if all g, $h \in G$ are in the same connected component of $\mathcal{G}$.

   a) Show that the class of all connected graphs is not elementary.

   b) Deduce that there is no $\mathcal{L}$-formula $\varphi(x, y)$ with the property that for all graphs $\mathcal{G} = (G, R^{\mathcal{G}})$ and $g, h \in G$: $\mathcal{G} \models \varphi(g, h) \Longleftrightarrow g$ and h are in the same connected component of $\mathcal{G}$.

6. Let I be a non-empty set.

   a) Show that a filter $\mathcal{U}$ on I is a principal ultrafilter if and only if $\mathcal{U} = \langle b \rangle$ for some $b \in I$.

b) Suppose that I is infinite, and let $\mathcal{F}$ be the Fréchet filter of cofinite subsets of I. Show that $\mathcal{F}$ is not principal and not an ultrafilter.

c) Show that an ultrafilter $\mathcal{U}$ on an infinite set I is non-principal if and only if $\mathcal{U} \supseteq \mathcal{F}$.

7. Let $\mathcal{L}$ be a language and $\{\mathcal{A}_i\}_{i\in I}$ be a family of $\mathcal{L}$-structures, $I \neq \emptyset$. Let $\mathcal{U}$ be a principal ultrafilter on I. Show that there exists $j \in I$ such that $\mathcal{A}_j \cong \prod_{i\in I} \mathcal{A}_i/\mathcal{U}$.

8. Let $\{K_i\}_{i\in I}$ be a family of fields, considered as $\mathcal{L}$-structures in the language $\mathcal{L} = \{0, 1, +, -, \cdot\}$, where $I \neq \emptyset$. Put $R := \prod_{i\in I} K_i$, and for an ultrafilter $\mathcal{U}$ on I consider

$$\mathfrak{M}_{\mathcal{U}} := \{r \in R : \|r = 0\| \in \mathcal{U}\}.$$

By Łos' Theorem, the ultraproduct $\prod_{i\in I} K_i/\mathcal{U}$ is a field. Show the following:

a) $R/\mathfrak{M}_{\mathcal{U}} \cong \prod_{i\in I} K_i/\mathcal{U}$. (Hence $\mathfrak{M}_{\mathcal{U}}$ is a maximal ideal of R.)

b) For every maximal ideal $\mathfrak{m}$ of R there exists an ultrafilter $\mathcal{U}$ on I with $\mathfrak{m} = \mathfrak{M}_{\mathcal{U}}$.