Problem Set 2

Due February 20

Model Theory

Math 506, Spring 2004.

1. Prove: if $\mathcal{F}$ is a filter on a set $I \neq \emptyset$ such that $\bigcap \mathcal{F} = \emptyset$, then every ultrafilter $\mathcal{U} \supseteq \mathcal{F}$ on I is non-principal. (Hint: use problem 6. (a) on Problem Set 1.)

   Solution. If $\mathcal{U}$ is a principal ultrafilter on I, then $\mathcal{U} = \langle b \rangle$ for some $b \in I$, by problem 6. (a) on Problem Set 1. Hence if $\mathcal{F} \subseteq \mathcal{U}$ is a filter on I, then $b \in \bigcap \mathcal{F}$.

2. Let $\mathcal{L}$ be a language and let T and T' be $\mathcal{L}$-theories. Suppose that for every model $\mathcal{M}$ of T there exists $\sigma \in T'$ such that $\mathcal{M} \models \sigma$. Show that there exists a finite subset $\{\sigma_1, ..., \sigma_n\}$ of T' such that $T \models \sigma_1 \vee \cdots \vee \sigma_n$.

   Solution. We consider the $\mathcal{L}$-theory $T'' := T \cup \{\neg\sigma : \sigma \in T'\}$. By assumption, T'' is not satisfiable. Hence by the Compactness Theorem, some finite subset of T'' is not satisfiable. Therefore there exist $\sigma_1, ..., \sigma_n \in T'$ such that $T \cup \{\neg\sigma_1, ..., \neg\sigma_n\}$ is not satisfiable. This is equivalent with $T \models \sigma_1 \vee \cdots \vee \sigma_n$.

3. Let $\mathcal{L}$ be a language and $\mathcal{M} \subseteq \mathcal{N}$ be $\mathcal{L}$-structures.

   a) Suppose that for every finite subset A of M and every $b \in N$ there exists an automorphism f of $\mathcal{N}$ which fixes A pointwise (i.e., $f(a) = a$ for all $a \in A$) and such that $f(b) \in M$. Show that then $\mathcal{M} \preccurlyeq \mathcal{N}$.

   b) Now suppose $\mathcal{L} = \{<\}$ with a binary relation symbol $<$. We consider $\mathcal{M} = (\mathbb{Q}, <)$ and $\mathcal{N} = (\mathbb{R}, <)$ as $\mathcal{L}$-structures in the natural way. Use (a) to show that $(\mathbb{Q}, <) \preccurlyeq (\mathbb{R}, <)$.

   c) Show that the converse in (a) does not hold in general. (Hint: consider $\mathcal{M} = (\mathbb{N}, <)$.)

   d) [Optional.] Let R be a commutative ring and let X and Y be infinite sets of indeterminates over R, with $X \subseteq Y$. Show that $R[X] \preccurlyeq R[Y]$, considered as structures in the language $\mathcal{L} = \{0, 1, +, \cdot\}$ of rings.

   Solution.

   a) We use the Tarski-Vaught test. Let $\varrho(x, y_1, ..., y_n)$ be an $\mathcal{L}$-formula and $a = (a_1, ..., a_n) \in M^n$ such that for some $b \in N$ we have $\mathcal{N} \models \varrho(b, a_1, ..., a_n)$. We have to show that there is $c \in M$ with $\mathcal{N} \models \varrho(c, a_1, ..., a_n)$. For this, we choose an automorphism f of $\mathcal{N}$ which fixes $A = \{a_1, ..., a_n\}$ pointwise and such that $f(b) \in M$. Since f is elementary, we obtain $\mathcal{N} \models \varrho(f(b), f(a_1), ..., f(a_n))$. Hence with $c := f(b)$ we have $\mathcal{N} \models \varrho(c, a_1, ..., a_n)$ as required.

b) By (a) we have to show that given $a_1, \ldots, a_n \in \mathbb{Q}$ and $b \in \mathbb{R}$ there exists an automorphism f of $(\mathbb{R}, <)$ with $f(a_i) = a_i$ for all i and $f(b) \in \mathbb{Q}$. If $b = a_i$ for some i, then the identity automorphism $f = \mathrm{id}_{\mathbb{R}}$ does the job. Suppose $b \neq a_i$ for all i. After reordering the a_i we may assume $a_1 < \cdots < a_n$. Moreover we may assume that $a_i < b < a_{i+1}$ for some i. (Why?) Choose a rational number c with $a_i < c < a_{i+1}$. We define $f: \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) := \begin{cases} x & \text{if } x \in \mathbb{R} \setminus [a_i, a_{i+1}] \\ \dfrac{c - a_i}{b - a_i}(x - b) + c & \text{if } x \in [a_i, b] \\ \dfrac{a_{i+1} - c}{a_{i+1} - b}(x - b) + c & \text{if } x \in [b, a_{i+1}] \end{cases}$$

Then f is bijective and $x < y \Longleftrightarrow f(x) < f(y)$ for all $x, y \in \mathbb{R}$, that is, f is an automorphism of $(\mathbb{R}, <)$. Clearly $f(a_j) = a_j$ for all j, and $f(b) = c$ as required.

c) By the Compactness Theorem (or Löwenheim-Skolem "upwards") there exists a proper elementary extension $\mathcal{N} = (N, <^{\mathcal{N}})$ of $\mathcal{M} = (\mathbb{N}, <)$, that is, $\mathcal{M} \preccurlyeq \mathcal{N}$ and $\mathbb{N} \neq N$. Let $a \in N \setminus \mathbb{N}$. We claim that for every automorphism f of $\mathcal{N}$ we have $f(n) = n$ for all $n \in \mathbb{N}$. (This implies immediately that there cannot be an automorphism f of $\mathcal{N}$ with $f(a) \in \mathbb{N}$.) To see this, note first that for all $n \in \mathbb{N}$, since $\mathcal{M} \models \varphi(n, n+1)$ for $\varphi(x, y) = \forall z(z > x \to z = y \vee z > y)$, we have $\mathcal{N} \models \varphi(n, n+1)$, because $\mathcal{N}$ is an elementary extension of $\mathcal{M}$. Similarly it follows that there is no $a \in N$ with $a <^{\mathcal{N}} 0$. Hence there are no $a \in N \setminus \mathbb{N}$ with $a <^{\mathcal{N}} n$ for some $n \in \mathbb{N}$. This implies that $f(n) = n$ for all $n \in \mathbb{N}$.

d) According to (a) it suffices to show that for any $a_1, \ldots, a_m \in R[X]$ and for any $b \in R[Y]$ there is an automorphism of $R[Y]$ which fixes $a_1, \ldots, a_m$ and maps b into $R[X]$. Let $X_1, \ldots, X_k$ be all the indeterminates from X which occur in $a_1, \ldots, a_m, b$, and let $Y_1, \ldots, Y_l$ be all the indeterminates from $Y \setminus X$ which occur in b. Define a bijection $f: Y \to Y$ which fixes $X_1, \ldots, X_n$ pointwise and which maps each Y_i to some indeterminate in X. (Since X is infinite, such a bijection must exist.) The bijection f has a natural extension F: $R[Y] \to R[Y]$ to an R-algebra automorphism. Then F fixes $a_1, \ldots, a_m$ and maps b into $R[X]$, as required.

4. We say that an $\mathcal{L}$-theory T has **definable Skolem functions** if for every formula $\varphi(x_1, \ldots, x_n, y)$ there exists a formula $\psi(x_1, \ldots, x_n, y)$ such that

a) $T \models \forall x_1 \cdots \forall x_n \exists y \psi(x_1, \ldots, x_n, y)$

b) $T \models \forall x_1 \cdots \forall x_n \forall y \forall y'(\psi(x_1, \ldots, x_n, y) \wedge \psi(x_1, \ldots, x_n, y') \to y = y')$

c) $T \models \forall x_1 \cdots \forall x_n(\exists y \varphi(x_1, \ldots, x_n, y) \to \exists y(\psi(x_1, \ldots, x_n, y) \wedge \varphi(x_1, \ldots, x_n, y)))$.

In other words, in every model $\mathcal{N}$ of T, ψ defines the graph of a function f: $N^n \to N$ such that $\mathcal{N} \models \varphi(a, f(a))$ for all $a \in N^n$ for which $\mathcal{N} \models \exists y \varphi(a, y)$.

a) Show that if T has built-in Skolem functions, then T has definable Skolem functions.

b) Let $\mathcal{L} = \{0, +\}$ where $+$ is a binary function symbol and 0 is a constant symbol. Show that $T = \mathrm{Th}(\mathbb{N}, 0, +)$ has definable Skolem functions.

Solutions.

a) Suppose that T has built-in Skolem function, that is, for every formula $\varphi(x_1, \ldots, x_n, y)$ there exists an n-ary function symbol f_φ such that $T \models \forall x_1 \cdots \forall x_n(\exists y(\varphi(x_1, \ldots, x_n, y) \to \varphi(x_1, \ldots, x_n, f_\varphi(x_1, \ldots, x_n)))$. Then $\psi(x_1, \ldots, x_n, y) =$ "$f_\varphi(x_1, \ldots, x_n) = y$" satisfies conditions (a)–(b).

b) We write $\mathcal{N} = (\mathbb{N}, 0, +)$. Let $\varrho(y, z) := \exists w(w \neq 0 \wedge w + z = y)$. Then $\mathcal{N} \models \varrho(n, m) \Longleftrightarrow n > m$. Given an $\mathcal{L}$-formula $\varphi(x, y)$ with $x = (x_1, \ldots, x_n)$, the $\mathcal{L}$-formula

$$\psi = (\neg\exists z(\varphi(x, z)) \to y = 0) \vee (\varphi(x, y) \wedge \forall z(\varrho(y, z) \to \neg\varphi(x, z)))$$

satisfies conditions (a)–(b) from above. Hence T has definable Skolem functions.

5. An $\mathcal{L}$-theory T is called (absolutely) **categorical** if it is satisfiable and any two models of T are isomorphic.

a) Show that if T is categorical, then its unique model must be finite.

b) Let $\mathcal{L} = \{f\}$ where f is a unary function symbol. Give an example of a finite $\mathcal{L}$-theory T all of whose models are infinite. (Hence T is not categorical.)

c) Suppose that $\mathcal{L}$ is finite, and let $\mathcal{M}$ be an $\mathcal{L}$-structure whose universe is finite. Show that there exists an $\mathcal{L}$-sentence φ with the property that $\mathcal{N} \models \varphi \Longleftrightarrow \mathcal{M} \cong \mathcal{N}$ for every $\mathcal{L}$-structure $\mathcal{N}$. (In particular, $\mathcal{M} \equiv \mathcal{N} \Longleftrightarrow \mathcal{M} \cong \mathcal{N}$; thus $\mathrm{Th}(\mathcal{M})$ is categorical.)

d) [Optional.] Show that if $\mathcal{L}$ is an arbitrary language, and $\mathcal{M}$ an $\mathcal{L}$-structure whose universe is finite, then for all $\mathcal{L}$-structures $\mathcal{N}$ we have $\mathcal{M} \equiv \mathcal{N} \Longleftrightarrow \mathcal{M} \cong \mathcal{N}$.

Solutions.

a) Suppose that T has an infinite model $\mathcal{M}$. Then by compactness (or Löwenheim-Skolem "upwards") T has an infinite model of cardinality $> |M|$. Hence T cannot be categorical.

b) There are many possible solutions. Here is one: let φ be the $\mathcal{L}$-sentence

$$\forall x \forall y(f(x) = f(y) \to x = y) \wedge \exists z \forall x(\neg f(x) = z).$$

Then an $\mathcal{L}$-structure $\mathcal{M} = (M, f^{\mathcal{M}})$ satisfies φ if and only if $f^{\mathcal{M}}: M \to M$ is injective but not surjective. But a non-empty set S is finite if and only if every injective map $S \to S$ is surjective. Hence $T = \{\varphi\}$ only has infinite models.

c) Suppose that $M = \{a_0, \ldots, a_n\}$ with pairwise distinct $a_0, \ldots, a_n$ ($n \in \mathbb{N}$). Let $\psi_0, \ldots, \psi_k$ enumerate all the atomic formulas $\psi(x_0, \ldots, x_n)$ of $\mathcal{L}$ such that if t is a terms which occurs in ψ, then t contains at most one occurrence of a function symbol. There are only finitely many such formulas ψ. (Why?) For $i = 0, \ldots, k$ let

$$\varphi_i = \begin{cases} \psi_i & \text{if } \mathcal{M} \models \psi_i(a_0, \ldots, a_n) \\ \neg\psi_i & \text{if } \mathcal{M} \models \neg\psi_i(a_0, \ldots, a_n) \end{cases}$$

Let now

$$\theta(x_0, \ldots, x_n) = \forall y \left(\bigvee_{0 \leqslant i \leqslant n} y = x_i \right).$$

Then

$$\varphi = \exists x_0 \cdots \exists x_n(\theta \wedge \varphi_0 \wedge \cdots \wedge \varphi_k)$$

has the required property.

d) If $\mathcal{L}$ is finite, this follows from (c). Suppose that $\mathcal{L}$ is infinite. It is enough to show the direction $\Rightarrow$. So assume for a contradiction that $\mathcal{M} \equiv \mathcal{N}$ but $\mathcal{M} \not\cong \mathcal{N}$. Hence no bijection $\pi: M \to N$ is an isomorphism $\mathcal{M} \to \mathcal{N}$. By the definition of isomorphism of $\mathcal{L}$-structures, the failure of a bijection $\pi: M \to N$ being an isomorphism is witnessed by finitely many of the relation symbols, function symbols, and constant symbols of $\mathcal{L}$: π is not an isomorphism if and only if there exists a relation symbol R of $\mathcal{L}$ and $a_1, \ldots, a_n \in M$ with $R^{\mathcal{M}}(a_1, \ldots, a_n)$, but not $R^{\mathcal{N}}(\pi(a_1), \ldots, \pi(a_n))$, or if there exists a function symbol f of $\mathcal{L}$ and $a_1, \ldots, a_n \in M$ with $\pi(f^{\mathcal{M}}(a_1, \ldots, a_n)) \neq f^{\mathcal{N}}(\pi(a_1), \ldots, \pi(a_n))$, or a constant symbol c with $\pi(c^{\mathcal{M}}) \neq c^{\mathcal{N}}$. Hence for any bijection $\pi: M \to N$ there exists a finite sublanguage $\mathcal{L}'$ of $\mathcal{L}$ such that π is not an isomorphism of the $\mathcal{L}'$-structures $\mathcal{M}|\mathcal{L}'$ and $\mathcal{N}|\mathcal{L}'$. Note that this implies that for any finite sublanguage $\mathcal{L}''$ of $\mathcal{L}$ which contains $\mathcal{L}'$, π will also not be an isomorphism of the $\mathcal{L}'$-structures $\mathcal{M}|\mathcal{L}''$ and $\mathcal{N}|\mathcal{L}''$. Since there are only finitely many bijections $\pi: M \to N$ (since M and N are finite sets), this means that there is some finite subset $\mathcal{L}''$ of $\mathcal{L}$ such that $\mathcal{M}|\mathcal{L}''$ and $\mathcal{N}|\mathcal{L}''$ are not isomorphic. But since $\mathcal{M} \equiv \mathcal{N}$, we clearly have $\mathcal{M}|\mathcal{L}'' \equiv \mathcal{N}|\mathcal{L}''$. Hence by (b) we get $\mathcal{M}|\mathcal{L}'' \cong \mathcal{N}|\mathcal{L}''$, a contradiction.

6. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures with $\mathcal{M} \equiv \mathcal{N}$. Show (without using the Keisler-Shelah theorem) that there exists an ultrafilter $\mathcal{U}$ on some index set I and an elementary embedding $\mathcal{N} \to \mathcal{M}^I/\mathcal{U}$.

Solution. We let $T = \mathrm{Diag}_{\mathrm{el}}(\mathcal{N})$, an $\mathcal{L}(N)$-theory. We claim that for every finite non-empty subset $\Phi = \{\varphi_1, \ldots, \varphi_m\}$ of T we can expand $\mathcal{M}$ to a model $\mathcal{M}_\Phi$ of Φ. To see this, write $\varphi_i = \psi_i(c)$ for each i, where $c = (c_1, \ldots, c_n)$ are new constant symbols in $\mathcal{L}(N) \setminus \mathcal{L}$ and the $\psi_i(x)$ are $\mathcal{L}$-formulas, $x = (x_1, \ldots, x_n)$. Now clearly $\mathcal{N}$ satisfies the $\mathcal{L}$-sentence $\sigma := \exists x(\psi_1 \wedge \cdots \wedge \psi_m)$ (the elements of N corresponding to the $c_1, \ldots, c_n$ serve as witnesses). Hence $\mathcal{M} \models \sigma$, since $\mathcal{M} \equiv \mathcal{N}$ by assumption. Interpreting the c_j by witnesses for $\psi_1 \wedge \cdots \wedge \psi_m$ in M, and the other constant symbols in $\mathcal{L}(N) \setminus \mathcal{L}$ arbitrarily, yields an expansion of $\mathcal{M}$ to a model of the $\mathcal{L}(N)$-theory Φ as claimed. Now as in the proof of the Compactness Theorem we obtain an ultrafilter $\mathcal{U}$ on the set I of finite non-empty subsets of T such that the $\mathcal{L}(N)$-structure $\mathcal{M}^* := \prod_{\Phi \in I} \mathcal{M}_\Phi/\mathcal{U}$ is a model of T. The reduct of each $\mathcal{M}_\Phi$ to $\mathcal{L}$ is $\mathcal{M}$. Hence the reduct of $\mathcal{M}^*$ to $\mathcal{L}$ is $\mathcal{M}^I/\mathcal{U}$. By the Diagram Lemma it follows that there exists an elementary embedding $\mathcal{N} \to \mathcal{M}^I/\mathcal{U}$.