# Problem Set 3

Due March 5

*Model Theory*

Math 506, Spring 2004.

1. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A$ a non-empty subset of its domain. Show that the set

$$B := \{t^{\mathcal{M}}(a_1, ..., a_n) : t(x_1, ..., x_n)\ \mathcal{L}\text{-term}, n \in \mathbb{N}, a_1, ..., a_n \in A\}$$

contains $A$ and is the domain of a substructure of $\mathcal{M}$, called the **substructure of $\boldsymbol{\mathcal{M}}$ generated by $\boldsymbol{A}$** and denoted by $\langle A\rangle_{\mathcal{M}}$. Show that $\langle A\rangle_{\mathcal{M}}$ is the smallest substructure of $\mathcal{M}$ which contains $A$, that is, whenever $\mathcal{C}$ is a substructure of $\mathcal{M}$ whose domain $C$ contains $A$ as a subset, then $\mathcal{C} \supseteq \langle A\rangle_{\mathcal{M}}$.

**Solution.** Clearly $B$ contains $A$, since $a = t^{\mathcal{M}}(a)$ for the $\mathcal{L}$-term $t(x) = x$. Similarly we have $c^{\mathcal{M}} \in B$ for every constant symbol $c$ of $\mathcal{L}$ (use the $\mathcal{L}$-term $t = c$). Let $f$ be an $m$-ary function symbol of $\mathcal{L}$ and $b_1, ..., b_m \in B$. Then there are $a_1, ..., a_n \in A$ and $\mathcal{L}$-terms $t_1(x_1, ..., x_n), ..., t_m(x_1, ..., x_n)$ such that $b_i = t_i^{\mathcal{M}}(a_1, ..., a_n)$ for all $i$. Hence $f^{\mathcal{M}}(b_1, ..., b_m) = s^{\mathcal{M}}(a_1, ..., a_n) \in B$, where $s$ is the $\mathcal{L}$-term $s(x_1, ..., x_n) = f(t_1(x_1, ..., x_n), ..., t_m(x_1, ..., x_n))$. Therefore by setting

$$\begin{aligned} f^{\langle A\rangle_{\mathcal{M}}} &:= f^{\mathcal{M}}|B^m && \text{for an } m\text{-ary function symbol } f \text{ of } \mathcal{L}, \\ c^{\langle A\rangle_{\mathcal{M}}} &:= c^{\mathcal{M}} && \text{for a constant symbol } c \text{ of } \mathcal{L}, \\ R^{\langle A\rangle_{\mathcal{M}}} &:= R^{\mathcal{M}} \cap B^m && \text{for an } m\text{-ary relation symbol } R \text{ of } \mathcal{L} \end{aligned}$$

we obtain an $\mathcal{L}$-substructure $\langle A\rangle_{\mathcal{M}}$ of $\mathcal{M}$ with domain $B$. Now let $\mathcal{C}$ be any substructure of $\mathcal{M}$ whose domain contains $A$ as a subset. Then $t^{\mathcal{M}}(c_1, ..., c_n) \in C$ for all $\mathcal{L}$-terms $t(x_1, ..., x_n)$ and all $c_1, ..., c_n \in C$. In particular $t^{\mathcal{M}}(a_1, ..., a_n) \in C$ for all $\mathcal{L}$-terms $t(x_1, ..., x_n)$ and all $a_1, ..., a_n \in A$, hence $B \subseteq C$. Therefore $\mathcal{C} \supseteq \langle A\rangle_{\mathcal{M}}$.

2. Let $\mathcal{L} = \{0, 1, +, -, \cdot, <\}$ and $\mathcal{Q} = (\mathbb{Q}, 0, 1, +, -, \cdot, <)$ (the ordered field $\mathbb{Q}$).

   a) Show that for every subset $S$ of $\mathbb{Q}$ definable in $\mathcal{Q}$ by a quantifier-free $\mathcal{L}$-formula there exists $q \in \mathbb{Q}$ such that $(q, \infty) \subseteq S$ or $(q, \infty) \cap S = \emptyset$.

   b) Use (a) to show that $\mathrm{Th}(\mathcal{Q})$ does *not* admit quantifier elimination.

**Solution.**

a) Let $\mathfrak{S}$ be the collection of all subsets $S \subseteq \mathbb{Q}$ such that there exists $q \in \mathbb{Q}$ such that $(q, \infty) \subseteq S$ or $(q, \infty) \cap S = \emptyset$. We will show that all quantifier-free definable subsets of $\mathbb{Q}$ belong to $\mathfrak{S}$. Let first $\varphi(x)$ be an atomic $\mathcal{L}$-formula. Then $\varphi(x)$ is equivalent (in $\mathrm{Th}(\mathcal{Q})$) to a formula of the form "$P(x) = 0$" or "$P(x) < 0$" for some polynomial $P(X) \in \mathbb{Z}[X]$. In the first case, $\varphi(x)$ defines a finite set (if $P \neq 0$) or $\mathbb{Q}$ (if $P = 0$), hence the set defined by $\varphi(x)$ belongs to $\mathfrak{S}$. In the second case, the set $A := \{r \in \mathbb{R} : P(r) < 0\}$ is a finite union of open intervals in $\mathbb{R}$, hence the set $A \cap \mathbb{Q}$ defined by $\varphi(x)$ belongs to $\mathfrak{S}$. Observe that by the symmetry in the definition of $\mathfrak{S}$, we have that $S \in \mathfrak{S} \Rightarrow \mathbb{Q} \setminus S \in \mathfrak{S}$. Now suppose $S_1, S_2 \in \mathfrak{S}$, and let $q_i \in \mathbb{Q}$ such that $(q_i, \infty) \subseteq S_i$ or $(q_i, \infty) \cap S_i = \emptyset$ for $i = 1, 2$. We distinguish two cases:

   i. For some $i \in \{1, 2\}$ we have $(q_i, \infty) \subseteq S_i$. Then $(q_i, \infty) \subseteq S_1 \cup S_2$ and thus $S_1 \cup S_2 \in \mathfrak{S}$.

ii. For all $i \in \{1, 2\}$ we have $(q_i, \infty) \subseteq \mathbb{Q} \setminus M_i$. Then we have, with $q := \max\{q_1, q_2\}$:

$$(q, \infty) \subseteq (\mathbb{Q} \setminus S_1) \cap (\mathbb{Q} \setminus S_2) = \mathbb{Q} \setminus (S_1 \cup S_2),$$

so $S_1 \cup S_2 \in \mathfrak{S}$.

It follows that $\mathfrak{S}$ contains all quantifier-free definable subsets of $\mathbb{Q}$, as claimed.

b) The subset $S := \{q^2 : q \in \mathbb{Q}\}$ of $\mathbb{Q}$ is definable in $\mathcal{Q}$ by the formula $\varphi(x) = \exists y(y^2 = x)$. By (a), $S$ cannot be defined by a quantifier-free $\mathcal{L}$-formula. Hence $\mathrm{Th}(\mathcal{Q})$ does not admit quantifier elimination.

3. Let $K$ be a field and let $K^{\mathrm{alg}}$ be an algebraic closure of $K$. A polynomial $f \in K[X_1, ..., X_n]$ is called **absolutely irreducible** if $f$ is irreducible when considered as an element of the polynomial ring $K^{\mathrm{alg}}[X_1, ..., X_n]$. (For example, the polynomial $X^2 + 1 \in \mathbb{Q}[X]$ is not absolutely irreducible, since $X^2 + 1 = (X - i)(X + i)$ in $\mathbb{Q}^{\mathrm{alg}}[X]$, whereas $XY - 1 \in \mathbb{Q}[X, Y]$ is absolutely irreducible.) For a polynomial $f \in \mathbb{Z}[X_1, ..., X_n]$ and a prime number $p$ we denote by $f_p$ the polynomial in $\mathbb{F}_p[X_1, ..., X_n]$ obtained by reducing the coefficients of $f$ modulo $p$. Show: $f \in \mathbb{Z}[X_1, ..., X_n]$ is absolutely irreducible if and only if $f_p$ is absolutely irreducible for all but finitely many primes $p$. (This fact is known as the Noether-Ostrowski Irreducibility Theorem.)

**Solution.** We let $\lambda$, $\mu$, $\nu$ range over $\mathbb{N}^n$, and for $\nu = (\nu_1, ..., \nu_n) \in \mathbb{N}^n$ we write $|\nu| := \nu_1 + \cdots + \nu_n$ and $X^\nu = X_1^{\nu_1} \cdots X_n^{\nu_n}$. Let $f \in \mathbb{Z}[X_1, ..., X_n]$ be a polynomial of degree $d$. Write $f = \sum_\nu a_\nu X^\nu$ with $a_\nu \in \mathbb{Z}$. We let $y = (y_\mu)_{|\mu| \leqslant d}$ and $z = (z_\lambda)_{|\lambda| \leqslant d}$ be (finite) tuples of variables. Consider the sentence $\sigma$ in the language $\mathcal{L} = \{0, 1, +, -, \cdot\}$ of rings:

$$\sigma := \forall y \forall z \left( \left( \bigwedge_{|\nu| \leqslant d} \sum_{\mu + \lambda = \nu} y_\mu z_\lambda = a_\nu \right) \rightarrow \left( \bigwedge_{|\mu| > 0} y_\mu = 0 \right) \vee \left( \bigwedge_{|\lambda| > 0} z_\lambda = 0 \right) \right)$$

Then $K \models \sigma$ if and only if the image of the polynomial $f$ under the natural homomorphism $\mathbb{Z}[X_1, ..., X_n] \rightarrow K[X_1, ..., X_n]$ is irreducible, for every field $K$. In particular $\mathbb{Q}^{\mathrm{alg}} \models \sigma$ if and only if $f$ is absolutely irreducible, and $\mathbb{F}_p^{\mathrm{alg}} \models \sigma$ if and only if $f_p$ is absolutely irreducible. The claim now follows from the Lefschetz Principle.

4. Let $T$ be an $\mathcal{L}$-theory and $T_\forall := \{\varphi : \varphi \text{ universal sentence}, T \models \varphi\}$. Show that an $\mathcal{L}$-structure $\mathcal{A}$ is a model of $T_\forall$ if and only if there exists a model $\mathcal{M}$ of $T$ such that $\mathcal{A} \subseteq \mathcal{M}$.

**Solution.** If $\mathcal{A} \subseteq \mathcal{M}$ are $\mathcal{L}$-structures and $\mathcal{M} \models T$, then every universal sentence which holds in $\mathcal{M}$ will also hold in $\mathcal{A}$. This shows the "if" direction. For the converse, suppose that the $\mathcal{L}$-structure $\mathcal{A}$ is a model of $T_\forall$. By the Diagram Lemma it is enough to show that the $\mathcal{L}(A)$-theory $T \cup \mathrm{Diag}(\mathcal{A})$ is satisfiable. By Compactness, we need to show that every finite subset of it is satisfiable. For this, let $\psi_1(x), ..., \psi_m(x)$ with $x = (x_1, ..., x_n)$ be quantifier-free $\mathcal{L}$-formulas and $a = (a_1, ..., a_n) \in A^n$ such that $\psi_i(a) \in \mathrm{Diag}(\mathcal{A})$ for every $i$. It suffices to show that $T \cup \{\psi_1(a), ..., \psi_m(a)\}$ has a model. Suppose not. Then $T \models \forall x(\neg\psi_1(x) \vee \cdots \vee \neg\psi_m(x))$. Now $\varphi := \forall x(\neg\psi_1(x) \vee \cdots \vee \neg\psi_m(x))$ is a universal sentence in $T_\forall$, hence $\mathcal{A} \models \varphi$. Therefore $\mathcal{A} \models \neg\psi_i(a)$ for some $i$, a contradiction to $\psi_i(a) \in \mathrm{Diag}(\mathcal{A})$.

5. We say that an $\mathcal{L}$-structure $\mathcal{M}$ is **existentially closed** in an $\mathcal{L}$-structure $\mathcal{N}$ with $\mathcal{M} \subseteq \mathcal{N}$ if for every quantifier-free $\mathcal{L}$-formula $\varphi(x, y)$ with $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_m)$, $\psi(y) = \exists x(\varphi(x, y))$, and $a = (a_1, ..., a_m) \in M^m$, we have $\mathcal{N} \models \psi(a) \Rightarrow \mathcal{M} \models \psi(a)$. (It is enough that this holds for $n = 1$.) Let $K$ be an infinite field and $t$ an indeterminate over $K$. Show that $K$ is existentially closed in $K[t]$ (in the language $\mathcal{L} = \{0, 1, +, \cdot, -\}$).

**Solution.** Every quantifier-free formula $\varphi(x, y)$ in the language $\mathcal{L} = \{0, 1, +, \cdot, -\}$ is equivalent (in the theory of integral domains) to a disjunction of formulas of the form

$$P_1(x, y) = 0 \wedge \cdots \wedge P_k(x, y) = 0 \wedge Q(x, y) \neq 0 \qquad (1)$$

where the $P_j(x, y)$ and $Q(x, y)$ are polynomials with integer coefficients. If $K[t] \models \varphi(a, b(t))$ for some $a \in K^n$ and $b(t) = (b_1(t), ..., b_m(t)) \in K[t]^m$ then one of this disjuncts has to hold when $a$ and $b(t)$ are substituted for $x$ and $y$, respectively. Hence we may assume that our formula $\varphi(x, y)$ has the form (1). Then each $P_j(a, b(t)) \in K[t]$ is the zero polynomial in $K[t]$, and $Q(a, b(t)) \neq 0$ in $K[t]$. A non-zero polynomial in one indeterminate has only finitely many zeros in $K$. Hence, since $K$ is infinite, there is $\tau \in K$ such that $Q(a, b(\tau)) \neq 0$, and thus $K \models \varphi(a, b(\tau))$.

6. Let $(I, <)$ be a totally ordered set, $I \neq \emptyset$, and suppose that for every $i \in I$ we are given an $\mathcal{L}$-structure $\mathcal{M}_i$. We say that $(\mathcal{M}_i)_{i \in I}$ is a **chain** of $\mathcal{L}$-structures if $\mathcal{M}_i \subseteq \mathcal{M}_j$ for all $i < j$ in $I$. Show that there exists a unique $\mathcal{L}$-structure $\mathcal{M} := \bigcup_{i \in I} \mathcal{M}_i$ (the **union** of the chain) with the following properties: $\mathcal{M}_i \subseteq \mathcal{M}$ for all $i$, and if $\mathcal{N}$ is any $\mathcal{L}$-structure with $\mathcal{M}_i \subseteq \mathcal{N}$ for all $i$, then $\mathcal{M} \subseteq \mathcal{N}$.

   **Solution.** We define an $\mathcal{L}$-structure $\mathcal{M}$ as follows. Its universe will be the union $M := \bigcup_i M_i$ of the universes of the $M_i$. If $c$ is a constant symbol in $\mathcal{L}$ then $c^{\mathcal{M}_i} = c^{\mathcal{M}_j}$ for all $i, j \in I$; we put $c^{\mathcal{M}} := c^{\mathcal{M}_i}$ where $i$ is arbitrary. Suppose that $a_1, ..., a_n \in M$ and $f$ is an $n$-ary function symbol of $\mathcal{L}$. Since $I$ is linearly ordered we find $i \in I$ such that $a_1, ..., a_n \in M_i$. If $i < j$ then $f^{\mathcal{M}_i}(a_1, ..., a_n) = f^{\mathcal{M}_j}(a_1, ..., a_n)$, hence there is a well-defined function $f^{\mathcal{M}}: M^n \to M$ whose graph is the union of the graphs of the $f^{\mathcal{M}_i}$. Similarly, if $R$ is an $n$-ary relation symbol of $\mathcal{L}$ and $i < j$ then $(a_1, ..., a_n) \in R^{\mathcal{M}_i}$ if and only if $(a_1, ..., a_n) \in R^{\mathcal{M}_j}$. We let $R^{\mathcal{M}} := \bigcup_{i \in I} R^{\mathcal{M}_i}$. It is easy to check that the structure thus defined has the required properties.

7. An $\mathcal{L}$-theory $T$ is a **∀∃-theory** if $T$ consists only of sentences of the form $\forall x_1 \cdots \forall x_n \exists y_1 \cdots \exists y_m(\varphi)$, where $\varphi(x_1, ..., x_n, y_1, ..., y_m)$ is a quantifier-free $\mathcal{L}$-formula. Show that the union of a chain of models of a ∀∃-theory $T$ is again a model of $T$.

   **Solution.** Let $(\mathcal{M}_i)_{i \in I}$ be a chain of $\mathcal{L}$-structures and $\mathcal{M} = \bigcup_{i \in I} \mathcal{M}_i$. Let $\varphi(x_1, ..., x_n, y_1, ..., y_m)$ be a quantifier-free $\mathcal{L}$-formula such that $\mathcal{M}_i \models \forall x_1 \cdots \forall x_n \exists y_1 \cdots \exists y_m(\varphi)$ for all $i$. We need to show that $\mathcal{M} \models \forall x_1 \cdots \forall x_n \exists y_1 \cdots \exists y_m(\varphi)$. For this let $a_1, ..., a_n \in M$. Then there is $i_k \in I$ with $a_k \in M_{i_k}$ for every $k$. Let $i = \max \{i_1, ..., i_n\}$. Then $a_k \in M_i$ for all $k$. Since $\mathcal{M}_i \models \forall x_1 \cdots \forall x_n \exists y_1 \cdots \exists y_m(\varphi)$ we find $b_1, ..., b_m \in M_i$ such that $\mathcal{M}_i \models \varphi(a_1, ..., a_n, b_1, ..., b_m)$. Hence $\mathcal{M} \models \varphi(a_1, ..., a_n, b_1, ..., b_m)$ since the formula $\varphi$ is quantifier-free. This shows $\mathcal{M} \models \forall x_1 \cdots \forall x_n \exists y_1 \cdots \exists y_m(\varphi)$ as required.