Problem Set 4

Solutions

Model Theory

Math 506, Spring 2004.

1. [Optional] We say that an $\mathcal{L}$-structure $\mathcal{M}\vDash T$ is an **existentially closed** model of T if $\mathcal{M}$ is existentially closed in every $\mathcal{N}\vDash T$ with $\mathcal{M}\subseteq\mathcal{N}$.

   a) Suppose that T is a $\forall\exists$–theory. Show that for every model $\mathcal{M}$ of T there exists a model $\mathcal{M}^*\vDash T$ with $\mathcal{M}\subseteq\mathcal{M}^*$, having the following property: for every quantifier-free $\mathcal{L}$-formula $\varphi(x, y_1, ..., y_m)$ and $a_1, ..., a_m\in M$, if there exists some $\mathcal{N}\vDash T$ with $\mathcal{M}^*\subseteq\mathcal{N}$ and $b\in N$ such that $\mathcal{N}\vDash\varphi(b, a_1, ..., a_m)$, then there exists $c\in M^*$ such that $\mathcal{M}^*\vDash\varphi(c, a_1, ..., a_m)$.

   b) Use (a) to show that for any model $\mathcal{M}$ of a $\forall\exists$-theory T there is some existentially closed $\mathcal{N}\vDash T$ with $\mathcal{M}\subseteq\mathcal{N}$ and $|N|=\max\{|M|, |\mathcal{L}|, \aleph_0\}$.

   Solution.

   a) Let $(\varphi_\lambda)_{\lambda<\kappa}$ be an enumeration of all $\mathcal{L}(M)$-formulas of the form $\exists x\varphi(x, a_1, ..., a_m)$ where $\varphi(x, y_1, ..., y_m)$ is a quantifier-free $\mathcal{L}$-formula and $a_1, ..., a_m\in M$. (Here κ is some limit ordinal.) We define recursively an increasing sequence $(\mathcal{M}_\lambda)_{\lambda<\kappa}$ of models of T as follows: $\mathcal{M}_0 := \mathcal{M}$; if λ is a limit ordinal, then we put $\mathcal{M}_\lambda := \bigcup_{\mu<\lambda}\mathcal{M}_\mu$; if there exists a model $\mathcal{N}\vDash T$ with $\mathcal{M}_\lambda\subseteq\mathcal{N}$ and $(\mathcal{N}, M)\vDash\varphi_\lambda$, then we put $\mathcal{M}_{\lambda+1} := \mathcal{N}$, and otherwise we let $\mathcal{M}_{\lambda+1} := \mathcal{M}_\lambda$. Since T is $\forall\exists$ we have $\mathcal{M}_\lambda\vDash T$ for all λ (by an earlier homework problem). Put $\mathcal{M}^* := \bigcup_{\lambda<\kappa}\mathcal{M}_\lambda$. Then $\mathcal{M}^*\vDash T$, and if $\mathcal{N}\vDash T$ with $\mathcal{M}^*\subseteq\mathcal{N}$ such that $(\mathcal{N}, M)\vDash\varphi_\lambda$, then $\mathcal{M}_\lambda\subseteq\mathcal{N}$ and hence $(\mathcal{M}_{\lambda+1}, M)\vDash\varphi_\lambda$ by construction; therefore $(\mathcal{M}^*, M)\vDash\varphi_\lambda$. This shows that $\mathcal{M}^*$ has the required property. Note that $|M^*|=\max\{|M|, |\mathcal{L}|, \aleph_0\}$.

   b) We define an increasing chain $(\mathcal{M}_n)_{n\in\mathbb{N}}$ of models of T as follows: put $\mathcal{M}_0 := \mathcal{M}$ and $\mathcal{M}_{n+1} := \mathcal{M}_n^*$ for all n. Let $\mathcal{N} := \bigcup_n\mathcal{M}_n$, a model of T (since T is $\forall\exists$) and $|N| = \max\{|M|, |\mathcal{L}|, \aleph_0\}$. We claim that $\mathcal{N}$ is an existentially closed model of T. To see this, let $\mathcal{N}'\supseteq\mathcal{N}$ be a model of T and φ an existential $\mathcal{L}(N)$-sentence with one existential quantifier, such that $(\mathcal{N}', N)\vDash\varphi$. There is some n such that φ is a sentence of $\mathcal{L}(M_n)$ (since only finitely many constant symbols from N appear in φ), so $(\mathcal{N}', M_n)\vDash\varphi$. By choice of $\mathcal{M}_{n+1}$ (cf. (a)!) we have $(\mathcal{M}_{n+1}, M_n)\vDash\varphi$ and hence $(\mathcal{N}, N)\vDash\varphi$, as required.

2. Let T be an $\mathcal{L}$-theory.

   a) Show that if $\mathcal{M}\subseteq\mathcal{N}$ are models of T and $\mathcal{M}$ is an existentially closed model of T, then there is $\mathcal{M}_1\vDash T$ such that $\mathcal{M}\subseteq\mathcal{N}\subseteq\mathcal{M}_1$ with $\mathcal{M}\preccurlyeq\mathcal{M}_1$. (Hint: Diagram Lemma.)

   b) Show that T is model-complete if and only if every model of T is existentially closed. (Hint for "$\Leftarrow$": suppose that $\mathcal{M}_0\subseteq\mathcal{N}_0$ are models of T; use (a) to build a chain $\mathcal{M}_0\subseteq\mathcal{N}_0\subseteq\mathcal{M}_1\subseteq\mathcal{N}_1\subseteq\mathcal{M}_2\subseteq\cdots$ of models of T such that $\mathcal{M}_i\preccurlyeq\mathcal{M}_{i+1}$ and $\mathcal{N}_i\preccurlyeq\mathcal{N}_{i+1}$.)

   Solution.

   a) By the Diagram Lemma it suffices to show that the $\mathcal{L}(N)$-theory $\mathrm{Diag_{el}}(\mathcal{M})\cup\mathrm{Diag}(\mathcal{N})$ is satisfiable. For this, let $\varphi(x, y)$ be a quantifier-free $\mathcal{L}$-formula, $x=(x_1, ..., x_n)$, $y=(y_1, ..., y_m)$, and $a\in M^n$, $b\in(N\setminus M)^m$, with $\mathcal{N}\vDash\varphi(a, b)$. Then $\mathcal{N}\vDash\exists y(\varphi(a,$

$y)$), hence $\mathcal{M} \models \exists y(\varphi(a, y))$, so $\mathcal{M} \models \varphi(a, c)$ for some $c \in M^m$. This shows that the $\mathcal{L}(M)$-structure $(\mathcal{M}, M)$ can be expanded to a model of $\mathrm{Diag}_{\mathrm{el}}(\mathcal{M}) \cup \{\varphi(a, b)\}$. Hence the $\mathcal{L}(N)$-theory $\mathrm{Diag}_{\mathrm{el}}(\mathcal{M}) \cup \mathrm{Diag}(\mathcal{N})$, being finitely satisfiable, is satisfiable, by compactness.

b) The forward direction is trivial. For the converse, suppose that $\mathcal{M}_0 \subseteq \mathcal{N}_0$ are models of T. As in the hint we use (a) to build a chain $\mathcal{M}_0 \subseteq \mathcal{N}_0 \subseteq \mathcal{M}_1 \subseteq \mathcal{N}_1 \subseteq \mathcal{M}_2 \subseteq \cdots$ of models of T such that $\mathcal{M}_i \preccurlyeq \mathcal{M}_{i+1}$ and $\mathcal{N}_i \preccurlyeq \mathcal{N}_{i+1}$. Let $\mathcal{M} = \bigcup_i \mathcal{M}_i$. Note that then $\mathcal{M} = \bigcup_i \mathcal{N}_i$. By Proposition 2.3.11 in the textbook we have $\mathcal{M}_0 \preccurlyeq \mathcal{M}$ and $\mathcal{N}_0 \preccurlyeq \mathcal{M}$. Hence $\mathcal{M}_0 \preccurlyeq \mathcal{N}_0$ as required.

3. Let T be an $\mathcal{L}$-theory.

a) Show that T admits quantifier-elimination if and only if for all $\mathcal{M} \models T_\forall$ the $\mathcal{L}(M)$-theory $T \cup \mathrm{Diag}(\mathcal{M})$ is complete.

b) Show that T is model-complete if and only if for all $\mathcal{M} \models T$, the $\mathcal{L}(M)$-theory $T \cup \mathrm{Diag}(\mathcal{M})$ is complete. (This explains the origin of the term "model-complete.")

Solution.

a) A model of the $\mathcal{L}(M)$-theory $T \cup \mathrm{Diag}(\mathcal{M})$ is essentially a model of T which contains $\mathcal{M}$ as a submodel (by the Diagram Lemma). Using this observation, the q.e. test discussed in class easily translates into the criterion given here. (Because of this characterization of q.e., some people speak of a "substructure complete theory" when they mean "a theory which admits q.e.".)

b) By definition, T is model-complete if $\mathcal{M} \subseteq \mathcal{N} \Rightarrow \mathcal{M} \preccurlyeq \mathcal{N}$ for all models $\mathcal{M}$ and $\mathcal{N}$ of T. Therefore if T is model-complete, then $T \cup \mathrm{Diag}(\mathcal{M})$ is complete for all $\mathcal{M} \models T$ (using the observation made in part (a) above). Conversely, suppose that the $\mathcal{L}(M)$-theory $T \cup \mathrm{Diag}(\mathcal{M})$ is complete, and let $\mathcal{M} \subseteq \mathcal{N}$ be models of T. Then both $(\mathcal{M}, M)$ and $(\mathcal{N}, M)$ are models of $T \cup \mathrm{Diag}(\mathcal{M})$, hence $(\mathcal{M}, M) \equiv (\mathcal{N}, M)$. This means that $\mathcal{M} \preccurlyeq \mathcal{N}$.

4. An $\mathcal{L}$-structure $\mathcal{M}$ is called **ultra-homogeneous** if every isomorphism between finitely generated substructures of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$.

a) Let $\mathcal{M}$ be a finite $\mathcal{L}$-structure. Show that $\mathrm{Th}(\mathcal{M})$ admits quantifier-elimination if and only if $\mathcal{M}$ is ultra-homogeneous.

b) Show that the finite abelian group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ (construed as a structure in the language $\mathcal{L} = \{0, +, -\}$) is not ultra-homogeneous.

Solution.

a) Suppose first that $T = \mathrm{Th}(\mathcal{M})$ has q.e., and let $\mathcal{A}$ and $\mathcal{B}$ be isomorphic substructures of $\mathcal{M}$, say with isomorphism $\sigma\colon \mathcal{A} \to \mathcal{B}$. We can expand $\mathcal{M}$ to an $\mathcal{L}(A)$-structure in two ways: by interpreting (the constant symbol corresponding to) a as a, for all $a \in A$, or by interpreting a as $\sigma(a)$, for all $a \in A$. The first structure is just $\mathcal{M}^* := (\mathcal{M}, A)$; we denote the other one by $\mathcal{M}^{**}$. Note that then both $\mathcal{M}^*$ and $\mathcal{M}^{**}$ are models of the $\mathcal{L}(A)$-theory $T \cup \mathrm{Diag}(\mathcal{A})$. Since T has q.e., this yields $\mathcal{M}^* \equiv \mathcal{M}^{**}$ by Problem 3.(a). Since these are finite $\mathcal{L}(A)$-structures, there exists an isomorphism $h\colon \mathcal{M}^* \to \mathcal{M}^{**}$. (By Problem 5.(d) on Homework Set 2.) Then h is an automorphism of $\mathcal{M}$ which extends σ, as required. Conversely, suppose that $\mathcal{M}$ is ultra-homogeneous, and let $\mathcal{N} \models T$ and let $\mathcal{A}$ be a common substructure of $\mathcal{M}$ and $\mathcal{N}$. Since $\mathcal{M} \equiv \mathcal{N}$ and $\mathcal{M}$ is finite, there exists an isomorphism $\sigma\colon \mathcal{N} \to \mathcal{M}$. Then $\sigma|\mathcal{A}\colon \mathcal{A} \to \mathcal{B}$ is an isomorphism between the substructures $\mathcal{A}$ and $\mathcal{B} := \sigma(\mathcal{A})$ of $\mathcal{M}$. By ultrahomogeneity there exists an automorphism h of $\mathcal{M}$ which extends σ. Then $\sigma^{-1} \circ h$ is an isomorphism $(\mathcal{M}, A) \to (\mathcal{N}, A)$ of $\mathcal{L}(A)$-structures. Thus $(\mathcal{M}, A) \equiv (\mathcal{N}, A)$, showing that T has q.e.

b) Consider the elements $e := (1, 0)$ and $f := (0, 1)$ of $G := \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. The isomorphism between the subgroups A and B of G generated by e and $2f$ cannot be extended to an automorphism of the group G, since $G/A \cong \mathbb{Z}/4\mathbb{Z}$ and $G/B \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ are not isomorphic.

5. Let K be a field and let $\mathcal{L}$ be the language of vector spaces over K, consisting of the binary function symbol $+$, the unary function symbol $-$, a constant symbol 0, and for each $a \in K$ a unary function symbol μ_a. We construe each K-vector space V as an $\mathcal{L}$-structure by interpreting $+$, $-$ and 0 as usual and μ_a by scalar multiplication $v \mapsto a\, v$ by a. Let T be the theory of infinite K-vector spaces in this language. Show that T admits quantifier elimination, and use this to show that T is complete.

Solution. This is very similar to the proof, done in class, that DAG has q.e. and is complete.

6. Let $\mathcal{L} = \{<\}$ and let T be a satisfiable theory containing the axioms for linearly ordered sets which contain at least two elements. Show that if T admits quantifier-elimination, then $\mathrm{Mod}(T) = \mathrm{Mod}(\mathrm{DLO})$.

Solution. By completeness of DLO it is enough to show that every model $\mathcal{A} = (A, <^{\mathcal{A}})$ of T is a model of DLO. Let $a < b$ be elements of A, and consider the $\mathcal{L}$-formula $\varphi(x, y) = \neg\exists z(x < z \wedge z < y)$. Suppose that $\mathcal{A} \models \varphi(a, b)$. Then, for all $c < d$ in a model $\mathcal{B}$ of T, we must have $\mathcal{B} \models \varphi(c, d)$, by our q.e. test from class (since the substructures of $\mathcal{A}$ and $\mathcal{B}$ with universes $\{a, b\}$ and $\{c, d\}$, respectively, are isomorphic). Therefore every model of T has only two elements. This is impossible: the $\mathcal{L}$-formula $\varphi(x) = \exists y(x < y)$ holds for a in $\mathcal{A}$, but not for b; this contradicts the q.e. criterion. Hence the ordering $<^{\mathcal{A}}$ is dense. Similarly one shows that $<^{\mathcal{A}}$ does not have endpoints.