Problem Set 4

Due March 29

Model Theory

Math 506, Spring 2004.

1. [Optional] We say that an $\mathcal{L}$-structure $\mathcal{M} \models T$ is an **existentially closed** model of T if $\mathcal{M}$ is existentially closed in every $\mathcal{N} \models T$ with $\mathcal{M} \subseteq \mathcal{N}$.

   a) Suppose that T is a $\forall\exists$–theory. Show that for every model $\mathcal{M}$ of T there exists a model $\mathcal{M}^* \models T$ with $\mathcal{M} \subseteq \mathcal{M}^*$, having the following property: for every quantifier-free $\mathcal{L}$-formula $\varphi(x, y_1, ..., y_m)$ and $a_1, ..., a_m \in M$, if there exists some $\mathcal{N} \models T$ with $\mathcal{M}^* \subseteq \mathcal{N}$ and $b \in N$ such that $\mathcal{N} \models \varphi(b, a_1, ..., a_m)$, then there exists $c \in M^*$ such that $\mathcal{M}^* \models \varphi(c, a_1, ..., a_m)$.

   b) Use (a) to show that for any model $\mathcal{M}$ of a $\forall\exists$-theory T there is some existentially closed $\mathcal{N} \models T$ with $\mathcal{M} \subseteq \mathcal{N}$ and $|N| = \max\{|M|, |\mathcal{L}|, \aleph_0\}$.

2. Let T be an $\mathcal{L}$-theory.

   a) Show that if $\mathcal{M} \subseteq \mathcal{N}$ are models of T and $\mathcal{M}$ is an existentially closed model of T, then there is $\mathcal{M}_1 \models T$ such that $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{M}_1$ with $\mathcal{M} \preccurlyeq \mathcal{M}_1$. (Hint: Diagram Lemma.)

   b) Show that T is model-complete if and only if every model of T is existentially closed. (Hint for "$\Leftarrow$": suppose that $\mathcal{M}_0 \subseteq \mathcal{N}_0$ are models of T; use (a) to build a chain $\mathcal{M}_0 \subseteq \mathcal{N}_0 \subseteq \mathcal{M}_1 \subseteq \mathcal{N}_1 \subseteq \mathcal{M}_2 \subseteq \cdots$ of models of T such that $\mathcal{M}_i \preccurlyeq \mathcal{M}_{i+1}$ and $\mathcal{N}_i \preccurlyeq \mathcal{N}_{i+1}$.)

3. Let T be an $\mathcal{L}$-theory.

   a) Show that T admits quantifier-elimination if and only if for all $\mathcal{M} \models T_\forall$ the $\mathcal{L}(M)$-theory $T \cup \mathrm{Diag}(\mathcal{M})$ is complete.

   b) Show that T is model-complete if and only if for all $\mathcal{M} \models T$, the $\mathcal{L}(M)$-theory $T \cup \mathrm{Diag}(\mathcal{M})$ is complete. (This explains the origin of the term "model-complete.")

4. An $\mathcal{L}$-structure $\mathcal{M}$ is called **ultra-homogeneous** if every isomorphism between finitely generated substructures of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$.

   a) Let $\mathcal{M}$ be a finite $\mathcal{L}$-structure. Show that $\mathrm{Th}(\mathcal{M})$ admits quantifier-elimination if and only if $\mathcal{M}$ is ultra-homogeneous.

   b) Show that the finite abelian group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ (construed as a structure in the language $\mathcal{L} = \{0, +, -\}$) is not ultra-homogeneous.

5. Let K be a field and let $\mathcal{L}$ be the language of vector spaces over K, consisting of the binary function symbol $+$, the unary function symbol $-$, a constant symbol 0, and for each $a \in K$ a unary function symbol μ_a. We construe each K-vector space V as an $\mathcal{L}$-structure by interpreting $+$, $-$ and 0 as usual and μ_a by scalar multiplication $v \mapsto a\, v$ by a. Let T be the theory of infinite K-vector spaces in this language. Show that T admits quantifier elimination, and use this to show that T is complete.

6. Let $\mathcal{L} = \{<\}$ and let T be a satisfiable theory containing the axioms for linearly ordered sets which contain at least two elements. Show that if T admits quantifier-elimination, then $\mathrm{Mod}(T) = \mathrm{Mod}(\mathrm{DLO})$.